Discrete Mathematics 1

Chapter 4: Induction and Recursions

Department of Mathematics
The FPT university

Chapter 4: Introduction

Topics covered:

Topics covered:

4.1 Mathematical Induction

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Topics covered:

4.1 Mathematical Induction

4.2 Strong Induction and Well-Ordering

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4.3 Recursive Definitions and Structural Induction

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4.3 Recursive Definitions and Structural Induction

4.4 Recursive Algorithms

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4.2 Strong Induction and Well-Ordering

4.3 Recursive Definitions and Structural Induction

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4.5 Program Correctness

4.1 Mathematical Induction

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Example 1. Prove that for all positive integers n we have
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Example 4. Let n be a positive integer. Prove that every checkerboard of size $2^n \times 2^n$ with one square removed can be titled by triominoes.

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Example 2. Prove that every postage of 12 cents or more can be formed using only 4-cent and 5-cent stamps.

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Any nonempty set of non-negative integers has a least element.

4.3 Recursive Definitions and Structural Induction

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Recursively Defined Sets and Structures

Example 1. Determine the set S defined by:

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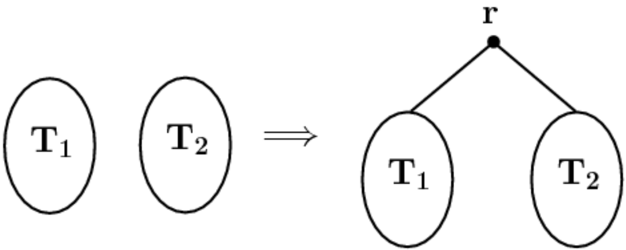

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Basic step: Prove that P is true for elements of S defined in the basic step.

Recursive step: Show that if the property P is true for the elements used to construct new elements in the recursive step of the definition of S, then the property P is also true for these new elements.

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Prove that $a_{m,n} = m + n(n+1)/2$ for all $m, n \geq 0$.

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Procedure power (n: non-negative)
if $n = 0$ then $power(0) := 1$
else $power(n) := power(n-1) * 5$

Example 2. Write a recursive algorithm to compute $n!$.

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Example 3. Write a recursive algorithm to compute the greatest common divisor of two non-negative integers.

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Recursion and Iteration

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if $n = 0$ then $y := 0$
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  x:=0
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  for $i := 1$ to $n - 1$ do
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Procedure Fib (n)
if $n = 0$ then $Fib(0) := 0$
else if $n = 1$ then $Fib(1) := 1$
else
$Fib(n) := Fib(n-1) + Fib(n-2)$

Merge Sort Algorithm

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Procedure mergesort $(L = a_1, a_2, \ldots, a_n)$
if $n > 1$ **then**

$m := \lfloor n/2 \rfloor$

$L_1 = a_1, a_2, \ldots, a_m$

$L_2 = a_{m+1}, a_{m+2}, \ldots, a_n$

$L := merge(mergesort(L_1), mergersort(L_2))$

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Theorem

The number of comparisons needed to merge sort a list of n elements is $O(n \log n)$.

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- Prove that if the program terminates then the answer is correct (this part is called partial correctness).
- Show that the program always terminates.

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- correct with respect to the initial assertion $p : x = 1$ and the final assertion $q : z = 3$

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- correct with respect to the initial assertion $p : x = 1$ and the final assertion $q : z = 3$
- is not correct with respect to the initial assertion $p : x < 5$ and the final assertion $q : z > 10$

Rules of Inference for Condition Statements

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(p \wedge \text{ condition})\{S\}q \\
(p \wedge \neg \text{ condition}) \rightarrow q \\
\hline
\therefore p\{\textbf{if } \text{condition } \textbf{then } S\}q
\end{array}
$$

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$$\frac{(p \wedge \text{ condition})\{S\}p}{\therefore p\{\textbf{while} \text{ condition } \textbf{then } S\}\ (\neg \text{ condition } \wedge p)}$$

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while condition
    S

To develope a rule of inference for this type of program, we must choose a statement that remains true each time S is executed. Such a statement is called a loop invariant.

If a loop invariant p is chosen then the rule of inference for this program segment is:

$$\frac{(p \wedge \text{ condition})\{S\}p}{\therefore\ p\{\textbf{while}\ \text{ condition } \textbf{then}\ S\}\ (\neg \text{ condition } \wedge p)}$$

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$i := 1$
factorial $:= 1$
while $i < n$
begin
    $i := i + 1$
    factorial $:=$ *factorial* $* i$
end

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Example 2. Find a loop invariant for the program segment

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$power := 1$
$i := 1$
while $i \leq n$
begin
    $power := power * x$
    $i := i + 1$
end