§ 4 Time Independent Point Games

Idea: given $p_B[1,0] + p_A[0,1] \xrightarrow{\text{trans}} 1[\beta,\alpha]$

$\Downarrow$ trivial (it gets cancelled from the inequalities

$$p_B[1,0] + p_A[0,1] + \sum_i w_i[x_i, y_i]$$

$$\Big\downarrow \text{trans}$$

$$1[\beta,\alpha] + \sum_i w_i[x_i, y_i]$$

with $w_i \geq 0$.

Claim: Inverse is also true! (will prove shortly)

Consequences: (a) We can allow <u>-ve prob.</u> in our transitions in between, $\because$ we "fill in" with the "catalyst state".

This motivates a

Def^n 21: (Valid f^n) $p : \mathbb{R}^+ \to \infty$ is valid $\Leftrightarrow \sum_z p(z) = 0$ &

$$\sum_z \left(\frac{-1}{\lambda + z}\right) p(z) \geq 0 \quad \forall \lambda \geq 0.$$

Remarks: Def^n implies p is in the dual to the cone of operator mon. f^ns.

Recall: e.g. $C : \{X \mid a^T X a \geq 0 \ \forall a \in \mathbb{R}\}$

(dual) $D : \{Y \mid \mathrm{tr}(YX) \geq 0 \ \forall X \in C\}$

$X = bb^T$

$Y_1 - Y_2 \succeq 0 \Leftrightarrow Y_1 - Y_2 \in D$

similarly $C : \{f \mid f(X) < f(Y) \ \forall X < Y\}$

(dual) $D : \{p \mid \mathrm{tr}(f(X)\, p(X)) > 0 \ \forall f \in C\}$

$p_2 - p_1 \succeq 0 \Leftrightarrow$

$\sum (p_2 f - p_1 f) > 0$

Lemma 22. Following are valid functions

- Point raise: $-p[z] + p[z']$ $(z \leq z')$

- Point merging: $-p_1[z_1] - p_2[z_2] + (p_1+p_2)\left[\frac{p_1 z_1 + p_2 z_2}{p_1+p_2}\right]$
- Point splitting: $-(p_1+p_2)\left[\frac{p_1+p_2}{p_1 w_1 + p_2 w_2}\right] + p_1\left[\frac{1}{w_1}\right] + p_2\left[\frac{1}{w_2}\right]$

(b) We can <u>combine transitions</u>; all horizontal & vertical $\because$ def^n stated above. After first horizontal (for e.g.) may have -ve. prob. That's alright.

This motivates.

Def^n 23. A f^n $p : \mathbb{R}^+ \otimes \mathbb{R}^+ \to \mathbb{R}$ is valid iff

either

- $\forall\, c \in \mathbb{R}^+$ $\quad p(z, c)$ is valid $\quad\leftrightarrow$ horizontal

or

- $\forall\, c \in \mathbb{R}^+$ $\quad p(z, c)$ is valid. $\quad\leftrightarrow$ vertical

NB: We needn't specify which is done first.

Def^n 24: A Time independent point game (TIPG).

consists of f^{n}s $h, v : \mathbb{R}^+ \otimes \mathbb{R}^+ \to \mathbb{R}$ s.t.

- h is a valid horizontal f^n
- v is a valid vertical f^n
- $h + v = 1[\beta, \alpha] - p_B[1, 0] - p_A[0, 1]$

where $[\beta, \alpha]$ is called the final point of the TIPG.

§ 4.1.1 Relating TDPGs & TIPGs.

TDPG $\Rightarrow$ TIPG.

proof: $\quad h = \sum_{i \in H} (p_i - p_{i-1}) \quad ; \quad v = \sum_{i \in V} (p_i - p_{i-1})$

$$h + v = p_n - p_0 = 1[\beta, \alpha] - p_B[1, 0] - p_A[0, 1]$$

$\square$

TIPG $\Rightarrow$ TDPG

Assume h, v & $[\beta, \alpha]$ are known.

$$v^-(x,y) = -\min\left(v(x,y), 0\right) \geq 0$$ Magnitude of the neg. part.

NB*: $P_B[1,0] + P_A[0,1] + v^- \xrightarrow{\text{valid}} P_B[1,0] + P_A[0,1] + v^- + v$

$$\xrightarrow{\text{valid}} P_B[1,0] + P_A[0,1] + v^- + v + h$$

$$= [\beta, \alpha] + v^-$$

Remarks: If we can somehow show that v^- can be added & removed, we're done.

Lemma 25. For $r : \mathbb{R}^+ \otimes \mathbb{R}^+ \to \mathbb{R}^+$ s.t. $r(0,0) = 0$

$\exists$ $c > 0$ & $r' : \mathbb{R}^+ \otimes \mathbb{R}^+ \to \mathbb{R}^+$ s.t.

$$c P_B[1,0] + c P_A[0,1] \longrightarrow r + r'$$

is transitively valid (assumed $P_A, P_B \geq 0$).

(will prove shortly)

Lemma 26. given $\epsilon > 0$ & a f^n $r'' : \mathbb{R}^+ \otimes \mathbb{R}^+ \to \mathbb{R}^+$

s.t. $\sum_{x,y} r''(x,y) = 1$

$\exists$ $1 > \delta > 0$ s.t.

$$(1-\delta)[\beta, \alpha] + \delta r'' \to 1[\beta+\epsilon, \alpha+\epsilon]$$

is transitively valid.

(will prove soon)

proof of TIPG $\Rightarrow$ TDPG:

$$\delta\left(P_A[0,1] + P_B[1,0]\right) \xrightarrow[\text{with } r = v^-]{\text{lem 25}} \delta\left(\frac{v^-}{c} + \frac{r'}{c}\right)$$

NB: $\sum \frac{v^-}{c} + \frac{r'}{c} = P_A + P_B = 1$

Add $(1-\delta)\left(P_A[0,1] + P_B[1,0]\right)$

$$\Rightarrow \quad P_A[0,1] + P_B[1,0] \xrightarrow{\text{trans}} (1-\delta)\left(P_A[0,1] + P_B[1,0]\right) + \delta\left(\frac{v^-}{c} + \frac{\rho'}{c}\right)$$

$$\xrightarrow{\text{NB* iteratively until}} (1-\delta)\,[\beta, \alpha] + \delta\left(\frac{v^-}{c} + \frac{\rho'}{c}\right)$$

$$\xrightarrow[\text{with } \rho'' = \frac{v^- + \rho'}{c}]{\text{lemma 26}} [\beta + \epsilon, \alpha + \epsilon]$$

Lemma 25

proof: Start with assuming ρ has support at a single point.

Let $\rho = q[x, y]$ $\quad q, x > 0$ while $y \geq 0$.

For $x \geq 1$

$$\frac{q}{P_B} P_B[1,0] + \frac{q}{P_B} P_A[0,1] \longrightarrow q[x,y] + \frac{q}{P_B} P_A[0,1]$$

is transitively valid $\because$ we're just raising

$$\Rightarrow \quad c = \frac{q}{P_B} \quad \& \quad \rho' = q\frac{P_A}{P_B}[0,1]$$

For $x < 1$

$$P_B[1,0] + P_A[0,1] \xrightarrow{\text{raise}} P_B[1,y] + P_A[0,1]$$

$$\xrightarrow{\text{split}} \frac{x}{2} P_B[x,y] + \left(1 - \frac{x}{2}\right) P_B[2-x, y] + P_A[0,1]$$

as we can always scale with a +ve #, using $c = \frac{2q}{x P_B}$

$$\rho' = c\left(1 - \frac{x}{2}\right) P_B[2-x, y] + c P_A[0,1].$$

For the general case where $r = \sum q_i [x_i, y_i]$, let

c_i & r_i' be as chosen above

$$\Rightarrow c_i P_B [1,0] + c_i P_A [0,1] \xrightarrow{\text{trans.}} q_i [x_i, y_i] + r_i'$$

Now

$$\sum_{i=1}^{k} \left(c_i P_B [1,0] + c_i P_A [0,1] \right) \xrightarrow{\text{trans}} \sum_{i=2}^{k} \left(c_i P_B [1,0] + c_i P_A [0,1] \right)$$
$$+ q_1 [x_1, y_1] + r_1'$$

$$\xrightarrow{\text{tran}} \dots \xrightarrow{\text{trans}} \sum_{i=j}^{k} c_i \left(P_B [1,0] + P_A [0,1] \right) +$$
$$\sum_{i=1}^{j} \left(q_i [x_i, y_i] + r_i' \right)$$

$$\xrightarrow{\text{tran}} r + \sum_{i=1}^{k} r_i'$$

Thus the lemma holds for $c = \sum c_i$, $r' = \sum r_i'$

Lemma 26

Proof: let x'' max x-coordinate in r''
y'' similarly.

Then $r'' \xrightarrow{\text{raise}} [x'', y'']$ so valid.

(assume also $x'' > \beta + \epsilon$, $y'' > \alpha + \epsilon$, else raise more).

Now

$$(1-\delta) [\beta, \alpha] + \delta [x'', y''] \xrightarrow{\text{raise}} (1-\delta') \underbrace{[\beta, \alpha]}_{\text{raise}} + (\delta' - \delta) [\beta, y''] + \delta [x'', y'']$$ merge

$$\xrightarrow{\text{raise + merge}} (1-\delta') [\beta + \epsilon, \alpha] + \delta' [\beta + \epsilon, y'']$$

merge

$$\longrightarrow [\beta + \epsilon, \alpha + \epsilon]$$

& conditions of merge validity enforce

$$(\delta' - \delta) \beta + \delta x'' = \delta' (\beta + \epsilon) \Leftrightarrow \delta (x'' - \beta) = \delta' \epsilon$$

The second merge: $\delta'(y''-\alpha)=\epsilon$.

Both δ & δ' can be found s.t. $1>\delta'>\delta>0$.

e.g. $\delta' = \dfrac{\epsilon}{(y''-\alpha)} \Rightarrow \delta = \dfrac{\epsilon^2}{(x''-\beta)(y''-\alpha)}$

$\llcorner$

Rough

For one vari transitions

$$\sum p(z)\,[z] + \sum w_i\,[x_i] \longrightarrow \sum p'(z)\,[z] + \sum w_i\,[x_i]$$

Let $p_1 = 2p^*$

$z_i = z^*$

$-p^*[z^*] \to -p^*[z^*]$ not valid.