I-Tensors
Formalization of Tensors

Foundations of Neuro-Symbolic AI

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Summer Term 2026

Motivation: Factored Systems

We think of tensors as an alignment of real numbers, where each axis is one direction of the alignment.

- Vectors (Order-1 Tensors): Alignment in one direction
- Matrices (Order-2 Tensors): Alignment in two directions
- Order-3 Tensors: Alignment in three directions, for example:

$$\epsilon_{(0,2,1)} = \epsilon_0 \otimes \epsilon_2 \otimes \epsilon_1 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \otimes \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix} \otimes \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}$$

When encoding more than three variables we need a more abstract graphical notation!

Formalization: Tensors are Maps

Definition (Tensor)

A tensor of order $d \in \mathbb{N}$ and leg dimensions $\{m_k \,:\, k \in [d]\}$ is a map

$$\tau : \bigtimes_{k \in [d]} [m_k] \to \mathbb{R} \,.$$

- We refer to $\tau(x_0, \ldots, x_{d-1}) \in \mathbb{R}$ by the coordinate of τ with the indices $x_0, \ldots, x_{d-1}$.
- By scalar multiplation and addition of maps, tensors build linear spaces called tensor spaces

$$\tau \in \bigotimes_{k \in [d]} \mathbb{R}^{m_k} \,.$$

Example

Example: The one-hot encoding of the state $(0, 2, 1)$ is the tensor

$$\epsilon_{(0,2,1)} \in \bigotimes_{k \in [3]} \mathbb{R}^4$$

with the coordinates

$$\epsilon_{(0,2,1)}(x_0, x_1, x_2) = \begin{cases} 1 & \text{if } x_0 = 0,\ x_1 = 2 \text{ and } x_2 = 1 \\ 0 & \text{else} \end{cases}$$

Depiction of Tensors

We will depict

- tensors by blocks
- affected categorical variables by open legs

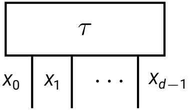
For example, given the categorical variables

$$\{X_k \,:\, k \in [d]\}$$

with respective dimensions $m_k \in \mathbb{N}$ we depict a tensor

$$\tau \in \bigotimes_{k \in [d]} \mathbb{R}^{m_k}$$

as



Example: Depiction of Vectors

For example, a vector

$$V \in \mathbb{R}^m$$

with an index variable $X \in [m]$ is depicted by a block with a single open leg