Mat 648

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Deformation of algebras and Hochschild Cohomology

k is a field.

bilinear map

A is a k-algebra $(A \times A \longrightarrow A)$

Def: Let R be a ring with unit, and $s: k \to R$, a hom. such that $s(1) = 1_R$. A map $\boxed{\epsilon : R \to k}$ is an augmentation map if $\epsilon s = 1_k$.

The two sided ideal $\operatorname{Ker} \epsilon$ is called the augmentation ideal.

Examples

(1) $R = k[[t]] = \{ \sum_{i \geq 0} \lambda_i t^i \mid \lambda_i \in k \}$

$\epsilon : R \longrightarrow k$

$\left(\sum_{i \geq 0} \lambda_i t^i \mapsto \lambda_0 \right)$

(2) $R = k[G]$, where G is a finite group.

$$\left(\sum_{g \in G} \lambda_g g\right)\left(\sum_{h \in G} \beta_h h\right) = \sum_{gh = z} c_z z$$

$\epsilon : R \longrightarrow k$

$\sum_{g \in G} \lambda_g g \mapsto \sum \lambda_g$

Multiplication on Augmented Rings

Ex 1: Let $A = k[x,y]$.

$$* : A[t] \times A[t] \longrightarrow A[t]$$

defined by $x^i * x^j = x^{i+j}$ $\quad x^i * y^j = x^i y^j$

$y^i * y^j = y^{i+j}$ $\quad y * x = xy + t$

{Can also be seen as a ring of differential operators with poly. coefficients.}

and require that $*$ is associative.

What we get is a family of algebras denoted A_t.

when $t = 1$; $A_1 = k_1[x,y] = k\langle x,y \rangle / \langle yx - xy - 1 \rangle$. } Weyl algebra.

$$\{A = k[x,y]\} \longleftrightarrow \left\{ \begin{array}{c} \text{Deformation} \\ \text{of algebra} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Hochschild} \\ \text{cohomology} \end{array} \right\}$$

e.g. Weyl $\qquad$ 2-cocycles

Def: Let A be a k-algebra and R an augmented unital ring. An R-deformation of A is an associative R-algebra B and an isomorphism $\phi : \bar{B} \to A$, where $\bar{B}$ is the "reduction" of B. (restriction of B)

$$A \xrightarrow{R} (A_t, *),$$ think of B as $\{A_t\}_t$ family of algebras and $\bar{B} = A_t \big|_{t=0}$.

Remark:

(1) An R deformation of an algebra A is also denoted by $(B, \bar{B} \xrightarrow{\phi} A)$.

(2) The multiplication ϕ on B $(= * \text{ on } A_t)$ is determined by its restriction to A.

(3) Equivalence of deformations: Two R-deformations $(B_1, \bar{B}_1 \xrightarrow{\phi_1} A)$, $(B_2, \bar{B}_2 \xrightarrow{\phi_2} A)$ are equivalent if there is $w : B_1 \to B_2$ s.t. the following diagram commutes;

$$\begin{tikzcd} \bar{B}_1 \subset B \arrow[r, "\phi_1"] \arrow[d, "w"'] & A \arrow[d, equal] \\ \bar{B}_2 \subset B_2 \arrow[r, "\phi_2"] & A \end{tikzcd}$$

$w = \phi_2^{-1} \phi_1$.

Def: Let A be a k-algebra. R any ring.

An R-deformation of A is called a formal deformation whenever $R = k[[t]]$.

It is called an infinitesimal deformation if $R = k[t]/(t^2)$.

Remark: $A = k[x,y]$

In the example $\boxed{y * x = xy + t}$

take $f = y^2$, $g = x^2$

$$f * g = y^2 * x^2 = y * (y * x) * x = \cdots$$

$$= x^2 y^2 + 4xyt + 2t^2 \longrightarrow x^2y^2 + \left[\frac{\partial}{\partial x}(g) \cdot \frac{\partial}{\partial y}(f)\right] t + \frac{1}{2!}\left[\frac{\partial^2}{\partial x^2}(g) \cdot \frac{\partial^2}{\partial x^2}(f)\right] t^2$$

$$= \mu_0(f,g) + \mu_1(f,g)t + \mu_2(f,g)t^2$$

For maps $\mu_i : A \times A \longrightarrow A$. $\quad i = 0,1,2$.

This is showing the fact that whenever you have an R-deformation of an algebra $(A_t, *)$, there are maps $\mu_i : A \times A \to A$ such that $\forall\, a, b \in A$

$$\boxed{a * b = \sum_{i \geq 0} \mu_i(a,b) t^i}$$

Theorem:

Let $(A_t, *)$ be a formal deformation of the algebra A. Then A_t is given by a family of maps $\{\mu_i : A \times A \to A \mid i \in A\}$ such that $\forall\, a,b,c \in A$, $\mu_0(a,b) = ab$.

$$\boxed{\sum_{\substack{i+j=k \\ i,j \geq 0}} \mu_i(\mu_j(a,b),c) = \sum_{\substack{i+j=k \\ i,j \geq 0}} \mu_i(a, \mu_j(b,c))} \quad (D_k)$$

$$A \longrightarrow \{(A_t, *)\} \longrightarrow \{\mu_i : A \times A \to A \mid i \in \mathbb{N}\}$$

Proof: From the associativity of $*$ we obtain the relations (D_k)

Remark: D_0: $(i,j) = (0,0)$

$\mu_0(\mu_0(a,b),c) = \mu_0(a, \mu_0(b,c))$

$(ab)c = a(bc) \implies$ associativity in A.

D_1: $(i,j) = (1,0)$, $(0,1)$. So we get

$$\mu_0(\mu_1(a,b),c) + \mu_1(\mu_0(a,b),c) = \mu_0(a, \mu_1(b,c)) + \mu_1(a, \mu_0(b,c))$$

$$\boxed{\mu_1(a,b)c + \mu_1(ab,c) = a\mu_1(b,c) + \mu_1(a,bc)}$$

Hochschild Cohomology

$$\mathbb{B}_\cdot : \cdots \to A^{\otimes(n+2)} \xrightarrow{d_n} A^{\otimes(n+1)} \to \cdots \to A^{\otimes 2} (\to A)$$

$$\begin{array}{ccc} A^{\otimes(n+2)} & & A^{\otimes(n+1)} \\ \| & & \| \\ \mathbb{B}_n & \longrightarrow & \mathbb{B}_{n-1} \end{array}$$

$$d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

Apply the functor $\operatorname{Hom}_{A^e}(\mathbb{B}_\cdot, A)$ where $A^e = A \otimes A^{op}$, take the cohomology of the resulting complex

$$\operatorname{Hom}_{A^e}(\mathbb{B}_{n-1}, A) \xrightarrow{d_n^*} \operatorname{Hom}_{A^e}(\mathbb{B}_n, A) \xrightarrow{d_{n+1}^*} \operatorname{Hom}_{A^e}(\mathbb{B}_{n+1}, A)$$

$$H^n(\operatorname{Hom}_{A^e}(\mathbb{B}_n, A)) =: \frac{\operatorname{Ker} d_{n+1}^*}{\operatorname{Im} d_n^*}$$

$$HH^*(A) = \bigoplus_{n \geq 0} H^n(\operatorname{Hom}_{A^e}(\mathbb{B}_\cdot, A))$$

$$= \bigoplus_{n \geq 0} \operatorname{Ext}^n_{A^e}(A, A)$$

Consider a module homomorphism $f : \mathbb{B}_2 \longrightarrow A$

$f : A^{\otimes 4} \longrightarrow A$ (module homo over A^e)

that is a cocycle. i.e. $fd = 0$

$$\begin{tikzcd} \cdots \arrow[r] & A^{\otimes 5} \arrow[r, "d"] \arrow[dr] & A^{\otimes 4} \arrow[r] \arrow[d, "f"] & A^{\otimes 3} \arrow[r] & \cdots \\ & & A & & \end{tikzcd}$$

$\| $ (module homo over k)

$f(x \otimes y \otimes z \otimes p) \to x f(y \otimes z) p$

$$fd(1 \otimes a \otimes b \otimes c \otimes 1) = f\big[a \otimes b \otimes c \otimes 1 - 1 \otimes ab \otimes c \otimes 1 + 1 \otimes a \otimes bc \otimes 1 - 1 \otimes a \otimes b \otimes c\big]$$

$$= f(a \otimes b \otimes c \otimes 1) - f(1 \otimes ab \otimes c \otimes 1) + f(1 \otimes a \otimes bc \otimes 1) - f(1 \otimes a \otimes b \otimes c)$$

$$= af(b \otimes c) - f(ab \otimes c) + f(a \otimes bc) - f(a \otimes b)c = 0$$

$$\boxed{af(b \otimes c) + f(a \otimes bc) = f(ab \otimes c) + f(a \otimes b)c.}$$

$$\boxed{\mu_1(a,b)c + \mu_1(ab,c) = a\mu_1(b,c) + \mu_1(a,bc)}$$

Theorem: There is a 1-1 correspondence between the space of Hochschild 2-cocycles and the equivalence classes of infinitesimal deformation of an algebra A.

Sketch of Proof: $A \xrightarrow{\text{deformations}} (B_1, \bar{B}_1 \xrightarrow{\phi_1} A)$

$\qquad\qquad (B_2, \bar{B}_2 \xrightarrow{\phi_2} A)$

$\forall\, x, y \in A$

$\phi_1(x,y) = xy + t\mu(x,y)$

$\phi_2(x,y) = xy + t\bar{\mu}(x,y)$

Here μ and $\bar{\mu}$ are Hochschild 2 cocycles.

Use the property that B_1 and B_2 are equivalent to show that there is a map $w : B_1 \longrightarrow B_2$ satisfying

$$w(\phi_1(x,y)) = \phi_2(w(x,y))$$

w is determined by its restriction A. In particular,

$w(a) = a + t\bar{w}(a)$, where $\bar{w} : A \longrightarrow A$

We eventually have

$$\mu(x,y) - \bar{\mu}(x,y) = \bar{w}\, d(x,y)$$

$\implies \mu$ and $\bar{\mu}$ differ by a coboundary. $\boxtimes$

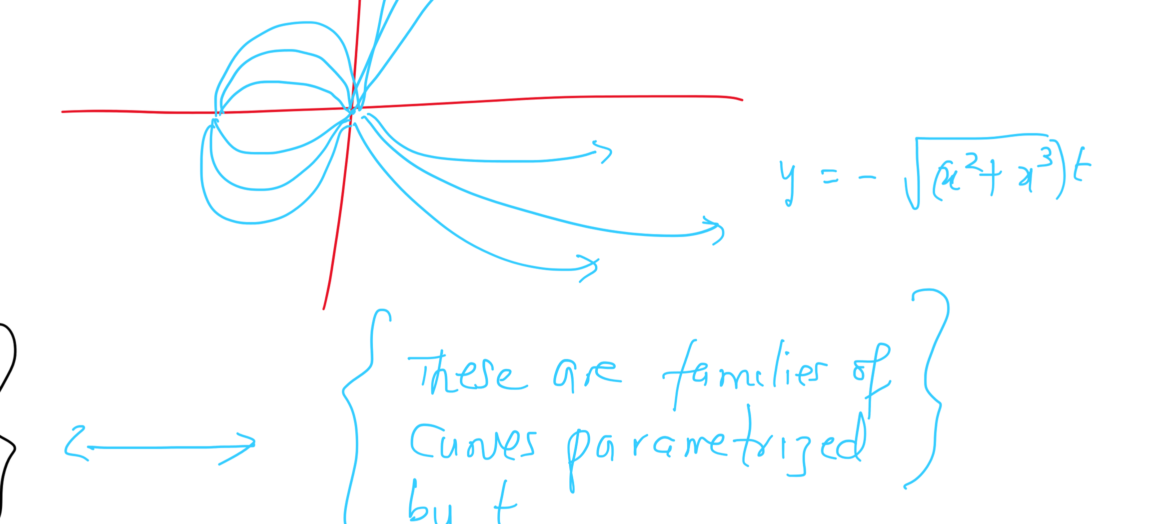
Drawing Connections to geometry.

$$A = k[x,y]/(y^2).$$

$$* : A[t] \times A[t] \longrightarrow A[t]$$

$$y * y = (x^2 + x^3)t$$

$$A_t = k\langle x,y \rangle / (y^2 - (x^2 + x^3)t)$$



{these are families of algebras or deformations of A.} $\longleftrightarrow$ {These are families of curves parametrized by t}

$\Updownarrow \qquad\qquad \Updownarrow$

{the space of Hochschild 2 cocycles} $\overset{??}{\longleftrightarrow}$ {encoding topologically distinct singularity types in the Spectrum of A_t}