§5 Towards zero bias

Goal: Describe protocols that achieve $\to 0$ bias in Kitaev's 2nd form

Recall: $p(z)$ is valid $\Leftrightarrow$ $\sum_z p(z)$, $\sum_z \frac{-1}{\lambda+z} p(z) \geq 0 \quad \forall \lambda \geq 0.$

e.g. point raises, point merges & splits

$\therefore$ $h(x,y)$ is valid as a f^n of x $\quad \forall y \geq 0$

$v(x,y)$ is valid as a f^n of y $\quad \forall x \geq 0$

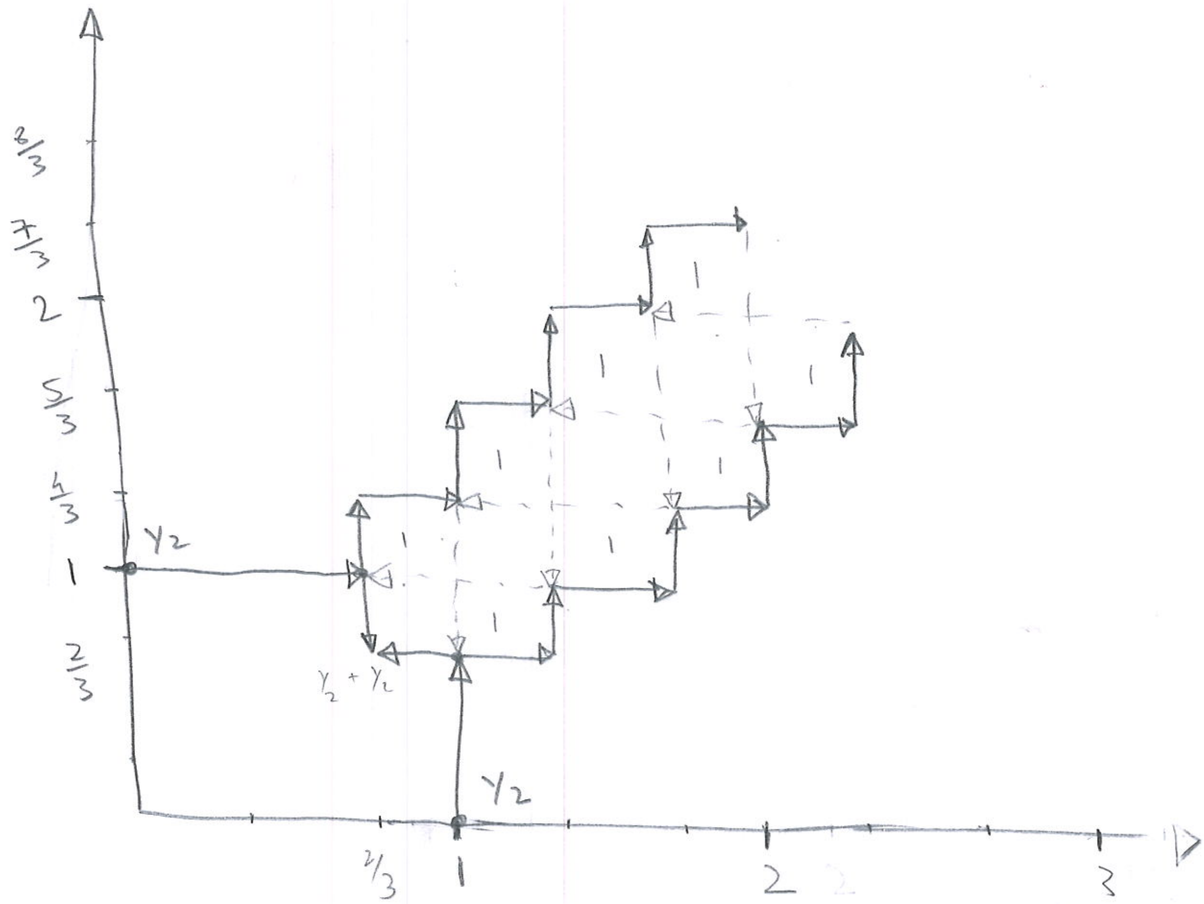
$\therefore$ TIPG: is a valid h & v s.t. $h+v = 1[\beta,\alpha] - P_B[1,0] - P_A[0,1]$

$\therefore$ yields a CF protocol with $P_A^* \leq \alpha$ & $P_B^* \leq \beta$.

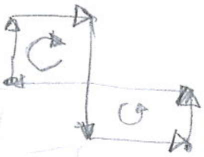
§5.1 Guiding Principles

Goal: Analyse a TIPG with 1/6 bias

Observe: (the same game as before with a new convention)



$\therefore$ Prob cons^n "$\Rightarrow$" each arrow $\leftrightarrow$ # ; Prob carried from base to head.

: The prob. must go around in loops e.g. 

: Label each loop with probs. that go around the loop.

Using the Algebraic notation,

$$h = \frac{1}{2}\left[\frac{2}{3},\frac{2}{3}\right] - \frac{3}{2}\left[1,\frac{2}{3}\right] + 1\left[\frac{4}{3},\frac{2}{3}\right]$$

$$-\frac{1}{2}\left[0,1\right] + \frac{3}{2}\left[\frac{2}{3},1\right] - 2\left[\frac{4}{3},1\right] + 1\left[\frac{5}{3},1\right]$$

$$+\sum_{k=4}^{\infty}\left(-1\left[\frac{k-2}{3},\frac{k}{3}\right] + 2\left[\frac{k-1}{3},\frac{k}{3}\right]\right.$$

$$\left. -2\left[\frac{k+1}{3},\frac{k}{3}\right] + 1\left[\frac{k+2}{3},\frac{k}{3}\right]\right)$$

Def^n: The last term represents a $\sum$ ladder rungs.

Remark: More generally, a pattern around the diagonal will be called a ladder.

Also, if (def^n) $v(x,y) = h(y,x)$, by construction

$$h + v = 1\left[\frac{2}{3},\frac{2}{3}\right] - \frac{1}{2}\left[1,0\right] - \frac{1}{2}\left[0,1\right]$$

Lets check: Is h valid (by symmetry v will also be valid)

(a) ✓ NB: $\forall\, y,\ \sum_x h(x,y) = 0$.

(b) ? : $\sum_x \frac{-1}{\lambda + x} h(x,y) \geq 0 \quad \forall\ \lambda > 0\ \&\ y \geq 0$

⌈ Lets start with $y = \frac{k}{3} \geq 4/3$.

$$\sum_x \frac{-1}{\lambda + x}\, h\left(x, \frac{k}{3}\right) = \frac{1}{\lambda + \frac{k-2}{3}} - \frac{2}{\lambda + \frac{k-1}{3}} + \frac{2}{\lambda + \frac{k+1}{3}} - \frac{1}{\lambda + \frac{k+2}{3}}$$

$$\left[\text{NB}: \quad \frac{1}{\lambda + x_1} - \frac{1}{\lambda + x_2} = \frac{x_2 - x_1}{(\lambda + x_1)(\lambda + x_2)}\right]$$

$$= \frac{\frac{1}{3}}{\left(\lambda + \frac{k-2}{3}\right)\left(\lambda + \frac{k-1}{3}\right)} - \frac{\frac{2}{3}}{\left(\lambda + \frac{k-1}{3}\right)\left(\lambda + \frac{k+1}{3}\right)}$$

$$+ \frac{\frac{1}{3}}{\left(\lambda + \frac{k+1}{3}\right)\left(\lambda + \frac{k+2}{3}\right)}$$

(using the same technique)

$$= \frac{\frac{1}{3}}{\left(\lambda + \frac{k-2}{3}\right)\left(\lambda + \frac{k-1}{3}\right)\left(\lambda + \frac{k-1}{3}\right)} - \frac{\frac{1}{3}}{\left(\lambda + \frac{k-1}{3}\right)\left(\lambda + \frac{k+1}{3}\right)\left(\lambda + \frac{k+2}{3}\right)}$$

(one last time)

$$= \frac{\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)}{\left(\lambda + \frac{k-2}{3}\right)\left(\lambda + \frac{k-1}{3}\right)\left(\lambda + \frac{k-1}{3}\right)\left(\lambda + \frac{k+2}{3}\right)} \geq 0$$

Interpret^n: 1st line : Standard sum over <u>points</u> with num = prob.

2nd line : Sum over <u>arrows</u>, numerator $\leftrightarrow$ prob $\times$ dist

3rd line : Sum over <u>pairs of arrows</u> with zero net momentum

For $y=1$,

$$-\frac{1}{2}[0]+\frac{3}{2}\left[\frac{2}{3}\right]-2\left[\frac{4}{3}\right]+1\left[\frac{5}{3}\right]$$

$$=\underbrace{\left(-\frac{1}{2}[0]+1\left[\frac{1}{3}\right]-\frac{1}{2}\left[\frac{2}{3}\right]\right)}_{\text{point merge.}}+\left(-1\left[\frac{1}{3}\right]+2\left[\frac{2}{3}\right]-2\left[\frac{4}{3}\right]+1\left[\frac{5}{3}\right]\right)$$

[proof $\frac{1}{\cancel{2}}\cdot 0+\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}=1\cdot\frac{1}{3}$]

$\therefore$ sum of 2 valid terms, it is valid.

ladder like terms with $y=1 \Leftrightarrow (k=3)$.

$\Downarrow$

valid.

For $y=\frac{2}{3}$, $\frac{1}{2}\left[\frac{2}{3}\right]-\frac{3}{2}[1]+1\left[4/3\right]=0$

which is a point split & thus valid.

(proof: $\frac{3}{2}/1 \stackrel{?}{=} 1/2 \Big/ \frac{2}{3}+1\Big/\frac{4}{3}$

$\Rightarrow \frac{3}{2} \stackrel{\text{yup}}{=} 2\cdot\frac{3}{4}$)

Remark: Other than the infinite point issue, we've established the $1/6$ protocol (in the TIPG fmwk)

Trouble: The infinite points carry infinite prob. $\Rightarrow$ The catalyst state would have to carry infinite probability.

Intuition Part: Read page 46 before 5.1.1 (can proceed without it as well)

§5.1.1 Obtaining non-negative numerators.

Motivation: For analysing the validity of functions $p(x)$ we would need expressions of the form

$$\sum_i \left(\frac{-1}{\lambda+x_i}\right) p(x_i) = \frac{f(-\lambda)}{\pi_i(\lambda+x_i)}$$

where $f(-\lambda)$ is a polynomial whose coefficients depend on the values of $p(x_i)$.

Remark: $f(-\lambda)$ as opposed to $f(\lambda)$? Will be clarified soon.

NB: $p(x)$ is valid $\Leftrightarrow$ $f(-\lambda) \geq 0$ $\forall$ $\lambda > 0$.

Remark: Combining terms of $p(x)$ in general to obtain $f(-\lambda)$ can be involved.

: Constructing polynomials is relatively easy. (e.g. specify it as a product of its roots).

$\Downarrow$

Approach: use $f(-\lambda)$ to compute $p(x)$ over $\{x_i\}$ chosen earlier.

(result!): we'll show $p(x_i) = \dfrac{-f(x_i)}{\prod_{j \neq i} (x_j - x_i)}$

which also conserves prob, granted $f(-\lambda)$ has deg. $\leq \underbrace{n}_{\#\text{ points}}-2$

Lemma 29. Let $n \geq 2$ & $x_1 \ldots x_n \in \mathbb{R}$ be distinct.
Then $\displaystyle\sum_{i=1}^{n} \prod_{\substack{j=1 \\ i \neq j}}^{n} \frac{1}{(x_j - x_i)} = 0$

proof: We use induction.
For $n = 2$,

$$\frac{1}{x_2 - x_1} + \frac{1}{x_1 - x_2} \left(= \frac{x_1 - x_2 + x_2 - x_1}{(x_2 - x_1)(x_1 - x_2)} \right) = 0$$

For $n > 2$,

NB: $\forall\ i$ s.t. $1 < i < n$

$$\frac{1}{(x_1 - x_i)(x_n - x_i)} = \frac{1}{(x_n - x_1)} \underbrace{\left(\frac{1}{(x_1 - x_i)} - \frac{1}{(x_n - x_i)} \right)}_{}$$

$$\because \quad \frac{x_n \cancel{- x_i} - x_1 \cancel{+ x_i}}{(x_1 - x_i)(x_n - x_i)}$$

which we use to write

$$\underbrace{\sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{(x_j - x_i)}} = \sum_{i=1}^{n} \frac{1}{(x_2 - x_i)(x_3 - x_i) \ldots (x_{n-1} - x_i)(x_i - x_n)}$$

$$\cdot \left[\frac{1}{(x_1 - x_i)} - \frac{1}{(x_n - x_i)} \right]$$

$$= \frac{1}{(x_1 - x_n)} \left[\sum_{i=1}^{n-1} \prod_{i \neq j}^{n-1} \frac{1}{(x_j - x_i)} - \sum_{i=2}^{n} \prod_{i \neq j} \frac{1}{(x_j - x_i)} \right]$$

I can iteratively apply this into and eventually have two terms of the form $\frac{1}{x_j - x_i} - \frac{1}{x_j - x_i} = 0$.

viz. by induction, both terms inside the parenthesis are zero.

Lemma 30. Let $n \geq 2$ & $x_1, \ldots x_n \in \mathbb{R}$ be distinct.

For $f(x)$, a polynomial of degree $k \leq n-2$

$$\sum_{i=1}^{n} \frac{f(x_i)}{\prod_{j \neq i} (x_j - x_i)} = 0.$$

Proof: Again, we'll use induction (on k, the degree of $f(x)$).

For $k = 0$, $\displaystyle\sum_{i=1}^{n} \prod_{j \neq i} \frac{1}{(x_j - x_i)} \overset{\text{lemma 29}}{=} 0$.

For $k > 0$, one can always write

$$f(x) = c \prod_{j=1}^{k} (x_j - x) + \underbrace{g(x)}_{\deg < K}$$

$\because$ c can be chosen to match the coefficient of x^k & everything else is absorbed in $g(x)$.

$$\Rightarrow \sum_{i=1}^{n} \frac{f(x_i)}{\prod_{j \neq i} (x_j - x_i)} = c \sum_{i=k+1}^{n} \prod_{\substack{j=k+1 \\ j \neq i}}^{n} \frac{1}{(x_j - x_i)} + \sum_{i=1}^{n} \frac{g(x_i)}{\prod_{j \neq i} (x_j - x_i)}$$

where the first term is zero & the second by induction is also zero.

Lemma 31. Let $x_1, \ldots x_n \in [0, \infty)$ distinct & let $f(-\lambda)$ be a polynomial with deg. $k \leq n-2$ s.t. $f(-\lambda) \geq 0$ $\forall$ $\lambda \geq 0$. Then

$$p = \sum_{i} \left(\frac{-f(x_i)}{\prod_{j \neq i} (x_j - x_i)} \right) [x_i]$$

is a valid $f^{\wedge}$

Proof: Apply the previous lemma with an appended point $x_{n+1} = -\lambda$ to get

$$\sum_{i=1}^{n} \left(\frac{-1}{\lambda + x_i}\right)\left(\frac{f(x_i)}{\prod_{j\neq i}(x_j - x_i)}\right) + \frac{f(-\lambda)}{\prod_i(\lambda + x_i)} = 0.$$

NB: This immediately proves the condition for validity

$$\sum_i \left(\frac{-1}{\lambda + x_i}\right) p(x_i) \geq 0$$

$$\text{if } p = -\frac{f(x_i)}{\prod(x_j - x_i)}$$

except prob. conservation.

: The aforesaid holds for f with deg., k, $\leq (n+1)-2$.

we

$$\sum_i p(x_i) = \lim_{\lambda\to\infty} \sum_{i=1}^{n} \left(\frac{\lambda}{\lambda + x_i}\right)\left(\frac{-f(x_i)}{\prod_{j\neq i}(x_j - x_i)}\right)$$

$$= \lim_{\lambda\to\infty} \frac{-\lambda f(-\lambda)}{\prod_i(\lambda + x_i)} \quad \begin{matrix} \text{— deg } n-1 \\ \text{— deg } n \end{matrix}$$

which converges to 0 if f has deg $k \leq n-2$

§ 5.1.2 Truncating the ladder

Recall: A single rung of the ladder has the form

$$a\left[\frac{k-2}{3}, \frac{k}{3}\right] + b\left[\frac{k-1}{3}, \frac{k}{3}\right]$$

$$+ \; c\left[\frac{k+1}{3}, \frac{k}{3}\right] + d\left[\frac{k+2}{3}, \frac{k}{3}\right].$$

Recall: $\quad p(x_i) = -\dfrac{f(x_i)}{\prod_{j\neq i}(x_j - x_i)}$ makes P valid.

We therefore must set

$$a = \frac{-f\left(+\frac{k-2}{3}\right)}{\left(\frac{1}{3}\right)\left(\frac{3}{3}\right)\left(\frac{4}{3}\right)} = -\frac{9\, f\left(\frac{k-2}{3}\right)}{4}$$

& similarly $\quad b = \dfrac{9\, f\left(\frac{k-1}{3}\right)}{2}$, $\quad c = -\dfrac{9 f\left(\frac{k+1}{3}\right)}{2}$

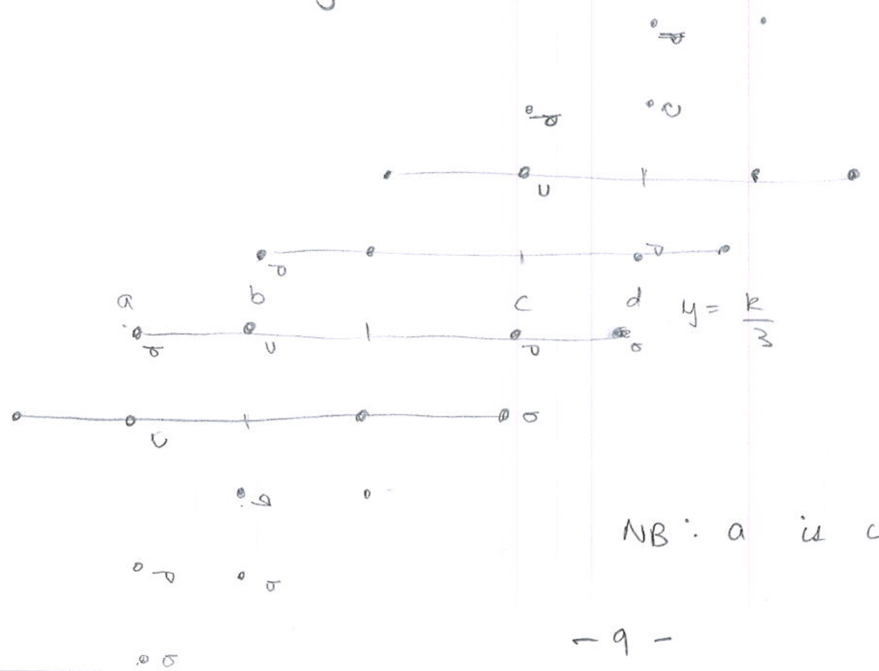
& $d = \dfrac{9\ f\left(\frac{k+2}{3}\right)}{4}$. (I didn't verify c & d myself).

Remark: The ladder we used for the Y_6 protocol can be obtained by setting $f(-\lambda) = \frac{4}{9}$.

Approach: We want to set different heights; we can use till a quadratic f^n $f(x,y)$ s.t.

$f(-\lambda, y) \geq 0 \quad \forall \quad \lambda \geq 0$ & y in the ladder. (non-zero!)

: We must also enforce
$h(y,x) = v(x,y)$ s.t. $h+v$ cancel the ladder;
viz. $h(x,y) = -h(y,x)$.



$a_{y=\frac{k}{3}} = -d_{y=\frac{k-2}{3}}$

$b_{y=\frac{k}{3}} = -c_{y=\frac{k-1}{3}}$

NB: a is close to the $x=0$ (or $y=0$ when flipped)

$$\Rightarrow \begin{cases} -\dfrac{9f\left(\frac{k-2}{3}, \frac{k}{3}\right)}{4} = -\dfrac{9f\left(\frac{k}{3}, \frac{k-2}{3}\right)}{4} \\[2ex] \dfrac{9f\left(\frac{k-1}{3}, \frac{k}{3}\right)}{2} = \dfrac{9f\left(\frac{k}{3}, \frac{k-1}{3}\right)}{2} \end{cases}$$

We now choose f to stop at a certain height $y = \frac{\Gamma}{3}$ by setting

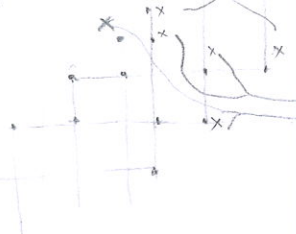
$$f(x,y) = C\left(\frac{\Gamma+1}{3} - x\right)\left(\frac{\Gamma+2}{3} - x\right)\left(\frac{\Gamma+1}{3} - y\right)\left(\frac{\Gamma+2}{3} - y\right)$$

where C & Γ are determined soon.

NB: We could've stopped the ladder for some height but that would make h assymmetric.

$$\because \; f\left(\frac{\Gamma+2}{3}, \frac{\Gamma}{3}\right) = f\left(\frac{\Gamma+1}{3}, \frac{\Gamma}{3}\right) = f\left(\frac{\Gamma+1}{3}, \frac{\Gamma-1}{3}\right) = 0$$

$\Rightarrow$ we can stop & still retain $h(x,y) = -h(y,x)$.



NB: The ladder part would then become

$$h_{lad} = \sum_{k=3}^{\Gamma} \left(-\frac{9}{4} f\left(\frac{k-2}{3}, \frac{k}{3}\right)\left[\frac{k-2}{3}, \frac{k}{3}\right] + \right.$$

$$\frac{9f\left(\frac{k-1}{3}, \frac{k}{3}\right)}{2}\left[\frac{k-1}{3}, \frac{k}{3}\right] - \frac{9}{2} f\left(\frac{k+1}{3}, \frac{k}{3}\right)\left[\frac{k+1}{3}, \frac{k}{3}\right]$$

$$\left. + \frac{9f\left(\frac{k+2}{3}, \frac{k}{3}\right)}{4}\left[\frac{k+2}{3}, \frac{k}{3}\right] \right)$$

NB 2: Lead is valid $\because$ $f(-\lambda, y) \geq 0$ $\forall$ $\lambda > 0$ & $y \leq \Gamma/3$.
& its quadratic in λ.

NB 3: For large Γ, close to the bottom of the ladder,
$f \simeq C\Gamma^4/3^4$. We choose $C \simeq 36/\Gamma^4$ so
that we approximately recover the $f = 4/9$ ladder.

NB 4: Details of merging the bottom of the ladder with the remaining structure has been skipped for now.
No crucial new ideas are used for this.
(citing Mochon)

§ 5.1.3 Building Better Ladders

Observe:

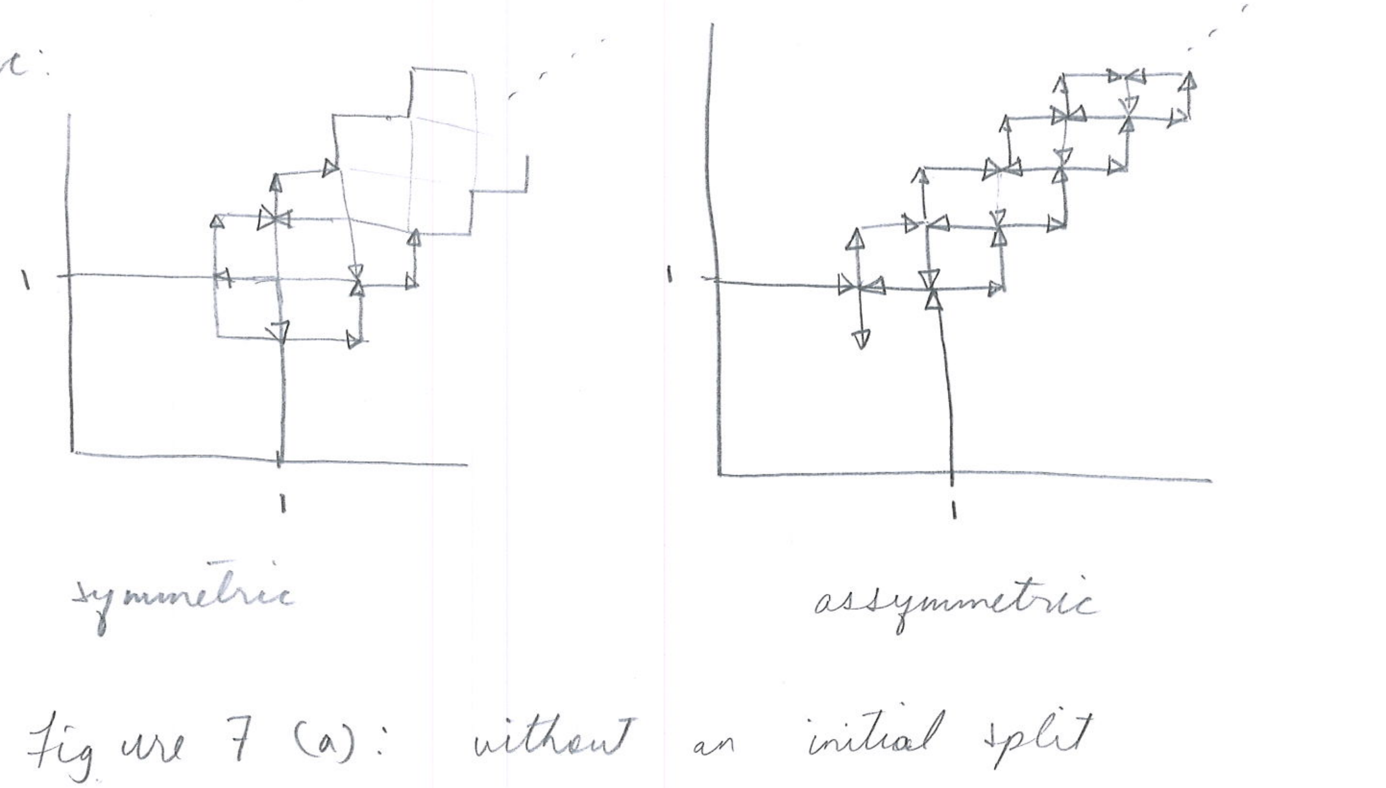


Figure 7 (a): without an initial split

NB: $\because$ the space of TIPG is a cone, an asymmetric TIPG can always be made symmetric by taking in addition the reflection of it.

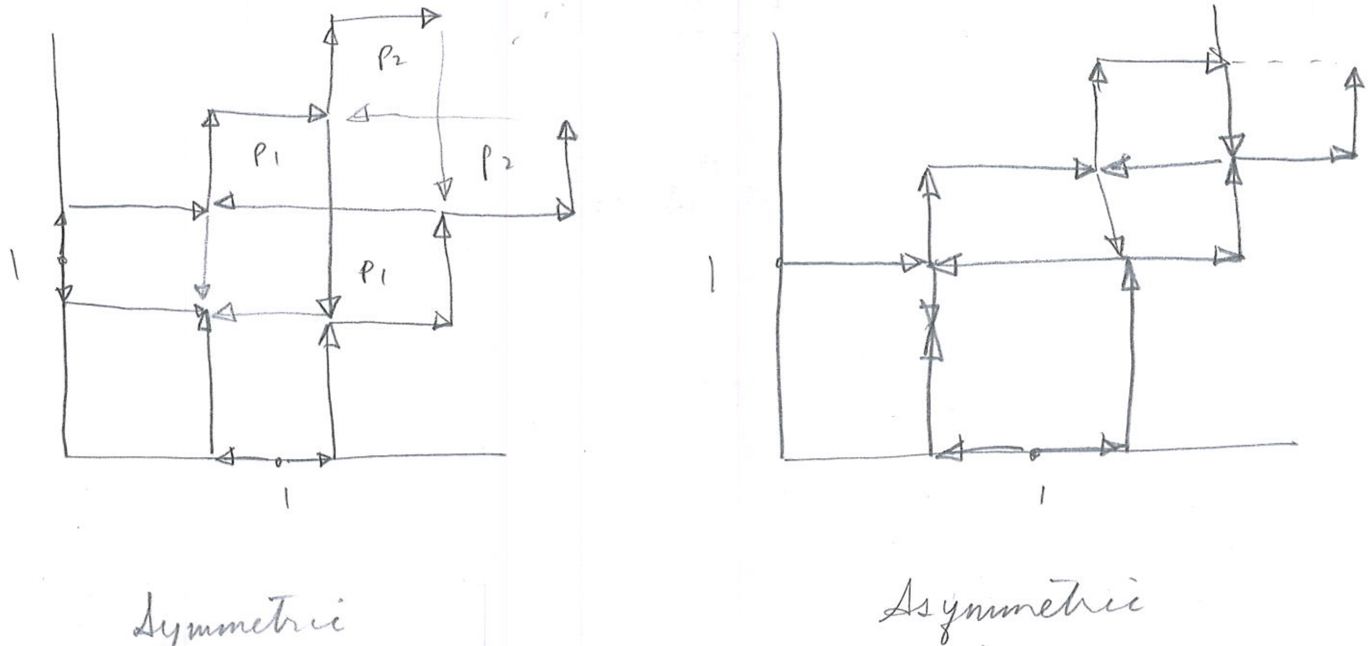


Fig 7 (b): with initial split.

General Remarks: Symmetric TIPGs ⇒ checking validity of h (say) is sufficient.
"however" ⇓ usually more complicated.

∴ In this, we consider only symmetric TIPGs. (asymmetric only for comparison).

Remarks: <u>Numerical Optimization</u> (allowing variable step size)

∴ show (a) no-split ladders (type fig. 7(a))

$$p^*_A = p^*_B \approx 0.64$$

(b) Initial split ladders

$$p^*_A = p^*_B \approx 0.57$$

∴ Initial point split ladders seem to be better however they're also more complicated (their analytic form).

Claim : Fig 7 (type of ladders) can't achieve arbitrarily small bias, can be proven analytically.