CLIMBING THE LATTER OF CAUSATION WITH tnreason

RESEARCH NOTES IN THE ENEXA AND QROM PROJECTS

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ABSTRACT

We here describe how to represent Causal AI models in the tnreason formalism.

We find tensor network models for the different rungs of the *latter of causation* Pearl and Mackenzie (2018)

- **Rung 1: Association** – captured by Bayesian Networks
- **Rung 2: Intervention** – captured by do-calculus, for which we introduce intervention variables
- **Rung 3: Counterfactuals** – captured by functional causal models (see (Koller and Friedman, 2009, Chapter 21)) involving response variables

Following the formalism of Chapter 21 in Koller and Friedman (2009):

- Mutilation of the Bayesian network is captured by contractions with one-hot encodings to do-variables D_k.
- Intervention queries are formalized by contractions with one-hot encodings to all do-variables
- Response variables are formalized by selection variables
- Counterfactual queries are modeled as intervention queries on counterfactual twin networks (see (Koller and Friedman, 2009, Chapter 21))

1 Rung 1: Association

See main report: Bayesian Networks are tensor networks and probabilistic queries can be answered by their contractions.

2 Rung 2: Intervention

So far, there are several hypergraphs to model a probability distributions by a Bayesian Network. We now want to understand the directed hyperedges as causal relationships between the variables, beyond encoding of conditional independencies. To this end, we model the effects of interventions, which will distinguish between multiple Bayesian Networks modeling the same joint distribution.

Interventions on Bayesian Networks are modeled by the do-calculus **?**, which we capture here based on do-variables.

2.1 Do calculus

We introduce the do-variables D_k taking values in $[m_k + 1]$ for each variable X_k of dimension m_k, which model the effects of interventions. The states are interpreted as

- $d_k \in [m_k]$: An intervention setting X_k to the value d_k
- $d_k = m_k$: No intervention on X_k

Definition 1. *The causally augmented conditional probability core is defined as*

$$\begin{aligned}\mathbb{P}^c\left[X_k \middle| D_k, X_{\mathrm{Pa}(k)}\right] =& \mathbb{P}\left[X_k \middle| X_{\mathrm{Pa}(k)}\right] \otimes \epsilon_{m_k}\left[D_k\right] \\ &+ \sum_{d_k \in [m_k]} \epsilon_{d_k}\left[X_k\right] \otimes \epsilon_{d_k}\left[D_k\right] \otimes \mathbb{I}\left[X_{\mathrm{Pa}(k)}\right] .\end{aligned}$$

The causally augmented Bayesian Network is the tensor network

$$\left\langle \left\{ \mathbb{P}^c\left[X_k \middle| D_k, X_{\mathrm{Pa}(k)}\right] \,:\, k \in [d] \right\} \right\rangle_{\left[X_{[d]}, D_{[d]}\right]} .$$

The Bayesian Networks investigated so far are retrieved by contractions with $\epsilon_{m_k}\left[D_k\right]$. This corresponds with the situation, where no interventions are performed on any variable.

Lemma 1. *For any Bayesian Network $\mathbb{P}\left[X_{[d]}\right]$ on a directed acyclic hypergraph $\mathcal{G} = ([d], \mathcal{E})$ we have*

$$\mathbb{P}\left[X_{[d]}\right] = \left\langle \left\{ \mathbb{P}^c\left[X_k \middle| D_k, X_{\mathrm{Pa}(k)}\right] \,:\, k \in [d] \right\} \cup \left\{ \epsilon_{m_k}\left[D_k\right] \,:\, k \in [d] \right\} \right\rangle_{\left[X_{[d]}\right]} .$$

Proof. By construction we have

$$\mathbb{P}^c\left[X_k \middle| D_k = m_k, X_{\mathrm{Pa}(k)}\right] = \mathbb{P}\left[X_k \middle| X_{\mathrm{Pa}(k)}\right] .$$

Performing this on each causally augmented conditional distribution retrieves thus the original Bayesian Network. □

2.2 Intervention Queries

In a Bayesian Network, when intervening on a variable X_v, the corresponding conditional probability gets trivialized. The probability tensor is then captured by

$$\mathbb{P}\left[X_{\mathcal{U}} \middle| \mathrm{do}\left(X_{\mathcal{W}} = x_{\mathcal{W}}\right)\right] = \left\langle \left\{ \mathbb{P}\left[X_v \middle| X_{\mathrm{Pa}(v)}\right] \,:\, v \notin \mathcal{W} \right\} \cup \left\{ \frac{1}{m_v} \mathbb{I}\left[X_v, X_{\mathrm{Pa}(v)}\right] \,:\, v \in \mathcal{W} \right\} \cup \left\{ \epsilon_{x_v}\left[X_v\right] \,:\, v \in \mathcal{W} \right\} \right\rangle_{\left[X_{\mathcal{U}}\right]} .$$

Since $\frac{1}{m_v} \mathbb{I}\left[X_v, X_{\mathrm{Pa}(v)}\right]$ is directed with $X_{\mathrm{Pa}(v)}$ incoming and X_v outgoing, the partition function of the network stays 1 and the normalization is captured by the contraction.

2.3 Simplifications

The back-door and front-door criterion provide conditions for intervention queries being equal to conditional queries. We can proof them in the tensor network formalism based on network separations, which contribution to contractions are scalar multiplications, which therefore are dropped in normalizations.

An example for a simplification is given by the randomized controlled trial.

Example 1 (Randomized controlled trial)**.** *The randomized controlled trial is a joint distribution of variables*

- *Confounding variable X_C summarizing other factors affecting the outcome.*
- *Treatment assignment X_T with values in $\{0, 1\}$. Here, 1 indicates assignment to the treatment group and 0 to the control group. The experiment is conducted such that the treatment variable is independent of the confounding variable X_C.*
- *Outcome X_O .*

We want to estimate the causal effect of the treatment on the outcome, namely calculate the intervention query

$$\mathbb{P}\left[X_O \middle| D_T = 1\right] .$$

By design of the randomized control, the confounding variable has no causal effect on X_T. Thus, we have

$$\mathbb{P}\left[X_O \middle| D_T = 1\right] = \mathbb{P}\left[X_O \middle| X_T = 1\right]$$

that is, the intervention query is equal to the conditional probability.

3 Rung 3: Counterfactuals

Following the formalism of Chapter 21 in Koller and Friedman (2009), counterfactual queries are modeled as intervention queries on counterfactual twin networks. Counterfactual twin networks are two copies of the original causal model, one representing the factual world and one representing the counterfactual world. They further share response variables, which capture the functional relationships between variables in the causal model. These response variables are modeled as selection variables in the $\mathrm{tnreason}$ formalism.

4 Learning

Given a parametrization of hypothesis distribution by the states of selection variables, we can include the intervention variables. The cores to be trivialized by intervention can depend on the state of the selection variables. For example, a selection variable might select whether the cores direction is from X_0 to X_1 or in the other direction. In both cases only one of the to intervention variables D_0 and D_1 influences the core.

References

Daphne Koller and Nir Friedman. *Probabilistic Graphical Models: Principles and Techniques*. The MIT Press, Cambridge, Mass., 1. edition edition, July 2009. ISBN 978-0-262-01319-2.

Judea Pearl and Dana Mackenzie. *The Book of Why: The New Science of Cause and Effect*. Basic Books, New York, May 2018. ISBN 978-0-465-09760-9.