Number Theory Problems of 2016 Competitions

Amir Hossein Parvardi

October 8, 2016

Problem 1 (AIME 2016)**.** For $-1 < r < 1$, let $S(r)$ denote the sum of the geometric series

$$12 + 12r + 12r^2 + 12r^3 + \dots.$$

Let a between -1 and 1 satisfy $S(a)S(-a) = 2016$. Find $S(a) + S(-a)$. Link

Problem 2 (AIME 2016)**.** For a permutation $p = (a_1, a_2, \dots, a_9)$ of the digits $1, 2, \dots, 9$, let $s(p)$ denote the sum of the three 3-digit numbers $a_1a_2a_3$, $a_4a_5a_6$, and $a_7a_8a_9$. Let m be the minimum value of $s(p)$ subject to the condition that the units digit of $s(p)$ is 0. Let n denote the number of permutations p with $s(p) = m$. Find $|m - n|$. Link

Problem 3 (AIME 2016)**.** A strictly increasing sequence of positive integers $a_1, a_2, a_3, \dots$ has the property that for every positive integer k, the subsequence $a_{2k-1}, a_{2k}, a_{2k+1}$ is geometric and the subsequence $a_{2k}, a_{2k+1}, a_{2k+2}$ is arithmetic. Suppose that $a_{13} = 2016$. Find a_1. Link

Problem 4 (AIME 2016)**.** Find the least positive integer m such that $m^2 - m + 11$ is a product of at least four not necessarily distinct primes. Link

Problem 5 (AIME 2016)**.** Let x, y and z be real numbers satisfying the system

$$\begin{aligned}
\log_2(xyz - 3 + \log_5 x) &= 5 \\
\log_3(xyz - 3 + \log_5 y) &= 4 \\
\log_4(xyz - 3 + \log_5 z) &= 4.
\end{aligned}$$

Find the value of $|\log_5 x| + |\log_5 y| + |\log_5 z|$. Link

Problem 6 (AIME 2016)**.** For polynomial $P(x) = 1 - \frac{1}{3}x + \frac{1}{6}x^2$, define

$$Q(x) = P(x)P(x^3)P(x^5)P(x^7)P(x^9) = \sum_{i=0}^{50} a_i x^i.$$

Then $\sum\limits_{i=0}^{50} |a_i| = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$. Link

Problem 7 (AIME 2016)**.** Find the number of sets $\{a, b, c\}$ of three distinct positive integers with the property that the product of $a, b,$ and c is equal to the product of $11, 21, 31, 41, 51,$ and 61. Link

Problem 8 (AIME 2016)**.** The sequences of positive integers $1, a_2, a_3, \ldots$ and $1, b_2, b_3, \ldots$ are an increasing arithmetic sequence and an increasing geometric sequence, respectively. Let $c_n = a_n + b_n$. There is an integer k such that $c_{k-1} = 100$ and $c_{k+1} = 1000$. Find c_k. Link

Problem 9 (AIME 2016)**.** For positive integers N and k, define N to be k-nice if there exists a positive integer a such that a^k has exactly N positive divisors. Find the number of positive integers less than 1000 that are neither 7-nice nor 8-nice.

Link

Problem 10 (All-Russian Olympiads 2016, Grade 11)**.** Let n be a positive integer and let $k_0, k_1, \ldots, k_{2n}$ be nonzero integers such that $k_0+k_1+\cdots+k_{2n} \neq 0$. Is it always possible to a permutation $(a_0, a_1, \ldots, a_{2n})$ of $(k_0, k_1, \ldots, k_{2n})$ so that the equation

$$a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_0 = 0$$

has not integer roots? Link

Problem 11 (APMO 2016)**.** A positive integer is called *fancy* if it can be expressed in the form

$$2^{a_1} + 2^{a_2} + \cdots + 2^{a_{100}},$$

where $a_1, a_2, \cdots, a_{100}$ are non-negative integers that are not necessarily distinct. Find the smallest positive integer n such that no multiple of n is a fancy number.

Link

Problem 12 (Azerbaijan TST 2016)**.** The set A consists of natural numbers such that these numbers can be expressed as $2x^2 + 3y^2$, where x and y are integers. $(x^2 + y^2 \neq 0)$

1. Prove that there is no perfect square in the set A.

2. Prove that multiple of odd number of elements of the set A cannot be a perfect square.

Link

Problem 13 (Azerbaijan Junior Mathematical Olympiad 2016)**.** Given

$$34! = 295232799039a041408476186096435b0000000,$$

in decimal representation, find the numbers a and b. Link

Problem 14 (Azerbaijan Junior Mathematical Olympiad 2016)**.** Prove that if for a real number a, $a+\frac{1}{a}$ is integer then $a^n+\frac{1}{a^n}$ is also integer for any positive integer n. Link

Problem 15 (Azerbaijan Junior Mathematical Olympiad 2016)**.** A quadruple (p,a,b,c) of positive integers is called a *good quadruple* if

(a) p is odd prime,

(b) a,b,c are distinct,

(c) $ab+1, bc+1$, and $ca+1$ are divisible by p.

Prove that for all good quadruple $p+2 \leq \frac{a+b+c}{3}$, and show the equality case.

Link

Problem 16 (Balkan 2016)**.** Find all monic polynomials f with integer coefficients satisfying the following condition: there exists a positive integer N such that p divides $2(f(p)!)+1$ for every prime $p>N$ for which $f(p)$ is a positive integer.
Note: A monic polynomial has a leading coefficient equal to 1. Link

Problem 17 (Bay Area Olympiad 2016)**.** Let $A=2^k-2$ and $B=2^k\cdot A$, where k is an integer ($k\geq 2$). Show that, for every integer k greater than or equal to 2,

1. A and B have the same set of distinct prime factors.
2. $A+1$ and $B+1$ have the same set of distinct prime factors.

Link

Problem 18 (Bay Area Olympiad 2016)**.** Find a positive integer N and $a_1, a_2, \cdots, a_N$ where $a_k=1$ or $a_k=-1$, for each $k=1,2,\cdots,N$, such that

$$a_1\cdot 1^3+a_2\cdot 2^3+a_3\cdot 3^3\cdots+a_N\cdot N^3=20162016$$

or show that this is impossible. Link

Problem 19 (Benelux 2016)**.** Find the greatest positive integer N with the following property: there exist integers $x_1,\ldots,x_N$ such that $x_i^2-x_ix_j$ is not divisible by 1111 for any $i\neq j$. Link

Problem 20 (Benelux 2016)**.** Let n be a positive integer. Suppose that its positive divisors can be partitioned into pairs (i.e. can be split in groups of two) in such a way that the sum of each pair is a prime number. Prove that these prime numbers are distinct and that none of these are a divisor of n. Link

Problem 21 (Bosnia and Herzegovina TST 2016)**.** For an infinite sequence $a_1<a_2<a_3<\ldots$ of positive integers we say that it is nice if for every positive integer n holds $a_{2n}=2a_n$. Prove the following statements:

(a) If there is given a nice sequence and prime number $p > a_1$, there exist some term of the sequence which is divisible by p.

(b) For every prime number $p > 2$, there exist a nice sequence such that no terms of the sequence are divisible by p.

Link

Problem 22 (Bosnia and Herzegovina TST 2016)**.** Determine the largest positive integer n which cannot be written as the sum of three numbers bigger than 1 which are pairwise coprime. Link

Problem 23 (Canada National Olympiad 2016)**.** Find all polynomials $P(x)$ with integer coefficients such that $P(P(n)+n)$ is a prime number for infinitely many integers n. Link

Problem 24 (Canadian Mathematical Olympiad Qualification 2016)**.**

(a) Find all positive integers n such that $11|(3^n + 4^n)$.

(b) Find all positive integers n such that $31|(4^n + 7^n + 20^n)$.

Link

Problem 25 (Canadian Mathematical Olympiad Qualification 2016)**.** Determine all ordered triples of positive integers (x, y, z) such that $\gcd(x+y, y+z, z+x) > \gcd(x, y, z)$. Link

Problem 26 (CCA Math Bonanza 2016)**.** Let $f(x) = x^2 + x + 1$. Determine the ordered pair (p, q) of primes satisfying $f(p) = f(q) + 242$. Link

Problem 27 (CCA Math Bonanza 2016)**.** Let $f(x) = x^2 + x + 1$. Determine the ordered pair (p, q) of primes satisfying $f(p) = f(q) + 242$. Link

Problem 28 (CCA Math Bonanza 2016)**.** Compute

$$\sum_{k=1}^{420} \gcd(k, 420).$$

Link

Problem 29 (CCA Math Bonanza 2016)**.** Pluses and minuses are inserted in the expression

$$\pm 1 \pm 2 \pm 3 \cdots \pm 2016$$

such that when evaluated the result is divisible by 2017. Let there be N ways for this to occur. Compute the remainder when N is divided by 503. Link

Problem 30 (CCA Math Bonanza 2016)**.** What is the largest integer that must divide $n^5 - 5n^3 + 4n$ for all integers n? Link

Problem 31 (CCA Math Bonanza 2016)**.** Determine the remainder when

$$2^6 \cdot 3^{10} \cdot 5^{12} - 75^4 \left(26^2 - 1\right)^2 + 3^{10} - 50^6 + 5^{12}$$

is divided by 1001. Link

Problem 32 (CentroAmerican 2016)**.** Find all positive integers n that have 4 digits, all of them perfect squares, and such that n is divisible by $2, 3, 5$, and 7. Link

Problem 33 (CentroAmerican 2016)**.** We say a number is *irie* if it can be written in the form $1 + \dfrac{1}{k}$ for some positive integer k. Prove that every integer $n \geq 2$ can be written as the product of r distinct irie numbers for every integer $r \geq n - 1$. Link

Problem 34 (China Girls Mathematical Olympiad 2016)**.** Let m and n are relatively prime integers and $m > 1, n > 1$. Show that there are positive integers a, b, c such that $m^a = 1 + n^b c$, and n and c are relatively prime. Link

Problem 35 (China National Olympiad 2016)**.** Let p be an odd prime and $a_1, a_2, ..., a_p$ be integers. Prove that the following two conditions are equivalent:

1. There exists a polynomial $P(x)$ with degree $\leq \frac{p-1}{2}$ such that $P(i) \equiv a_i \pmod{p}$ for all $1 \leq i \leq p$.
2. For any natural $d \leq \frac{p-1}{2}$,

$$\sum_{i=1}^{p} (a_{i+d} - a_i)^2 \equiv 0 \pmod{p},$$

where indices are taken modulo p.

Link

Problem 36 (China TST2016)**.** Let $c, d \geq 2$ be positive integers. Let $\{a_n\}$ be the sequence satisfying $a_1 = c, a_{n+1} = a_n^d + c$ for $n = 1, 2, \ldots$. Prove that for any $n \geq 2$, there exists a prime number p such that $p \mid a_n$ and $p \nmid a_i$ for $i = 1, 2, \ldots, n - 1$. Link

Problem 37 (China TST 2016)**.** Set positive integer $m = 2^k \cdot t$, where k is a non-negative integer, t is an odd number, and let $f(m) = t^{1-k}$. Prove that for any positive integer n and for any positive odd number $a \leq n$, $\prod_{m=1}^{n} f(m)$ is a multiple of a. Link

Problem 38 (China TST 2016)**.** Does there exist two infinite positive integer sets S, T, such that any positive integer n can be uniquely expressed in the form

$$n = s_1 t_1 + s_2 t_2 + \cdots + s_k t_k,$$

where k is a positive integer dependent on n, $s_1 < s_2 < \cdots < s_k$ are elements of S, $t_1, \ldots, t_k$ are elements of T? Link

Problem 39 (China TST 2016)**.** Let a, b, b', c, m, q be positive integers, where $m > 1, q > 1, |b - b'| \geq a$. It is given that there exist a positive integer M such that

$$S_q(an + b) \equiv S_q(an + b') + c \pmod{m}$$

holds for all integers $n \geq M$. Prove that the above equation is true for all positive integers n. (Here $S_q(x)$ is the sum of digits of x taken in base q). Link

Problem 40 (China Western Mathematical Olympiad 2016)**.** For an n-tuple of integers, define a transformation to be:

$$(a_1, a_2, \ldots, a_{n-1}, a_n) \to (a_1 + a_2, a_2 + a_3, \ldots, a_{n-1} + a_n, a_n + a_1)$$

Find all ordered pairs of integers (n, k) with $n, k \geq 2$, such that for any n-tuple of integers $(a_1, a_2, \ldots, a_{n-1}, a_n)$, after a finite number of transformations, every element in the of the n-tuple is a multiple of k. Link

Problem 41 (China Western Mathematical Olympiad 2016)**.** Prove that there exist infinitely many positive integer triples (a, b, c) such that a, b, c are pairwise relatively prime ,and $ab + c, bc + a, ca + b$ are pairwise relatively prime. Link

Problem 42 (China Hong Kong Mathematical Olympiad 2016)**.** Find all integral ordered triples (x, y, z) such that $\sqrt{\dfrac{2015}{x+y}} + \sqrt{\dfrac{2015}{y+z}} + \sqrt{\dfrac{2015}{x+z}}$ are positive integers. Link

Problem 43 (Croatia TST 2016)**.** Find all pairs (p, q) of prime numbers such that

$$p(p^2 - p - 1) = q(2q + 3).$$

Link

Problem 44 (Croatia TST 2016)**.** Let $p > 10^9$ be a prime number such that $4p + 1$ is also prime. Prove that the decimal expansion of $\frac{1}{4p+1}$ contains all the digits $0, 1, \ldots, 9$. Link

Problem 45 (European Girls' Mathematical Olympiad 2016)**.** Let S be the set of all positive integers n such that n^4 has a divisor in the range $n^2 + 1, n^2 + 2, ..., n^2 + 2n$. Prove that there are infinitely many elements of S of each of the forms $7m, 7m + 1, 7m + 2, 7m + 5, 7m + 6$ and no elements of S of the form $7m + 3$ and $7m + 4$, where m is an integer. Link

Problem 46 (Turkey TST for European Girls' Mathematical Olympiad 2016)**.** Prove that for every square-free integer $n > 1$, there exists a prime number p and an integer m satisfying

$$p \mid n \quad \text{and} \quad n \mid p^2 + p \cdot m^p.$$

Link

Problem 47 (ELMO 2016)**.** Cookie Monster says a positive integer n is *crunchy* if there exist $2n$ real numbers $x_1, x_2, \ldots, x_{2n}$, not all equal, such that the sum of any n of the x_i's is equal to the product of the other n of the x_i's. Help Cookie Monster determine all crunchy integers. Link

Problem 48 (ELMO2016)**.** Big Bird has a polynomial P with integer coefficients such that n divides $P(2^n)$ for every positive integer n. Prove that Big Bird's polynomial must be the zero polynomial. Link

Problem 49 (Germany TST 2016)**.** The positive integers $a_1, a_2, \ldots, a_n$ are aligned clockwise in a circular line with $n \geq 5$. Let $a_0 = a_n$ and $a_{n+1} = a_1$. For each $i \in \{1, 2, \ldots, n\}$ the quotient

$$q_i = \frac{a_{i-1} + a_{i+1}}{a_1}$$

is an integer. Prove

$$2n \leq q_1 + q_2 + \cdots + q_n < 3n.$$

Link

Problem 50 (Germany TST 2016, Taiwan TST First Round 2016)**.** Determine all positive integers M such that the sequence $a_0, a_1, a_2, \cdots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \cdots$$

contains at least one integer term. Link

Problem 51 (Greece 2016)**.** Find all triplets of nonnegative integers (x, y, z) and $x \leq y$ such that

$$x^2 + y^2 = 3 \cdot 2016^z + 77.$$

Link

Problem 52 (Greece TST 2016)**.** Given is the sequence $(a_n)_{n \geq 0}$ which is defined as follows:$a_0 = 3$ and $a_{n+1} - a_n = n(a_n - 1)$, $\forall n \geq 0$. Determine all positive integers m such that $\gcd(m, a_n) = 1$, $\forall n \geq 0$. Link

Problem 53 (The Harvard-MIT Math Tournament 2016)**.** Denote by $\mathbb{N}$ the positive integers. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that, for any $w, x, y, z \in \mathbb{N}$,

$$f(f(f(z)))f(wxf(yf(z))) = z^2 f(xf(y))f(w).$$

Show that $f(n!) \geq n!$ for every positive integer n. Link

Problem 54 (Hong Kong TST 2016)**.** Find all natural numbers n such that n, $n^2 + 10$, $n^2 - 2$, $n^3 + 6$, and $n^5 + 36$ are all prime numbers. Link

Problem 55 (Hong Kong TST 2016)**.** Find all triples (m, p, q) such that

$$2^m p^2 + 1 = q^7,$$

where p and q are ptimes and m is a positive integer. Link

Problem 56 (Hong Kong TST 2016)**.** Find all prine numbers p and q such that $p^2|q^3+1$ and $q^2|p^6-1$. Link

Problem 57 (Hong Kong TST 2016)**.** Let p be a prime number greater than 5. Suppose there is an integer k satisfying that k^2+5 is divisible by p. Prove that there are positive integers m and n such that $p^2 = m^2+5n^2$. Link

Problem 58 (IberoAmerican 2016)**.** Find all prime numbers p, q, r, k such that $pq + qr + rp = 12k + 1$. Link

Problem 59 (IberoAmerican 2016)**.** Let k be a positive integer and $a_1, a_2, \ldots, a_k$ digits. Prove that there exists a positive integer n such that the last $2k$ digits of 2^n are, in the following order, $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k$, for certain digits $b_1, b_2, \ldots, b_k$. Link

Problem 60 (IMO 2016)**.** A set of postive integers is called fragrant if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2+n+1$. What is the least possible positive integer value of b such that there exists a non-negative integer a for which the set

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

is fragrant? Link

Problem 61 (India IMO Training Camp 2016)**.** Given that n is a natural number such that the leftmost digits in the decimal representations of 2^n and 3^n are the same, find all possible values of the leftmost digit. Link

Problem 62 (India IMO Practice Test 2016)**.** We say a natural number n is perfect if the sum of all the positive divisors of n is equal to $2n$. For example, 6 is perfect since its positive divisors $1, 2, 3, 6$ add up to $12 = 2 \times 6$. Show that an odd perfect number has at least 3 distinct prime divisors. Link

Problem 63 (India TST 2016, Taiwan TST Second Round 2016)**.** Let m and n be positive integers such that $m > n$. Define $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1 x_2 \ldots x_{n+1} - 1$ is divisible by an odd prime.

Link

Problem 64 (India TST 2016)**.** Let n be a natural number. We define sequences $\langle a_i \rangle$ and $\langle b_i \rangle$ of integers as follows. We let $a_0 = 1$ and $b_0 = n$. For $i > 0$, we let

$$(a_i, b_i) = \begin{cases} (2a_{i-1}+1, b_{i-1}-a_{i-1}-1) & \text{if } a_{i-1} < b_{i-1}, \\ (a_{i-1}-b_{i-1}-1, 2b_{i-1}+1) & \text{if } a_{i-1} > b_{i-1}, \\ (a_{i-1}, b_{i-1}) & \text{if } a_{i-1} = b_{i-1}. \end{cases}$$

Given that $a_k = b_k$ for some natural number k, prove that $n+3$ is a power of two. Link

Problem 65. Let $\mathbb{N}$ denote the set of all natural numbers. Show that there exists two nonempty subsets A and B of $\mathbb{N}$ such that

1. $A \cap B = \{1\}$;
2. every number in $\mathbb{N}$ can be expressed as the product of a number in A and a number in B;
3. each prime number is a divisor of some number in A and also some number in B;
4. one of the sets A and B has the following property: if the numbers in this set are written as $x_1 < x_2 < x_3 < \cdots$, then for any given positive integer M there exists $k \in \mathbb{N}$ such that $x_{k+1} - x_k \geq M$;
5. Each set has infinitely many composite numbers.

Link

Problem 66 (India National Olympiad 2016)**.** Let $\mathbb{N}$ denote the set of natural numbers. Define a function $T : \mathbb{N} \to \mathbb{N}$ by $T(2k) = k$ and $T(2k+1) = 2k+2$. We write $T^2(n) = T(T(n))$ and in general $T^k(n) = T^{k-1}(T(n))$ for any $k > 1$.

(i) Show that for each $n \in \mathbb{N}$, there exists k such that $T^k(n) = 1$.

(ii) For $k \in \mathbb{N}$, let c_k denote the number of elements in the set $\{n : T^k(n) = 1\}$. Prove that $c_{k+2} = c_{k+1} + c_k$, for $k \geq 1$.

Link

Problem 67 (India National Olympiad 2016)**.** Consider a non-constant arithmetic progression $a_1, a_2, \cdots, a_n, \cdots$. Suppose there exist relatively prime positive integers $p > 1$ and $q > 1$ such that a_1^2, a_{p+1}^2 and a_{q+1}^2 are also the terms of the same arithmetic progression. Prove that the terms of the arithmetic progression are all integers. Link

Problem 68 (International Zhautykov Olympiad 2016)**.** $a_1, a_2, ..., a_{100}$ are permutation of $1, 2, ..., 100$. $S_1 = a_1, S_2 = a_1 + a_2, ..., S_{100} = a_1 + a_2 + ... + a_{100}$Find the maximum number of perfect squares from S_i. Link

Problem 69 (International Zhautykov Olympiad 2016)**.** We call a positive integer q a *convenient denominator* for a real number α if

$$|\alpha - \frac{p}{q}| < \frac{1}{10q}$$

for some integer p. Prove that if two irrational numbers α and β have the same set of convenient denominators then either $\alpha + \beta$ or $\alpha - \beta$ is an integer. Link

Problem 70 (Iran Third Round National Olympiad 2016)**.** Let F be a subset of the set of positive integers with at least two elements and P be a polynomial with integer coefficients such that for any two elements of F like a and b, the following two conditions hold

(i) $a+b \in F$, and

(ii) $\gcd(P(a), P(b)) = 1$.

Prove that $P(x)$ is a constant polynomial. Link

Problem 71 (Iran Third Round National Olympiad 2016)**.** Let P be a polynomial with integer coefficients. We say P is *good* if there exist infinitely many prime numbers q such that the set

$$X = \{P(n) \mod q : \quad n \in \mathbb{N}\}$$

has at least $\frac{q+1}{2}$ members. Prove that the polynomial $x^3 + x$ is good. Link

Problem 72 (Iran Third Round National Olympiad 2016)**.** Let m be a positive integer. The positive integer a is called a *golden residue* modulo m if $\gcd(a, m) = 1$ and $x^x \equiv a \pmod m$ has a solution for x. Given a positive integer n, suppose that a is a golden residue modulo n^n. Show that a is also a golden residue modulo n^{n^n}. Link

Problem 73 (Iran Third Round National Olympiad 2016)**.** Let p, q be prime numbers (q is odd). Prove that there exists an integer x such that

$$q|(x+1)^p - x^p$$

if and only if

$$q \equiv 1 \pmod p.$$

Link

Problem 74 (Iran Third Round National Olympiad 2016)**.** We call a function g *special* if $g(x) = a^{f(x)}$ (for all x) where a is a positive integer and f is polynomial with integer coefficients such that $f(n) > 0$ for all positive integers n.

A function is called an *exponential polynomial* if it is obtained from the product or sum of special functions. For instance, $2^x 3^{x^2+x-1} + 5^{2x}$ is an exponential polynomial.

Prove that there does not exist a non-zero exponential polynomial $f(x)$ and a non-constant polynomial $P(x)$ with integer coefficients such that

$$P(n)|f(n)$$

for all positive integers n. Link

Problem 75 (Iran Third Round National Olympiad 2016)**.** A sequence $P = \{a_n\}_{n=1}^{\infty}$ is called a *permutation* of natural numbers if for any natural number m, there exists a unique natural number n such that $a_n = m$.

We also define $S_k(P)$ as $S_k(P) = a_1 + a_2 + \cdots + a_k$ (the sum of the first k elements of the sequence).

Prove that there exists infinitely many distinct permutations of natural numbers like $P_1, P_2, \ldots$ such that

$$\forall k, \forall i < j : S_k(P_i)|S_k(P_j).$$

Link

Problem 76 (Iran TST 2016)**.** Let $p \neq 13$ be a prime number of the form $8k+5$ such that 39 is a quadratic non-residue modulo p. Prove that the equation

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 \equiv 0 \pmod{p}$$

has a solution in integers such that $p \nmid x_1x_2x_3x_4$. Link

Problem 77 (Italy National Olympiad 2016)**.** Determine all pairs of positive integers (a, n) with $a \geq n \geq 2$ for which $(a+1)^n + a - 1$ is a power of 2. Link

Problem 78 (Japan Mathematical Olympiad Preliminary 2016)**.** For $1 \leq n \leq 2016$, how many integers n satisfying the condition: the reminder divided by 20 is smaller than the one divided by 16. Link

Problem 79 (Japan Mathematical Olympiad Preliminary 2016)**.** Determine the number of pairs (a, b) of integers such that $1 \leq a, b \leq 2015$, a is divisible by $b+1$, and $2016 - a$ is divisible by b.Link

Problem 80 (Japan Mathematical Olympiad Finals 2016)**.** Let p be an odd prime number. For positive integer k satisfying $1 \leq k \leq p-1$, the number of divisors of $kp+1$ between k and p exclusive is a_k. Find the value of $a_1 + a_2 + \ldots + a_{p-1}$. Link

Problem 81 (Turkey TST for Junior Balkan Mathematical Olympiad 2016)**.** Let n be a positive integer, p and q be prime numbers such that

$$pq \mid n^p + 2 \quad \text{and} \quad n + 2 \mid n^p + q^p.$$

Prove that there exists a positive integer m satisfying $q \mid 4^m \cdot n + 2$. Link

Problem 82 (Turkey TST for Junior Balkan Mathematical Olympiad 2016)**.** Find all pairs (p, q) of prime numbers satisfying

$$p^3 + 7q = q^9 + 5p^2 + 18p.$$

Link

Problem 83 (Junior Balkan Mathematical Olympiad 2016)**.** Find all triplets of integers (a, b, c) such that the number

$$N = \frac{(a-b)(b-c)(c-a)}{2} + 2$$

is a power of 2016. Link

Problem 84 (Serbia TST for Junior Balkan Mathematical Olympiad 2016)**.** Find minimal number of divisors that can number $|2016^m - 36^n|$ have,where m and n are natural numbers. Link

Problem 85 (Romania TST for Junior Balkan Mathematical Olympiad 2016)**.** Let M be the set of natural numbers k for which there exists a natural number n such that

$$3^n \equiv k \pmod{n}.$$

Prove that M has infinitely many elements. Link

Problem 86 (Romania TST for Junior Balkan Mathematical Olympiad 2016)**.** Let n be an integer greater than 2 and consider the set

$$A = \{2^n - 1, 3^n - 1, \ldots, (n-1)^n - 1\}.$$

Given that n does not divide any element of A, prove that n is a square-free number. Does it necessarily follow that n is a prime? Link

Problem 87 (Romania TST for Junior Balkan Mathematical Olympiad 2016)**.** Let n be a positive integer and consider the system

$$S(n) : \begin{cases} x^2 + ny^2 = z^2 \\ nx^2 + y^2 = t^2 \end{cases},$$

where x, y, z, and t are naturals. If

- $M_1 = \{n \in \mathbb{N} : \text{system } S(n) \text{ has infinitely many solutions}\}$, and
- $M_1 = \{n \in \mathbb{N} : \text{system } S(n) \text{ has no solutions}\}$,

prove that

(a) $7 \in M_1$ and $10 \in M_2$.

(b) sets M_1 and M_2 are infinite.

Link

Problem 88 (Korean Summer Program Practice Test 2016)**.** A infinite sequence $\{a_n\}_{n\geq 0}$ of real numbers satisfy $a_n \geq n^2$. Suppose that for each $i, j \geq 0$ there exist k, l with $(i, j) \neq (k, l)$, $l - k = j - i$, and $a_l - a_k = a_j - a_i$. Prove that $a_n \geq (n + 2016)^2$ for some n. Link

Problem 89 (Korean Summer Program Practice Test 2016)**.** A finite set S of positive integers is given. Show that there is a positive integer N dependent only on S, such that any $x_1, \ldots, x_m \in S$ whose sum is a multiple of N, can be partitioned into groups each of whose sum is exactly N. (The numbers $x_1, \ldots, x_m$ need not be distinct.) Link

Problem 90 (Korea Winter Program Practice Test 2016)**.** $p(x)$ is an irreducible polynomial with integer coefficients, and q is a fixed prime number. Let a_n be a number of solutions of the equation $p(x) \equiv 0 \mod q^n$. Prove that we can find M such that $\{a_n\}_{n\geq M}$ is constant. Link

Problem 91 (Korea Winter Program Practice Test 2016)**.** Find all $\{a_n\}_{n\geq 0}$ that satisfies the following conditions.

1. $a_n \in \mathbb{Z}$,
2. $a_0 = 0, a_1 = 1$,
3. For infinitely many m, $a_m = m$, and
4. For every $n \geq 2$, $\{2a_i - a_{i-1} | i = 1, 2, 3, \cdots, n\} \equiv \{0, 1, 2, \cdots, n-1\}$ mod n.

Link

Problem 92 (Korea Winter Program Practice Test 2016)**.** Find all positive integers a, b, m, and n such that

$$a^2 + b^2 = m^2 - n^2, \text{ and}$$
$$ab = 2mn.$$

Link

Problem 93 (Korea Winter Program Practice Test 2016)**.** Find all pairs of positive integers (n, t) such that $6^n + 1 = n^2 t$, and $(n, 29 \times 197) = 1$. Link

Problem 94 (Korea National Olympiad Final Round 2016)**.** Prove that for all rationals x, y, $x - \frac{1}{x} + y - \frac{1}{y} = 4$ is not true. Link

Problem 95 (Kosovo TST 2016)**.** Show that for any $n \geq 2$, the number $2^{2^n} + 1$ ends with 7. Link

Problem 96 (Latvia National Olympiad 2016)**.**

1. Given positive integers x and y such that xy^2 is a perfect cube, prove that $x^2 y$ is also a perfect cube. Link
2. Given that x and y are positive integers such that xy^{10} is perfect 33rd power of a positive integer, prove that $x^{10} y$ is also a perfect 33rd power. Link
3. Given that x and y are positive integers such that xy^{433} is a perfect 2016-power of a positive integer, prove that $x^{433} y$ is also a perfect 2016-power. Link
4. Given that x, y and z are positive integers such that $x^3 y^5 z^6$ is a perfect 7th power of a positive integer, show that also $x^5 y^6 z^3$ is a perfect 7th power. Link

Problem 97 (Latvia National Olympiad 2016)**.** Prove that among any 18 consecutive positive 3 digit numbers, there is at least one that is divisible by the sum of its digits. Link

Problem 98 (Latvia National Olympiad 2016)**.** Two functions are defined by equations: $f(a) = a^2 + 3a + 2$ and $g(b, c) = b^2 - b + 3c^2 + 3c$. Prove that for any positive integer a there exist positive integers b and c such that $f(a) = g(b, c)$.

Link

Problem 99 (Macedonian National Olympiad 2016)**.** Solve the equation in the set of natural numbers $1 + x^z + y^z = \text{lcm}(x^z, y^z)$. Link

Problem 100 (Macedonian National Olympiad 2016)**.** Solve the equation in the set of natural numbers $xyz + yzt + xzt + xyt = xyzt + 3$. Link

Problem 101 (Mediterranean Mathematics Olympiad 2016)**.** Determine all integers $n \geq 1$ for which the number $n^8 + n^6 + n^4 + 4$ is prime. Link

Problem 102 (Middle European Mathematical Olympiad 2016)**.** Find all $f : \mathbb{N} \to \mathbb{N}$ such that $f(a) + f(b)$ divides $2(a + b - 1)$ for all $a, b \in \mathbb{N}$. Link

Problem 103 (Middle European Mathematical Olympiad 2016)**.** A positive integer n is Mozart if the decimal representation of the sequence $1, 2, \ldots, n$ contains each digit an even number of times. Prove that:

1. All Mozart numbers are even.
2. There are infinitely many Mozart numbers.

Link

Problem 104 (Middle European Mathematical Olympiad 2016)**.** For a positive integer n, the equation $a^2 + b^2 + c^2 + n = abc$ is given in the positive integers. Prove that:

1. There does not exist a solution (a, b, c) for $n = 2017$.
2. For $n = 2016$, a is divisible by 3 for all solutions (a, b, c).
3. There are infinitely many solutions (a, b, c) for $n = 2016$.

Link

Problem 105 (Netherlands TST 2016)**.** Find all positive integers k for which the equation:

$$\text{lcm}(m, n) - \gcd(m, n) = k(m - n)$$

has no solution in integers positive (m, n) with $m \neq n$. Link

Problem 106 (Pan-African Mathematical Olympiad 2016)**.** For any positive integer n, we define the integer $P(n)$ by

$$P(n) = n(n + 1)(2n + 1)(3n + 1) \ldots (16n + 1)$$

Find the greatest common divisor of the integers $P(1), P(2), P(3), \ldots, P(2016)$. Link

Problem 107 (Polish Mathematical Olympiad 2016)**.** Let p be a certain prime number. Find all non-negative integers n for which polynomial $P(x) = x^4 - 2(n+p)x^2 + (n-p)^2$ may be rewritten as product of two quadratic polynomials $P_1,\ P_2 \in \mathbb{Z}[X]$. Link

Problem 108 (Polish Mathematical Olympiad 2016)**.** Let k, n be odd positive integers greater than 1. Prove that if there a exists natural number a such that $k|2^a+1,\ n|2^a-1$, then there is no natural number b satisfying $k|2^b-1,\ n|2^b+1$. Link

Problem 109 (Polish Mathematical Olympiad 2016)**.** There are given two positive real number $a < b$. Show that there exist positive integers $p,\ q,\ r,\ s$ satisfying following conditions:

1. $a < \frac{p}{q} < \frac{r}{s} < b$.
2. $p^2 + q^2 = r^2 + s^2$.

Link

Problem 110 (Romania TST 2016)**.** Given a prime p, prove that the sum $\sum\limits_{k=1}^{\lfloor \frac{q}{p} \rfloor} k^{p-1}$ is not divisible by q for all but finitely many primes q. Link

Problem 111 (Romania TST 2016)**.** Determine the positive integers expressible in the form $\frac{x^2+y}{xy+1}$, for at least two pairs (x, y) of positive integers. Link

Problem 112 (Romanian Masters in Mathematics 2016)**.** A *cubic sequence* is a sequence of integers given by $a_n = n^3 + bn^2 + cn + d$, where b, c and d are integer constants and n ranges over all integers, including negative integers.

(a) Show that there exists a cubic sequence such that the only terms of the sequence which are squares of integers are a_{2015} and a_{2016}.

(b) Determine the possible values of $a_{2015} \cdot a_{2016}$ for a cubic sequence satisfying the condition in part (a).

Link

Problem 113 (San Diego Math Olympiad 2016)**.** Let a, b, c, d be four integers. Prove that

$$(b-a)(c-a)(d-a)(d-c)(d-b)(c-b)$$

is divisible by 12. Link

Problem 114 (San Diego Math Olympiad 2016)**.** Quadratic equation $x^2+ax+b+1 = 0$ have 2 positive integer roots, for integers a, b. Show that $a^2 + b^2$ is not a prime. Link

Problem 115 (Selection round of Kiev team to UMO 2016). Find all numbers n such, that in $[1; 1000]$ there exists exactly 10 numbers with digit sum equal to n. Link

Problem 116 (Selection round of Kiev team to UMO 2016). Number 125 is written as the sum of several pairwise distinct and relatively prime numbers, greater than 1. What is the maximal possible number of terms in this sum? Link

Problem 117 (Selection round of Kiev team to UMO 2016). Given prime number p and different natural numbers m, n such that $p^2 = \frac{m^2+n^2}{2}$. Prove that $2p - m - n$ is either square or doubled square of an integer number. Link

Problem 118 (Selection round of Kiev team to UMO 2016). Solve the equation $n(n^2 + 19) = m(m^2 - 10)$ in positive integers. Link

Problem 119 (Serbia Additional TST 2016). Let $w(x)$ be largest odd divisor of x. Let a, b be natural numbers such that $(a, b) = 1$ and $a + w(b + 1)$ and $b + w(a + 1)$ are powers of two. Prove that $a + 1$ and $b + 1$ are powers of two. Link

Problem 120 (Serbia National Olympiad 2016). Let $n > 1$ be an integer. Prove that there exist $m > n^n$ such that $\frac{n^m - m^n}{m+n}$ is a positive integer. Link

Problem 121 (Serbia National Olympiad 2016). Let $a_1, a_2, \ldots, a_{2^{2016}}$ be positive integers not bigger than 2016. We know that for each $n \leq 2^{2016}$, $a_1 a_2 \ldots a_n +$ 1 is a perfect square. Prove that for some i , $a_i = 1$. Link

Problem 122 (South Africa National Olympiad 2016). Let k and m be integers with $1 < k < m$. For a positive integer i, let L_i be the least common multiple of $1, 2, \ldots, i$. Prove that k is a divisor of

$$L_i \cdot \left[\binom{m}{i} - \binom{m-k}{i} \right]$$

for all $i \geq 1$. Link

Problem 123 (China South East Mathematical Olympiad 2016). Let n be a positive integer and let D_n be the set of all positive divisors of n. Define $f(n) = \sum\limits_{d \in D_n} \frac{1}{1+d}$. Prove that for any positive integer m,

$$\sum_{i=1}^{m} f(i) < m.$$

Link

Problem 124 (China South East Mathematical Olympiad 2016). Let $\{a_n\}$ be a sequence consisting of positive integers such that $n^2 \mid \sum_{i=1}^{n} a_i$ and $a_n \leq (n + 2016)^2$ for all $n \geq 2016$. Define $b_n = a_{n+1} - a_n$. Prove that the sequence $\{b_n\}$ is eventually constant. Link

Problem 125 (China South East Mathematical Olympiad 2016)**.** Define the sets

$$A = \{a^3 + b^3 + c^3 - 3abc : a, b, c \in \mathbb{N}\},$$
$$B = \{(a + b - c)(b + c - a)(c + a - b) : a, b, c \in \mathbb{N}\},$$
$$P = \{n : n \in A \cap B, 1 \leq n \leq 2016\}.$$

Find the number of elements of P. Link

Problem 126 (Spain National Olympiad 2016)**.** Two real number sequences are guiven, one arithmetic $(a_n)_{n\in\mathbb{N}}$ and another geometric sequence $(g_n)_{n\in\mathbb{N}}$ none of them constant. Those sequences verifies $a_1 = g_1 \neq 0$, $a_2 = g_2$ and $a_{10} = g_3$. Find with proof that, for every positive integer p, there is a positive integer m, such that $g_p = a_m$. Link

Problem 127 (Spain National Olympiad 2016)**.** Given a positive prime number p. Prove that there exist a positive integer α such that $p|\alpha(\alpha - 1) + 3$, if and only if there exist a positive integer β such that $p|\beta(\beta - 1) + 25$. Link

Problem 128 (Spain National Olympiad 2016)**.** Let m be a positive integer and a and b be distinct positive integers strictly greater than m^2 and strictly less than $m^2 + m$. Find all integers d such that $m^2 < d < m^2 + m$ and d divides ab. Link

Problem 129 (Taiwan TST First Round 2016)**.** Find all ordered pairs (a, b) of positive integers that satisfy $a > b$ and the equation $(a - b)^{ab} = a^b b^a$. Link

Problem 130 (Taiwan TST Second Round 2016)**.** Let a and b be positive integers such that $a! + b!$ divides $a!b!$. Prove that $3a \geq 2b + 2$. Link

Problem 131 (Taiwan TST Second Round 2016)**.** Let $\langle F_n \rangle$ be the Fibonacci sequence, that is, $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ holds for all nonnegative integers n. Find all pairs (a, b) of positive integers with $a < b$ such that $F_n - 2na^n$ is divisible by b for all positive integers n. Link

Problem 132 (Taiwan TST Third Round 2016)**.** Let n be a positive integer. Find the number of odd coefficients of the polynomial $(x^2 - x + 1)^n$. Link

Problem 133 (Taiwan TST Third Round 2016)**.** Let k be a positive integer. A sequence $a_0, a_1, \ldots, a_n$ $(n > 0)$ of positive integers satisfies the following conditions:

(i) $a_0 = a_n = 1$;

(ii) $2 \leq a_i \leq k$ for each $k = 1, 2, \ldots, n - 1$

(iii) For each $j = 2, 3, \ldots, k$, the number j appears $\varphi(j)$ times in the sequence $a_0, a_1, \ldots, a_n$ ($\varphi(j)$ is the number of positive integers that do not exceed j and are coprime to j);

(iv) For any $i = 1, 2, \ldots, n-1$, $\gcd(a_{i-1}, a_i) = 1 = \gcd(a_i, a_{i+1})$, and a_i divides $a_{i-1} + a_{i+1}$

There is another sequence $b_0, b_1, \ldots, b_n$ of integers such that

$$\frac{b_{i+1}}{a_{i+1}} > \frac{b_i}{a_i}$$

for all $i = 0, 1, \ldots, n-1$. Find the minimum value for $b_n - b_0$. Link

Problem 134 (Taiwan TST Third Round 2016)**.** Let $f(x)$ be the polynomial with integer coefficients ($f(x)$ is not constant) such that

$$(x^3 + 4x^2 + 4x + 3)f(x) = (x^3 - 2x^2 + 2x - 1)f(x+1)$$

Prove that for each positive integer $n \geq 8$, $f(n)$ has at least five distinct prime divisors. Link

Problem 135 (USA TSTST 2016)**.** Decide whether or not there exists a nonconstant polynomial $Q(x)$ with integer coefficients with the following property: for every positive integer $n > 2$, the numbers

$$Q(0),\ Q(1), Q(2),\ \ldots,\ Q(n-1)$$

produce at most $0.499n$ distinct residues when taken modulo n. Link

Problem 136 (USA TSTST 2016)**.** Suppose that n and k are positive integers such that

$$1 = \underbrace{\varphi(\varphi(\ldots\varphi(}_{k\text{ times}}n)\ldots)).$$

Prove that $n \leq 3^k$. Link

Problem 137 (Turkey TST 2016)**.** p is a prime. Let K_p be the set of all polynomials with coefficients from the set $\{0, 1, \ldots, p-1\}$ and degree less than p. Assume that for all pairs of polynomials $P, Q \in K_p$ such that $P(Q(n)) \equiv n \pmod p$ for all integers n, the degrees of P and Q are equal. Determine all primes p with this condition. Link

Problem 138 (Turkmenistan Regional Olympiad 2016)**.** Find all distinct prime numbers p, q, r, s such that

$$1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r} - \frac{1}{s} = \frac{1}{pqrs}.$$

Link

Problem 139 (USA TST 2016)**.** Let $\sqrt{3} = 1.b_1b_2b_3\ldots_{(2)}$ be the binary representation of $\sqrt{3}$. Prove that for any positive integer n, at least one of the digits b_n, b_{n+1}, $\ldots$, b_{2n} equals 1. Link

Problem 140 (USAJMO 2016). Prove that there exists a positive integer $n < 10^6$ such that 5^n has six consecutive zeros in its decimal representation. Link

Problem 141 (USAMO 2016). Prove that for any positive integer k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer. Link

Problem 142 (USAMO 2016).

(a) Prove that if n is an odd perfect number then n has the following form

$$n = p^s m^2$$

where p is prime has form $4k+1$, s is positive integers has form $4h+1$, and $m \in \mathbb{Z}^+$, m is not divisible by p.

(b) Find all $n \in \mathbb{Z}^+$, $n > 1$ such that $n-1$ and $\frac{n(n+1)}{2}$ is perfect number.

Link