Hochschild 2-cocycles as the space of infinitesimal deformations

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1 Introduction

This note explores the relationship between Hochschild cohomology and the theory of deformation of algebras. We explore ideas showing how you can draw a one-to-one correspondence between equivalent classes of infinitesimal deformations and Hochschild 2-cocycles. These ideas can be found in several literatures such as [1, 2] . The definition of an infinitesimal deformation is given as Definition 18 with reference to Definition 7 and the main theorem is Theorem 19. This note is NOT self contained. Feel free to contact/email me if you have any questions or if you find any typos!

2 Preliminaries

Let k be a field of characteristics 0 and R a ring with unit 1_R. By a ring, we mean a commutative associative k-algebra. We take $\otimes = \otimes_k$ unless otherwise specified.

Definition 1. *Let R be a ring with unit and $s : k \to R$, a homomorphism with $s(1) = 1_R$. A map $\epsilon : R \to k$ is an augmentation of R if $\epsilon s = 1_k$.*

Examples

1. The ring $R = k[[t]]$ of formal power series with coefficients in k is augmented with $\epsilon : R \to k$ given by $\sum_i r_i t^i \mapsto r_0$.

2. Let A be an algebra, and let $R = A[[t]]$ be the ring of formal power series with coefficients in A. R is augmented with $\epsilon : R \to k$ given by $\sum_i a_i t^i \mapsto a_0$.

3. The group ring $k[G]$ of a finite group G is the space of all formal linear combination $\sum_{g\in G} a_g g$, $a_g \in k$ with multiplication
$$(\sum_{g\in G} a_g g)(\sum_{g\in G} b_g g) = \sum_{g\in G}\sum_{xy=g} a_x b_y g.$$
$k[G]$ is augmented with $\epsilon : k[G] \to k$ given by $\sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g$.

Remark 2. *The subspace $\mathrm{Ker}(\epsilon)$ is a two sided ideal of R called the augmentation ideal.*

Observe that k is an R bimodule with the structure induced by ϵ. Since R an associative k-algebra, R is automatically a k-module. We will now discuss how to pass from k-modules to R-modules and vice versa.

Definition 3. *Let V be a k-vector space and R a unital ring (or a k-algebra). The free R-module generated by V is an R-module $R\langle V\rangle$ together with a k-linear map $\iota : V \to R\langle V\rangle$ with the property that for every R-module and module homomorphism $V \xrightarrow{f} W$ there exists a module homomorphism $R\langle V\rangle \xrightarrow{\phi} W$ for which the following diagram commutes.*

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "f"'] & R\langle V\rangle \arrow[d, "\psi"] \\ & W \end{tikzcd}$$

Remark 4. *This universal property determines the free module $R\langle V\rangle$ uniquely up to isomorphism. A concrete model which will later be used is the module $R\langle V\rangle = R \otimes V$.*

The above describes how we pass from a k-vector space V to an R-module $R\langle V\rangle$ (with model in mind being $R\otimes V$). How then do we pass from an R-module to a vector space over k.

Definition 5. *Let W be an R-module. The reduction of W is the k-module $\overline{W} := k\otimes_R W$, with a k-action given by*

$$t(t'\otimes w) = tt'\otimes w \quad \textit{for all} \quad t,t'\in k,\ w\in W$$

It is clear that there are k-module isomorphisms $\overline{W}\cong W/\overline{R}W$ and $\overline{R\langle V\rangle}\cong V$. This reduction clearly defines a functor from the category of R-modules to the category of k-modules. Note that for the R-module R, the reduction $\overline{R} := k\otimes_R R\cong k$. Let the map $W\to\overline{W}$ be defined by $rx\mapsto\overline{r}\otimes x$. Since this tensor product is taken over R we realize that $\overline{r}\otimes x = 1\otimes\overline{r}x$, so the kernel of this map is the set $\overline{R}W := \{\overline{r}w \mid r\in R, w\in W\}$. Notice that for any $\lambda\in k$, $s(\lambda)\in R$, so $\lambda\otimes w\mapsto s(\lambda)w$ implies the surjectivity of the map $W\to\overline{W}$. The first isomorphism theorem implies that $\overline{W}\cong W/\overline{R}W$. It is also straight forward to see that $\overline{R\langle V\rangle}\cong\overline{R\otimes V}\cong\overline{R}\otimes V\cong k\otimes_k V\cong V$.

Proposition 6. *If B is an associative R-algebra, then the reduction $\overline{B}$ is a k-algebra with the structure induced by the algebra structure on B.*

Proof. $\overline{B}$ by definition is a k-algebra. Since $\overline{B}\cong B/\overline{R}B$, it is enough to show that $\overline{R}B$ is a two sided ideal of B, so it inherits the algebra structure on B. For any $r\in\overline{R}, b, b_1\in B$, the multiplication $(rb, b_1) = r(b, b_1)$ showing that $(\overline{R}B, B)\subseteq\overline{R}B$. □

3 Deformation theory

We now delve into the classical definition of deformation theory and how this connects with Hochschild 2-cocycles.

Definition 7. *Let A be an associative k-algebra and R an augmental unital ring. An R-deformation of A is an associative R-algebra B and a module homomorphism ϕ such that $\phi:\overline{B}\to A$ is an isomorphism of k-algebras.*

The following is a pictorial representation of the information given in Definition 7.

$$\begin{tikzcd}
A \arrow[r, "passing\ to"'] & R\langle A\rangle (e.g. \cong R\otimes A) \arrow[d, "deformation\ problem"] \\
\overline{B} \arrow[u, "\phi\ (isomorphism)"'] & B \arrow[l, "reduction\ module"]
\end{tikzcd}$$

We denote a deformation of the algebra A by the pair $(B, \overline{B}\xrightarrow{\phi} A)$. There is a notion of equivalence classes of deformations of an algebra A. Two R-deformations of an algebra A given by $(B_1, \overline{B_1}\xrightarrow{\phi_1} A)$ and $(B_2, \overline{B_2}\xrightarrow{\phi_2} A)$ are equivalent if there is a map $w: B_1\to B_2$ such that $w = (\phi_2)^{-1}\phi_1$

Remark 8.

- *In the foregoing, we assume that $R\langle A\rangle$ of the above diagram is $R\otimes A$ and is isomorphic to B as an R-module.*
- *The isomorphism $B\cong R\otimes A$ identifies A with the submodule $1\otimes A$ of B and $A\otimes A$ with the submodule $(1\otimes A)\otimes(1\otimes A)$ of $B\otimes B$.*

Lemma 9. *For an R-deformation $(B, \overline{B}\xrightarrow{\phi} A)$ the multiplication ϕ in B is determined by its restriction to $A\otimes A\subset B\otimes B$. Every deformation $w: B\to B'$ that is equivalent to ϕ is determined by its restriction to $A\subset B$.*

Proof. Using the fact that $B \cong R \otimes A$, each element of B is a finite sum of elements of the form $ra, r \in R, a \in A$. Since $\phi : \overline{B} \to A$, is an isomorphism of k-algebras, for all $\lambda_i \in k$, $x, y \in \overline{B}$, k-bilinearity of ϕ as a map on $\overline{B}$ implies that $\phi(\lambda_1 x, \lambda_2 y) = \lambda_1 \lambda_2 \phi(x, y)$. Now suppose $x = ra_1, y = sa_2, r, s \in R, a_1, a_2 \in A$, R-bilinearity of ϕ as a multiplicative map $B \times B \to B$, implies that $\phi(x, y) = \phi(ra_1, sa_2) = rs\phi(a_1, a_2)$, showing that with the multiplication $\phi(a_1, a_2)$ in $A \otimes A$, we can determine the multiplication of ϕ in B. The equivalence of w and ϕ implies there is a deformation $(B', \overline{B'} \xrightarrow{\phi'} A)$ such that $w = (\phi')^{-1}\phi$. Since the multiplication of ϕ and ϕ' in B are determined by their restriction to $A \otimes A$, w is determined by it's restriction to $A \otimes A$. □

4 Formal and Infinitesimal deformations

Definition 10. *By a formal deformation, we mean a deformation in the sense of Definition 7 over the complete local augmented ring $R = k[[t]]$.*

Notice that in this case if $(A_t, \overline{A_t} \xrightarrow{\mu} A)$ is a formal deformation of A, then we identify the reduction of A_t $\overline{A_t}$ with $A \otimes A_t$ so that according to Lemma 9 the multiplication of μ in A_t is determined by its restriction to $A \otimes A$. All such formal deformation are given by the map $\mu : A \otimes A \to A$ satisfying certain conditions.

Theorem 11. *A formal deformation A_t of a k-algebra A is given by the family*

$$\{\mu_i : A \otimes A \to A |\ i \in \mathbb{N}\}$$

such that $\mu_0(a, b) = ab$ (the multiplication on A) and

$$(D_k) \qquad \sum_{\substack{i+j=k \\ i,j \geq 0}} \mu_i(\mu_j(a, b), c) = \sum_{\substack{i+j=k \\ i,j \geq 0}} \mu_i(a, \mu_j(b, c))$$

for all $a, b, c \in A$ and $k \geq 1$.

Proof. By Lemma 9, the multiplication of μ_t is determined by its restriction to $A \otimes A$. For $a, b \in A$, expanding $\mu(a, b)$ into the power series

$$\mu(a, b) = \mu_0(a, b) + t\mu_1(a, b) + t^2\mu_2(a, b) + \cdots \tag{12}$$

for some k-bilinear functions μ_i. It is obvious that μ_0 corresponds with multiplication in A. It is also verifiable that μ is associative if and only if (D_k) are satisfied for each $k \geq 1$. □

Remark 13. *We observe that by taking $\mu_0(a, b) = 0$, the associativity of μ implies the associativity in A (that is D_0) and (D_1) means that*

$$a\mu_1(b, c) - \mu_1(ab, c) + \mu_1(a, bc) - \mu_1(a, b)c = 0 \tag{14}$$

Hochschild cohomology: For a k-algebra A, Hochschild cohomology was originally defined using the following projective resolution known as the bar resolution.

$$B(A)_\bullet := \qquad \cdots \to A^{\otimes(n+2)} \xrightarrow{\delta_n} A^{\otimes(n+1)} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} A^{\otimes 3} \xrightarrow{\delta_1} A^{\otimes 2} \ (\to A) \tag{15}$$

where the differentials δ_i are given by

$$\delta_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

for each element $a_i \in A$, for all $i = 0, 1, 2, \ldots, n+1$. Let $f \in \mathrm{Hom}_{A^e}(A^{m+2}, A) \cong \mathrm{Hom}_k(A^{\otimes m}, A)$, satisfying $\delta^* f = 0$, where $\delta^* f(a_0 \otimes a_1 \otimes \cdots \otimes a_{m+2}) = f(\delta(a_0 \otimes a_1 \otimes \cdots \otimes a_{m+2}))$.

Definition 16. *An Hochschild 2-cocycle is a map $f : A \otimes A \to A$, (which can also be viewed as a module homomorphism $f \in \mathrm{Hom}_{A^e}(A^4, A)$) satisfying $\delta^* f(1 \otimes a \otimes b \otimes c \otimes 1) = 0$. This means that*

$$fd(1 \otimes a \otimes b \otimes c \otimes 1) = f(a \otimes b \otimes c \otimes 1) - f(1 \otimes ab \otimes c \otimes 1)$$
$$+ f(1 \otimes a \otimes bc \otimes 1) - f(1 \otimes a \otimes b \otimes c) = 0$$

which is equivalent to

$$af(b \otimes c) - f(ab \otimes c) + f(a \otimes bc) - f(a \otimes b)c = 0. \tag{17}$$

Definition 18. *By an infinitesimal deformation, we mean a deformation in the sense of Definition 7 over the complete local augmented ring $R = k[[t]]/(t^2)$.*

Theorem 19. *There is a one-to-one correspondence between the space of equivalence classes of infinitesimal deformations of A and the space of Hochschild 2-cocycles of A with coefficients in A.*

Proof. By an infinitesimal deformation, the multiplication $*$ in B reduces to

$$a * b = \mu_0(a,b) + t\mu_1(a,b) = ab + t\mu_1(a,b).$$

Then associativity of $*$ implies Equation 14, that is

$$a\mu_1(b,c) - \mu_1(ab,c) + \mu_1(a,bc) - \mu_1(a,b)c = 0$$

where $\mu_0(a,b) = ab$. Comparing this with Equation (17), we realize that μ_1 is a Hochschild 2-cocycle with $\mu_1(x,y) = \mu_1(x \otimes y)$.

Now suppose that $(B, \overline{B} \xrightarrow{*} A)$ and $(B', \overline{B'} \xrightarrow{\bar{*}} A)$ are two equivalent deformations. Lets associate Hochschild 2-cocycles $\mu_1, \bar{\mu_1}$ to each deformation $*$ and $\bar{*}$ and express for $a, b \in A$,

$$\begin{aligned} a * b &= ab + t\mu_1(a,b) \\ a \bar{*} b &= ab + t\bar{\mu_1}(a,b). \end{aligned} \tag{20}$$

By Lemma 9 every equivalence of deformations $w : B \to B'$ is determined by its restriction to A. In this case $w = (\bar{*}^{-1})(*)$, is determined by a k-linear map $w_1 : A \to A$ satisfying $w(a) = a + tw_1(a)$ for all $a \in A$. w is invertible and satisfies

$$w(a * b) = w(a) \bar{*} w(b) \tag{21}$$

Substituting Equation (20) into (21), we obtain

$$\begin{aligned} &w(ab + t\mu_1(a,b)) = (a + tw_1(a)) \bar{*} (b + tw_1(b)) \\ &ab + t\mu_1(a,b) + tw_1(ab + t\mu_1(a,b)) = (a + tw_1(a))(b + tw_1(b)) + t\bar{\mu_1}(a + tw_1(a), b + tw_1(b)) \\ &ab + t\mu_1(a,b) + tw_1(ab) + t^2 w_1\mu_1(a,b)) = ab + t[aw_1(b) + w_1(a)b)] \\ &\ + t[\bar{\mu_1}(a,b) + t\bar{\mu_1}(a, w_1(b)) + t\bar{\mu_1}(w_1(a), b) + t^2\bar{\mu_1}(w_1(a), w_1(b))] \end{aligned}$$

since $t^2 = 0$, we obtain

$$ab + t[\mu_1(a,b) + w_1(ab)] = ab + t[aw_1(b) + w_1(a)b) + \bar{\mu_1}(a,b)]$$

showing that

$$\mu_1(a,b) - \bar{\mu_1}(a,b) = aw_1(b) - w_1(ab) + w_1(a)b) = \delta^* w(a,b)$$

that is, the two infinitesimal deformations $\mu_1, \bar{\mu_1}$ are equivalent if and only if they differ by a coboundary. □

References

[1] M. Doubek, M. Markl, P. Zima, *Deformation Theory (Lecture notes)*, arXiv:0705.3719.

[2] S. Witherspoon, *Hochschild Cohomology for Algebras*, Graduate Studies in Mathematics, American Mathematical Society.