Discrete Mathematics 1

Chapter 7: Advanced Counting Techniques

Department of Mathematics
The FPT university

Chapter 7: Introduction

Topics covered:

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7.1 Recurrence Relations

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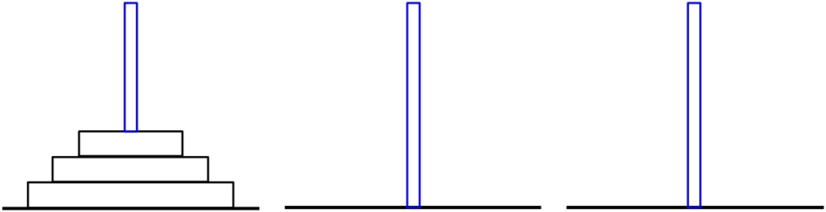
Example 2. A young pair of rabits (one of each sex) is placed on an island. A pair of rabbits does not breed untill they are 2 month old. After they are 2 month old, each month they produce a pair. Find the recurrence relation for the number of pairs of rabbits after n months (that is, at the end of the nth month).

Example 2. (The tower of Hanoi)



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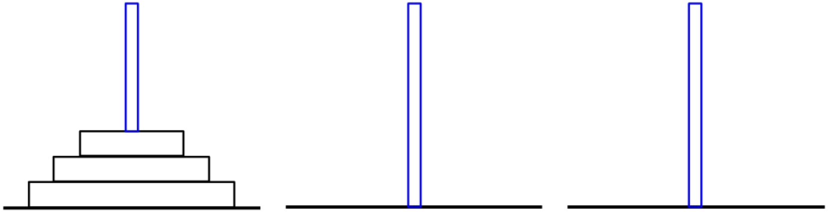

64 disks are placed on the first of three pegs in order of size (as shown in the picture). A disk is allowed to move from one peg to another as long as a disk is never placed on a disk of smaller size. Find the least number of moves required to move all disks to another peg.

Example 3. How many bit strings of length 10 that do not have 2 consecutive 0s?

7.3 Divide-and-Conquer Algorithms and Recurrence Relations

Recall the Merge sort algorithm.

Procedure mergesort $(L = a_1, a_2, \ldots, a_n)$
if $n > 1$ **then**
    $m := \lfloor n/2 \rfloor$
    $L_1 = a_1, a_2, \ldots, a_m$
    $L_2 = a_{m+1}, a_{m+2}, \ldots, a_n$
    $L := merge(mergesort(L_1), mergersort(L_2))$
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Let $f(n)$ be the number of comparisons used in the algorithm. Then $f(1) = 1$ and $f(n) = 2f(n/2) + n$.

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Master Theorem

Let f be an increasing function that satisfies

$$f(n) = af(n/b) + cn^d,$$

for all $n = b^k$, where k is a positive integer, $a \geq 1$ and $b > 1$ be positive integers, and c, d are positive real numbers. then

$$f(n) = \begin{cases} O(n^d) & \text{if} \quad a < b^d, \\ O(n^d \log n) & \text{if} \quad a = b^d, \\ O(n^{\log_b a}) & \text{if} \quad a > b^d. \end{cases}$$

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Let $a = 2^nA_1 + A_0$ and $b = 2^nB_1 + B_0$. Then

$$ab = (2^{2n} + 2^n)A_1B_1 + 2^n(A_1 - A_0)(B_0 - B_1) + (2^n + 1)A_0B_0.$$

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Using the Master Theorem we obtain $f(n) \approx O(n^{1.6})$.

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Example 1. 1000 students are taking Math, Computer Science and English classes. There are 600 students taking Math classes and 300 students taking Computer Science class. Among those, 200 are taking both Math and Computer Science classes, 50 taking all classes, 200 taking Math and English, and 80 taking Computer Science and English. How many students are taking English classes?

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Example 2. Find the number of bit strings of length 10, starting with a 0 or ending with a 1 or has exactly three 1s?