II-Probabilities
Probabilistic Reasoning

Foundations of Neuro-Symbolic AI

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Example: Being at a dentist

We are reasoning about a factored system with three variables:

- Toothache $i \in [2]$, whether your tooth hurts
- Cavity $j \in [2]$, whether there is a cavity in your tooth
- Catch $k \in [2]$, whether the dentist catches in your tooth

	toothache		$\neg$*toothache*	
	catch	$\neg$*catch*	*catch*	$\neg$*catch*
cavity	0.108	0.012	0.072	0.008
$\neg$*cavity*	0.016	0.064	0.144	0.576

Formal Definition of Probability Tensors

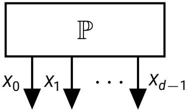
Definition (Probability Tensor)

Let there be a factored system $X_{[d]}$ defined by variables X_k taking values in $[m_k]$. A probability distribution over the states of $X_{[d]}$ is a map

$$\mathbb{P} : \bigtimes_{k \in [d]} [m_k] \to [0, \infty)$$

such that $\sum_{x_0,\dots,x_{d-1}} \mathbb{P}(x_0, \dots, x_{d-1}) = 1$.

We depict



Directed Tensors

We depict the condition, that coordinate sums are one, by directions on the legs.

Definition (Directed Tensor)

A tensor

$$\tau \in \bigotimes_{v \in \mathcal{V}} \mathbb{R}^{m_v}$$

is said to be directed with incoming variables $\mathcal{V}^{\mathrm{in}}$ and outgoing variables $\mathcal{V}^{\mathrm{out}}$, where $\mathcal{V} = \mathcal{V}^{\mathrm{in}} \dot\cup \mathcal{V}^{\mathrm{out}}$, when

$$\langle \{\tau\} \rangle_{[\mathcal{V}^{\mathrm{in}}]} = \mathbb{I}^{\mathcal{V}^{\mathrm{in}}}$$

where $\mathbb{I}^{\mathcal{V}^{\mathrm{in}}}$ denoted the trivial tensor in $\bigotimes_{v \in \mathcal{V}^{\mathrm{in}}} \mathbb{R}^{m_v}$ which coordinates are all 1.

Example: Marginal Distributions

What is the probability that there is a cavity?

$$\mathbb{P}^{\text{Dentist,Cavity}}[\text{Cavity}] = \sum_{j \in [2]} \sum_{k \in [2]} \mathbb{P}^{\text{Dentist}}(:, j, k) \tag{1}$$

This is called a marginal distribution.

Exercise

Calculate the marginal probability of Cavity given the probability tensor $\mathbb{P}^{\text{Dentist}}$.

	toothache		*¬toothache*	
	catch	*¬catch*	*catch*	*¬catch*
cavity	0.108	0.012	0.072	0.008
¬cavity	0.016	0.064	0.144	0.576

Formal Definition of Marginal Distributions

Definition (Marginal Probability)

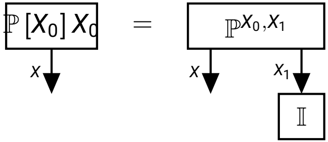
Given a distribution $\mathbb{P}$ of the categorical variables X_0 and X_1 the marginal distribution of the categorical variable X_0 is defined for each x_{X_0} as

$$\mathbb{P}\left[X_0 = x_{X_0}\right] X_0 = \sum_{x_{X_1} \in [m_{X_1}]} \mathbb{P}^{X_0 = x_{X_0}, X_1 = x_{X_1}} .$$

Marginal probabilities are contractions

$$\mathbb{P}^{X_0} = \langle \mathbb{P} \rangle_{[X_0]}$$

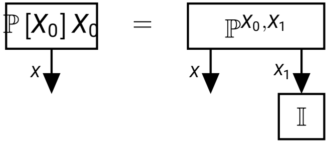
depicted by



Marginal Distributions in Contraction Formalism

Contractions are useful to

- Specify the Probability Tensor, or a decomposition of it
- Specify the variables to marginalize over as the ones left open



Directed notation preserved: Marginal probabilities are again probability distributions, since

$$\sum_{i\in[2]} \mathbb{P}^{\text{Dentist,Cavity}}[i] = \sum_{i\in[2]}\sum_{j\in[2]}\sum_{k\in[2]} \mathbb{P}^{\text{Dentist}}_{i,j,k} = 1\,. \tag{2}$$

Example: Conditional Distributions

What is the probability of having a cavity when having a toothache?

$$\mathbb{P}^{\text{Dentist}}[\text{Cavity}|\text{Toothache}] = \frac{\sum_{j \in [2]} \mathbb{P}^{\text{Dentist}}_{:,j,:}}{\sum_{j,k \in [2]} \mathbb{P}^{\text{Dentist}}_{:,j,k}} \qquad (3)$$

This is called a conditional distribution.

Exercise

Calculate the probability of Cavity conditioned on Toothache.

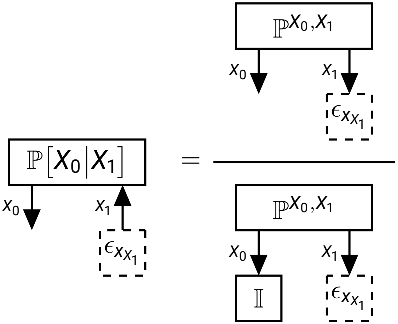
	toothache		*¬toothache*	
	catch	*¬catch*	*catch*	*¬catch*
cavity	0.108	0.012	0.072	0.008
¬cavity	0.016	0.064	0.144	0.576

Formal Definition of Conditional Distributions

Definition (Conditional Probability)

Given a distribution $\mathbb{P}$ of the categorical variables X_0 and Y, the conditioned distribution of X_0 is defined by

$$\mathbb{P}\left[X_0 = x_{X_0} \middle| Y = x_Y\right] = \frac{\mathbb{P}^{X_0 = x_{X_0}, Y = x_Y}}{\mathbb{P}^{Y = x_Y}} .$$



Normations

Definition (Normation of Tensor Networks)

A tensor network $\tau^{\mathcal{G}}$ on variables $\mathcal{V}$ can be normed on $\mathcal{U}$, if the coordinates of no slice with respect to $\mathcal{U}$ sum to 0. Then we define the normed tensor

$$\left\langle \tau^{\mathcal{G}} \right\rangle_{[\mathcal{V}^{\mathrm{out}}|\mathcal{V}^{\mathrm{in}}]} \in \left(\bigotimes_{v \in \mathcal{V}^{\mathrm{in}}} \mathbb{R}^{m_v} \right) \otimes \left(\bigotimes_{v \in \mathcal{V}^{\mathrm{out}}} \mathbb{R}^{m_v} \right)$$

by

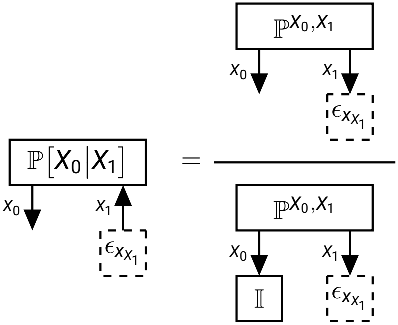
$$\left\langle \tau^{\mathcal{G}} \right\rangle_{[\mathcal{V}^{\mathrm{out}}|\mathcal{V}^{\mathrm{in}}]} = \sum_{x_{\mathcal{V}^{\mathrm{in}}} \in \times_{v \in \mathcal{V}^{\mathrm{in}}} m_v} \epsilon_{x_{\mathcal{V}^{\mathrm{in}}}} \otimes \frac{\left\langle \tau^{\mathcal{G}} \cup \left\{ \epsilon_{x_{\mathcal{V}^{\mathrm{in}}}} \right\} \right\rangle_{[\mathcal{V}^{\mathrm{out}}]}}{\left\langle \tau^{\mathcal{G}} \cup \left\{ \epsilon_{x_{\mathcal{V}^{\mathrm{in}}}} \right\} \right\rangle_{[\varnothing]}} \, .$$

Conditional Distributions by Normations

Conditioning is the normation

$$\mathbb{P}\left[X_0 \middle| X_1\right] = \langle \mathbb{P} \rangle_{[X_0|X_1]}$$

depicted by



The directed notation highlights Conditions by incoming legs and Distributions by outgoing legs.

The Bayes Theorem

The Bayes Theorem relates conditional probabilities:

Theorem (Bayes Theorem)

For any joint distribution of two categorical variables X_0 and X_1 it holds that

$$\mathbb{P}\left[X_0 \middle| X_1\right] = \frac{\mathbb{P}^{X_0,X_1}}{\mathbb{P}^{X_1}} = \frac{\mathbb{P}\left[X_1 \middle| X_0\right] \mathbb{P}^{X_0}}{\mathbb{P}^{X_1}} .$$

Bayes Theorem in the Dentist Example

Directions of Reasoning

- Causal direction: Toothache is caused by cavity
- Diagnostic direction: Cavity is probable because of toothache

Bayes Theorem allows us to reason in diagnostic direction, given an underlying causal influence:

$$\begin{aligned}&\mathbb{P}^{\text{Dentist}}[\text{Cavity}|\text{Toothache}]\\&\qquad = \mathbb{P}^{\text{Dentist}}[\text{Toothache}|\text{Cavity}]\frac{\mathbb{P}^{\text{Dentist}}[\text{Cavity}]}{\mathbb{P}^{\text{Dentist}}[\text{Toothache}]}\end{aligned}$$

Contraction Calculus for Probability Tensors

Probabilistic queries can be answered by contractions

- Marginal probabilities

$$\mathbb{P}^{X_0} = \langle \mathbb{P} \rangle_{[X_0]}$$

- Conditional probabilities

$$\mathbb{P}\left[X_0 \middle| X_1\right] = \langle \mathbb{P} \rangle_{[X_0|X_1]}$$

Outlook: Tensor network decompositions of $\mathbb{P}$ increase the execution efficiency!