CUP PRODUCT ON HOCHSCHILD COHOMOLOGY OF A FAMILY OF QUIVER ALGEBRAS

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Abstract. We present the cup product structure on Hochschild cohomology of a family of quiver algebras. We use the formula to determine the set of homogeneous nilpotent Hochschild cocycles and construct a canonical isomorphism between Hochschild cohomology modulo nilpotents and a subalgebra of $k[x, y]$ that is not finitely generated.

1. Introduction

The theory of support varieties has been well developed for finite groups using group cohomology. Several efforts were made to develop similar theories for finitely generated modules over finite dimensional algebras using Hochschild cohomology. Hochschild cohomology $\mathrm{HH}^*(\Lambda)$ of a k-algebra Λ is graded commutative. If the characteristics of the field k is different from 2, then every homogeneous element of odd degree is nilpotent. Let $\mathcal{N}$ be the set of nilpotent elements of $\mathrm{HH}^*(\Lambda)$, Hochschild cohomology modulo nilpotents $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ is therefore a commutative k-algebra. For some finite dimensional algebras, it is well known that the Hochschild cohomology ring modulo nilpotents is finitely generated as an algebra. N. Snashall described many classes of such algebras in [7, section 3]. Before the expository paper [7], it was conjectured in [8] that Hochschild cohomology modulo nilpotents is always finitely generated as an algebra for finite dimensional algebras. The first counterexample to this conjecture appeared in [11] where F. Xu used certain techniques in category theory to construct a seven-dimensional category algebra whose Hochschild cohomology ring modulo nilpotents is not finitely generated. Some authors have presented several constructions of different counterexamples to this conjecture. While it is of great use to produce a counterexample, it is equally important to understand the cohomology ring structure of these algebras. We give a brief summary of a variation of the F. Xu counterexample which was presented in [7].

A quiver is a directed graph where loops and multiple arrows (also called paths) between two vertices are possible. The path algebra kQ, is the k-vector space generated by all paths in the quiver Q. By taking multiplication of two paths x and y to be the concatenation xy if the terminal vertex $t(x)$ of x and the origin vertex $o(y)$ of y are equal, and otherwise 0, kQ becomes an associative k-algebra. Let I be an ideal of kQ. The quotient $\Lambda = kQ/I$

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is called a quiver algebra. Consider the quiver;

$$\begin{tikzcd} 1 \arrow[loop above, "a"] \arrow[loop below, "b"] \arrow[r, "c"] & 2 \end{tikzcd}$$

and let

$$\left\{\Lambda_q = kQ/I_q\right\}_{q\in k}, \quad I_q = \langle a^2, b^2, ab - qba, ac\rangle, \tag{1.1}$$

be the family of quiver algebras generated by Q for each $q \in k$. We give the following remarks to summarize some of what has been done with respect to this family.

Remark 1.2. *From* [6, 7]*, we have that for each* q*,*

- Λ_q *is finitely generated since* Q *is a finite quiver i.e. has finite vertices and arrows.*
- Λ_q *is a graded Koszul quiver algebra.*
- *Let* $\Lambda_q = \oplus_{i\geq 0}(\Lambda_q)_i$ *be a grading. The Koszul dual* $\Lambda_q^!$ *of* Λ_q *and the Yoneda algebra* $E(\Lambda_q)$ *are related by the following equation;*
$$E(\Lambda_q) = Ext^*_{\Lambda_q}((\Lambda_q)_0, (\Lambda_q)_0) \cong \Lambda_q^! = kQ^{op}/I_q^{\perp} \tag{1.3}$$
where Q^{op} *is the quiver with opposite arrows,* $I_q^{\perp} := \langle a^o b^0 + q^{-1} b^0 a^0, b^0 c^0\rangle$*, with* v^0 *the correponding arrow in the opposite quiver algebra* kQ^{op} *for any* $v \in KQ$*. Note also that* $\Lambda_q^!$ *is generated in degrees 0 and 1.*
- *The case where* $q = \pm 1$*,* I_q *belongs to a class of (anti-)commutative ideals studied by E. Gawell and Q.R. Xantcha. There is an associated generator graph (of the orthogonal ideal* $I_q^{\perp}$ *of* I_q*) which has no directed cycles. This means that the ideal* I_q *is admissible* [4]*.*
- *For* $q = 1$*, the graded center of the Yoneda algebra* $Z_{gr}(E(\Lambda_q))$ *is given by the following*
$$Z_{gr}(E(\Lambda_1)) = \begin{cases} k \oplus k[a,b]b, & \textit{if } char(k) = 2 \\ k \oplus k[a^2,b^2]b^2, & \textit{if } char(k) \neq 2 \end{cases}$$
where the degree of b *is 1, and that of* ab *is 2.*

The following result shows that Λ_1 is a counterexample to the Snashall-Solberg finite generation conjecture.

Theorem. [7, Theorem 4.5] *Let* k *be a field and* Λ_1 *be a member of quiver algebras given in Equation* (1.1)*. Let* $\mathcal{N}$ *be the set of nilpotent elements of* $\mathrm{HH}^*(\Lambda_1)$*, then*

$$\mathrm{HH}^*(\Lambda_1)/\mathcal{N} \cong Z_{gr}(E(\Lambda_1)) = \begin{cases} k \oplus k[a,b]b, & \textit{if } char(k) = 2 \\ k \oplus k[a^2,b^2]b^2, & \textit{if } char(k) \neq 2 \end{cases}$$

where the degree of b *is 1, and that of* ab *is 2.*

Our result: In this paper, we study the Hochschild cohomology ring of the family Λ_q of quiver algebras of Equation (1.1). We present a comultiplicative map on a resolution $\mathbb{K}$

for this family and determine a cup product formula using the comultiplicative map. We also present a formula for the dimension of the space of Hochschild cocycles $\operatorname{Ker} d^*$ where $d^* : \operatorname{Hom}_{\Lambda^e}(\mathbb{K}_*, \Lambda) \to \operatorname{Hom}_{\Lambda^e}(\mathbb{K}_*, \Lambda)$. Furthermore, with the generalized cup product formula, we show that many of these cocycles are nilpotent with respect to the cup product and all cocycles of odd homological degrees are nilpotent. We want to point out that the comultiplicative formula we derived will be very useful in determining the bracket structure on Hochschild cohomology of this family; the bracket structure is currently unknown.

As an application, we present a different approach to show that not just Λ_1 but $\Lambda_q, q = \pm1$, have Hochschild cohomology modulo nilpotents that is not finitely generated: so they are counterexamples to the Snashall-Solberg finite generation conjecture. The usual way many authors present this proof is by looking at the graded center of the Yoneda algebra $Z_{gr}(E(\Lambda))$ of Λ and considering the isomorphism $\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong Z_{gr}(E(\Lambda))$. However, our approach involves a direct computation and the use of the generalized cup product formula. Furthermore, for even integers $n > 0$, we give explicit presentation of elements of $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ corresponding to $x^{2n-r}y^r$ in $k[x, y]$. We noted that these elements cannot be generated from other elements of lower homological degrees. Our main results are the following;

Theorem. *Let $\phi : \mathbb{K}_m \to \Lambda_q$, and $\mu : \mathbb{K}_n \to \Lambda_q$, be two Hochschild cocycles. Let $\{\varepsilon_r^m\}_{r=0}^{t_m}$ be free basis elements of $\mathbb{K}_m$ such that for each $0 \leq r \leq t_m$, $\phi(\varepsilon_r^m) = \phi_r^m \in \Lambda_q$. Then the following gives a formula for the cup product on Hochschild cohomology;*

$$(\phi \smile \mu)(\varepsilon_r^{m+n}) = (\phi\mu)_r^{m+n} = \begin{cases} (-1)^{mn}\phi_0^m\mu_0^n, & \text{when } r = 0, \\ (-1)^{mn}T_r^{m+n} & \text{when } 0 < r < m+n, \\ (-1)^{mn}\phi_m^m\mu_n^n, & \text{when } r = m+n, \\ (-1)^{mn}\phi_0^m\mu_{n+1}^n, & \text{when } r = m+n+1, \end{cases}$$

where $$T_r^{m+n} = \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)}\phi_j^m\mu_{r-j}^n, \qquad 0 < r < m+n.$$

Theorem. *Let k be a field of characteristics different from 2. Let $\Lambda_q = kQ/I_q$ be the family of quiver algebras of (1.1) and $\mathcal{N}$ the set of nilpotent elements of $\mathrm{HH}^*(\Lambda_q)$, then*

$$\mathrm{HH}^*(\Lambda_q)/\mathcal{N} = \begin{cases} \mathrm{HH}^0(\Lambda_q)/\mathcal{N} \cong k, & \text{if } q \neq \pm1 \\ Z^0(\Lambda_q, \Lambda_q) \oplus k[x^2, y^2]y^2 \cong k \oplus k[x^2, y^2]y^2, & \text{if } q = \pm1 \end{cases}$$

where the degree of y^2 is 1, and that of x^2y^2 is 4.

2. Minimal projective resolution for Koszul quiver algebras

We recall that a quiver is a directed graph with the allowance of loops and multiple arrows. A quiver Q is sometimes denoted as a quadruple (Q_0, Q_1, o, t) where Q_0 is the set of vertices in Q, Q_1 is the set of arrows in Q, and $o, t : Q_1 \longrightarrow Q_0$ are maps which assign to each arrow $a \in Q_1$, its origin vertex $o(a)$ and terminal vertex $t(a)$ in Q_0. A path in Q is a sequence of arrows $a = a_1a_2 \cdots a_{n-1}a_n$ such that the terminal vertex of a_i is the same as the origin vertex of a_{i+1}, using the convention of concatenating paths from left to right. The path algebra kQ is defined as a vector space having all paths in Q as a basis. Vertices are regarded as paths of length 0, an arrow is a path of length 1, and so on. We take multiplication on kQ as concatenation of paths. Two paths a and b satisfy $ab = 0$ if

$t(a) \neq o(b)$. This multiplication defines an associative algebra over k. By taking kQ_i to be the k-vector subspace of kQ with paths of length i as basis, $kQ = \bigoplus_{i \geq 0} kQ_i$ can be viewed as an $\mathbb{N}$-graded algebra. A relation on a quiver Q is a linear combination of paths from Q each having the same origin vertex and terminal vertex. A quiver together with a set of relations is called a quiver with relations. Let I be an ideal of kQ generated by some relations. Recall that the quotient $\Lambda = kQ/I$ is called the quiver algebra associated with (Q, I).

Before we present a construction of the resolution $\mathbb{K}$ that we use later to determine Hochschild cohomology, we present the following definition.

Definition 2.1. *Let Λ be a quadratic graded quiver algebra. A graded Λ-module M is Koszul if it has a (minimal) graded projective resolution*

$$\mathbb{P}_\bullet : \qquad \cdots \to \mathbb{P}_n \xrightarrow{d_n} \mathbb{P}_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \mathbb{P}_1 \xrightarrow{d_1} \mathbb{P}_0 (\to M)$$

i.e. for each n, $\mathbb{P}_n$ is graded and the differentials have entries in $\Lambda_+ = \bigoplus_{i>0} \Lambda_i$.

Construction of the minimal projective resolution $\mathbb{K}$: Let $\Lambda = kQ/I$ be a graded Koszul algebra. Then Λ_0 has a graded (minimal) projective resolution $\mathbb{L}$ as a right Λ-module. An algorithmic approach to find such a minimal projective resoluton $\mathbb{L} \to \Lambda_0$ of right Λ-modules was given in [3]. The resolution was shown to have a "comultiplicative structure" and this structure was used to find a minimal projective resolution $\mathbb{K} \to \Lambda$ of modules over the enveloping algebra of Λ in [5]. We now describe these resolutions.

Take J to be the ideal of kQ generated by all arrows and suppose further that $I \subseteq J^2$ is an admissible ideal, that is, $J^m \subseteq I \subseteq J^2$ for some m and set $\mathfrak{r} = J/I$. A non-zero element $x \in kQ$ is called uniform if there exist vertices u, v such that $x = uxv = ux = xv$, where u is the common origin vertex and v is the common terminal vertex of each of the paths summing up to x. For $R = kQ$, it was shown that there are integers $\{t_n\}_{n \geq 0}$ and uniform elements $\{f_i^n\}_{i=0}^{t_n}$ such that the minimal right projective resolution $\mathbb{L}$ of $\Lambda_0 \cong \Lambda/\mathfrak{r}$, is obtained from a filtration of R. The element f_i^n for each i, is a path of length n. The filtration is given by the following nested family of right ideals:

$$\cdots \subseteq \bigoplus_{i=0}^{t_n} f_i^n R \subseteq \bigoplus_{i=0}^{t_{n-1}} f_i^{n-1} R \subseteq \cdots \subseteq \bigoplus_{i=0}^{t_1} f_i^1 R \subseteq \bigoplus_{i=0}^{t_0} f_i^0 R = R,$$

where for each n, $\mathbb{L}_n = \bigoplus_{i=0}^{t_n} f_i^n R / \bigoplus_{i=0}^{t_n} f_i^n I$ and the differentials d^L on $\mathbb{L}$ are induced by the inclusions $\bigoplus_{i=0}^{t_n} f_i^n R \subseteq \bigoplus_{i=0}^{t_{n-1}} f_i^{n-1} R$. These inclusions imply that there are elements $h_{ji}^{n-1,n}$ in R such that

$$f_i^n = \sum_{j=0}^{t_{n-1}} f_j^{n-1} h_{ji}^{n-1,n}$$

for all $i = 0, 1, \ldots, t_n$ and all $n \geq 1$. The differentials $d_n^L : \mathbb{L}_n \to \mathbb{L}_{n-1}$ are given by

$$d_n^L(f_i^n) = \begin{pmatrix} h_{0i}^{n-1,n} & h_{1i}^{n-1,n} & \cdots & h_{t_{n-1}i}^{n-1,n} \end{pmatrix}$$

for all $n \geq 1$.

Furthermore, it was shown that, with some choice of scalars, the elements $\{f_i^n\}_{i=0}^{t_n}$ satisfy a comultiplicative structure given below in (2.2). That is, for $0 \leq i \leq t_n$, there are scalars $c_{pq}(n,i,r)$ such that

$$f_i^n = \sum_{p=0}^{t_r} \sum_{q=0}^{t_{n-r}} c_{pq}(n,i,r) f_p^r f_q^{n-r}. \tag{2.2}$$

To set up this equation in practice, we can take $\{f_i^0\}_{i=0}^{t_0}$ to be the set of vertices, $\{f_i^1\}_{i=0}^{t_1}$ to be the set of arrows, $\{f_i^2\}_{i=0}^{t_2}$ to be the generators of I, and define $\{f_i^n\}(n \geq 3)$ recursively in terms of f_i^{n-1} and f_j^1. The resolution $\mathbb{L}$ and the comultiplicative structure (2.2) were used to construct a minimal projective resolution $\mathbb{K} \to \Lambda$ of modules over the enveloping algebra $\Lambda^e = \Lambda \otimes \Lambda^{op}$. The minimal projective resolution $\mathbb{K}$ of Λ^e-modules was given to be

$$\mathbb{K}: \qquad \cdots \xrightarrow{d_{n+1}} \mathbb{K}_n \xrightarrow{d_n} \mathbb{K}_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \mathbb{K}_1 \xrightarrow{d_1} \mathbb{K}_0 \ (\xrightarrow{\mu} \Lambda) \tag{2.3}$$

where μ is the multiplication map, and for each n,

$$\mathbb{K}_n = \bigoplus_{i=0}^{t_n} \Lambda o(f_i^n) \otimes_k t(f_i^n)\Lambda.$$

Since each f_i^n is a uniform element, the notations $o(f_i^n), t(f_i^n)$ are well defined. Let $\varepsilon_i^n = (0, \ldots, 0, o(f_i^n) \otimes_k t(f_i^n), 0, \ldots, 0)$, $0 \leq i \leq t_n$ where $o(f_i^n) \otimes_k t(f_i^n)$ is in the i-th position. Then for each n and i, $\{\varepsilon_i^n\}_{i=0}^{t_n}$ is a free basis of $\mathbb{K}_n$ as a Λ^e-module. The differentials as expressed in [5, Theorem 2.1] and [2, page 5], are given by

$$d_n(\varepsilon_i^n) = \sum_{j=0}^{t_{n-1}} \Big(\sum_{p=0}^{t_1} c_{p,j}(n,i,1) f_p^1 \varepsilon_j^{n-1} + (-1)^n \sum_{q=0}^{t_1} c_{j,q}(n,i,n-1) \varepsilon_j^{n-1} f_q^1 \Big) \tag{2.4}$$

where the scalars $c_{p,j}(n,i,r)$ are those appearing in Equation (2.2) and $\overline{f_*^1}$, which by abuse of notation has been written as f_*^1 in Equation (2.4), is the residue class of f_*^1 in $\bigoplus_{i=0}^{t_1} f_i^1 R / \bigoplus_{i=0}^{t_n} f_i^1 I$.

We define Hochschild cohomology using this resolution. That is, we apply the functor $\mathrm{Hom}_{\Lambda^e}(\cdot, \Lambda)$ to the resolution $\mathbb{K}$ and take a direct sum of the cohomology groups in each degree:

$$\mathrm{HH}^*(\Lambda) := \bigoplus_{n \geq 0} \mathrm{H}^n(\mathrm{Hom}_{\Lambda^e}(\mathbb{K}_n, \Lambda)) \cong \bigoplus_{n \geq 0} \mathrm{Ext}^n_{\Lambda^e}(\Lambda, \Lambda) = \mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda)$$

Using the comultiplicative structure of Equation (2.2), it was shown in [2] that the cup product on the Hochschild cohomology ring of Koszul algebras defined by quivers and relations has the following description:

Theorem 2.5 (See [2], Theorem 2.3)**.** *Let $\Lambda = kQ/I$ be a Koszul algebra over a field k, where Q is a finite quiver and $I \subseteq J^2$. Suppose that $\eta : \mathbb{K}_n \to \Lambda$ and $\theta : \mathbb{K}_m \to \Lambda$ represent elements in $\mathrm{HH}^*(\Lambda)$ and are given by $\eta(\varepsilon_i^n) = \lambda_i$ for $i = 0, 1, \ldots, t_n$ and $\theta(\varepsilon_i^m) = \lambda'_i$ for $i = 0, 1, \ldots, t_m$. Then $\eta \smile \theta : \mathbb{K}_{n+m} \to \Lambda$ can be expressed as*

$$(\eta \smile \theta)(\varepsilon_j^{n+m}) = \sum_{p=0}^{t_n} \sum_{q=0}^{t_m} c_{pq}(n+m, i, n) \lambda_p \lambda'_q,$$

for $j = 0, 1, 2, \ldots, t_{n+m}$.

The reduced bar resolution of $\Lambda = kQ/I$ as presented in [2, Section 1]: We recall the definition of the reduced bar resolution of algebras defined by quivers and relations. If Λ_0 is isomorphic to m copies of k, take $\{e_1, e_2, \dots, e_m\}$ to be a complete set of primitive orthogonal central idempotents of Λ_0. In this case Λ is not necessarily an algebra over Λ_0. Define the reduced bar resolution $(\mathcal{B}, \delta)$ to be $\mathcal{B}_n = \Lambda^{\otimes_{\Lambda_0}(n+2)}$, the $(n+2)$-fold tensor product of Λ over Λ_0 with differentials δ given by:

$$\delta_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}. \tag{2.6}$$

If $\Lambda_0 \cong k$, then $\mathcal{B} = \mathbb{B}$, the usual bar resolution, as Λ is an algebra over Λ_0. The resolution $\mathbb{K}$ can be embedded naturally into the reduced bar resolution $\mathcal{B}$. There is a map $\iota : \mathbb{K} \to \mathcal{B}$ defined by $\iota(\varepsilon_r^n) = 1 \otimes \widetilde{f_r^n} \otimes 1$ such that $\delta\iota = \iota d$, with

$$\widetilde{f_j^n} = \sum c_{j_1 j_2 \cdots j_n} f_{j_1}^1 \otimes f_{j_2}^1 \otimes \cdots \otimes f_{j_n}^1 \quad \text{if} \quad f_j^n = \sum c_{j_1 j_2 \cdots j_n} f_{j_1}^1 f_{j_2}^1 \cdots f_{j_n}^1 \tag{2.7}$$

for some scalar $c_{j_1 j_2 \cdots j_n}$. See [2, Proposition 2.1] for a proof that ι is indeed an embedding. Let $\Delta : \mathcal{B} \to \mathcal{B} \otimes_\Lambda \mathcal{B}$ be comultiplicative map (also called the diagonal map) on the bar resolution given explicitly by Equation (3.7). It was also shown in [2, Proposition 2.2] that there is a comultiplicative map $\Delta_{\mathbb{K}} : \mathbb{K} \to \mathbb{K} \otimes_\Lambda \mathbb{K}$ on the complex $\mathbb{K}$ compatible with ι. This means that $(\iota \otimes \iota)\Delta_{\mathbb{K}} = \Delta\iota$ where $(\iota \otimes \iota)(\mathbb{K} \otimes_\Lambda \mathbb{K}) = \iota(\mathbb{K}) \otimes_\Lambda \iota(\mathbb{K}) \subseteq \mathcal{B} \otimes_\Lambda \mathcal{B}$. The comultiplicative map on $\mathbb{K}$ is given in general by

$$\Delta_{\mathbb{K}}(\varepsilon_r^n) = \sum_{v=0}^{n} \sum_{p=0}^{t_v} \sum_{q=0}^{t_{n-v}} c_{p,q}(n, r, v) \varepsilon_p^v \otimes_\Lambda \varepsilon_q^{n-v}. \tag{2.8}$$

We present a specific comultiplicative map for the family $\{\Lambda_q\}_{q \in k}$ under study in Remark 3.11 and use it to determine the structure of Hochschild cohomology of this family. Furthermore, there are recent techniques such as [9] for computing the bracket structure on Hochschild cohomology which relies on comultiplicative maps such as this.

3. Cup product structure

In this section, we will study the Hochschild cohomology of the family of quiver algebras of Equation (1.1) i.e.

$$\{\Lambda_q = \frac{kQ}{I_q}\}, \quad I_q = \langle a^2, b^2, ab - qba, ac \rangle.$$

and present the cup product structure on their Hochschild cohomology. For this family, the resolution $\mathbb{K} \to \Lambda_q$ has free basis elements $\{\varepsilon_i^n\}_{i=0}^{t_n}$ such that for each n, $\varepsilon_i^n = (0, \dots, 0, o(f_i^n) \otimes_k t(f_i^n), 0, \dots, 0)$. To concretely define the free basis ε_i^n for each module $\mathbb{K}_n$, we start by using kQ_0, the subspace of kQ generated by the vertices of Q with basis $\{e_1, e_2\}$. We define $\{f_0^0 = e_1, f_1^0 = e_2\}$. Next, since kQ_1 is the subspace generated by paths of length 1 with free basis $\{a, b, c\}$, we define $\{f_0^1 = a, f_1^1 = b, f_2^1 = c\}$. We let f_j^2, $j = 0, 1, 2, 3$ to be the set of paths of length 2 generated by the ideal I, that is, $\{f_0^2 = a^2, f_1^2 = ab - qba, f_2^2 = b^2, f_3^2 = ac\}$. We continue in this way and define for each

$n > 2$,

$$\text{(3.1)} \qquad \begin{cases} f_0^n = a^n, \\ f_s^n = f_{s-1}^{n-1} b + (-q)^s f_s^{n-1} a, \quad (0 < s < n), \\ f_n^n = b^n, \\ f_{n+1}^n = a^{(n-1)} c, \end{cases}$$

We recall that each f_i^n is a uniform relation therefore the origin vertex $0(f_i^n)$ and the terminal vertex $t(f_i^n)$ exist. Therefore the notation $o(f_i^n) \otimes_k t(f_i^n)$ in the definition of ε_i^n makes sense. The differentials on $\mathbb{K}_n$ are given explicitly for this family by

$$\begin{aligned} d_1(\varepsilon_2^1) &= c\varepsilon_1^0 - \varepsilon_0^0 c \\ d_n(\varepsilon_r^n) &= (1 - \partial_{n,r})[a\varepsilon_r^{n-1}) + (-1)^{n-r} q^r \varepsilon_r^{n-1} a] \\ &\quad + (1 - \partial_{r,0})[(-q)^{n-r} b\varepsilon_{r-1}^{n-1} + (-1)^n \varepsilon_{r-1}^{n-1} b], \text{ for } r \leq n \\ d_n(\varepsilon_{n+1}^n) &= a\varepsilon_n^{n-1} + (-1)^n \varepsilon_0^{n-1} c, \text{ when } n \geq 2, \end{aligned} \tag{3.2}$$

where $\partial_{r,s} = 1$ when $r = s$ and 0 when $r \neq s$. We give below, a proof that the differentials satisfy $d^2 = 0$. For a general proof that the resolution we obtain using these descriptions, and it's general form presented in Equation (2.3) is a minimal projective resolution, see [5, Theorem 2.1].

A proof that $d^2 = 0$.

Proof.

$$\begin{aligned} d_0 d_1((\varepsilon_2^1) &= \mu d_1((\varepsilon_2^1) = \mu(c\varepsilon_1^0 - \varepsilon_0^0 c) = \mu(c(e_2 \otimes e_2) - (e_1 \otimes e_1)c) \\ &= \mu(c \otimes e_2 - e_1 \otimes c) = ce_2 - e_1 c = c - c = 0. \end{aligned}$$

Now for $r \leq n$, we take $(1 - \partial_{n,r}) = \bar{\partial}_{n,r}$, so that

$$d_n(\varepsilon_r^n) = \bar{\partial}_{n,r}[a\varepsilon_r^{n-1}) + (-1)^{n-r} q^r \varepsilon_r^{n-1} a] + \bar{\partial}_{r,0}[(-q)^{n-r} b\varepsilon_{r-1}^{n-1} + (-1)^n \varepsilon_{r-1}^{n-1} b].$$

Now we apply the differential again,

$$\begin{aligned} & d_{n-1} d_n(\varepsilon_r^n) \\ &= d_{n-1}\big\{\bar{\partial}_{n,r}[a\varepsilon_r^{n-1} + (-1)^{n-r} q^r \varepsilon_r^{n-1} a] + \bar{\partial}_{r,0}[(-q)^{n-r} b\varepsilon_{r-1}^{n-1} + (-1)^n \varepsilon_{r-1}^{n-1} b]\big\} \\ &= \bar{\partial}_{n,r}[a d_{n-1}(\varepsilon_r^{n-1}) + (-1)^{n-r} q^r d_{n-1}(\varepsilon_r^{n-1}) a] + \bar{\partial}_{r,0}[(-q)^{n-r} b d_{n-1}(\varepsilon_{r-1}^{n-1}) + (-1)^n d_{n-1}(\varepsilon^{n-1})_{r-1} b] \\ \\ &= \bar{\partial}_{n,r} a\big\{\bar{\partial}_{n-1,r}[a\varepsilon_r^{n-2} + (-1)^{n-r-1} q^r \varepsilon_r^{n-2} a] + \bar{\partial}_{r,0}[(-q)^{n-r-1} b\varepsilon_{r-1}^{n-2} + (-1)^{n-1} \varepsilon_{r-1}^{n-2} b]\big\} \\ &+ (-1)^{n-r} q^r \bar{\partial}_{n,r}\big\{\bar{\partial}_{n-1,r}[a\varepsilon_r^{n-2} + (-1)^{n-r-1} q^r \varepsilon_r^{n-2} a] + \bar{\partial}_{r,0}[(-q)^{n-r-1} b\varepsilon_{r-1}^{n-2} + (-1)^{n-1} \varepsilon_{r-1}^{n-2} b]\big\} a \\ &+ (-q)^{n-r} \bar{\partial}_{r,0} b\big\{\bar{\partial}_{n-1,r-1}[a\varepsilon_{r-1}^{n-2} + (-1)^{n-r} q^{r-1} \varepsilon_{r-1}^{n-2} a] + \bar{\partial}_{r-1,0}[(-q)^{n-r} b\varepsilon_{r-2}^{n-2} + (-1)^{n-1} \varepsilon_{r-2}^{n-2} b]\big\} \\ &+ (-1)^n \bar{\partial}_{r,0}\big\{\bar{\partial}_{n-1,r-1}[a\varepsilon_{r-1}^{n-2} + (-1)^{n-r} q^{r-1} \varepsilon_{r-1}^{n-2} a] + \bar{\partial}_{r-1,0}[(-q)^{n-r} b\varepsilon_{r-2}^{n-2} + (-1)^{n-1} \varepsilon_{r-2}^{n-2} b]\big\} b \end{aligned}$$

$$
\begin{aligned}
&= \bar{\partial}_{n,r}\bar{\partial}_{n-1,r}a^2\varepsilon_r^{n-2} + (-1)^{2(n-r)-1}q^{2r}\bar{\partial}_{n,r}\bar{\partial}_{n-1,r}\varepsilon_r^{n-2}a^2\\
&+ (-q)^{2(n-r)}\bar{\partial}_{r,0}\bar{\partial}_{r-1,0}b^2\varepsilon_{r-2}^{n-2} + (-1)^{2n-1}\bar{\partial}_{r,0}\bar{\partial}_{r-1,0}\varepsilon_{r-2}^{n-2}b^2\\
&+ [(-1)^{-1}+1](-1)^{n-r}q^r\bar{\partial}_{n,r}\bar{\partial}_{n-1,r}a\varepsilon_r^{n-2}a + [(-1)^{-1}+1](-1)^n(-q)^{n-r}\bar{\partial}_{r,0}\bar{\partial}_{r-1,0}b\varepsilon_{r-2}^{n-2}b\\
&+ [-\bar{\partial}_{n,r}+\bar{\partial}_{n-1,r-1}](-1)^{2(n-r)}q^{n-1}\bar{\partial}_{r,0}b\varepsilon_{r-1}^{n-2}a + [-\bar{\partial}_{n,r}+\bar{\partial}_{n-1,r-1}]\bar{\partial}_{r,0}a\varepsilon_{r-1}^{n-2}b\\
&+ (-q)^{n-2}\bar{\partial}_{r,0}[\bar{\partial}_{n,r}ab-\bar{\partial}_{n-1,r-1}qba]\varepsilon_{r-1}^{n-2} + (-1)^{2n-r}(-q)^{r-1}\bar{\partial}_{r,0}\varepsilon_{r-1}^{n-2}[\bar{\partial}_{n,r}-qba+\bar{\partial}_{n-1,r-1}ab]\\
&= 0.
\end{aligned}
$$

In the last equality, we have used the fact that $a^2 = b^2 = 0$, $\bar{\partial}_{n,r}$ and $\bar{\partial}_{n-1,r-1}$ have the same sign, implying that $[\bar{\partial}_{n,r}ab - \bar{\partial}_{n-1,r-1}qba] = [ab - qba] = 0$ and $[\bar{\partial}_{n,r} - qba + \bar{\partial}_{n-1,r-1}ab] = [-qba + ab] = 0$. Lastly we have

$$
\begin{aligned}
d_{n-1}d_n(\varepsilon_{n+1}^n) &= d_{n-1}[a\varepsilon_n^{n-1} + (-1)^n\varepsilon_0^{n-1}c]\\
&= a\big\{a\varepsilon_{n-1}^{n-2} + (-1)^{n-1}\varepsilon_0^{n-2}c\big\} + (-1)^n\big\{a\varepsilon_0^{n-2} + (-1)^{n-1}\varepsilon_0^{n-2}a\big\}c\\
&\text{after eliminating terms with coefficients } a^2 = ac = 0,\\
&= (-1)^{n-1}a\varepsilon_0^{n-2}c + (-1)^n a\varepsilon_0^{n-2}c = [(-1)^{-1}+1](-1)^n a\varepsilon_0^{n-2}c = 0
\end{aligned}
$$

□

Recall that for each member Λ_q of the family, the resolution $\mathbb{K}$ can be embedded into the reduced bar resolution $\mathcal{B}$ via ι. The embedding map $\iota : \mathbb{K}_n \to \mathcal{B}_n$ is defined by $\varepsilon_r^n \mapsto 1 \otimes \widetilde{f_r^n} \otimes 1$, where each $\widetilde{f_r^n}$ is viewed as a sum of tensor products of paths of length 1 as given in Equation (2.7). For example, for this family, $\widetilde{f_0^2} = f_0^1 \otimes f_0^1 = a \otimes a$, $\widetilde{f_1^2} = f_0^1 \otimes f_1^1 - q f_1^1 \otimes f_0^1 = a \otimes b - qb \otimes a$. It is clear from Equation (3.1) that the following holds;

(3.3)
$$
\widetilde{f_s^n} = \begin{cases}
f_0^1 \otimes f_0^1 \otimes \cdots \otimes f_0^1, \quad (n\ \ times) & \text{when } s = 0,\\
\widetilde{f_{s-1}^{n-1}} \otimes f_1^1 + (-q)^s \widetilde{f_s^{n-1}} \otimes f_0^1, & \text{when } (0 < s < n),\\
f_1^1 \otimes f_1^1 \otimes \cdots \otimes f_1^1, \quad (n\ \ times) & \text{when } s = n,\\
f_0^1 \otimes f_0^1 \otimes \cdots \otimes f_0^1 \otimes f_2^1, \quad (f_0^1 \text{ appears } (n-1)\ \ times), & \text{when } s = n+1,
\end{cases}
$$

In case $0 < s < n$, it was shown in [1] that
$f_s^n = \sum_{j=max\{0,r+t-n\}}^{min\{t,s\}} (-q)^{j(n-s+j-t)} f_j^t f_{s-j}^{n-t}$, hence,

$$
\widetilde{f_s^n} = \sum_{j=max\{0,r+t-n\}}^{min\{t,s\}} (-q)^{j(n-s+j-t)} \widetilde{f_j^t} \otimes \widetilde{f_{s-j}^{n-t}}. \tag{3.4}
$$

We are now ready to present the cup product formula. We first present the following alternate definition of the cup product.

Definition 3.5. *Let A be a k-algebra. Let $\Delta : \mathbb{B} \to \mathbb{B} \otimes_A \mathbb{B}$ be the comultiplicative map lifting the identity map $A \cong A \otimes_A A$. Let $f \in \operatorname{Hom}_{A^e}(\mathbb{B}_m, A) \cong \operatorname{Hom}_k(A^{\otimes m}, A)$ and $g \in \operatorname{Hom}_k(A^{\otimes n}, A)$ be cocycles of degree m and n respectively. The cup product $f \smile g$ at the chain level is an element of $\operatorname{Hom}_k(A^{\otimes(m+n)}, A)$ given by*

$$
f \smile g = \pi(f \otimes g)\Delta, \tag{3.6}
$$

where π is multiplication, and Δ is given by

$$\Delta(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (a_0 \otimes \cdots \otimes a_i \otimes 1) \otimes_{\Lambda} (1 \otimes a_{i+1} \otimes \cdots \otimes a_{n+1}). \tag{3.7}$$

For homogeneous elements a, b of degrees m and n respectively, the map $f \otimes g$ is taken to be $(f \otimes g)(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b)$, where the degree of g is $|g| = n$. We recall that for any member Λ_q of the family, if $\phi \in \mathrm{Hom}_{\Lambda_q^e}(\mathbb{K}_m, \Lambda_q)$, and $\eta \in \mathrm{Hom}_{\Lambda_q^e}(\mathbb{K}_n, \Lambda_q)$ are two cocycles, we can use Definition 3.5 on the resolution $\mathbb{K}$ provided we have an explicit presentation of the comultiplication $\Delta_{\mathbb{K}}$. This definition is presented using the following composition of maps;

$$\phi \smile \eta : \mathbb{K} \xrightarrow{\Delta_{\mathbb{K}}} \mathbb{K} \otimes_{\Lambda_q} \mathbb{K} \xrightarrow{\phi \otimes \eta} \Lambda_q \otimes_{\Lambda_q} \Lambda_q \overset{\pi}{\simeq} \Lambda_q$$

where π is multiplication, $(\phi \otimes \eta)(\varepsilon_i^m \otimes \varepsilon_j^n) = (-1)^{mn} \phi(\varepsilon_r^m) \otimes \eta(\varepsilon_j^n)$, and the comultiplicative map $\Delta_{\mathbb{K}}$ is such that following diagram:

$$\begin{array}{ccc} \mathbb{K} & \xrightarrow{\Delta_{\mathbb{K}}} & \mathbb{K} \otimes_{\Lambda_q} \mathbb{K} \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\iota \otimes \iota} \\ \mathcal{B} & \xrightarrow{\Delta} & \mathcal{B} \otimes_{\Lambda_q} \mathcal{B}. \end{array}$$

is commutative i.e.

$$(\iota \otimes \iota)\Delta_{\mathbb{K}} = \Delta \iota. \tag{3.8}$$

Notice that we do not distinquish between $\mathbb{B}$ and $\mathcal{B}$ when using the map Δ. We are able to present explicit definition of $\Delta_{\mathbb{K}}$ in Remark 3.11 after providing a proof of Theorem 3.9 which relies on Equations (3.7) and (3.8).

Let $\phi : \mathbb{K}_m \to \Lambda_q$ and $\eta : \mathbb{K}_n \to \Lambda_q$ be two cocycles of homological degrees m and n respectively. Suppose that ϕ takes ε_0^m to ϕ_0^m, ε_1^m to ϕ_1^m and so on until ε_{m+1}^m to ϕ_{m+1}^m, we use the following standard notation $\phi = \begin{pmatrix} \phi_0^m & \phi_1^m & \cdots & \phi_m^m & \phi_{m+1}^m \end{pmatrix}$ and $\eta = \begin{pmatrix} \eta_0^n & \eta_1^n & \eta_2^n & \cdots & \eta_n^n & \eta_{n+1}^n \end{pmatrix}$. We denote the cup product of ϕ and η by

$$\phi \smile \eta := \begin{pmatrix} (\phi\eta)_0^{m+n} & (\phi\eta)_1^{m+n} & (\phi\eta)_2^{m+n} & \cdots & (\phi\eta)_{m+n}^{m+n} & (\phi\eta)_{m+n+1}^{m+n} \end{pmatrix},$$

that is, $(\phi \smile \eta)(\varepsilon_i^{m+n}) = (\phi\eta)_i^{m+n}, i = 0, 1, \ldots, m+n+1$.

Theorem 3.9. *Let $\phi : \mathbb{K}_m \to \Lambda_q$, and $\eta : \mathbb{K}_n \to \Lambda_q$ be two representatives of elements of $\mathrm{HH}^m(\Lambda_q)$ and $\mathrm{HH}^n(\Lambda_q)$. Then the following gives the cup product $\phi \smile \eta : \mathbb{K}_{m+n} \to \Lambda_q$ of ϕ and η:*

$$(\phi \smile \eta)(\varepsilon_i^{m+n}) = (\phi\eta)_i^{m+n} = \begin{cases} (-1)^{mn} \phi_0^m \eta_0^n, & \text{when } i = 0 \\ (-1)^{mn} T_i^{m+n} & \text{when } 0 < i < m+n \\ (-1)^{mn} \phi_m^m \eta_n^n, & \text{when } i = m+n \\ (-1)^{mn} \phi_0^m \eta_{n+1}^n, & \text{when } i = m+n+1 \end{cases} \tag{3.10}$$

where

$$T_i^{m+n} = \sum_{j=max\{0,i-n\}}^{min\{m,i\}} (-q)^{j(n-i+j)} \phi_j^m \eta_{i-j}^n, \qquad 0 < i < m+n.$$

Proof. We will find an explicit description of the comultiplicative map $\Delta_{\mathbb{K}}$ for which Equation (3.8) holds. It is enough to find the image of the basis elements $\{\varepsilon_r^{m+n}\}_{r=0}^{m+n+1}$ under this map. We will look at the case when $r = 0, r = m+n, r = m+n+1$. We last consider the case where $0 < r < m+n$ then use the formula $(\phi \smile \eta)(\varepsilon_r^{m+n}) = \pi(\phi \otimes \eta)\Delta_{\mathbb{K}}(\varepsilon_r^{m+n})$ as the definition of the cup product.

When $r = 0$, we have that

$$\begin{aligned}
&(\iota \otimes \iota)\Delta_{\mathbb{K}}(\varepsilon_0^{m+n}) = \Delta\iota(\varepsilon_0^{m+n}) \\
&= \Delta(1 \otimes \widetilde{f_0^{m+n}} \otimes 1) = \Delta(1 \otimes \overbrace{f_0^1 \otimes f_0^1 \otimes \cdots \otimes f_0^1}^{m+n\ times} \otimes 1) \\
&= \sum_{s=0}^{m+n} (1 \otimes \widetilde{f_0^s} \otimes 1) \otimes (1 \otimes \widetilde{f_0^{m+n-s}} \otimes 1) = (\iota \otimes \iota)(\sum_{s=0}^{m+n} \varepsilon_0^s \otimes \varepsilon_0^{m+n-s}).
\end{aligned}$$

Notice that by the usual definition of the comultiplicative map on the bar resolution, $1 \otimes \widetilde{f_0^0} \otimes 1 = 1 \otimes 1$. Hence $\boxed{\Delta_{\mathbb{K}}(\varepsilon_0^{m+n}) = (\sum_{s=0}^{m+n} \varepsilon_0^s \otimes \varepsilon_0^{m+n-s}).}$ Since ϕ is a cocycle of degree m, we can evaluate $\phi(\varepsilon_*^m)$, and in a similar way evaluate $\eta(\varepsilon_*^n)$ to obtain

$$\begin{aligned}
(\phi \smile \eta)(\varepsilon_0^{m+n}) &= \pi(\phi \otimes \eta)\Delta_{\mathbb{K}}(\varepsilon_0^{m+n}) = \pi(\phi \otimes \eta)(\sum_{r=0}^{m+n} \varepsilon_0^r \otimes \varepsilon_0^{m+n-r}) \\
&= \pi((-1)^{mn}\phi(\varepsilon_0^m) \otimes \eta(\varepsilon_0^n)) = (-1)^{mn}\phi_0^m\eta_0^n
\end{aligned}$$

In case $r = m+n$

$$\begin{aligned}
&(\iota \otimes \iota)\Delta_{\mathbb{K}}(\varepsilon_{m+n}^{m+n}) = \Delta\iota(\varepsilon_{m+n}^{m+n}) \\
&= \Delta(1 \otimes \widetilde{f_{m+n}^{m+n}} \otimes 1) = \Delta(1 \otimes \overbrace{f_1^1 \otimes f_1^1 \otimes \cdots \otimes f_1^1}^{m+n\ times} \otimes 1) \\
&= \sum_{s=0}^{m+n} (1 \otimes \widetilde{f_s^s} \otimes 1) \otimes (1 \otimes \widetilde{f_{m+n-s}^{m+n-s}} \otimes 1) = (\iota \otimes \iota)(\sum_{s=0}^{m+n} \varepsilon_s^s \otimes \varepsilon_{m+n-s}^{m+n-s}),
\end{aligned}$$

so $\boxed{\Delta_{\mathbb{K}}(\varepsilon_{m+n}^{m+n}) = \sum_{s=0}^{m+n} \varepsilon_s^s \otimes \varepsilon_{m+n-s}^{m+n-s},}$ and

$$(\phi \smile \eta)(\varepsilon_{m+n}^{m+n}) = \pi((-1)^{mn}\phi(\varepsilon_m^m) \otimes \eta(\varepsilon_n^n)) = (-1)^{mn}\phi_m^m\eta_n^n.$$

A similar result holds with $r = m+n+1$, i.e.

$$\begin{aligned}
&(\iota \otimes \iota)\Delta_{\mathbb{K}}(\varepsilon_{m+n+1}^{m+n}) = \Delta(1 \otimes \widetilde{f_{m+n+1}^{m+n}} \otimes 1) = \Delta(1 \otimes \overbrace{f_0^1 \otimes f_0^1 \otimes \cdots \otimes f_0^1}^{m+n-1\ times} \otimes f_2^1 \otimes 1) \\
&= \sum_{s=0}^{m+n-1} (1 \otimes \widetilde{f_0^s} \otimes 1) \otimes (1 \otimes \widetilde{f_{m+n-s+1}^{m+n-s}} \otimes 1) + (1 \otimes f_0^1 \otimes f_0^1 \otimes \cdots \otimes f_0^1 \otimes f_2^1 \otimes 1) \otimes (1 \otimes 1) \\
&= (\iota \otimes \iota)(\sum_{s=0}^{m+n-1} \varepsilon_0^s \otimes \varepsilon_{m+n-s+1}^{m+n-s} + \varepsilon_{m+n+1}^{m+n} \otimes \varepsilon_0^0),
\end{aligned}$$

hence $\boxed{\Delta_{\mathbb{K}}(\varepsilon^{m+n}_{m+n+1}) = (\sum_{s=0}^{m+n-1} \varepsilon^s_0 \otimes \varepsilon^{m+n-s}_{m+n-s+1}) + \varepsilon^{m+n}_{m+n+1} \otimes \varepsilon^0_0}$. Therefore, when $s = m + n + 1$, we obtain $(\phi \smile \eta)(\varepsilon^{m+n}_{m+n+1}) = \pi((-1)^{mn}\phi(\varepsilon^m_0) \otimes \eta(\varepsilon^n_{n+1})) = (-1)^{mn}\phi^m_0\eta^n_{n+1}$. It was shown in [1] that for $0 < r < m + n$,

$$f^{m+n}_r = \sum_{j=max\{0,r+t-n-m\}}^{min\{t,r\}} (-q)^{j(m+n-r+j-t)} f^t_j f^{m+n-t}_{r-j},$$

therefore $\iota(\varepsilon^{m+n}_r) = 1 \otimes \Big[\sum_{j=max\{0,r+t-m-n\}}^{min\{t,r\}} (-q)^{j(m+n-r+j-t)} \widetilde{f^t_j} \otimes \widetilde{f^{m+n-t}_{r-j}} \Big] \otimes 1$, and by letting $t = m$, the above expression becomes $\sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)} 1 \otimes \widetilde{f^m_j} \otimes \widetilde{f^n_{r-j}} \otimes 1$. When we apply $\Delta_{\mathbb{K}}$ to the above expression, we obtain many terms including terms such as $\sum_{j=max\{0,r-n-1\}}^{min\{m-1,r\}} (-q)^{j(n-r+j+1)} (1 \otimes \widetilde{f^{m-1}_j} \otimes 1) \otimes (1 \otimes \widetilde{f^{n+1}_{r-j}} \otimes 1)$. Since we are only interested in terms which are non-zero under $\phi \otimes \eta$, we get

$$\begin{aligned}(\Delta\iota)(\varepsilon^{m+n}_r) &= \cdots + \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)} (1 \otimes \widetilde{f^m_j} \otimes 1) \otimes (1 \otimes \widetilde{f^n_{r-j}} \otimes 1) + \cdots \\ &= \cdots + \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)} (\iota \otimes \iota)(\varepsilon^m_j \otimes \varepsilon^n_{r-j}) + \cdots\end{aligned}$$

using the relation that $(\iota \otimes \iota)\Delta_{\mathbb{K}} = \Delta\iota$

$$(\iota \otimes \iota)\Delta_{\mathbb{K}}(\varepsilon^{m+n}_r) = (\iota \otimes \iota)\Big[\cdots + \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)} (\varepsilon^m_j \otimes \varepsilon^n_{r-j}) + \cdots \Big].$$

Hence $\boxed{\Delta_{\mathbb{K}}(\varepsilon^{m+n}_r) = \cdots + \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)} \varepsilon^m_j \otimes \varepsilon^n_{r-j} + \cdots}$. After applying $\phi \otimes \eta$ and multiplication π, we have

$$\begin{aligned}(\phi \smile \eta)(\varepsilon^{m+n}_r) &= (-1)^{mn} \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)} \phi(\varepsilon^m_j)\eta(\varepsilon^n_{r-j}) \\ &= (-1)^{mn} \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)} \phi^m_j \eta^n_{r-j} \\ &= (-1)^{mn} T^{m+n}_r,\end{aligned}$$

which is the result. □

Remark 3.11. *By some change of variables, we can infer from all the boxed equations in the proof of Theorem 3.9 that the explicit definition of the comultiplication* $\Delta_{\mathbb{K}} : \mathbb{K} \to$

$\mathbb{K} \otimes_{\Lambda_q} \mathbb{K}$ *is the following;*

$$\Delta_{\mathbb{K}}(\varepsilon^n_s) = \begin{cases} \sum_{r=0}^{n} \varepsilon^r_0 \otimes \varepsilon^{n-r}_0, & s=0 \\ \varepsilon^0_0 \otimes \varepsilon^n_s + \Big[\sum_{j=max\{0,s+w-n\}}^{min\{w,s\}} (-q)^{j(n-s+j-w)} \varepsilon^w_j \otimes \varepsilon^{n-w}_{s-j} \Big] + \varepsilon^n_s \otimes \varepsilon^0_0, & 0<s<n \\ \sum_{t=0}^{n} \varepsilon^t_t \otimes \varepsilon^{n-t}_{n-t}, & s=n \\ \Big[\sum_{t=0}^{n} \varepsilon^t_0 \otimes \varepsilon^{n-t}_{n-t+1} \Big] + \varepsilon^n_{n+1} \otimes \varepsilon^0_0, & s=n+1 \end{cases}$$

where in the expansion of $\Delta_{\mathbb{K}}(\varepsilon^n_s), 0 < s < n$, the index w is choosen so there are no repeated terms.

4. Hochschild cohomology modulo nilpotents not finitely generated

Recall from the introduction that there was an attempt to develop the theory of support varieties for finitely generated modules of finite dimensional algebras using Hochschild cohomology. The idea of this theory is the following:

Let A be a finite dimensional algebra. Let M, N be two A-modules and $\mathrm{Ext}^*_A(M,N)$ be the extension group. There is an action of Hochschild cohomology on this group:

$$\mathrm{HH}^*(A) \times \mathrm{Ext}^*_A(M,N) \to \mathrm{Ext}^*_A(M,N)$$

defined by taking any pair (f,g) to $\Phi(f)g$, where $\Phi(f)$ is to be defined: for $\mathbb{P} \to A$, a projective resolution of A, and any representative $f \in \mathrm{HH}^m(A)$, we can think of f as a representative of an equivalence class of m-extensions of A by A, that is, $f \in \mathrm{Ext}^m_A(A,A)$. So $\Phi(f) = f \otimes 1_M \in \mathrm{Ext}^m_A(M,M)$ and $\Phi(f)g$ is the Yoneda product. Recall that for some finite dimensional algebras, it is well known that the Hochschild cohomology ring modulo nilpotents is finitely generated as an algebra. Furthermore, when M, N are finite-dimensional modules and H a subalgebra of $\mathrm{HH}^*(A)$, define

$$I_H(M,N) = \{f \in H \mid \Phi(f)g = 0, \text{ for all } g \in \mathrm{Ext}^*_A(M,N)\}$$

to be the annihilator of H in $\mathrm{Ext}^*_A(M,N)$. $I_H(M,N)$ is obviously an ideal of H. This theory of support varieties is built on the following definition of a variety.

Definition 4.1. *Let M, N be finite-dimensional A-modules. The support variety of the pair M, N is*

$$V_H(M,N) = V_H(I_H(M,N)) \cong Max(H/I_H(M,N))$$

the maximal ideal spectrum of the quotient ring $H/I_H(M,N)$. The variety of M is defined as $V_H(M) = V_H(M,M)$.

For this theory to have all the nice properties that one would like, (i) H has to be a finitely generated algebra and (ii) $\mathrm{Ext}^*_A(A/\mathfrak{r}, A/\mathfrak{r})$ has to be finitely generated as an H-module. This leads to the conjecture in [8] that Hochschild cohomology modulo nilpotents is always finitely generated as an algebra. For instance, we can take $H = \mathrm{HH}^{ev}(A)$ the subalgebra of $\mathrm{HH}^*(A)$ generated by homogeneous elements of even degrees.

The first counterexample to this conjecture appeared in [11] where F. Xu used certain techniques in category theory to construct a seven-dimensional category algebra whose Hochschild cohomology ring modulo nilpotents is not finitely generated. There are other constructions as well e.g. see [12]. Using the generalized cup product formula of Theorem 3.9, we now prove that for $q = \pm 1$, $\mathrm{HH}^*(\Lambda_q)/\mathcal{N}$ is not finitely generated.

The 0-th Hochschild cohomology ($\mathrm{HH}^0(\Lambda_q) = \frac{\ker d_1^*}{\operatorname{Im} 0}$).
Let $\phi \in \ker d_1^* \subseteq \hat{\mathbb{K}}_0 = \mathrm{Hom}_{\Lambda^e}(\mathbb{K}_0, \Lambda)$, such that $\phi = (\lambda_0^0\ \lambda_1^0)$, for some $\lambda_1^0, \lambda_1^0 \in \Lambda$. We solve for the λ_i^0 $(i = 0, 1)$ for which $d_1^*\phi(\varepsilon_i^1) = 0$ as follows

$$\begin{aligned} d_1^*\phi(\varepsilon_0^1) &= \phi d_1(\varepsilon_0^1) = \phi(a(\varepsilon_0^0) + (-1)^1 q^0(\varepsilon_0^0)a) = a\lambda_0^0 - \lambda_0^0 a = 0 \\ d_1^*\phi(\varepsilon_1^1) &= \phi d_1(\varepsilon_1^1) = \phi((-q)^0 b(\varepsilon_0^0) - (\varepsilon_0^0)b) = b\lambda_0^0 - \lambda_0^0 b = 0 \\ d_1^*\phi(\varepsilon_2^1) &= \phi d_1(\varepsilon_2^1) = \phi(c(\varepsilon_1^0) - (\varepsilon_0^0)c) = c\lambda_1^0 - \lambda_0^0 c = 0 \end{aligned}$$

If $q = 1$, then $ab - ba = 0$, we get the following set of solutions: $\phi = (a\ 0)$, $(ab\ 0)$,$(0\ a)$, $(0\ b)$, $(e_1\ e_2)$ or $(0\ e_1)$. By identifying each solution $(\lambda_0^0\ \ \lambda_1^0)$ with $(o(f_0^0)\lambda_0^0 t(f_0^0)\ \ o(f_1^0)\lambda_1^0 t(f_1^0)) = (e_1\lambda_0^0 e_1\ \ e_2\lambda_1^0 e_2)$, we need to have $o(\lambda_0^0) = t(\lambda_0^0) = e_1$ and $o(\lambda_1^0) = t(\lambda_1^0) = e_2$. This leads us to eliminate some solutions in order to have the following set of solutions; $\phi_1 = (a\ 0), \phi_2 = (ab\ 0)$ and $\phi_3 = (e_1\ e_2)$.
If $q = -1$, then $ab + ba = 0$, we get the same unique set of solutions: $\phi_1 = (a\ 0)$ (if $char(k) = 2), \phi_2 = (ab\ 0)$ and $\phi_3 = (e_1\ e_2)$.
If $q \neq \pm 1$, then $ab - qba = 0$, we get $\phi_2 = (ab\ 0)$ and $\phi_3 = (e_1\ e_2)$. Therefore, the Λ^e-module homomorphisms ϕ_1, ϕ_2, ϕ_3 form a basis for the kernel of d_1^* as a k-vector space. That is,

$$\ker d_1^* = span_k\{\phi_1, \phi_2, \phi_3\}.$$

In summary we obtain,

$$\mathrm{HH}^0(\Lambda_q) = \frac{\ker d_1^*}{\operatorname{Im} 0} = \begin{cases} span_k\{(a\ 0), (ab\ 0), (e_1\ e_2)\}, & \text{if } q = 1 \\ span_k\{(ab\ 0), (e_1\ e_2)\}, & \text{if } q = -1 \\ span_k\{(ab\ 0), (e_1\ e_2)\}, & \text{for every other } q \neq \pm 1. \end{cases}$$

Notice that if the characteristics of k is 2, then $q = 1 = -1$, so we obtain the first case.

Remark 4.2. *We note that the Hochschild 0-cocycles $\phi = (a\ 0), q = 1$ and $\phi = (ab\ 0), q = \pm 1$ corresponds to elements a and ab respectively of the center of the algebra Λ_q. As we will see later, these elements are nilpotent with respect to the cup product but the 0-cocycle $\phi = (e_1\ \ e_2)$ is not, since e_1 and e_2 are idempotent elements. We then identify $span_k\{(e_1\ \ e_2)\}$ to be the generator of $\mathrm{HH}^*(\Lambda_q)/\mathcal{N}$, and it is isomorphic to k because $e_1 + e_2 = 1_{\Lambda_q}$. This brings us to make the following deduction for any $q \in k$:*

$$\mathrm{HH}^0(\Lambda)/\mathcal{N} = \frac{\ker d_1^*}{\operatorname{Im} 0} = span_k\{(e_1\ e_2)\} \cong k. \tag{4.3}$$

We now give the following counting proposition about $\dim(\ker d^*)$, the dimension of the kernels of the differentials $d_{n+1}^* : \hat{\mathbb{K}}_n \to \hat{\mathbb{K}}_{n+1}$.

Proposition 4.4. *Let k be a field and let $\{\Lambda_q\}_{q \in k}$ be the family of quiver algebras of Equation (1.1). For the Hochschild cohomology ring $\mathrm{HH}^n(\Lambda) = \frac{ker d_{n+1}^*}{Im\ d_n^*}$, $n \neq 0$, the following* *holds;*

$$\dim(\ker d^*_{n+1}) = \begin{cases} 2(n+2), & q=1,\ \text{n is odd},\\ \frac{5n}{2}+4, & q=1,\ \text{n is even},\\ 2(n+2), & q=-1,\ \text{n is even},\\ \frac{5n}{2}+4, & q=-1,\ \text{n is odd},\\ n+2, & q\neq \pm 1,\ \text{n is any integer}, \end{cases} \tag{4.5}$$

as a k-vector space.

Proof. Let $\phi \in ker\ d^*_{n+1}$, with $\phi = \begin{pmatrix} \phi^n_0 & \phi^n_1 & \cdots & \phi^n_n & \phi^n_{n+1} \end{pmatrix}$. The elements $\phi^n_i = \phi(\varepsilon^n_i), i = 0, \cdots, n+1$ are obtained by equating the following sets of equations equal to 0:
For any n ***or*** q

$$d^*_{n+1}\phi(\varepsilon^{n+1}_0) = a\phi(\varepsilon^n_0) + (-1)^{n+1}\phi(\varepsilon^n_0)a = a\phi^n_0 \pm \phi^n_0 a \qquad \text{and}$$
$$d^*_{n+1}\phi(\varepsilon^{n+1}_{n+2}) = a\phi(\varepsilon^n_{n+1}) + (-1)^{n+1}\phi(\varepsilon^n_0)c = a\phi^n_{n+1} \pm \phi^n_0 c.$$

For this set of equations to be 0, we should have $\phi^n_0 \in span_k\{a, c, ab, bc\}$ and $\phi^n_{n+1} \in span_k\{a, c, ab, bc\}$. But we recall that $\phi^n_0 \in e_1\Lambda_q e_1$, and $\phi^n_{n+1} \in e_1\Lambda_q e_2$. These constraints make us obtain the following $\phi^n_0 \in span_k\{a, ab\}$ and $\phi^n_{n+1} \in span_k\{c, bc\}$. The rest of this proof involves obtaining the values of ϕ^n_r when you set the following equations

$$\begin{aligned} d^*_{n+1}\phi(\varepsilon^{n+1}_r) &= a\phi(\varepsilon^n_r) + (-1)^{n+1-r}q^r\phi(\varepsilon^n_r)a + (-q)^{n+1-r}b\phi(\varepsilon^n_{r-1}) + (-1)^{n+1}\phi(\varepsilon^n_{r-1})b\\ &= a\phi^n_r + (-1)^{n+1-r}q^r\phi^n_r a + (-q)^{n+1-r}b\phi^n_{r-1} + (-1)^{n+1}\phi^n_{r-1}b\\ d^*_{n+1}\phi(\varepsilon^{n+1}_{r+1}) &= a\phi^n_{r+1} + (-1)^{n-r}q^{r+1}\phi^n_{r+1}a + (-q)^{n-r}b\phi^n_r + (-1)^{n+1}\phi^n_r b \end{aligned}$$

equal to 0 for different values of n, r and q. We recall that $q = \pm 1$ implies $ab \mp ab = 0$.
When n ***is even,*** r ***is even,*** $q = 1$, we obtain ϕ^n_r by setting $\phi^n_{r-1} = \phi^n_{r+1} = 0$. Then solving $d^*_{n+1}\phi(\varepsilon^{n+1}_r) = a\phi^n_r - \phi^n_r a = 0$ and $d^*_{n+1}\phi(\varepsilon^{n+1}_{r+1}) = b\phi^n_r - \phi^n_r b = 0$, we obtain $\phi^n_r \in span_k\{a, b, ab, bc, e_1\}$. Again we recall that $\phi^n_r \in e_1\Lambda_q e_1$, so $\phi^n_r \in span_k\{a, b, ab, e_1\}$.
When n ***is even,*** r ***is odd,*** $q = 1$, we obtain ϕ^n_r by setting $\phi^n_{r-1} = \phi^n_{r+1} = 0$. Then solving $d^*_{n+1}\phi(\varepsilon^{n+1}_r) = a\phi^n_r + \phi^n_r a = 0$ and $d^*_{n+1}\phi(\varepsilon^{n+1}_{r+1}) = -b\phi^n_r - \phi^n_r b = 0$ to obtain $\phi^n_r \in span_k\{ab, bc\}$. So $\phi^n_r \in span_k\{ab\}$.
When n ***is odd,*** r ***is even,*** $q = 1$, we obtain ϕ^n_r by setting $\phi^n_{r-1} = \phi^n_{r+1} = 0$. After solving $d^*_{n+1}\phi(\varepsilon^{n+1}_r) = a\phi^n_r + \phi^n_r a = 0$ and $d^*_{n+1}\phi(\varepsilon^{n+1}_{r+1}) = -b\phi^n_r + \phi^n_r b = 0$, we get $\phi^n_r \in span_k\{a, ab, bc\}$ and finally we get $\phi^n_r \in span_k\{a, ab\}$.
When n ***is odd,*** r ***is odd,*** $q = 1$, we obtain ϕ^n_r by setting $\phi^n_{r-1} = \phi^n_{r+1} = 0$. Then solve $d^*_{n+1}\phi(\varepsilon^{n+1}_r) = a\phi^n_r - \phi^n_r a = 0$ and $d^*_{n+1}\phi(\varepsilon^{n+1}_{r+1}) = b\phi^n_r + \phi^n_r b = 0$, to get $\phi^n_r \in span_k\{ab, bc, b\}$. Like before we obtain $\phi^n_r \in span_k\{b, ab\}$.
We continue in this fashion and obtain the following results as well.
When n ***is even,*** r ***is even,*** $q = -1$, we obtain $\phi^n_r \in span_k\{ab, e_1\}$.
When n ***is even,*** r ***is odd,*** $q = -1$,we obtain $\phi^n_r \in span_k\{ab, b\}$.
When n ***is odd,*** r ***is even,*** $q = -1$, we obtain $\phi^n_r \in span_k\{a, b, ab\}$.
When n ***is odd,*** r ***is odd,*** $q = -1$, we get $\phi^n_r \in span_k\{a, ab\}$.
For any other $q \neq \pm 1$ and n even, r even, we obtain after solving $d^*_{n+1}\phi(\varepsilon^{n+1}_r) = a\phi^n_r - q^r\phi^n_r a = 0$ and $d^*_{n+1}\phi(\varepsilon^{n+1}_{r+1}) = q^{n-r}b\phi^n_r - \phi^n_r b = 0$, $\phi^n_r \in span_k\{ab\}$. In case n is even and r is odd or even, we obtain the same $\phi^n_r \in span_k\{ab\}$.

The following table summarize the set of all solutions:

$q=1$				
	n is even		n is odd	
	r is even	r is odd	r is even	r is odd
ϕ_0^n	a, ab		a, ab	
ϕ_r^n	a, b, ab, e_1	ab	a, ab	b, ab
ϕ_{n+1}^n	c, bc		c, bc	
$q=-1$				
	n is even		n is odd	
	r is even	r is odd	r is even	r is odd
ϕ_0^n	a, ab		a, ab	
ϕ_r^n	ab, e_1	b, ab	a, b, ab	a, ab
ϕ_{n+1}^n	c, bc		c, bc	

$q\neq\pm1$				
	n is even		n is odd	
	r is even	r is odd	r is even	r is odd
ϕ_0^n	a, ab		a, ab	
ϕ_r^n	ab	ab	ab	ab
ϕ_{n+1}^n	c, bc		c, bc	

From all these tables, we make the following deductions;

$$(n \text{ is even and } q=+1):\ \dim(\operatorname{Ker} d_{n+1}^*) = 2+(\overset{(odd-positions)}{\frac{n}{2}\times 1} + \overset{(even-positions)}{\frac{n}{2}\times 4})+2 = 5(\frac{n}{2})+4$$

$$(n \text{ is odd and } q=+1):\ \dim(\operatorname{Ker} d_{n+1}^*) = 2+(\overset{(odd-positions)}{\frac{n}{2}\times 2} + \overset{(even-positions)}{\frac{n}{2}\times 2})+2 = 2(n+2)$$

$$(n \text{ is even and } q=-1):\ \dim(\operatorname{Ker} d_{n+1}^*) = 2+(\overset{(odd-positions)}{\frac{n}{2}\times 2} + \overset{(even-positions)}{\frac{n}{2}\times 2})+2 = 2(n+2)$$

$$(n \text{ is odd and } q=-1):\ \dim(\operatorname{Ker} d_{n+1}^*) = 2+(\overset{(odd-positions)}{\frac{n}{2}\times 2} + \overset{(even-positions)}{\frac{n}{2}\times 3})+2 = 5(\frac{n}{2})+4$$

$$(\text{ for any } n,\ q\neq\pm1):\ \dim(\operatorname{Ker} d_{n+1}^*) = 2+(\overset{(odd-positions)}{\frac{n}{2}\times 1} + \overset{(even-positions)}{\frac{n}{2}\times 1})+2 = n+4$$

which is the result. □

Remark 4.6. *We note from the above result that these dimensions grow linearly as the homological dimension n grows. We also observe that there are Hochschild n-cocycles of the form $\phi = \begin{pmatrix}0 & \cdots & 0 & e_1 & 0 & \cdots & 0\end{pmatrix}$ i.e. $\phi_i^n = 0$ for all i except at some position r. We will sometimes use the notation $\phi = \begin{pmatrix}0 & \cdots & 0 & (e_1)^{(r)} & 0 & \cdots & 0\end{pmatrix}$ when we want to emphasize that for the n-cocycle ϕ, the e_1 is in the r-th position. Our next result shows that ϕ is non-nilpotent and that cocycles of this nature appear (obvious from the tables) whenever both n and r are even and $q=\pm1$. Observe also that whenever $r=0$ or $r=n+1$, $\phi_r^n \neq e_1$.*

Lemma 4.7. *If* $\phi = \begin{pmatrix} 0 & \cdots & 0 & \phi_r^n & 0 & \cdots & 0 \end{pmatrix}$ *is any cocyle such that both* n *and* r *are even (except* 0 *and* $n+1$*),* $q = \pm 1$*, then* ϕ *is not nilpotent.*

Proof. We have from Theorem 3.9 that when $0 < r < m+n$,

$$(\phi \smile \phi)(\varepsilon_r^{m+n}) = (-1)^{mn} \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)} \phi_j^m \phi_{r-j}^n \tag{4.8}$$

where $\phi_j^m \phi_{r-j}^n$ is a product of any two elements in the set $\{a, b, ab, c, bc\}$ which is equal to 0 in the algebra. In general, if it is not a zero e.g ab, we simply take a triple cup product using the following;

$$\begin{aligned}
&(\phi \smile \phi \smile \phi)(\varepsilon_r^{n+n+n}) \\
&= (\mu \smile \phi)(\varepsilon_r^{m+n}) \quad (\text{take } \mu = \phi \smile \phi, m = n+n) \\
&= (-1)^{mn} \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)} \mu(\varepsilon_j^m) \phi(\varepsilon_{r-j}^n) \\
&= (-1)^{mn} \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)} [\phi \smile \phi(\varepsilon_j^{n+n})] \phi(\varepsilon_{r-j}^n) \\
&= (-1)^{mn} \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)} \Big[(-1)^{n^2} \sum_{i=max\{0,l-n\}}^{min\{n,l\}} (-q)^{i(n-l+i)} \phi(\varepsilon_i^n) \phi(\varepsilon_{l-i}^n)\Big] \phi(\varepsilon_{r-j}^n) \\
&= (-1)^{3n^2} \sum_{j=max\{0,r-n\}}^{min\{m,r\}} \sum_{i=max\{0,l-n\}}^{min\{n,l\}} (-q)^{ij(n-r+j)(n-l+i)} \phi(\varepsilon_i^n) \phi(\varepsilon_{l-i}^n) \phi(\varepsilon_{r-j}^n).
\end{aligned}$$

The product $\phi(\varepsilon_i^n)\phi(\varepsilon_{l-i}^n)\phi(\varepsilon_{r-j}^n) = \phi_i^n \phi_{l-i}^n \phi_{r-j}^n$ is always 0 in Λ_q by the defining relations in I_q. Therefore a cocycle $\phi : \mathbb{K}_m \to \Lambda$ is non-nilpotent if and only if $\phi_i^m = \phi_{l-i}^m = \phi_{r-j}^m = e_1$ for some i, j, l, r. Accordingly, this is the case if and only if $q = \pm 1, n$ is even and i, l, r are even. □

We now present the following corollary.

Corollary 4.9. *Let* $\phi : \mathbb{K}_n \to \Lambda_q$*, be an* n*-cocycle. Then* ϕ *is non-nilpotent if, and only if* $q = \pm 1, n$ *and* r *are even and* $\phi = \begin{pmatrix} 0 & \cdots & 0 & (e_1)^{(r)} & 0 & \cdots & 0 \end{pmatrix}$.

Proof. Follows from Lemma 4.7. □

Let $\mathrm{HH}^n(\Lambda_q, \Lambda_q) = H^n(\mathrm{Hom}_{\Lambda_q^e}(\mathbb{K}_n, \Lambda_q))$ be the Λ_q^e-module generated by all n-Hochschild cochains. Denote by $Z^n(\Lambda_q, \Lambda_q) = \mathrm{HH}^n(\Lambda_q, \Lambda_q)/\mathcal{N}^n$, where $\mathcal{N}^n$ is the set of homogeneous nilpotent elements of degree n. For each n, the non-nilpotent cocycles $Z^n(\Lambda_q, \Lambda_q)$, are those given in Corollary 4.9.

These cocycles do not differ by a coboundary. Therefore, they constitute their own equivalence class. This is because for a fixed n, let ϕ, β be two distinct $2n$-cocycles such that $\phi(\varepsilon_r^{2n}) = \phi_r^{2n} = e_1$, $\beta(\varepsilon_s^{2n}) = \beta_s^{2n} = e_1$ where $r < s$ are both even. Suppose there is an α such that $d^*(\alpha) = \phi - \beta = \begin{pmatrix} 0 & \cdots & 0 & e_1 & 0 & \cdots & 0 & -e_1 & 0 & \cdots & 0 \end{pmatrix}$ where the idempotent e_1 is in the r-th and s-th positions. This α does not exist. This is because by considering for example at the position r,

$$\begin{aligned}&e_1 = (\phi - \beta)(\varepsilon_r^{2n}) = d^*(\alpha)(\varepsilon_r^{2n}), \quad \text{implying that}\\ &\alpha(d(\varepsilon_r^{2n})) = a\alpha(\varepsilon_r^{2n-1}) + (-1)^{2n-r}q^r\alpha(\varepsilon_r^{2n-1})a + (-q)^{2n-r}b\alpha(\varepsilon_{r-1}^{2n-1}) + (-1)^{2n}\alpha(\varepsilon_{r-1}^{2n-1})b.\end{aligned}$$

There is no way to define $\alpha(\varepsilon_r^{2n-1})$ and $\alpha(\varepsilon_{r-1}^{2n-1})$ so that equality hold in the above expression. Therefore each n-cocycle is in a distinct class and do not differ by a coboundary.

We now define a canonical map from $Z^*(\Lambda_q, \Lambda_q) = \bigoplus_{n>0} Z^n(\Lambda_q, \Lambda_q)$ to the polynomial ring in two indeterminates $k[x, y]$. Recall that ϕ_0^n and ϕ_{n+1}^n are never equal to e_1 in order for ϕ to be non-nilpotent. We therefore define this map by

$$\begin{aligned}\begin{pmatrix}0 & 0 & (e_1)^2 & 0 & \cdots & 0\end{pmatrix} &\mapsto x^{2(n-1)}y^2,\\ \begin{pmatrix}0 & 0 & 0 & 0 & (e_1)^4 & 0 & \cdots & 0\end{pmatrix} &\mapsto x^{2(n-2)}y^4,\\ &\vdots\\ \begin{pmatrix}0 & \cdots & 0 & (e_1)^r & 0 & \cdots & 0\end{pmatrix} &\mapsto x^{2n-r}y^r,\\ &\vdots\\ \begin{pmatrix}0 & 0 & \cdots & 0 & (e_1)^{2n} & 0\end{pmatrix} &\mapsto y^{2n}\end{aligned}$$

This map is well defined as the kernel contains only the zero map. Under this map, the image of $Z^*(\Lambda_q, \Lambda_q)$ is the subalgebra $k[x^2, y^2]y^2$ which is not finitely generated as an algebra. This is because for each n, $x^{2(n-1)}y^2$ cannot be generated by lower degree elements. Also note how the cup product corresponds with multiplication in $k[x, y]$, that is, given even positive integers r, s, we have

$$\begin{tikzcd}
(\,0 \ \cdots \ 0 \ (e_1)^r \ 0 \ \cdots \ 0\,) \smile (\,0 \ \cdots \ 0 \ (e_1)^s \ 0 \ \cdots \ 0\,) \arrow[r, "="] \arrow[d, dashed, "="'] & (x^{2n-r}y^r)\cdot(x^{2m-s}y^s) \arrow[d, dashed, "="'] \\
(\,0 \ \cdots \ 0 \ (e_1)^{r+s} \ 0 \ \cdots \ 0\,) \arrow[r, "="] & x^{2(n+m)-(r+s)}y^{r+s}
\end{tikzcd}$$

At each degree n, the element $\begin{pmatrix}0 & 0 & e_1 & 0 & \cdots & 0\end{pmatrix}$ identified with $x^{2(n-1)}y^2$ cannot be generated as a cup product of any two elements of lower homological degrees. Since this map is 1-1, we conclude that $Z^*(\Lambda_q, \Lambda_q) \cong k[x^2, y^2]y^2$. The next proposition formalizes this idea whereas the next example is an illustration.

Proposition 4.10. *$Z^*(\Lambda_q, \Lambda_q), q = \pm 1$ is graded with respect to the cup product and is canonically isomorphic to a subalgebra $k[x^2, y^2]y^2$ of $k[x, y]$, that is*

$$Z^*(\Lambda_q, \Lambda_q) \cong k[x^2, y^2]y^2$$

where the degree of y^2 is 2 and that of x^2y^2 is 4.

Example 4.11. *To show that*

$$\begin{aligned}x^2y^2 \cdot y^2 &= (0 \ \ 0 \ \ e_1 \ \ 0 \ \ 0 \ \ 0) \smile (0 \ \ 0 \ \ e_1 \ \ 0)\\ &= (0 \ \ 0 \ \ 0 \ \ 0 \ \ e_1 \ \ 0 \ \ 0 \ \ 0) = x^2 \cdot y^4\end{aligned}$$

Take $\phi = x^2y^2 = (\phi_0^4\ \phi_1^4\ \phi_2^4\ \phi_3^4\ \phi_4^4\ \phi_5^4)$ *and* $\mu = y^2 = (\phi_0^2\ \phi_1^2\ \phi_2^2\ \phi_3^2)$.

$$(\phi \smile \mu)(\varepsilon_0^6) = \phi_0^4\mu_0^2 = 0$$

$$(\phi \smile \mu)(\varepsilon_1^6) = \sum_{j=0}^{1}(-1)^{j(1+j)}\phi_j^4\mu_{1-j}^2 = \phi_0^4\mu_1^2 + \phi_1^4\mu_0^2 = 0$$

$$(\phi \smile \mu)(\varepsilon_2^6) = \sum_{j=0}^{2}(-1)^{j^2}\phi_j^4\mu_{2-j}^2 = \phi_0^4\mu_2^2 - \phi_1^4\mu_1^2 + \phi_2^4\mu_0^2 = 0$$

$$(\phi \smile \mu)(\varepsilon_3^6) = \sum_{j=1}^{3}(-1)^{j(-1+j)}\phi_j^4\mu_{3-j}^2 = \phi_1^4\mu_2^2 + \phi_2^4\mu_1^2 + \phi_3^4\mu_0^2 = 0$$

$$(\phi \smile \mu)(\varepsilon_4^6) = \sum_{j=2}^{4}(-1)^{j(-2+j)}\phi_j^4\mu_{4-j}^2 = \phi_2^4\mu_2^2 - \phi_3^4\mu_1^2 + \phi_4^4\phi_0^2 = e_1$$

$$(\phi \smile \mu)(\varepsilon_5^6) = \sum_{j=3}^{4}(-1)^{j(-3+j)}\phi_j^4\mu_{5-j}^2 = \phi_3^4\mu_2^2 + \phi_4^4\mu_1^2 = 0$$

$$(\phi \smile \mu)(\varepsilon_6^6) = \phi_4^4\mu_4^2 = 0$$

$$(\phi \smile \mu)(\varepsilon_7^6) = \phi_0^4\mu_3^2 = 0$$

Theorem 4.12. *Let k ($char(k) \neq 2$) be a field and $\Lambda_q = kQ/I_q$ be the family of quiver algebras of* (1.1). *Let $\mathcal{N}$ be the set of nilpotent elements of* $\mathrm{HH}^*(\Lambda_q)$, *then*

$$\mathrm{HH}^*(\Lambda_q)/\mathcal{N} = \begin{cases} \mathrm{HH}^0(\Lambda_q)/\mathcal{N} \cong k, & \text{if } q \neq \pm 1 \\ Z^0(\Lambda_q,\Lambda_q) \oplus k[x^2,y^2]y^2 \cong k \oplus k[x^2,y^2]y^2, & \text{if } q = \pm 1 \end{cases}$$

where the degree of y^2 is 2, and that of x^2y^2 is 4.

Proof. If $q \neq \pm 1$, and $n > 0$, then all cocycles $\phi : \mathbb{K}_n \to \Lambda_q$ are nilpotent by Lemma 4.7. From Remark 4.2, we have then that

$$\mathrm{HH}^*(\Lambda_q)/\mathcal{N} = \mathrm{HH}^0(\Lambda_q)/\mathcal{N} \cong Z^0(\Lambda_q,\Lambda_q) \cong k.$$

If $q = \pm 1$, then the only non-nilpotent elements are those of Corollary 4.9. From Remark 4.2 and Proposition 4.10 we have that Hochschild cohomology ring modulo homogeneous nilpotent elements of $\{\Lambda_q\}_{q=\pm 1}$ is spanned by graded pieces of sets containing cocycles given by Corollary 4.9. That means that

$$\begin{aligned} &\mathrm{HH}^*(\Lambda_q)/\mathcal{N} = Z^0(\Lambda_q,\Lambda_q) \oplus Z^*(\Lambda_q,\Lambda_q) \\ &\cong k \oplus \Big(\bigoplus_{n>0} span_k \Big\{ \phi : \mathbb{K}_{2n} \to \Lambda_q \mid \phi = \begin{pmatrix} 0 & \cdots & 0 & (e_1)^{(r)} & 0 & \cdots & 0 \end{pmatrix}, r \text{ is even} \Big\} \Big) \\ &= k \oplus k[x^2,y^2]y^2. \end{aligned}$$

□

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