A Generalization of Erdös's Proof of Bertrand-Chebyshev Theorem

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Abstract

Legendre's conjecture states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer n. We consider the following question : for all integer $n > 1$ and a fixed integer $k \leq n$ does there exist a prime number such that $kn < p < (k+1)n$? Bertrand-Chebyshev theorem answers this question affirmatively for $k = 1$. A positive answer for $k = n$ would prove Legendre's conjecture. In this paper, we show that one can determine explicitly a number N_k such that for all $n \geq N_k$, there is at least one prime between kn and $(k+1)n$. Our proof is a generalization of Erdös's proof of Bertrand-Chebyshev theorem [2] and uses elementary combinatorial techniques without appealing to the prime number theorem.

Keywords: Bertrand's Postulate, Bertrand-Chebyshev theorem, distribution of prime numbers, Landau's problems, Legendre's conjecture.

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1 Introduction

Bertrand's postulate states that for every positive integer n, there is always at least one prime p such that $n < p < 2n$. This was first proved by Chebyshev in 1850 and hence the postulate is also called the Bertrand-Chebyshev theorem. Ramanujan gave a simpler proof by using the properties of the Gamma function [4], which resulted in the concept of Ramanujan primes. In 1932, Erdös published a simpler proof using the Chebyshev function and properties of binomial coefficients [2].

Legendre's conjecture states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer n. It is one of the four Landau's problems, considered as four basic problems about prime numbers. The other three problems are (i) Goldbach's conjecture : every even integer $n > 2$ can be written as the sum of two primes (ii) Twin prime conjecture : there are infinitely many primes p such that $p+2$ is prime (iii) are there infinitely many primes p such that $p-1$ is a perfect square ? All these problems are open till date.

We consider a generalization of the Bertrand's postulate : for all integer $n > 1$ and a fixed integer $k \leq n$ does there exist a prime number such that $kn < p < (k+1)n$? This question was first posed by Bachraoui [1]. He provided an affirmative answer for $k = 2$ and observed that a positive answer for $k = n$ would prove Legendre's conjecture. Bertrand-Chebyshev theorem answers this question affirmatively for $k = 1$. In this paper, we show that one can determine explicitly a number N_k such that for all $n \geq N_k$, there is at least one prime between kn and $(k+1)n$.

The existence of such N_k can be proved using the prime number theorem. Using sophisticated techniques several strong results are known towards proving Legendre's conjecture. Several papers improved the value of c such that "for any $N > 1$ there is a prime between N and $N + CN^{(1-1/c)}$", where $c > 2$ is some constant and $C > 0$ is a large constant depending on c. Writing $N = kn$ this shows that there is a prime between kn and $(k+1)n$ once $n > C^c.k^{(c-1)}$. The existence of such $c > 2$ (and some $C > 0$) was proved by Hoheisel in 1930, Ingham in 1937 showed that any $c > 8/3$ is good, Huxley in 1972 showed that any $c > 12/5$ is good, Heath-Brown and Iwaniec showed that any $c > 20/9$ is good, and the current record is due to Baker-Harman-Pintz who showed in 2000 that any $c > 40/19$ is good.

The main goal of this paper is to generalize Erdös's proof of Bertrand-Chebyshev theorem [2] and show that elementary combinatorial techniques can be used to obtain a bound on N_k.

Let $\pi(x)$ denote the number of prime numbers not greater than x. Let $\ln(x)$ denote the logarithm with base e of x. We write $k|n$ when k divides n. We let n run through the natural numbers and p through the primes. Let $\phi(a,b)$ denote the product of all primes greater than a and not greater than b, i.e.,

$$\phi(a,b) = \prod_{a<p\leq b} p$$

2 Lemmas

In this section, we present several lemmas which are used in the proof of our main theorem, presented in the next section.

Lemma 2.1. *If $k|n$ then*

$$\binom{\frac{(k+1)n}{k}}{n} < \left(\frac{(k+1)^{(k+1)}}{k^k}\right)^{\frac{n}{k}}$$

If $k|(n+l)$, $0 < l < k$ and $n > (k+1)^k$ then

$$\binom{\frac{(k+1)n+l}{k}}{n} < \left(\frac{(k+1)^{(k+1)}}{k^k}\right)^{\frac{n+l}{k}}$$

Proof. We prove this lemma for $l = 0$. The case $0 < l < k$ is similar. We use induction on n. It is easy to see that

$$\binom{k+1}{k} < \frac{(k+1)^{(k+1)}}{k^k}$$

Let the inequality hold for $\binom{(k+1)n}{kn}$. Then

$$\begin{aligned}\binom{(k+1)n+(k+1)}{kn+k} &= \binom{(k+1)n}{kn}\frac{((k+1)n+1)\ldots((k+1)n+(k+1))}{(n+1)(kn+1)\ldots(kn+k)} \\ &= \binom{(k+1)n}{kn}\frac{(k+1)((k+1)n+1)\ldots((k+1)n+k)}{(kn+1)\ldots(kn+k)}\end{aligned}$$

Comparing the coefficients of n^k and n^{k-1} in the numerator and the denominator we have, for all $n > k$

$$\frac{(k+1)((k+1)n+1)\ldots((k+1)n+k)}{(kn+1)\ldots(kn+k)} < \frac{(k+1)^{(k+1)}}{k^k}$$

□

Lemma 2.2. *If $k|n$ and $n \geq k(k+1)^{(k+1)}$ then*

$$\binom{\frac{(k+1)n}{k}}{n} > \left(\frac{(k+1)^{(k+1)}-1}{k^k}\right)^{\frac{n}{k}}$$

Proof. It is easy to prove that the inequality holds for $n = k(k+1)^{(k+1)}$. Let S_k denote the sum of integers from 1 to k, i.e., $S_k = \sum_{i=1}^{k} i$. Following the previous proof and comparing the coefficients of n^k and n^{k-1} in the numerator and the denominator, for all n such that $nk^k > S_k(k^{k-1}((k+1)^{k+1}-1) - k^k(k+1)^k)$ we have

$$\frac{(k+1)((k+1)n+1)\ldots((k+1)n+k)}{(kn+1)\ldots(kn+k)} > \frac{(k+1)^{(k+1)}-1}{k^k}$$

□

Lemma 2.3. *Let $N_k = k(k+1)^{2k+2}$. If $n \geq N_k$ and $k > 1$ then*

$$\left(\frac{(k+1)^{(k+1)}-1}{k^k}\right)^{n}\left(\frac{1}{(k+1)^{(k+1)}}\right)^{\frac{n}{k}} > ((k+1)n)^{\frac{\sqrt{(k+1)n}}{k}}$$

Proof. The following inequalities are equivalent:

$$\left(\frac{(k+1)^{(k+1)}-1}{k^k}\right)^n\left(\frac{1}{(k+1)^{(k+1)}}\right)^{\frac{n}{k}} > ((k+1)n)^{\frac{\sqrt{(k+1)n}}{k}}$$

$$\frac{k}{\sqrt{(k+1)}}\ln\left(\left(\frac{(k+1)^{(k+1)}-1}{k^k}\right)\left(\frac{1}{(k+1)^{(k+1)}}\right)^{\frac{1}{k}}\right) > \frac{\ln((k+1)n)}{\sqrt{n}}$$

The function $\frac{\ln((k+1)x)}{\sqrt{x}}$ is decreasing and the above inequality holds for $n = N_k$ $\square$

Lemma 2.4. *If $k|n$ then*

$$\phi\left(\frac{n}{k}, \frac{(k+1)n}{(k+2)}\right)\phi\left(n, \frac{(k+1)n}{k}\right) < \binom{\frac{(k+1)n}{k}}{n}$$

If $k|(n+l)$, $0 < l < k$, then

$$\phi\left(\frac{n+l}{k}, \frac{(k+1)n}{(k+2)}\right)\phi\left(n, \frac{(k+1)n+l}{k}\right) < \binom{\frac{(k+1)n+l}{k}}{n}$$

Proof. We prove this lemma for $l = 0$. The case $0 < l < k$ is similar. We have

$$\binom{\frac{(k+1)n}{k}}{n} = \frac{(n+1)\dots\frac{(k+1)n}{k}}{\frac{n}{k}!} \tag{1}$$

Clearly $\phi\left(n, \frac{(k+1)n}{k}\right)$ divides $\binom{\frac{(k+1)n}{k}}{n}$. If $\frac{n}{k} < p \le \frac{(k+1)n}{(k+2)}$ then kp occurs in the numerator of (1) but p does not occur in the denominator. After simplification of kp with a number of the form αk from the denominator we get the prime factor p in $\binom{\frac{(k+1)n}{k}}{n}$. Hence $\phi\left(\frac{n}{k}, \frac{(k+1)n}{(k+2)}\right)$ divides $\binom{\frac{(k+1)n}{k}}{n}$ too and the lemma follows. $\square$

3 The proof of main theorem

Theorem 3.1. *For any integer $1 < k < n$, there exists a number N_k such that for all $n \ge N_k$, there is at least one prime between kn and $(k+1)n$.*

Proof. The product of primes between kn and $(k+1)n$, if there are any, divides $\binom{(k+1)n}{kn}$. For a fixed prime p, let $\beta(p)$ be the highest number x, such that p^x divides $\binom{(k+1)n}{kn}$. Let $\binom{(k+1)n}{kn} = P_1P_2P_3$, such that,

$$P_1 = \prod_{p\le\sqrt{(k+1)n}} p^{\beta(p)}, \qquad P_2 = \prod_{\sqrt{(k+1)n}<p\le kn} p^{\beta(p)}, \qquad P_3 = \prod_{kn+1<p\le(k+1)n} p^{\beta(p)}$$

To prove the theorem we have to show that $P_3 > 1$ for $n \ge N_k$. Clearly, $P_1 < ((k+1)n)^{\pi(\sqrt{(k+1)n})}$. From Lemma 4.2, we have $P_2 < ((k+1)^{(k+1)})^{\frac{n}{k}}$. From Lemmas 3.1 and 3.2, we have

$$\left(\frac{(k+1)^{(k+1)}-1}{k^k}\right)^n < P_1P_2P_3 < ((k+1)n)^{\pi(\sqrt{(k+1)n})}((k+1)^{(k+1)})^{\frac{n}{k}}P_3$$

Using Lemma 3.3 and $\pi(\sqrt{(k+1)n}) \leq \frac{\sqrt{(k+1)n}}{2}$ we have

$$P_3 > \left(\frac{(k+1)^{(k+1)}-1}{k^k}\right)^n \left(\frac{1}{(k+1)^{(k+1)}}\right)^{\frac{n}{k}} \frac{1}{((k+1)n)^{\pi(\sqrt{(k+1)n})}} > 1$$

□

Lemma 3.2. *Let P_2 be as defined in the proof of Theorem 4.1. Then $P_2 < ((k+1)^{(k+1)})^{\frac{n}{k}}$.*

Proof. We have

$$\binom{(k+1)n}{kn} = \frac{(kn+1)(kn+2)\cdots(k+1)n}{1\cdot 2\cdots n}. \tag{2}$$

The prime decomposition [3] of $\binom{(k+1)n}{kn}$ implies that the powers of primes in P_2 are less than 2. Clearly, a prime p satisfying $\frac{(k+1)n}{k+2} < p \leq n$ appears in the denominator of (2) but $2p$ does not, and $(k+1)p$ appears in the numerator of (2) but $(k+2)p$ does not. Hence the powers of such primes in P_2 is 0. Also if a prime p satisfies $\frac{(k+1)n}{k} < p \leq kn$ then its power in P_2 is 0 because it appears neither in the denominator nor in the numerator of (2). We have

$$\begin{aligned} P_2 &< \phi\left(\sqrt{(k+1)n}, \frac{n}{k}\right) \phi\left(\frac{n}{k}, \frac{(k+1)n}{(k+2)}\right) \phi\left(n, \frac{(k+1)n}{k}\right) \\ &< 4^{\frac{n}{k}} \binom{\frac{(k+1)n}{k}}{n} \\ &< 4^{\frac{n}{k}} \left(\frac{(k+1)^{(k+1)}}{k^k}\right)^{\frac{n}{k}} \\ &< ((k+1)^{(k+1)})^{\frac{n}{k}} \end{aligned}$$

We used Lemmas 3.4, 3.1 and the fact that $\prod_{p \leq x} p < 4^x$. Similarly we get the same bound when $0 < l < k$ in Lemmas 3.4, 3.1. □

References

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