# CUP PRODUCT AND GERSTENHABER BRACKET ON HOCHSCHILD COHOMOLOGY OF A FAMILY OF QUIVER ALGEBRAS

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ABSTRACT. We present a cup product formula on the Hochschild cohomology of a family of quiver algebras. We use the formula to determine the set of homogeneous non-nilpotent Hochschild cocycles and construct a canonical isomorphism between Hochschild cohomology modulo nilpotents and a subalgebra of $k[x,y]$ that is not finitely generated. For some members of the family, we present the Gerstenhaber ideal of homogeneous nilpotent cocycles using homotopy lifting technique. We then determine their Hochschild cohomology modulo the weak Gerstenhaber ideal generated by nilpotent elements, thereby providing an answer to a question of Reiner Hermann.

## 1. INTRODUCTION

The theory of support varieties has been well developed for finite groups using group cohomology. Several efforts were made to develop similar theories for finitely generated modules over finite dimensional algebras using Hochschild cohomology. Hochschild cohomology $\mathrm{HH}^*(\Lambda)$ of a $k$-algebra $\Lambda$ is graded commutative. If the characteristics of the field $k$ is different from 2, then every homogeneous element of odd degree is nilpotent. Let $\mathcal{N}$ be the set of nilpotent elements of $\mathrm{HH}^*(\Lambda)$, Hochschild cohomology modulo nilpotents $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ is therefore a commutative $k$-algebra. For some finite dimensional algebras, it is well known that the Hochschild cohomology ring modulo nilpotents is finitely generated as an algebra. N. Snashall described many classes of such algebras in [10, section 3]. Before the expository paper [10], it was conjectured in [11] that Hochschild cohomology modulo nilpotents is always finitely generated as an algebra for finite dimensional algebras. The first counterexample to this conjecture appeared in [14] where F. Xu used certain techniques in category theory to construct a seven-dimensional category algebra whose Hochschild cohomology ring modulo nilpotents is not finitely generated. Some authors have presented several constructions of different counterexamples to this conjecture. While it is of great use to produce a counterexample, it is equally important to understand the cohomology ring structure of these algebras. We give a brief summary of a variation of the F. Xu counterexample which was presented in [10].

A quiver is a directed graph where loops and multiple arrows (also called paths) between two vertices are possible. The path algebra $kQ$, is the $k$-vector space generated by all paths in the quiver $Q$. By taking multiplication of two paths $x$ and $y$ to be the concatenation $xy$ if the terminal vertex $t(x)$ of $x$ and the origin vertex $o(y)$ of $y$ are equal, and otherwise 0, $kQ$ becomes an associative $k$-algebra. Let $I$ be an ideal of $kQ$. The quotient $\Lambda = kQ/I$

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is called a quiver algebra. Consider the quiver $Q$:

$$\begin{tikzcd} 1 \arrow[loop above, "a"] \arrow[loop below, "b"] \arrow[r, "c"] & 2 \end{tikzcd}$$

and let

$$\left\{\Lambda_q = kQ/I_q\right\}_{q\in k}, \quad I_q = \langle a^2, b^2, ab - qba, ac\rangle, \tag{1.1}$$

be the family of quiver algebras generated by $Q$ for each $q \in k$. We give the following remarks to summarize some of what has been done with respect to this family.

**Remark 1.2.** *From* [7, 10], *we have that for each* $q$*,*

- $\Lambda_q$ *is finitely generated since* $Q$ *is a finite quiver i.e. has finite vertices and arrows.*
- $\Lambda_q$ *is a graded Koszul quiver algebra.*
- *Let* $\Lambda_q = \oplus_{i\geq 0}(\Lambda_q)_i$ *be a grading. The Koszul dual* $\Lambda_q^!$ *of* $\Lambda_q$ *and the Yoneda algebra* $E(\Lambda_q)$ *are related by the following equation;*

$$E(\Lambda_q) = Ext^*_{\Lambda_q}((\Lambda_q)_0, (\Lambda_q)_0) \cong \Lambda_q^! = kQ^{op}/I_q^{\perp} \tag{1.3}$$

  *where* $Q^{op}$ *is the quiver with opposite arrows,* $I_q^{\perp} := \langle a^o b^0 + q^{-1}b^0 a^0, b^0 c^0\rangle$*, with* $v^0$ *the correponding arrow in the opposite quiver algebra* $kQ^{op}$ *for any* $v \in kQ$*. Note also that* $\Lambda_q^!$ *is generated in degrees 0 and 1.*
- *The case where* $q = \pm 1$*,* $I_q$ *belongs to a class of (anti-)commutative ideals studied by E. Gawell and Q.R. Xantcha. There is an associated generator graph (of the orthogonal ideal* $I_q^{\perp}$ *of* $I_q$*) which has no directed cycles. This means that the ideal* $I_q$ *is admissible* [4]*.*
- *For* $q = 1$*, the graded center of the Yoneda algebra* $Z_{gr}(E(\Lambda_q))$ *is given by the following*

$$Z_{gr}(E(\Lambda_1)) = \begin{cases} k \oplus k[a,b]b, & \text{if } char(k) = 2 \\ k \oplus k[a^2, b^2]b^2, & \text{if } char(k) \neq 2 \end{cases}$$

  *where the degree of* $b$ *is 1, and that of* $ab$ *is 2.*

The following result shows that $\Lambda_1$ is a counterexample to the Snashall-Solberg finite generation conjecture.

**Theorem.** [10, Theorem 4.5] *Let* $k$ *be a field and* $\Lambda_1$ *be a member of quiver algebras given in Equation* (1.1)*. Let* $\mathcal{N}$ *be the set of nilpotent elements of* $\mathrm{HH}^*(\Lambda_1)$*, then*

$$\mathrm{HH}^*(\Lambda_1)/\mathcal{N} \cong Z_{gr}(E(\Lambda_1)) = \begin{cases} k \oplus k[a,b]b, & \text{if } char(k) = 2 \\ k \oplus k[a^2, b^2]b^2, & \text{if } char(k) \neq 2 \end{cases}$$

*where the degree of* $b$ *is 1, and that of* $ab$ *is 2.*

Let $A$ and $B$ be $k$-algebras, $A^e, B^e$ their enveloping algebra and $Mod(A^e)$ the category of $A^e$-modules. It is natural to ask when an exact functor $Mod(A^e) \to Mod(B^e)$ gives rise to a graded homomorphism $\mathrm{HH}^*(A) \to \mathrm{HH}^*(B)$ between the Hochschild cohomology of $A$ and $B$. A recollement of module categories can be thought of as an "exact sequence" of categories with maps between them being adjunct functors. R. Hermann in [6] showed that recollements of module categories give rise to homomorphisms between the associated Hochschild cohomology algebras preserving the strict Gerstenhaber structure. This led to a formulation of another variation of the Snashall-Solberg finite generation conjecture which asks whether Hochschild cohomology modulo the weak Gerstenhaber ideal generated by homogeneous nilpotent elements is finitely generated. In particular, it is unknown whether or not $\mathrm{HH}^*(\Lambda_q)/G(\mathcal{N})$ is finitely generated when $q = 1$.

**Our result:** In this paper, we study the Hochschild cohomology ring of the family $\Lambda_q$ of quiver algebras of Equation (1.1). We present a comultiplicative map on a projective resolution $\mathbb{K}$ for this family and determine a cup product formula using the comultiplicative map. With the generalized cup product formula, we completely determine homogeneous nilpotent and non-nilpotent Hochschild cocycles. Our description and calculations agrees with the general notion that all cocycles of odd homological degrees are nilpotent.

Furthermore, we show that whenever $q = \pm 1$, Hochschild cohomology modulo the ideal generated by homogeneous nilpotent elements is not finitely generated. We use the idea of homotopy lifting introduced by Y. Volkov in [12] to completely determine the Gerstenhaber ideal generated by both homogeneous nilpotent and non-nilpotent cocycles. We show that for $\Lambda_q, q = \pm 1$, Hochschild cohomology ring modulo the weak Gerstenhaber ideal generated by homogeneous nilpotent cocycles is not finitely generated. Some of our results include the following;

**Theorem.** *Let $\phi : \mathbb{K}_m \to \Lambda_q$, and $\mu : \mathbb{K}_n \to \Lambda_q$, be two Hochschild cocycles. Let $\{\varepsilon_j^m\}_{j=0}^{m+1}$ be free basis elements of $\mathbb{K}_m$ such that $\phi(\varepsilon_j^m) = \phi_j^m \in \Lambda_q$ and let $\{\varepsilon_i^n\}_{i=0}^{n+1}$ be free basis elements of $\mathbb{K}_n$ such that $\mu(\varepsilon_i^n) = \mu_i^n \in \Lambda_q$ . Then the following gives a formula for the cup product on Hochschild cohomology;*

$$(\phi \smile \mu)(\varepsilon_r^{m+n}) = (\phi\mu)_r^{m+n} = \begin{cases} (-1)^{mn}\phi_0^m\mu_0^n, & \text{when } r = 0, \\ (-1)^{mn}T_r^{m+n} & \text{when } 0 < r < m+n, \\ (-1)^{mn}\phi_m^m\mu_n^n, & \text{when } r = m+n, \\ (-1)^{mn}\phi_0^m\mu_{n+1}^n, & \text{when } r = m+n+1, \end{cases}$$

$$\text{where } T_r^{m+n} = \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)}\phi_j^m\mu_{r-j}^n, \qquad 0 < r < m+n.$$

**Theorem.** *Let $k$ be a field of characteristics different from 2. Let $\Lambda_q = kQ/I_q$ be the family of quiver algebras of* (1.1) *and $\mathcal{N}$ the set of homogeneous nilpotent elements of* $\mathrm{HH}^*(\Lambda_q)$, *then*

$$\mathrm{HH}^*(\Lambda_q)/\mathcal{N} = \begin{cases} \mathrm{HH}^0(\Lambda_q)/\mathcal{N} \cong k, & \text{if } q \neq \pm 1 \\ Z^0(\Lambda_q, \Lambda_q) \oplus k[x^2, y^2]y^2 \cong k \oplus k[x^2, y^2]y^2, & \text{if } q = \pm 1 \end{cases}$$

*where the degree of $y^2$ is 1, and that of $x^2y^2$ is 4.*

**Theorem.** *Let $k$ be a field and $\Lambda_q, q = \pm 1$ be members of the family of quiver algebras of* (1.1)*. Let $\mathcal{N}$ be the set of homogeneous nilpotent elements of* $\mathrm{HH}^*(\Lambda_q)$*, and $G(\mathcal{N})$ the weak Gerstenhaber ideal generated by $\mathcal{N}$. Then* $\mathrm{HH}^*(\Lambda_q)/G(\mathcal{N}) \cong \mathrm{HH}^*(\Lambda_q)/\mathcal{N}$*.*

## 2. Minimal projective resolution for Koszul quiver algebras

We recall that a quiver is a directed graph with the allowance of loops and multiple arrows. A quiver $Q$ is sometimes denoted as a quadruple $(Q_0, Q_1, o, t)$ where $Q_0$ is the set of vertices in $Q$, $Q_1$ is the set of arrows in $Q$, and $o, t : Q_1 \longrightarrow Q_0$ are maps which assign to each arrow $a \in Q_1$, its origin vertex $o(a)$ and terminal vertex $t(a)$ in $Q_0$. A path in $Q$ is a sequence of arrows $a = a_1 a_2 \cdots a_{n-1} a_n$ such that the terminal vertex of $a_i$ is the same as the origin vertex of $a_{i+1}$, using the convention of concatenating paths from left to right. The path algebra $kQ$ is defined as a vector space having all paths in $Q$ as a basis. Vertices are regarded as paths of length 0, an arrow is a path of length 1, and so on. We take multiplication on $kQ$ as concatenation of paths. Two paths $a$ and $b$ satisfy $ab = 0$ if $t(a) \neq o(b)$. This multiplication defines an associative algebra over $k$. By taking $kQ_i$ to be the $k$-vector subspace of $kQ$ with paths of length $i$ as basis, $kQ = \bigoplus_{i \geq 0} kQ_i$ can be viewed as an $\mathbb{N}$-graded algebra. A relation on a quiver $Q$ is a linear combination of paths from $Q$ each having the same origin and terminal vertex. A quiver together with a set of relations is called a quiver with relations. Let $I$ be an ideal of $kQ$ generated by some relations. Recall that the quotient $\Lambda = kQ/I$ is called the quiver algebra associated with $(Q, I)$.

We now present a construction of the resolution $\mathbb{K}$ that we use later to determine Hochschild cohomology.

**Construction of the minimal projective resolution $\mathbb{K}$:** Let $\Lambda = kQ/I$ be a graded Koszul algebra. Then $\Lambda_0$ has a graded (minimal) projective resolution $\mathbb{L}$ as a right $\Lambda$-module. An algorithmic approach to find such a minimal projective resoluton $\mathbb{L} \to \Lambda_0$ of right $\Lambda$-modules was given in [3]. The resolution was shown to have a "comultiplicative structure" and this structure was used to find a minimal projective resolution $\mathbb{K} \to \Lambda$ of modules over the enveloping algebra of $\Lambda$ in [5]. We now describe these resolutions.

Take $J$ to be the ideal of $kQ$ generated by all arrows and suppose further that $I \subseteq J^2$ is an admissible ideal, that is, $J^m \subseteq I \subseteq J^2$ for some $m$ and set $\mathfrak{r} = J/I$. A non-zero element $x \in kQ$ is called uniform if there exist vertices $u, v$ such that $x = uxv = ux = xv$, where $u$ is the common origin vertex and $v$ is the common terminal vertex of each of the paths summing up to $x$. For $R = kQ$, it was shown that there are integers $\{t_n\}_{n \geq 0}$ and uniform elements $\{f_i^n\}_{i=0}^{t_n}$ such that the minimal right projective resolution $\mathbb{L}$ of $\Lambda_0 \cong \Lambda/\mathfrak{r}$, is obtained from a filtration of $R$. The element $f_i^n$ for each $i$, is a path of length $n$. The filtration is given by the following nested family of right ideals:

$$\cdots \subseteq \bigoplus_{i=0}^{t_n} f_i^n R \subseteq \bigoplus_{i=0}^{t_{n-1}} f_i^{n-1} R \subseteq \cdots \subseteq \bigoplus_{i=0}^{t_1} f_i^1 R \subseteq \bigoplus_{i=0}^{t_0} f_i^0 R = R,$$

where for each $n$, $\mathbb{L}_n = \bigoplus_{i=0}^{t_n} f_i^n R / \bigoplus_{i=0}^{t_n} f_i^n I$ and the differentials $d^L$ on $\mathbb{L}$ are induced by the inclusions $\bigoplus_{i=0}^{t_n} f_i^n R \subseteq \bigoplus_{i=0}^{t_{n-1}} f_i^{n-1} R$. The existence of these inclusions imply that there are elements $h_{ji}^{n-1,n}$ in $R$ such that

$$f_i^n = \sum_{j=0}^{t_{n-1}} f_j^{n-1} h_{ji}^{n-1,n}$$

for all $i = 0, 1, \ldots, t_n$ and all $n \geq 1$. The differentials $d_n^L : \mathbb{L}_n \to \mathbb{L}_{n-1}$ are given by

$$d_n^L(f_i^n) = \begin{pmatrix} h_{0i}^{n-1,n} & h_{1i}^{n-1,n} & \cdots & h_{t_{n-1}i}^{n-1,n} \end{pmatrix}$$

for all $n \geq 1$.

Furthermore, it was shown in [5] that with some choice of scalars, the elements $\{f_i^n\}_{i=0}^{t_n}$ satisfy a comultiplicative structure given below in (2.1). That is, for $0 \leq i \leq t_n$ and some positive integer $r$, there are scalars $c_{pq}(n, i, r)$ such that

$$f_i^n = \sum_{p=0}^{t_r} \sum_{q=0}^{t_{n-r}} c_{pq}(n, i, r) f_p^r f_q^{n-r}. \tag{2.1}$$

To set up this equation in practice, we can take $\{f_i^0\}_{i=0}^{t_0}$ to be the set of vertices, $\{f_i^1\}_{i=0}^{t_1}$ to be the set of arrows, $\{f_i^2\}_{i=0}^{t_2}$ to be the generators of $I$, and define $\{f_i^n\}(n \geq 3)$ recursively in terms of $f_i^{n-1}$ and $f_j^1$. The resolution $\mathbb{L}$ and the comultiplicative structure of Equation (2.1) were used to construct a minimal projective resolution $\mathbb{K} \to \Lambda$ of modules over the enveloping algebra $\Lambda^e = \Lambda \otimes \Lambda^{op}$. The minimal projective resolution $\mathbb{K}$ is given by the following theorem.

**Theorem 2.2.** [5, Theorem 2.1] *Let $\Lambda = KQ/I$ be a Koszul algebra, and let $\{f_i^n\}_{i=0}^{t_n}$ define a minimal resolution of $\Lambda_0$ as a right $\Lambda$-module. A minimal projective resolution $(\mathbb{K}, d)$ of $\Lambda$ over $\Lambda^e$ is given by*

$$\mathbb{K}_n = \bigoplus_{i=0}^{t_n} \Lambda o(f_i^n) \otimes_k t(f_i^n) \Lambda$$

*for $n \geq 0$, where the differential $d_n : \mathbb{K}_n \to \mathbb{K}_{n-1}$ applied to $\varepsilon_i^n = (0, \ldots, 0, o(f_i^n) \otimes_k t(f_i^n), 0, \ldots, 0)$, $0 \leq i \leq t_n$ where $o(f_i^n) \otimes_k t(f_i^n)$ is in the $i$-th position is given by*

$$d_n(\varepsilon_i^n) = \sum_{j=0}^{t_{n-1}} \Big( \sum_{p=0}^{t_1} c_{p,j}(n, i, 1) f_p^1 \varepsilon_j^{n-1} + (-1)^n \sum_{q=0}^{t_1} c_{j,q}(n, i, n-1) \varepsilon_j^{n-1} f_q^1 \Big) \tag{2.3}$$

*and $d_0 : K_0 \to \Lambda$ is the multiplication map. In particular, $\Lambda$ is a linear module over $\Lambda^e$.*

Since each $f_i^n$ is a uniform element, the notations $o(f_i^n), t(f_i^n)$ are well defined. The scalars $c_{p,j}(n, i, r)$ are those appearing in Equation (2.1) and $\overline{f_*^1}$, which by abuse of notation has been written as $f_*^1$ in Equation (2.3), is the residue class of $f_*^1$ in $\bigoplus_{i=0}^{t_1} f_i^1 R / \bigoplus_{i=0}^{t_n} f_i^1 I$.

We define Hochschild cohomology using this resolution. That is, we apply the functor $\mathrm{Hom}_{\Lambda^e}(\cdot, \Lambda)$ to the resolution $\mathbb{K}$ and take a direct sum of the cohomology groups in each degree:

$$\mathrm{HH}^*(\Lambda) := \bigoplus_{n \geq 0} \mathrm{H}^n(\mathrm{Hom}_{\Lambda^e}(\mathbb{K}_n, \Lambda))$$

Using the comultiplicative structure of Equation (2.1), it was shown in [2] that the cup product on the Hochschild cohomology ring of a Koszul algebra defined by a quiver and relations has the following description:

**Theorem 2.4** (See [2], Theorem 2.3)**.** *Let $\Lambda = kQ/I$ be a Koszul algebra over a field $k$, where $Q$ is a finite quiver and $I \subseteq J^2$. Suppose that $\eta : \mathbb{K}_n \to \Lambda$ and $\theta : \mathbb{K}_m \to \Lambda$ represent elements in $\mathrm{HH}^*(\Lambda)$ and are given by $\eta(\varepsilon_i^n) = \lambda_i$ for $i = 0, 1, \ldots, t_n$ and $\theta(\varepsilon_i^m) = \lambda_i'$ for $i = 0, 1, \ldots, t_m$. Then $\eta \smile \theta : \mathbb{K}_{n+m} \to \Lambda$ can be expressed as*

$$(\eta \smile \theta)(\varepsilon_j^{n+m}) = \sum_{p=0}^{t_n} \sum_{q=0}^{t_m} c_{pq}(n+m, i, n)\lambda_p \lambda_q',$$

*for $j = 0, 1, 2, \ldots, t_{n+m}$.*

**The reduced bar resolution of** $\Lambda = kQ/I$ as presented in [2, Section 1]: We recall the definition of the reduced bar resolution of algebras defined by quivers and relations. If $\Lambda_0$ is isomorphic to $m$ copies of $k$, take $\{e_1, e_2, \ldots, e_m\}$ to be a complete set of primitive orthogonal central idempotents of $\Lambda_0$. In this case $\Lambda$ is not necessarily an algebra over $\Lambda_0$. Define the reduced bar resolution $(\mathcal{B}, \delta)$ to be $\mathcal{B}_n = \Lambda^{\otimes_{\Lambda_0}(n+2)}$, the $(n+2)$-fold tensor product of $\Lambda$ over $\Lambda_0$ with differentials $\delta$ given by:

$$\delta_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}. \tag{2.5}$$

If $\Lambda_0 \cong k$, then $\mathcal{B} = \mathbb{B}$, the usual bar resolution, as $\Lambda$ is an algebra over $\Lambda_0$. The resolution $\mathbb{K}$ can be embedded naturally into the reduced bar resolution $\mathcal{B}$. There is a map $\iota : \mathbb{K} \to \mathcal{B}$ defined by $\iota(\varepsilon_r^n) = 1 \otimes \widetilde{f_r^n} \otimes 1$ such that $\delta\iota = \iota d$, with

$$\widetilde{f_j^n} = \sum c_{j_1 j_2 \cdots j_n} f_{j_1}^1 \otimes f_{j_2}^1 \otimes \cdots \otimes f_{j_n}^1 \quad \text{if} \quad f_j^n = \sum c_{j_1 j_2 \cdots j_n} f_{j_1}^1 f_{j_2}^1 \cdots f_{j_n}^1 \tag{2.6}$$

for some scalar $c_{j_1 j_2 \cdots j_n}$. See [2, Proposition 2.1] for a proof that $\iota$ is indeed an embedding. Let $\Delta : \mathcal{B} \to \mathcal{B} \otimes_\Lambda \mathcal{B}$ be a comultiplicative map (also called the diagonal map) on the bar resolution given explicitly by Equation (3.6). It was also shown in [2, Proposition 2.2] that there is a comultiplicative map $\Delta_{\mathbb{K}} : \mathbb{K} \to \mathbb{K} \otimes_\Lambda \mathbb{K}$ on the complex $\mathbb{K}$ compatible with $\iota$. This means that $(\iota \otimes \iota)\Delta_{\mathbb{K}} = \Delta\iota$ where $(\iota \otimes \iota)(\mathbb{K} \otimes_\Lambda \mathbb{K}) = \iota(\mathbb{K}) \otimes_\Lambda \iota(\mathbb{K}) \subseteq \mathcal{B} \otimes_\Lambda \mathcal{B}$. The comultiplicative map on $\mathbb{K}$ is given in general by

$$\Delta_{\mathbb{K}}(\varepsilon_r^n) = \sum_{v=0}^{n} \sum_{p=0}^{t_v} \sum_{q=0}^{t_{n-v}} c_{p,q}(n, r, v) \varepsilon_p^v \otimes_\Lambda \varepsilon_q^{n-v}. \tag{2.7}$$

We present a specific comultiplicative map for the family $\{\Lambda_q\}_{q \in k}$ under study in Remark 3.9 and use it to determine the structure of Hochschild cohomology of this family. Furthermore, there are recent techniques such as [12] for computing the bracket structure on Hochschild cohomology which relies on comultiplicative maps such as this.

## 3. Cup product structure

In this section, we will study the Hochschild cohomology of the family of quiver algebras of Equation (1.1) i.e.

$$\{\Lambda_q = \frac{kQ}{I_q}\}, \quad I_q = \langle a^2, b^2, ab - qba, ac \rangle.$$

and present the cup product structure on their Hochschild cohomology. For this family, the resolution $\mathbb{K} \to \Lambda_q$ has free basis elements $\{\varepsilon_i^n\}_{i=0}^{t_n}$ such that for each $n$, $\varepsilon_i^n = (0,\ldots,0,o(f_i^n)\otimes_k t(f_i^n),0,\ldots,0)$. To concretely define the free basis $\varepsilon_i^n$ for each module $\mathbb{K}_n$, we start by using $kQ_0$, the subspace of $kQ$ generated by the vertices of $Q$ with basis $\{e_1,e_2\}$. We define $\{f_0^0=e_1, f_1^0=e_2\}$. Next, since $kQ_1$ is the subspace generated by paths of length 1 with free basis $\{a,b,c\}$, we define $\{f_0^1=a, f_1^1=b, f_2^1=c\}$. We let $f_j^2$, $j=0,1,2,3$ to be the set of paths of length 2 generated by the ideal $I$, that is, $\{f_0^2=a^2, f_1^2=ab-qba, f_2^2=b^2, f_3^2=ac\}$. We continue in this way and define for each $n>2$,

$$\begin{cases} f_0^n = a^n, \\ f_s^n = f_{s-1}^{n-1}b + (-q)^s f_s^{n-1}a, \quad (0<s<n), \\ f_n^n = b^n, \\ f_{n+1}^n = a^{(n-1)}c. \end{cases} \tag{3.1}$$

We recall that each $f_i^n$ is a uniform relation therefore the origin vertex $o(f_i^n)$ and the terminal vertex $t(f_i^n)$ exist. Therefore the notation $o(f_i^n)\otimes_k t(f_i^n)$ in the definition of $\varepsilon_i^n$ makes sense. The differentials on $\mathbb{K}_n$ are given explicitly for this family by

$$\begin{aligned} d_1(\varepsilon_2^1) &= c\varepsilon_1^0 - \varepsilon_0^0 c \\ d_n(\varepsilon_r^n) &= (1-\partial_{n,r})[a\varepsilon_r^{n-1}) + (-1)^{n-r}q^r\varepsilon_r^{n-1}a] \\ &\quad + (1-\partial_{r,0})[(-q)^{n-r}b\varepsilon_{r-1}^{n-1} + (-1)^n\varepsilon_{r-1}^{n-1}b], \text{ for } r\le n \\ d_n(\varepsilon_{n+1}^n) &= a\varepsilon_n^{n-1} + (-1)^n\varepsilon_0^{n-1}c, \text{ when } n\ge 2, \end{aligned} \tag{3.2}$$

where $\partial_{r,s}=1$ when $r=s$ and $0$ when $r\ne s$. We give below, a proof that the differentials satisfy $d^2=0$. For a general proof that the resolution we obtain using these descriptions and its general form presented in Theorem (2.2) is a minimal projective resolution, see [5, Theorem 2.1].

**A proof that $d^2 = 0$.**

*Proof.*

$$\begin{aligned} d_0d_1((\varepsilon_2^1) &= \mu d_1((\varepsilon_2^1) = \mu(c\varepsilon_1^0 - \varepsilon_0^0 c) = \mu(c(e_2\otimes e_2) - (e_1\otimes e_1)c) \\ &= \mu(c\otimes e_2 - e_1\otimes c) = ce_2 - e_1c = c - c = 0. \end{aligned}$$

Now for $r\le n$, we set $\bar{\partial}_{n,r} = (1-\partial_{n,r})$, so that

$$d_n(\varepsilon_r^n) = \bar{\partial}_{n,r}[a\varepsilon_r^{n-1}) + (-1)^{n-r}q^r\varepsilon_r^{n-1}a] + \bar{\partial}_{r,0}[(-q)^{n-r}b\varepsilon_{r-1}^{n-1} + (-1)^n\varepsilon_{r-1}^{n-1}b].$$

Now we apply the differential again,

$$\begin{aligned} &d_{n-1}d_n(\varepsilon_r^n) \\ &= d_{n-1}\{\bar{\partial}_{n,r}[a\varepsilon_r^{n-1} + (-1)^{n-r}q^r\varepsilon_r^{n-1}a] + \bar{\partial}_{r,0}[(-q)^{n-r}b\varepsilon_{r-1}^{n-1} + (-1)^n\varepsilon_{r-1}^{n-1}b]\} \\ &= \bar{\partial}_{n,r}[ad_{n-1}(\varepsilon_r^{n-1}) + (-1)^{n-r}q^r d_{n-1}(\varepsilon_r^{n-1})a] + \bar{\partial}_{r,0}[(-q)^{n-r}bd_{n-1}(\varepsilon_{r-1}^{n-1}) + (-1)^n d_{n-1}(\varepsilon^{n-1})_{r-1}b] \end{aligned}$$

$$
\begin{aligned}
&= \bar{\partial}_{n,r} a\big\{\bar{\partial}_{n-1,r}[a\varepsilon_r^{n-2} + (-1)^{n-r-1}q^r\varepsilon_r^{n-2}a] + \bar{\partial}_{r,0}[(-q)^{n-r-1}b\varepsilon_{r-1}^{n-2} + (-1)^{n-1}\varepsilon_{r-1}^{n-2}b]\big\}\\
&+ (-1)^{n-r}q^r\bar{\partial}_{n,r}\big\{\bar{\partial}_{n-1,r}[a\varepsilon_r^{n-2} + (-1)^{n-r-1}q^r\varepsilon_r^{n-2}a] + \bar{\partial}_{r,0}[(-q)^{n-r-1}b\varepsilon_{r-1}^{n-2} + (-1)^{n-1}\varepsilon_{r-1}^{n-2}b]\big\}a\\
&+ (-q)^{n-r}\bar{\partial}_{r,0}b\big\{\bar{\partial}_{n-1,r-1}[a\varepsilon_{r-1}^{n-2} + (-1)^{n-r}q^{r-1}\varepsilon_{r-1}^{n-2}a] + \bar{\partial}_{r-1,0}[(-q)^{n-r}b\varepsilon_{r-2}^{n-2} + (-1)^{n-1}\varepsilon_{r-2}^{n-2}b]\big\}\\
&+ (-1)^{n}\bar{\partial}_{r,0}\big\{\bar{\partial}_{n-1,r-1}[a\varepsilon_{r-1}^{n-2} + (-1)^{n-r}q^{r-1}\varepsilon_{r-1}^{n-2}a] + \bar{\partial}_{r-1,0}[(-q)^{n-r}b\varepsilon_{r-2}^{n-2} + (-1)^{n-1}\varepsilon_{r-2}^{n-2}b]\big\}b
\end{aligned}
$$

$$
\begin{aligned}
&= \bar{\partial}_{n,r}\bar{\partial}_{n-1,r}a^2\varepsilon_r^{n-2} + (-1)^{2(n-r)-1}q^{2r}\bar{\partial}_{n,r}\bar{\partial}_{n-1,r}\varepsilon_r^{n-2}a^2\\
&+ (-q)^{2(n-r)}\bar{\partial}_{r,0}\bar{\partial}_{r-1,0}b^2\varepsilon_{r-2}^{n-2} + (-1)^{2n-1}\bar{\partial}_{r,0}\bar{\partial}_{r-1,0}\varepsilon_{r-2}^{n-2}b^2\\
&+ [(-1)^{-1}+1](-1)^{n-r}q^r\bar{\partial}_{n,r}\bar{\partial}_{n-1,r}a\varepsilon_r^{n-2}a + [(-1)^{-1}+1](-1)^n(-q)^{n-r}\bar{\partial}_{r,0}\bar{\partial}_{r-1,0}b\varepsilon_{r-2}^{n-2}b\\
&+ [-\bar{\partial}_{n,r} + \bar{\partial}_{n-1,r-1}](-1)^{2(n-r)}q^{n-1}\bar{\partial}_{r,0}b\varepsilon_{r-1}^{n-2}a + [-\bar{\partial}_{n,r} + \bar{\partial}_{n-1,r-1}]\bar{\partial}_{r,0}a\varepsilon_{r-1}^{n-2}b\\
&+ (-q)^{n-2}\bar{\partial}_{r,0}[\bar{\partial}_{n,r}ab - \bar{\partial}_{n-1,r-1}qba]\varepsilon_{r-1}^{n-2} + (-1)^{2n-r}(-q)^{r-1}\bar{\partial}_{r,0}\varepsilon_{r-1}^{n-2}[\bar{\partial}_{n,r} - qba + \bar{\partial}_{n-1,r-1}ab]\\
&= 0.
\end{aligned}
$$

In the last equality, we have used the fact that $a^2 = b^2 = 0$, $\bar{\partial}_{n,r}$ and $\bar{\partial}_{n-1,r-1}$ have the same sign, implying that $(\bar{\partial}_{n,r}ab - \bar{\partial}_{n-1,r-1}qba) = ab - qba = 0$ and $(\bar{\partial}_{n,r} - qba + \bar{\partial}_{n-1,r-1}ab) = -qba + ab = 0$. Lastly we have

$$
\begin{aligned}
d_{n-1}d_n(\varepsilon_{n+1}^n) &= d_{n-1}[a\varepsilon_n^{n-1} + (-1)^n\varepsilon_0^{n-1}c]\\
&= a[a\varepsilon_{n-1}^{n-2} + (-1)^{n-1}\varepsilon_0^{n-2}c] + (-1)^n[a\varepsilon_0^{n-2} + (-1)^{n-1}\varepsilon_0^{n-2}a]c\\
&\text{after eliminating terms with coefficients } a^2 = ac = 0,\\
&= (-1)^{n-1}a\varepsilon_0^{n-2}c + (-1)^n a\varepsilon_0^{n-2}c = [(-1)^{-1}+1](-1)^n a\varepsilon_0^{n-2}c = 0
\end{aligned}
$$

□

Recall that for each member $\Lambda_q$ of the family, the resolution $\mathbb{K}$ can be embedded into the reduced bar resolution $\mathcal{B}$ via $\iota$. The embedding map $\iota : \mathbb{K}_n \to \mathcal{B}_n$ is defined by $\varepsilon_r^n \mapsto 1 \otimes \widetilde{f_r^n} \otimes 1$, where each $\widetilde{f_r^n}$ is viewed as a sum of tensor products of paths of length 1 as given in Equation (2.6). For example, for this family, $\widetilde{f_0^2} = f_0^1 \otimes f_0^1 = a \otimes a$, $\widetilde{f_1^2} = f_0^1 \otimes f_1^1 - qf_1^1 \otimes f_0^1 = a \otimes b - qb \otimes a$. It is clear from Equation (3.1) that the following holds;
(3.3)

$$
\widetilde{f_s^n} = \begin{cases}
f_0^1 \otimes f_0^1 \otimes \cdots \otimes f_0^1, \quad (n \;\; times) & \text{when } s = 0,\\
\widetilde{f_{s-1}^{n-1}} \otimes f_1^1 + (-q)^s \widetilde{f_s^{n-1}} \otimes f_0^1, & \text{when } (0 < s < n),\\
f_1^1 \otimes f_1^1 \otimes \cdots \otimes f_1^1, \quad (n \;\; times) & \text{when } s = n,\\
f_0^1 \otimes f_0^1 \otimes \cdots \otimes f_0^1 \otimes f_2^1, \quad (f_0^1 \text{ appears } (n-1) \;\; times), & \text{when } s = n+1,
\end{cases}
$$

In case $0 < s < n$, it was shown in [1] that
$f_s^n = \sum_{j=max\{0,r+t-n\}}^{min\{t,s\}} (-q)^{j(n-s+j-t)} f_j^t f_{s-j}^{n-t}$, hence,

$$
\widetilde{f_s^n} = \sum_{j=max\{0,r+t-n\}}^{min\{t,s\}} (-q)^{j(n-s+j-t)} \widetilde{f_j^t} \otimes \widetilde{f_{s-j}^{n-t}}. \tag{3.4}
$$

We are now ready to present the cup product formula. We first present the following alternate definition of the cup product.

**Definition 3.5.** *Let $A$ be a $k$-algebra. Let $\Delta : \mathbb{B} \to \mathbb{B} \otimes_A \mathbb{B}$ be the comultiplicative map lifting the identity map on $A \cong A \otimes_A A$. Let $f \in \mathrm{Hom}_{A^e}(\mathbb{B}_m, A) \cong \mathrm{Hom}_k(A^{\otimes m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$ be cocycles of degree $m$ and $n$ respectively. The cup product $f \smile g$ at the chain level is an element of $\mathrm{Hom}_k(A^{\otimes(m+n)}, A)$ given by*

$$f \smile g = \pi(f \otimes g)\Delta,$$

*where $\pi$ is multiplication, and $\Delta$ is given by*

$$\Delta(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (a_0 \otimes \cdots \otimes a_i \otimes 1) \otimes_A (1 \otimes a_{i+1} \otimes \cdots \otimes a_{n+1}). \tag{3.6}$$

For homogeneous elements $a, b$ of degrees $m$ and $n$ respectively, the map $f \otimes g$ is taken to be $(f \otimes g)(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b)$, where the degree of g is $|g| = n$. We recall that for any member $\Lambda_q$ of the family, if $\phi \in \mathrm{Hom}_{\Lambda_q^e}(\mathbb{K}_m, \Lambda_q)$, and $\eta \in \mathrm{Hom}_{\Lambda_q^e}(\mathbb{K}_n, \Lambda_q)$ are two cocycles, we can use Definition 3.5 on the resolution $\mathbb{K}$ provided we have an explicit presentation of the comultiplication $\Delta_{\mathbb{K}}$. This definition is presented using the following composition of maps;

$$\phi \smile \eta : \mathbb{K} \xrightarrow{\Delta_{\mathbb{K}}} \mathbb{K} \otimes_{\Lambda_q} \mathbb{K} \xrightarrow{\phi\otimes\eta} \Lambda_q \otimes_{\Lambda_q} \Lambda_q \overset{\pi}{\simeq} \Lambda_q$$

where $\pi$ is multiplication, $(\phi\otimes\eta)(\varepsilon_i^m \otimes \varepsilon_j^n) = (-1)^{mn}\phi(\varepsilon_r^m) \otimes \eta(\varepsilon_j^n)$, and the comultiplicative map $\Delta_{\mathbb{K}}$ is such that the diagram

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "\Delta_{\mathbb{K}}"] \arrow[d, "\iota"'] & \mathbb{K} \otimes_{\Lambda_q} \mathbb{K} \arrow[d, "\iota\otimes\iota"] \\
\mathcal{B} \arrow[r, "\Delta"] & \mathcal{B} \otimes_{\Lambda_q} \mathcal{B}.
\end{tikzcd}$$

is commutative i.e.

$$(\iota \otimes \iota)\Delta_{\mathbb{K}} = \Delta\iota. \tag{3.7}$$

Notice that we do not distinquish between $\mathbb{B}$ and $\mathcal{B}$ when using the map $\Delta$. We are able to present explicit definition of $\Delta_{\mathbb{K}}$ in Remark 3.9 after providing a proof of Theorem 4.7 which relies on Equations (3.6) and (3.7).

Let $\phi : \mathbb{K}_m \to \Lambda_q$ and $\eta : \mathbb{K}_n \to \Lambda_q$ be two cocycles of homological degrees $m$ and $n$ respectively. Suppose that $\phi$ takes $\varepsilon_i^m$ to $\phi_i^m$, for $i = 0, 1, \ldots, m+1$, we use the following standard notation $\phi = \begin{pmatrix} \phi_0^m & \phi_1^m & \cdots & \phi_m^m & \phi_{m+1}^m \end{pmatrix}$ and $\eta = \begin{pmatrix} \eta_0^n & \eta_1^n & \eta_2^n & \cdots & \eta_n^n & \eta_{n+1}^n \end{pmatrix}$. We denote the cup product of $\phi$ and $\eta$ by

$$\phi \smile \eta := \begin{pmatrix} (\phi\eta)_0^{m+n} & (\phi\eta)_1^{m+n} & (\phi\eta)_2^{m+n} & \cdots & (\phi\eta)_{m+n}^{m+n} & (\phi\eta)_{m+n+1}^{m+n} \end{pmatrix},$$

that is, $(\phi \smile \eta)(\varepsilon_i^{m+n}) = (\phi\eta)_i^{m+n}, i = 0, 1, \ldots, m+n+1$.

**Theorem 3.8.** *Let $\phi : \mathbb{K}_m \to \Lambda_q$ and $\eta : \mathbb{K}_n \to \Lambda_q$, be two cocycles representing two classes in cohomology. Let $\{\varepsilon_j^m\}_{j=0}^{m+1}$ be free basis elements of $\mathbb{K}_m$ such that $\phi(\varepsilon_j^m) = \phi_j^m \in \Lambda_q$ and let $\{\varepsilon_i^n\}_{i=0}^{n+1}$ be free basis elements of $\mathbb{K}_n$ such that $\eta(\varepsilon_i^n) = \eta_i^n \in \Lambda_q$ . Then the following gives a formula for the cup product on Hochschild cohomology:*

$$(\phi \smile \eta)(\varepsilon_r^{m+n}) = (\phi\eta)_r^{m+n} = \begin{cases} (-1)^{mn}\phi_0^m\eta_0^n, & \textit{when } r = 0, \\ (-1)^{mn}T_r^{m+n} & \textit{when } 0 < r < m+n, \\ (-1)^{mn}\phi_m^m\eta_n^n, & \textit{when } r = m+n, \\ (-1)^{mn}\phi_0^m\eta_{n+1}^n, & \textit{when } r = m+n+1, \end{cases}$$

$$\textit{where } T_r^{m+n} = \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)}\phi_j^m\mu_{r-j}^n, \qquad 0 < r < m+n.$$

*Proof.* We will find an explicit description of the comultiplicative map $\Delta_{\mathbb{K}}$ on $\{\varepsilon_r^{m+n}\}_{r=0}^{m+n+1}$ for which Equation (3.7) holds. We will then use the formula $(\phi \smile \eta)(\varepsilon_r^{m+n}) = \pi(\phi \otimes \eta)\Delta_{\mathbb{K}}(\varepsilon_r^{m+n})$ as the definition of the cup product. We start with the case when $r = 0, r = m+n, r = m+n+1$ and last consider the case where $0 < r < m+n$.

When $r = 0$, we have that

$$\begin{aligned}
&(\iota \otimes \iota)\Delta_{\mathbb{K}}(\varepsilon_0^{m+n}) = \Delta\iota(\varepsilon_0^{m+n}) \\
&= \Delta(1 \otimes \widetilde{f_0^{m+n}} \otimes 1) = \Delta(1 \otimes \overbrace{f_0^1 \otimes f_0^1 \otimes \cdots \otimes f_0^1}^{m+n \textit{ times}} \otimes 1) \\
&= \sum_{s=0}^{m+n} (1 \otimes \widetilde{f_0^s} \otimes 1) \otimes (1 \otimes \widetilde{f_0^{m+n-s}} \otimes 1) = (\iota \otimes \iota)(\sum_{s=0}^{m+n} \varepsilon_0^s \otimes \varepsilon_0^{m+n-s}).
\end{aligned}$$

Notice that by the usual definition of the comultiplicative map on the bar resolution, $1 \otimes \widetilde{f_0^0} \otimes 1 = 1 \otimes 1$. Hence $\boxed{\Delta_{\mathbb{K}}(\varepsilon_0^{m+n}) = (\sum_{s=0}^{m+n} \varepsilon_0^s \otimes \varepsilon_0^{m+n-s}).}$ Since $\phi$ is a cocycle of degree $m$, we can evaluate $\phi(\varepsilon_*^m)$, and in a similar way evaluate $\eta(\varepsilon_*^n)$ to obtain

$$\begin{aligned}
(\phi \smile \eta)(\varepsilon_0^{m+n}) &= \pi(\phi \otimes \eta)\Delta_{\mathbb{K}}(\varepsilon_0^{m+n}) = \pi(\phi \otimes \eta)(\sum_{r=0}^{m+n} \varepsilon_0^r \otimes \varepsilon_0^{m+n-r}) \\
&= \pi((-1)^{mn}\phi(\varepsilon_0^m) \otimes \eta(\varepsilon_0^n)) = (-1)^{mn}\phi_0^m\eta_0^n.
\end{aligned}$$

In case $r = m+n$

$$\begin{aligned}
&(\iota \otimes \iota)\Delta_{\mathbb{K}}(\varepsilon_{m+n}^{m+n}) = \Delta\iota(\varepsilon_{m+n}^{m+n}) \\
&= \Delta(1 \otimes \widetilde{f_{m+n}^{m+n}} \otimes 1) = \Delta(1 \otimes \overbrace{f_1^1 \otimes f_1^1 \otimes \cdots \otimes f_1^1}^{m+n \textit{ times}} \otimes 1) \\
&= \sum_{s=0}^{m+n} (1 \otimes \widetilde{f_s^s} \otimes 1) \otimes (1 \otimes \widetilde{f_{m+n-s}^{m+n-s}} \otimes 1) = (\iota \otimes \iota)(\sum_{s=0}^{m+n} \varepsilon_s^s \otimes \varepsilon_{m+n-s}^{m+n-s}),
\end{aligned}$$

so $\boxed{\Delta_{\mathbb{K}}(\varepsilon_{m+n}^{m+n}) = \sum_{s=0}^{m+n} \varepsilon_s^s \otimes \varepsilon_{m+n-s}^{m+n-s},}$ and

$$(\phi \smile \eta)(\varepsilon_{m+n}^{m+n}) = \pi((-1)^{mn}\phi(\varepsilon_m^m) \otimes \eta(\varepsilon_n^n)) = (-1)^{mn}\phi_m^m\eta_n^n.$$

A similar result holds with $r = m+n+1$, i.e.

$$(\iota\otimes\iota)\Delta_{\mathbb{K}}(\varepsilon^{m+n}_{m+n+1}) = \Delta(1\otimes \widetilde{f^{m+n}_{m+n+1}}\otimes 1) = \Delta(1\otimes f^1_0 \otimes \overbrace{f^1_0\otimes\cdots\otimes f^1_0}^{m+n-1\ times}\otimes f^1_2\otimes 1)$$
$$= \sum_{s=0}^{m+n-1}(1\otimes \widetilde{f^s_0}\otimes 1)\otimes(1\otimes \widetilde{f^{m+n-s}_{m+n-s+1}}\otimes 1) + (1\otimes f^1_0\otimes f^1_0\otimes\cdots\otimes f^1_0\otimes f^1_2\otimes 1)\otimes(1\otimes 1)$$
$$= (\iota\otimes\iota)(\sum_{s=0}^{m+n-1}\varepsilon^s_0\otimes\varepsilon^{m+n-s}_{m+n-s+1} + \varepsilon^{m+n}_{m+n+1}\otimes\varepsilon^0_0),$$

hence $\boxed{\Delta_{\mathbb{K}}(\varepsilon^{m+n}_{m+n+1}) = (\sum_{s=0}^{m+n-1}\varepsilon^s_0\otimes\varepsilon^{m+n-s}_{m+n-s+1}) + \varepsilon^{m+n}_{m+n+1}\otimes\varepsilon^0_0}$. Therefore, when $s = m+n+1$, we obtain $(\phi\smile\eta)(\varepsilon^{m+n}_{m+n+1}) = \pi((-1)^{mn}\phi(\varepsilon^m_0)\otimes\eta(\varepsilon^n_{n+1})) = (-1)^{mn}\phi^m_0\eta^n_{n+1}$. It was shown in [1] that for $0<r<m+n$,

$$f^{m+n}_r = \sum_{j=max\{0,r+t-n-m\}}^{min\{t,r\}}(-q)^{j(m+n-r+j-t)}f^t_j f^{m+n-t}_{r-j},$$

therefore $\iota(\varepsilon^{m+n}_r) = 1\otimes\Big[\sum_{j=max\{0,r+t-m-n\}}^{min\{t,r\}}(-q)^{j(m+n-r+j-t)}\widetilde{f^t_j}\otimes\widetilde{f^{m+n-t}_{r-j}}\Big]\otimes 1$, and by letting $t=m$, the above expression becomes $\sum_{j=max\{0,r-n\}}^{min\{m,r\}}(-q)^{j(n-r+j)}1\otimes\widetilde{f^m_j}\otimes\widetilde{f^n_{r-j}}\otimes 1$. When we apply $\Delta$ to the above expression, we obtain

$$(\Delta\iota)(\varepsilon^{m+n}_r) = \sum_{u=-m}^{n}\sum_{j=max\{0,r-n+u\}}^{min\{m+u,r\}}(-q)^{j(n-u-r+j)}(1\otimes\widetilde{f^{m+u}_j}\otimes 1)\otimes(1\otimes\widetilde{f^{n-u}_{r-j}}\otimes 1)$$
$$= \sum_{u=-m}^{n}\sum_{j=max\{0,r-n+u\}}^{min\{m-u,r\}}(-q)^{j(n-u-r+j)}(\iota\otimes\iota)(\varepsilon^{m+u}_j\otimes\varepsilon^{n-u}_{r-j})$$

using the relation that $(\iota\otimes\iota)\Delta_{\mathbb{K}} = \Delta\iota$, we obtain

$$(\iota\otimes\iota)\Delta_{\mathbb{K}}(\varepsilon^{m+n}_r) = (\iota\otimes\iota)\Big[\sum_{u=-m}^{n}\sum_{j=max\{0,r-n+u\}}^{min\{m-u,r\}}(-q)^{j(n-u-r+j)}(\varepsilon^{m+u}_j\otimes\varepsilon^{n-u}_{r-j})\Big].$$

To apply $\pi(\phi\otimes\eta)\Delta_{\mathbb{K}}(\varepsilon^{m+n}_r)$, set $u=0$ to get

$$(\phi\smile\eta)(\varepsilon^{m+n}_r) = (-1)^{mn}\sum_{j=max\{0,r-n\}}^{min\{m,r\}}(-q)^{j(n-r+j)}\phi(\varepsilon^m_j)\eta(\varepsilon^n_{r-j})$$
$$= (-1)^{mn}\sum_{j=max\{0,r-n\}}^{min\{m,r\}}(-q)^{j(n-r+j)}\phi^m_j\eta^n_{r-j}$$
$$= (-1)^{mn}T^{m+n}_r,$$

which is the result. $\square$

**Remark 3.9.** *By some change of variables, we can infer from all the boxed equations in the proof of Theorem 4.7 that the explicit definition of the comultiplication* $\Delta_{\mathbb{K}} : \mathbb{K} \to \mathbb{K} \otimes_{\Lambda_q} \mathbb{K}$ *is the following;*

$$\Delta_{\mathbb{K}}(\varepsilon_s^n) = \begin{cases} \displaystyle\sum_{r=0}^{n} \varepsilon_0^r \otimes \varepsilon_0^{n-r}, & s = 0 \\ \displaystyle\sum_{w=0}^{n} \sum_{j=max\{0,s+w-n\}}^{min\{w,s\}} (-q)^{j(n-s+j-w)} \varepsilon_j^w \otimes \varepsilon_{s-j}^{n-w}, & 0 < s < n \\ \displaystyle\sum_{t=0}^{n} \varepsilon_t^t \otimes \varepsilon_{n-t}^{n-t}, & s = n \\ \Big[\displaystyle\sum_{t=0}^{n} \varepsilon_0^t \otimes \varepsilon_{n-t+1}^{n-t}\Big] + \varepsilon_{n+1}^n \otimes \varepsilon_0^0, & s = n+1. \end{cases}$$

*where in the expansion of* $\Delta_{\mathbb{K}}(\varepsilon_s^n), 0 < s < n$, *the index* $w$ *is such that there are no repeated terms.*

## 4. Hochschild cohomology modulo nilpotents not finitely generated

Let us recall from the introduction that there was an attempt to develop the theory of support varieties for finitely generated modules of finite dimensional algebras using Hochschild cohomology. The idea of this theory is the following:

Let $A$ be a finite dimensional algebra. Let $M, N$ be two $A$-modules and $\mathrm{Ext}_A^*(M, N)$ their extension group. There is an action of Hochschild cohomology on the extension group defined as follows. Let $\mathbb{P} \to A$ be a projective resolution of $A$. Let $f \in \mathrm{HH}^m(A)$ be a representative. We can think of $f$ as a representative of an equivalence class of $m$-extensions of $A$ by $A$ that is $f \in \mathrm{Ext}_{A^e}^m(A, A)$. Now define a map $\Phi : \mathrm{Ext}_{A^e}^m(A, A) \to \mathrm{Ext}_A^m(M, M)$ taking the equivalence class of $f$ to the equivalence class $f \otimes 1_M$. For any $g \in \mathrm{Ext}_A^n(M, N)$, the Yoneda product of $f \otimes 1_M$ and $g$ gives an element of $\mathrm{Ext}_A^{m+n}(M, N)$. This induces the left action

$$\mathrm{HH}^*(A) \times \mathrm{Ext}_A^*(M, N) \to \mathrm{Ext}_A^*(M, N)$$

defined by taking any pair $(f, g)$ to the Yoneda product of $\Phi(f)$ and $g$. For some finite dimensional algebras, it is well known that Hochschild cohomology ring modulo nilpotents is finitely generated as an algebra. Furthermore, when $M, N$ are finite-dimensional modules and $H$ a subalgebra of $\mathrm{HH}^*(A)$, define

$$I_H(M, N) = \{f \in H \mid \Phi(f)g = 0, \text{ for all } g \in \mathrm{Ext}_A^*(M, N)\}$$

to be the annihilator of $\mathrm{Ext}_A^*(M, N)$ in $H$. $I_H(M, N)$ is obviously an ideal of $H$.

**Definition 4.1.** *Let* $M, N$ *be finite-dimensional* $A$*-modules. The support variety of the pair* $M, N$ *is*

$$V_H(M, N) = V_H(I_H(M, N)) \cong Max(H/I_H(M, N))$$

*the maximal ideal spectrum of the quotient ring* $H/I_H(M, N)$. *The variety of* $M$ *is defined as* $V_H(M) = V_H(M, M)$.

For this theory to have all the nice properties that one would like, (i) $H$ has to be a finitely generated algebra and (ii) $\mathrm{Ext}^*_A(A/\mathfrak{r}, A/\mathfrak{r})$ has to be finitely generated as an $H$-module. This leads to the conjecture in [11] that Hochschild cohomology modulo nilpotents is always finitely generated as an algebra. For instance, we can take $H = \mathrm{HH}^{ev}(A)$ the subalgebra of $\mathrm{HH}^*(A)$ generated by homogeneous elements of even degrees.

The first counterexample to this conjecture appeared in [14] where F. Xu used certain techniques in category theory to construct a seven-dimensional category algebra whose Hochschild cohomology ring modulo nilpotents is not finitely generated. There are other constructions as well e.g. see [15]. The rest of this section is devoted to identifying nilpotent and non-nilpotent Hochschild cocycles for the family of quiver algebras under study after which we determine their Hochschild cohomology modulo homogeneous nilpotent elements. We start with cocycles of degree 0.

**The 0-th Hochschild cohomology ($\mathrm{HH}^0(\Lambda_q) = \frac{\ker d_1^*}{\mathrm{Im}\, 0}$).**
Let $\phi \in \ker d_1^* \subseteq \hat{\mathbb{K}}_0 = \mathrm{Hom}_{\Lambda^e}(\mathbb{K}_0, \Lambda)$, such that $\phi = (\lambda_0^0\ \lambda_1^0)$, for some $\lambda_1^0, \lambda_1^0 \in \Lambda$. We solve for the $\lambda_i^0$ $(i = 0, 1)$ for which $d_1^*\phi(\varepsilon_i^1) = 0$ as follows

$$d_1^*\phi(\varepsilon_0^1) = \phi d_1(\varepsilon_0^1) = \phi(a(\varepsilon_0^0) + (-1)^1 q^0(\varepsilon_0^0)a) = a\lambda_0^0 - \lambda_0^0 a = 0$$
$$d_1^*\phi(\varepsilon_1^1) = \phi d_1(\varepsilon_1^1) = \phi((-q)^0 b(\varepsilon_0^0) - (\varepsilon_0^0)b) = b\lambda_0^0 - \lambda_0^0 b = 0$$
$$d_1^*\phi(\varepsilon_2^1) = \phi d_1(\varepsilon_2^1) = \phi(c(\varepsilon_1^0) - (\varepsilon_0^0)c) = c\lambda_1^0 - \lambda_0^0 c = 0$$

**If** $q = 1$, then $ab - ba = 0$, we get the following set of solutions: $\phi = (a\ 0)$, $(ab\ 0)$,$(0\ a)$, $(0\ b)$, $(e_1\ e_2)$ or $(0\ e_1)$. By identifying each solution $(\lambda_0^0\ \ \lambda_1^0)$ with $(o(f_0^0)\lambda_0^0 t(f_0^0)\ \ o(f_1^0)\lambda_1^0 t(f_1^0)) = (e_1\lambda_0^0 e_1\ \ e_2\lambda_1^0 e_2)$, we need to have $o(\lambda_0^0) = t(\lambda_0^0) = e_1$ and $o(\lambda_1^0) = t(\lambda_1^0) = e_2$. This leads us to eliminate some solutions in order to have the following set of solutions; $\phi_1 = (a\ 0), \phi_2 = (ab\ 0)$ and $\phi_3 = (e_1\ e_2)$.
**If** $q = -1$, then $ab + ba = 0$, we get the solutions set: $\phi_2 = (ab\ 0)$ and $\phi_3 = (e_1\ e_2)$.
**If** $q \neq \pm 1$, then $ab - qba = 0$, we get $\phi_2 = (ab\ 0)$ and $\phi_3 = (e_1\ e_2)$. Therefore, the $\Lambda^e$-module homomorphisms $\phi_1, \phi_2, \phi_3$ form a basis for the kernel of $d_1^*$ as a $k$-vector space. We write,

$$\ker d_1^* = span_k\{\phi_1, \phi_2, \phi_3\}.$$

In summary we obtain,

$$\mathrm{HH}^0(\Lambda_q) = \frac{\ker d_1^*}{\mathrm{Im}\, 0} = \begin{cases} span_k\{(a\ 0), (ab\ 0), (e_1\ e_2)\}, & \text{if } q = 1 \\ span_k\{(ab\ 0), (e_1\ e_2)\}, & \text{if } q \neq 1 \end{cases}$$

Notice that if the characteristics of $k$ is 2, then $q = 1 = -1$, so we obtain the first case.

**Remark 4.2.** *We note that the Hochschild 0-cocycles $\phi = (a\ 0)$ and $\phi = (ab\ 0)$ correspond to elements $a$ and $ab$ respectively. These elements are in the center of the algebra $\Lambda_q$. As we will see later, these elements are nilpotent with respect to the cup product. The 0-cocycle $\phi = (e_1\ e_2)$ is not nilpotent, since $e_1$ and $e_2$ are idempotent elements. It is obvious that $\phi$ generates $\mathrm{HH}^0(\Lambda_q)/\mathcal{N}$. This brings us to make the following deduction for any $q \in k$ :*

$$\mathrm{HH}^0(\Lambda)/\mathcal{N} = \frac{\ker d_1^*}{\mathrm{Im}\, 0} = span_k\{(e_1\ e_2)\} \cong k, \tag{4.3}$$

since $e_1 + e_2 = 1_{\Lambda_q}$. We now give the following counting proposition about $\dim(\ker d^*)$, the dimension of the kernels of the differentials $d_{n+1}^* : \hat{\mathbb{K}}_n \to \hat{\mathbb{K}}_{n+1}$.

**Proposition 4.4.** *Let $k$ be a field and let $\{\Lambda_q\}_{q\in k}$ be the family of quiver algebras of Equation* (1.1)*. For the Hochschild cohomology ring* $\mathrm{HH}^n(\Lambda) = \frac{ker d^*_{n+1}}{Im\, d^*_n}$, $n \neq 0$, *the following holds;*

$$\dim(\ker d^*_{n+1}) = \begin{cases} 2(n+2), & q=1, \text{ n is odd,} \\ \frac{5n}{2}+4, & q=1, \text{ n is even,} \\ 2(n+2), & q=-1, \text{ n is even,} \\ \frac{5n}{2}+4, & q=-1, \text{ n is odd,} \\ n+2, & q\neq \pm 1, \text{ n is any integer,} \end{cases} \tag{4.5}$$

*as a $k$-vector space.*

*Proof.* Let $\phi \in ker\ d^*_{n+1}$, with $\phi = \begin{pmatrix} \phi^n_0 & \phi^n_1 & \cdots & \phi^n_n & \phi^n_{n+1}\end{pmatrix}$. The elements $\phi^n_i = \phi(\varepsilon^n_i), i = 0, \cdots, n+1$ are obtained by setting the following sets of equations to 0:
***For any*** $n$ ***or*** $q$

$$d^*_{n+1}\phi(\varepsilon^{n+1}_0) = a\phi(\varepsilon^n_0) + (-1)^{n+1}\phi(\varepsilon^n_0)a = a\phi^n_0 \pm \phi^n_0 a \qquad \text{and}$$
$$d^*_{n+1}\phi(\varepsilon^{n+1}_{n+2}) = a\phi(\varepsilon^n_{n+1}) + (-1)^{n+1}\phi(\varepsilon^n_0)c = a\phi^n_{n+1} \pm \phi^n_0 c.$$

For this set of equations to be 0, we should have $\phi^n_0 \in span_k\{a, c, ab, bc\}$ and $\phi^n_{n+1} \in span_k\{a, c, ab, bc\}$. But we recall that $\phi^n_0 \in e_1\Lambda_q e_1$, and $\phi^n_{n+1} \in e_1\Lambda_q e_2$. These constraints make us obtain the following $\phi^n_0 \in span_k\{a, ab\}$ and $\phi^n_{n+1} \in span_k\{c, bc\}$. The rest of this proof involves obtaining the values of $\phi^n_r$ when you set the following equations

$$\begin{aligned} d^*_{n+1}\phi(\varepsilon^{n+1}_r) &= a\phi(\varepsilon^n_r) + (-1)^{n+1-r}q^r\phi(\varepsilon^n_r)a + (-q)^{n+1-r}b\phi(\varepsilon^n_{r-1}) + (-1)^{n+1}\phi(\varepsilon^n_{r-1})b \\ &= a\phi^n_r + (-1)^{n+1-r}q^r\phi^n_r a + (-q)^{n+1-r}b\phi^n_{r-1} + (-1)^{n+1}\phi^n_{r-1}b \\ d^*_{n+1}\phi(\varepsilon^{n+1}_{r+1}) &= a\phi^n_{r+1} + (-1)^{n-r}q^{r+1}\phi^n_{r+1}a + (-q)^{n-r}b\phi^n_r + (-1)^{n+1}\phi^n_r b \end{aligned}$$

equal to 0 for different values of $n, r$ and $q$. We recall that $q = \pm 1$ implies $ab \mp ab = 0$.
***When*** $n$ ***is even,*** $r$ ***is even,*** $q = 1$, we obtain $\phi^n_r$ by setting $\phi^n_{r-1} = \phi^n_{r+1} = 0$. Then solving $d^*_{n+1}\phi(\varepsilon^{n+1}_r) = a\phi^n_r - \phi^n_r a = 0$ and $d^*_{n+1}\phi(\varepsilon^{n+1}_{r+1}) = b\phi^n_r - \phi^n_r b = 0$, we obtain $\phi^n_r \in span_k\{a, b, ab, bc, e_1\}$. Again we recall that $\phi^n_r \in e_1\Lambda_q e_1$, so $\phi^n_r \in span_k\{a, b, ab, e_1\}$.
***When*** $n$ ***is even,*** $r$ ***is odd,*** $q = 1$, we obtain $\phi^n_r$ by setting $\phi^n_{r-1} = \phi^n_{r+1} = 0$. Then solving $d^*_{n+1}\phi(\varepsilon^{n+1}_r) = a\phi^n_r + \phi^n_r a = 0$ and $d^*_{n+1}\phi(\varepsilon^{n+1}_{r+1}) = -b\phi^n_r - \phi^n_r b = 0$ to obtain $\phi^n_r \in span_k\{ab, bc\}$. So $\phi^n_r \in span_k\{ab\}$.
***When*** $n$ ***is odd,*** $r$ ***is even,*** $q = 1$, we obtain $\phi^n_r$ by setting $\phi^n_{r-1} = \phi^n_{r+1} = 0$. After solving $d^*_{n+1}\phi(\varepsilon^{n+1}_r) = a\phi^n_r + \phi^n_r a = 0$ and $d^*_{n+1}\phi(\varepsilon^{n+1}_{r+1}) = -b\phi^n_r + \phi^n_r b = 0$, we get $\phi^n_r \in span_k\{a, ab, bc\}$ and finally we get $\phi^n_r \in span_k\{a, ab\}$.
***When*** $n$ ***is odd,*** $r$ ***is odd,*** $q = 1$, we obtain $\phi^n_r$ by setting $\phi^n_{r-1} = \phi^n_{r+1} = 0$. Then solve $d^*_{n+1}\phi(\varepsilon^{n+1}_r) = a\phi^n_r - \phi^n_r a = 0$ and $d^*_{n+1}\phi(\varepsilon^{n+1}_{r+1}) = b\phi^n_r + \phi^n_r b = 0$, to get $\phi^n_r \in span_k\{ab, bc, b\}$. Like before we obtain $\phi^n_r \in span_k\{b, ab\}$.
We continue in this fashion and obtain the following results as well.
***When*** $n$ ***is even,*** $r$ ***is even,*** $q = -1$, we obtain $\phi^n_r \in span_k\{ab, e_1\}$.
***When*** $n$ ***is even,*** $r$ ***is odd,*** $q = -1$,we obtain $\phi^n_r \in span_k\{ab, b\}$.
***When*** $n$ ***is odd,*** $r$ ***is even,*** $q = -1$, we obtain $\phi^n_r \in span_k\{a, b, ab\}$.
***When*** $n$ ***is odd,*** $r$ ***is odd,*** $q = -1$, we get $\phi^n_r \in span_k\{a, ab\}$.
**For any other** $q \neq \pm 1$ and $n$ even, $r$ even, we obtain after solving $d^*_{n+1}\phi(\varepsilon^{n+1}_r) =$

$a\phi^n_r - q^r\phi^n_r a = 0$ and $d^*_{n+1}\phi(\varepsilon^{n+1}_{r+1}) = q^{n-r}b\phi^n_r - \phi^n_r b = 0$, $\phi^n_r \in span_k\{ab\}$. In case $n$ is even and $r$ is odd or even, we obtain the same $\phi^n_r \in span_k\{ab\}$.
The following table summarizes the set of all solutions:

| | | | | |
|---|---|---|---|---|
| $q = 1$ | $n$ is even | | $n$ is odd | |
| | $r$ is even | $r$ is odd | $r$ is even | $r$ is odd |
| $\phi^n_0$ | $a, ab$ | | $a, ab$ | |
| $\phi^n_r$ | $a, b, ab, e_1$ | $ab$ | $a, ab$ | $b, ab$ |
| $\phi^n_{n+1}$ | $c, bc$ | | $c, bc$ | |

| | | | | |
|---|---|---|---|---|
| $q = -1$ | $n$ is even | | $n$ is odd | |
| | $r$ is even | $r$ is odd | $r$ is even | $r$ is odd |
| $\phi^n_0$ | $a, ab$ | | $a, ab$ | |
| $\phi^n_r$ | $ab, e_1$ | $b, ab$ | $a, b, ab$ | $a, ab$ |
| $\phi^n_{n+1}$ | $c, bc$ | | $c, bc$ | |

| | | | | |
|---|---|---|---|---|
| $q \neq \pm 1$ | $n$ is even | | $n$ is odd | |
| | $r$ is even | $r$ is odd | $r$ is even | $r$ is odd |
| $\phi^n_0$ | $a, ab$ | | $a, ab$ | |
| $\phi^n_r$ | $ab$ | $ab$ | $ab$ | $ab$ |
| $\phi^n_{n+1}$ | $c, bc$ | | $c, bc$ | |

From all these tables, we make the following deductions;

$$(n \text{ is even and } q = +1): \ \dim(\operatorname{Ker} d^*_{n+1}) = 2 + \Big( \overset{(odd-positions)}{\frac{n}{2} \times 1} + \overset{(even-positions)}{\frac{n}{2} \times 4} \Big) + 2 = 5(\frac{n}{2}) + 4$$

$$(n \text{ is odd and } q = +1): \ \dim(\operatorname{Ker} d^*_{n+1}) = 2 + \Big( \overset{(odd-positions)}{\frac{n}{2} \times 2} + \overset{(even-positions)}{\frac{n}{2} \times 2} \Big) + 2 = 2(n+2)$$

$$(n \text{ is even and } q = -1): \ \dim(\operatorname{Ker} d^*_{n+1}) = 2 + \Big( \overset{(odd-positions)}{\frac{n}{2} \times 2} + \overset{(even-positions)}{\frac{n}{2} \times 2} \Big) + 2 = 2(n+2)$$

$$(n \text{ is odd and } q = -1): \ \dim(\operatorname{Ker} d^*_{n+1}) = 2 + \Big( \overset{(odd-positions)}{\frac{n}{2} \times 2} + \overset{(even-positions)}{\frac{n}{2} \times 3} \Big) + 2 = 5(\frac{n}{2}) + 4$$

$$(\text{ for any } n,\ q \neq \pm 1): \ \dim(\operatorname{Ker} d^*_{n+1}) = 2 + \Big( \overset{(odd-positions)}{\frac{n}{2} \times 1} + \overset{(even-positions)}{\frac{n}{2} \times 1} \Big) + 2 = n + 4$$

which is the result. ☐

**Remark 4.6.** *From Proposition 4.4, note that the dimension of* $\ker d^*_n$ *increases as* $n$ *increases. We also observe that there are Hochschild* $n$*-cocycles of the form* $\phi = \begin{pmatrix} 0 & \cdots & 0 & e_1 & 0 & \cdots & 0 \end{pmatrix}$ *i.e.* $\phi^n_i = 0$ *for all* $i$ *except at some position* $r$*. These cocycles are denoted by* $\phi := \begin{pmatrix} 0 & \cdots & 0 & (e_1)^{(r)} & 0 & \cdots & 0 \end{pmatrix}$ *when we want to emphasize that* $e_1$ *is in the* $r$*-th position. Our next result shows that cocycles like this are non-nilpotent. Reading from the table of solutions in Proposition 4.4 they occur whenever both* $n$ *and* $r$ *are even and* $q = \pm 1$*. Observe that for any* $n$*, whenever* $r = 0$ *or* $r = n + 1$*,* $\phi^n_r \neq e_1$*. We also observe that whenever* $q \neq \pm 1$ *all* $n$*-cocycles are nilpotent except in degree* $0$*.*

**Lemma 4.7.** *If $\phi = \begin{pmatrix} 0 & \cdots & 0 & \phi_r^n & 0 & \cdots & 0 \end{pmatrix}$ is any cocyle such that $\phi_r^n \neq e_1$, $\phi$ is nilpotent.*

*Proof.* We have from Theorem 3.8 that when $0 < r < m + n$,

$$(\phi \smile \phi)(\varepsilon_r^{m+n}) = (-1)^{mn} \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)} \phi_j^m \phi_{r-j}^n \tag{4.8}$$

where $\phi_j^m \phi_{r-j}^n$ is a product of any two elements from the set $\{a, b, ab, c, bc\}$ which is equal to 0 in the algebra except $ab$ and $bc$. In general, if it is not a zero, we simply take a triple cup product using the following;

$$\begin{aligned}
&(\phi \smile \phi \smile \phi)(\varepsilon_r^{n+n+n}) \\
&= (\mu \smile \phi)(\varepsilon_r^{m+n}) \quad (\text{take } \mu = \phi \smile \phi, m = n + n) \\
&= (-1)^{mn} \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)} \mu(\varepsilon_j^m) \phi(\varepsilon_{r-j}^n) \\
&= (-1)^{mn} \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)} [\phi \smile \phi(\varepsilon_j^{n+n})] \phi(\varepsilon_{r-j}^n) \\
&= (-1)^{mn} \sum_{j=max\{0,r-n\}}^{min\{m,r\}} (-q)^{j(n-r+j)} \Big[(-1)^{n^2} \sum_{i=max\{0,l-n\}}^{min\{n,l\}} (-q)^{i(n-l+i)} \phi(\varepsilon_i^n) \phi(\varepsilon_{l-i}^n)\Big] \phi(\varepsilon_{r-j}^n) \\
&= (-1)^{3n^2} \sum_{j=max\{0,r-n\}}^{min\{m,r\}} \sum_{i=max\{0,l-n\}}^{min\{n,l\}} (-q)^{ij(n-r+j)(n-l+i)} \phi(\varepsilon_i^n) \phi(\varepsilon_{l-i}^n) \phi(\varepsilon_{r-j}^n).
\end{aligned}$$

The product $\phi(\varepsilon_i^n)\phi(\varepsilon_{l-i}^n)\phi(\varepsilon_{r-j}^n) = \phi_i^n \phi_{l-i}^n \phi_{r-j}^n$ is always 0 in $\Lambda_q$ by the defining relations in $I_q$ except when $\phi_i^m = \phi_{l-i}^m = \phi_{r-j}^m = e_1$ for some $i, j, l, r$. Accordingly, this is the case if and only if $q = \pm 1, n$ is even and $i, l, r$ are even. □

We now present the following corollary to Lemma 4.7.

**Corollary 4.9.** *Let $\phi : \mathbb{K}_n \to \Lambda_q$, be an $n$-cocycle. Then $\phi$ is non-nilpotent if, and only if $q = \pm 1, n$ and $r$ are even, $r \neq 0$ and $\phi = \begin{pmatrix} 0 & \cdots & 0 & (e_1)^{(r)} & 0 & \cdots & 0 \end{pmatrix}$.*

*Proof.* Follows from Lemma 4.7 and the tables of solutions in Proposition 4.4. □

Let $\mathrm{HH}^n(\Lambda_q, \Lambda_q) = H^n(\mathrm{Hom}_{\Lambda_q^e}(\mathbb{K}_n, \Lambda_q))$ be the Hochschild cohomology class of $n$ cocycles and denote by $Z^n(\Lambda_q, \Lambda_q) := \mathrm{HH}^n(\Lambda_q, \Lambda_q)/\mathcal{N}$ the class of non-nilpotent Hochschild n-cocycles. For each $n$, the non-nilpotent cocycles $Z^n(\Lambda_q, \Lambda_q)$, are those given by Corollary 4.9.

Furthermore, these non-nilpotent cocycles do not differ by a coboundary. Therefore, they constitute their own equivalence class. This is because for a fixed $n$, let $\phi, \beta$ be two distinct $2n$-cocycles such that $\phi(\varepsilon_r^{2n}) = \phi_r^{2n} = e_1$, $\beta(\varepsilon_s^{2n}) = \beta_s^{2n} = e_1$ where $r < s$ are both even. Suppose there is an $\alpha$ such that

$$d^*(\alpha) = \phi - \beta = \begin{pmatrix} 0 & \cdots & 0 & e_1 & 0 & \cdots & 0 & -e_1 & 0 & \cdots & 0 \end{pmatrix}$$

where the idempotent $e_1$ is in the $r$-th and $s$-th positions. This $\alpha$ does not exist because by considering for example at the position $r$,

$$e_1 = (\phi - \beta)(\varepsilon_r^{2n}) = d^*(\alpha)(\varepsilon_r^{2n}), \quad \text{implies that}$$
$$\alpha(d(\varepsilon_r^{2n})) = a\alpha(\varepsilon_r^{2n-1}) + (-1)^{2n-r}q^r\alpha(\varepsilon_r^{2n-1})a + (-q)^{2n-r}b\alpha(\varepsilon_{r-1}^{2n-1}) + (-1)^{2n}\alpha(\varepsilon_{r-1}^{2n-1})b.$$

There is no way to define $\alpha(\varepsilon_r^{2n-1})$ and $\alpha(\varepsilon_{r-1}^{2n-1})$ so that equality hold in the above expression. Another way to look at this is that if $d^*(\alpha) = \phi - \beta$ for some $\alpha$, then $\alpha$ has to be a non-nilpotent element of odd homological degree. But there are no non-nilpotents of odd degree. Therefore there is no such $\alpha$. Therefore each non-nilpotent $n$-cocycle is in a distinct class and they do not differ by a coboundary.

We now define a canonical map from $Z^*(\Lambda_q, \Lambda_q) = \bigoplus_{n>0} Z^n(\Lambda_q, \Lambda_q)$ to the polynomial ring in two indeterminates $k[x, y]$. We can recall from Lemma 4.7 that $\phi_0^n$ and $\phi_{n+1}^n$ are never equal to $e_1$ whenever $\phi$ is a non-nilpotent cocycle. We define this map by

$$\begin{aligned}
\begin{pmatrix}0 & 0 & (e_1)^2 & 0 & \cdots & 0\end{pmatrix} &\mapsto x^{2(n-1)}y^2,\\
\begin{pmatrix}0 & 0 & 0 & 0 & (e_1)^4 & 0 & \cdots & 0\end{pmatrix} &\mapsto x^{2(n-2)}y^4,\\
&\vdots\\
\begin{pmatrix}0 & \cdots & 0 & (e_1)^r & 0 & \cdots & 0\end{pmatrix} &\mapsto x^{2n-r}y^r,\\
&\vdots\\
\begin{pmatrix}0 & 0 & \cdots & 0 & (e_1)^{2n} & 0\end{pmatrix} &\mapsto y^{2n}
\end{aligned}$$

This map is well defined as the kernel contains only the zero map. Under this map, the image of $Z^*(\Lambda_q, \Lambda_q)$ is the subalgebra $k[x^2, y^2]y^2$ which is not finitely generated as an algebra. This is because for each $n$, $x^{2(n-1)}y^2$ cannot be generated by lower degree elements. Also note how the cup product corresponds with multiplication in $k[x, y]$, that is, given even positive integers $r, s$, we have

$$\begin{tikzcd}
(\,0\ \cdots\ 0\ (e_1)^r\ 0\ \cdots\ 0\,) \smile (\,0\ \cdots\ 0\ (e_1)^s\ 0\ \cdots\ 0\,) \arrow[r, dashed, "="] \arrow[d, dashed, "="'] & (x^{2n-r}y^r)\cdot(x^{2m-s}y^s) \arrow[d, dashed, "="'] \\
(\,0\ \cdots\ 0\ (e_1)^{r+s}\ 0\ \cdots\ 0\,) \arrow[r, "="] & x^{2(n+m)-(r+s)}y^{r+s}
\end{tikzcd}$$

At each degree $n$, the element $\begin{pmatrix}0 & 0 & e_1 & 0 & \cdots & 0\end{pmatrix}$ identified with $x^{2(n-1)}y^2$ cannot be generated as a cup product of any two elements of lower homological degrees. Since this map is 1-1, we conclude that $Z^*(\Lambda_q, \Lambda_q) \cong k[x^2, y^2]y^2$. The next proposition formalizes this idea whereas the next example is an illustration.

**Proposition 4.10.** *$Z^*(\Lambda_q, \Lambda_q), q = \pm 1$ is graded with respect to the cup product and is canonically isomorphic to the subalgebra $k[x^2, y^2]y^2$ of $k[x, y]$, that is $Z^*(\Lambda_q, \Lambda_q) \cong k[x^2, y^2]y^2$ where the degree of $y^2$ is 2 and that of $x^2y^2$ is 4.*

**Example 4.11.** *To show that*

$$\begin{aligned}
x^2y^2 \cdot y^2 &= (0\ \ 0\ \ e_1\ \ 0\ \ 0\ \ 0) \smile (0\ \ 0\ \ e_1\ \ 0)\\
&= (0\ \ 0\ \ 0\ \ 0\ \ e_1\ \ 0\ \ 0\ \ 0) = x^2 \cdot y^4
\end{aligned}$$

*Take* $\phi = x^2y^2 = (\phi_0^4\ \phi_1^4\ \phi_2^4\ \phi_3^4\ \phi_4^4\ \phi_5^4)$ *and* $\mu = y^2 = (\phi_0^2\ \phi_1^2\ \phi_2^2\ \phi_3^2)$.

$$(\phi \smile \mu)(\varepsilon_0^6) = \phi_0^4\mu_0^2 = 0$$
$$(\phi \smile \mu)(\varepsilon_1^6) = \sum_{j=0}^{1}(-1)^{j(1+j)}\phi_j^4\mu_{1-j}^2 = \phi_0^4\mu_1^2 + \phi_1^4\mu_0^2 = 0$$
$$(\phi \smile \mu)(\varepsilon_2^6) = \sum_{j=0}^{2}(-1)^{j^2}\phi_j^4\mu_{2-j}^2 = \phi_0^4\mu_2^2 - \phi_1^4\mu_1^2 + \phi_2^4\mu_0^2 = 0$$
$$(\phi \smile \mu)(\varepsilon_3^6) = \sum_{j=1}^{3}(-1)^{j(-1+j)}\phi_j^4\mu_{3-j}^2 = \phi_1^4\mu_2^2 + \phi_2^4\mu_1^2 + \phi_3^4\mu_0^2 = 0$$
$$(\phi \smile \mu)(\varepsilon_4^6) = \sum_{j=2}^{4}(-1)^{j(-2+j)}\phi_j^4\mu_{4-j}^2 = \phi_2^4\mu_2^2 - \phi_3^4\mu_1^2 + \phi_4^4\phi_0^2 = e_1$$
$$(\phi \smile \mu)(\varepsilon_5^6) = \sum_{j=3}^{4}(-1)^{j(-3+j)}\phi_j^4\mu_{5-j}^2 = \phi_3^4\mu_2^2 + \phi_4^4\mu_1^2 = 0$$
$$(\phi \smile \mu)(\varepsilon_6^6) = \phi_4^4\mu_4^2 = 0$$
$$(\phi \smile \mu)(\varepsilon_7^6) = \phi_0^4\mu_3^2 = 0$$

**Theorem 4.12.** *Let* $k$ *(*$char(k) \neq 2$*) be a field and* $\Lambda_q = kQ/I_q$ *be the family of quiver algebras of* (1.1). *Let* $\mathcal{N}$ *be the set of homogeneous nilpotent elements of* $\mathrm{HH}^*(\Lambda_q)$, *then*

$$\mathrm{HH}^*(\Lambda_q)/\mathcal{N} = \begin{cases} \mathrm{HH}^0(\Lambda_q)/\mathcal{N} \cong k, & \text{if } q \neq \pm 1 \\ Z^0(\Lambda_q, \Lambda_q) \oplus k[x^2, y^2]y^2 \cong k \oplus k[x^2, y^2]y^2, & \text{if } q = \pm 1 \end{cases}$$

*where the degree of* $y^2$ *is 2, and that of* $x^2y^2$ *is 4.*

*Proof.* If $q \neq \pm 1$, and $n > 0$, then all cocycles $\phi : \mathbb{K}_n \to \Lambda_q$ are nilpotent by Lemma 4.7. From Remark 4.2, we have then that

$$\mathrm{HH}^*(\Lambda_q)/\mathcal{N} = \mathrm{HH}^0(\Lambda_q)/\mathcal{N} \cong Z^0(\Lambda_q, \Lambda_q) \cong k.$$

If $q = \pm 1$, then the only non-nilpotent elements are those of Corollary 4.9. From Remark 4.2 and Proposition 4.10 we have that Hochschild cohomology ring modulo homogeneous nilpotent elements of $\{\Lambda_q\}_{q=\pm 1}$ is spanned by graded pieces of sets containing cocycles given by Corollary 4.9. That means that

$$\begin{aligned} &\mathrm{HH}^*(\Lambda_q)/\mathcal{N} = Z^0(\Lambda_q, \Lambda_q) \oplus Z^*(\Lambda_q, \Lambda_q) \\ &\cong k \oplus \Big( \bigoplus_{n>0} span_k \Big\{ \phi : \mathbb{K}_{2n} \to \Lambda_q \mid \phi = \begin{pmatrix} 0 & \cdots & 0 & (e_1)^{(r)} & 0 & \cdots & 0 \end{pmatrix}, r \text{ is even } \Big\} \Big) \\ &= k \oplus k[x^2, y^2]y^2. \end{aligned}$$

∎

## 5. Gerstenhaber Ideal of Nilpotent Cocycles

We denote by $\mathcal{N}^c$, the set of homogeneous non-nilpotent elements of $\mathrm{HH}^*(\Lambda_q)$ given in Corollary 4.9. In this section we compute the Gerstenhaber bracket of homogeneous non-nilpotent cocycles $[\mathcal{N}^c, \mathcal{N}^c]$ and show that it is zero. We showed that for $\Lambda_q, q = \pm 1$, the Gerstenhaber ideal of homogeneous nilpotent cocycles is the same as the set of homogeneous nilpotent cocycles $\mathcal{N}$. We use the idea of homotopy lifting which we briefly introduce to handle the Gerstenhaber bracket structure. We computed some examples of homotopy lifting maps as well as the bracket of two nilpotent cocycles. We next summarize the techniques from [12, 13] for computing Gerstenhaber brackets on Hochschild cohomology.

**Homotopy lifting:** Let $\mathbb{P} \xrightarrow{\mu_P} A$ be a projective resolution of $A$ as an $A^e$-module with differential $d^P$ and augmentation map $\mu_P$. We take $\mathbf{d}$ to be the differential on the Hom complex $\mathrm{Hom}_{\Lambda^e}(\mathbb{P}, \mathbb{P})$ defined for any degree $n$ map $g : \mathbb{P} \to \mathbb{P}[-n]$ as

$$\mathbf{d}(g) := d^P g - (-1)^n g d^P$$

where $\mathbb{P}[-n]$ is a shift in homological dimension with $(\mathbb{P}[-n])_m = \mathbb{P}_{m-n}$. In the following definition, the notation $\sim$ is used for two cocycles that are cohomologous, that is, they differ by a coboundary.

**Definition 5.1.** *Let $\Delta_{\mathbb{P}}$ be a chain map lifting the identity map on $A \cong A \otimes_A A$ and suppose that $\eta \in \mathrm{Hom}_{A^e}(\mathbb{P}_n, A)$ is a cocycle. A module homomorphism $\psi_\eta : \mathbb{P} \to \mathbb{P}[1-n]$ is called a **homotopy lifting** map of $\eta$ with respect to $\Delta_{\mathbb{P}}$ if*

$$\begin{aligned} &\boldsymbol{d}(\psi_\eta) = (\eta \otimes 1_P - 1_P \otimes \eta)\Delta_{\mathbb{P}} \qquad \textit{and} \\ &\mu_P \psi_\eta \sim\ (-1)^{n-1} \eta \psi \end{aligned} \tag{5.2}$$

*for some $\psi : \mathbb{P} \to \mathbb{P}[1]$ for which $\boldsymbol{d}(\psi) = (\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_{\mathbb{P}}$.*

**Remark 5.3.** *For Koszul algebras, the resolution $\mathbb{K}$ is furnished with the differential graded coalgebra property i.e. $(\Delta_{\mathbb{K}} \otimes 1_{\mathbb{K}})\Delta_{\mathbb{K}} = (1_{\mathbb{K}} \otimes \Delta_{\mathbb{K}})\Delta_{\mathbb{K}}$ and $(d \otimes 1 + 1 \otimes d)\Delta_{\mathbb{K}} = \Delta_{\mathbb{K}} d$. Furthermore, the augmentation map $d_0 = \mu : \mathbb{K} \to \Lambda$, which can be thought of as a counit makes $(\mu \otimes 1_{\mathbb{K}})\Delta_{\mathbb{K}} - (1_{\mathbb{K}} \otimes \mu)\Delta_{\mathbb{K}} = 0$. We can therefore take $\psi = 0$, so that we have $\mu\psi_\eta \sim 0$. Next, we set $\psi_\eta(\mathbb{K}_{n-1}) = 0$ and the second hypothesis of Definition 5.1 is satisfied. We now give a theorem of Y. Volkov which is equivalent to the original definition of the Gerstenhaber bracket on Hochschild cohomology* [13]*.*

**Theorem 5.4.** [12, Theorem 4] *Let $(\mathbb{P}, \mu_P)$ be a $\Lambda^e$-projective resolution of $\Lambda$, and let $\Delta_{\mathbb{P}} : \mathbb{P} \to \mathbb{P} \otimes_\Lambda \mathbb{P}$ be a diagonal map. Let $\eta : \mathbb{P}_n \to \Lambda$ and $\theta : \mathbb{P}_m \to \Lambda$ represent some cocycles. Suppose that $\psi_\eta$ and $\psi_\theta$ are homotopy liftings for $\eta$ and $\theta$ respectively. Then the Gerstenhaber bracket of the classes of $\eta$ and $\theta$ can be represented by the class of the element*

$$[\eta, \theta]_{\Delta_{\mathbb{P}}} = \eta\psi_\theta - (-1)^{(m-1)(n-1)} \theta\psi_\eta.$$

We present the following theorem describing homotopy lifting maps for non-nilpotent cocycles. We follow up with a remark showing that Gerstenhaber bracket of any two non-nilpotent cocycles is 0.

**Theorem 5.5.** *Let $k$ be a field and $\Lambda_q = kQ/I_q$, $q = \pm 1$. Suppose that $\eta : \mathbb{K}_n \to \Lambda$ is a non-nilpotent $n$ cocycle i.e. $\eta = \begin{pmatrix} 0 & \cdots & 0 & (e_1)^{(r)} & 0 & \cdots & 0 \end{pmatrix}$, where $n$ and $r$ are even. The associated homotopy lifting map $\psi_\eta : \mathbb{K}_m \to \mathbb{K}_{m-n+1}$ satisfies*

$$d\psi_\eta - (-1)^{n-1} \psi_\eta d = 0.$$

*Proof.* From Remark 3.9, the diagonal map $\Delta_{\mathbb{K}}$ on $\varepsilon^m_s$ whenever $0 < s < m$ is given by $\Delta_{\mathbb{K}}(\varepsilon^m_s) = \sum_{w=0}^{m} \sum_{j=max\{0,s+w-m\}}^{min\{w,s\}} (-q)^{j(m-s+j-w)} \varepsilon^w_j \otimes \varepsilon^{m-w}_{s-j}$. The right hand side of Equation (5.2) is given by

$$\begin{aligned}
&(\eta \otimes 1 - 1 \otimes \eta)\Delta_{\mathbb{K}}(\varepsilon^m_r) \\
&= (\eta \otimes 1 - 1 \otimes \eta)\Big[\sum_{w=0}^{m} \sum_{j=max\{0,s+w-m\}}^{min\{w,s\}} (-q)^{j(m-s+j-w)} \varepsilon^w_j \otimes \varepsilon^{m-w}_{s-j}\Big].
\end{aligned}$$

Since $\eta$ is an $n$-cocycle, we consider only the case $(\eta \otimes 1)((-q)^{r(m-s+r-n)}\varepsilon^n_r \otimes \varepsilon^{m-n}_{s-r})$ that is $w = n, j = r$ and $(1 \otimes \eta)((-q)^{(s-r)(n-r)}\varepsilon^{m-n}_{s-r} \otimes \varepsilon^n_r)$ that is $m - w = n, s - j = r$. The last expression therefore becomes

$$\begin{aligned}
&(-q)^{r(m-s+r-n)}\eta(\varepsilon^n_r)\varepsilon^{m-n}_{s-r} - (-1)^{n(m-n)}(-q)^{(s-r)(n-r)}\varepsilon^{m-n}_{s-r}\eta(\varepsilon^n_r) \\
&= (-q)^{r(m-s+r-n)}e_1\varepsilon^{m-n}_{s-r} - (-1)^{n(m-n)}(-q)^{(s-r)(n-r)}\varepsilon^{m-n}_{s-r}e_1.
\end{aligned}$$

We recall that since $s - r \neq m - n + 1$, $o(f^{m-n}_{s-r}) = t(f^{m-n}_{s-r}) = e_1$. Therefore

$$\begin{aligned}
e_1\varepsilon^{m-n}_{s-r} &= e_1(0, \cdots, 0, o(f^{m-n}_{s-r}) \otimes_k t(f^{m-n}_{s-r}), 0, \cdots, 0) \\
&= (0, \cdots, 0, e_1o(f^{m-n}_{s-r}) \otimes_k t(f^{m-n}_{s-r}), 0, \cdots, 0) \\
&= (0, \cdots, 0, e_1^2 \otimes_k t(f^{m-n}_{s-r}), 0, \cdots, 0) \\
&= (0, \cdots, 0, e_1 \otimes_k t(f^{m-n}_{s-r}), 0, \cdots, 0) \\
&= (0, \cdots, 0, o(f^{m-n}_{s-r}) \otimes_k t(f^{m-n}_{s-r}), 0, \cdots, 0) = \varepsilon^{m-n}_{s-r}
\end{aligned}$$

and $\varepsilon^{m-n}_{s-r}e_1 = \varepsilon^{m-n}_{s-r}$. Therefore, Equation (5.2) becomes

$$(d\psi_\eta - (-1)^{(n-1)}\psi_\eta d)(\varepsilon^m_r) = (-q)^{r(m-s+r-n)}\varepsilon^{m-n}_{s-r} - (-1)^{n(m-n)}(-q)^{(s-r)(n-r)}\varepsilon^{m-n}_{s-r} = 0.$$

since both $n$ and $r$ are even and $q = \pm 1$. □

**Remark 5.6.** *We recall from Remark 5.3 that for Koszul algebras, we can take the first homotopy lifting map $(\psi_\eta)_{n-1} : \mathbb{K}_{n-1} \to \mathbb{K}_0$ to be the zero map. From the result of Theorem 5.5, $d(\psi_\eta)_n = (-1)^{n-1}(\psi_\eta)_{n-1}d = 0$, so we see that we can define all homotopy lifting maps $\psi_\eta = 0$ for all $n$. This means that if $\eta$ and $\bar{\eta}$ are two non-nilpotent cocycles their Gerstenhaber bracket $[\eta, \bar{\eta}] = \eta\psi_{\bar{\eta}} + \bar{\eta}\psi_\eta = 0$. Refer to* [8] *on the general Gerstenhaber algebra structure of Koszul algebras defined by quivers and relations.*

**Definition 5.7.** *Lie subalgebras.*

1. *(1) Let $(\mathfrak{g}, [\cdot,\cdot]_g)$ be a Lie algebra. A subspace $\mathfrak{a}$ of $\mathfrak{g}$ is said to be a Lie subalgebra if it is closed under the Lie bracket that is $[x, y]_g \in \mathfrak{a}$ for all $x, y \in \mathfrak{a}$.*
2. *(2) The subalgebra $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ if $[x, z]_g \in \mathfrak{a}$ for all $x \in \mathfrak{a}, z \in \mathfrak{g}$.*

**Definition 5.8.** *Gerstenhaber ideal: Let $S \subset (\mathrm{HH}^*(\Lambda_q), \smile, [\cdot,\cdot])$ be a subset of homogeneous elements. The weak Gerstenhaber ideal generated by $S$ is the smallest homogeneous ideal $G(S)$ containing $S$ such that $[\gamma, \gamma'] \in G(S)$ for all $\gamma, \gamma' \in G(S)$ and $\gamma \smile \phi \in G(S)$ for all $\phi \in \mathrm{HH}^*(\Lambda_q)$. It is a (strong) Gerstenhaber ideal generated by $S$ if in addition $[\gamma, \phi] \in G(S)$ for all $\phi \in \mathrm{HH}^*(\Lambda_q)$.*

The next result shows that for $\Lambda_q, q = \pm 1$, Hochschild cohomology modulo the weak Gerstenhaber ideal generated by homogeneous nilpotent elements is not finitely generated thus providing an answer to a question of R. Hermann in [6, Question 9.8].

**Theorem 5.9.** *Let $k$ be a field and $\Lambda_q, q = \pm 1$ be members of the family of quiver algebras of* (1.1)*. Let $\mathcal{N}$ be the set of homogeneous nilpotent elements of* $\mathrm{HH}^*(\Lambda_q)$*, and $G(\mathcal{N})$ the weak Gerstenhaber ideal generated by $\mathcal{N}$. Then* $\mathrm{HH}^*(\Lambda_q)/G(\mathcal{N}) \cong \mathrm{HH}^*(\Lambda_q)/\mathcal{N}$*.*

*Proof.* We will show that $G(\mathcal{N}) = \mathcal{N}$ for $\Lambda_{\pm 1}$. Let $\chi$ be a nilpotent $n$-cocycle and $\theta$ a nilpotent $m$-cocycle. We now consider

$$[\chi, \theta](\varepsilon_r^{m+n-1}) = \chi\psi_\theta(\varepsilon_r^{m+n-1}) - (-1)^{(m-1)(n-1)}\theta\psi_\eta(\varepsilon_r^{m+n-1}).$$

The homotopy lifting map $\psi_\theta : \mathbb{K}_{m-n+1} \to \mathbb{K}_n$ is a $\Lambda^e$-module homomorphism, and would therefore take a basis element $\varepsilon_r^{m+n-1}$ a $\Lambda_q^e$-linear combination of basis elements. These combinations can take any of the following form: $\varepsilon_s^n, f_i^1\varepsilon_s^n, \varepsilon_s^n f_j^1, f_i^1\varepsilon_s^n f_j^1, f_i^2\varepsilon_s^n, \varepsilon_s^n f_j^2$ or $f_i^2\varepsilon_s^n f_j^2$ for some $s, i, j$. Since $\chi$ is nilpotent, by Corollary 4.9, $\chi\psi_\theta(\varepsilon_s^{m+n-1}) \neq e_1$ for any $s$. Applying the same reasoning, it is obvious that $\theta\psi_\chi(\varepsilon_s^{m+n-1}) \neq e_1$ either. This implies that $[\chi, \theta]$ is nilpotent. Since the choice of $\chi$ and $\theta$ were arbitrary, we conclude in this case that $[\mathcal{N}, \mathcal{N}] \subseteq \mathcal{N}$. Now suppose that $\theta = \begin{pmatrix} 0 & \cdots & 0 & (w)^s & 0 & \cdots & 0 \end{pmatrix}$, that is $\theta(\varepsilon_s^m) = w \neq e_1$ for some $w \in \Lambda_q$ whose terminal vertex is $e_1$ and $\phi$ is an $n$-cocycle defined by $\phi = \begin{pmatrix} 0 & \cdots & 0 & (e_1)^r & 0 & \cdots & 0 \end{pmatrix}$, we have seen from Theorem 3.8 and Lemma 4.7 that $(\theta \smile \phi)(\varepsilon_i^{m+n}) = 0$ or

$$(\theta \smile \phi)(\varepsilon_i^{m+n}) = \sum_{j=max\{0,i-n\}}^{min\{m,i\}} (-q)^{j(n-i+j)}\theta(\varepsilon_j^m)\phi(\varepsilon_{i-j}^n) = (-q)^{j(n-i+j)}we_1 \neq e_1$$

whenever $j = s$ and $i - j = r$. This means that $(\theta \smile \phi)$ is nilpotent for any $\phi$. Therefore $\mathcal{N}$ is an ideal with respect to the cup product. The weak Gerstenhaber ideal of $\mathcal{N}$ is therefore $\mathcal{N}$. We want to point out that $G(\mathcal{N})$ is not a (strong) Gerstenhaber ideal because the bracket of a nilpotent and a non-nilpotent cocycle can yield a non-nilpotent cocycle. □

**Some bracket computations:** We now calculate the bracket for some nilpotent cocycles. We refer to [8, Examples section] for more examples of homotopy lifting maps for quiver algebras and [9] for more examples of homotopy lifting maps for twisted tensor product algebras. Consider the nilpotent 2-cocycle $\chi$ and the nilpotent 1-cocycle $\theta$:

$$\chi = \begin{pmatrix} a & 0 & 0 & 0 \end{pmatrix}, \ \theta = \begin{pmatrix} a & 0 & 0 \end{pmatrix}$$

Calculations show that the first three homotopy lifting maps $\chi_i : \mathbb{K}_i \to \mathbb{K}_{i-1}, \theta_j : \mathbb{K}_j \to \mathbb{K}_j, i, j = 1, 2, 3$ associated to $\chi$ and $\theta$ are given by

$$\begin{array}{lll}
\chi_1(\varepsilon_0^1) = 0, \theta_1(\varepsilon_0^1) = \varepsilon_0^1 & \chi_2(\varepsilon_0^2) = \varepsilon_0^1, \theta_2(\varepsilon_0^2) = 2\varepsilon_0^2 & \chi_3(\varepsilon_0^3) = 0, \theta_3(\varepsilon_0^3) = 3\varepsilon_0^3 \\
\chi_1(\varepsilon_1^1) = 0, \theta_1(\varepsilon_1^1) = 0 & \chi_2(\varepsilon_1^2) = 0, \theta_2(\varepsilon_1^2) = \varepsilon_1^2 & \chi_3(\varepsilon_1^3) = \varepsilon_1^2, \theta_3(\varepsilon_1^3) = 2\varepsilon_1^3 \\
\chi_1(\varepsilon_2^1) = 0, \theta_1(\varepsilon_2^1) = 2\varepsilon_2^1 & \chi_2(\varepsilon_2^2) = 0, \theta_2(\varepsilon_2^2) = 0 & \chi_3(\varepsilon_2^3) = 0, \theta_3(\varepsilon_2^3) = 1\varepsilon_2^3 \\
 & \chi_2(\varepsilon_3^2) = 0 & \chi_3(\varepsilon_3^3) = 0, \theta_2(\varepsilon_2^2) = 0 \\
 & & \chi_3(\varepsilon_4^3) = \varepsilon_3^2, \theta_3(\varepsilon_4^3) = 4\varepsilon_4^3.
\end{array}$$

These maps satisfy Equation (5.2). We verify Equation (5.2) for $\chi$ only as follows:

$$(d_2\chi_3 + \chi_2 d_3)(\varepsilon_0^3) = d_2(0) + \chi_2(a\varepsilon_0^2 - \varepsilon_0^2 a) = a\varepsilon_0^1 - \varepsilon_0^1 a$$
$$(d_2\chi_3 + \chi_2 d_3)(\varepsilon_1^3) = d_2(\varepsilon_1^2) + \chi_2(a\varepsilon_1^2 + q\varepsilon_1^2 a + q^2 b\varepsilon_0^2 - \varepsilon_0^2 b) = a\varepsilon_1^1 - \varepsilon_1^1 a$$
$$(d_2\chi_3 + \chi_2 d_3)(\varepsilon_2^3) = d_2(0) + \chi_2(a\varepsilon_2^2 + q^2\varepsilon_2^2 a - qb\varepsilon_1^2 - \varepsilon_1^2 b) = 0$$
$$(d_2\chi_3 + \chi_2 d_3)(\varepsilon_3^3) = d_2(0) + \chi_2(b\varepsilon_2^2 - \varepsilon_2^2 b) = 0$$
$$(d_2\chi_3 + \chi_2 d_3)(\varepsilon_4^3) = d_2(\varepsilon_3^2) + \chi_2(a\varepsilon_3^2 - \varepsilon_0^2 c) = a\varepsilon_2^1$$

On the other hand, using Koszul signs in the expansion of $(1 \otimes \chi)(\varepsilon_r^n \otimes \varepsilon_s^m)$ to obtain $(-1)^{|\eta|n}\varepsilon_r^n \eta(\varepsilon_s^m)$, we get

$$(\chi \otimes 1 - 1 \otimes \chi)\Delta_{\mathbb{K}}(\varepsilon_0^3) = (\chi \otimes 1 - 1 \otimes \chi)\big[\varepsilon_0^0 \otimes \varepsilon_0^3 + \varepsilon_0^1 \otimes \varepsilon_0^2 + \varepsilon_0^2 \otimes \varepsilon_0^1 + \varepsilon_0^3 \otimes \varepsilon_0^0\big] = a\varepsilon_0^1 - \varepsilon_0^1 a$$
$$\begin{aligned}(\chi \otimes 1 - 1 \otimes \chi)\Delta_{\mathbb{K}}(\varepsilon_0^3) &= (\chi \otimes 1 - 1 \otimes \chi)\big[\varepsilon_0^0 \otimes \varepsilon_1^3 + \varepsilon_0^1 \otimes \varepsilon_1^2 + q^2\varepsilon_1^1 \otimes \varepsilon_0^2 + \varepsilon_0^2 \otimes \varepsilon_1^1 - q\varepsilon_1^2 \otimes \varepsilon_0^1 + \varepsilon_1^3 \otimes \varepsilon_0^0\big] \\ &= a\varepsilon_1^1 - q^2\varepsilon_1^1 a\end{aligned}$$
$$(\chi \otimes 1 - 1 \otimes \chi)\Delta_{\mathbb{K}}(\varepsilon_0^3) = (\chi \otimes 1 - 1 \otimes \chi)\big[\varepsilon_0^0 \otimes \varepsilon_2^3 + \varepsilon_0^1 \otimes \varepsilon_2^2 + q\varepsilon_1^1 \otimes \varepsilon_1^2 - \varepsilon_1^2 \otimes \varepsilon_1^1 + \varepsilon_2^2 \otimes \varepsilon_0^1 + \varepsilon_2^3 \otimes \varepsilon_0^0\big] = 0$$
$$(\chi \otimes 1 - 1 \otimes \chi)\Delta_{\mathbb{K}}(\varepsilon_0^3) = (\chi \otimes 1 - 1 \otimes \chi)\big[\varepsilon_0^0 \otimes \varepsilon_3^3 + \varepsilon_1^1 \otimes \varepsilon_2^2 + \varepsilon_2^2 \otimes \varepsilon_1^1 + \varepsilon_3^3 \otimes \varepsilon_0^0\big] = 0$$
$$(\chi \otimes 1 - 1 \otimes \chi)\Delta_{\mathbb{K}}(\varepsilon_0^3) = (\chi \otimes 1 - 1 \otimes \chi)\big[\varepsilon_0^0 \otimes \varepsilon_4^3 + \varepsilon_0^1 \otimes \varepsilon_3^2 + \varepsilon_0^2 \otimes \varepsilon_2^1 + \varepsilon_4^3 \otimes \varepsilon_0^0\big] = a\varepsilon_2^1$$

So we see that $(d_2\chi_3 + \chi_2 d_3)(\varepsilon_i^3) = (\chi \otimes 1 - 1 \otimes \chi)\Delta_{\mathbb{K}}(\varepsilon_i^3), \quad i = 0, 1, 2, 3, 4$. It is easy to verify as well that $[\chi, \theta] = \chi$ that is

$$[\chi, \theta](\varepsilon_0^2) = \chi\theta_2(\varepsilon_0^2) - \theta\chi_2(\varepsilon_0^2) = \chi(2\varepsilon_0^2) - \theta(\varepsilon_0^1) = 2a - a = a$$
$$[\chi, \theta](\varepsilon_i^2) = \chi\theta_2(\varepsilon_i^2) - \theta\chi_2(\varepsilon_i^2) = 0, \text{for } i = 1, 2, 3.$$

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