Zadaca 2

22. listopada 2023.   22:35

Task 1 (10 pts.). Prove the following statement: $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

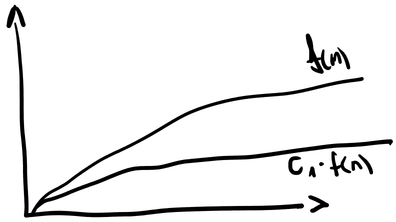
$$f(n) = \Theta(g(n)) \Leftrightarrow f(n) = O(g(n)) \text{ i } f(n) = \Omega(g(n))$$

$\boxed{\Leftarrow}$

$f(n) = \Theta(g(n))$ ako $\exists c_1 > 0, c_2 > 0, n_0 > 0$ t.d.

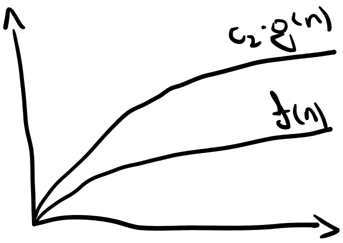
$$0 \leq \underbrace{c_1 \cdot g(n) \leq \overbrace{f(n)}^{}}_{\text{I.}} \overbrace{\leq c_2 \cdot g(n)}^{\text{II.}}, \forall n \geq n_0$$

I. $c_1 \cdot g(n) \leq f(n); \forall n \geq n_0$



$\Rightarrow f(n) = \Omega(g(n)); \forall n \geq n_0$

II. $f(n) \leq c_2 \cdot g(n)$



$\Rightarrow f(n) = O(g(n))$

$\boxed{\Rightarrow}$ Ako je $f(n) = O(n)$ i $f(n) = \Omega(n)$, onda je $f(n) = \Theta(n)$

$f(n) = O(g(n)) \Rightarrow \exists c_1 > 0$ t.d $f(n) \leq c_1 \cdot g(n); \forall n \geq \underline{n_0}$

$f(n) = \Omega(g(n)) \Rightarrow \exists c_2 > 0$ t.d. $c_2 \cdot g(n) \leq f(n); \forall n \geq \underline{n_1}$

$\Rightarrow \max = \{n_0, n_1\} = n_2$

$$\Rightarrow f(n) = O(g(n)) \,;\, \forall n \geq n_2 \quad \left.\begin{matrix} \\ \\ \end{matrix}\right\} \Rightarrow c_2 \cdot g(n) \leq f(n) \leq c_1 \cdot g(n) \,;$$
$$f(n) = \Omega(g(n)) \,;\, \forall n \geq n_2 \qquad \forall n \geq n_2$$
$$\Rightarrow f(n) = \Theta(g(n)) \checkmark$$

Task 2 (25 pts.). Use the master theorem to asymptotically bound the following functions:

a) $T(n) = 16T(\frac{n}{4}) + n$.

$$T(n) = \underbrace{16}_{a} \cdot T(\frac{n}{\underbrace{4}_{b}}) + n$$

$a = 16 \,,\, b = 4 \,,\, f(n) = n$

$$n^{\log_4 16} = n^2$$

1\. SLUČAJ: Ako je $f(n) = O(n^{\log_b a - \varepsilon})$, za $\forall \varepsilon > 0$,
tada $T(n) = \Theta(n^{\log_b a})$

$$n = O(n^{2-\varepsilon}) \; (\varepsilon = 1) \checkmark \Rightarrow T(n) = \Theta(n^2)$$

b) $T(n) = 3T(\frac{n}{6}) + n^{0.500001}$.

$a = 3 \,,\, b = 6 \,,\, f(n) = n^{0.500001}$

$$n^{\log_b a} = n^{\log_6 3} = n^{0.613147}$$

1\. SLUČAJ: $f(n) = O(n^{\log_b a})$, za $\forall \varepsilon > 0$,
tada $T(n) = \Theta(n^{\log_b a})$

$$n^{0.500001} = O(n^{0.613147 - \varepsilon}) \; (\varepsilon = 0{,}1) \checkmark$$

$$\Rightarrow T(n) = \Theta(n^{0.613147})$$

c) $T(n) = 3T(\frac{n}{4}) + n\log n.$

$a = 3$, $b = 4$, $f(n) = n\log n$

$n^{\log_b a} = n^{\log_4 3} = n^{0.792481}$

SLUČAJ 3:

1° $f(n) = n\log n = \Omega\left(n^{0.792481+\varepsilon}\right)$, za neki $\varepsilon > 0$

npr. $\varepsilon = 0.5$ ✓

2° $a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n)$, za neki $c > 1$

$3 \cdot \frac{n}{4} \cdot \log\left(\frac{n}{4}\right) \leq c \cdot n\log n \left(\log\frac{n}{4} \leq \log n, \forall n \in \mathbb{N}\right)$

$\frac{3}{4} \cdot \cancel{n\log n} \leq c \cdot \cancel{n\log n}$

$\frac{3}{4} \leq c \Rightarrow c \in \left[\frac{3}{4}, 1\right\rangle$ ✓

$\Rightarrow T(n) = \Theta(n\log n)$

d) $T(n) = 4T(\frac{n}{2}) + n^2.$

$a = 4$, $b = 2$, $f(n) = n^2$

$n^{\log_b a} = n^{\log_2 4} = n^2$

SLUČAJ 2: $f(n) = n^2 = \Theta(n^2)$

$\Rightarrow T(n) = \Theta(n^2 \log n)$

e) $T(n) = 4T(\frac{n}{2}) - n^2.$

$a = 4 \quad b = 2 \quad f(n) = -n^2$

- $f(n)$ je negativna f
- master theorem se ne može primjenit

Task 3 (10 pts.). Use the recursion tree to determine a good asymptotic upper bound on the following recurrence:

$$T(n) = 3T\left(\frac{n}{6}\right) + n^2.$$

$$T(n) = 3T\left(\frac{n}{6}\right) + n^2$$