Hochschild 2-cocycles and equivalent classes of split short exact sequence

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1 Some Motivation

This is a short note that shows how to view Hochschild 2-cocycles as equivalent classes of split short exact sequence. These split short exact sequences are also called split abelian extensions and they give rise to square-zero extensions. It was motivated as result of my discussion with mathematicians during the AMS sectional meeting at the University of Florida in Gainesville.
Throughout, take k be a field and A a k-algebra.

Definition 1.1. *A square-zero extension of an algebra A is another algebra R such that $A \cong R/I$ for some ideal I of R satisfying $I^2 = 0$.*

Remark 1.2. .

1. *I is viewed as an A-bimodule by taking for any $a = r + I \in A$, $r \in R$ and $x \in I$, let $ax = (r + I)x = rx \in I$ and similarly $xa = xr$. Hence we have a split short exact sequence i.e. $R \cong I \oplus A$.*
$$I \to R \to A$$
2. *Every A-bimodule M determines a square-zero extension R of A. Let $R = A \oplus M$. Then $(R, +, \cdot)$ is a ring with the addition and multiplication defined as*
$$(a_1 \oplus m_1) + (a_2 \oplus m_2) = (a_1 + a_2) \oplus (m_1 + m_2)$$
$$(a_1 \oplus m_1) \cdot (a_2 \oplus m_2) = (a_1 a_2) \oplus (a_1 m_2 + m_1 a_2)$$
Again, we have a split short exact sequence
$$M \to R \to A$$
3. *This means that every square-zero extension of A determines a split short exact sequence and for every A-bimodule M, there is a square-zero extension of A given by $R = A \oplus M$ such that $M^2 = 0$.*

2 Hochschild cohomology

Definition 2.1. *Let*

$$\mathbb{B}(A) := \cdots \longrightarrow A^{\otimes(n+2)} \xrightarrow{d_n} A^{\otimes(n+1)} \to \cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A^{\otimes 2} \longrightarrow 0,$$

be the bar resolution with differentials given by

$$d_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1},$$

for all $n \geq 1$.

Definition 2.2. *We define the space* $\mathrm{HH}^2(A, M)$ *of Hochschild 2-cocycles with coefficients in* M *as the set of all* $f \in \mathrm{Hom}_k(A^{\otimes 2}, M) \cong \mathrm{Hom}_{A \otimes A^{op}}(A^{\otimes 4}, M)$ *such that* $d^* f = 0$ *modulo the set of all* $g \in \mathrm{Hom}_k(A^{\otimes 2}, M)$ *such that* $d^* h = g$ *for some* $h \in \mathrm{Hom}_k(A^{\otimes 3}, M)$. *Essentially, this is the* $\ker d^* / \operatorname{Im} d^*$, *without paying attention to the indices. The isomorphism* $\mathrm{Hom}_{A \otimes A^{op}}(A^{\otimes 4}, M) \cong \mathrm{Hom}_k(A^{\otimes 2}, M)$ *above is defined using*

$$f(a_0 \otimes a_1 \otimes a_2 \otimes a_3) \mapsto a_0 f(a_1 \otimes a_2) a_3. \tag{2.3}$$

We take $f \in \mathrm{HH}^2(A, M)$, *a representative of a class and observe that*

$$\begin{aligned}
&d^* f(1 \otimes a_1 \otimes a_2 \otimes a_3 \otimes 1) = 0, \qquad \textit{implies that} \\
&f d(1 \otimes a_1 \otimes a_2 \otimes a_3 \otimes 1) = f(a_1 \otimes a_2 \otimes a_3 \otimes 1) - f(1 \otimes a_1 a_2 \otimes a_3 \otimes 1) \\
&+ f(1 \otimes a_1 \otimes a_2 a_3 \otimes 1) - f(1 \otimes a_1 \otimes a_2 \otimes a_3) = 0 \\
&\ \textit{using the isomorphism of } (2.3) \textit{ we must have} \\
&a_1 f(a_2 \otimes a_3) - f(a_1 a_2 \otimes a_3) + f(a_1 \otimes a_2 a_3) - f(a_1 \otimes a_2) a_3 = 0.
\end{aligned}$$

Theorem 2.4. *Let* M *be an* A*-bimodule, there is a* $1-1$ *correspondence between* $HH^2(A, M)$ *and the equivalence classes of split short exact sequence (OR square-zero extensions of* A*, OR split abelian extension of* A *)*

We have not mentioned what it means for short exact sequences to be equivalent. This is a common term in homological algebra and can be found in many homological algebra texts.

Proof. Suppose that $f \in HH^2(A, M)$ is a Hochschild 2-cocycle, that is

$$a_1 f(a_2 \otimes a_3) + f(a_1 \otimes a_2 a_3) = f(a_1 a_2 \otimes a_3) + f(a_1 \otimes a_2) a_3$$

The following is a split short exact sequence

$$M \to R \to A$$

where $A \oplus M \cong (R, +, \cdot)$ is a ring with the addition and multiplication defined as

$$(a_1 \oplus m_1) + (a_2 \oplus m_2) = (a_1 + a_2) \oplus (m_1 + m_2)$$
$$(a_1 \oplus m_1) \cdot (a_2 \oplus m_2) = (a_1 a_2) \oplus (a_1 m_2 + m_1 a_2 + f(a_1 \otimes a_2))$$

We note that
(1) $M^2 = 0$ as an A- bi submodule of R. This is true since $M \cong 0 + M$

$$(0 \oplus m_1) \cdot (0 \oplus m_2) = 0 \oplus (0m_2 + m_1 0 + f(0 \otimes 0)) = 0$$

(2) Associativity of the multiplication uses the property that f is a Hochschild 2-cocycle. For all $a_1, a_2, a_3 \in A, m_1, m_2, m_3 \in M$,

$$\begin{aligned}
&(a_1 \oplus m_1) \cdot [(a_2 \oplus m_2) \cdot (a_3 \oplus m_3)] \\
&= (a_1 \oplus m_1)[(a_2 a_3) \oplus (a_2 m_3 + m_2 a_3 + f(a_2 \otimes a_3))] \\
&= (a_1 a_2 a_3) \oplus (a_1(a_2 m_3 + m_2 a_3 + f(a_2 \otimes a_3)) + m_1(a_2 a_3 + f(a_1 \otimes a_2 a_3) \\
&= (a_1 a_2 a_3) \oplus (a_1 a_2 m_3 + a_1 m_2 a_3 + m_1 a_2 a_3 + a_1 f(a_2 \otimes a_3) + f(a_1 \otimes a_2 a_3)
\end{aligned}$$

$$\begin{aligned}
&[(a_1 \oplus m_1) \cdot (a_2 \oplus m_2)] \cdot (a_2 \oplus m_2) \\
&= (a_1 a_2) \oplus (a_1 m_2 + m_1 a_2 + f(a_1 \otimes a_2))] \cdot (a_3 \oplus m_3) \\
&= (a_1 a_2 a_3) \oplus (a_1 a_2 m_3 + (a_1 m_2 + m_1 a_2 + f(a_1 \otimes a_2))a_3 + f(a_1 a_2 \otimes a_3) \\
&= (a_1 a_2 a_3) \oplus (a_1 a_2 m_3 + a_1 m_2 a_2 + m_1 a_2 a_3 + f(a_1 \otimes a_2)a_3 + f(a_1 a_2 \otimes a_3)
\end{aligned}$$

and we have equality.
Conversely, Suppose $M \overset{w}{\to} R \overset{\pi}{\to} A$ is a split short exact sequence (or a square-zero extension of A), then there is a map $\tau : A \to R$ such that $\pi\tau = 1_A$. Since we want equivalent classes of split short exact sequence, the map τ is unique to a class. We will find a Hochschild 2-cocycle with respect to τ. Let

$$f_\tau : A \otimes A \to M$$

be defined by $f_\tau(a_1 \otimes a_2) = \tau(a_1 a_2) - \tau(a_1)\tau(a_2)$ Observe that since A, M are A-bimodules, R is also an A-bimodule, hence τ an A-bimodule homomorphism i.e $\tau(a_1 a_2) = a_1\tau(a_2) = \tau(a_1)a_2$. We will now show that f_τ is an Hochschild 2-cocycle.

$$\begin{aligned}
&d^* f_\tau(1 \otimes a_1 \otimes a_2 \otimes a_3 \otimes 1) = f_\tau d(1 \otimes a_1 \otimes a_2 \otimes a_3 \otimes 1) \\
&= f_\tau(a_1 \otimes a_2 \otimes a_3 \otimes 1) - f_\tau(1 \otimes a_1 a_2 \otimes a_3 \otimes 1) \\
&+ f_\tau(1 \otimes a_1 \otimes a_2 a_3 \otimes 1) - f_\tau(1 \otimes a_1 \otimes a_2 \otimes a_3) \\
&\text{using the isomorphism of (2.3) we have} \\
&= a_1 f_\tau(a_2 \otimes a_3) - f_\tau(a_1 a_2 \otimes a_3) + f_\tau(a_1 \otimes a_2 a_3) - f_\tau(a_1 \otimes a_2)a_3 \\
&= a_1\tau(a_2 a_3) - a_1\tau(a_2)\tau(a_3) - \tau(a_1 a_2 a_3) + \tau(a_1 a_2)\tau(a_3) \\
&+ \tau(a_1 a_2 a_3) - \tau(a_1)\tau(a_2 a_3) - \tau(a_1 a_2)a_3 + \tau(a_1)\tau(a_2)a_3 = 0.
\end{aligned}$$

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References

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[2] S. Witherspoon, *Hochschild Cohomology for Algebras*, Graduate Studies in Mathematics, American Mathematical Society.