Discrete Mathematics 2

Chapter 9: Graphs

Department of Mathematics
The FPT university

Chapter 9: Introduction

Topics covered:

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9.1 Graphs and Graph Models

9.2 Graph Terminologies and Special Types of Graphs

9.3 Representing Graphs and Graph Isomorphism

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9.3 Representing Graphs and Graph Isomorphism

9.4 Connectivity

9.5 Euler and Hamilton Paths

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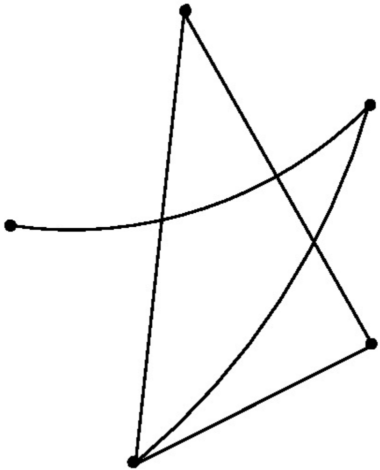

Multigraphs:

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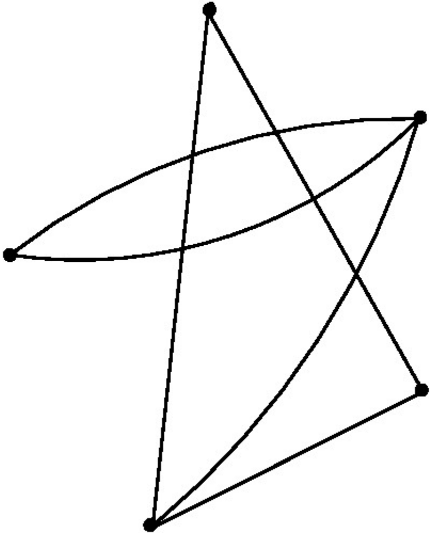

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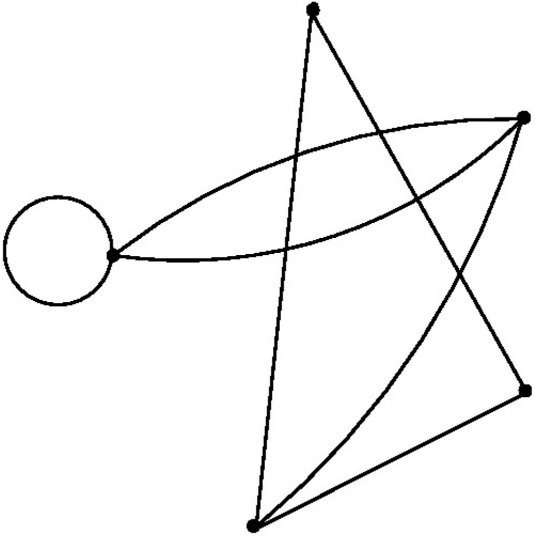

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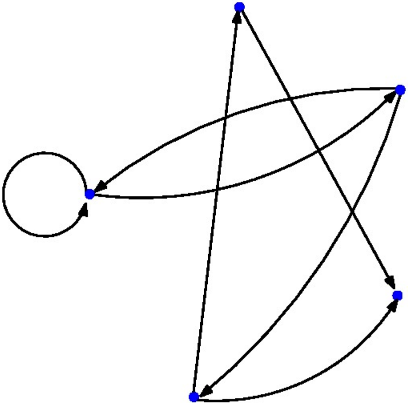

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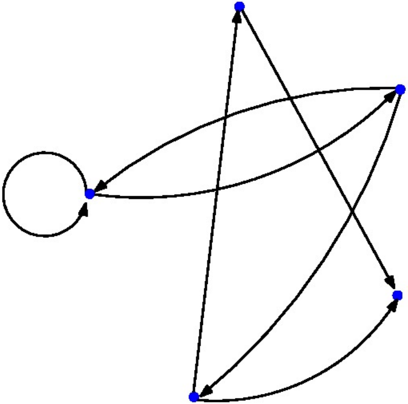

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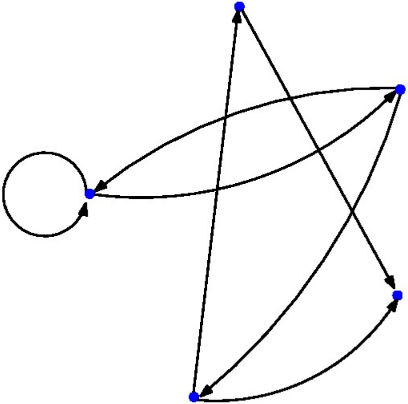

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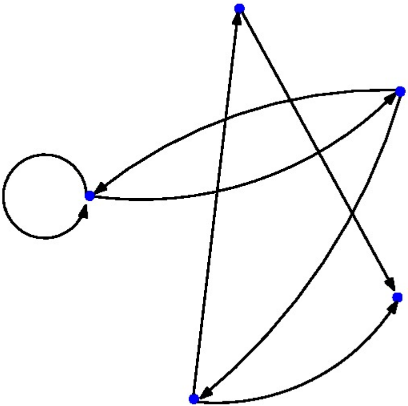

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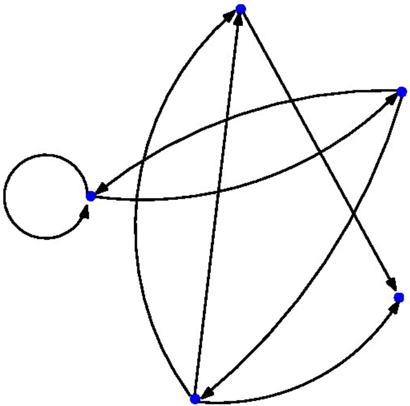

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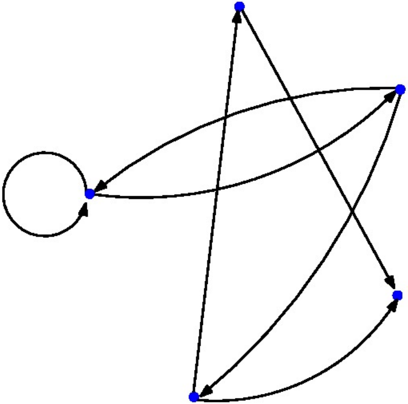

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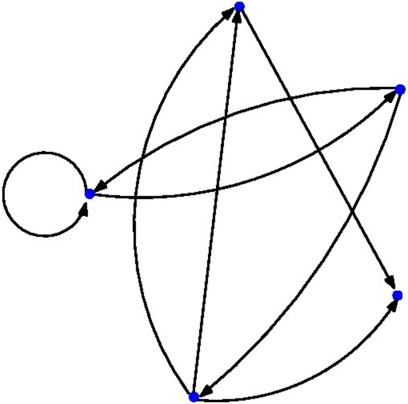

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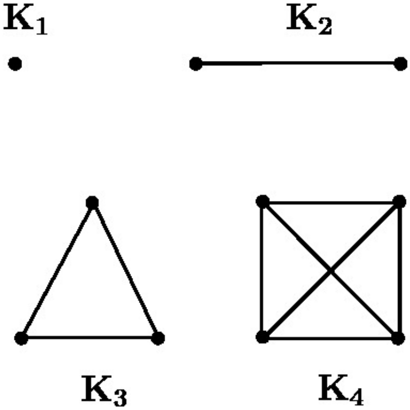

Cycles C_n, $n \geq 3$:

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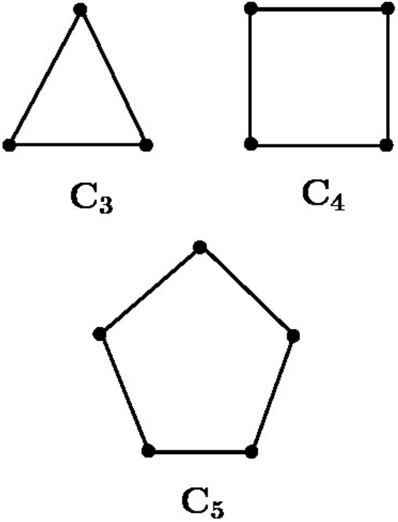

Wheels W_n, $n \geq 3$:

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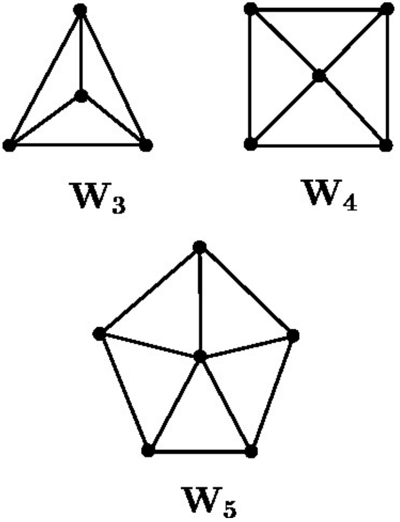

n-**Cubes** Q_n, $n \geq 1$:

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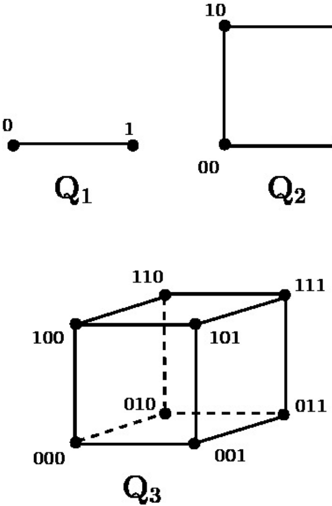

Question. How many edges each of the graphs K_n, C_n, W_n, Q_n has?

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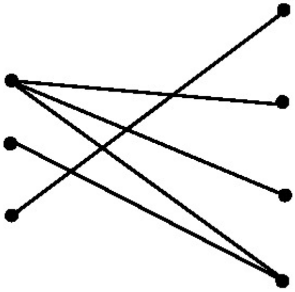

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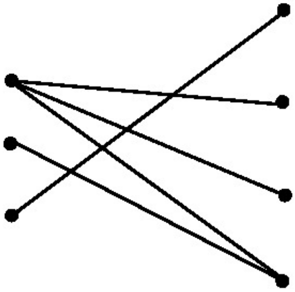

Question. Which graphs K_n, C_n, W_n, Q_n are bipartite?

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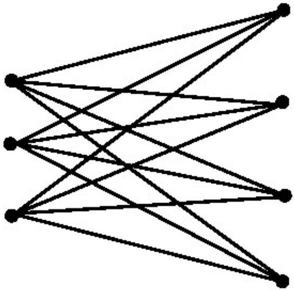

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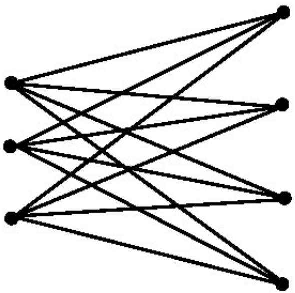

$\mathbf{K_{3,4}}$

Question. How many edges does the graph K_{mn} have?

New Graphs from Olds

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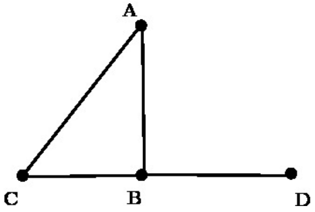

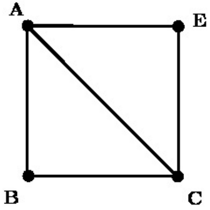

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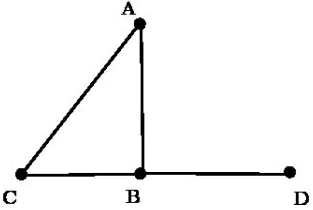

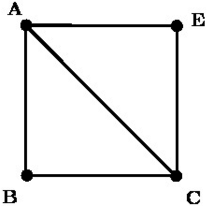

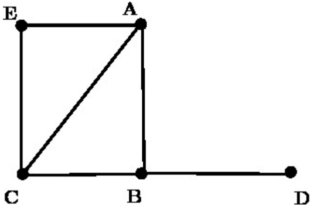

is the graph

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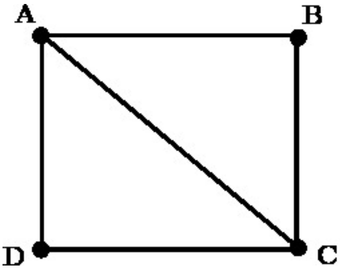
Example. Given the graph 

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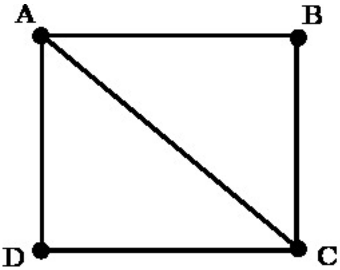

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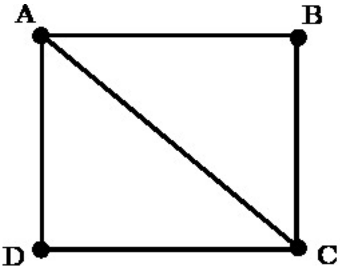

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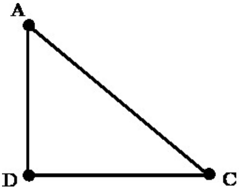

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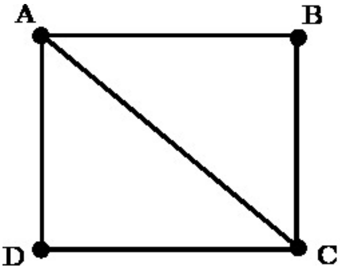

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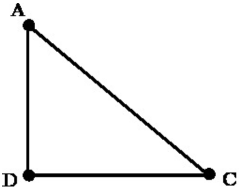

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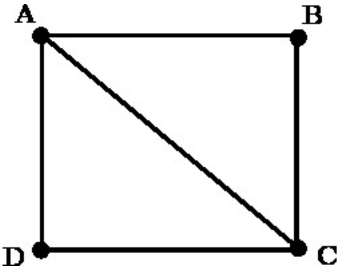

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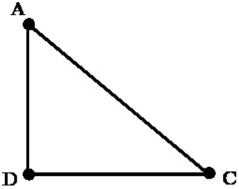

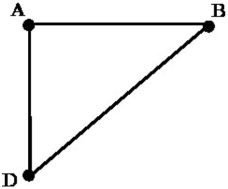

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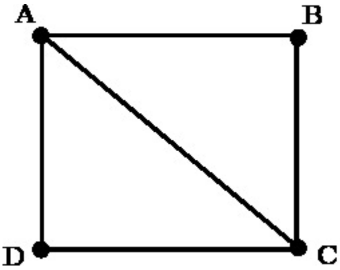

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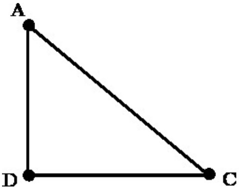

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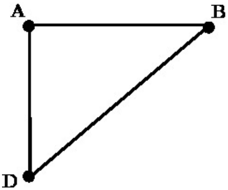

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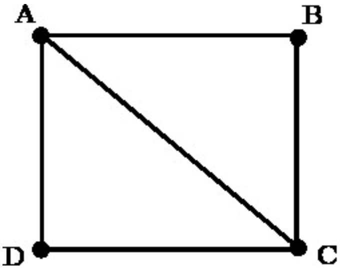

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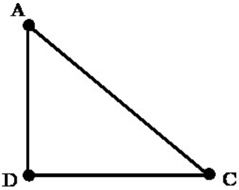

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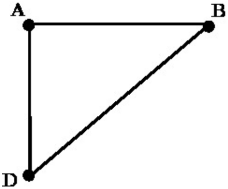

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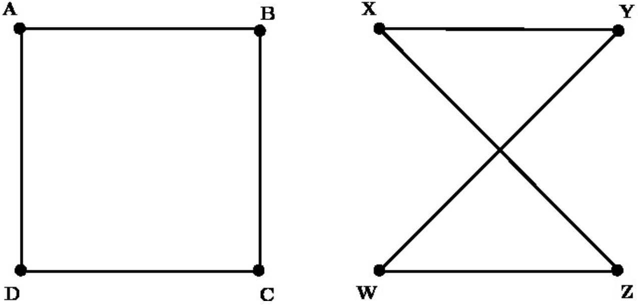

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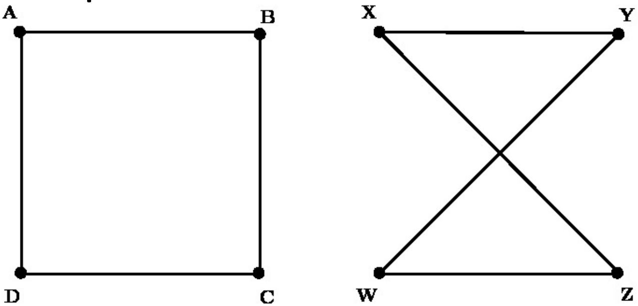

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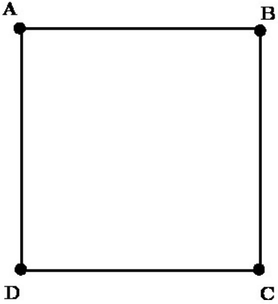

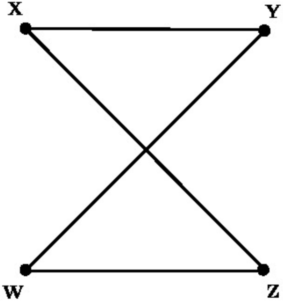

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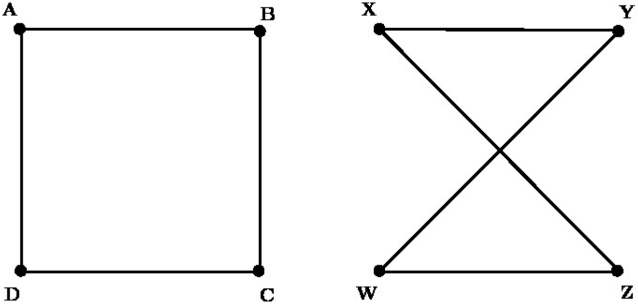

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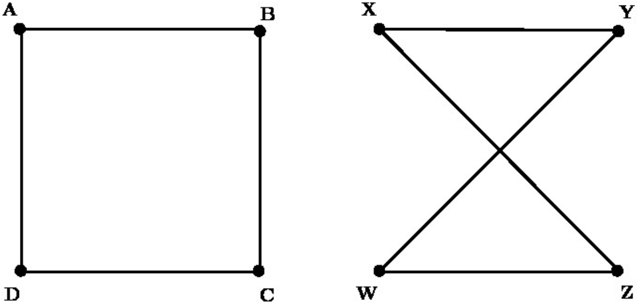

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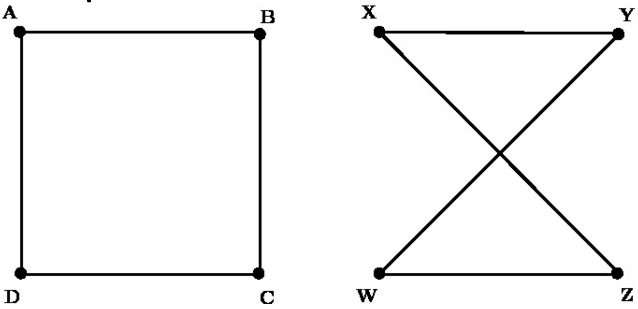

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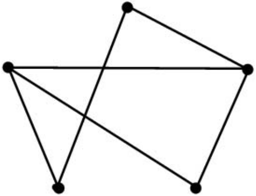

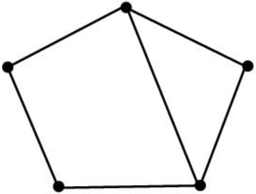

Problem.

Example 2. Determine if the two graphs are isomorphic:

Problem. Find an algorithm to check if two graphs are isomorphic.

Graph Invariants

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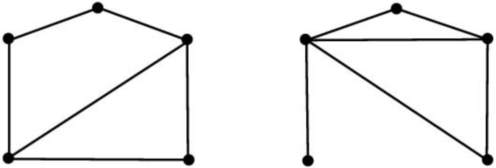

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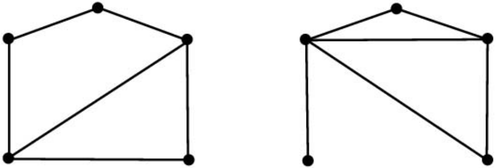

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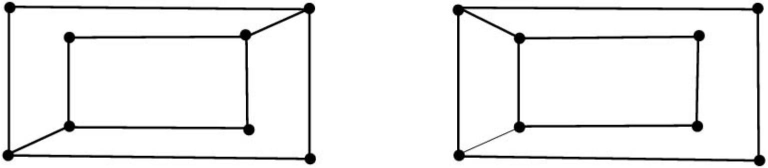

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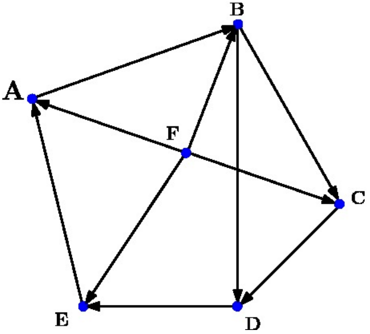

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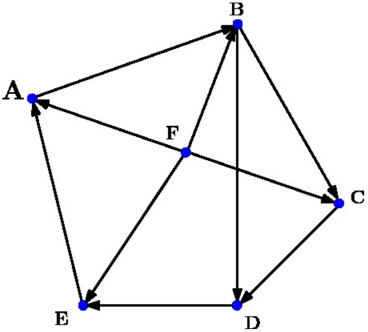

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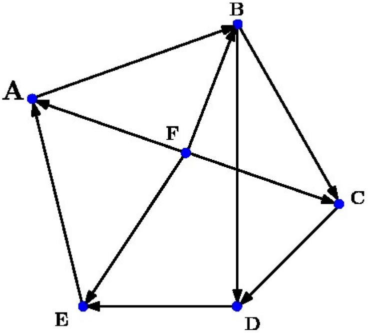

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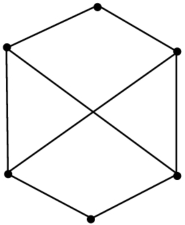
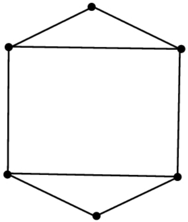
(a)  

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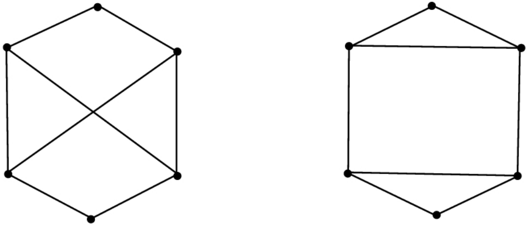

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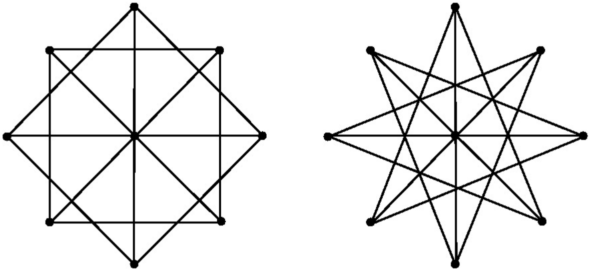

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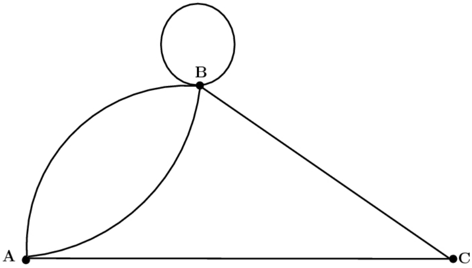

9.5 Euler and Hamilton Paths

The 7 bridges problem.

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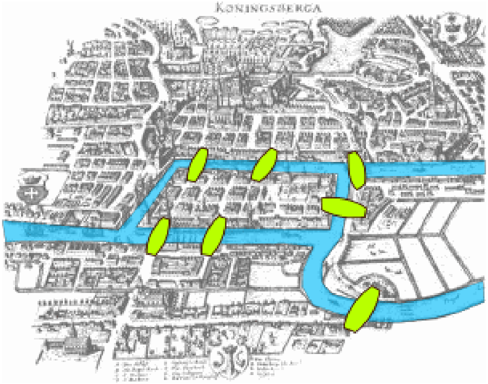

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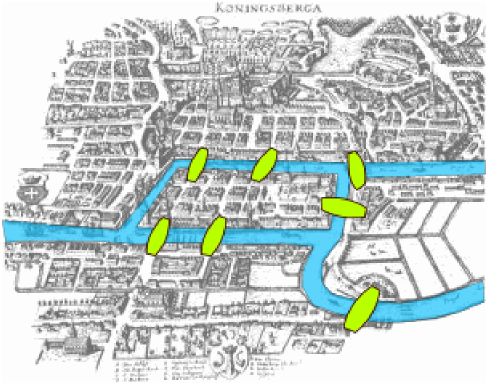

Question. Is this possible to start at some location, travel across all bridges without crossing any bridge twice, then return to the starting point?

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- A connected multigraph G has Euler circuits if and only if every vertex has even degree.

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- A connected multigraph G has Euler circuits if and only if every vertex has even degree.
- If G does not have Euler circuits, then it has Euler paths if and only if it has exactly two vertices of odd degrees.

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Conditions for existence of Euler paths and circuits

- A connected multigraph G has Euler circuits if and only if every vertex has even degree.
- If G does not have Euler circuits, then it has Euler paths if and only if it has exactly two vertices of odd degrees.

Example. The answer to the 7 bridges problem is NO.

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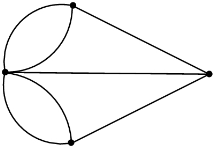

Question.

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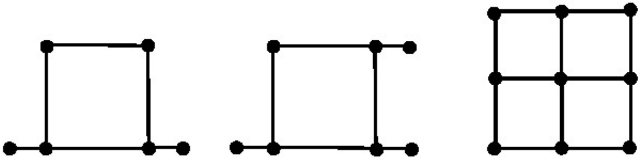

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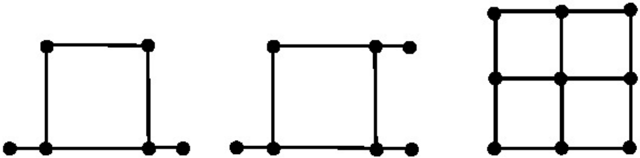

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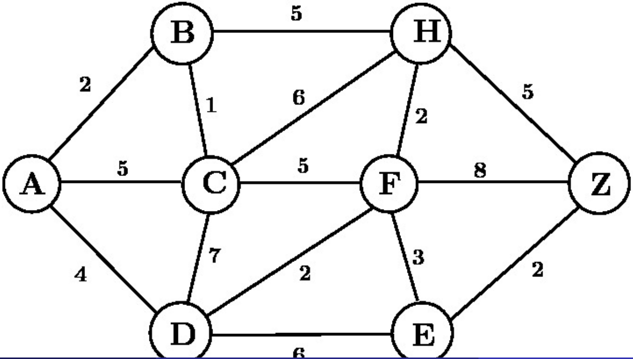

Dijkstra's Algorithm

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- Continue the process until Z is reached.

Example.

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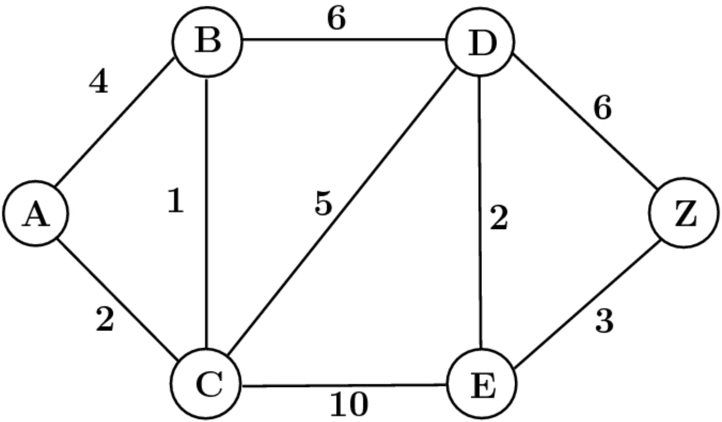

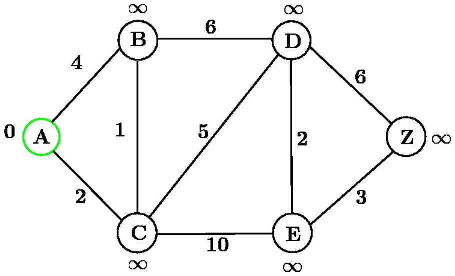

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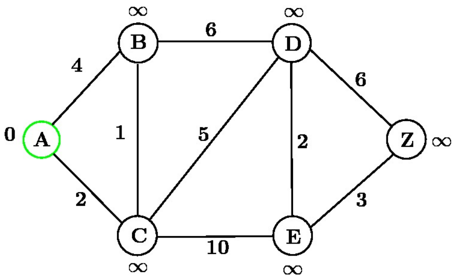
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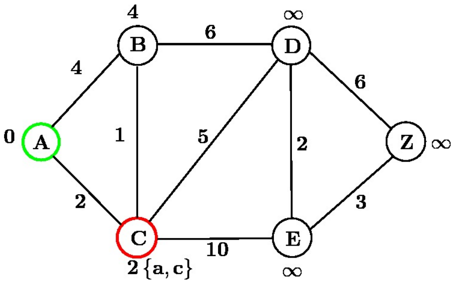
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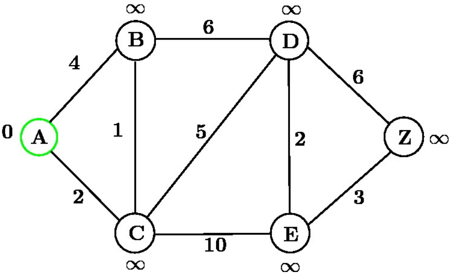
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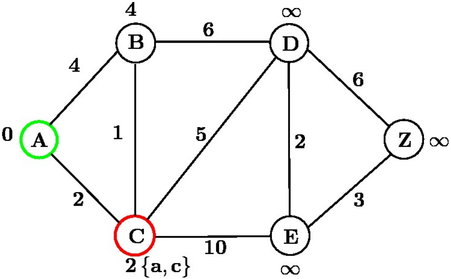
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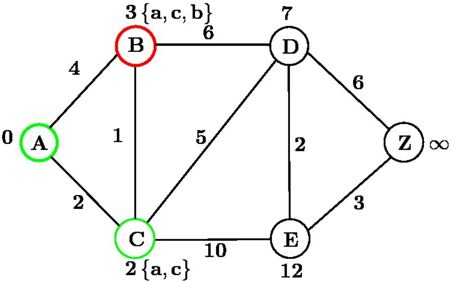
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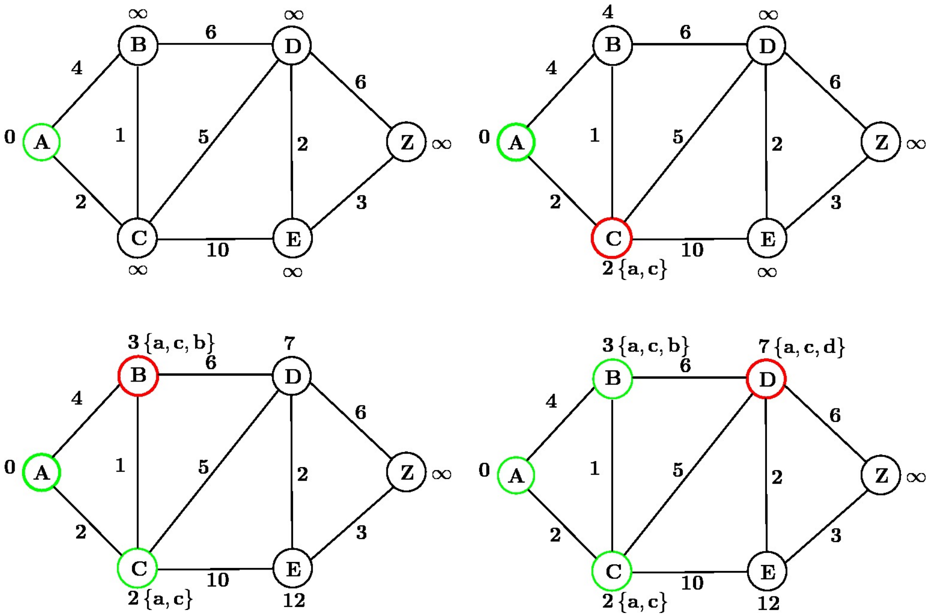
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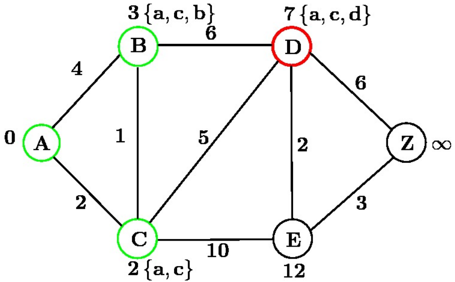
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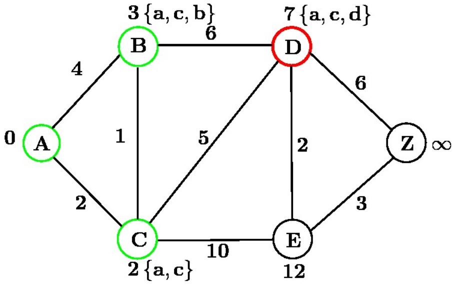
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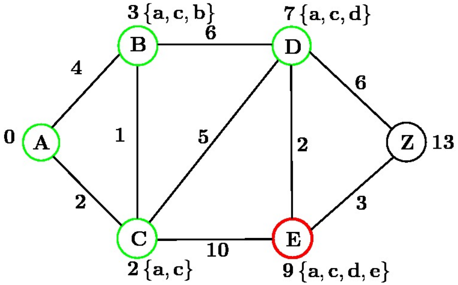
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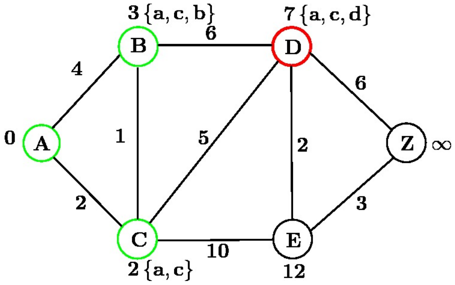
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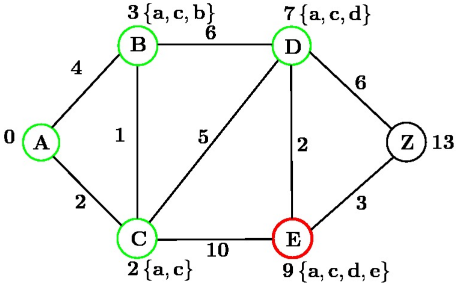
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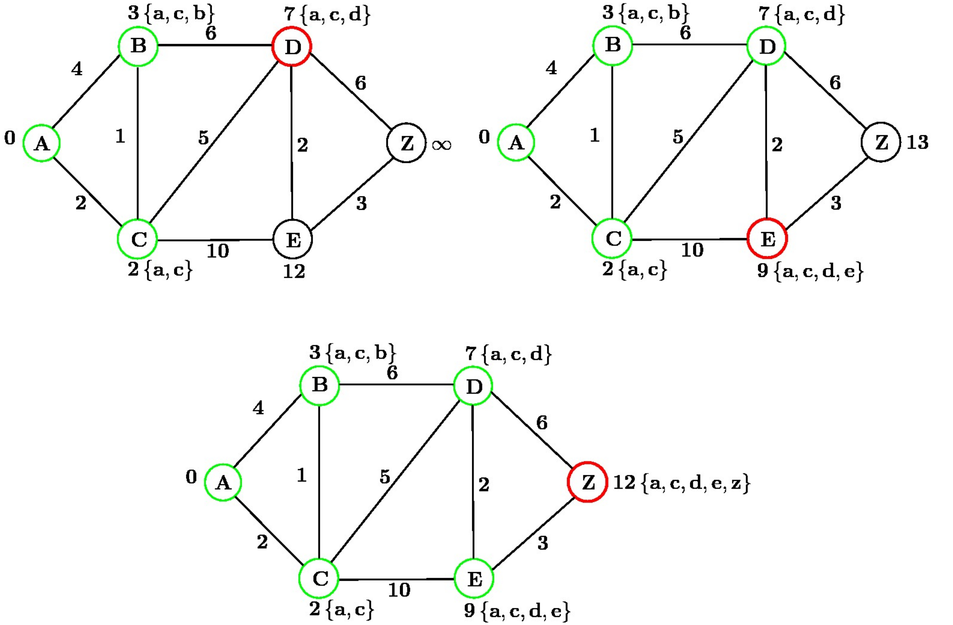
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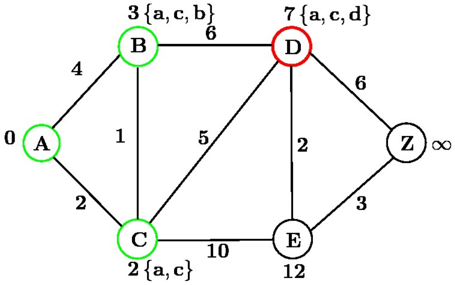
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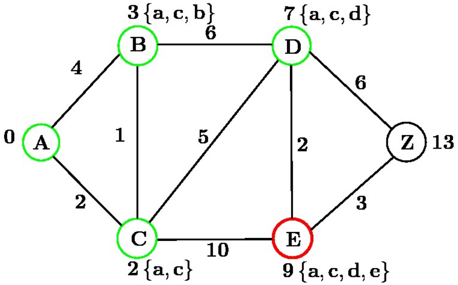
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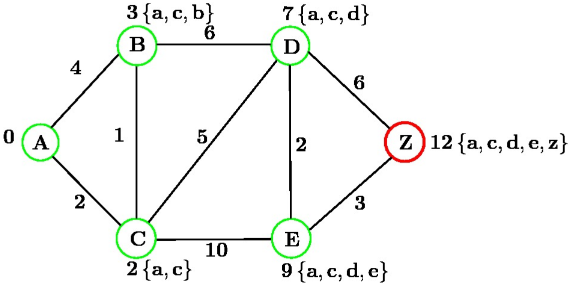
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Procedure Dijkstra(G: weighted connected simple graph with n vertices $v_1, v_2, \dots, v_n$)

for $i := 1$ **to** n

  $L(v_i) := \infty$

$L(A) := 0$

$S := \emptyset$

while $Z \notin S$

begin

  $u :=$ vertex not in S with minimum label

  $S := S \cup \{u\}$

  for all vertices v not in S

   $L(v) := \min\{L(v), L(u) + distance(u, v)\}$

end