Recurrence relations on scalars associated to homotopy lifting maps on Hochschild cohomology and connections to deformation of algebras

Tolulope Oke

Abstract. We describe the Gerstenhaber bracket structure on Hochschild cohomology of Koszul quiver algebras in terms of homotopy lifting maps. There is a projective bimodule resolution of Koszul quiver algebras that admits a comultiplicative structure. Introducing new scalars, we describe homotopy lifting maps associated to Hochschild cocycles using the comultiplicative structure. We show that the scalars can be described by some recurrence relations and we give several examples where these scalars appear in the literature. In particular, for a member of a family of quiver algebras, we describe Hochschild 2-cocycles and their associated homotopy lifting maps and determine the Maurer-Cartan elements of the quiver algebra in two ways: (i) by the use of homotopy lifting maps and (ii) by the use of a combinatorial star product that arises from the deformation of algebras using reduction systems.

1. Introduction

The Hochschild cohomology $\mathrm{HH}^*(\Lambda)$ of an associative algebra Λ possesses a multiplicative map called the cup product making it into a graded commutative ring. The ring structure of Hochschild cohomology of certain path algebras were determined using quiver techniques. For instance, if a path algebra is Koszul, its resolution possesses a comultiplicative structure and the cup product structure on its Hochschild cohomology can be presented using this comultiplicative structure. This is the work of E. L. Green, G. Hartman, E. N. Marcos and Ø. Solberg in [**6**].

In addition to the cup product on Hochschild cohomology ring is the Gerstenhaber bracket making $\mathrm{HH}^*(\Lambda)$ into a graded Lie/Gerstenhaber algebra. The bracket plays an important role in the theory of deformation of algebras. The theory of deformation of algebras employs techniques in algebraic and noncommutative geometry to describe variations of the associative multiplicative structure on any algebra. Of recent is the work of S. Barmeier and Z. Wang [**1**] in which deformations of algebras is proven to be equivalent to deformations of a reduction systems. Reduction systems were introduced by Bergman [**2**] in the late seventies. In particular, for an algebra $\Lambda = kQ/I$ where Q is a finite quiver,

Key words and phrases: Hochschild cohomology, Gerstenhaber brackets, quiver algebras, homotopy liftings, reduction system, Maurer-Cartan equation
MSC: 16E40; 16S37; 16S80, 16S15.

there is associated a reduction system R useful in determining a projective bimodule resolution of Λ reminiscent of the Bardzell resolution [**4**]. It was shown in [**1**] that there is an equivalence of formal deformations between (i) deformations of the associative algebra Λ, (ii) deformations of the reduction system R and (iii) deformations of the relations in I.

The Gerstenhaber bracket can be difficult to compute in general settings. Several works have been carried out to interpret the bracket as well as make computations of the bracket accessible for a large class of algebras for instance in [**9, 7**]. In [**16**], Y. Volkov introduced a method in which the bracket is defined in terms of a homotopy lifting map. This method works for arbitrary projective bimodule resolution of the algebra. In earlier works [**13**], we present a general formula for homotopy lifting maps associated to cocycles on Hochschild cohomology of Koszul path algebras.

The resolution introduced in [**6**] had scalars $c_{p,j}(n,i,r)$ appearing in the definition of the differentials on the resolution. These scalars made it possible to give a closed formula for the cup product structure on Hochschild cohomology. In Section 3, we present new scalars $b_{m,r}(m-n+1,s)$ associated to homotopy lifting maps on Hochschild cocycles using the scalars $c_{p,j}(n,i,r)$ of the comultiplicative relations. We show that the scalars $b_{m,r}(m-n+1,s)$ can be described using some recurrence relations and present the Gerstenhaber bracket structure using these scalars. We give several examples where these scalars appear in the literature in Section 4. In Section 5, we introduce a quiver algebra A_1 that is a counterexample to the Snashall-Solberg and R. Herman finite generation conjecture. We find Hochschild 2-cocycles of the algebra A_1 and present homotopy lifting maps associated to these cocycles. We showed that $\mathrm{HH}^2(A_1)$ is generated by five Maurer-Cartan elements.

Relevant important results from the theory of deformation of algebras using reduction system were recalled in the preliminaries. The deformation of an algebra involves altering the associative multiplicative structure on the algebra. The altered structure can be described using a combinatorial star product $(\star)$[**1**] and this product can be used to describe Maurer-Cartan elements. In Section 6, we give a characterization using $(\star)$ that shows that $\mathrm{HH}^2(A_1)$ is five dimensional.

2. Preliminaries

The Hochschild cohomology of an associative k-algebra Λ was originally defined using the following projective resolution known as the bar resolution.

$$\mathbb{B}_\bullet: \qquad \cdots \to \Lambda^{\otimes(n+2)} \xrightarrow{\delta_n} \Lambda^{\otimes(n+1)} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} \Lambda^{\otimes 3} \xrightarrow{\delta_1} \Lambda^{\otimes 2} \; (\xrightarrow{\mu} \Lambda) \tag{2.1}$$

where μ is multiplication and the differentials δ_n are given by

$$\delta_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \tag{2.2}$$

for all $a_0, a_1, \ldots, a_{n+1} \in \Lambda$. This resolution consists of Λ-bimodules or left modules over the enveloping algebra $\Lambda^e = \Lambda \otimes \Lambda^{op}$, where Λ^{op} is the opposite algebra. The resolution is sometimes written $\mathbb{B}_\bullet \xrightarrow{\mu} \Lambda$ with μ referred to as the augmentation map. Let M be a finitely generated left Λ^e-module, the Hochschild cohomology of Λ with coefficients in M denoted $\mathrm{HH}^*(\Lambda, M)$ is obtained by applying the functor $\mathrm{Hom}_{\Lambda^e}(-, M)$ to the complex $\mathbb{B}_\bullet$, and then taking the cohomology of the resulting cochain complex. That is

$$\mathrm{HH}^*(\Lambda, M) := \bigoplus_{n\geq 0} \mathrm{HH}^n(\Lambda, M) = \bigoplus_{n\geq 0} \mathrm{H}^n(\mathrm{Hom}_{\Lambda^e}(\mathbb{B}_n, M)).$$

If we let $M = \Lambda$, we then define $\mathrm{HH}^*(\Lambda) := \mathrm{HH}^*(\Lambda, \Lambda)$ to be the Hochschild cohomology of Λ. An element $\chi \in \mathrm{Hom}_{\Lambda^e}(\mathbb{B}_m, \Lambda)$ is a cocycle if $(\delta^*(\chi))(\cdot) := \chi\delta(\cdot) = 0$. There is an isomorphism of the abelian groups $\mathrm{Hom}_{\Lambda^e}(\mathbb{B}_m, \Lambda) \cong \mathrm{Hom}_k(\Lambda^{\otimes m}, \Lambda)$, so we can also view χ as an element of $\mathrm{Hom}_k(\Lambda^{\otimes m}, \Lambda)$.

The Gerstenhaber bracket of two cocycles $\chi \in \mathrm{Hom}_k(\Lambda^{\otimes m}, \Lambda)$ and $\theta \in \mathrm{Hom}_k(\Lambda^{\otimes n}, \Lambda)$ at the chain level is given by

$$[\chi, \theta] = \chi \circ \theta - (-1)^{(m-1)(n-1)}\theta \circ \chi \tag{2.3}$$

where $\chi \circ \theta = \sum_{j=1}^{m} (-1)^{(n-1)(j-1)} \chi \circ_j \theta$ with

$$\begin{aligned}&(\chi \circ_j \theta)(a_1 \otimes \cdots \otimes a_{m+n-1})\\ &\qquad = \chi(a_1 \otimes \cdots \otimes a_{j-1} \otimes \theta(a_j \otimes \cdots \otimes a_{j+n-1}) \otimes a_{j+n} \otimes \cdots \otimes a_{m+n-1}).\end{aligned}$$

This induces a well defined map $[\cdot\,,\cdot] : \mathrm{HH}^m(\Lambda) \times \mathrm{HH}^n(\Lambda) \to \mathrm{HH}^{m+n-1}(\Lambda)$ on cohomology.

Gerstenhaber bracket using homotopy lifting: We present an equivalent definition of the Gerstenhaber bracket presented by Y. Volkov in [**16**] and reformulated with a sign change by S. Witherspoon in Theorem (2.8). We assume that A is an algebra over the field k and take $\mathbb{P} \xrightarrow{\mu_P} A$ to be a projective resolution of A as an A^e-module with differential $d^{\mathbb{P}}$ and augmentation map $\mu_{\mathbb{P}}$. We take $\mathbf{d}$ to be the differential on the Hom complex $\mathrm{Hom}_{\Lambda^e}(\mathbb{P}, \mathbb{P})$ defined for any degree n map $g : \mathbb{P} \to \mathbb{P}[-n]$ as

$$\mathbf{d}(g) := d^{\mathbb{P}} g - (-1)^n g d^{\mathbb{P}}$$

where $\mathbb{P}[-n]$ is a shift in homological dimension with $(\mathbb{P}[-n])_m = \mathbb{P}_{m-n}$. In the following definition, the notation $\sim$ is used for two cocycles that are cohomologous, that is, they differ by a coboundary.

Definition 2.4. *Let $\Delta_{\mathbb{P}}$ be a chain map lifting the identity map on $A \cong A \otimes_A A$ and suppose that $\eta \in \mathrm{Hom}_{A^e}(\mathbb{P}_n, A)$ is a cocycle. A module homomorphism $\psi_\eta : \mathbb{P} \to \mathbb{P}[1-n]$ is called a **homotopy lifting** map of η with respect to $\Delta_{\mathbb{P}}$ if*

$$\begin{aligned} &\boldsymbol{d}(\psi_\eta) = (\eta \otimes 1_{\mathbb{P}} - 1_{\mathbb{P}} \otimes \eta)\Delta_{\mathbb{P}} \qquad \textit{and} \\ &\mu_{\mathbb{P}}\psi_\eta \sim\ (-1)^{n-1}\eta\psi \end{aligned} \tag{2.5}$$

for some $\psi : \mathbb{P} \to \mathbb{P}[1]$ for which $\boldsymbol{d}(\psi) = (\mu_{\mathbb{P}} \otimes 1_{\mathbb{P}} - 1_{\mathbb{P}} \otimes \mu_{\mathbb{P}})\Delta_{\mathbb{P}}$.

Example 2.6. *Let us consider a homotopy lifting formula for a cocycle β using the bar resolution $\mathbb{B}$. Suppose that $\beta \in \mathrm{Hom}_{\Lambda^e}(\mathbb{B}_n, A) \cong \mathrm{Hom}_k(A^{\otimes n}, A)$, then one way to define a homotopy lifting map $\psi_\beta : \mathbb{B}_{m+n-1} \longrightarrow \mathbb{B}_m$ for the cocycle β is the following:*

$$\begin{aligned}&\psi_\beta(1 \otimes a_1 \otimes \cdots \otimes a_{m+n-1} \otimes 1)\\ &= \sum_{i=1}^{m} (-1)^{(m-1)(i-1)} 1 \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+n-1}) \otimes a_{i+n} \otimes \cdots \otimes a_{m+n-1} \otimes 1.\end{aligned}$$

We compute an example in which $\beta \in \mathrm{Hom}_k(\Lambda^{\otimes 2}, \Lambda)$ *and* $m = 2$, *so,* $\psi_\beta : \mathbb{B}_3 \to \mathbb{B}_2$. *Using the differentials on the bar resolution given in Equation* (2.2) *and the diagonal map* $\Delta_{\mathbb{B}}$ *later given in Equation* (2.13), *we have*

$$\delta\psi_\beta(1 \otimes a_1 \otimes a_2 \otimes a_3 \otimes 1) = \beta(a_1 \otimes a_2) \otimes a_3 \otimes 1 - 1 \otimes \beta(a_1 \otimes a_2)a_3 \otimes 1 + 1 \otimes \beta(a_1 \otimes a_2) \otimes a_3$$
$$- a_1 \otimes \beta(a_2 \otimes a_3) \otimes 1 + 1 \otimes a_1\beta(a_2 \otimes a_3) \otimes 1 - 1 \otimes a_1 \otimes \beta(a_2 \otimes a_3) \qquad \textit{and}$$
$$\psi_\beta\delta(1 \otimes a_1 \otimes a_2 \otimes a_3 \otimes 1) = a_1 \otimes \beta(a_2 \otimes a_3) \otimes 1 - 1 \otimes \beta(a_1a_2 \otimes a_3) \otimes 1$$
$$+ 1 \otimes \beta(a_1 \otimes a_2a_3) \otimes 1 - 1 \otimes \beta(a_1 \otimes a_2) \otimes a_3.$$

Therefore $(\delta\psi_\beta + \psi_\beta\delta)(1 \otimes a_1 \otimes a_2 \otimes a_3 \otimes 1) = \beta(a_1 \otimes a_2) \otimes a_3 \otimes 1 - 1 \otimes a_1 \otimes \beta(a_2 \otimes a_3)$. *On the other hand,*

$$(\beta \otimes 1 - 1 \otimes \beta)\Delta_{\mathbb{B}}(1 \otimes a_1 \otimes a_2 \otimes a_3 \otimes 1) = (\beta \otimes 1 - 1 \otimes \beta)\Big((1 \otimes 1) \otimes_\Lambda (1 \otimes a_1 \otimes a_2 \otimes a_3 \otimes 1) +$$
$$(1 \otimes a_1 \otimes 1) \otimes_\Lambda (1 \otimes a_2 \otimes a_3 \otimes 1) + (1 \otimes a_1 \otimes a_2 \otimes 1) \otimes_\Lambda (1 \otimes a_3 \otimes 1)$$
$$+ (1 \otimes a_1 \otimes a_2 \otimes a_3 \otimes 1) \otimes_\Lambda (1 \otimes 1)\Big) = \beta(a_1 \otimes a_2) \otimes a_3 \otimes 1 - 1 \otimes a_1 \otimes \beta(a_2 \otimes a_3).$$

So we see that Equation (2.5) *holds in degree 3 i.e.*

$$\delta\psi_\beta - (-1)^{2-1}\psi_\beta\delta = (\beta \otimes 1 - 1 \otimes \beta)\Delta_{\mathbb{B}}.$$

REMARK 2.7. *Suppose that* $\mathbb{K}$ *is the Koszul resolution, then it is a differential graded coalgebra i.e.* $(\Delta_{\mathbb{K}} \otimes 1_{\mathbb{K}})\Delta_{\mathbb{K}} = (1_{\mathbb{K}} \otimes \Delta_{\mathbb{K}})\Delta_{\mathbb{K}}$ *and* $(d \otimes 1 + 1 \otimes d)\Delta_{\mathbb{K}} = \Delta_{\mathbb{K}}d$. *Furthermore, the augmentation map* $\mu : \mathbb{K} \to \Lambda$ *makes* $(\mu \otimes 1_{\mathbb{K}})\Delta_{\mathbb{K}} - (1_{\mathbb{K}} \otimes \mu)\Delta_{\mathbb{K}} = 0$. *We can therefore set* $\psi = 0$ *in the second part of Equation* (2.5), *so that we have* $\mu\psi_\eta \sim 0$. *Next, we set* $\psi_\eta(\mathbb{K}_{n-1}) = 0$ *and the second relations of Equation* (2.5) *is satisfied. To check if a map is a homotopy lifting map, it is sufficient to verify the first equation in* (2.5) *if the resolution is Koszul.*

The following is a theorem of Y. Volkov which is equivalent to the definition of the bracket presented earlier in Equation (2.3).

THEOREM 2.8. [**16**, Theorem 4] *Let* $(\mathbb{P}, \mu_{\mathbb{P}})$ *be a* A^e*-projective resolution of the algebra* A, *and let* $\Delta_{\mathbb{P}} : \mathbb{P} \to \mathbb{P} \otimes_A \mathbb{P}$ *be a diagonal map. Let* $\eta : \mathbb{P}_n \to A$ *and* $\theta : \mathbb{P}_m \to A$ *be cocycles representing two classes. Suppose that* ψ_η *and* ψ_θ *are homotopy liftings for* η *and* θ *respectively. Then the Gerstenhaber bracket of the classes of* η *and* θ *can be represented by the class of the element*

$$[\eta, \theta]_{\Delta_{\mathbb{P}}} = \eta\psi_\theta - (-1)^{(m-1)(n-1)}\theta\psi_\eta.$$

Quiver algebras: A quiver is a directed graph with the allowance of loops and multiple arrows. A quiver Q is sometimes denoted as a quadruple (Q_0, Q_1, o, t) where Q_0 is the set of vertices in Q, Q_1 is the set of arrows in Q, and $o, t : Q_1 \longrightarrow Q_0$ are maps which assign to each arrow $a \in Q_1$, its origin vertex $o(a)$ and terminal vertex $t(a)$ in Q_0. A path in Q is a sequence of arrows $a = a_1a_2 \cdots a_{n-1}a_n$ such that the terminal vertex of a_i is the same as the origin vertex of a_{i+1}, using the convention of concatenating paths from left to right. The quiver algebra or path algebra kQ is defined as a vector space having all paths in Q as a basis. Vertices are regarded as paths of length 0, an arrow is a path of length 1, and so on. We take multiplication on kQ as concatenation of paths. Two paths a and b satisfy $ab = 0$ if $t(a) \neq o(b)$. This multiplication defines an associative algebra over k. By taking kQ_i to be the k-vector subspace of kQ with paths of length i as basis, $kQ = \bigoplus_{i\geq 0} kQ_i$ can be viewed as an $\mathbb{N}$-graded vector space. Two paths are parallel if they have the same origin and terminal vertex. A relation on a quiver Q is a linear combination of parallel paths in Q. A quiver together with a set of relations is called a quiver with relations. Letting I be an ideal of the path algebra kQ, we denote by (Q, I) the quiver Q with relations I. The quotient $\Lambda = kQ/I$ is called the quiver algebra associated with (Q, I). Let $\Lambda = \bigoplus_{i\geq 0} \Lambda_i$ be a grading on Λ. If Λ is Koszul and Λ_0 is isomorphic to k or copies of k, Λ_0 has a linear graded projective resolution $\mathbb{L}$ as a right Λ-module [**15, 8**].

An algorithmic approach to finding such a minimal projective resolution $\mathbb{L}$ of Λ_0 was given in [**5**]. The modules $\mathbb{L}_n$ are right Λ-modules for each n. There is a "comultiplicative structure" on $\mathbb{L}$ and this structure was used to find a minimal projective resolution $\mathbb{K} \to \Lambda$ of modules over the enveloping algebra of Λ in [**6**]. A non-zero element $x \in kQ$ is called uniform if it is a linear combination of paths each having the same origin vertex and the same terminal vertex: In other words, $x = \sum_j c_j w_j$ with scalars $c_j \neq 0$ for all j and each path w_j are of equal length having the same origin vertex and the same terminal vertex. For $R = kQ$, it was shown in [**5**] that there are integers t_n and uniform elements $\{f_i^n\}_{i=0}^{t_n}$ such that the right projective resolution $\mathbb{L} \to \Lambda_0$ is obtained from a filtration of R. This filtration is given by the following nested family of right ideals:

$$\cdots \subseteq \bigoplus_{i=0}^{t_n} f_i^n R \subseteq \bigoplus_{i=0}^{t_{n-1}} f_i^{n-1} R \subseteq \cdots \subseteq \bigoplus_{i=0}^{t_1} f_i^1 R \subseteq \bigoplus_{i=0}^{t_0} f_i^0 R = R$$

where for each n, $\mathbb{L}_n = \bigoplus_{i=0}^{t_n} f_i^n R / \bigoplus_{i=0}^{t_n} f_i^n I$ and the differentials on $\mathbb{L}$ are induced by the inclusions $\bigoplus_{i=0}^{t_n} f_i^n R \subseteq \bigoplus_{i=0}^{t_{n-1}} f_i^{n-1} R$. Furthermore, it was shown in [**5**] that with some choice of scalars, the $\{f_i^n\}_{i=0}^{t_n}$ satisfying the comultiplicative equation of (2.9) make $\mathbb{L}$ minimal. In other words, for $0 \leq i \leq t_n$, there are scalars $c_{pq}(n, i, r)$ such that

$$f_i^n = \sum_{p=0}^{t_r} \sum_{q=0}^{t_{n-r}} c_{pq}(n, i, r) f_p^r f_q^{n-r} \tag{2.9}$$

holds and $\mathbb{L}$ is a minimal resolution. To construct the above multiplicative equation for example, we can take $\{f_i^0\}_{i=0}^{t_0}$ to be the set of vertices, $\{f_i^1\}_{i=0}^{t_1}$ to be the set of arrows, $\{f_i^2\}_{i=0}^{t_2}$ to be the set of uniform relations generating the ideal I, and define $\{f_i^n\}_{i=0}^{t_n} (n \geq 3)$ recursively, that is in terms of f_i^{n-1} and f_j^1. We presented the comultiplicative structure of a family of quiver algebras in [**12**] and use the homotopy lifting technique to show that for some members of the family, the Hochschild cohomology ring modulo the weak Gerstenhaber ideal generated by homogeneous nilpotent elements is not finitely generated.

The resolution $\mathbb{L}$ and the comultiplicative structure (2.9) were used to construct a minimal projective resolution $\mathbb{K} \to \Lambda$ of modules over the enveloping algebra $\Lambda^e = \Lambda \otimes \Lambda^{op}$ on which we now define Hochschild cohomology. This minimal projective resolution $\mathbb{K}$ of Λ^e-modules associated to Λ was given in [**6**] and now restated with slight notational changes below.

THEOREM 2.10. [**6**, Theorem 2.1] *Let $\Lambda = kQ/I$ be a Koszul algebra, and let $\{f_i^n\}_{i=0}^{t_n}$ define a minimal resolution of Λ_0 as a right Λ-module. A minimal projective resolution*

$(\mathbb{K}, d)$ of Λ over Λ^e is given by

$$\mathbb{K}_n = \bigoplus_{i=0}^{t_n} \Lambda o(f_i^n) \otimes_k t(f_i^n)\Lambda$$

for $n \geq 0$, where the differential $d_n : \mathbb{K}_n \to \mathbb{K}_{n-1}$ applied to $\varepsilon_i^n = (0, \ldots, 0, o(f_i^n) \otimes_k t(f_i^n), 0, \ldots, 0)$, $0 \leq i \leq t_n$ with $o(f_i^n) \otimes_k t(f_i^n)$ in the i-th position, is given by

$$d_n(\varepsilon_i^n) = \sum_{j=0}^{t_{n-1}} \Big(\sum_{p=0}^{t_1} c_{p,j}(n, i, 1) f_p^1 \varepsilon_j^{n-1} + (-1)^n \sum_{q=0}^{t_1} c_{j,q}(n, i, n-1) \varepsilon_j^{n-1} f_q^1 \Big) \tag{2.11}$$

and $d_0 : \mathbb{K}_0 \to \Lambda$ is the multiplication map. In particular, Λ is a linear module over Λ^e.

We note that for each n and i, $\{\varepsilon_i^n\}_{i=0}^{t_n}$ is a basis of $\mathbb{K}_n$ as a Λ^e-module. The scalars $c_{p,j}(n, i, r)$ are those appearing in (2.9) and $f_*^1 := \overline{f_*^1}$ is the residue class of f_*^1 in $\bigoplus_{i=0}^{t_1} f_i^1 R / \bigoplus_{i=0}^{t_n} f_i^1 I$. Using the comultiplicative structure of Equation (2.9), a cup product formula on Hochschild cohomology of Koszul quiver algebra was presented in [**3**].

We recall the definition of the reduced bar resolution of algebras defined by quivers and relations. If Λ_0 is isomorphic to m copies of k, take $\{e_1, e_2, \ldots, e_m\}$ to be a complete set of primitive orthogonal central idempotents of Λ. In this case Λ is not necessarily an algebra over Λ_0. If Λ_0 is isomorphic to k, then Λ is an algebra over Λ_0. For convenience, we use the same notation $\mathbb{B}$ for both the bar resolution and the reduced bar resolution. The reduced bar resolution $(\mathbb{B}, \delta)$, where $\mathbb{B}_n := \Lambda^{\otimes_{\Lambda_0}(n+2)}$ is the $(n+2)$-fold tensor product of Λ over Λ_0 and uses the same differential as the usual bar resolution presented in Equation (2.2). The resolution $\mathbb{K}$ can be embedded naturally into the reduced bar resolution $\mathbb{B}$. There is a map $\iota : \mathbb{K} \to \mathbb{B}$ defined by $\iota(\varepsilon_r^n) = 1 \otimes \widetilde{f_r^n} \otimes 1$ such that $\delta\iota = \iota d$, where

$$\widetilde{f_j^n} = \sum c_{j_1 j_2 \cdots j_n} f_{j_1}^1 \otimes f_{j_2}^1 \otimes \cdots \otimes f_{j_n}^1 \quad \text{if} \quad f_j^n = \sum c_{j_1 j_2 \cdots j_n} f_{j_1}^1 f_{j_2}^1 \cdots f_{j_n}^1 \tag{2.12}$$

for some scalar $c_{j_1 j_2 \cdots j_n}$. It was shown in [**3**, Proposition 2.1] that ι is indeed an embedding. By taking $\Delta_{\mathbb{B}} : \mathbb{B} \to \mathbb{B} \otimes_\Lambda \mathbb{B}$ to be the following comultiplicative map (or diagonal map) on the bar resolution,

$$\Delta_{\mathbb{B}}(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (a_0 \otimes \cdots \otimes a_i \otimes 1) \otimes_\Lambda (1 \otimes a_{i+1} \otimes \cdots \otimes a_{n+1}). \tag{2.13}$$

it was also shown in [**3**, Proposition 2.2] that the diagonal map $\Delta_{\mathbb{K}} : \mathbb{K} \to \mathbb{K} \otimes_\Lambda \mathbb{K}$ on the complex $\mathbb{K}$ has the following form.

$$\Delta_{\mathbb{K}}(\varepsilon_r^n) = \sum_{v=0}^{n} \sum_{p=0}^{t_v} \sum_{q=0}^{t_{n-v}} c_{p,q}(n, r, v) \varepsilon_p^v \otimes_\Lambda \varepsilon_q^{n-v}. \tag{2.14}$$

The compatibility of $\Delta_{\mathbb{K}}, \Delta_{\mathbb{B}}$ and ι means that $(\iota \otimes \iota)\Delta_{\mathbb{K}} = \Delta_{\mathbb{B}}\iota$ where $(\iota \otimes \iota)(\mathbb{K} \otimes_\Lambda \mathbb{K}) = \iota(\mathbb{K}) \otimes_\Lambda \iota(\mathbb{K}) \subseteq \mathbb{B} \otimes_\Lambda \mathbb{B}$.

Deformation of algebras using reduction system: The theory of deformation of algebras was introduced by M. Gerstenhaber with the Gerstenhaber bracket playing a major role in this theory. For the algebra $\Lambda = kQ/I$ where Q is a finite quiver, there is associated a reduction system R and the the Gerstenhaber bracket equips Hochschild cohomology $\mathrm{HH}^*(\Lambda)$ with a DG Lie algebra structure which controls the theory of deformation of R. It was shown in [**1**] that the deformations of Λ is equivalent to the deformations of the reduction system R which is also equivalent to the deformation of the relations in I. Suppose that (Λ_t, μ_t) is a formal deformation of the associative multiplication on Λ, the deformed multiplication μ_t can be described by a combinatorial star product given by Equation (2.26). There is a projective bimodule resolution $\mathfrak{p}(Q, R)$ arising from a reduction system R and the combinatorial star product can be used to describe Maurer-Cartan elements in $\mathfrak{p}(Q, R) \otimes \mathfrak{m}$, where $(N, \mathfrak{m})$ is a complete local Noetherian k-algebra. In Section 6, we use the combinatorial star product to determine Maurer-Cartan elements of $\mathrm{HH}^2(A_1)$ thus determining a family of deformations of the algebra A_1. In the future, it will be interesting to find a meaningful way to describe Maurer-Cartan elements that were obtained using the star product in terms of those obtained by the homotopy lifting technique and vice versa. We begin with a result from [**2**] on reduction systems.

DEFINITION 2.15. *Let $\Lambda = kQ/I$ be a path algebra with finite quiver Q. A reduction system R for kQ is a set of pairs*

$$R = \{(s, f_s) \mid s \in S, f_s \in kQ\}$$

where

- *S is a subset of paths of length greater than or equal to 2 such that s is not a subpath of s' when $s \neq s' \in S$*
- *s and f_s are parallel paths*
- *for each $(s, f_s) \in R$, f_s is irreducible or a linear combination of irreducible paths.*

We say a path is *irreducible* if it does not contain elements in S as a subpath.

DEFINITION 2.16. *Given a two-sided ideal I of kQ, we say that a reduction system R satisfies the diamond condition $(\diamond)$ for I if*

- *I is equal to the two-sided ideal generated by the set $\{s - f_s\}_{(s,f_s)\in R}$ and*
- *every path is reduction unique.*

We call a reduction system R finite if R is a finite set.

DEFINITION 2.17. *Let R be a reduction system for kQ and p, q, r be paths of length at least 1. A path pqr of length at least 3 is an overlap ambiguity of R if $pq, qr \in S$.*

Let $\pi : kQ \to \Lambda$ be the projection map. We now state the diamond lemma as provided in [**2**].

THEOREM 2.18. [**2**, Thm 1.2] *Let $R = \{(s, f_s)\}$ be a reduction system for kQ and let $S = \{s \mid (s, f_s) \in R\}$. Denote by $I = \langle s - f_s\rangle_{s\in S}$ the corresponding two-sided ideal of $\Lambda = kQ/I$. If R is reduction finite, the following are equivalent:*

- *all overlap ambiguities of R are resolvable*
- *R is reduction unique, that is R satisfies $(\diamond)$ for I*
- *the image of the set of irreducible paths under the projection $\pi : kQ \to A$ forms a k-basis for A.*

It is known that there is always a choice of a reduction system R satisfying the diamond condition $(\diamond)$ of Definition 2.16 for any quiver algebra $\Lambda = kQ/I$. S. Chouhy and A. Solotar showed in [**4**, Prop 2.7, Thm 4.1, Thm 4.2] that for any two-sided ideal $I \subset kQ$, there exists a reduction system R satisfying the diamond condition. Furthermore, there is a projective bimodule resolution $\mathfrak{p}(Q, R)$ associated to the algebra $\Lambda = kQ/I$.

Let B be a k-algebra and let τ be an indeterminate. The ring $B[[\tau]]$ is the ring of formal power series in τ with coefficients in B. The ring $B[[\tau]]$ is a $k[[\tau]]$-module if we identify k with subalgebra $k \cdot 1$ of B. The multiplication in B is usually denoted by concatenation while the multiplication in $B[[\tau]]$ is given as

$$(\sum_{i\geq 0} a_i\tau^i)(\sum_{j\geq 0} b_j\tau^j) = \sum_{m\geq 0}(\sum_{i+j=m} a_ib_j\tau^m)$$

We are interested in a new associative structure on B. All such associative structure provides a deformation B_τ of the algebra B.

Definition 2.19. *A formal deformation (B_τ, μ_τ) of B (also called a deformation of A over $k[[\tau]]$) is an associative bilinear multiplication $\mu_\tau : B[[\tau]]\otimes B[[\tau]] \to B[[\tau]]$ such that in the quotient by the ideal (τ), the multiplication $\mu_\tau(b_1, b_2)$ coincides with the multiplication in B for all $b_1, b_2 \in B[[\tau]]$.*

The multiplication μ_τ above is determined by product of pairs of elements of B, so that for every $a, b \in B$

$$\mu_\tau(a, b) = ab + \mu_1(a, b)\tau + \mu_2(a, b)\tau^2 + \mu_3(a, b)\tau^3 + \cdots \tag{2.20}$$

where ab is the usual multiplication in B and $\mu_i : B \otimes_k B \to B$. If we denote the usual multiplication in B by μ, we may denote a deformation of (B, μ) by (B_τ, μ_τ) where

$$\mu_\tau = \mu + \mu_1\tau + \mu_2\tau^2 + \mu_3\tau^3 + \cdots$$

Remark 2.21.

- *A first order deformation (B_τ, μ_τ) of B is a deformation over $k[\tau]/(\tau^2)$ given by*

$$\mu_\tau = \mu + \mu_1\tau$$

- *If (B_τ, μ_τ) is a first order deformation, μ_τ is associative implies that μ_1 is a Hochschild 2-cocycle. In order words, Hochschild cohomology controls the equivalence classes of first-order deformations of an algebra.*
- *It is possible to extend first order deformations to formal deformation.*

The idea of making a formal deformation into a first-order deformation is called the *algebraization of formal deformations* and were examined in detail by S. Barmeier and Z. Wang in [**1**]. One of their main results is the following theorem.

Theorem 2.22. [**1**, Thm 7.1] *Given any finite quiver Q and any two-sided ideal of relations I, let $\Lambda = kQ/I$ be the quotient algebra and let R be any reduction system satisfying the diamond condition $(\diamond)$ for I. There is an equivalence of formal deformation problems between*

(i) deformations of the associative algebra structure on Λ
(ii) deformations of the reduction system R
(iii) deformations of the relations I

In [**1**, Section 5], it was established that are comparison morphisms $F_\bullet, G_\bullet$ between the bar resolution $\mathbb{B}$ and the resolution $\mathfrak{p}(Q, R)$ coming from reduction systems.

$$\mathfrak{p}_\bullet \xrightarrow{F_\bullet} \mathbb{B}_\bullet \xrightarrow{G_\bullet} \mathfrak{p}_\bullet$$

Let (B_τ, μ_τ) be a formal deformation of the path algebra $B = kQ/I$. We may write $\mu_\tau = \mu + \widetilde{\mu}$ where

$$\widetilde{\mu} = \widetilde{\mu_1}\tau + \widetilde{\mu_2}\tau^2 + \widetilde{\mu_3}\tau^3 + \cdots$$

so that we can view $\widetilde{\mu} \in \mathrm{Hom}(kS, B) \otimes (\tau)$, since each $\widetilde{\mu_i} : B \otimes_k B \to B$ and $S \subseteq B \otimes_k B$. The following theorem about a combinatorial star product was a major tool used to prove Theorem (2.22).

THEOREM 2.23. [**1**, Thm. 7.21] *Let $\widetilde{\mu} \in \mathrm{Hom}(kS, B) \otimes (\tau)$ be arbitrary and let $a, b \in B$ and define $G_0^\bullet(a \otimes b) := ab$. For any $i \geq 0$, we have*

$$a \star b := a \star_{\widetilde{\mu}}^{i} b = G_0(a \otimes b) + \sum_{i \geq 1} G_i(\widetilde{\mu}^{\otimes i})(a \otimes b) \tag{2.24}$$

REMARK 2.25.

(1) *Under certain conditions, the combinatorial star product $\star$ defines an associative structure on the algebra (B_τ, μ_τ) where $\mu_\tau(a, b) = a \star b$.*
(2) *For a quotient algebra $\Lambda = kQ/I$ and a reduction system R satisfying $(\diamond)$ for I, the space of Hochschild 2-cocycles $\mathrm{HH}^2(\Lambda)$ is isomorphic to the space of first-order deformations of R modulo equivalence.*
(3) *Suppose that $uvw \in S_3$ such that $uv, vw \in S$, then $\widetilde{\mu}$ satisfies the Maurer-Cartan equation if and only if*

$$(\pi(u) \star \pi(v)) \star \pi(w) = \pi(u) \star (\pi(v) \star \pi(w)). \tag{2.26}$$

(4) *It is also known that an Hochschild 2-cocycle μ satisfies the Maurer-Cartan equation if $d^*\mu + \frac{1}{2}[\mu, \mu] = 0$, where $[\cdot, \cdot]$ is the Gerstenhaber bracket.*

3. Main Results

In what follows, we consider a finite quiver Q and a quiver algebra $\Lambda = kQ/I$ that is Koszul i.e. I is an admissible ideal generated by paths of length 2. We assume the quiver Q has arrows labelled $f_1^1, f_2^1, \ldots, f_{t_1}^1$ for some integer t_1. We suppose further that for each n, there are uniform elements $f_1^n, f_2^n, \ldots, f_{t_n}^n$, for some integer t_n defining a minimal projective resolution $\mathbb{K}$ of Λ as given by Theorem (2.10). Let $\eta : \mathbb{K}_n \to \Lambda$ be a Hochschild cocycle such that for some index i, $\eta(\varepsilon_i^n) = f_w^1$ (resp. $\eta(\varepsilon_i^n) = f_w^1 f_{w'}^1$) with $0 \leq w, w' \leq t_1$ and $\eta(\varepsilon_j^n) = 0$ for all $i \neq j$. We also write $\eta = \begin{pmatrix} 0 & \cdots & 0 & (f_w^1)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$ for this type of cocycle (resp. $\eta = \begin{pmatrix} 0 & \cdots & 0 & (f_w^1 f_{w'}^1)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$). Let $\Delta_{\mathbb{K}} : \mathbb{K} \to \mathbb{K} \otimes_\Lambda \mathbb{K}$ be the diagonal map. Results from [**16, 15**] establishes that there exists maps $\psi_\eta : \mathbb{K}_m \to \mathbb{K}_{m-n+1}$ such that

$$d\psi_\eta - (-1)^{1-n}\psi_\eta d = (\eta \otimes 1 - 1 \otimes \eta)\Delta_{\mathbb{K}} \tag{3.1}$$

for Koszul algebras. These maps are called *homotopy lifting* maps for η. How would we define such a map explicitly in terms of the basis elements ε_r^n? can we give a closed formula or expression of the Gerstenhaber bracket using an explicity described version of these maps? These are among the questions we address in this section.

In order to distinguish the index n which is the degree of the cocycle η, we will index the resolution $\mathbb{K}$ by m so that each $\mathbb{K}_m$ is free and generated by $\{\varepsilon_r^m\}_{r=0}^{t_m}$. For an n-cocycle, the map $\psi_\eta : \mathbb{K}_\bullet \to \mathbb{K}_\bullet$ associated to η shifts the degree of the resolution $\mathbb{K}$ by $1 - n$ so that for a fixed m, $\psi_\eta : \mathbb{K}_m \to \mathbb{K}_{m-n+1}$. Suppose that $\mathbb{K}_{m-n+1}$ is generated by

$\{\varepsilon_{r'}^{m-n+1}\}_{r'=0}^{t_{m-n+1}}$, fundamental results from linear algebra means such a map is a $t_{m-n+1} \times t_m$ matrix when the modules are considered as left Λ^e-modules. In particular, for $\eta = \begin{pmatrix} 0 & \cdots & 0 & (f_w^1)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$, such a map is defined on ε_r^m by

$$\psi_\eta(\varepsilon_r^m) = \sum_{j=0}^{t_{m-n+1}} \lambda_j(m,r)\varepsilon_j^{m-n+1} \tag{3.2}$$

and for $\eta = \begin{pmatrix} 0 & \cdots & 0 & (f_w^1 f_{w'}^1)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$, such a map is defined on ε_r^m by

$$\psi_\eta(\varepsilon_r^m) = \sum_{j=0}^{t_{m-n+1}} \lambda_j(m,r) f_w^1 \varepsilon_j^{m-n+1} + \lambda_j'(m,r)\varepsilon_j^{m-n+1} f_{w'}^1 \tag{3.3}$$

where $\lambda_j(m,r), \lambda_j'(m,r) \in \Lambda^e$ in general. For details about these maps, see [**13**]. We now restrict Equation (3.2) to the special case where for some $j = r'$, $\lambda_j(m,r) = b_{m,r}(m-n+1, r')$ is a scalar and $\lambda_j(m,r) = 0$ for all $j \neq r'$, that is

$$\psi_\eta(\varepsilon_r^m) = b_{m,r}(m-n+1, r')\varepsilon_{r'}^{m-n+1} \tag{3.4}$$

and restrict Equation (3.3) to the special case where all $\lambda_j(m,r), \lambda_j^{'}(m,r)$ are zero except for some indices s and s' with $\lambda_j(m,r) = \lambda_{m,r}(m-n+1, s) \neq 0$ and $\lambda_j^{'}(m,r) = \lambda_{m,r}(m-n+1, s') \neq 0$. That is,

$$\psi_\eta(\varepsilon_r^m) = \lambda_{m,r}(m-n+1, s) f_w^1 \varepsilon_s^{m-n+1} + \lambda_{m,r}(m-n+1, s')\varepsilon_{s'}^{m-n+1} f_{w'}^1. \tag{3.5}$$

It was shown in [**13**] that Equation (3.1) holds under certain conditions on the scalars $b_{m,r}(m-n+1, r')$, and therefore the special maps given by Equations (3.4) are indeed homotopy lifting maps for the associated cocycles. Similar argument holds for the map given by Equation (3.5).

Our motivation for defining the maps the way it was defined comes from several examples that were computed as well as what we note as appearing in the literature. The rest of this work is devoted to proving these claims - in particular, that the scalars $b_{m,r}(m-n+1, r')$ satisfy some recurrence relations. We observe that if $\psi_\eta(\varepsilon_{\bar{r}}^{m-1}) = b_{m-1,\bar{r}}(m-n, r'')\varepsilon_{r''}^{m-n}$, we can obtain the scalars $b_{m,r}(m-n+1, r')$ in terms of the scalars $b_{m-1,\bar{r}}(m-n, r'')$ and the scalars $c_{pq}(n, i, r)$ coming from the comultiplicative structure on $\mathbb{K}$. The following diagram is not commutative but gives a pictures of this idea:

$$\begin{array}{ccccccccc}
\mathbb{K} := \cdots & \longrightarrow & \mathbb{K}_{m+1} & \longrightarrow & \mathbb{K}_m & \longrightarrow & \mathbb{K}_{m-1} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \psi_{\bar{\eta}}} & & \downarrow{\scriptstyle \psi_{\bar{\eta}}} & & \downarrow{\scriptstyle \psi_{\bar{\eta}}} & & \downarrow{\scriptstyle \psi_{\bar{\eta}}} & & \\
\mathbb{K}[1-n] := \cdots & \longrightarrow & \mathbb{K}_{m-n+2} & \longrightarrow & \mathbb{K}_{m-n+1} & \longrightarrow & \mathbb{K}_{m-n} & \longrightarrow & \cdots
\end{array}$$

• We can obtain $\psi_{\bar{\eta}}|_{\mathbb{K}_{m+1}}$ from $\psi_{\bar{\eta}}|_{\mathbb{K}_m}$ for every m using the scalars $b_{m,r}(m-n+1, r*)$. From Remark 2.7, the scalars $b_{n-1,r^*}(0, r^{**}) = 0$ for all r^*, r^{**} since $\psi_{\bar{\eta}}|_{\mathbb{K}_{n-1}}$ is the zero map.

We start with the following Lemma.

Lemma 3.6. *Let Q be a finite quiver and $\Lambda = kQ/I$ a quiver algebra that is Koszul. Denote by $\{f_r^n\}_{r=0}^{t_n}$ elements of kQ defining a minimal projective resolution of Λ_0 as a right Λ-module. Suppose that $\eta : \mathbb{K}_n \to \Lambda$ is a cocycle such that $\eta = \begin{pmatrix} 0 & \cdots & 0 & (f_w^1)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$, $0 \leq w \leq t_1$. For an integer m, let $\mathbb{K}$ be the projective bimodule resolution of Λ with free* *basis* $\{\varepsilon_r^m\}_{r=0}^{t_m}$ *for* $\mathbb{K}_m$, $\{\varepsilon_{\bar{r}}^{m-1}\}_{\bar{r}=0}^{t_{m-1}}$ *for* $\mathbb{K}_{m-1}$, $\{\varepsilon_{r''}^{m-n}\}_{r''=0}^{t_{m-n}}$ *for* $\mathbb{K}_{m-n}$, *and* $\{\varepsilon_{r'}^{m-n+1}\}_{r'=0}^{t_{m-n+1}}$ *for* $\mathbb{K}_{m-n+1}$. *For the indices* r, r', *let* $b_{m,r}(m-n+1, r')$ *be scalars such that Equation* (3.4) *holds that is* $\psi_\eta(\varepsilon_r^m) = b_{m,r}(m-n+1, r')\varepsilon_{r'}^{m-n+1}$. *If* $d\psi_\eta - (-1)^{1-n}\psi_\eta d = (\eta\otimes 1 - 1\otimes\eta)\Delta_{\mathbb{K}}$, *the following recurrence relations hold.*

$$(i)\ b_{m,r}(m-n+1, r')c_{rr''}(m-n+1, r', 1) = (-1)^{m-1}b_{m-1,\bar{r}}(m-n, r'')c_{r\bar{r}}(m, r, 1) + c_{ir''}(m, r, n),$$

$$b_{m,r}(m-n+1, r')c_{r''r}(m-n+1, r', m-n) = (-1)^{m-1}b_{m-1,\bar{r}}(m-n, r'')c_{\bar{r}r}(m, r, m-1) - (-1)^{n(m-n)}c_{r''i}(m, r, m-n),$$

and for every pair of indices $(p, q) \neq (r, r')$, $(p, q) \neq (r'', r)$,
(ii) $b_{m,r}(m-n+1, r')c_{pq}(m-n+1, r', *) = (-1)^{m-1}b_{m-1,\bar{*}}(m-n, *'')c_{pq}(m, r, *)$ *whenever* $i + r'' = r$.

Proof. We prove this result in the following way; having set indices for basis for the free modules $\mathbb{K}_{m-1}, \mathbb{K}_m, \mathbb{K}_{m-n}, \mathbb{K}_{m-n+1}$, we define the special case of the map ψ_η given by Equation (3.4). We then use the left and right hand side of (3.1) to derive the recurrence relations.

Let us suppose we have a quiver Q generated by two arrows $\{f_r^1, f_s^1\}$ and each $\mathbb{K}_n$ is free of rank 2. For each m let $\{\varepsilon_{\bar{r}}^{m-1}, \varepsilon_{\bar{s}}^{m-1}\}$, $\{\varepsilon_r^m, \varepsilon_s^m\}$, $\{\varepsilon_{r'}^{m-n+1}, \varepsilon_{s'}^{m-n+1}\}$, and $\{\varepsilon_{r''}^{m-n}, \varepsilon_{s''}^{m-n}\}$ be a basis for $\mathbb{K}_{m-1}$, $\mathbb{K}_m$, $\mathbb{K}_{m-n+1}$, and $\mathbb{K}_{m-n}$ respectively. A possible example of this scenario is given in Example (4.4). The differential given by (2.11) on ε_r^m for this special case for instance, is given by

$$\begin{aligned} d(\varepsilon_r^m) = {} & c_{r\bar{r}}(m, r, 1)f_r^1\varepsilon_{\bar{r}}^{m-1} + c_{\bar{r}r}(m, r, m-1)\varepsilon_{\bar{r}}^{m-1}f_r^1 \\ & + c_{r\bar{s}}(m, r, 1)f_r^1\varepsilon_{\bar{s}}^{m-1} + c_{\bar{r}s}(m, r, m-1)\varepsilon_{\bar{r}}^{m-1}f_s^1 + c_{s\bar{r}}(m, r, 1)f_s^1\varepsilon_{\bar{r}}^{m-1} \\ & + c_{\bar{s}r}(m, r, m-1)\varepsilon_{\bar{s}}^{m-1}f_r^1 + c_{s\bar{s}}(m, r, 1)f_s^1\varepsilon_{\bar{s}}^{m-1} + c_{\bar{s}s}(m, r, m-1)\varepsilon_{\bar{s}}^{m-1}f_s^1 \end{aligned}$$

and a similar formula can be written for $d(\varepsilon_s^m)$.
Let us recall that $\eta = \begin{pmatrix} 0 & \cdots & 0 & (f_w^1)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$ means that $\eta(\varepsilon_i^n) = f_w^1$ with $w = r$ or $w = s$ and $\eta(\varepsilon_j^n) = 0$ for all $j \neq i$. From the hypothesis, we define $\psi_\eta : \mathbb{K}_m \to \mathbb{K}_{m-n+1}$ by $\psi_\eta(\varepsilon_r^m) = b_{m,r}(m-n+1, r')\varepsilon_{r'}^{m-n+1}$, and $\psi_\eta(\varepsilon_s^m) = b_{m,s}(m-n+1, s')\varepsilon_{s'}^{m-n+1}$, and $\psi_\eta : \mathbb{K}_{m-1} \to \mathbb{K}_{m-n}$ is defined by $\psi_\eta(\varepsilon_{\bar{r}}^{m-1}) = b_{m-1,\bar{r}}(m-n, r'')\varepsilon_{r''}^{m-n}$, and $\psi_\eta(\varepsilon_{\bar{s}}^{m-1}) = b_{m-1,\bar{s}}(m-n, s'')\varepsilon_{s''}^{m-n}$.

Using Equation(3.1), the expression $(d\psi_\eta - (-1)^{m-1}\psi_\eta d)(\varepsilon_r^m)$ becomes $d(b_{m,r}(m-n+1,r')\varepsilon_{r'}^{m-n+1}) - (-1)^{m-1}\psi_\eta d(\varepsilon_r^m)$ and its equal to

$$\begin{aligned}
&b_{m,r}(m-n+1,r')\Big(c_{rr''}(m-n+1,r',1)f_r^1\varepsilon_{r''}^{m-n} + c_{r''r}(m-n+1,r',m-n)\varepsilon_{r''}^{m-n}f_r^1\\
&+ c_{rs''}(m-n+1,r',1)f_r^1\varepsilon_{s''}^{m-n} + c_{r''s}(m-n+1,r',m-n)\varepsilon_{r''}^{m-n}f_s^1\\
&+ c_{sr''}(m-n+1,r',1)f_s^1\varepsilon_{r''}^{m-n} + c_{s''r}(m-n+1,r',m-n)\varepsilon_{s''}^{m-n}f_r^1\\
&+ c_{ss''}(m-n+1,r',1)f_s^1\varepsilon_{s''}^{m-n} + c_{s''s}(m-n+1,r',m-n)\varepsilon_{s''}^{m-n}f_s^1\Big)\\
&- (-1)^{m-1}\Big(b_{m-1,\bar r}(m-n,r'')c_{r\bar r}(m,r,1)f_r^1\varepsilon_{r''}^{m-n} + b_{m-1,\bar r}(m-n,r'')c_{\bar r r}(m,r,m-1)\varepsilon_{r''}^{m-n}f_r^1\\
&+ b_{m-1,\bar s}(m-n,s'')c_{r\bar s}(m,r,1)f_r^1\varepsilon_{s''}^{m-n} + b_{m-1,\bar r}(m-n,r'')c_{\bar r s}(m,r,m-1)\varepsilon_{r''}^{m-n}f_s^1\\
&+ b_{m-1,\bar r}(m-n,r'')c_{s\bar r}(m,r,1)f_s^1\varepsilon_{r''}^{m-n} + b_{m-1,\bar s}(m-n,s'')c_{\bar s r}(m,r,m-1)\varepsilon_{s''}^{m-n}f_r^1\\
&+ b_{m-1,\bar s}(m-n,s'')c_{s\bar s}(m,r,1)f_s^1\varepsilon_{s''}^{m-n} + b_{m-1,\bar s}(m-n,s'')c_{\bar s s}(m,r,m-1)\varepsilon_{s''}^{m-n}f_s^1\Big).
\end{aligned}$$

On the other hand, the diagonal map is given by $\Delta_{\mathbb{K}}(\varepsilon_r^m) = \displaystyle\sum_{x+y=r}\sum_{u+v=m} c_{x,y}(m,r,u)\varepsilon_x^u \otimes_\Lambda \varepsilon_y^v$.

We obtain a non-zero in the expansion of $(\eta\otimes 1 - 1\otimes\eta)\Delta_{\mathbb{K}}(\varepsilon_r^m)$ whenever $x=i$ and $y=i$. This means that

$$\begin{aligned}
&(\eta\otimes 1 - 1\otimes\eta)\sum_{x+y=r}\sum_{u+v=m} c_{x,y}(m,r,u)\varepsilon_x^u\otimes_\Lambda\varepsilon_y^v\\
&= (\eta\otimes 1)(c_{i,y}(m,r,n)\varepsilon_i^n\otimes_\Lambda\varepsilon_y^{m-n}) - (1\otimes\eta)(c_{x,i}(m,r,m-n)\varepsilon_x^{m-n}\otimes_\Lambda\varepsilon_i^n)\\
&= c_{i,y}(m,r,n)f_w^1\varepsilon_y^{m-n} - (-1)^{n(m-n)}c_{xi}(m,r,m-n)\varepsilon_x^{m-n}f_w^1,
\end{aligned}$$

for $\{x,y\} = \{r'',s''\}$ with $i+y=r, x+i=r$ and some arrow f_w^1. After collecting common terms, the expression $(d\psi_\eta - (-1)^{m-1}\psi_\eta d)(\varepsilon_r^m)$ which is the left hand side of Equation (3.1) becomes

$$\begin{aligned}
&\Big(b_{m,r}(m-n+1,r')c_{rr''}(m-n+1,r',1) - (-1)^{m-1}b_{m-1,\bar r}(m-n,r'')c_{r\bar r}(m,r,1)\Big)f_r^1\varepsilon_{r''}^{m-n}\\
&+\Big(b_{m,r}(m-n+1,r')c_{r''r}(m-n+1,r',m-n) - (-1)^{m-1}b_{m-1,\bar r}(m-n,r'')c_{\bar r r}(m,r,m-1)\Big)\varepsilon_{r''}^{m-n}f_r^1\\
&+\Big(b_{m,r}(m-n+1,r')c_{rs''}(m-n+1,r',1) - (-1)^{m-1}b_{m-1,\bar s}(m-n,s'')c_{r\bar s}(m,r,1)\Big)f_r^1\varepsilon_{s''}^{m-n}\\
&+\Big(b_{m,r}(m-n+1,r')c_{r''s}(m-n+1,r',m-n) - (-1)^{m-1}b_{m-1,\bar r}(m-n,r'')c_{\bar r s}(m,r,m-1)\Big)\varepsilon_{r''}^{m-n}f_s^1\\
&+\Big(b_{m,r}(m-n+1,r')c_{sr''}(m-n+1,r',1) - (-1)^{m-1}b_{m-1,\bar r}(m-n,r'')c_{s\bar r}(m,r,1)\Big)f_s^1\varepsilon_{r''}^{m-n}\\
&+\Big(b_{m,r}(m-n+1,r')c_{s''r}(m-n+1,r',m-n) - (-1)^{m-1}b_{m-1,\bar s}(m-n,s'')c_{\bar s r}(m,r,m-1)\Big)\varepsilon_{s''}^{m-n}f_r^1\\
&+\Big(b_{m,r}(m-n+1,r')c_{ss''}(m-n+1,r',1) - (-1)^{m-1}b_{m-1,\bar s}(m-n,s'')c_{s\bar s}(m,r,1)\Big)f_s^1\varepsilon_{s''}^{m-n}\\
&+\Big(b_{m,r}(m-n+1,r')c_{s''s}(m-n+1,r',m-n) - (-1)^{m-1}b_{m-1,\bar s}(m-n,s'')c_{\bar s s}(m,r,m-1)\Big)\varepsilon_{s''}^{m-n}f_s^1.
\end{aligned}$$

The expression $(\eta\otimes 1 - 1\otimes\eta)\Delta_{\mathbb{K}}(\varepsilon_r^m)$ which is the right hand side of Equation (3.1) still remains $c_{i,y}(m,r,n)f_w^1\varepsilon_y^{m-n} - (-1)^{n(m-n)}c_{xi}(m,r,m-n)\varepsilon_x^{m-n}f_w^1$. We observe the following about indices w,x,y. The index w is either r or s, the index y is either r'' or s'' and

the index x is either r'' or s''. We notice that the additional constraint that $i+y=r$ and $i+x=r$ implies that whenever $y=r''$ we must have $x=r''$ and whenever $y=s''$, we must have $x=s''$. We therefore have the following four cases:
Case I: Whenever $w=r$, $y=x=r''$, $i+r''=r$, we have the following set of *recurrence* relations on the scalars $b_{m,r}(m-n+1,r')$,

$$b_{m,r}(m-n+1,r')c_{rr''}(m-n+1,r',1)-(-1)^{m-1}b_{m-1,\bar{r}}(m-n,r'')c_{r\bar{r}}(m,r,1)=c_{ir''}(m,r,n)$$
$$b_{m,r}(m-n+1,r')c_{r''r}(m-n+1,r',m-n)-(-1)^{m-1}b_{m-1,\bar{r}}(m-n,r'')c_{\bar{r}r}(m,r,m-1)$$
$$=-(-1)^{n(m-n)}c_{r''i}(m,r,m-n)$$
$$b_{m,r}(m-n+1,r')c_{rs''}(m-n+1,r',1)-(-1)^{m-1}b_{m-1,\bar{s}}(m-n,s'')c_{r\bar{s}}(m,r,1)=0$$
$$b_{m,r}(m-n+1,r')c_{r''s}(m-n+1,r',m-n)-(-1)^{m-1}b_{m-1,\bar{r}}(m-n,r'')c_{\bar{r}s}(m,r,m-1)=0$$
$$b_{m,r}(m-n+1,r')c_{sr''}(m-n+1,r',1)-(-1)^{m-1}b_{m-1,\bar{r}}(m-n,r'')c_{s\bar{r}}(m,r,1)=0$$
$$b_{m,r}(m-n+1,r')c_{s''r}(m-n+1,r',m-n)-(-1)^{m-1}b_{m-1,\bar{s}}(m-n,s'')c_{\bar{s}r}(m,r,m-1)=0$$
$$b_{m,r}(m-n+1,r')c_{ss''}(m-n+1,r',1)-(-1)^{m-1}b_{m-1,\bar{s}}(m-n,s'')c_{s\bar{s}}(m,r,1)=0$$
$$b_{m,r}(m-n+1,r')c_{s''s}(m-n+1,r',m-n)-(-1)^{m-1}b_{m-1,\bar{s}}(m-n,s'')c_{\bar{s}s}(m,r,m-1)=0.$$

We note that all the equations of Case (I) above can be expressed more succinctly to mean that whenever $w=r$, $i+r''=r$ and for all $s\neq r$

$$b_{m,r}(m-n+1,r')c_{rr''}(m-n+1,r',1)=(-1)^{m-1}b_{m-1,\bar{r}}(m-n,r'')c_{r\bar{r}}(m,r,1)+c_{ir''}(m,r,n),$$
$$b_{m,r}(m-n+1,r')c_{r''r}(m-n+1,r',m-n)=(-1)^{m-1}b_{m-1,\bar{r}}(m-n,r'')c_{\bar{r}r}(m,r,m-1)$$
$$-(-1)^{n(m-n)}c_{r''i}(m,r,m-n),$$

and for every indices such that $(p,q)\neq(r,r'')$ and $(p,q)\neq(r'',r)$, $b_{m,r}(m-n+1,r')c_{pq}(m-n+1,r',*)=(-1)^{m-1}b_{m-1,\bar{*}}(m-n,*'')c_{pq}(m,r,*)$.
Case II: Whenever $w=s$, $y=x=r''$, $i+r''=r$, we have the following set of *recurrence* relations on the scalars $b_{m,r}(m-n+1,r')$,

$$b_{m,r}(m-n+1,r')c_{sr''}(m-n+1,r',1)=(-1)^{m-1}b_{m-1,\bar{r}}(m-n,r'')c_{r\bar{r}}(m,r,1)+c_{ir''}(m,r,n),$$
$$b_{m,r}(m-n+1,r')c_{r''s}(m-n+1,r',m-n)=(-1)^{m-1}b_{m-1,\bar{r}}(m-n,r'')c_{\bar{r}r}(m,r,m-1)$$
$$-(-1)^{n(m-n)}c_{r''i}(m,r,m-n),$$

and for every indices $(p,q)\neq(s,r''),(p,q)\neq(r'',s)$,
$b_{m,r}(m-n+1,r')c_{pq}(m-n+1,r',*)=(-1)^{m-1}b_{m-1,\bar{*}}(m-n,*'')c_{pq}(m,r,*)$.
Case III: Whenever $w=r$, $y=x=s''$, $i+s''=r$, we have the following set of *recurrence* relations on the scalars $b_{m,r}(m-n+1,r')$,

$$b_{m,r}(m-n+1,r')c_{rs''}(m-n+1,r',1)=(-1)^{m-1}b_{m-1,\bar{s}}(m-n,s'')c_{r\bar{s}}(m,r,1)+c_{is''}(m,r,n),$$
$$b_{m,r}(m-n+1,r')c_{s''r}(m-n+1,r',m-n)=(-1)^{m-1}b_{m-1,\bar{s}}(m-n,s'')c_{\bar{s}r}(m,r,m-1)$$
$$-(-1)^{n(m-n)}c_{s''i}(m,r,m-n),$$

and for every indices $(p,q)\neq(r,s''),(p,q)\neq(s'',r)$
$b_{m,r}(m-n+1,r')c_{pq}(m-n+1,r',*)=(-1)^{m-1}b_{m-1,\bar{*}}(m-n,*'')c_{pq}(m,r,*)$.

Case IV: Whenever $w = s$, $y = x = s''$, $i + s'' = r$, we have the following set of *recurrence* relations on the scalars $b_{m,r}(m-n+1, r')$,

$$b_{m,r}(m-n+1, r')c_{ss''}(m-n+1, r', 1) = (-1)^{m-1}b_{m-1,\bar{s}}(m-n, s'')c_{s\bar{s}}(m, r, 1)+c_{is''}(m, r, n),$$
$$\begin{aligned} b_{m,r}(m-n+1, r')c_{s''s}(m-n+1, r', m-n) = (-1)^{m-1}&b_{m-1,\bar{s}}(m-n, s'')c_{\bar{s}s}(m, r, m-1) \\ &- (-1)^{n(m-n)}c_{s''i}(m, r, m-n), \end{aligned}$$

and for every indices $(p, q) \neq (s, s''), (p, q) \neq (s'', s)$
$b_{m,r}(m-n+1, r')c_{pq}(m-n+1, r', *) = (-1)^{m-1}b_{m-1,\bar{*}}(m-n, *'')c_{pq}(m, r, *)$.

More generally, if $\mathbb{K}_m$ is generated by $\{\varepsilon_r^m\}_{r=1}^{t_m}$, $\mathbb{K}_{m-1}$ generated by $\{\varepsilon_{\bar{r}}^{m-1}\}_{\bar{r}=1}^{t_{m-1}}$, $\mathbb{K}_{m-n}$ by $\{\varepsilon_{r''}^{m-n}\}_{r''=1}^{t_{m-n}}$, and $\mathbb{K}_{m-n+1}$ by $\{\varepsilon_{r'}^{m-n+1}\}_{r'=1}^{t_{m-n+1}}$, the following relations hold for all r, r', r'' and $\bar{r}$

$$(i)\ b_{m,r}(m-n+1, r')c_{rr''}(m-n+1, r', 1) = (-1)^{m-1}b_{m-1,\bar{r}}(m-n, r'')c_{r\bar{r}}(m, r, 1)+c_{ir''}(m, r, n),$$
$$\begin{aligned} b_{m,r}(m-n+1, r')c_{r''r}(m-n+1, r', m-n) = (-1)^{m-1}&b_{m-1,\bar{r}}(m-n, r'')c_{\bar{r}r}(m, r, m-1) \\ &- (-1)^{n(m-n)}c_{r''i}(m, r, m-n), \end{aligned}$$

and for every pair of indices $pq \neq rr''$, $pq \neq r''r$,
(ii) $b_{m,r}(m-n+1, r')c_{pq}(m-n+1, r', *) = (-1)^{m-1}b_{m-1,\bar{*}}(m-n, *'')c_{pq}(m, r, *)$. □

THEOREM 3.7. *Let Q be a finite quiver and $\Lambda = kQ/I$ a quiver algebra that is Koszul. Denote by $\{f_r^n\}_{r=0}^{t_n}$ elements of kQ defining a minimal projective resolution of Λ_0 as a right Λ-module. Suppose that $\eta : \mathbb{K}_n \to \Lambda$ is a cocycle such that $\eta = \begin{pmatrix}0 & \cdots & 0 & (f_w^1)^{(i)} & 0 & \cdots & 0\end{pmatrix}$, $0 \leq w \leq t_1$. For an integer m, let $\mathbb{K}$ be the projective bimodule resolution of Λ with free basis $\{\varepsilon_r^m\}_{r=0}^{t_m}$ for $\mathbb{K}_m$, $\{\varepsilon_{\bar{r}}^{m-1}\}_{\bar{r}=0}^{t_{m-1}}$ for $\mathbb{K}_{m-1}$, $\{\varepsilon_{r''}^{m-n}\}_{r''=0}^{t_{m-n}}$ for $\mathbb{K}_{m-n}$, and $\{\varepsilon_{r'}^{m-n+1}\}_{r'=0}^{t_{m-n+1}}$ for $\mathbb{K}_{m-n+1}$. There are scalars $\lambda_{m,r}(m-n+1, r')$ such that the map $\psi_\eta : \mathbb{K}_m \to \mathbb{K}_{m-n+1}$ associated to η and defined by*

$$\psi_\eta(\varepsilon_r^m) = \lambda_{m,r}(m-n+1, r')\varepsilon_{r'}^{m-n+1}$$

is a homotopy lifting map for η.

PROOF. Let $\lambda_{m,r}(m-n+1, r') = b_{m,r}(m-n+1, r')$. By Lemma 3.6, the equation $d\psi_\eta - (-1)^{1-n}\psi_\eta d = (\eta \otimes 1 - 1 \otimes \eta)\Delta_{\mathbb{K}}$ holds, so ψ_η is a homotopy lifting map. □

THEOREM 3.8. *Let Q be a finite quiver and let $\Lambda = kQ/I$ be a quiver algebra that is Koszul. Denote by $\{f_r^m\}_{r=0}^{t_m}$ elements of kQ defining a minimal projective resolution of Λ_0 as a right Λ-module. Let $\mathbb{K}$ be the projective bimodule resolution of Λ with basis of $\mathbb{K}_m$ consisting of all $\{\varepsilon_r^m\}_{r=0}^{t_m}$. Suppose that $\eta : \mathbb{K}_n \to \Lambda$ is a cocycle such that $\eta = \begin{pmatrix}0 & \cdots & 0 & (f_w^1 f_{w'}^1)^{(i)} & 0 & \cdots & 0\end{pmatrix}$ for some $0 \leq w, w' \leq t_1$. Then there exist scalars $\lambda_{m,r}(m-n+1, s)$ and $\lambda_{m,r}(m-n+1, s')$ such that $\psi_\eta : \mathbb{K}_m \to \mathbb{K}_{m-n+1}$ defined by*

$$\psi_\eta(\varepsilon_r^m) = \lambda_{m,r}(m-n+1, s)f_w^1\varepsilon_s^{m-n+1} + \lambda_{m,r}(m-n+1, s')\varepsilon_{s'}^{m-n+1}f_{w'}^1$$

for all ε_r^m is a homotopy lifting map for η.

PROOF. Although we do not write them here, we do say that there are recurrence relations the scalars given in Equation (3.5) satisfy so that $d\psi_\eta - (-1)^{1-n}\psi_\eta d = (\eta \otimes 1 - 1 \otimes \eta)\Delta_{\mathbb{K}}$ holds true. See [**14**, Lemma 5.20] and [**14**, Theorem 5.23] for details about this. □

3.1. A special case of Theorem 3.7. We consider a special case in which each Λ^e-module $\mathbb{K}_n$ is free of rank 1. This case arises for example, from a quiver with one arrow (a loop) on a vertex e_1. We also give a concrete example in Example (4.1). Let $I = (x^n)$ be an ideal of the path algebra kQ. The quiver algebra of interest here is Morita equivalent the truncated polynomial ring $A = k[x]/(x^n)$. This is the case where $f_r^n = x^n$ where $r = 0$ for all n and $\varepsilon_r^n = 1 \otimes \widetilde{f_r^n} \otimes 1$. From the Preliminaries (2), there are scalars $c_{p,q}(m, r, u)$ for which the diagonal map is given by

$$\Delta_{\mathbb{K}}(\varepsilon_r^m) = \sum_{u+v=m} c_{i,j}(m, r, u)\varepsilon_i^u \otimes_\Lambda \varepsilon_j^v, \tag{3.9}$$

with $i = j = r = 0$. Also with $p = r = 0$, the differential takes the form

$$d(\varepsilon_r^m) = c_{p,r}(m, r, 1)f_p^1\varepsilon_r^{m-1} + (-1)^m c_{r,p}(m, r, m-1)\varepsilon_r^{m-1}f_p^1.$$

Let $\chi : \mathbb{K}_n \to A$ be an n-cocycle defined by $\chi(\varepsilon_r^n) = f_r^1$. According to Theorem (3.7), a homotopy lifting map for χ can be given by

$$\psi_{\chi_m}(\varepsilon_r^m) = b_{m,r}(m-n+1, r)\varepsilon_r^{m-n+1}, \qquad r = 0.$$

We can determine $b_{m,r}(m-n+1, r)$ from the previous scalar $b_{m-1,r}(m-n, r)$. In order words, the conditions (i) and (ii) of Theorem (3.7) is a *recurrence relation.* We know from Defintion (2.4) that homotopy lifting maps satisfy

$$\begin{aligned}
&(d\psi_\chi - (-1)^{m-1}\psi_\chi d)(\varepsilon_r^m) = (\chi \otimes 1 - 1 \otimes \chi)\Delta_{\mathbb{K}}(\varepsilon_r^m), \qquad \text{so then}\\
&d(b_{m,r}(m-n+1, r)\varepsilon_r^{m-n+1}) - (-1)^{m-1}\psi_\chi\Big(c_{p,r}(m, r, 1)f_p^1\varepsilon_r^{m-1} + (-1)^m c_{r,p}(m, r, m-1)\varepsilon_r^{m-1}f_p^1\Big)\\
&= (\chi \otimes 1 - 1 \otimes \chi)\sum_{u+v=m} c_{i,j}(m, r, u)\varepsilon_i^u \otimes_\Lambda \varepsilon_j^v.
\end{aligned}$$

The modules $\mathbb{K}$ are free of rank 1 so we get

$$\begin{aligned}
b_{m,r}(m-n+1, r)c_{p,r}(m-n+1, r, 1)f_p^1\varepsilon_r^{m-n} + (-1)^{m-n+1}b_{m,r}(m-n+1, r)&\\
c_{r,p}(m-n+1, r, m-n)\varepsilon_r^{m-n}f_p^1 - (-1)^{m-1}b_{m-1,r}(m-n, r)c_{p,r}(m, r, 1)f_p^1\varepsilon_r^{m-n}&\\
+ b_{m-1,r}(m-n, r)c_{r,p}(m, r, m-1)\varepsilon_r^{m-n}f_p^1&\\
= c_{r,j}(m, r, n)f_r^1\varepsilon_j^{m-n} + (-1)^{n(m-n)}c_{i,r}(m, r, m-n)\varepsilon_i^{m-n}f_r^1&
\end{aligned}$$

We would obtain the following expressions for the above equality to hold,

$$\begin{aligned}
&b_{m,r}(m-n+1, r)c_{p,r}(m-n+1, r, 1) - (-1)^{m-1}b_{m-1,r}(m-n, r)c_{p,r}(m, r, 1) = c_{p,r}(m, r, n) \quad \text{and}\\
&(-1)^{m-n+1}b_{m,r}(m-n+1, r)c_{r,p}(m-n+1, r, m-n) + b_{m-1,r}(m-n, r)c_{r,p}(m, r, m-1)\\
&= (-1)^{n(m-n)}c_{r,p}(m, r, m-n).
\end{aligned}$$

The scalars $c_{p,r}(m-n+1, r, *)$ come from the differentials on the resolution $\mathbb{K}$, so they are not equal to 0 for all r. In case $c_{p,r}(m-n+1, r, *) \neq 0$ for all r, the first equality in the last expression yields

$$b_{m,r}(m-n+1, r) = \frac{(-1)^{m-1}b_{m-1,r}(m-n, r)c_{p,r}(m, r, 1) + c_{p,r}(m, r, n)}{c_{p,r}(m-n+1, r, 1)} \tag{3.10}$$

while the second one yields

(3.11)

$$b_{m,r}(m-n+1,r) = \frac{b_{m-1,r}(m-n,r)c_{r,p}(m,r,m-1) + (-1)^{n(m-n)+1}c_{r,p}(m,r,m-n)}{(-1)^{m-n}c_{r,p}(m-n+1,r,m-n)}.$$

We now present the Gerstenhaber bracket structure on Hochschild cohomology using these scalars.

Theorem 3.12. *Let $\Lambda = kQ/I$ be a quiver algebra that is Koszul. Denote by $\{f_r^m\}_{r=0}^{t_m}$ elements of kQ defining a minimal projective resolution of Λ_0 as a right Λ-module. Let $\mathbb{K}$ be the projective bimodule resolution of Λ with $\mathbb{K}_m$ having basis $\{\varepsilon_r^m\}_{r=0}^{t_m}$. Assume that $\eta : \mathbb{K}_n \to \Lambda$ and $\theta : \mathbb{K}_m \to \Lambda$ represent elements in $\mathrm{HH}^*(\Lambda)$ and are given by $\eta(\varepsilon_i^n) = \lambda_i$ for $i = 0, 1, \ldots, t_n$ and $\theta(\varepsilon_j^m) = \beta_j$ for $j = 0, 1, \ldots, t_m$, with each λ_i and β_j paths of length of 1. Then the class of the bracket $[\eta, \theta] : \mathbb{K}_{n+m-1} \to \Lambda$ can be expressed on the r-th basis element ε_r^{m+n-1} as*

$$[\eta,\theta](\varepsilon_r^{m+n-1}) = \sum_{i=0}^{t_n}\sum_{j=0}^{t_m} b_{m-n+1,r}(n,i)\lambda_i - (-1)^{(m-1)(n-1)}(b_{m-n+1,r}(m,j)\beta_j$$

for some scalars $b_{m-n+1,r}(n,i)$ and $b_{m-n+1,r}(m,j)$ associated with homotopy lifting maps $\psi_{\theta(j)}$ and $\psi_{\eta(i)}$ respectively.

Proof. This is same as [**13**, Theorem 3.15] and proved therein. $\square$

4. Some computations and examples

In this section, we give examples in which the scalars $b_{m,r}(m-n+1,*)$ are obtained from $b_{m-1,r}(m-n,**)$ using the recurrence relations of Theorem (3.7), Equations (3.10) and (3.11). In most of the examples, we described the scalars $c_{p,q}(m,r,n)$ which are also used in the recurrence relations.

Example 4.1. *Let's consider the following quiver*

$$Q := \begin{tikzcd} 1 \arrow[loop left, "x"] \end{tikzcd},$$

and take $A = k[x]/(x^n)$ to be the truncated polynomial ring. The idempotent $f_0^0 = 1$, generates $\mathbb{K}_0$, $f_0^1 = x$ generates $\mathbb{K}_1$ and $f_0^n = x^n$ generates $\mathbb{K}_n$. Notice that we can write $f_0^n = f_0^1 f_0^{n-1} = f_0^{n-1} f_0^1$, so then $c_{0,0}(n,0,1) = c_{0,0}(n,0,n-1) = 1$. If we ***assume that*** *$n = 2$, that is, we mode out by the ideal $I = (x^2)$, some calculations show that using $\varepsilon_0^n = 1 \otimes f_0^n \otimes 1$,*

$$\begin{aligned}
d(\varepsilon_0^1) &= d(1\otimes x\otimes 1) = x(1\otimes 1) - (1\otimes 1)x = x\varepsilon_0^0 - \varepsilon_0^0 x\\
d(\varepsilon_0^2) &= d(1\otimes x\otimes x\otimes 1) = x(1\otimes x\otimes 1) - 1\otimes x^2\otimes 1 + (1\otimes x\otimes 1)x = x\varepsilon_0^1 + \varepsilon_0^1 x\\
d(\varepsilon_0^3) &= d(1\otimes x\otimes x\otimes x\otimes 1) = x(1\otimes x\otimes x\otimes 1) - (1\otimes x\otimes x\otimes 1)x\\
&= x\varepsilon_0^1 - \varepsilon_0^1 x \quad \textit{and more generally}\\
d(\varepsilon_0^n) &= x\varepsilon_0^{n-1} - (-1)^{n-1}\varepsilon_0^{n-1}x = f_0^1\varepsilon_0^{n-1} - (-1)^{n-1}\varepsilon_0^{n-1}f_0^1,
\end{aligned}$$

Let $\eta : \mathbb{K}_1 \to A$ be defined by $\eta(\varepsilon_0^1) = x$. Also let $\chi : \mathbb{K}_2 \to A$ be defined by $\chi(\varepsilon_0^2) = x$. A short calculation shows that η and χ are cocycles. A diagonal map for this particular resolution is given by $\Delta_{\mathbb{K}}(\varepsilon_0^m) = \sum_{i+j=m} \varepsilon_0^i \otimes \varepsilon_0^j$. It can be verified by direct evaluation of Equation (2.5) that the map

$$\psi_\eta : \mathbb{K}_m \to \mathbb{K}_m \qquad \text{defined by} \qquad \psi_\eta(\varepsilon_0^m) = m\varepsilon_0^m$$

is a homotopy lifting map for η that is

(4.2)

$$\begin{aligned}
(d\psi_\eta - (-1)^0\psi_\eta d)(\varepsilon_0^m) &= d(m\varepsilon_0^m) - \psi_\eta(x\varepsilon_0^{m-1} - (-1)^{m-1}\varepsilon_0^{m-1}x) \\
&= mx\varepsilon_0^{m-1} - (-1)^{m-1}m\varepsilon_0^{m-1}x - (m-1)x\varepsilon_0^{m-1} + (-1)^{m-1}(m-1)\varepsilon_0^{m-1}x \\
&= x\varepsilon_0^{m-1} - (-1)^{m-1}\varepsilon_0^{m-1}x \qquad \text{is equal to} \\
(\eta\otimes 1 - 1\otimes\eta)\Delta_{\mathbb{K}}(\varepsilon_0^m) &= (\eta\otimes 1 - 1\otimes\eta)\sum_{i+j=m}\varepsilon_0^i\otimes\varepsilon_0^j \\
&= \eta\otimes 1(\varepsilon_0^1\otimes\varepsilon_0^{m-1}) - (-1)^{m-1}1\otimes\eta(\varepsilon_0^{m-1}\otimes\varepsilon_0^1) \\
&= x\varepsilon_0^{m-1} - (-1)^{m-1}\varepsilon_0^{m-1}x,
\end{aligned}$$

where the Koszul sign convention has been employed in the expansion of $(1\otimes\eta)(\varepsilon_0^{m-1}\otimes\varepsilon_0^1) = (-1)^{degree(\eta)\cdot(m-1)}\varepsilon_0^{m-1}\eta(\varepsilon_0^1)$. We note that by the general definition given in Theorem 3.7, the map $\psi_\eta : \mathbb{K}_{m-1} \to \mathbb{K}_{m-1}$ defined by $\psi_\eta(\varepsilon_0^{m-1}) = (m-1)\varepsilon_0^{m-1}$ implies that $b_{m-1,0}(m-1,0) = m-1$. The map η is a 1-cocycle so $n = 1$, $r = p = 0$.

We can use the expression of (3.11) to verify that

$$\begin{aligned}
b_{m,r}(m-n+1,r) &= \frac{b_{m-1,r}(m-n,r)c_{r,p}(m,r,m-1) + (-1)^{n(m-n)+1}c_{r,p}(m,r,m-n)}{(-1)^{m-n}c_{r,p}(m-n+1,r,m-n)} \\
b_{m,0}(m,0) &= \frac{b_{m-1,0}(m-1,0)c_{0,0}(m,0,m-1) + (-1)^m c_{0,0}(m,0,m-1)}{(-1)^{m-1}c_{0,0}(m,0,m-1)} \\
&= \frac{m-1+(-1)^m}{1} = m, \qquad \text{when } m \text{ is even,}
\end{aligned}$$

and the expression of (3.10) to verify that

$$\begin{aligned}
b_{m,0}(m,0) &= \frac{(-1)^{m-1}b_{m-1,0}(m-1,0)c_{0,0}(m,0,1) + c_{0,0}(m,0,1)}{c_{0,0}(m,0,1)} \\
&= \frac{m-1+1}{1} = m, \qquad \text{when } m \text{ is odd.}
\end{aligned}$$

Similarly, it is a straightforward calculation (same calculations as (4.2)) to verify that the map $\psi_\chi : \mathbb{K}_m \to \mathbb{K}_{m-1}$ defined by

$$\psi_\chi(\varepsilon_0^m) = b_{m,0}(m-1,0)\varepsilon_0^{m-1} = \begin{cases} \varepsilon_0^{m-1}, & \text{when m is even} \\ 0, & \text{when m is odd} \end{cases}$$

is a homotopy lifting map for χ. In this case $b_{m,0}(m-1,0) = 1$ when m is even and 0 when m is odd. But we can also use the expression of (3.10) to verify that when m is even,

$$b_{m+1,0}(m,0) = \frac{(-1)^{m-1}b_{m,0}(m-1,0)c_{0,0}(m,0,1) + c_{0,0}(m,0,2)}{c_{0,0}(m-n+1,0,1)} = -1+1 = 0,$$

and when m is odd,

$$b_{m+1,0}(m,0) = \frac{(-1)^{m-1} b_{m,0}(m-1,0) c_{0,0}(m,0,1) + c_{0,0}(m,0,2)}{c_{0,0}(m-n+1,0,1)} = 0 + 1 = 1.$$

EXAMPLE 4.3. *The following example of a homotopy lifting map was first given in* [**10**, Example 4.7.2]. *We will now verify that the recurrence relations also hold. Let k be a field and $A = k[x]/(x^3)$. Consider the following projective bimodule resolution of A:*

$$\mathbb{P}_\bullet : \qquad \cdots \to A \otimes A \xrightarrow{\cdot u} A \otimes A \xrightarrow{\cdot u} \cdots \xrightarrow{\cdot v} A \otimes A \xrightarrow{\cdot u} A \otimes A \; (\xrightarrow{\mu} A)$$

where $u = x \otimes 1 - 1 \otimes x$ and $v = x^2 \otimes 1 + x \otimes x + 1 \otimes x^2$. We consider the following elements $e_m := 1 \otimes 1$, $r = 0$ for all m in the m-th module $P_m := A \otimes A$. A diagonal map $\Delta_{\mathbb{P}} : \mathbb{P} \to \mathbb{P} \otimes_A \mathbb{P}$ for this resolution is given by

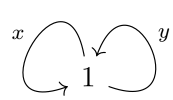
$$\Delta_{\mathbb{P}}(e_m) = \sum_{j+l=m} (-1)^j e_j \otimes e_l.$$

By comparing $\Delta_{\mathbb{P}}(e_m)$ with Equation (2.14)*, the scalars $c_{rr}(m,r,j) = (-1)^j$ for all m, r. Consider the Hochschild 1-cocycle $\alpha : \mathbb{P}_1 \to A$ defined by $\alpha(e_1) = x$ and $\alpha(e_m) = 0$ for all $m \neq 1$. With a slight notation, it was shown in* [**10**, Example 4.7.2] *that the following $\psi_\alpha : \mathbb{P}_{2m} \to \mathbb{P}_{2m}$ defined by $\psi_\alpha(e_{2m}) = -3m \cdot e_{2m}$ is a homotopy lifting map for α. We note that the map ψ_α was regarded as an A_∞-coderivation in* [**10**]. *It can be also verified that ψ_α is a homotopy lifting map. We can use the recurrence relations of Equation* (3.11) *to obtain $b_{2m+1,r}(2m+1,r)$ from $b_{2m,r}(2m,r) = -3m$. That is*

$$\begin{aligned} b_{2m+1,r}(2m+1,r) &= \frac{b_{2m,r}(2m,r) c_{r,r}(2m,r,2m) + (-1)^{2m+1} c_{r,r}(2m+1,r,2m)}{(-1)^{2m} c_{r,r}(2m+1,r,2m)} \\ &= \frac{-3m(-1)^{2m} + (-1)^{2m+1}(-1)^{2m}}{(-1)^{2m}(-1)^{2m-1+1}} = \frac{-3m-1}{1}, \end{aligned}$$

so it follows that $\psi_\alpha : \mathbb{P}_{2m+1} \to \mathbb{P}_{2m+1}$ is defined by $\psi_\alpha(e_{2m+1}) = (-3m-1)e_{2m+1}$.

EXAMPLE 4.4. *Let k be a field of characteristics different from 2. Consider the quiver algebra $A = kQ/I$ (also examined in* [**3**, Example 5]*) defined using the following finite quiver:*



with one vertex and two arrows x, y. We denote by e_1 the idempotent associated with the only vertex. Let I, an ideal of the path algebra kQ be defined by $I = \langle x^2, xy + yx \rangle$. Since $\{x^2, xy + yx\}$ is a quadratic Grobner basis for the ideal generated by relations under the length lexicographich order with $x > y > 1$, the algebra is Koszul.

In order to define a comultiplicative structure, we take $t_0 = 0, t_n = 1$ for all n, $f_0^0 = e_1, f_1^0 = 0, f_0^1 = x, f_1^1 = y, f_0^2 = x^2, f_1^2 = xy + yx, f_0^3 = x^3, f_1^3 = x^2y + xyx + yx^2$, and in general $f_0^n = x^n, f_1^n = \sum_{i+j=n-1} x^i y x^j$. We also see that $f_0^n = f_0^r f_0^{n-r}$ and $f_1^n = f_0^r f_1^{n-r} + f_1^r f_0^{n-r}$ so $c_{00}(n,0,r) = c_{01}(n,1,r) = c_{10}(n,1,r) = 1$ and all other $c_{pq}(n,i,r) = 0$.

With the above stated, we can construct the resolution $\mathbb{K}$ for the algebra A. A calculation shows that

$$d_1(\varepsilon_0^1) = x\varepsilon_0^0 - \varepsilon_0^0 x, \qquad d_1(\varepsilon_1^1) = y\varepsilon_0^0 - \varepsilon_0^0 y$$
$$d_2(\varepsilon_0^2) = x\varepsilon_0^1 + \varepsilon_0^1 x, \qquad d_2(\varepsilon_1^2) = y\varepsilon_0^1 + \varepsilon_0^1 y + x\varepsilon_1^1 + \varepsilon_1^1 x.$$

Consider the following map $\theta : \mathbb{K}_1 \to A$ defined by $\theta = (0 \quad y)$. With the following calculations

$$\theta d_2(\varepsilon_0^2) = \theta(x\varepsilon_0^1 + \varepsilon_0^1 x) = 0, \quad \theta d_2(\varepsilon_1^2) = \theta(y\varepsilon_0^1 + \varepsilon_0^1 y + x\varepsilon_1^1 + \varepsilon_1^1 x) = 0 + xy + yx = 0,$$

θ is a cocycle. The comultiplicative map $\Delta : \mathbb{K} \to \mathbb{K} \otimes_A \mathbb{K}$ on ε_0^1, $\varepsilon_1^1, \varepsilon_0^2, \varepsilon_1^2$ is given by

$$\Delta(\varepsilon_0^1) = c_{00}(1,0,0)\varepsilon_0^0 \otimes \varepsilon_0^1 + c_{00}(1,0,1)\varepsilon_0^1 \otimes \varepsilon_0^0 = \varepsilon_0^0 \otimes \varepsilon_0^1 + \varepsilon_0^1 \otimes \varepsilon_0^0,$$
$$\Delta(\varepsilon_1^1) = \varepsilon_0^0 \otimes \varepsilon_1^1 + \varepsilon_1^1 \otimes \varepsilon_0^0,$$
$$\Delta(\varepsilon_0^2) = \varepsilon_0^0 \otimes \varepsilon_0^2 + \varepsilon_0^1 \otimes \varepsilon_0^1 + \varepsilon_0^2 \otimes \varepsilon_0^0,$$
$$\Delta(\varepsilon_1^2) = \varepsilon_0^0 \otimes \varepsilon_1^2 + \varepsilon_0^1 \otimes \varepsilon_1^1 + \varepsilon_1^1 \otimes \varepsilon_0^1 + \varepsilon_1^2 \otimes \varepsilon_0^0.$$

From Theorems (3.7), it can be verified by direct calculations using Equation (4.2) that the first, second and third degrees of the homotopy lifting maps ψ_θ associated θ is the following:

$$\psi_{\theta_0}(\varepsilon_i^0) = 0, \qquad \psi_{\theta_1}(\varepsilon_0^1) = 0, \qquad \psi_{\theta_1}(\varepsilon_1^1) = \varepsilon_1^1 \qquad \psi_{\theta_2}(\varepsilon_0^2) = 0.$$

The scalars $b_{1,1}(1,1) = 1$ and for other $(m,r) \neq (1,1),(2,1)$, $b_{m,r}(m,r) = 0$. Since θ is a 1-cocycle, $n = 1$. Also, $\theta = (0\ y)$ and when compared with $\eta = \begin{pmatrix} 0 & \cdots & 0 & (f_w^1)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$ as given in Theorem (3.7), $f_w^1 = y$ and $i = 1$. To obtain $b_{2,1}(2,1)$ from $b_{1,r}(1,s)$ for some s, we take $m = 2, r = 1$, so that $m - n = 1$. Since $t_{m-n} = t_1$, we would have $r'' = 0$ or $r'' = 1$. From the statement of the theorem, we must have $i + r'' = r$, so $r'' = 0$ and $c_{r,r''}(m-n+1, r', 1) = c_{10}(2, r', 1) = 1$. It then follows from the first recurrence relations in Theorem (3.7) that

$$b_{m,r}(m-n+1, r') = \frac{(-1)^{m-1} b_{m-1,\bar{r}}(m-n, r'') c_{r,\bar{r}}(m,r,1) + c_{i,r''}(m,r,n)}{c_{r,r''}(m-n+1, r', 1)}.$$
$$b_{2,1}(2,1) = \frac{-b_{1,\bar{r}}(1,0) c_{1,\bar{r}}(2,1,1) + c_{1,0}(2,1,1)}{c_{1,0}(2,1,1)} = \frac{0+1}{1} = 1,$$

so $\psi_{\theta_2}(\varepsilon_1^2) = b_{2,1}(2,1)\varepsilon_1^2 = \varepsilon_1^2$.

5. Finding a Maurer-Cartan element

In this section, we find explicitly the Maurer-Cartan elements of a quiver algebra. We first recall the definition of a Maurer-Cartan element.

Definition 5.1. *An Hochschild 2-cocycle η is said to satisfy the Maurer-Cartan equation if*

$$d(\eta) + \frac{1}{2}[\eta, \eta] = 0. \tag{5.2}$$

Applying the definition of the bracket using homotopy lifting, we obtain the following version of the Maurer-Cartan equation for the resolution $\mathbb{K}$. $d_3^*(\eta) + \frac{1}{2}(\eta\psi_\eta + \eta\psi_\eta) = d_3^*(\eta) + \eta\psi_\eta = 0$.

We begin with the following finite quiver:

$$Q := \begin{tikzcd} 1 \arrow[loop above, "a"] \arrow[loop below, "b"] \arrow[r, "c"] & 2 \end{tikzcd}$$

with two vertices and three arrows a, b, c. We denote by e_1 and e_2 the idempotents associated with vertices 1 and 2. Let kQ be the path algebra associated with Q and take for each $q \in k$, $I_q \subseteq kQ$ to be an admissible ideal of kQ generated as follows

$$I_q = \langle a^2, b^2, ab - qba, ac \rangle.$$

These family of quiver algebras have been well studied in [**12, 13**] and [**14**]. We simply recall the main tools needed to find Maurer-Cartan elements. To define a set of free basis for the resolution $\mathbb{K}$ we start by letting kQ_0 to be the ideal of kQ generated by the vertices of Q with basis $f_0^0 = e_1, f_1^0 = e_2$. Next, set kQ_1 to be the ideal generated by paths with basis $f_0^1 = a, f_1^1 = b$ and $f_2^1 = c$. Set f_j^2, $j = 0, 1, 2, 3$ to be the set of paths of length 2 that generates the ideal I, that is $f_0^2 = a^2, f_1^2 = ab - qba, f_2^2 = b^2, f_3^2 = ac$, and define a comultiplicative equation on the paths of length $n > 2$ in the following way.

$$\begin{cases} f_0^n = a^n, \\ f_s^n = f_{s-1}^{n-1} b + (-q)^s f_s^{n-1} a, \quad (0 < s < n), \\ f_n^n = b^n, \\ f_{n+1}^n = a^{(n-1)} c, \end{cases}$$

The resolution $\mathbb{K} \to \Lambda_q$ has free basis elements $\{\varepsilon_i^n\}_{i=0}^{t_n}$ such that for each i, we have $\varepsilon_i^n = (0, \ldots, 0, o(f_i^n) \otimes_k t(f_i^n), 0, \ldots, 0)$. The differentials on $\mathbb{K}_n$ are given explicitly for this family by

$$\begin{aligned} d_1(\varepsilon_2^1) &= c\varepsilon_1^0 - \varepsilon_0^0 c \\ d_n(\varepsilon_r^n) &= (1 - \partial_{n,r})[a\varepsilon_r^{n-1}) + (-1)^{n-r} q^r \varepsilon_r^{n-1} a] \\ &\quad + (1 - \partial_{r,0})[(-q)^{n-r} b\varepsilon_{r-1}^{n-1} + (-1)^n \varepsilon_{r-1}^{n-1} b], \text{ for } r \leq n \\ d_n(\varepsilon_{n+1}^n) &= a\varepsilon_n^{n-1} + (-1)^n \varepsilon_0^{n-1} c, \text{ when } n \geq 2, \end{aligned}$$

where $\partial_{r,s} = 1$ when $r = s$ and 0 when $r \neq s$.

solutions	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
λ_0	a	ab	*0*	*0*	*0*	*0*	*0*	*0*	*0*
λ_1	*0*	*0*	*0*	*0*	*0*	*0*	ab	*0*	*0*
λ_2	*0*	*0*	a	b	ab	e_1	*0*	*0*	*0*
λ_3	*0*	*0*	*0*	*0*	*0*	*0*	*0*	c	bc

TABLE 1. Possible values of $\eta(\varepsilon_r^2) = \lambda_r$ for different r.

Calculations from [**12**] show that for this family, the comultiplicative map can be expressed in the following way

$$\Delta_{\mathbb{K}}(\varepsilon_s^n) = \begin{cases} \displaystyle\sum_{r=0}^{n} \varepsilon_0^r \otimes \varepsilon_0^{n-r}, & s = 0 \\ \displaystyle\sum_{w=0}^{n} \sum_{j=max\{0,s+w-n\}}^{min\{w,s\}} (-q)^{j(n-s+j-w)} \varepsilon_j^w \otimes \varepsilon_{s-j}^{n-w}, & 0 < s < n \\ \displaystyle\sum_{t=0}^{n} \varepsilon_t^t \otimes \varepsilon_{n-t}^{n-t}, & s = n \\ \varepsilon_0^0 \otimes \varepsilon_{n+1}^n + \Big[\displaystyle\sum_{t=0}^{n} \varepsilon_0^t \otimes \varepsilon_{n-t+1}^{n-t}\Big] + \varepsilon_{n+1}^n \otimes \varepsilon_0^0, & s = n+1. \end{cases}$$

EXAMPLE 5.3. *Let $A_1 = kQ/I$ be a member of the family where $I = I_1 = \langle a^2, b^2, ab - ba, ac\rangle$. We now find Hochschild 2-cocycles that satisfy the Maurer-Cartan equation of 5.2. Suppose that the A_1^e-module homomorphism $\eta : \mathbb{K}_2 \to A_1$ defined by $\eta = \begin{pmatrix}\lambda_0 & \lambda_1 & \lambda_2 & \lambda_3\end{pmatrix}$ is a cocycle, that is $d^*\eta = 0$, with $\lambda_i \in \Lambda_q$ for all i. Since $d^*\eta : \mathbb{K}_3 \to A_1$, we obtain using $d^*\eta(\varepsilon_r^3) = \eta d(\varepsilon_r^3)$,*

$$\eta \cdot d(\varepsilon_i^3) = \eta\Bigg(\begin{cases} a\varepsilon_0^2 - \varepsilon_0^2 a & \text{if } i = 0 \\ a\varepsilon_1^2 + \varepsilon_1^2 a + b\varepsilon_0^2 - \varepsilon_0^2 b & \text{if } i = 1 \\ a\varepsilon_2^2 - \varepsilon_2^2 a - b\varepsilon_1^2 - \varepsilon_1^2 b & \text{if } i = 2 \\ b\varepsilon_2^2 - \varepsilon_2^2 b & \text{if } i = 3 \\ a\varepsilon_3^2 - \varepsilon_0^2 c & \text{if } i = 4 \end{cases} \Bigg)$$

$\eta \cdot d$ may then be identified with the matrix

$$\begin{pmatrix} a\lambda_0 - \lambda_0 a, & a\lambda_1 + q\lambda_1 a + q^2 b\lambda_0 - \lambda_0 b, & a\lambda_2 - q^2\lambda_2 a - qb\lambda_1 - \lambda_1 b, & b\lambda_2 - \lambda_2 b, & a\lambda_3 - \lambda_0 c\end{pmatrix}$$

which will be equated to $\begin{pmatrix}0 & 0 & 0 & 0 & 0\end{pmatrix}$ and solved. We solve this system of equations with the following in mind. There is an isomorphism of A_1^e-modules $\mathrm{Hom}_{A_1^e}(Ao(f_i^n) \otimes_k t(f_i^n)A,)A \simeq o(f_i^n)A\, t(f_i^n)$ ensuring that

$$\begin{aligned} o(f_i^2)\lambda_i t(f_i^2) &= o(f_i^2)\eta(\varepsilon_i^2)t(f_i^2) = o(f_i^2)\eta(o(f_i^2) \otimes_k t(f_i^2))t(f_i^2) \\ &= \phi(o(f_i^2)^2 \otimes_k t(f_i^2)^2) = \phi(o(f_i^2) \otimes_k t(f_i^2)) = \lambda_i. \end{aligned}$$

This means that for $i = 0, 1, 2$ each λ_i should satisfy $e_1\lambda_i e_1 = \lambda_i$ since the origin and terminal vertex of f_0^2, f_1^2, f_2^2 is e_1 and $e_1\lambda_3 e_2 = \lambda_3$. We obtain 9 solutions presented in Table 1.

Now suppose that there is some $\phi : \mathbb{K}_1 \to A_1$ such that $\phi d_2(\varepsilon_i^2) = \eta(\varepsilon_i^2)$, i=0,1,2,3. If $\phi = \begin{pmatrix} 0 & \frac{1}{2}a & 0 \end{pmatrix}$, we get $\eta = \begin{pmatrix} 0 & 0 & ab & 0 \end{pmatrix}$, so $\eta = \begin{pmatrix} 0 & 0 & ab & 0 \end{pmatrix} \in \mathrm{Im}(d_2^*)$. If ϕ is equal to $\begin{pmatrix} 0 & \frac{1}{2}e_1 & 0 \end{pmatrix}$, $\begin{pmatrix} e_1 & 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} b & 0 & 0 \end{pmatrix}$, we obtain the following for η; $\begin{pmatrix} 0 & 0 & b & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 & c \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 & bc \end{pmatrix}$ respectively. Therefore

$$\mathrm{HH}^2(A_1) = \frac{\mathrm{Ker}\, d_3^*}{\mathrm{Im}\, d_2^*} = \langle \begin{pmatrix} a & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} ab & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & ab & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & a & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & e_1 & 0 \end{pmatrix} \rangle$$

The following are the first, second and third degree homotopy lifting maps associated to each elements of $\mathrm{HH}^2(A_1)$. It can be easily verified using the homotopy lifting equation in Definition (2.5) that these indeed are homotopy lifting maps.

	For $\eta = \begin{pmatrix} a & 0 & 0 & 0 \end{pmatrix}$, *we get*
	$\psi_{\eta_1}(\varepsilon_i^1) = 0,\ i = 0,1,2,\ \psi_{\eta_2}(\varepsilon_i^2) = \begin{cases} \varepsilon_0^1, & \text{if } i = 0 \\ 0, & \text{if } i = 1,2,3 \end{cases},\quad \psi_{\eta_3}(\varepsilon_i^3) = \begin{cases} 0, & \text{if } i = 0 \\ \varepsilon_1^2, & \text{if } i = 1 \\ 0, & \text{if } i = 2 \\ 0, & \text{if } i = 3 \\ \varepsilon_3^2, & \text{if } i = 4 \end{cases}$
	For $\chi = \begin{pmatrix} 0 & 0 & a & 0 \end{pmatrix}$, *we get*
	$\psi_{\chi_1}(\varepsilon_i^1) = 0,\ i = 0,1,2,\ \psi_{\chi_2}(\varepsilon_i^2) = \begin{cases} 0, & \text{if } i = 0,1,3 \\ \varepsilon_0^1, & \text{if } i = 2 \end{cases},,\quad \psi_{\chi_3}(\varepsilon_i^3) = \begin{cases} 0, & \text{if } i = 0 \\ 0, & \text{if } i = 1 \\ 0, & \text{if } i = 2 \\ \varepsilon_1^2, & \text{if } i = 3 \\ 0, & \text{if } i = 4 \end{cases}$
	For $\bar{\eta} = \begin{pmatrix} ab & 0 & 0 & 0 \end{pmatrix}$, *we get*
	$\psi_{\bar{\eta}_1}(\varepsilon_i^1) = 0,\ i = 0,1,2,\ \psi_{\bar{\eta}_2}(\varepsilon_i^2) = \begin{cases} a\varepsilon_1^1 + \varepsilon_0^1 b & \text{if } i = 0 \\ 0, & \text{if } i = 1 \\ 0 & \text{if } i = 2 \\ 0 & \text{if } i = 3 \end{cases},\quad \psi_{\bar{\eta}_3}(\varepsilon_i^3) = \begin{cases} -a\varepsilon_1^2, & \text{if } i = 0 \\ 0, & \text{if } i = 1 \\ 0, & \text{if } i = 2 \\ 0, & \text{if } i = 3 \\ b\varepsilon_3^2 + \varepsilon_1^2 c, & \text{if } i = 4 \end{cases}$
	For $\bar{\chi} = \begin{pmatrix} 0 & ab & 0 & 0 \end{pmatrix}$, *we get*
	$\psi_{\bar{\chi}_1}(\varepsilon_i^1) = 0,\ i = 0,1,2,\ \psi_{\bar{\chi}_2}(\varepsilon_i^2) = \begin{cases} 0 & \text{if } i = 0 \\ a\varepsilon_1^1 + \varepsilon_0^1 b, & \text{if } i = 1 \\ 0 & \text{if } i = 2 \\ 0 & \text{if } i = 3 \end{cases},\quad \psi_{\bar{\chi}_3}(\varepsilon_i^3) = \begin{cases} 0, & \text{if } i = 0 \\ a\varepsilon_1^2 - 2\varepsilon_0^2 b, & \text{if } i = 1 \\ \varepsilon_1^2 b, & \text{if } i = 2 \\ 0, & \text{if } i = 3 \\ 0, & \text{if } i = 4 \end{cases}.$
	For $\sigma = \begin{pmatrix} 0 & 0 & e_1 & 0 \end{pmatrix}$, *we get*
	$\psi_{\sigma_1}(\varepsilon_i^1) = 0,\ i = 0,1,2,$ $\psi_{\sigma_2}(\varepsilon_i^2) = 0,\ i = 0,1,2,3$ $\psi_{\sigma_3}(\varepsilon_i^3) = 0,\ i = 0,1,2,3,4$

The following Lemma follows immediately.

Lemma 5.4. *Let $A_1 = kQ/I$ be a member of the family of quiver algebras where $I = \langle a^2, b^2, ab - ba, ac\rangle$. The Hochschild 2-cocycles $\eta = \begin{pmatrix} a & 0 & 0 & 0\end{pmatrix}$, $\chi = \begin{pmatrix} 0 & 0 & a & 0\end{pmatrix}$, $\bar{\eta} = \begin{pmatrix} ab & 0 & 0 & 0\end{pmatrix}$, $\bar{\chi} = \begin{pmatrix} 0 & ab & 0 & 0\end{pmatrix}$ and $\sigma = \begin{pmatrix} 0 & 0 & e_1 & 0\end{pmatrix}$ are Maurer-Cartan elements.*

Proof. Let γ be any of those elements of $\mathrm{HH}^2(A_1)$. We make use of Equation (5.2). Since they are all cocycles, $d_3^*(\gamma) = 0$. Also observe that $\gamma\psi_{\gamma_3}(\varepsilon_i^3) = 0$ for all $\gamma \in \mathrm{HH}^2(A_1)$, therefore $d_3^*(\gamma) + \gamma\psi_\gamma = 0$. □

6. Deformation of the algebra using reduction system

Let $A_1 = kQ/I$ be a member of family of quiver algebras introduced in Section 5, we now show using the combinatorial star product of Equation (2.26) that $\mathrm{HH}^2(A)$ has 5 elements satisfying the Maurer-Cartan equation drawing parallels with the earlier result obtained in Example 5.3. We are interested in first order deformations of A_1.

Example 6.1. *Recall that for $A_1 = kQ/I$, $I = \langle a^2, b^2, ab - ba, ac\rangle$. If we use the set $\{(a^2, 0), (b^2, 0), (ab, ba), (ac, 0)\}$ as the reduction system, this system is reduction finite and reduction unique. All the one overlaps given by S_3 resolves to 0 uniquely. We therefore use the following reduction system*

$$R = \{(a^2, 0), (b^2, 0), (ab, ba), (ac, 0)\}$$

which satisfies the diamond condition. The set S and the set Irr_S of irreducible paths in the algebra are given respectively by

$$S = \{a^2, b^2, ab, ac\} \qquad \text{and} \qquad Irr_S = \{e_1, e_2, a, b, c, ba, bc\}$$

The one-overlaps or the set of 1-ambiguities is given as

$$S_3 = \{a^3, b^3, a^2b, ab^2, a^2c\}.$$

Notice that in the quiver Q, the path $a^2, b^2, ab \in S$ are all parallel to the irreducible paths $e_1 = e, a, b, ba$ and the path $ac \in S$ is parallel to c and bc. Any element $\widetilde{\varphi} : kS \to A \cong kIrr_S$ has the following general form

$$\begin{aligned}
\widetilde{\varphi}(a^2) &= \lambda_e + \lambda_a a + \lambda_b b + \lambda_{ba} ba \\
\widetilde{\varphi}(b^2) &= \mu_e + \mu_a a + \mu_b b + \mu_{ba} ba \\
\widetilde{\varphi}(ab) &= \nu_e + \nu_a a + \nu_b b + \nu_{ba} ba \\
\widetilde{\varphi}(ac) &= w_c c + w_{bc} bc
\end{aligned}$$

*for scalars $\lambda_e, \lambda_a, \cdots, w_c, w_{bc} \in k$. We have here that $\varphi(a^2) = 0, \varphi(b^2) = 0, \varphi(ab) = ba$ and $\varphi(ac) = 0$ is given from the reduction system R. By [**1**, Corollary 7.37] and Remark 2.25, $\widetilde{\varphi}$ is a Maurer-Cartan element if and only if for each $uvw \in S_3$ with $uv, vw \in S$, Equation (2.26) holds. That is*

$$(\pi(u) \star \pi(v)) \star \pi(w) = \pi(u) \star (\pi(v) \star \pi(w)) (mod\ t^2)$$

since we are considering first order deformations. We now check conditions on the scalars for the associativity of the star product. This product is explicitly defined by $a \star b = \varphi(a, b) +$

$\widetilde{\varphi}(a,b)t$. We check for all elements of S_3. For instance, using $a^3 = a\star(a\star a) = (a\star a)\star a$, we obtain $a\star(a\star a)$

$$
\begin{aligned}
&= a\star(\varphi(a^2)+\widetilde{\varphi}(a^2)t) = a\star(\lambda_e+\lambda_a a+\lambda_b b+\lambda_{ba}ba)t\\
&= (\lambda_e\varphi(a)+\lambda_a\varphi(a^2)+\lambda_b\varphi(ab)+\lambda_{ba}\varphi(aba))t+[\lambda_e\widetilde{\varphi}(a)+\lambda_a\widetilde{\varphi}(a^2)+\lambda_b\widetilde{\varphi}(ab)+\lambda_{ba}\widetilde{\varphi}(aba)]t^2\\
&= (\lambda_e a+\lambda_b ba)t
\end{aligned}
$$

equals to $(a\star a)\star a$

$$
\begin{aligned}
&= (\varphi(a^2)+\widetilde{\varphi}(a^2)t)\star a = (\lambda_e+\lambda_a a+\lambda_b b+\lambda_{ba}ba)t\star a\\
&= (\lambda_e\varphi(a)+\lambda_a\varphi(a^2)+\lambda_b\varphi(ba)+\lambda_{ba}\varphi(ba^2))t+[\lambda_e\widetilde{\varphi}(a)+\lambda_a\widetilde{\varphi}(a^2)+\lambda_b\widetilde{\varphi}(ba)+\lambda_{ba}\widetilde{\varphi}(ba^2))t^2\\
&= (\lambda_e a+\lambda_b ba)t.
\end{aligned}
$$

So no new equation. If we use $a^2b = a\star(a\star b) = (a\star a)\star b$, we obtain $a\star(a\star b) = (\nu_e a+\nu_b ba)t$ and $(a\star a)\star b = (\lambda_e b+\lambda_a ba)t$, so we get $\nu_e = \lambda_e = 0$ and $\nu_b = \lambda_a$. Equivalent calculations for ab^2 and a^2c yields $\mu_e = \nu_e = 0$ and $\mu_b = \nu_a$ and $\lambda_e = \lambda_b = 0$. We can now rewrite

$$
\begin{aligned}
\widetilde{\varphi}(a^2) &= \lambda_a a+\lambda_{ba}ba\\
\widetilde{\varphi}(b^2) &= \mu_a a+\mu_b b+\mu_{ba}ba\\
\widetilde{\varphi}(ab) &= \mu_b a+\lambda_a b+\nu_{ba}ba\\
\widetilde{\varphi}(ac) &= w_c c+w_{bc}bc
\end{aligned}
$$

so the cocycle $\widetilde{\varphi}$ is parametrized by $(\lambda_a,\lambda_{ba},\mu_a,\mu_b,\mu_{ba},\nu_{ba},w_c,w_{bc})\in k^8$. From [**1**, Corollary 7.44]*, two cocycles $\widetilde{\varphi}$ and $\widetilde{\varphi}'$ are cohomologous or satisfy $\widetilde{\varphi}-\widetilde{\varphi} = \langle\Theta\rangle$, $\Theta\in\mathrm{Hom}(kQ_1, kIrr_S)$ if*

$$T(\varphi(s))+\widetilde{\varphi}'(s)t = T(s_1)\star\cdots\star T(s_m)(mod\ t^2)$$

for some $T: kIrr_S[t]/(t^2)\to Irr_S[t]/(t^2)$ defined by $T(x) = x+\Theta(x)$ with $s = s_1s_2\cdots s_m$ a path of length m. Any $\Theta\in\mathrm{Hom}(kQ_1, kIrr_S)$ has a general form

$$
\begin{aligned}
\Theta(a) &= \alpha_e+\alpha_a a+\alpha_b b+\alpha_{ba}ba\\
\Theta(b) &= \beta_e+\beta_a a+\beta_b b+\beta_{ba}ba\\
\Theta(c) &= \gamma_c c
\end{aligned}
$$

For instance take $s = a^2$, then $T(\varphi(a^2))+\widetilde{\varphi}'(a^2)t = T(a)\star T(a)$ yields $\lambda'_a a+\lambda'_{ba}ba = \lambda_a a+\lambda_{ba}ba+2\alpha_e a+2\alpha_b ba$ or $\lambda'_a-\lambda_a = 2\alpha_e$ and $\lambda'_{ba}-\lambda_{ba} = 2\alpha_b$. With other similar equivalent calculations on b^2, ab, ac, we get

$$
\begin{array}{lll}
\lambda'_a-\lambda_a = 2\alpha_e & \lambda'_{ba}-\lambda_{ba} = 2\alpha_b & \\
\mu'_a-\mu_a = \beta_e & \mu'_b-\mu_b = \alpha_e & \mu'_{ba}-\mu_{ba} = \alpha_a+\beta_b\\
w'_c-w_c = \alpha_e & w'_{bc}-w_{bc} = \alpha_b &
\end{array}
$$

This implies that the three variables in the parametric definition of $\widetilde{\varphi}$ can be eliminated. Therefore $\widetilde{\varphi}$ is in k^5 or equivalently the dimension of $\mathrm{HH}^2(A_1) = 5$.

Acknowledgement The author thanks Severin Barmeier for useful discussions and for helping out with the calculations in Section 6.

References

[1] S. Barmeier, Z. Wang, *Deformation of path algebras of quivers with relations,* ArXiv: 2002.10001v4.

[2] G.M. Bergman, *The Diamond Lemma for ring theory,* Adv. Math. 29 (1978), 178-218.

[3] R.O. Buchweitz, E.L. Green, N. Snashall, Ø. Solberg, *Multiplicative structures for Koszul algebras,* The Quarterly Journal of Mathematics 59 (2008), no. 4, 441-454.

[4] S. Chouhy, A. Solotar, *Projective resolutions of associative algebras and ambiguities,* J. Algebra 432 (2015), 22-61.

[5] E.L. Green, Ø. Solberg, *An Algorithmic approach to resolutions*, J. Symbolic Comput., 42 (2007), 1012–1033.

[6] E. L. Green, G. Hartman, E. N. Marcos, Ø. Solberg, *Resolutions over Koszul algebras* Arch. Math. 85 (2005), 118–127.

[7] T. Karadag, D. McPhate, P. S. Ocal, T. Oke, and S. Witherspoon, *Gerstenhaber brackets on Hochschild cohomology of general twisted tensor products*, J. Pure and Applied Algebra, 225 (2021) no.6, 106597

[8] R. Martínez-Villa, *Introduction to Koszul algebras*, Rev. Un. Mat. Argentina 58 (2008), no. 2, 67-95.

[9] C. Negron and S. Witherspoon, *An alternate approach to the Lie bracket on Hochschild cohomology,* Homology, Homotopy and Applications 18 (2016), no. 1, 265-285.

[10] C. Negron, Y. Volkov and S. Witherspoon, *A_∞-coderivations and the Gerstenhaber bracket on Hochschild cohomology,* J. Noncommutative Geometry 14 (2020), no. 2, 531-565.

[11] N. Snashall, *Support varieties and the Hochschild cohomology ring modulo nilpotence*, Proceedings of the 41st Symposium on Ring Theory and Representation Theory, Tsukuba (2009) 68–82.

[12] T. Oke, *Cup product and Gerstenhaber bracket on Hochschild cohomology of a family of quiver algebras,* Communications in algebra, 50 (2022), no. 5, 1821-1841.

[13] T. Oke, *Bracket structure on Hochschild cohomology of Koszul quiver algebras using homotopy liftings,* arXiv:2103.12331.

[14] T. Oke, *On the Lie algebra structure on Hochschild cohomology of Koszul quiver algebras,* Texas A&M PhD Dissertation, August 2021.

[15] S. Witherspoon, *Hochschild Cohomology for Algebras*, Graduate Studies in Mathematics 204, American Mathematical Society (2019).

[16] Y. Volkov, *Gerstenhaber bracket on the Hochschild cohomology via an arbitrary resolution,* Proc. of the Edinburgh Math. Soc., 62 (2019), no. 3, 817-836.