
MARKOV NETWORKS

RESEARCH NOTES IN THE ENEXA AND QROM PROJECTS

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ABSTRACT

We investigate families of Markov Networks as examples of Hybrid Logic Networks.

In general, we have the equivalence:

- There are non cube-like faces in the mean polytope to the family of Hybrid Logic Networks
- There are realizable mean parameters, which are not realizable by any member Hybrid Logic Network

1 Minimal connected hypergraphs

In the case of minimal connected hypergraphs, as we show in the following, all realizable mean parameters can be realized by a Markov Network.

Define a root hyperedge in the hypergraph and direct the hypergraph from the root to the edges. At the root hyperedge we define the activation core by the mean parameter.

For all other hyperedges e , we define activation cores $\alpha^e[X_e]$ as follows depending on the mean parameter of e and of the hyperedge \tilde{e} sharing the incoming variables (which is an edge sharing \mathcal{V}^{in} with e). For each x_e if $\langle \mu_{\tilde{e}}[X_{\tilde{e}}] \rangle [X_{\mathcal{V}^{\text{in}}} = x_{\mathcal{V}^{\text{in}}}] = 0$ then we define

$$\alpha^e[X_e = x_e] = \frac{1}{\prod_{v \in e / \{\mathcal{V}^{\text{in}}\}} m_v}.$$

If this is not the case, then

$$\alpha^e[X_e = x_e] = \frac{\mu_e[X_e = x_e]}{\langle \mu_{\tilde{e}}[X_{\tilde{e}}] \rangle [X_{\mathcal{V}^{\text{in}}} = x_{\mathcal{V}^{\text{in}}}]}$$

By definition, α^e is directed with \mathcal{V}^{in} incoming and $e / \{\mathcal{V}^{\text{in}}\}$ outgoing, since for any $x_{\mathcal{V}^{\text{in}}}$

$$\sum_{x_e / \{\mathcal{V}^{\text{in}}\}} \frac{\mu_e[X_e = x_e]}{\langle \mu_{\tilde{e}}[X_{\tilde{e}}] \rangle [X_{\mathcal{V}^{\text{in}}} = x_{\mathcal{V}^{\text{in}}}]^2} = \frac{\langle \mu_e \rangle [X_{\mathcal{V}^{\text{in}}} = x_{\mathcal{V}^{\text{in}}}]^2}{\langle \mu_{\tilde{e}} \rangle [X_{\mathcal{V}^{\text{in}}} = x_{\mathcal{V}^{\text{in}}}]^2} = 1$$

where we used the local consistency property of the hyperedge mean parameters. Note that at the leafs the activation cores are trivial.

Lemma 1. *The contraction of the above constructed activation cores is a Markov Network reproducing the mean parameters, and therefore the maximum entropy distribution reproducing the mean parameters.*

Proof. By directionality of the cores, we show that the marginal distribution at any hyperedge is the contraction of the activation cores of the directed ancestors of that hyperedge. One can show with induction that this contraction is When leaving the \square

As a consequence we have the following theorem.

Theorem 1. Let \mathcal{G} be a minimal connected hypergraph. Then $\mathcal{M}_{\mathcal{S}^{\mathcal{G}}}$ is cube-like and all maximum entropy distributions with respect to the statistic $\mathcal{S}^{\mathcal{G}}$ are Markov Networks over \mathcal{G} .

Proof. Since we can construct to each mean parameter a Markov Network, which is a Hybrid Logic Network, any mean parameter lies on the relative interior of a cube-like face. Any Hybrid Logic Network is furthermore a maximum entropy distribution. \square

2 Loopy hypergraphs

Main question: Does the failure to characterize the mean polytope correspond with the existence of non-realizable mean parameters?

When loops exist in the hypergraph, we have:

- The existence of mean parameters in the local consistency polytope, which are not in the mean polytope, i.e. not realizable by any distribution.
- When loops exist there exist distributions, which are not positive, satisfy all conditional independences suggested by the hypergraph, but are not representable by maximum entropy distributions.