Number Theory Problems of 2016 Competitions

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Problem 1 (APMO 2016)**.** A positive integer is called *fancy* if it can be expressed in the form

$$2^{a_1} + 2^{a_2} + \cdots + 2^{a_{100}},$$

where $a_1, a_2, \cdots, a_{100}$ are non-negative integers that are not necessarily distinct. Find the smallest positive integer n such that no multiple of n is a fancy number.

Link

Problem 2 (Argentina Intercollegiate Olympiad First Level 2016)**.** Find all positive integers a, b, c, and d, all less than or equal to 6, such that

$$\frac{a}{b} = \frac{c}{d} + 2.$$

Problem 3 (Argentina Intercollegiate Olympiad Second Level 2016)**.** Find all positive integers x and y which satisfy the following conditions:

1. x is a $4-$digit palindromic number, and
2. $y = x + 312$ is a $5-$digit palindromic number.

Note. A palindromic number is a number that remains the same when its digits are reversed. For example, 16461 is a palindromic number.

Problem 4 (Argentina Intercollegiate Olympiad Third Level 2016)**.** Find a number with the following conditions:

1. it is a perfect square,
2. when 100 is added to the number, it equals a perfect square plus 1, and
3. when 100 is again added to the number, the result is a perfect square.

Problem 5 (Argentina Intercollegiate Olympiad Third Level 2016)**.** Let $a_1, a_2, \ldots, a_{15}$ be an arithmetic progression. If the sum of all 15 terms is twice the sum of the first 10 terms, find $\frac{d}{a_1}$, where d is the common difference of the progression.

Problem 6 (Austria Federal Competition for Advanced Students Final Round 2016)**.** Determine all composite positive integers n with the following property: If $1 = d_1 < d_2 < \cdots < d_k = n$ are all the positive divisors of n, then

$$(d_2 - d_1) : (d_3 - d_2) : \cdots : (d_k - d_{k-1}) = 1 : 2 \cdots : (k-1).$$

Problem 7 (Austria National Competition Final Round 2016)**.** Let a, b, and c be integers such that

$$\frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a}$$

is an integer. Prove that each of the numbers

$$\frac{ab}{c}, \frac{ac}{b}, \text{ and } \frac{bc}{a}$$

is an integer.

Problem 8 (Austria Beginners' Competition 2016)**.** Determine all nonnegative integers n having two distinct positive divisors with the same distance from $n/3$.

Problem 9 (Austria Regional Competition 2016)**.** Determine all positive integers k and n satisfying the equation

$$k^2 - 2016 = 3^n.$$

Problem 10 (Azerbaijan TST 2016)**.** The set A consists of natural numbers such that these numbers can be expressed as $2x^2 + 3y^2$, where x and y are integers. $(x^2 + y^2 \neq 0)$

1. Prove that there is no perfect square in the set A.
2. Prove that multiple of odd number of elements of the set A cannot be a perfect square.

Link

Problem 11 (Azerbaijan Junior Mathematical Olympiad 2016)**.** Given

$$34! = 295232799039a041408476186096435b0000000,$$

in decimal representation, find the numbers a and b. Link

Problem 12 (Azerbaijan Junior Mathematical Olympiad 2016)**.** Prove that if for a real number a, $a + \frac{1}{a}$ is integer then $a^n + \frac{1}{a^n}$ is also integer for any positive integer n. Link

Problem 13 (Azerbaijan Junior Mathematical Olympiad 2016)**.** A quadruple (p, a, b, c) of positive integers is called a *good quadruple* if

(a) p is odd prime,

(b) a, b, c are distinct,

(c) $ab + 1, bc + 1$, and $ca + 1$ are divisible by p.

Prove that for all good quadruple $p + 2 \leq \frac{a+b+c}{3}$, and show the equality case.

Link

Problem 14 (Balkan 2016). Find all monic polynomials f with integer coefficients satisfying the following condition: there exists a positive integer N such that p divides $2(f(p)!) + 1$ for every prime $p > N$ for which $f(p)$ is a positive integer.
Note: A monic polynomial has a leading coefficient equal to 1. Link

Problem 15 (Bay Area Olympiad 2016). Let $A = 2^k - 2$ and $B = 2^k \cdot A$, where k is an integer ($k \geq 2$). Show that, for every integer k greater than or equal to 2,

1. A and B have the same set of distinct prime factors.
2. $A + 1$ and $B + 1$ have the same set of distinct prime factors.

Link

Problem 16 (Bay Area Olympiad 2016). Find a positive integer N and $a_1, a_2, \cdots, a_N$ where $a_k = 1$ or $a_k = -1$, for each $k = 1, 2, \cdots, N$, such that

$$a_1 \cdot 1^3 + a_2 \cdot 2^3 + a_3 \cdot 3^3 \cdots + a_N \cdot N^3 = 20162016$$

or show that this is impossible. Link

Problem 17 (Belgium Flanders Math Olympiad Final Round 2016). Find the smallest positive integer n which does not divide 2016!.

Problem 18 (Belgium National Olympiad Final Round 2016). Solve the equation

$$2^{2m+1} + 9 \cdot 2^m + 5 = n^2$$

for integers m and n.

Problem 19 (Benelux 2016). Find the greatest positive integer N with the following property: there exist integers $x_1, \ldots, x_N$ such that $x_i^2 - x_i x_j$ is not divisible by 1111 for any $i \neq j$. Link

Problem 20 (Benelux 2016). Let n be a positive integer. Suppose that its positive divisors can be partitioned into pairs (i.e. can be split in groups of two) in such a way that the sum of each pair is a prime number. Prove that these prime numbers are distinct and that none of these are a divisor of n. Link

Problem 21 (Bosnia and Herzegovina TST 2016). For an infinite sequence $a_1 < a_2 < a_3 < \ldots$ of positive integers we say that it is nice if for every positive integer n holds $a_{2n} = 2a_n$. Prove the following statements:

(a) If there is given a nice sequence and prime number $p > a_1$, there exist some term of the sequence which is divisible by p.

(b) For every prime number $p > 2$, there exist a nice sequence such that no terms of the sequence are divisible by p.

Link

Problem 22 (Bosnia and Herzegovina TST 2016)**.** Determine the largest positive integer n which cannot be written as the sum of three numbers bigger than 1 which are pairwise coprime. Link

Problem 23 (Bulgaria National Olympiad 2016)**.** Find all positive integers m and n such that $\left(2^{2^m}+1\right)\left(2^{2^n}+1\right)$ is divisible by mn.

Problem 24 (Bulgaria National Olympiad 2016)**.** Determine whether there exists a positive integer $n < 10^9$ such that n can be expressed as a sum of three squares of positive integers by more than 1000 distinct ways.

Problem 25 (Canadian Mathematical Olympiad Qualification 2016)**.**

(a) Find all positive integers n such that $11|(3^n + 4^n)$.

(b) Find all positive integers n such that $31|(4^n + 7^n + 20^n)$.

Link

Problem 26 (Canadian Mathematical Olympiad Qualification 2016)**.** Determine all ordered triples of positive integers (x, y, z) such that $\gcd(x+y, y+z, z+x) > \gcd(x, y, z)$. Link

Problem 27 (Canada National Olympiad 2016)**.** Find all polynomials $P(x)$ with integer coefficients such that $P(P(n) + n)$ is a prime number for infinitely many integers n. Link

Problem 28 (CCA Math Bonanza 2016)**.** Let $f(x) = x^2 + x + 1$. Determine the ordered pair (p, q) of primes satisfying $f(p) = f(q) + 242$. Link

Problem 29 (CCA Math Bonanza 2016)**.** Let $f(x) = x^2 + x + 1$. Determine the ordered pair (p, q) of primes satisfying $f(p) = f(q) + 242$. Link

Problem 30 (CCA Math Bonanza 2016)**.** Compute

$$\sum_{k=1}^{420} \gcd(k, 420).$$

Link

Problem 31 (CCA Math Bonanza 2016)**.** Pluses and minuses are inserted in the expression

$$\pm 1 \pm 2 \pm 3 \cdots \pm 2016$$

such that when evaluated the result is divisible by 2017. Let there be N ways for this to occur. Compute the remainder when N is divided by 503. Link

Problem 32 (CCA Math Bonanza 2016)**.** What is the largest integer that must divide $n^5 - 5n^3 + 4n$ for all integers n? Link

Problem 33 (CCA Math Bonanza 2016)**.** Determine the remainder when

$$2^6 \cdot 3^{10} \cdot 5^{12} - 75^4 \left(26^2 - 1\right)^2 + 3^{10} - 50^6 + 5^{12}$$

is divided by 1001. Link

Problem 34 (CentroAmerican 2016)**.** Find all positive integers n that have 4 digits, all of them perfect squares, and such that n is divisible by $2, 3, 5$, and 7. Link

Problem 35 (CentroAmerican 2016)**.** We say a number is *irie* if it can be written in the form $1 + \dfrac{1}{k}$ for some positive integer k. Prove that every integer $n \geq 2$ can be written as the product of r distinct irie numbers for every integer $r \geq n - 1$. Link

Problem 36 (Chile 2016)**.** Determine all triples of positive integers (p, n, m) with p a prime number, which satisfy the equation:

$$p^m - n^3 = 27.$$

Problem 37 (Chile 2016[1])**.** Find all prime numbers that do not have a multiple ending in 2015.

Problem 38 (Chile 2016)**.** Find the number of different numbers of the form $\lfloor \frac{i^2}{2015} \rfloor$, where $i = 1, 2, \ldots, 2015$.

Problem 39 (China Girls Mathematical Olympiad 2016)**.** Let m and n are relatively prime integers and $m > 1, n > 1$. Show that there are positive integers a, b, c such that $m^a = 1 + n^b c$, and n and c are relatively prime. Link

Problem 40 (China National Olympiad 2016)**.** Let p be an odd prime and $a_1, a_2, ..., a_p$ be integers. Prove that the following two conditions are equivalent:

1. There exists a polynomial $P(x)$ with degree $\leq \frac{p-1}{2}$ such that $P(i) \equiv a_i \pmod{p}$ for all $1 \leq i \leq p$.
2. For any natural $d \leq \frac{p-1}{2}$,

$$\sum_{i=1}^{p} (a_{i+d} - a_i)^2 \equiv 0 \pmod{p},$$

   where indices are taken modulo p.

Link

[1] Thanks to Kamal Kamrava and Behnam Sajadi for translating the problem.

Problem 41 (China South East Mathematical Olympiad 2016)**.** Let n be a positive integer and let D_n be the set of all positive divisors of n. Define $f(n) = \sum\limits_{d \in D_n} \frac{1}{1+d}$. Prove that for any positive integer m,

$$\sum_{i=1}^{m} f(i) < m.$$

Link

Problem 42 (China South East Mathematical Olympiad 2016)**.** Let $\{a_n\}$ be a sequence consisting of positive integers such that $n^2 \mid \sum_{i=1}^{n} a_i$ and $a_n \leq (n+2016)^2$ for all $n \geq 2016$. Define $b_n = a_{n+1} - a_n$. Prove that the sequence $\{b_n\}$ is eventually constant. Link

Problem 43 (China South East Mathematical Olympiad 2016)**.** Define the sets

$$\begin{aligned}
A &= \{a^3 + b^3 + c^3 - 3abc : a, b, c \in \mathbb{N}\},\\
B &= \{(a+b-c)(b+c-a)(c+a-b) : a, b, c \in \mathbb{N}\},\\
P &= \{n : n \in A \cap B, 1 \leq n \leq 2016\}.
\end{aligned}$$

Find the number of elements of P. Link

Problem 44 (China TST2016)**.** Let $c, d \geq 2$ be positive integers. Let $\{a_n\}$ be the sequence satisfying $a_1 = c, a_{n+1} = a_n^d + c$ for $n = 1, 2, \ldots$. Prove that for any $n \geq 2$, there exists a prime number p such that $p \mid a_n$ and $p \nmid a_i$ for $i = 1, 2, \ldots, n-1$. Link

Problem 45 (China TST 2016)**.** Set positive integer $m = 2^k \cdot t$, where k is a non-negative integer, t is an odd number, and let $f(m) = t^{1-k}$. Prove that for any positive integer n and for any positive odd number $a \leq n$, $\prod_{m=1}^{n} f(m)$ is a multiple of a. Link

Problem 46 (China TST 2016)**.** Does there exist two infinite positive integer sets S, T, such that any positive integer n can be uniquely expressed in the form

$$n = s_1 t_1 + s_2 t_2 + \cdots + s_k t_k,$$

where k is a positive integer dependent on n, $s_1 < s_2 < \cdots < s_k$ are elements of S, $t_1, \ldots, t_k$ are elements of T? Link

Problem 47 (China TST 2016)**.** Let a, b, b', c, m, q be positive integers, where $m > 1, q > 1, |b - b'| \geq a$. It is given that there exist a positive integer M such that

$$S_q(an + b) \equiv S_q(an + b') + c \pmod{m}$$

holds for all integers $n \geq M$. Prove that the above equation is true for all positive integers n. (Here $S_q(x)$ is the sum of digits of x taken in base q). Link

Problem 48 (China Western Mathematical Olympiad 2016)**.** For an n-tuple of integers, define a transformation to be:

$$(a_1, a_2, \ldots, a_{n-1}, a_n) \rightarrow (a_1 + a_2, a_2 + a_3, \ldots, a_{n-1} + a_n, a_n + a_1)$$

Find all ordered pairs of integers (n, k) with $n, k \geq 2$, such that for any n-tuple of integers $(a_1, a_2, \ldots, a_{n-1}, a_n)$, after a finite number of transformations, every element in the of the n-tuple is a multiple of k. Link

Problem 49 (China Western Mathematical Olympiad 2016)**.** Prove that there exist infinitely many positive integer triples (a, b, c) such that a, b, c are pairwise relatively prime ,and $ab + c, bc + a, ca + b$ are pairwise relatively prime. Link

Problem 50 (Croatia First Round Competition 2016)**.** Can the sum of squares of three consecutive integers be divisible by 2016?

Problem 51 (Croatia First Round Competition 2016)**.** Let $a = 123456789$ and $N = a^3 - 2a^2 - 3a$. Prove that N is a multiple of 540.

Problem 52 (Croatia First Round Competition 2016)**.** Find all pairs (a, b) of positive integers such that $1 < a, b \leq 100$ and

$$\frac{1}{\log_a 10} + \frac{1}{\log_b 10}$$

is a positive integer.

Problem 53 (Croatia First Round Competition 2016)**.** A sequence (a_n) is given: $a_1 = a_2 = 1$, and

$$a_{n+1} = \frac{a_2^2}{a_1} + \frac{a_3^2}{a_2} + \cdots + \frac{a_n^2}{a_{n-1}} \quad \text{for } n \geq 2.$$

Find a_{2016}.

Problem 54 (Croatia First Round Competition 2016)**.** Let a, b, and c be integers. If $4a + 5b - 3c$ is divisible by 19, prove that $6a - 2b + 5c$ is also divisible by 19.

Problem 55 (Croatia First Round Competition 2016)**.** Determine all pairs of positive integers (x, y) such that $x^2 - y! = 2016$.

Problem 56 (Croatia Second Round Competition 2016)**.**

(a) Prove that there are no two positive integers such that the difference of their squares is 987654.

(b) Prove that there are no two positive integers such that the difference of their cubes is 987654.

Problem 57 (Croatia Second Round Competition 2016)**.** How many ordered pairs (m, k) of positive integers satisfy

$$20m = k(m - 15k)?$$

Problem 58 (Croatia Second Round Competition 2016). Determine all pairs (a,b) of positive integers such that

$$a^3 - 3b = 15, b^2 - a \qquad = 13.$$

Problem 59 (Croatia Second Round Competition 2016). Prove that, for every positive integer $n > 3$, there are n different positive integers whose reciprocals add up to 1.

Problem 60 (Croatia Second Round Competition 2016). Determine all pairs (a,b) of integers such that $(7a-b)^2 = 2(a-1)b^2$.

Problem 61 (Croatia Final Round National Competition 2016). Determine the sum

$$\frac{2^2+1}{2^2-1} + \frac{3^2+1}{3^2-1} + \cdots + \frac{100^2+1}{100^2-1}.$$

Problem 62 (Croatia Final Round National Competition 2016). Let a, b, and c be positive integers such that

$$c = a + \frac{b}{a} - \frac{1}{b}.$$

Prove that c is the square of an integer.

Problem 63 (Croatia Final Round National Competition 2016). Determine all pairs (m,n) of positive integers for which exist integers a, b, and c that satisfy

$$a+b+c = 0 \quad \text{and} \quad a^2+b^2+c^2 = 2^m \cdot 3^n.$$

Problem 64 (Croatia Final Round National Competition 2016). Prove that there does not exist a positive integer k such that $k+4$ and k^2+5k+2 are cubes of positive integers.

Problem 65 (Croatia Final Round National Competition 2016). Determine all triples (m,n,k) of positive integers such that $3^m + 7^n = k^2$.

Problem 66 (Croatian Mathematical Olympiad 2016). Find all pairs (p,q) of prime numbers such that

$$p(p^2-p-1) = q(2q+3).$$

Link

Problem 67 (Croatian TST for MEMO 2016, Sweden 2014). Find all pairs (m,n) of positive integers such that

$$3 \cdot 5^m - 2 \cdot 6^n = 3.$$

Problem 68 (Croatia IMO TST 2016). Prove that for every positive integer n there exist integers a and b such that n divides $4a^2 + 9b^2 - 1$.

Problem 69 (Croatia IMO TST 2016, Bulgaria TST 2016)**.** Let $p > 10^9$ be a prime number such that $4p+1$ is also prime. Prove that the decimal expansion of $\frac{1}{4p+1}$ contains all the digits $0, 1, \dots, 9$. Link

Problem 70 (Denmark Georg Mohr Contest Second Round 2016)**.** Find all possible values of the number

$$\frac{a+b}{c} + \frac{a+c}{b} + \frac{b+c}{a},$$

where a, b, and c are positive integers, and $\frac{a+b}{c}$, $\frac{a+c}{b}$, and $\frac{b+c}{a}$ are also positive integers.

Problem 71 (ELMO 2016)**.** Cookie Monster says a positive integer n is *crunchy* if there exist $2n$ real numbers $x_1, x_2, \dots, x_{2n}$, not all equal, such that the sum of any n of the x_i's is equal to the product of the other n of the x_i's. Help Cookie Monster determine all crunchy integers. Link

Problem 72 (ELMO2016)**.** Big Bird has a polynomial P with integer coefficients such that n divides $P(2^n)$ for every positive integer n. Prove that Big Bird's polynomial must be the zero polynomial. Link

Problem 73 (Estonia IMO TST First Stage 2016)**.** Let p be a prime. Find all integers (not necessarily positive) a, b, and c such that

$$a^b b^c c^a = p.$$

Problem 74 (Estonia IMO TST First Stage 2016)**.** Prove that for every positive integer $n \geq 3$,

$$2 \cdot \sqrt{3} \cdot \sqrt[3]{4} \dots \sqrt[n-1]{n} > n.$$

Problem 75 (Estonia IMO TST Second Stage 2016)**.** Find all positive integers n such that

$$\left(n^2 + 11n - 4\right) \cdot n! + 33 \cdot 13^n + 4$$

is a perfect square.

Problem 76 (Estonia National Olympiad Tenth Grade 2016)**.** Find all pairs of integers (a, b) which satisfy

$$3(a^2 + b^2) - 7(a + b) = -4.$$

Problem 77 (Estonia National Olympiad Eleventh Grade 2016)**.** Find the greatest positive integer n for which $3^{2016} - 1$ is divisible by 2^n.

Problem 78 (Estonia National Olympiad Eleventh Grade 2016)**.** Let n be a positive integer. Let $\delta(n)$ be the number of positive divisors of n and let $\sigma(n)$ be their sum. Prove that

$$\sigma(n) > \frac{\left(\delta(n)\right)^2}{2}.$$

Problem 79 (Estonia Regional Olympiad Tenth Grade 2016)**.** Does the equation

$$x^2 + y^2 + z^2 + w^2 = 3 + xy + yz + zw$$

has a solution in which x, y, z, and w are different integers?

Problem 80 (Estonia Regional Olympiad Twelfth Grade 2016)**.** Determine whether the logarithm of 6 in base 10 is larger or smaller than $\frac{7}{9}$.

Problem 81 (Estonia Regional Olympiad Twelfth Grade 2016)**.** Find the largest positive integer n so that one can select n primes $p_1, p_2, \ldots, p_n$ (not necessarily distinct) such that

$$p_1, p_1 + p_2, \ldots, p_1 + p_2 + \cdots + p_n$$

are all primes.

Problem 82 (European Girls' Mathematical Olympiad 2016)**.** Let S be the set of all positive integers n such that n^4 has a divisor in the range $n^2+1, n^2+2, \ldots, n^2+2n$. Prove that there are infinitely many elements of S of each of the forms $7m, 7m+1, 7m+2, 7m+5, 7m+6$ and no elements of S of the form $7m+3$ and $7m+4$, where m is an integer. Link

Problem 83 (European Mathematical Cup Seniors 2016)**.** $A = \{a, b, c\}$ is a set containing three positive integers. Prove that we can find a set $B \subset A$, say $B = \{x, y\}$, such that for all odd positive integers m and n,

$$10 \mid x^m y^n - x^n y^m.$$

Problem 84 (European Mathematical Cup Juniors 2016)**.** Let $d(n)$ denote the number of positive divisors of n. For a positive integer n we define $f(n)$ as

$$f(n) = d(k_1) + d(k_2) + \cdots + d(k_m),$$

where $1 = k_1 < k_2 < \cdots < k_m = n$ are all divisors of the number n. We call an integer $n > 1$ *almost perfect* if $f(n) = n$. Find all almos perfect numbers.

Problem 85 (Finland MAOL Competition 2016)**.** Let n be a positive integer. Find all pairs (x, y) of positive integers such that

$$(4a - b)(4b - a) = 1770^n.$$

Problem 86 (Germany National Olympiad First Round Ninth/Tenth Grade, 2016)**.** (A) Prove that there exists an integer $a > 1$ such that the number

$$82 \cdot \left(a^8 - a^4\right)$$

is divisible by the product of three consecutive positive integers each of which has at least two digits.

(B) Determine the smallest prime number a with at least two digits such that the number

$$82 \cdot \left(a^8 - a^4\right)$$

is divisible by the product of three consecutive positive integers each of which has at least two digits.

(C) Determine the smallest integer $a > 1$ such that the number

$$82 \cdot \left(a^8 - a^2\right)$$

is divisible by the product of three consecutive positive integers each of which has at least two digits.

Problem 87 (Germany National Olympiad First Round Eleventh/Twelfth Grade, 2016)**.** Consider the following system of equations:

$$\begin{aligned} 2(z-1) - x &= 55, \\ 4xy - 8z &= 12, \\ a(y+z) &= 11. \end{aligned}$$

Find two largest real values for a for which there are positive integers x, y, and z that satisfy the system of equations. In each of these solutions, determine xyz.

Problem 88 (Germany National Olympiad First Round Eleventh/Twelfth Grade, 2016)**.** Find all pairs (a, b) of positive integers for which $(a+1)(b+1)$ is divisible by ab.

Problem 89 (Germany National Olympiad Second Round Tenth Grade, 2016)**.** For each of the following cases, determine whether there exist prime numbers x, y, and z such that the given equality holds

(a) $y = z^2 - x^2$.

(b) $x^2 + y = z^4$.

(c) $x^2 + y^3 = z^4$.

Problem 90 (Germany National Olympiad Second Round Eleventh/Twelfth Grade, 2016)**.** The sequence $x_1, x_2, x_3, \ldots$ is defined as $x_1 = 1$ and

$$x_{k+1} = x_k + y_k \quad \text{for } k = 1, 2, 3, \ldots$$

where y_k is the last digit of decimal representation of x_k. Prove that the sequence $x_1, x_2, x_3, \ldots$ contains all powers of 4. That is, for every positive integer n, there exists some natural k for which $x_k = 4^n$.

Problem 91 (Germany National Olympiad Third Round Eleventh/Twelfth Grade, 2016)**.** Find all positive integers a and b which satisfy

$$\binom{ab+1}{2} = 2ab(a+b).$$

Problem 92 (Germany National Olympiad Third Round Eleventh/Twelfth Grade, 2016)**.** Let m and n be two positive integers. Prove that for every positive integer k, the following statements are equivalent:

1. $n+m$ is a divisor of n^2+km^2.
2. $n+m$ is a divisor of $k+1$.

Problem 93 (Germany National Olympiad Fourth Round Ninth Grade, 2016)**.** Find all triples (a,b,c) of integers which satisfy

$$\begin{aligned} a^3+b^3 &= c^3+1, \\ b^2-a^2 &= a+b, \\ 2a^3-6a &= c^3-4a^2. \end{aligned}$$

Problem 94 (Germany National Olympiad Fourth Round Tenth Grade, 2016[2])**.** A sequence of positive integers $a_1, a_2, a_3, \dots$ is defined as follows: a_1 is a 3 digit number and a_{k+1} (for $k \geq 1$) is obtained by

$$a_{k+1} = a_k + 2 \cdot Q(a_k),$$

where $Q(a_k)$ is the sum of digits of a_k when represented in decimal system. For instance, if one takes $a_1 = 358$ as the initial term, the sequence would be

$$\begin{aligned} a_1 &= 358, \\ a_2 &= 358 + 2 \cdot 16 = 390, \\ a_3 &= 390 + 2 \cdot 12 = 414, \\ a_4 &= 414 + 2 \cdot 9 = 432, \\ &\vdots \end{aligned}$$

Prove that no matter what we choose as the starting number of the sequence,

(a) the sequence will not contain 2015.

(b) the sequence will not contain 2016.

Problem 95 (Germany National Olympiad Fourth Round Eleventh Grade, 2016)**.** Find all positive integers m and n with $m \leq 2n$ which satisfy

$$m \cdot \binom{2n}{n} = \binom{m^2}{2}.$$

Problem 96 (Germany TST 2016)**.** The positive integers $a_1, a_2, \dots, a_n$ are aligned clockwise in a circular line with $n \geq 5$. Let $a_0 = a_n$ and $a_{n+1} = a_1$. For each $i \in \{1, 2, \dots, n\}$ the quotient

$$q_i = \frac{a_{i-1} + a_{i+1}}{a_1}$$

is an integer. Prove

$$2n \leq q_1 + q_2 + \cdots + q_n < 3n.$$

[2]Thanks to Arian Saffarzadeh for translating the problem.

Link

Problem 97 (Germany TST 2016, Taiwan TST First Round 2016). Determine all positive integers M such that the sequence $a_0, a_1, a_2, \cdots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \cdots$$

contains at least one integer term. Link

Problem 98 (Greece 2016). Find all triplets of nonnegative integers (x, y, z) and $x \leq y$ such that

$$x^2 + y^2 = 3 \cdot 2016^z + 77.$$

Link

Problem 99 (Greece TST 2016). Given is the sequence $(a_n)_{n \geq 0}$ which is defined as follows:$a_0 = 3$ and $a_{n+1} - a_n = n(a_n - 1)$, $\forall n \geq 0$. Determine all positive integers m such that $\gcd(m, a_n) = 1$, $\forall n \geq 0$. Link

Problem 100 (Harvard-MIT Math Tournament 2016). Denote by $\mathbb{N}$ the positive integers. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that, for any $w, x, y, z \in \mathbb{N}$,

$$f(f(f(z)))f(wxf(yf(z))) = z^2 f(xf(y))f(w).$$

Show that $f(n!) \geq n!$ for every positive integer n. Link

Problem 101 (Hong Kong (China) Mathematical Olympiad 2016). Find all integral ordered triples (x, y, z) such that $\sqrt{\dfrac{2015}{x+y}} + \sqrt{\dfrac{2015}{y+z}} + \sqrt{\dfrac{2015}{x+z}}$ are positive integers. Link

Problem 102 (Hong Kong Preliminary Selection Contest 2016). Find the remainder when

$$19^{17^{15^{\cdot^{\cdot^{\cdot^{3^1}}}}}}$$

is divided by 100.

Problem 103 (Hong Kong Preliminary Selection Contest 2016). Let k be an integer. If the equation

$$kx^2 + (4k - 2)x + (4k - 7) = 0$$

has an integral root, find the sum of all possible values of k.

Problem 104 (Hong Kong Preliminary Selection Contest 2016). Let n be a positive integer. If the two numbers $(n+1)(2n+15)$ and $n(n+5)$ have exactly the same prime factors, find the greatest possible value of n.

Problem 105 (Hong Kong Preliminary Selection Contest 2016). An arithmetic sequence with 10 terms has common difference $d > 0$. If the absolute value of each term is a prime number, find the smallest possible value of d.

Problem 106 (Hong Kong Preliminary Selection Contest 2016)**.** Let $a_1 = \frac{2}{3}$ and

$$a_{n+1} = \frac{a_n}{4} + \sqrt{\frac{24a_n + 9}{256}} - \frac{9}{48}$$

for all integers $n \geq 1$. Find the value of

$$a_1 + a_2 + a_3 + \dots$$

Problem 107 (Hong Kong TST 2016)**.** Find all natural numbers n such that n, n^2+10, n^2-2, n^3+6, and n^5+36 are all prime numbers. Link

Problem 108 (Hong Kong TST 2016)**.** Find all triples (m, p, q) such that

$$2^m p^2 + 1 = q^7,$$

where p and q are primes and m is a positive integer. Link

Problem 109 (Hong Kong TST 2016)**.** Find all prime numbers p and q such that $p^2 | q^3 + 1$ and $q^2 | p^6 - 1$. Link

Problem 110 (Hong Kong TST 2016)**.** Let p be a prime number greater than 5. Suppose there is an integer k satisfying that $k^2 + 5$ is divisible by p. Prove that there are positive integers m and n such that $p^2 = m^2 + 5n^2$. Link

Problem 111 (IberoAmerican 2016)**.** Find all prime numbers p, q, r, k such that $pq + qr + rp = 12k + 1$. Link

Problem 112 (IberoAmerican 2016)**.** Let k be a positive integer and $a_1, a_2, \dots, a_k$ digits. Prove that there exists a positive integer n such that the last $2k$ digits of 2^n are, in the following order, $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k$, for certain digits $b_1, b_2, \dots, b_k$. Link

Problem 113 (IMO Shortlist 2015, India TST 2016, Taiwan TST Second Round 2016, Croatian Mathematical Olympiad 2016, Switzerland TST 2016)**.** Let m and n be positive integers such that $m > n$. Define

$$x_k = \frac{m+k}{n+k}$$

for $k = 1, 2, \dots, n+1$. Prove that if all the numbers $x_1, x_2, \dots, x_{n+1}$ are integers, then $x_1 x_2 \dots x_{n+1} - 1$ is divisible by an odd prime.

Link

Problem 114 (IMO 2016)**.** A set of postive integers is called fragrant if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2+n+1$. What is the least possible positive integer value of b such that there exists a non-negative integer a for which the set

$$\{P(a+1), P(a+2), \dots, P(a+b)\}$$

is fragrant? Link

Problem 115 (India IMO Training Camp 2016)**.** Given that n is a natural number such that the leftmost digits in the decimal representations of 2^n and 3^n are the same, find all possible values of the leftmost digit. Link

Problem 116 (India IMO Practice Test 2016)**.** We say a natural number n is perfect if the sum of all the positive divisors of n is equal to $2n$. For example, 6 is perfect since its positive divisors $1, 2, 3, 6$ add up to $12 = 2 \times 6$. Show that an odd perfect number has at least 3 distinct prime divisors. Link

Problem 117 (India TST 2016)**.** Let n be a natural number. We define sequences $\langle a_i \rangle$ and $\langle b_i \rangle$ of integers as follows. We let $a_0 = 1$ and $b_0 = n$. For $i > 0$, we let

$$(a_i, b_i) = \begin{cases} (2a_{i-1} + 1, b_{i-1} - a_{i-1} - 1) & \text{if } a_{i-1} < b_{i-1}, \\ (a_{i-1} - b_{i-1} - 1, 2b_{i-1} + 1) & \text{if } a_{i-1} > b_{i-1}, \\ (a_{i-1}, b_{i-1}) & \text{if } a_{i-1} = b_{i-1}. \end{cases}$$

Given that $a_k = b_k$ for some natural number k, prove that $n + 3$ is a power of two. Link

Problem 118 (India TST 2016)**.** Let $\mathbb{N}$ denote the set of all natural numbers. Show that there exists two nonempty subsets A and B of $\mathbb{N}$ such that

1. $A \cap B = \{1\}$;
2. every number in $\mathbb{N}$ can be expressed as the product of a number in A and a number in B;
3. each prime number is a divisor of some number in A and also some number in B;
4. one of the sets A and B has the following property: if the numbers in this set are written as $x_1 < x_2 < x_3 < \cdots$, then for any given positive integer M there exists $k \in \mathbb{N}$ such that $x_{k+1} - x_k \geq M$;
5. Each set has infinitely many composite numbers.

Link

Problem 119 (India National Olympiad 2016)**.** Let $\mathbb{N}$ denote the set of natural numbers. Define a function $T : \mathbb{N} \to \mathbb{N}$ by $T(2k) = k$ and $T(2k + 1) = 2k + 2$. We write $T^2(n) = T(T(n))$ and in general $T^k(n) = T^{k-1}(T(n))$ for any $k > 1$.

(i) Show that for each $n \in \mathbb{N}$, there exists k such that $T^k(n) = 1$.

(ii) For $k \in \mathbb{N}$, let c_k denote the number of elements in the set $\{n : T^k(n) = 1\}$. Prove that $c_{k+2} = c_{k+1} + c_k$, for $k \geq 1$.

Link

Problem 120 (India National Olympiad 2016)**.** Consider a non-constant arithmetic progression $a_1, a_2, \cdots, a_n, \cdots$. Suppose there exist relatively prime positive integers $p > 1$ and $q > 1$ such that a_1^2, a_{p+1}^2 and a_{q+1}^2 are also the terms of the same arithmetic progression. Prove that the terms of the arithmetic progression are all integers. Link

Problem 121 (Iran Third Round National Olympiad 2016)**.** Let F be a subset of the set of positive integers with at least two elements and P be a polynomial with integer coefficients such that for any two elements of F like a and b, the following two conditions hold

(i) $a + b \in F$, and

(ii) $\gcd(P(a), P(b)) = 1$.

Prove that $P(x)$ is a constant polynomial. Link

Problem 122 (Iran Third Round National Olympiad 2016)**.** Let P be a polynomial with integer coefficients. We say P is *good* if there exist infinitely many prime numbers q such that the set

$$X = \{P(n) \mod q : \quad n \in \mathbb{N}\}$$

has at least $\frac{q+1}{2}$ members. Prove that the polynomial $x^3 + x$ is good. Link

Problem 123 (Iran Third Round National Olympiad 2016)**.** Let m be a positive integer. The positive integer a is called a *golden residue* modulo m if $\gcd(a, m) = 1$ and $x^x \equiv a \pmod m$ has a solution for x. Given a positive integer n, suppose that a is a golden residue modulo n^n. Show that a is also a golden residue modulo n^{n^n}. Link

Problem 124 (Iran Third Round National Olympiad 2016)**.** Let p, q be prime numbers (q is odd). Prove that there exists an integer x such that

$$q|(x+1)^p - x^p$$

if and only if

$$q \equiv 1 \pmod p.$$

Link

Problem 125 (Iran Third Round National Olympiad 2016)**.** We call a function g *special* if $g(x) = a^{f(x)}$ (for all x) where a is a positive integer and f is polynomial with integer coefficients such that $f(n) > 0$ for all positive integers n.

A function is called an *exponential polynomial* if it is obtained from the product or sum of special functions. For instance, $2^x 3^{x^2+x-1} + 5^{2x}$ is an exponential polynomial.

Prove that there does not exist a non-zero exponential polynomial $f(x)$ and a non-constant polynomial $P(x)$ with integer coefficients such that

$$P(n)|f(n)$$

for all positive integers n. Link

Problem 126 (Iran Third Round National Olympiad 2016)**.** A sequence $P = \{a_n\}_{n=1}^{\infty}$ is called a *permutation* of natural numbers if for any natural number m, there exists a unique natural number n such that $a_n = m$.

We also define $S_k(P)$ as $S_k(P) = a_1 + a_2 + \cdots + a_k$ (the sum of the first k elements of the sequence).

Prove that there exists infinitely many distinct permutations of natural numbers like $P_1, P_2, \ldots$ such that

$$\forall k, \forall i < j : S_k(P_i) | S_k(P_j).$$

Link

Problem 127 (Iran TST 2016)**.** Let $p \neq 13$ be a prime number of the form $8k+5$ such that 39 is a quadratic non-residue modulo p. Prove that the equation

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 \equiv 0 \pmod p$$

has a solution in integers such that $p \nmid x_1 x_2 x_3 x_4$. Link

Problem 128 (Italy National Olympiad 2016)**.** Determine all pairs of positive integers (a, n) with $a \geq n \geq 2$ for which $(a+1)^n + a - 1$ is a power of 2. Link

Problem 129 (Japan Mathematical Olympiad Preliminary 2016)**.** For $1 \leq n \leq 2016$, how many integers n satisfying the condition: the reminder divided by 20 is smaller than the one divided by 16. Link

Problem 130 (Japan Mathematical Olympiad Preliminary 2016)**.** Determine the number of pairs (a, b) of integers such that $1 \leq a, b \leq 2015$, a is divisible by $b+1$, and $2016 - a$ is divisible by b.Link

Problem 131 (Japan Mathematical Olympiad Finals 2016)**.** Let p be an odd prime number. For positive integer k satisfying $1 \leq k \leq p-1$, the number of divisors of $kp+1$ between k and p exclusive is a_k. Find the value of $a_1 + a_2 + \ldots + a_{p-1}$. Link

Problem 132 (Junior Balkan Mathematical Olympiad 2016)**.** Find all triplets of integers (a, b, c) such that the number

$$N = \frac{(a-b)(b-c)(c-a)}{2} + 2$$

is a power of 2016. Link

Problem 133 (Korea Summer Program Practice Test 2016)**.** A infinite sequence $\{a_n\}_{n \geq 0}$ of real numbers satisfy $a_n \geq n^2$. Suppose that for each $i, j \geq 0$ there exist k, l with $(i, j) \neq (k, l)$, $l - k = j - i$, and $a_l - a_k = a_j - a_i$. Prove that $a_n \geq (n+2016)^2$ for some n. Link

Problem 134 (Korea Summer Program Practice Test 2016)**.** A finite set S of positive integers is given. Show that there is a positive integer N dependent only on S, such that any $x_1, \ldots, x_m \in S$ whose sum is a multiple of N, can be partitioned into groups each of whose sum is exactly N. (The numbers $x_1, \ldots, x_m$ need not be distinct.) Link

Problem 135 (Korea Winter Program Practice Test 2016)**.** $p(x)$ is an irreducible polynomial with integer coefficients, and q is a fixed prime number. Let a_n be a number of solutions of the equation $p(x) \equiv 0 \mod q^n$. Prove that we can find M such that $\{a_n\}_{n \geq M}$ is constant. Link

Problem 136 (Korea Winter Program Practice Test 2016)**.** Find all $\{a_n\}_{n \geq 0}$ that satisfies the following conditions.

1. $a_n \in \mathbb{Z}$,
2. $a_0 = 0, a_1 = 1$,
3. For infinitely many m, $a_m = m$, and
4. For every $n \geq 2$, $\{2a_i - a_{i-1} | i = 1, 2, 3, \cdots, n\} \equiv \{0, 1, 2, \cdots, n-1\} \mod n$.

Link

Problem 137 (Korea Winter Program Practice Test 2016)**.** Find all positive integers a, b, m, and n such that

$$a^2 + b^2 = m^2 - n^2, \text{ and}$$
$$ab = 2mn.$$

Link

Problem 138 (Korea Winter Program Practice Test 2016)**.** Find all pairs of positive integers (n, t) such that $6^n + 1 = n^2 t$, and $(n, 29 \times 197) = 1$. Link

Problem 139 (Korea National Olympiad Final Round 2016)**.** Prove that for all rationals x, y, $x - \frac{1}{x} + y - \frac{1}{y} = 4$ is not true. Link

Problem 140 (Kosovo TST 2016)**.** Show that for any $n \geq 2$, the number $2^{2^n} + 1$ ends with 7. Link

Problem 141 (Latvia National Olympiad 2016)**.**

1. Given positive integers x and y such that xy^2 is a perfect cube, prove that $x^2 y$ is also a perfect cube. Link
2. Given that x and y are positive integers such that xy^{10} is perfect 33rd power of a positive integer, prove that $x^{10} y$ is also a perfect 33rd power. Link
3. Given that x and y are positive integers such that xy^{433} is a perfect 2016-power of a positive integer, prove that $x^{433} y$ is also a perfect 2016-power. Link
4. Given that x, y and z are positive integers such that $x^3 y^5 z^6$ is a perfect 7th power of a positive integer, show that also $x^5 y^6 z^3$ is a perfect 7th power. Link

Problem 142 (Latvia National Olympiad 2016)**.** Prove that among any 18 consecutive positive 3 digit numbers, there is at least one that is divisible by the sum of its digits. Link

Problem 143 (Latvia National Olympiad 2016)**.** Two functions are defined by equations: $f(a) = a^2 + 3a + 2$ and $g(b, c) = b^2 - b + 3c^2 + 3c$. Prove that for any positive integer a there exist positive integers b and c such that $f(a) = g(b, c)$.

Link

Problem 144 (Macedonian National Olympiad 2016)**.** Solve the equation in the set of natural numbers $1 + x^z + y^z = \mathrm{lcm}(x^z, y^z)$. Link

Problem 145 (Macedonian National Olympiad 2016)**.** Solve the equation in the set of natural numbers $xyz + yzt + xzt + xyt = xyzt + 3$. Link

Problem 146 (Macedonian Junior Mathematical Olympiad 2016)**.** Solve the equation

$$x_1^4 + x_2^4 + \cdots + x_{14}^4 = 2016^3 - 1$$

in the set of integers.

Problem 147 (Macedonian Junior Mathematical Olympiad 2016)**.** Solve the equation

$$x + y^2 + (\gcd(x, y))^2 = xy \cdot \gcd(x, y)$$

in the set of positive integers.

Problem 148 (Mediterranean Mathematics Olympiad 2016)**.** Determine all integers $n \geq 1$ for which the number $n^8 + n^6 + n^4 + 4$ is prime. Link

Problem 149 (Middle European Mathematical Olympiad 2016)**.** Find all $f : \mathbb{N} \to \mathbb{N}$ such that $f(a) + f(b)$ divides $2(a + b - 1)$ for all $a, b \in \mathbb{N}$. Link

Problem 150 (Middle European Mathematical Olympiad 2016)**.** A positive integer n is Mozart if the decimal representation of the sequence $1, 2, \ldots, n$ contains each digit an even number of times. Prove that:

1. All Mozart numbers are even.
2. There are infinitely many Mozart numbers.

Link

Problem 151 (Middle European Mathematical Olympiad 2016)**.** For a positive integer n, the equation $a^2 + b^2 + c^2 + n = abc$ is given in the positive integers. Prove that:

1. There does not exist a solution (a, b, c) for $n = 2017$.

2. For $n = 2016$, a is divisible by 3 for all solutions (a, b, c).

3. There are infinitely many solutions (a, b, c) for $n = 2016$.

Link

Problem 152 (Netherlands TST 2016). Find all positive integers k for which the equation:

$$\operatorname{lcm}(m, n) - \gcd(m, n) = k(m - n)$$

has no solution in integers positive (m, n) with $m \neq n$. Link

Problem 153 (Nordic Mathematical Competition 2016). Determine all sequences $(a_n)_{n=1}^{2016}$ of non-negative integers such that all sequence elements are less than or equal to 2016 and

$$i + j \mid ia_i + ja_j$$

for all $i, j \in \{1, 2, \ldots, 2016\}$.

Problem 154 (Norway Niels Henrik Abel Mathematics Competition Final Round 2016). (a) Find all positive integers a, b, c, and d with $a \leq b$ and $c \leq d$ such that

$$a + b = cd, c + d \qquad = ab.$$

(b) Find all non-negative integers x, y, and z such that

$$x^3 + 2y^3 + 4z^3 = 9!.$$

Problem 155 (Pan-African Mathematical Olympiad 2016). For any positive integer n, we define the integer $P(n)$ by

$$P(n) = n(n + 1)(2n + 1)(3n + 1) \ldots (16n + 1)$$

Find the greatest common divisor of the integers $P(1), P(2), P(3), \ldots, P(2016)$. Link

Problem 156 (Philippine Mathematical Olympiad Area Stage 2016). Let a, b, and c be three consecutive even numbers such that $a > b > c$. What is the value of $a^2 + b^2 + c^2 - ab - bc - ac$?

Problem 157 (Philippine Mathematical Olympiad Area Stage 2016). Find the sum of all the prime factors of 27,000,001.

Problem 158 (Philippine Mathematical Olympiad Area Stage 2016). Find the largest number N so that

$$\sum_{n=5}^{N} \frac{1}{n(n-2)} < \frac{1}{4}.$$

Problem 159 (Philippine Mathematical Olympiad Area Stage 2016)**.** Let s_n be the sum of the digits of a natural number n. Find the smallest value of $\frac{n}{s_n}$ if n is a four-digit number.

Problem 160 (Philippine Mathematical Olympiad Area Stage 2016)**.** The 6 digit number $\overline{739ABC}$ is divisible by $7, 8$, and 9. What values can A, B, and C take?

Problem 161 (Polish Mathematical Olympiad 2016)**.** Let p be a certain prime number. Find all non-negative integers n for which polynomial $P(x) = x^4 - 2(n+p)x^2 + (n-p)^2$ may be rewritten as product of two quadratic polynomials $P_1,\ P_2 \in \mathbb{Z}[X]$. Link

Problem 162 (Polish Mathematical Olympiad 2016)**.** Let k, n be odd positive integers greater than 1. Prove that if there a exists natural number a such that $k|2^a+1,\ n|2^a-1$, then there is no natural number b satisfying $k|2^b-1,\ n|2^b+1$. Link

Problem 163 (Polish Mathematical Olympiad 2016)**.** There are given two positive real number $a < b$. Show that there exist positive integers $p,\ q,\ r,\ s$ satisfying following conditions:

1. $a < \frac{p}{q} < \frac{r}{s} < b$.
2. $p^2 + q^2 = r^2 + s^2$.

Link

Problem 164 (Romania Danube Mathematical Competition 2016)**.** Determine all positive integers n such that all positive integers less than or equal to n and prime to n are pairwise coprime.

Problem 165 (Romania Danube Mathematical Competition 2016)**.** Given an integer $n \geq 2$, determine the numbers that can be written in the form

$$\sum_{i=2}^{k} a_{i-1}a_i,$$

where k is an integer greater than or equal to 2, and $a_1, a_2, \ldots, a_k$ are positive integers that add up to n.

Problem 166 (Romania Imar Mathematical Competition 2016)**.** Determine all positive integers expressible, for every integer $n \geq 3$, in the form

$$\frac{(a_1+1)(a_2+1)\ldots(a_n+1)-1}{a_1a_2\ldots a_n},$$

where $a_1, a_2, \ldots, a_n$ are pairwise distinct positive integers.

Problem 167 (Romanian Masters in Mathematics 2016)**.** A *cubic sequence* is a sequence of integers given by $a_n = n^3 + bn^2 + cn + d$, where b, c and d are integer constants and n ranges over all integers, including negative integers.

(a) Show that there exists a cubic sequence such that the only terms of the sequence which are squares of integers are a_{2015} and a_{2016}.

(b) Determine the possible values of $a_{2015} \cdot a_{2016}$ for a cubic sequence satisfying the condition in part (a).

Link

Problem 168 (Romanian Mathematical Olympiad District Round Grade 5, 2016)**.** Find all three-digit numbers which decrease 13 times when the tens' digit is suppressed.

Problem 169 (Romanian Mathematical Olympiad District Round Grade 5, 2016)**.** If A and B are positive integers, then $\overline{AB}$ will denote the number obtained by writing, in order, the digits of B after the digits of A. For instance, if $A = 193$ and $B = 2016$, then $\overline{AB} = 1932016$. Prove that there are infinitely many perfect squares of the form $\overline{AB}$ in each of the following situations:

(a) A and B are perfect squares;

(b) A and B are perfect cubes;

(c) A is a perfect cube and B is a perfect square;

(d) A is a perfect square and B is a perfect cube.

Problem 170 (Romanian Mathematical Olympiad District Round Grade 6, 2016)**.** The positive integers m and n are such that $m^{2016} + m + n^2$ is divisible by mn.

(a) Give an example of such m and n, with $m > n$.

(b) Prove that m is a perfect square.

Problem 171 (Romanian Mathematical Olympiad District Round Grade 7, 2016)**.** Find all pairs of positive integers (x, y) such that

$$x + y = \sqrt{x} + \sqrt{y} + \sqrt{xy}.$$

Problem 172 (Romanian Mathematical Olympiad District Round Grade 7, 2016)**.** Let

$$M = \left\{ x_1 + 2x_2 + 3x_3 + \cdots + 2015x_{2015} : x_1, x_2, \ldots, x_{2015} \in \{-2, 3\} \right\}.$$

Prove that $2015 \in M$ but $2016 \notin M$.

Problem 173 (Romanian Mathematical Olympiad District Round Grade 8, 2016)**.** For each positive integer n denote x_n the number of the positive integers with n digits, divisible by 4, formed with digits $2, 0, 1$, or 6.

(a) Compute x_1, x_2, x_3, and x_4;

(b) Find n so that

$$1 + \left\lfloor \frac{x_2}{x_1} \right\rfloor + \left\lfloor \frac{x_3}{x_2} \right\rfloor + \cdots + \left\lfloor \frac{x_{n+1}}{x_n} \right\rfloor = 2016.$$

Problem 174 (Romanian Mathematical Olympiad District Round Grade 8, 2016)**.**

(a) Prove that, for every integer k, the equation $x^3 - 24x + k = 0$ has at most one integer solution.

(b) Prove that the equation $x^3 + 24x - 2016 = 0$ has exactly one integer solution.

Problem 175 (Romanian Mathematical Olympiad District Round Grade 9, 2016)**.** Let a and n be positive integers such that

$$\left\{ \sqrt{n + \sqrt{n}} \right\} = \left\{ \sqrt{a} \right\},$$

where $\{\cdot\}$ denotes the fractional part. Prove that $4a + 1$ is a perfect square.

Problem 176 (Romanian Mathematical Olympiad District Round Grade 9, 2016)**.** Let $a \geq 2$ be an integer. Prove that the following statements are equivalent:

(a) One can find positive integers b and c such that $a^2 = b^2 + c^2$.

(b) One can find a positive integer d such that the equations $x^2 - ax + d = 0$ and $x^2 - ax - d = 0$ have integer roots.

Problem 177 (Romanian Mathematical Olympiad Final Round Grade 5, 2016)**.** Two positive integers x and y are such that

$$\frac{2010}{2011} < \frac{x}{y} < \frac{2011}{2012}.$$

Find the smallest possible value of the sum $x + y$.

Problem 178 (Romanian Mathematical Olympiad Final Round Grade 5, 2016)**.** Find all the positive integers a, b, and c with the property $a + b + c = abc$.

Problem 179 (Romanian Mathematical Olympiad Final Round Grade 6, 2016)**.** We will call a positive integer *exquisite* if it is a multiple of the number of its divisors (for instance, 12 is exquisite because it has 6 divisors and 12 is a multiple of 6).

(a) Find the largest exquisite two digit number.

(b) Prove that no exquisite number has its last digit 3.

Problem 180 (Romanian Mathematical Olympiad Final Round Grade 6, 2016)**.** Find all positive integers a and b so that $\frac{a+1}{b}$ and $\frac{b+2}{a}$ are simultaneously positive integers.

Problem 181 (Romanian Mathematical Olympiad Final Round Grade 6, 2016)**.** Let a and b be positive integers so that there exists a prime number p with the property $[a, a+p] = [b, b+p]$. Prove that $a = b$. Here, $[x, y]$ denotes the least common multiple of x and y.

Problem 182 (Romanian Mathematical Olympiad Final Round Grade 7, 2016)**.** Find all non-negative integers n such that

$$\sqrt{n+3} + \sqrt{n + \sqrt{n+3}}$$

is an integer.

Problem 183 (Romanian Mathematical Olympiad Final Round Grade 7, 2016)**.** Find all the positive integers p with the property that the sum of the first p positive integers is a four-digit positive integer whose decomposition into prime factors is of the form $2^m 3^n (m+n)$, where m and n are non-negative integers.

Problem 184 (Romanian Mathematical Olympiad Final Round Grade 8, 2016)**.** Let n be a non-negative integer. We will say that the non-negative integers $x_1, x_2, \ldots, x_n$ have property (P) if

$$x_1 x_2 \ldots x_n = x_1 + 2x_2 + \cdots + nx_n.$$

(a) Show that for every non-negative integer n, there exists n positive integers with property (P).

(b) Find all integers $n \geq 2$ so that there exists n positive integers $x_1, x_2, \ldots, x_n$ with $x_< x_2 < \cdots < x_n$, having property (P).

Problem 185 (Romanian Mathematical Olympiad Final Round Grade 9, 2016)**.**

(a) Prove that 7 cannot be written as a sum of squares of three rational numbers.

(b) Let a be a rational number that can be written as a sum of squares of three rational numbers. Prove that a^m can be written as a sum of squares of three rational numbers, for any positive integer m.

Problem 186 (Romanian National Mathematical Olympiad Small Juniors Shortlist, 2016)**.** Find all non-negative integers n so that n^2-4n+2, $n^2-3n+13$ and $n^2-6n+19$ are simultaneously primes.

Problem 187 (Romanian National Mathematical Olympiad Small Juniors Shortlist, 2016)**.** We will call a number good if it is a positive integer with at least two digits and by removing one of its digits we get a number which is equal to the sum of its initial digits (for instance, 109 is good: remove 9 to get $10 = 1 + 0 + 9$).

(a) Find the smallest good number.

(b) Find how many numbers are good.

Problem 188 (Romanian National Mathematical Olympiad Small Juniors Shortlist, 2016)**.** Four positive integers a, b, c, and d are not divisible by 5 and the sum of their squares is divisible by 5. Prove that

$$N = (a^2 + b^2)(a^2 + c^2)(a^2 + d^2)(b^2 + c^2)(b^2 + d^2)(c^2 + d^2)$$

is divisible by 625.

Problem 189 (Romanian National Mathematical Olympiad Small Juniors Shortlist, 2016)**.** For a positive integer n denote $d(n)$ the number of its positive divisors and $s(n)$ their sum. It is known that $n + d(n) = s(n) + 1$, $m + d(m) = s(m) + 1$, and $nm + d(nm) + 2016 = s(nm)$. Find n and m.

Problem 190 (Romanian National Mathematical Olympiad Small Juniors Shortlist, 2016)**.** Prove that there are no positive integers of the form

$$n = \underbrace{\overline{aa\ldots a}}_{k \text{ times}} + 5a, \quad k > 1$$

divisible by 2016.

Problem 191 (Romanian National Mathematical Olympiad Small Juniors Shortlist, 2016)**.** Find the smallest positive integer of the form

$$n = \underbrace{\overline{aa\ldots a}}_{k \text{ times}} + a(a-2)^2, \quad k > 1$$

divisible by 2016.

Problem 192 (Romanian National Mathematical Olympiad Small Juniors Shortlist, 2016)**.** A positive integer k will be called of *type* n ($n \neq k$) if n can be obtained by adding to k the sum or the product of the digits of k.

(a) Show that there are at least two numbers of type 2016.

(b) Find all numbers of type 216.

Problem 193 (Romanian National Mathematical Olympiad Juniors Shortlist, 2016)**.**

(a) Prove that $2^n + 3^n + 5^n + 8^n$ is not a perfect square for any positive integer n.

(b) Find all positive integers n so that

$$1^n + 4^n + 6^n + 7^n = 2^n + 3^n + 5^n + 8^n.$$

Problem 194 (Romanian National Mathematical Olympiad Juniors Shortlist, 2016)**.**

(a) Find all perfect squares of the form $\overline{aabcc}$.

(b) Let n be a given positive integer. Prove that there exists a perfect square of the form

$$\overline{aab\underbrace{cc\cdots c}_{2n \text{ times}}}.$$

Problem 195 (Romanian National Mathematical Olympiad Juniors Shortlist, 2016)**.** Prove that $2n^2+27n+91$ is a perfect square for infinitely many positive integers n.

Problem 196 (Romanian National Mathematical Olympiad Seniors Shortlist, 2016)**.** Let p be a prime number and $n_1, n_2, \ldots, n_k \in \{1, 2, \ldots, p-1\}$ be positive integers. Show that the equation

$$x_1^{n_1} + x_2^{n_2} + \cdots + x_k^{n_k} = x_{k+1}^p$$

has infinitely many positive integer solutions.

Problem 197 (Romanian National Mathematical Olympiad Seniors Shortlist, 2016)**.** Let $n \geq 4$ be a positive integer and define $A_n = \{1, 2, \ldots, n-1\}$. Find the number of solutions in the set $A_n \times A_n \times A_n \times A_n$ of the system

$$\begin{cases} x + z &= 2y, \\ y + t &= 2z. \end{cases}$$

Problem 198 (Romanian Stars of Mathematics Junior Level 2016)**.** Show that there are positive odd integers $m_1 < m_2 < \ldots$ and positive integers $n_1 < n_2 < \ldots$ such that m_k and n_k are relatively prime, and $m_k^k 2n_k^4$ is a perfect square for each index k.

Problem 199 (Romanian Stars of Mathematics Junior Level 2016)**.** Given an integer $n \geq 3$ and a permutation $a_1, a_2, \ldots, a_n$ of the first n positive integers, show that at least $\sqrt{n}$ distinct residue classes modulo n occur in the list

$$a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_n.$$

Problem 200 (Romanian Stars of Mathematics Senior Level 2016)**.** Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be n positive integers. Show that

$$\sum_{k=1}^{n} \frac{\sqrt{a_k}}{1 + a_1 + a_2 + \cdots + a_k} < \sum_{k=1}^{n^2} \frac{1}{k}.$$

Problem 201 (Romania TST for Junior Balkan Mathematical Olympiad 2016)**.** Let M be the set of natural numbers k for which there exists a natural number n such that

$$3^n \equiv k \pmod{n}.$$

Prove that M has infinitely many elements. Link

Problem 202 (Romania TST for Junior Balkan Mathematical Olympiad 2016)**.** Let n be an integer greater than 2 and consider the set

$$A = \{2^n - 1, 3^n - 1, \dots, (n-1)^n - 1\}.$$

Given that n does not divide any element of A, prove that n is a square-free number. Does it necessarily follow that n is a prime? Link

Problem 203 (Romania TST for Junior Balkan Mathematical Olympiad 2016)**.** Let n be a positive integer and consider the system

$$S(n) : \begin{cases} x^2 + ny^2 = z^2 \\ nx^2 + y^2 = t^2 \end{cases},$$

where x, y, z, and t are naturals. If

- $M_1 = \{n \in \mathbb{N} : \text{system } S(n) \text{ has infinitely many solutions}\}$, and
- $M_1 = \{n \in \mathbb{N} : \text{system } S(n) \text{ has no solutions}\}$,

prove that

(a) $7 \in M_1$ and $10 \in M_2$.

(b) sets M_1 and M_2 are infinite.

Link

Problem 204 (Romania TST 2016)**.** Let n be a positive integer and let $a_1, a_2, \dots, a_n$ be pairwise distinct positive integers. Show that

$$\sum_{k=1}^{n} \frac{1}{[a_1, a_2, \dots, a_k]} < 4,$$

where $[a_1, a_2, \dots, a_k]$ is the least common multiple of the integers $a_1, a_2, \dots, a_k$.

Problem 205 (Romania TST 2016)**.** Determine the integers $k \geq 2$ for which the sequence

$$\binom{2n}{n} \pmod{k}, \quad n = 0, 1, 2, \dots,$$

is eventually periodic.

Problem 206 (Romania TST 2016)**.** Given positive integers k and m, show that m and $\binom{n}{k}$ are coprime for infinitely many integers $n \geq k$.

Problem 207 (Romania TST 2016)**.** Prove that:

(a) If $(a_n)_{n\geq 1}$ is a strictly increasing sequence of positive integers such that $(a_{2n1} + a_{2n})/a_n$ is constant as n runs through all positive integers, then this constant is an integer greater than or equal to 4; and

(b) Given an integer $N \geq 4$, there exists a strictly increasing sequence $(a_n)_{n\geq 1}$ of positive integers such that $(a_{2n1} + a_{2n})/a_n = N$ for all indices n.

Problem 208 (Romaina TST 2016)**.** Given a positive integer k and an integer $a \equiv 3 \pmod 8$, show that $a^m + a + 2$ is divisible by 2^k for some positive integer m.

Problem 209 (Romaina TST 2016)**.** Given a positive integer n, show that for no set of integers modulo n, whose size exceeds $1 + \sqrt{n+4}$, is it possible that the pairwise sums of unordered pairs be all distinct.

Problem 210 (Romania TST 2016)**.** Given a prime p, prove that the sum $\sum_{k=1}^{\lfloor q/p \rfloor} k^{p-1}$ is not divisible by q for all but finitely many primes q. Link

Problem 211 (Romania TST 2016)**.** Determine the positive integers expressible in the form $\frac{x^2+y}{xy+1}$, for at least two pairs (x, y) of positive integers. Link

Problem 212 (All-Russian Olympiads 2016, Grade 11)**.** Let n be a positive integer and let $k_0, k_1, \ldots, k_{2n}$ be nonzero integers such that $k_0+k_1+\cdots+k_{2n} \neq 0$. Is it always possible to a permutation $(a_0, a_1, \ldots, a_{2n})$ of $(k_0, k_1, \ldots, k_{2n})$ so that the equation

$$a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_0 = 0$$

has not integer roots? Link

Problem 213 (San Diego Math Olympiad 2016)**.** Let a, b, c, d be four integers. Prove that

$$(b-a)(c-a)(d-a)(d-c)(d-b)(c-b)$$

is divisible by 12. Link

Problem 214 (San Diego Math Olympiad 2016)**.** Quadratic equation $x^2+ax+b+1 = 0$ have 2 positive integer roots, for integers a, b. Show that a^2+b^2 is not a prime. Link

Problem 215 (Saudi Arabia Preselection Test 2016)**.** Let p be a given prime. For each prime r, we define

$$F(r) = \frac{(p^{rp}-1)(p-1)}{(p^r-1)(p^p-1)}.$$

1. Show that $F(r)$ is a positive integer for any prime $r \neq p$.

2. Show that $F(r)$ and $F(s)$ are coprime for any primes r and s such that $r \neq p, s \neq p$ and $r \neq s$.

3. Fix a prime $r \neq p$. Show that there is a prime divisor q of $F(r)$ such that $p|q-1$ but $p^2 \nmid q-1$.

Problem 216 (Saudi Arabia Preselection Test 2016). Let u and v be positive rational numbers with $u \neq v$. Assume that there are infinitely many positive integers n with the property that $u^n - v^n$ is an integer. Prove that u and v are integers.

Problem 217 (Saudi Arabia Preselection Test 2016). Let a and b be two positive integers such that

$$b+1 \mid a^2+1 \quad \text{and} \quad a+1 \mid b^2+1$$

Prove that both a and b are odd.

Problem 218 (Saudi Arabia Preselection Test 2016).

1. Prove that there are infinitely many positive integers n such that there exists a permutation of $1, 2, 3, \ldots, n$ with the property that the difference between any two adjacent numbers is equal to either 2015 or 2016.

2. Let k be a positive integer. Is the statement in part 1 still true if we replace the numbers 2015 and 2016 by k and $k + 2016$, respectively?

Problem 219 (Saudi Arabia Preselection Test 2016). Let n be a given positive integer. Prove that there are infinitely many pairs of positive integers (a, b) with $a, b > n$ such that

$$\prod_{i=1}^{2015}(a+i) \mid b(b+2016),$$
$$\prod_{i=1}^{2015}(a+i) \nmid b,$$
$$\prod_{i=1}^{2015}(a+i) \nmid (b+2016).$$

Problem 220 (Saudi Arabia TST for Gulf Mathematical Olympiad 2016). Find all positive integer n such that there exists a permutation $(a_1, a_2, \ldots, a_n)$ of $(1, 2, 3, \ldots, n)$ satisfying the condition:

$$a_1 + a_2 + \cdots + a_k \text{ is divisible by } k \text{ for each } k = 1, 2, 3, \ldots, n.$$

Problem 221 (Saudi Arabia TST for Balkan Mathematical Olympiad 2016). Show that there are infinitely many positive integers n such that n has at least two prime divisors and $20^n + 16^n$ is divisible by n^2.

Problem 222 (Saudi Arabia TST for Balkan Mathematical Olympiad 2016)**.** Let m and n be odd integers such that (n^2-1) is divisible by m^2+1-n^2. Prove that $|m^2+1-n^2|$ is a perfect square.

Problem 223 (Saudi Arabia TST for Balkan Mathematical Olympiad 2016)**.** Let $a > b > c > d$ be positive integers such that

$$a^2 + ac - c^2 = b^2 + bd - d^2.$$

Prove that $ab + cd$ is a composite number.

Problem 224 (Saudi Arabia TST for Balkan Mathematical Olympiad 2016)**.** For any positive integer n, show that there exists a positive integer m such that n divides $2016^m + m$.

Problem 225 (Saudi Arabia TST for Balkan Mathematical Olympiad 2016)**.** Let d be a positive integer. Show that for every integer S, there exist a positive integer n and a sequence $a_1, a_2, \dots, a_n \in \{1, 1\}$ such that

$$S = a_1(1+d)^2 + a_2(1+2d)^2 + \cdots + a_n(1+nd)^2.$$

Problem 226 (Saudi Arabia TST for Balkan Mathematical Olympiad 2016)**.** Let p and q be given primes and the sequence $(p_n)_{n\geq 1}$ defined recursively as follows: $p_1 = p$, $p_2 = q$, and p_{n+2} is the largest prime divisor of the number $(p_n + p_{n+1} + 2016)$ for all $n \geq 1$. Prove that this sequence is bounded. That is, there exists a positive real number M such that $a_n < M$ for all positive integers n.

Problem 227 (Saudi Arabia IMO TST 2016)**.** Let $n \geq 3$ be an integer and let $x_1, x_2, \dots, x_n$ be n distinct integers. Prove that

$$(x_1 - x_2)^2 + (x_2 - x_3)^2 + \cdots + (x_n - x_1)^2 \geq 4n - 6.$$

Problem 228 (Saudi Arabia IMO TST 2016)**.** Let k be a positive integer. Prove that there exist integers x and y, neither of which divisible by 7, such that

$$x^2 + 6y^2 = 7^k.$$

Problem 229 (Saudi Arabia IMO TST 2016)**.** Define the sequence $a_1, a_2, \dots a_s$ follows: $a_1 = 1$, and for every $n \geq 2$, $a_n = n-2$ if $a_{n-1} = 0$ and $a_n = a_{n-1} - 1$, otherwise. Find the number of $1 \leq k \leq 2016$ such that there are non-negative integers r and s and a positive integer n satisfying $k = r + s$ and $a_{n+r} = a_n + s$.

Problem 230 (Saudi Arabia IMO TST 2016)**.** Let a be a positive integer. Find all prime numbers p with the following property: there exist exactly p ordered pairs of integers (x, y), with $0 \leq x, y \leq p-1$, such that p divides $y^2 - x^3 - a^2x$.

Problem 231 (Saudi Arabia IMO TST 2016)**.** Find the number of permutations $(a_1, a_2, \dots, a_{2016})$ of the first 2016 positive integers satisfying the following two conditions:

1. $a_{i+1} - a_i \leq 1$ for all $i = 1, 2, \ldots, 2015$, and

2. There are exactly two indices $i < j$ with $1 \leq i < j \leq 2016$ such that $a_i = i$ and $a_j = j$.

Problem 232 (Saudi Arabia IMO TST 2016). Call a positive integer $N \geq 2$ *special* if for every k such that $2 \leq k \leq N$, N can be expressed as a sum of k positive integers that are relatively prime to N (although not necessarily relatively prime to each other). Find all special positive integers.

Problem 233 (Serbia Additional TST 2016). Let $w(x)$ be largest odd divisor of x. Let a, b be natural numbers such that $(a, b) = 1$ and $a + w(b+1)$ and $b + w(a+1)$ are powers of two. Prove that $a + 1$ and $b + 1$ are powers of two. Link

Problem 234 (Serbia National Olympiad 2016). Let $n > 1$ be an integer. Prove that there exist $m > n^n$ such that

$$\frac{n^m - m^n}{m + n}$$

is a positive integer. Link

Problem 235 (Serbia National Olympiad 2016). Let $a_1, a_2, \ldots, a_{2^{2016}}$ be positive integers not bigger than 2016. We know that for each $n \leq 2^{2016}$, $a_1 a_2 \ldots a_n + 1$ is a perfect square. Prove that for some i , $a_i = 1$. Link

Problem 236 (Serbia TST for Junior Balkan Mathematical Olympiad 2016). Find minimal number of divisors that can number $|2016^m - 36^n|$ have,where m and n are natural numbers. Link

Problem 237 (Slovakia Domestic Category B Mathematical Olympiad 2016). Let k, l, and m be positive integers such that

$$\frac{k + m + klm}{lm + 1} = \frac{2051}{44}.$$

Find all possible values for klm.

Problem 238 (Slovakia Domestic Category B Mathematical Olympiad 2016). A positive integer has the property that the number of its even divisors is 3 more than the number of its odd divisors. What is the ratio of sum of all even divisors over the sum of all odd divisors of this number? Find all possible answers.

Problem 239 (Slovakia Domestic Category C Mathematical Olympiad 2016). Find all possible values for the product pqr, where p, q, and r are primes satisfying

$$p^2 - (q + r)^2 = 637.$$

Problem 240 (Slovakia School Round Category C Mathematical Olympiad 2016)**.** Find all four digit numbers $\overline{abcd}$ such that

$$\overline{abcd} = 20 \cdot \overline{ab} + 16 \cdot \overline{cd}.$$

Problem 241 (Slovakia Regional Round Category B Mathematical Olympiad 2016)**.** Determine all positive integers k, l, and m such that

$$\frac{3l+1}{3kl+k+3} = \frac{lm+1}{5lm+m+5}.$$

Problem 242 (Slovakia Regional Round Category C Mathematical Olympiad 2016)**.** Find the least possible value of

$$3x^2 - 12xy + y^4,$$

where x and y are non-negative integers.

Problem 243 (Slovakia National Round Category A Mathematical Olympiad 2016)**.** Let $p > 3$ be a prime. Determine the number of all $6-$tuples (a, b, c, d, e, f) of positive integers with sum $3p$ such that

$$\frac{a+b}{c+d}, \frac{b+c}{d+e}, \frac{c+d}{e+f}, \frac{d+e}{f+a}, \frac{e+f}{a+b}$$

are all integers.

Problem 244 (Slovakia TST 2016)**.** Let n be a positive integer and let S_n be the set of all positive divisors of n (including 1 and n). Prove that the rightmost digit of more than half of the elements of S_n is 3.

Problem 245 (Slovakia TST 2016)**.** Find all odd integers M for which the sequence $a_0, a_1, a_2, \dots$ defined by $a_0 = \frac{1}{2}(2M+1)$ and $a_{k+1} = a_k \lfloor a_k \rfloor$ for $k = 0, 1, 2, \dots$ contains at least one integer.

Problem 246 (South Africa National Olympiad 2016)**.** Let k and m be integers with $1 < k < m$. For a positive integer i, let L_i be the least common multiple of $1, 2, \dots, i$. Prove that k is a divisor of

$$L_i \cdot \left[\binom{m}{i} - \binom{m-k}{i} \right]$$

for all $i \geq 1$. Link

Problem 247 (Slovenia National Math Olympiad First Grade 2016)**.** Find all relatively prime integers x and y that solve the equation

$$4x^3 + y^3 = 3xy^2.$$

Problem 248 (Slovenia National Math Olympiad Fourth Grade 2016)**.** Find all integers a, b, c, and d that solve the equation

$$a^2 + b^2 + c^2 = d + 13,$$
$$a + 2b + 3c = \frac{d}{2} + 13.$$

Problem 249 (Slovenia IMO TST 2016)**.** Let

$$N = 2^{15} \cdot 2015.$$

How many divisors of N^2 are strictly smaller than N and do not divide N?

Problem 250 (Slovenia IMO TST 2016, Philippine 2015)**.** Prove that for all positive integers $n \geq 2$,

$$\frac{1}{2} + \sqrt{\frac{1}{2}} + \sqrt[3]{\frac{2}{3}} + \cdots + \sqrt[n]{\frac{n-1}{n}} < \frac{n^2}{n+1}.$$

Problem 251 (Slovenia IMO TST 2016, Romania JBMO TST 2015)**.** Find all positive integers a, b, c, and d such that

$$4^a \cdot 5^b - 3^c \cdot 11^d = 1.$$

Problem 252 (Spain National Olympiad 2016)**.** Two real number sequences are guiven, one arithmetic $(a_n)_{n\in\mathbb{N}}$ and another geometric sequence $(g_n)_{n\in\mathbb{N}}$ none of them constant. Those sequences verifies $a_1 = g_1 \neq 0$, $a_2 = g_2$ and $a_{10} = g_3$. Find with proof that, for every positive integer p, there is a positive integer m, such that $g_p = a_m$. Link

Problem 253 (Spain National Olympiad 2016)**.** Given a positive prime number p. Prove that there exist a positive integer α such that $p|\alpha(\alpha-1)+3$, if and only if there exist a positive integer β such that $p|\beta(\beta-1)+25$. Link

Problem 254 (Spain National Olympiad 2016)**.** Let m be a positive integer and a and b be distinct positive integers strictly greater than m^2 and strictly less than m^2+m. Find all integers d such that $m^2 < d < m^2+m$ and d divides ab. Link

Problem 255 (Switzerland Preliminary Round 2016)**.** Determine all natural numbers n such that for all positive divisors d of n,

$$d + 1 \mid n + 1.$$

Problem 256 (Switzerland Final Round 2016)**.** Find all positive integers n for which primes p and q exist such that

$$p(p+1) + q(q+1) = n(n+1)$$

Problem 257 (Switzerland Final Round 2016)**.** Let a_n be a sequence of positive integers defined by $a_1 = m$ and $a_n = a_{n-1}^2 - 1$ for $n = 2, 3, 4, \ldots$. A pair (a_k, a_l) is called *interesting* if

(i) $0 < l - k < 2016$, and

(ii) a_k divides a_l.

Prove that there exists a positive integer m such that the sequence a_n contains no interesting pair.

Problem 258 (Switzerland TST 2016)**.** Let n be a positive integer. We call a pair of natural numbers *incompatible* if their greatest common divisor is equal to 1. Find the minimum value of incompatible pairs when one divides the set $\{1, 2, \ldots, 2n\}$ into n pairs.

Problem 259 (Switzerland TST 2016)**.** Let n be a positive integer. Show that $7^{7^n} + 1$ has at least $2n + 3$ prime divisors (not necessarily distinct).

Problem 260 (Switzerland TST 2016)**.** Find all positive integers n such that

$$\sum_{\substack{d|n \\ 1 \leq d \leq n}} d^2 = 5(n+1).$$

Problem 261 (Syria Central Round First Stage 2016)**.** A positive integer $n \geq 2$ is called *special* if n^2 can be written as sum of n consecutive positive integers (for instance, 3 is special since $3^2 = 2 + 3 + 4$).

(i) Prove that 2016 is not special.

(ii) Prove that the product of two special numbers is also special.

Problem 262 (Syria Central Round Second Stage 2016)**.** Find all integers a and b such that $a^3 - b^2 = 2$.

Problem 263 (Syria TST 2016)**.** Find all positive integers m and n such that

$$\frac{1}{m} + \frac{1}{n} = \frac{3}{2014}.$$

Problem 264 (Taiwan TST First Round 2016)**.** Find all ordered pairs (a, b) of positive integers that satisfy $a > b$ and the equation $(a - b)^{ab} = a^b b^a$. Link

Problem 265 (Taiwan TST Second Round 2016)**.** Let a and b be positive integers such that $a! + b!$ divides $a!b!$. Prove that $3a \geq 2b + 2$. Link

Problem 266 (Taiwan TST Second Round 2016)**.** Let $\langle F_n \rangle$ be the Fibonacci sequence, that is, $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ holds for all nonnegative integers n. Find all pairs (a, b) of positive integers with $a < b$ such that $F_n - 2na^n$ is divisible by b for all positive integers n. Link

Problem 267 (Taiwan TST Third Round 2016)**.** Let n be a positive integer. Find the number of odd coefficients of the polynomial $(x^2 - x + 1)^n$. Link

Problem 268 (Taiwan TST Third Round 2016)**.** Let k be a positive integer. A sequence $a_0, a_1, \ldots, a_n$ $(n > 0)$ of positive integers satisfies the following conditions:

(i) $a_0 = a_n = 1$;

(ii) $2 \leq a_i \leq k$ for each $k = 1, 2, \ldots, n-1$

(iii) For each $j = 2, 3, \ldots, k$, the number j appears $\varphi(j)$ times in the sequence $a_0, a_1, \ldots, a_n$ ($\varphi(j)$ is the number of positive integers that do not exceed j and are coprime to j);

(iv) For any $i = 1, 2, \ldots, n-1$, $\gcd(a_{i-1}, a_i) = 1 = \gcd(a_i, a_{i+1})$, and a_i divides $a_{i-1} + a_{i+1}$

There is another sequence $b_0, b_1, \ldots, b_n$ of integers such that

$$\frac{b_{i+1}}{a_{i+1}} > \frac{b_i}{a_i}$$

for all $i = 0, 1, \ldots, n-1$. Find the minimum value for $b_n - b_0$. Link

Problem 269 (Taiwan TST Third Round 2016)**.** Let $f(x)$ be the polynomial with integer coefficients ($f(x)$ is not constant) such that

$$(x^3 + 4x^2 + 4x + 3)f(x) = (x^3 - 2x^2 + 2x - 1)f(x+1)$$

Prove that for each positive integer $n \geq 8$, $f(n)$ has at least five distinct prime divisors. Link

Problem 270 (Turkey TST for European Girls' Mathematical Olympiad 2016)**.** Prove that for every square-free integer $n > 1$, there exists a prime number p and an integer m satisfying

$$p \mid n \quad \text{and} \quad n \mid p^2 + p \cdot m^p.$$

Link

Problem 271 (Turkey TST for Junior Balkan Mathematical Olympiad 2016)**.** Let n be a positive integer, p and q be prime numbers such that

$$pq \mid n^p + 2 \quad \text{and} \quad n + 2 \mid n^p + q^p.$$

Prove that there exists a positive integer m satisfying $q \mid 4^m \cdot n + 2$. Link

Problem 272 (Turkey TST for Junior Balkan Mathematical Olympiad 2016)**.** Find all pairs (p, q) of prime numbers satisfying

$$p^3 + 7q = q^9 + 5p^2 + 18p.$$

Link

Problem 273 (Turkey TST 2016). p is a prime. Let K_p be the set of all polynomials with coefficients from the set $\{0, 1, \ldots, p-1\}$ and degree less than p. Assume that for all pairs of polynomials $P, Q \in K_p$ such that $P(Q(n)) \equiv n \pmod{p}$ for all integers n, the degrees of P and Q are equal. Determine all primes p with this condition. Link

Problem 274 (Turkmenistan Regional Olympiad 2016). Find all distinct prime numbers p, q, r, s such that

$$1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r} - \frac{1}{s} = \frac{1}{pqrs}.$$

Link

Problem 275 (Tuymaada Senior League 2016). For each positive integer k determine the number of solutions of the equation

$$8^k = x^3 + y^3 + z^3 - 3xyz$$

in non-negative integers x, y, and z such that $0 \le x \le y \le z$.

Problem 276 (Tuymaada Senior League 2016). The ratio of prime numbers p and q does not exceed 2 ($p \neq q$). Prove that there are two consecutive positive integers such that the largest prime divisor of one of them is p and that of the other is q.

Problem 277 (Tuymaada Junior League 2016). Is there a positive integer $N > 10^{20}$ such that all its decimal digits are odd, the numbers of digits $1, 3, 5, 7, 9$ in its decimal representation are equal, and it is divisible by each $20-$digit number obtained from it by deleting digits? (Neither deleted nor remaining digits must be consecutive.)

Problem 278 (Ukraine TST for UMO 2016). Find all numbers n such, that in $[1; 1000]$ there exists exactly 10 numbers with digit sum equal to n. Link

Problem 279 (Ukraine TST for UMO 2016). Number 125 is written as the sum of several pairwise distinct and relatively prime numbers, greater than 1. What is the maximal possible number of terms in this sum? Link

Problem 280 (Ukraine TST for UMO 2016). Given prime number p and different natural numbers m, n such that $p^2 = \frac{m^2+n^2}{2}$. Prove that $2p - m - n$ is either square or doubled square of an integer number. Link

Problem 281 (Ukraine TST for UMO 2016). Solve the equation $n(n^2 + 19) = m(m^2 - 10)$ in positive integers. Link

Problem 282 (USA AIME 2016). For $-1 < r < 1$, let $S(r)$ denote the sum of the geometric series

$$12 + 12r + 12r^2 + 12r^3 + \ldots.$$

Let a between -1 and 1 satisfy $S(a)S(-a) = 2016$. Find $S(a) + S(-a)$. Link

Problem 283 (USA AIME 2016). For a permutation $p = (a_1, a_2, \ldots, a_9)$ of the digits $1, 2, \ldots, 9$, let $s(p)$ denote the sum of the three 3-digit numbers $a_1a_2a_3$, $a_4a_5a_6$, and $a_7a_8a_9$. Let m be the minimum value of $s(p)$ subject to the condition that the units digit of $s(p)$ is 0. Let n denote the number of permutations p with $s(p) = m$. Find $|m - n|$. Link

Problem 284 (USA AIME 2016). A strictly increasing sequence of positive integers $a_1, a_2, a_3, \ldots$ has the property that for every positive integer k, the subsequence $a_{2k-1}, a_{2k}, a_{2k+1}$ is geometric and the subsequence $a_{2k}, a_{2k+1}, a_{2k+2}$ is arithmetic. Suppose that $a_{13} = 2016$. Find a_1. Link

Problem 285 (USA AIME 2016). Find the least positive integer m such that $m^2 - m + 11$ is a product of at least four not necessarily distinct primes. Link

Problem 286 (USA AIME 2016). Let x, y and z be real numbers satisfying the system

$$\begin{aligned}\log_2(xyz - 3 + \log_5 x) &= 5\\ \log_3(xyz - 3 + \log_5 y) &= 4\\ \log_4(xyz - 3 + \log_5 z) &= 4.\end{aligned}$$

Find the value of $|\log_5 x| + |\log_5 y| + |\log_5 z|$. Link

Problem 287 (USA AIME 2016). For polynomial $P(x) = 1 - \frac{1}{3}x + \frac{1}{6}x^2$, define

$$Q(x) = P(x)P(x^3)P(x^5)P(x^7)P(x^9) = \sum_{i=0}^{50} a_i x^i.$$

Then $\sum\limits_{i=0}^{50} |a_i| = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$. Link

Problem 288 (USA AIME 2016). Find the number of sets $\{a, b, c\}$ of three distinct positive integers with the property that the product of $a, b,$ and c is equal to the product of $11, 21, 31, 41, 51,$ and 61. Link

Problem 289 (USA AIME 2016). The sequences of positive integers $1, a_2, a_3, \ldots$ and $1, b_2, b_3, \ldots$ are an increasing arithmetic sequence and an increasing geometric sequence, respectively. Let $c_n = a_n + b_n$. There is an integer k such that $c_{k-1} = 100$ and $c_{k+1} = 1000$. Find c_k. Link

Problem 290 (USA AIME 2016). For positive integers N and k, define N to be k-nice if there exists a positive integer a such that a^k has exactly N positive divisors. Find the number of positive integers less than 1000 that are neither 7-nice nor 8-nice.

Link

Problem 291 (USAJMO 2016). Prove that there exists a positive integer $n < 10^6$ such that 5^n has six consecutive zeros in its decimal representation. Link

Problem 292 (USAMO 2016)**.** Prove that for any positive integer k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer. Link

Problem 293 (USAMO 2016)**.**

(a) Prove that if n is an odd perfect number then n has the following form

$$n = p^s m^2$$

where p is prime has form $4k+1$, s is positive integers has form $4h+1$, and $m \in \mathbb{Z}^+$, m is not divisible by p.

(b) Find all $n \in \mathbb{Z}^+$, $n > 1$ such that $n-1$ and $\frac{n(n+1)}{2}$ is perfect number.

Link

Problem 294 (USA TSTST 2016)**.** Decide whether or not there exists a non-constant polynomial $Q(x)$ with integer coefficients with the following property: for every positive integer $n > 2$, the numbers

$$Q(0),\ Q(1), Q(2),\ \ldots,\ Q(n-1)$$

produce at most $0.499n$ distinct residues when taken modulo n. Link

Problem 295 (USA TSTST 2016)**.** Suppose that n and k are positive integers such that

$$1 = \underbrace{\varphi(\varphi(\dots\varphi(}_{k \text{ times}} n)\dots)).$$

Prove that $n \leq 3^k$. Link

Problem 296 (USA TST 2016)**.** Let $\sqrt{3} = 1.b_1 b_2 b_3 \dots_{(2)}$ be the binary representation of $\sqrt{3}$. Prove that for any positive integer n, at least one of the digits b_n, b_{n+1}, $\ldots$, b_{2n} equals 1. Link

Problem 297 (Venezuela Final Round Fourth Year 2106)**.** Find all pairs of prime numbers (p,q), with $p < q$, such that the numbers $p+2q, 2p+q$ and $p+q-22$ are also primes.

Problem 298 (Zhautykov Olympiad 2016)**.** $a_1, a_2, ..., a_{100}$ are permutation of $1, 2, ..., 100$. $S_1 = a_1, S_2 = a_1 + a_2, ..., S_{100} = a_1 + a_2 + ... + a_{100}$Find the maximum number of perfect squares from S_i. Link

Problem 299 (Zhautykov Olympiad 2016)**.** We call a positive integer q a *convenient denominator* for a real number α if

$$|\alpha - \frac{p}{q}| < \frac{1}{10q}$$

for some integer p. Prove that if two irrational numbers α and β have the same set of convenient denominators then either $\alpha + \beta$ or $\alpha - \beta$ is an integer. Link