CMPSCI 240: First Summary

1 Basic Probability Definitions

The basic language

1. *Experiment:* a process resulting in exactly one of several possible outcomes, e.g., rolling a dice
2. *Sample space:* the set of all possible outcomes of an experiment, e.g., $\Omega = \{1, 2, 3, 4, 5, 6\}$
3. *Event:* a subset of Ω, e.g., $A =$ "odd number" $= \{1, 3, 5\}$
4. *Atomic event:* event consisting of a single outcome, e.g., $\{1\}$
5. *Probability law:* A function $P(\cdot)$ that maps event to a number between 0 and 1 such that:
   (a) *Nonnegativity:* $P(A) \geq 0$ for every $A \subseteq \Omega$
   (b) *Normalization:* $P(\Omega) = 1$
   (c) *Additivity:* $P(A \cup B) = P(A) + P(B)$ if A and B are disjoint

Basic properties include:

1. *Probability of the Empty Event:* $P(\emptyset) = 0$
2. *Probability of Event Complement:* $P(A^c) = 1 - P(A)$
3. *Probability of Unions:* If $A_1, \ldots, A_N$ are mutually disjoint,
$$P(A_1 \cup \ldots \cup A_N) = P(A_1) + P(A_2) + \ldots + P(A_N) \ .$$
4. *Uniform Probability:* If Ω is finite and all outcomes are equally likely, then $P(A) = |A|/|\Omega|$

2 Conditional Probability Definitions

Given two events A and B, the probability of A conditioned on B is defined as
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$
and so $P(A|B) = |A \cap B|/|B|$ if the probability is uniform. All the properties of probability also hold for conditional probability. For example,
$$P(A|B) \geq 0 \quad , \quad P(\Omega|B) = 1 \quad , \quad P(A^c|B) = 1 - P(A|B) \quad , \quad \text{and} \quad P(\emptyset|B) = 0$$
and if $A_1, \ldots, A_N$ are mutually disjoint, $P(A_1 \cup \ldots \cup A_N|B) = P(A_1|B) + P(A_2|B) + \ldots + P(A_N|B)$. Other useful properties of conditional probability include:

1. *Multiplication Rule:* For any k events $A_1, A_2, \ldots, A_k$

$$P(A_1 \cap A_2 \cap \ldots \cap A_k) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \ldots P(A_k|A_1 \cap A_2 \cap \ldots \cap A_{k-1})$$

2. *Law of Total Probability:* Suppose $A_1, \ldots, A_k$ are partitioning[1] events then

$$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \ldots + P(A_k)P(B|A_k)$$

3. *Bayes' Rule:* Suppose $A_1, \ldots, A_k$ are partitioning events then

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \ldots + P(A_k)P(B|A_k)}$$

We say two events A and B and *independent* if $P(A \cap B) = P(A)P(B)$. We say three events A, B, C are independent if all the following inequalities are true.

$$P(A \cap B) = P(A)P(B) \quad , \quad P(A \cap C) = P(A)P(C)$$

$$P(B \cap C) = P(B)P(C) \quad , \quad P(A \cap B \cap C) = P(A)P(B)P(C) \ .$$

Note that it is possible for the first three equalities to be satisfied but the last one not to be true. We say two events A and B are *independent conditioned of event* C if $P(A \cap B|C) = P(A|C)P(B|C)$. Note that even if A and B are independent, it is not necessarily the case that A and B are independent conditioned on C. Similarly, even if A and B are independent conditioned on C, A and B need not be independent.

3 Counting

- Permutations: There are $n! = n \times (n-1) \times (n-2) \times \ldots \times 1$ ways to order n different objects.
- k Permutations: There are $n!/(n-k)! = n \times (n-1) \times (n-2) \times \ldots \times (n-k+1)$ ways to determine an ordering of k different objects amongst a set of n different elements.
- Combinations: There are $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ subsets of size k that can be chosen from a set of n distinct elements.

[1] Recall that events are partitioning when exactly one of them occurs, i.e., $A_1 \cup A_2 \cup \ldots \cup A_k = \Omega$ and $A_i \cap A_j = \emptyset$ for any $i \neq j$.