Six proofs of the infinity of primes

Chapter 1

It is only natural that we start these notes with probably the oldest Book Proof, usually attributed to Euclid (*Elements* IX, 20). It shows that the sequence of primes does not end.

■ **Euclid's Proof.** For any finite set $\{p_1, \ldots, p_r\}$ of primes, consider the number $n = p_1 p_2 \cdots p_r + 1$. This n has a prime divisor p. But p is not one of the p_i: otherwise p would be a divisor of n and of the product $p_1 p_2 \cdots p_r$, and thus also of the difference $n - p_1 p_2 \cdots p_r = 1$, which is impossible. So a finite set $\{p_1, \ldots, p_r\}$ cannot be the collection of *all* prime numbers. □

Before we continue let us fix some notation. $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ the set of integers, and $\mathbb{P} = \{2, 3, 5, 7, \ldots\}$ the set of primes.

In the following, we will exhibit various other proofs (out of a much longer list) which we hope the reader will like as much as we do. Although they use different view-points, the following basic idea is common to all of them: The natural numbers grow beyond all bounds, and every natural number $n \geq 2$ has a prime divisor. These two facts taken together force $\mathbb{P}$ to be infinite. The next proof is due to Christian Goldbach (from a letter to Leonhard Euler 1730), the third proof is apparently folklore, the fourth one is by Euler himself, the fifth proof was proposed by Harry Fürstenberg, while the last proof is due to Paul Erdős.

■ **Second Proof.** Let us first look at the *Fermat numbers* $F_n = 2^{2^n} + 1$ for $n = 0, 1, 2, \ldots$. We will show that any two Fermat numbers are relatively prime; hence there must be infinitely many primes. To this end, we verify the recursion

$$\prod_{k=0}^{n-1} F_k \;=\; F_n - 2 \qquad (n \geq 1),$$

from which our assertion follows immediately. Indeed, if m is a divisor of, say, F_k and F_n $(k < n)$, then m divides 2, and hence $m = 1$ or 2. But $m = 2$ is impossible since all Fermat numbers are odd.

F_0	$=$	3
F_1	$=$	5
F_2	$=$	17
F_3	$=$	257
F_4	$=$	65537
F_5	$=$	$641 \cdot 6700417$

The first few Fermat numbers

To prove the recursion we use induction on n. For $n = 1$ we have $F_0 = 3$ and $F_1 - 2 = 3$. With induction we now conclude

$$\prod_{k=0}^{n} F_k = \Big(\prod_{k=0}^{n-1} F_k\Big) F_n \;=\; (F_n - 2) F_n \;=$$

$$= (2^{2^n} - 1)(2^{2^n} + 1) \;=\; 2^{2^{n+1}} - 1 \;=\; F_{n+1} - 2. \qquad \square$$

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■ **Third Proof.** Suppose $\mathbb{P}$ is finite and p is the largest prime. We consider the so-called *Mersenne number* $2^p - 1$ and show that any prime factor q of $2^p - 1$ is bigger than p, which will yield the desired conclusion. Let q be a prime dividing $2^p - 1$, so we have $2^p \equiv 1 \pmod{q}$. Since p is prime, this means that the element 2 has order p in the multiplicative group $\mathbb{Z}_q \backslash \{0\}$ of the field $\mathbb{Z}_q$. This group has $q - 1$ elements. By Lagrange's theorem (see the box) we know that the order of every element divides the size of the group, that is, we have $p \mid q - 1$, and hence $p < q$. □

Lagrange's Theorem

If G is a finite (multiplicative) group and U is a subgroup, then $|U|$ divides $|G|$.

■ **Proof.** Consider the binary relation

$$a \sim b : \Longleftrightarrow \; ba^{-1} \in U.$$

It follows from the group axioms that $\sim$ is an equivalence relation. The equivalence class containing an element a is precisely the coset

$$Ua = \{xa : x \in U\}.$$

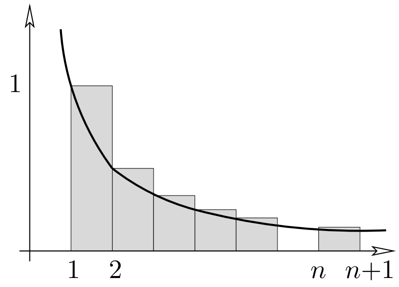
Since clearly $|Ua| = |U|$, we find that G decomposes into equivalence classes, all of size $|U|$, and hence that $|U|$ divides $|G|$. □

In the special case when U is a cyclic subgroup $\{a, a^2, \ldots, a^m\}$ we find that m (the smallest positive integer such that $a^m = 1$, called the *order* of a) divides the size $|G|$ of the group.

Now let us look at a proof that uses elementary calculus.

■ **Fourth Proof.** Let $\pi(x) := \#\{p \leq x : p \in \mathbb{P}\}$ be the number of primes that are less than or equal to the real number x. We number the primes $\mathbb{P} = \{p_1, p_2, p_3, \ldots\}$ in increasing order. Consider the natural logarithm $\log x$, defined as $\log x = \int_1^x \frac{1}{t} dt$.

Now we compare the area below the graph of $f(t) = \frac{1}{t}$ with an upper step function. (See also the appendix on page 10 for this method.) Thus for $n \leq x < n + 1$ we have



Steps above the function $f(t) = \frac{1}{t}$

$$\begin{aligned} \log x &\leq 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{n} \\ &\leq \sum \frac{1}{m}, \text{ where the sum extends over all } m \in \mathbb{N} \text{ which have only prime divisors } p \leq x. \end{aligned}$$

Since every such m can be written in a *unique* way as a product of the form $\prod\limits_{p \leq x} p^{k_p}$, we see that the last sum is equal to

$$\prod_{\substack{p \in \mathbb{P} \\ p \leq x}} \Big(\sum_{k \geq 0} \frac{1}{p^k} \Big).$$

The inner sum is a geometric series with ratio $\frac{1}{p}$, hence

$$\log x \; \leq \prod_{\substack{p \in \mathbb{P} \\ p \leq x}} \frac{1}{1 - \frac{1}{p}} \; = \prod_{\substack{p \in \mathbb{P} \\ p \leq x}} \frac{p}{p-1} \; = \prod_{k=1}^{\pi(x)} \frac{p_k}{p_k - 1}.$$

Now clearly $p_k \geq k + 1$, and thus

$$\frac{p_k}{p_k - 1} \; = 1 + \frac{1}{p_k - 1} \; \leq \; 1 + \frac{1}{k} \; = \; \frac{k+1}{k},$$

and therefore

$$\log x \; \leq \; \prod_{k=1}^{\pi(x)} \frac{k+1}{k} \; = \; \pi(x) + 1.$$

Everybody knows that $\log x$ is not bounded, so we conclude that $\pi(x)$ is unbounded as well, and so there are infinitely many primes. □