Number Theory Problems of 2015 Competitions

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1 2015 Contests

Problem 1 (Croatia First Round Competition 2015)**.** Does there exist a positive integer n such that $n^2 + 2n + 2015$ is a perfect square?

Problem 2 (Croatia First Round Competition 2015)**.** Determine the largest positive integer n such that

$$n + 5 \mid n^4 + 1395$$

Problem 3 (Croatia First Round Competition 2015)**.** Let n be a positive integer and define

$$S_n = \sum_{k=1}^{n} k!(k^2 + k + 1).$$

Find $\frac{S_n+1}{(n+1)!}$.

Problem 4 (Croatia First Round Competition 2015)**.** Let n be a positive integer. Each of the numbers $n, n+1, n+2, \ldots, 2n-1$ has a largest odd divisor. Determine the sum of these largest odd divisors.

Problem 5 (Croatia Second Round Competition 2015)**.** Positive integers a and b and prime number p satisfy the equation $a^2 + p^2 = b^2$. Prove that $2(b+p)$ is a perfect square.

Problem 6 (Croatia Second Round Competition 2015)**.** Determine all quadruples (a, b, c, d) of positive integers such that

$$a^3 = b^2, c^5 = d^4, \text{ and } a - c = 9.$$

Problem 7 (Croatia Second Round Competition 2015)**.** Let n be a positive integer larger than 1 such that both $2n-1$ and $3n-2$ are perfect squares. Prove that $10n - 7$ is composite.

Problem 8 (Croatia Second Round Competition 2015)**.** Let $a = \sqrt[2015]{2015}$ and (a_n) be a sequence such that $a_1 = a$ and $a_{n+1} = a^{a_n}$ for $n \geq 1$. Does there exist a positive integer n such that $a_n \geq 2015$?

Problem 9 (Croatia Second Round Competition 2015)**.** A positive integer is called *wacky* if its decimal representation contains 100 digits, and if by removing any of those digits one gets a 99-digit number divisible by 7. How many wacky positive integers are there?

Problem 10 (Croatia Final Round National Competition 2015)**.** Prove that there does not exist a positive integer n such that $7^n - 1$ is divisible by $6^n - 1$.

Problem 11 (Croatia Final Round National Competition 2015)**.** Determine all triples (p, m, n) of positive integers such that p is a prime number and

$$p^m - n^3 = 8.$$

Problem 12 (Croatia Final Round National Competition 2015)**.** Determine all triples (p, m, n) of positive integers such that p is a prime number and

$$2^m p^2 + 1 = n^5.$$

Problem 13 (Croatia Final Round National Competition 2015)**.** Determine all positive integers n for which there exists a divisor d of n such that

$$dn + 1 | d^2 + n^2.$$

Problem 14 (Croatian Mathematical Olympiad 2015, IMO Shortlist 2014)**.** Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k \geq 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd. (For a real number x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.)

Problem 15 (Croatian Mathematical Olympiad 2015)**.** Let $n > 2$ be a positive integer and p a prime number. If the number $p - 1$ is divisible by n, and the number $n^3 - 1$ is divisible by p, prove that $4p - 3$ is a square of an integer.

Problem 16 (Croatian TST for MEMO 2015)**.** Determine all positive integers x and y such that

$$x(x^2 + 19) = y(y^2 - 10).$$