Thesis Proposal

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1 Introduction

The Hochschild cohomology ring $\mathrm{HH}^*(A)$ of an associative algebra A has two binary operations, namely the cup product $\smile$ and a Lie bracket $[\cdot\ ,\cdot]$. These two binary operations make $(\mathrm{HH}^*(A), \smile, [\cdot\ ,\cdot])$ a Gerstenhaber algebra. We may think of the bracket as a generalized Poisson bracket which is useful in Hamiltonian mechanics and dynamical systems in physics. It also finds applications in algebraic deformation theory of algebras.

The bracket has been difficult to understand. Mathematicians have made several attempts to define it on an arbitrary projective resolution of A rather than on the bar resolution where it was originally defined. In particular, C. Negron and S. Witherspoon introduced the idea of contracting homotopy which works for resolutions that have the differential graded coalgebra property.

E.L. Green, G. Hartman, E.N. Marcos and Ø. Solberg introduced a resolution for Koszul algebras in 2005 [2]. This resolution has the differential graded coalgebra property. In [1], a formula for the cup product on Hochschild cohomology was presented using this resolution. My aim is to present a generalized formula for the Gerstenhaber bracket structure on Hochschild cohomology of Koszul algebras defined by quivers and relations using this resolution. Since there is an algorithm for computing this resolution and a closed formula for the cup product, it would be very useful to find an algorithm that can provide information about both the cup product and the Gerstenhaber algebra structure for quiver algebras using this resolution.

1.1 Algebras defined by quivers and relations

A quiver is a directed graph with the allowance of loops and multiple arrows (also called paths) between vertices. A quiver Q is sometimes denoted as a quadruple (Q_0, Q_1, o, t) where Q_0 is the set of vertices in Q, Q_1 is the set of arrows in Q, and $o, t : Q_1 \longrightarrow Q_0$ are maps which assign to each arrow $a \in Q_1$, its origin vertex $o(a)$ and terminal vertex $t(a)$ in Q_0.

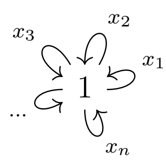
A path in Q is a sequence of arrows $a = a_1 a_2 \cdots a_{n-1} a_n$ such that the terminal vertex of a_i is the same as the origin vertex of a_{i+1}, using the convention of concatenating paths from left to right.

If k is a field, then the quiver algebra or path algebra kQ is defined as a vector space having all paths in Q as a basis. Vertices are regarded as paths of length 0, an arrow is a path of length 1, and so on. We take multiplication on kQ as concatenation of paths. Two paths a and b satisfy $ab = 0$ if $t(a) \neq o(b)$. In this case, we say the paths a and b cannot be concatenated. This multiplication defines an associative algebra over k. This algebra has a unit element if and only if the quiver has only finitely many vertices. By taking kQ_i to be the k-vector subspace of kQ with paths of length i as basis, $kQ = \bigoplus_{i \geq 0} kQ_i$ can be viewed as an $\mathbb{N}$-graded vector space.

A relation on a quiver Q is a linear combination of paths from Q. A quiver together with a set of relations is called a quiver with relations. Letting I be an ideal of the path algebra kQ, we denote by (Q, I) the quiver Q with relations I. The quotient $\Lambda = kQ/I$ is called the quiver algebra associated with (Q, I).

Example 1.1.

1. *The polynomial ring in n indeterminates $k[x_1, \cdots, x_n]$ is a graded quiver algebra with Q having one vertex and n arrows. The ideal I of relations is given by $I = \langle x_i x_j - x_j x_i | i, j = 1, 2, \cdots, n \rangle$.*



2. *Let V be a vector space with basis $\{v_1, \cdots, v_n\}$. The exterior algebra $\bigwedge(V)$ of V is a graded quiver algebra with Q as in the example 1 and the ideal I given by*
$$I = \langle x_i^2, x_i x_j + x_j x_i | i, j = 1, 2, \cdots, n \rangle.$$

3. *Let Q be the following quiver with two vertices and arrows a, b, c,*

$$\begin{tikzcd} 1 \arrow[loop above, "a"] \arrow[loop below, "b"] \arrow[r, "c"] & 2 \end{tikzcd}$$

and the ideal $I = \langle a^2, b^2, ab - qba, ac\rangle$, where $q \in k$. Then $\Lambda_q = \frac{kQ}{I}$ is a family of quiver algebras. This family is in fact an interesting example. It was shown in [7] that when $q = 1$, the Hochschild cohomology ring modulo the set of nilpotent cocycles is not finitely generated as an algebra. Using a generalized cup product formula on the Hochschild cohomology of this family, we gave a different proof of this result, and more generally showed that the Hochschild cohomology modulo the ideal generated by homogenous nilpotent elements is not finitely generated whenever $q = \pm 1$ [6].

2 Hochschild cohomology as an associative algebra

For a k-algebra Λ, Hochschild cohomology is defined on the following projective resolution,

$$\mathbb{B}_\bullet := \qquad \cdots \to \Lambda^{\otimes(n+2)} \xrightarrow{\delta_n} \Lambda^{\otimes(n+1)} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} \Lambda^{\otimes 3} \xrightarrow{\delta_1} \Lambda^{\otimes 2} \ (\to \Lambda)$$

where the differentials δ_i are given by

$$\delta_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

for each element $(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) \in \Lambda^{\otimes(n+2)}, n = 0, 1, 2,$ and so on. This resolution consists of $\Lambda-$bimodules or left modules over the enveloping algebra $\Lambda^e = \Lambda \otimes \Lambda^{op}$, where Λ^{op} is the opposite algebra. Letting M be a finitely generated left Λ^e-module, the space of Hochschild n-cochains $C^n(\Lambda, M)$ with coefficients in M is obtained by applying the functor $\mathrm{Hom}_{\Lambda^e}(-, M)$ to the complex $\mathbb{B}_\bullet$, and then taking the cohomology of the resulting complex. We define

$$HH^*(\Lambda, M) = Ext^*_{\Lambda^e}(\Lambda, M) = \bigoplus_{n \geq 0} H^n(\mathrm{Hom}_{\Lambda^e}(\mathbb{B}_\bullet, M)) = \bigoplus_{n \geq 0} C^n(\Lambda, M)$$

to be the Hochschild cohomology ring of Λ with coefficients in M. There are two binary operations, the cup product $\smile$ and the Gerstenhaber bracket $[\cdot\,,\cdot]$ on Hochschild cohomology making it into an associative algebra. Let $M = \Lambda$, and $HH^*(\Lambda, \Lambda) = HH^*(\Lambda)$ and take two maps $f \in Hom_{\Lambda^e}(\mathbb{B}_m, \Lambda) \cong Hom_k(\Lambda^{\otimes m}, \Lambda)$, and $g \in Hom_k(\Lambda^{\otimes n}, \Lambda)$ satisfying $\delta^* f = 0 = \delta^* g$, where $\delta^* f(a_0 \otimes a_1 \otimes \cdots \otimes a_{m+2}) = f(\delta(a_0 \otimes a_1 \otimes \cdots \otimes a_{m+2}))$. We define the

- cup product to be

$$f \smile g(a_1 \otimes \cdots \otimes a_{m+n}) = (-1)^{mn} f(a_1 \otimes \cdots \otimes a_m) \cdot g(a_{m+1} \otimes \cdots \otimes a_{m+n}) \quad (2.1)$$

which induces a map $\smile: HH^m(\Lambda) \times HH^n(\Lambda) \to HH^{m+n}(\Lambda)$, and define the

- Gerstenhaber bracket:

$$[f, g] = f \circ g - (-1)^{(m-1)(n-1)} g \circ f \quad (2.2)$$

$f \circ g = \sum_{j=1}^{m} (-1)^{(n-1)(j-1)} f \circ_j g \qquad$ where

$(f \circ_j g)(a_1 \otimes \cdots a_{m+n-1}) = f(a_1 \otimes \cdots \otimes a_{j-1} \otimes g(a_j \otimes \cdots \otimes a_{j+n-1}) \otimes a_{j+n} \otimes \cdots \otimes a_{m+n-1})$

which also induces a map $[\cdot\,,\cdot] : HH^m(\Lambda) \times HH^n(\Lambda) \to HH^{m+n-1}(\Lambda)$.

3 Background and problem

Let $\Lambda = kQ/I$ be an algebra defined by quiver and relations, as explained in subsection (1.1). An algorithmic approach to finding a minimal projective resoluton $\mathbb{F} \to \Lambda_0$ of graded semi-simple modules over Λ was given in [3]. The resolution was shown to have a "comultiplicative structure" and this structure was used to find a minimal projective resolution $\mathbb{K} \to \Lambda$ of Λ as a right module over the enveloping algebra in [2]. These results were used to describe a closed formula for the cup product on Hochschild cohomology. We present the resolution and the cup product formula next.

Let J be the ideal of kQ generated by all arrows. Suppose that $I \subseteq J^2$ is an admissible ideal i.e $J^m \subseteq I \subseteq J^2$ for some m and set $\mathfrak{r} = J/I$. A non-zero element $\alpha \in kQ$ is uniform if there exist vertices u, v such that $\alpha = u\alpha v = u\alpha = \alpha v$. For $R = kQ$, it was shown in [2] that there are integers t_n and uniform elements $\{f_i^n\}_{i=0}^{t_n}$ such that the minimal right projective resolution $\mathbb{F} \to \Lambda_0$ of $\Lambda_0 \cong \Lambda/\mathfrak{r}$, can be given in terms of filtration of right ideals,

$$\cdots \subseteq \bigoplus_{i=0}^{t_n} f_i^n R \subseteq \bigoplus_{i=0}^{t_{n-1}} f_i^{n-1} R \subseteq \cdots \subseteq \bigoplus_{i=0}^{t_1} f_i^1 R \subseteq \bigoplus_{i=0}^{t_0} f_i^0 R = R$$

in R.

Furthermore, it was shown that, with some choice of scalars, the $\{f_i^n\}_{i=0}^{t_n}$ satisfy a comultiplicative structure (3.1). That is, for $0 \leq i \leq t_n$, there are scalars $c_{pq}(n,i,r)$ such that

$$f_i^n = \sum_{p=0}^{t_r} \sum_{q=0}^{t_{n-r}} c_{pq}(n,i,r) f_p^r f_q^{n-r}. \tag{3.1}$$

For example, we can take $\{f_i^0\}_{i=0}^{t_0}$ to be the set of vertices, $\{f_i^1\}_{i=0}^{t_1}$ to be the set of arrows, $\{f_i^2\}_{i=0}^{t_2}$ to be the set of linear combination of paths of length 2 contained in I for each i, and define $\{f_i^n\}(n \geq 3)$ recursively, that is in terms of f_i^2, f_j^1. See [6] for the comultiplicative structure associated with the family of quiver algebras defined in Example (1.1)(3).

The resolution $\mathbb{F} \to \Lambda_0$ and the comultiplicative structure (3.1) were used to construct a minimal projective resolution $\mathbb{K}_\bullet \to \Lambda$ of Λ as a module over the enveloping algebra on which we now define Hochschild cohomology. This minimal projective resolution $\mathbb{K}$ of Λ^e-modules associated to $\Lambda = kQ/I$ was given by

$$\mathbb{K}: \qquad \cdots \xrightarrow{d_{n+1}} \mathbb{K}_n \xrightarrow{d_n} \mathbb{K}_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \mathbb{K}_1 \xrightarrow{d_1} \mathbb{K}_0 \; (\xrightarrow{\mu} \Lambda) \tag{3.2}$$

where for each n

$$\mathbb{K}_n = \bigoplus_{i=0}^{t_n} \Lambda o(f_i^n) \otimes_k t(f_i^n)\Lambda,$$

since each f_i^n are uniform elements, the notation $o(f_i^n), t(f_i^n)$ makes sense. Let $\varepsilon_i^n = (0, \ldots, 0, o(f_i^n) \otimes_k t(f_i^n), 0, \ldots, 0)$, $0 \leq i \leq t_n$ where $o(f_i^n) \otimes_k t(f_i^n)$ is in the i-th position. Note that for each n, $\{\varepsilon_i^n\}_{i=0}^{t_n}$ is a free basis of $\mathbb{K}_n$. The differentials are given by

$$d_n(\varepsilon_i^n) = \sum_{j=0}^{t_{n-1}} \Big(\sum_{p=0}^{t_1} c_{p,j}(n,i,1) f_p^1 \varepsilon_j^{n-1} + (-1)^n \sum_{q=0}^{t_1} c_{j,q}(n,i,n-1) \varepsilon_j^{n-1} f_q^1 \Big)$$

(see [2]Theorem 2.1, [1] page 5)

where μ is the multiplication map and $c_{p,j}(n,i,r) \in k$ for any n, r, i.

Using the comultiplicative structure of equation (3.1), it was shown in [1] that the multiplication/cup product on the Hochschild cohomology ring for algebras defined by quivers and relations have the following description.

Theorem 3.3 (See [1],Theorem 2.3)**.** *Let $\Lambda = kQ/I$ be a Koszul algebra over a field k, where Q is a finite quiver and $I \subseteq J^2$. Suppose that $\eta : \mathbb{K}_n \to \Lambda$ and $\theta : \mathbb{K}_m \to \Lambda$ represent elements in $\mathrm{HH}^*(\Lambda)$ and are given by $\eta(\varepsilon_i^n) = \lambda_i$ for $i = 0, 1, \cdots, t_n$ and $\theta(\varepsilon_i^m) = \lambda_i'$ for $i = 0, 1, \cdots, t_m$. Then* $\eta \smile \theta : \mathbb{K}_{n+m} \to \Lambda$

can be expressed as

$$(\eta \smile \theta)(\varepsilon_j^{n+m}) = \sum_{p=0}^{t_n} \sum_{q=0}^{t_m} c_{pq}(n+m, i, n)\lambda_p \lambda'_q,$$

for $j = 0, 1, 2, \cdots, t_{n+m}$.

3.1 Problem 1

As we have previously mentioned, the Lie/Gerstenhaber structure on Hochschild cohomology described in (2.2) is still not very well understood. In fact, this definition was described using the bar resolution, and it is only useful theoretically but not computationally. The interesting question therefore is whether we can describe the Gerstenhaber bracket structure on Hochschild cohomology for quiver algebras using the comultiplicative structure of (3.1) and the resolution $\mathbb{K}$ described in (3.2). Since the resolution $\mathbb{K}$ is computationally accessible, this may allow the use of computer programs to compute brackets on Hochschild cohomology. The goal of my thesis is to use recent ideas of derivation operators (by S. Alvarez [9]), contracting homotopy (by C. Negron and S. Witherspoon [5]) and homotopy lifting (by Y. Volkov [10]) to make progress on this problem. In the furture, I plan to develop and or add to existing computer algorithms that can compute brackets using these ideas.

3.2 Problem 2

The family of quiver algebras described in Example (1.1) exhibits a certain pathological behavior that makes them interesting to study. To fully develop the theory of support varieties for finitely generated modules over finite dimensional algebras, it was conjectured in [8] that for any finite dimensional algebra, Hochschild cohomology modulo nilpotents is always finitely generated as an algebra. But this is not true in general. In [7], it was shown that the Hochschild cohomology ring modulo nilpotents for the algebra $\Lambda_q(q = 1)$ is not finitely generated as an algebra, providing a counterexample.

A result: In [6], we used the resolution described in (3.2) to explicitly construct a generalized cup product formula and describe Hochschild cocycles that are nilpotent. Furthermore, using the generalized cup product formula, we showed that Hochschild cohomology modulo nilpotents for two members of the family of quiver algebras described in Example (1.1) is not finitely generated. These are $\Lambda_q, q = \pm 1$. There is a degree 0 cocycle θ_0 taking ε_1^0 to the first vertex e_1 and ε_2^0 to the second vertex e_2. Since $e_1 + e_2 = 1$, we may view the ideal generated by θ_0 as the ground field k. Whenever $q \neq \pm 1$, for all n, all degree n cocycles are nilpotent except θ_0, making Hochschild cohomology modulo the ideal generated by nilpotent elements for the family $\Lambda_q (q \neq \pm 1)$ isomorphic to the ground field k. We summarize with the following theorem.

Theorem 3.4 ([6] Theorem 3.10)**.** *Let k be a field, $char(k) \neq 2$, $\Lambda_q = kQ/I$ be the family of quiver algebras of (1.1), and $\mathcal{N}$ the set of nilpotent elements of* $\mathrm{HH}^*(\Lambda_q)$*. Then*

$$\mathrm{HH}^*(\Lambda_q)/\mathcal{N} = \begin{cases} \mathrm{HH}^0(\Lambda_q)/\mathcal{N} \cong Z(\Lambda_q)^* \cong k, & \text{if } q \neq \pm 1 \\ Z(\Lambda_q)^* \oplus k[x^2, y^2]y^2 \cong k \oplus k[x^2, y^2]y^2, & \text{if } q = \pm 1 \end{cases}$$

where $Z(\Lambda_q)$ is the center of the algebra Λ_q, the degree of y^2 is 2, and that of x^2y^2 is 4.

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