A SIMPLE CRITERION FOR IRRATIONALITY OF SOME REAL NUMBERS

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ABSTRACT. In this paper, we calculate asymptotic average of the decimals of some real numbers. For example, we show that the asymptotic average of the decimals of simply normal numbers is $9/2$. We also prove that if a real number r cannot be represented as a finite decimal and the asymptotic average of its decimals is zero, then r is irrational.

0. INTRODUCTION

A real number $r \in \mathbb{R}$ is *rational* if $r = \dfrac{m}{n}$, where m and n are integers and $n \neq 0$. Otherwise, the number r is *irrational*. The irrational numbers are further classified as *algebraic* if r is the root of an irreducible polynomial $f(X) \in \mathbb{Z}[X]$ of degree $\deg(f) \geq 2$, otherwise it is *transcendental* [5, 11]. The size of the set of transcendental numbers is much bigger than the size of the set of algebraic numbers: the set of algebraic numbers is countable, while the set of transcendental numbers is uncountable [3]. Recently, a "new" class of real numbers, called the *periods*, has been introduced. For a nice introduction to the periods, refer to [9].

One of the oldest and most beautiful topics in number theory is the theory of irrational numbers. Perhaps the numbers most easy to prove irrational are certain logarithms. For example, the proof of this statement that $\log_2^3$ is irrational is as follows:

Let $\log_2^3$ be rational. So, there are positive integers p and q such that $\log_2^3 = \frac{p}{q}$. This implies that $3^q = 2^p$, a contradiction, since 3^q is odd and 2^p is even [1, Lemma 6.2.1].

Note that the proof of some numbers to be rational or not can be very difficult. For example, we still don't know whether the Euler-Mascheroni constant

$$\gamma = \lim_{n \to +\infty} \left(-\ln n + \sum_{k=1}^{n} \frac{1}{k} \right)$$

is rational or not.

Let us recall that every real number to base b can be expressed by a decimal expansion, and this expansion can be performed in only one way [14, p. 38]. The single exception to this last statement, which depends on our conventions, occurs for numbers with a finite expansion. In order to resolve this, we can diminish the

last decimal by one unit and continue with an infinite series of $(b-1)$s. For instance, to base $b = 10$, we have the following:

$$1 = 0.\overline{9} = 0.999\ldots, \text{ and } 0.1397 = 0.1396\overline{9} = 0.1396999\ldots.$$

Note that a real number is *regular* with respect to some base number b when it can be expanded in the corresponding number system with a finite number of negative powers of b [13, p. 316]. A regular number is also called a real number with finite decimal [8, p. 25]. A number r with respect to some base number b is regular if and only if there are coprime integer numbers p and q such that $r = \dfrac{p}{q}$ and q contains no other prime factors than those that divide b [13, p. 316].

In this paper, if the fractional part of a regular number to base b is

$$\alpha = (0.r_1 r_2 \ldots r_n)_b,$$

we only consider the representation with an infinite series of $(b-1)$:

$$\alpha = (0.r_1 r_2 \ldots (r_n - 1)\overline{(b-1)} \ldots)_b.$$

Therefore, if we agree always to pick the non-terminating expansion in the case of regular numbers, then fractional part of each real number to base b corresponds uniquely to an infinite decimal $(0.r_1 r_2 r_3 \ldots r_n \ldots)_b$.

We add that a fractional part of a real number r, denoted by $\operatorname{frac}(r)$, is the non-negative real number $\operatorname{frac}(r) := |r| - \lfloor |r| \rfloor$, where $|r|$ is the absolute value of r and $\lfloor r \rfloor$ is the integer part of r.

In Definition 1.1, we denote the asymptotic average of the decimals of a real number r to base b by $\operatorname{Av}_b(r)$ and in Theorem 1.7, we show that if $\operatorname{Av}_b(r) = 0$, then r is irrational. By using this, we show the following:

If $(a_n)_{n\in\mathbb{N}}$ is a sequence of positive integers such that $\lim_{n\to+\infty} \frac{n}{a_n} = 0$, then $r = \sum_{n=1}^{+\infty} \frac{1}{10^{a_n}}$ is irrational (check Corollary 1.8).

Hardy and Wright (cf. [7, Theorem 137]) prove that the real number

$$r = 0.r_1 r_2 r_3 \ldots r_n \cdots = 0.011010100010\ldots,$$

where $r_n = 1$ if n is prime and $r_n = 0$ otherwise, is irrational. In Corollary 1.9, by using Corollay 1.8, we give an alternative proof for this fact.

Let us recall that a real number r is a simply normal number to base b if for the decimals $(r_n)_{n\in\mathbb{N}}$ of the fractional part $(0.r_1 r_2 r_3 \ldots r_n \ldots)_b$ of the real number r, we have the following property:

$$\lim_{n\to+\infty} \frac{\operatorname{card}\{j : 1 \le j \le n, r_j = d\}}{n} = \frac{1}{b},$$

where $d \in \{0, 1, 2, \ldots, b-1\}$ [2, Definition 4.1]. In Corollary 1.11, we prove that if r is a simply normal number to base b, then $\operatorname{Av}_b(r) = \dfrac{b-1}{2}$.

For a masterful exposition of some central results on irrational, transcendental, and normal numbers, refer to [11].

1. THE ASYMPTOTIC AVERAGE OF THE DECIMALS OF REAL NUMBERS

If $(r_n)_{n \in \mathbb{N}}$ is a sequence in real numbers, the sequence of the averages is defined as follows:

$$a_n = \frac{r_1 + r_2 + \cdots + r_n}{n}.$$

Definition 1.1. Let the fractional part of a real number r to base b be

$$(0.r_1 r_2 \ldots r_n \ldots)_b.$$

Then, we define

$$\text{Av}_b(r) = \lim_{n \to +\infty} \frac{r_1 + r_2 + \cdots + r_n}{n},$$

if it exists. Usually, we denote $\text{Av}_{10}(r)$ by $\text{Av}(r)$.

Remark 1.2. It is clear that if r is a real number such that $\text{Av}_b(r)$ defined in Definition 1.1 exists, then $0 \leq \text{Av}_b(r) \leq b - 1$.

Theorem 1.3. *Let the decimals $(r_n)_{n \in \mathbb{N}}$ of the fractional part*

$$(0.r_1 r_2 r_3 \ldots r_n \ldots)_b$$

of a real number r satisfy the following:

$$\lim_{n \to +\infty} \frac{\text{card}\{j : 1 \leq j \leq n, r_j = d\}}{n} = \omega_d,$$

where $d \in \{0, 1, 2, \ldots, (b-1)\}$, $0 \leq \omega_d \leq 1$, and $\sum_{d=0}^{b-1} \omega_d = 1$.

Then,

$$\text{Av}_b(r) = \sum_{d=1}^{b-1} (d \cdot \omega_d).$$

Proof. Put $A(d, n) = \{j : 1 \leq j \leq n, r_j = d\}$. By assumption,

$$\lim_{n \to +\infty} \frac{\text{card } A(d, n)}{n} = \omega_d,$$

that means that for any $\varepsilon > 0$, there is a natural number N_d such that if $n > N_d$, then

$$\left| \frac{\text{card } A(d, n)}{n} - \omega_d \right| < \varepsilon.$$

Now, if we define $N = \max\{N_d\}_{d=0}^{b-1}$, for each $n > N$, we have the following:

$$(1.4) \qquad \omega_d - \varepsilon < \frac{\text{card } A(d, n)}{n} < \omega_d + \varepsilon.$$

Since

$$\frac{r_1 + r_2 + \cdots + r_n}{n} = \frac{\sum_{d=0}^{b-1} \sum_{i \in A(d,n)} r_i}{n} = \frac{\sum_{d=1}^{b-1} d \, \text{card } A(d, n)}{n},$$

by using the inequality (1.4), we have the following:

$$\left| \frac{r_1 + r_2 + \cdots + r_n}{n} - \sum_{d=1}^{b-1} d \cdot \omega_d \right| < \frac{b(b-1)\varepsilon}{2}.$$

Hence,

$$\mathrm{Av}_b(r) = \sum_{d=1}^{b-1} d \cdot \omega_d$$

and the proof is complete. □

Let us recall that any rational number is expressible as a finite decimal (if it is regular) or an infinite periodic decimal; conversely, any decimal expansion which is either finite or infinite periodic is equal to some rational number [12, p. 32]. Since in this paper, we only consider the infinite decimal representation of a finite decimal number, we have the following:

Corollary 1.5. *Let r be a rational number. Then $\mathrm{Av}_b(r)$ exists and is a positive rational number. Moreover, if the fractional part of the rational number r to base b is*

$$(0.r_1 r_2 \ldots r_n \overline{p_1 p_2 \ldots p_m})_b,$$

then

$$\mathrm{Av}_b(r) = \frac{p_1 + p_2 + \cdots + p_m}{m}.$$

In particular, if r is regular to base b, then

$$\mathrm{Av}_b(r) = b - 1.$$

Example 1.6. (1) Since $\dfrac{1}{3} = 0.\overline{3}$ (to base 10), $\mathrm{Av}(\frac{1}{3}) = 3$.

(2) Since $\dfrac{1}{7} = 0.\overline{142857}$, $\mathrm{Av}(\frac{1}{7}) = \dfrac{1 + 4 + 2 + 8 + 5 + 7}{6} = \dfrac{27}{6} = \dfrac{9}{2}$.

(3) Since $\dfrac{1}{5} = 0.1\overline{9}$, $\mathrm{Av}(\frac{1}{5}) = 9$.

Theorem 1.7 (A Simple Criterion for Irrationality of Real Numbers)**.** *Let r be a real number such that $\mathrm{Av}_b(r)$ exists. If $\mathrm{Av}_b(r) = 0$, then r is irrational.*

Proof. If r is rational, by Corollary 1.5, $\mathrm{Av}_b(r) > 0$. On the other hand, as mentioned in Remark 1.2, $\mathrm{Av}_b(r) \geq 0$. So, if $\mathrm{Av}_b(r) = 0$, then r is irrational and the proof is complete. □

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Corollary 1.8. *Let $(a_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of positive integers such that*

$$\lim_{n \to +\infty} \frac{n}{a_n} = 0.$$

Define $r = \displaystyle\sum_{n=1}^{+\infty} \frac{b_n}{10^{a_n}}$, where $1 \leq b_n \leq 9$ is a positive integer, for each $n \in \mathbb{N}$. Then, r is irrational.

Proof. $\mathrm{Av}(r) \leq 9 \cdot \displaystyle\lim_{n \to +\infty} \frac{n}{a_n} = 0$. □

We give an alternative proof for the following nice result mentioned in the book by Hardy and Wright [7]:

Corollary 1.9. [7, Theorem 137] *The real number*

$$r = 0.r_1 r_2 r_3 \ldots r_n \cdots = 0.011010100010 \ldots,$$

where $r_n = 1$ if n is prime and $r_n = 0$ otherwise, is irrational.

Proof. Let $\mathbb{P}$ be the set of all prime numbers. It is clear that

$$\frac{r_1 + r_2 + r_3 + \cdots + r_n}{n} = \frac{\pi(n)}{n},$$

where $\pi(n) = \mathrm{card}\{p \in \mathbb{P} : p \leq n\}$ is the prime counting function. By Corollary 4.2.3 in [6], which states that $\lim_{n \to +\infty} \frac{\pi(n)}{n} = 0$, we have the following:

$$\mathrm{Av}(r) = \lim_{n \to +\infty} \frac{\pi(n)}{n} = 0.$$

Now by Corollary 1.8, r is irrational and this finishes the proof.   □

Remark 1.10. For proving Corollary 1.9, we only need to know [6, Corollary 4.2.3] that

$$\lim_{n \to +\infty} \frac{\pi(n)}{n} = 0.$$

This is, in fact, a corollary of Chebychev's estimate, which states that there exist positive constants A_1 and a_2 such that

$$A_1 \frac{n}{\ln n} < \pi(n) < A_2 \frac{n}{\ln n}.$$

for all $n \geq 2$ [6, Theorem 4.2.1].

Let us recall that a real number r is a simply normal number to base b if for the decimals $(r_n)_{n \in \mathbb{N}}$ of the fractional part $(0.r_1 r_2 r_3 \ldots r_n \ldots)_b$ of the real number r, we have the following property:

$$\lim_{n \to +\infty} \frac{\mathrm{card}\{j : 1 \leq j \leq n, r_j = d\}}{n} = \frac{1}{b},$$

where $d \in \{0, 1, 2, \ldots, b-1\}$ [2, Definition 4.1]. It is clear that a real number r is simply normal to base 2 if and only if $\mathrm{Av}_2(r) = \frac{1}{2}$.

Corollary 1.11. *Let r be a simply normal number to base b. Then $\mathrm{Av}_b(r) = \dfrac{b-1}{2}$.*

Let us recall that the real number, known as Champernowne number,

$$C_{10} = 0.12345678910111213141516171819\,20\ldots,$$

whose sequence of decimals is the increasing sequence of all positive integers, is simply normal to base 10 (cf. [4] and [2, Theorem 4.2]). Also recall that

$$\ell = \sum_{n=1}^{+\infty} \frac{1}{10^{n!}},$$

is called Liouville's constant [15, Theorem 6.6].

Proposition 1.12. *The following statements hold:*

   (1) There is a transcendental number r, for example Champernowne number C_{10}, such that $\mathrm{Av}(r) \neq 0$.
   (2) There is a transcendental number r, for example Liouville's constant, such that $\mathrm{Av}(r) = 0$.

Proof. (1): Since C_{10} is a normal number, by Corollary 1.11, $\mathrm{Av}(C_{10}) = 9/2$. On the other hand, Mahler proved that Champernowne number C_{10} is transcendental [10].

(2): Take

$$\ell = \sum_{n=1}^{+\infty} \frac{1}{10^{n!}},$$

to be the Liouville's constant. It is famous that ℓ is transcendental [15, Theorem 6.6]. Now by the proof of Corollary 1.8, we have $\mathrm{Av}(\ell) = 0$. □

Remark 1.13. Note that with the help of Corollary 1.5, it is easy to see that if $s \in (0,9] \cap \mathbb{Q}$, then there is a rational number r such that $\mathrm{Av}(r) = s$. For example, if $s = 2/3$, we should choose decimals p_1, p_2, and p_3 such that $(p_1 + p_2 + p_3)/3 = 2/3$. As an instance, it is clear that $\mathrm{Av}(0.\overline{101}) = 2/3$. Based on this discussion, the following questions arise:

Questions 1.14. (1) *Is there any irrational number r such that $\mathrm{Av}(r)$ exists and is irrational (transcendental)?*

(2) *Is there any transcendental number r such that $\mathrm{Av}(r)$ exists and is transcendental?*

(3) *Does $\mathrm{Av}(\sqrt{2})$ exist and if it exists, what is that? The same question arises for other irrational numbers such as e, π, and $\log_2^3$.*

(4) *Let $\gamma = \lim_{n \to +\infty} \left(-\ln n + \sum_{k=1}^{n} \frac{1}{k} \right)$ be the Euler-Mascheroni constant. Does $\mathrm{Av}(\gamma)$ exist and if it exists, what is that?*

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