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Some properties of Liouville's function

by
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Abstract

Several identities where Liouville's function $\lambda(n)$ occurs are presented. The proofs are based on a formula involving the summatory function for $\lambda(n)$ as well as on Shapiro's formula. The sum of the series $\sum_{n\geq 1} \frac{\lambda(n)}{n}$ is computed by using the classical result $\sum_{n\geq 1} \frac{\mu(n)}{n} = 0$.

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1 Introduction

Liouville introduced his function $\lambda(n)$ in the following way: $\lambda(1) = 1$ and $\lambda(n) = (-1)^{\alpha_1+\alpha_2+\cdots+\alpha_k}$ for $n \geq 2$, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. This is an example of totally multiplicative function, that is

$$\lambda(mn) = \lambda(m) \cdot \lambda(n). \tag{1}$$

Its summatory function has a very simple form:

$$S(n) = \sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if} \quad n = k^2, \\ 0 & \text{if} \quad n \neq k^2. \end{cases} \tag{2}$$

The proofs of these properties can be found e.g. in [7].

Denote $L(x) = \sum_{n\leq x} \lambda(n)$. In connection with this function, Pólya made the conjecture that $L(x) \leq 0$ for all $x \geq 2$. However this fact was disproved by C.B. Haselgrove in [3]. R.J. Anderson and H.M. Stark proved later in [1] a stronger result, namely $\limsup_{x\to\infty} \frac{L(x)}{\sqrt{x}} > 0.023$.

Denote $\alpha(x) = \sum\limits_{n \le x} \frac{\lambda(n)}{n}$. Related to this function, P. Turán made the conjecture that $\alpha(x) > 0$ whenever $x > 1$. In turn, this conjecture was disproved by C.B. Haselgrove in [3]. H. Gupta proved that $|\alpha(x)| \le 1 + \frac{1}{\sqrt{x}}$. Other properties are collected in [5] at page 162.

2 Identities involving the function $\lambda(n)$

One proves in [6] the identity

$$\sum_{k=1}^{n} h(k)F(k) = \sum_{k=1}^{n} h(k)f(k) \sum_{j=1}^{\left[\frac{n}{k}\right]} h(j), \tag{3}$$

where $h : \mathbb{N}^* \to \mathbb{R}$ is a totally multiplicative function, $f : \mathbb{N}^* \to \mathbb{R}$ and F is its summatory function, that is

$$F(n) = \sum_{d|n} f(d).$$

It is well-known Shapiro's theorem: for f and h as above, if

$$g(x) = \sum_{n \le x} h(n) f\left(\frac{x}{n}\right), \tag{4}$$

then

$$f(x) = \sum_{n \le x} \mu(n)h(n)g\left(\frac{x}{n}\right), \tag{5}$$

where μ is Möbius' function. We shall use the above formulas either for $f = \lambda$ or for $h = \lambda$.

PROPOSITION 1. *If $f : \mathbb{R} \to \mathbb{C}$, $x \ge 1$ and $g(x) = \sum\limits_{n \le x} \lambda(n) f\left(\frac{x}{n}\right)$, then*

$$f(x) = \sum_{n \le x} |\mu(n)| g\left(\frac{x}{n}\right).$$

Proof. Apply Shapiro's theorem for $h(n) = \lambda(n)$. By (4) we have $g(x) = \sum\limits_{n \le x} \lambda(n) f\left(\frac{x}{n}\right)$, while (5) implies $f(x) = \sum\limits_{n \le x} \mu(n)\lambda(n)g\left(\frac{x}{n}\right)$. If n is a squarefree number, then $\mu(n) = \lambda(n)$ hence $\mu(n)\lambda(n) = (\mu(n)^2 = |\mu(n)|$.

On the other hand, if n is not a squarefree number, then $\mu(n) = 0$ hence $\mu(n)\lambda(n) = |\mu(n)|$ and the desired conclusion follows. $\square$

Consequence 1. For $f = 1$ it follows $g(x) = \sum_{n\le x} L(x)$, so Proposition 1 implies

$$1 = \sum_{n\le x} |\mu(n)| L\left(\frac{x}{n}\right). \tag{6}$$

This can be written also as

$$\sum_{n \text{ squarefree} \le x} L\left(\frac{x}{n}\right) = 1. \tag{7}$$

Consequence 2. *For $f(x) = x$ one gets $g(x) = \sum_{n\le x} \lambda(n) \cdot \frac{x}{n} = x \sum_{n\le x} \frac{\lambda(n)}{n}$ $= x\alpha(x)$, hence the identity*

$$1 = \sum_{n\le x} \frac{|\mu(n)|}{n} \alpha\left(\frac{x}{n}\right) \tag{8}$$

follows.

We obtain now several interesting identities as applications of the formula (3). We mention that relation (9) below can be found in [7].

Proposition 2.1 *For $x \ge 1$, the relations*

$$\sum_{k\le x} \lambda(k) \left[\frac{x}{k}\right] = [\sqrt{x}] \tag{9}$$

and

$$\sum_{k\le x} \lambda(k) L\left(\frac{x}{k}\right) = \sum_{k\le x} \left[\frac{x}{k}\right] \tag{10}$$

hold.

Proof: By (2) and (3) applied for $f(k) = \lambda(k)$, $h = 1$ and $n = [\sqrt{x}]$, we get

$$[\sqrt{n}] = \sum_{k=1}^{n} \lambda(k) \left[\frac{n}{k}\right].$$

But $[\sqrt{n}] = [\sqrt{x}]$ and $\left[\frac{[x]}{k}\right] = \left[\frac{x}{k}\right]$, so (9) follows.

For proving (10), we apply (3) for $f = 1$ and $h(k) = \lambda(k)$. Then $F(k) = \tau(k)$, where $\tau(k) = \sum_{d|n} d$. By (3) we get

$$\sum_{k=1}^{n} \tau(k) = \sum_{k=1}^{n} \lambda(k) L\left(\frac{n}{k}\right). \tag{11}$$

The identity

$$\sum_{k=1}^{n} \tau(k) = \sum_{k=1}^{n} \left[\frac{n}{k}\right] \tag{12}$$

is well-known. Now (10) follows in view of the above remarks. □

Remark. Setting $h(n) = \lambda(n)$ and $f(x) = [x]$ $(x \geq 1)$ in Shapiro's Theorem, we obtain $g(x) = \sum\limits_{n \leq x} \lambda(n) \left[\frac{x}{n}\right]$ and furthermore $g(x) = [\sqrt{x}]$ by (9). Now we get

$$[x] = \sum_{n \leq x} |\mu(n)| \left[\sqrt{\frac{x}{n}}\right] \qquad \text{hence an already known result}$$

and

$$\sum_{n \text{ squarefree } \leq x} \left[\sqrt{\frac{x}{n}}\right] = [x].$$

The following identity is an immediate consequence of a formula due to Möbius.

Proposition 2.2 *For every integer $n \geq 1$ the formula*

$$\lambda(n) = \sum_{k^2 | n} \mu\left(\frac{n}{k^2}\right).$$

holds.

Proof: Consider $f(k) = \lambda(k)$. We have already seen that the summatory function of f is given by (2). By Möbius' inversion formula we deduce

$$\lambda(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) S(d) = \sum_{d=k^2|n} \mu\left(\frac{n}{d}\right) = \sum_{k^2|n} \mu\left(\frac{n}{k^2}\right).$$

□ □

3 The series $\sum\limits_{n \geq 1} \frac{\lambda(n)}{n}$

The result to be presented below is suggested by a classical one, namely $\sum\limits_{n \geq 1} \frac{\mu(n)}{n} = 0$ (see e.g. [4]). We shall prove:

Proposition 3.1 *The equality*

$$\sum_{n\geq 1} \frac{\lambda(n)}{n} = 0$$

holds.

Proof: Denote $s_k = \sum_{n=1}^{k} \frac{\mu(n)}{n}$ and $S_k = \sum_{n=1}^{k} \frac{\lambda(n)}{n}$. Then we have $\lim_{k\to\infty} s_k = 0$ and we must prove that $\lim_{k\to\infty} S_k = 0$.

Since every natural number n can be written as $n = hi^2$, where h is a squarefree number we have

$$S_k = \sum_{hi^2\leq k} \frac{\lambda(hi^2)}{hi^2} = \sum_{hi^2\leq k} \frac{\lambda(h)}{hi^2}.$$

We have $\lambda(h) = \mu(h)$, $s_k = \sum_{\substack{h=1 \\ h \text{ squarefree}}}^{k} \frac{\mu(h)}{h}$,

$$S_k = \sum_{\substack{hi^2\leq k \\ h \text{ squarefree}}} \frac{\mu(h)}{hi^2} = \frac{1}{1^2}s_k + \frac{1}{2^2}s_{[k/2^2]} + \cdots + \frac{1}{j^2}s_{[k/j^2]}$$

and $a = [\sqrt{k}]$.

Denoting $b = [\sqrt[3]{k}]$ we have

$$|S_k| \leq \frac{|s_k|}{1^2} + \frac{\left|s_{[k/2^2]}\right|}{2^2} + \cdots + \frac{\left|s_{[k/b^2]}\right|}{b^2} + \frac{\sum_{i=b+1}^{a} \left|s_{[k/i^2]}\right|}{b^2}. \tag{13}$$

Since $\lim_{k\to\infty} s_k = 0$, the sequence $(s_k)_{k\geq 1}$ is bounded, that is $\left|s_{[k/i^2]}\right| \leq M$. We have also $\frac{k}{b^2} = \frac{k}{[\sqrt[3]{k}]^2} > \sqrt[3]{k}$. Given $\varepsilon > 0$, we have for k sufficiently large $\left|s_{[k/i^2]}\right| < \varepsilon$ whenever $i \leq b$. The inequality (13) implies

$$|S_k| \leq \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{b^2}\right)\varepsilon + \frac{a-b}{b^2}M.$$

But $\lim_{k\to\infty} \frac{[\sqrt{k}]-[\sqrt[3]{k}]}{[\sqrt[3]{k}]^2} = 0$ and $\sum_{i\geq 1} = \frac{\pi^2}{6}$, so $0 \leq \lim_{k\to\infty} |S_k| \leq \varepsilon\cdot\frac{\pi^2}{6}$ for $\varepsilon > 0$ arbitrary. Consequently $\lim_{k\to\infty} S_k = 0$, that is $\sum_{n\geq 1} \frac{\mu(n)}{n} = 0$. □

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