Approach: construct ladders with more than four points across the diagonal.

Consider: A symmetric ladder, with lattice spacing ϵ & $2k$ points across each rung.

Recall/guess: A rung for an asymmetric ladder should have the form

$$\sum_{i=1}^{2k} \frac{-f(x+i\epsilon)}{\prod_{j\neq i}(j\epsilon - i\epsilon)}\,[x+i\epsilon]$$

$$= \sum_{i=1}^{2k} \frac{-f(x+i\epsilon)\,[x+i\epsilon]}{\epsilon^{2k-1}(2k-i)(2k-i-1)\cdots(1)(-1)(-2)\cdots(-(i-1))}$$

$$= \sum_{i=1}^{2k} \frac{\cancel{(-1)^{i-1}}\;(-1)^{i}\; f(x+i\epsilon)\,[x+i\epsilon]}{\epsilon^{2k-1}\,(2k-i)!\,(i-1)!}$$

1̶0 2
9 - 2
8 -> 2
7 -> 2
6 -> 2
5 -> 2
4 -> 2
3 -> 2
2̶
1 -> 2

NB: A symmetric ladder would have the same form (further conditions of f which we would derive soon) except that the centre point will be missing.
($\because$ $h(x,y) = -h(y,x)$ so for $x = y (=z)$ (diag.) $h(z,z) = -h(z,z) \Rightarrow h(z,z) = 0$.)

Thus we must have for a rung with $2k$ points

$$\sum_{\substack{i=-k\\ i\neq 0}}^{k} \frac{-f(x+i\epsilon)\,[x+i\epsilon]}{\prod_{j=-k}^{k}{}_{j\neq i,\, j\neq 0}\,(j\epsilon - i\epsilon)}$$

$$= \sum_{\substack{i=-k\\ i\neq 0}}^{k} \frac{f(x+i\epsilon)\,[x+i\epsilon]}{\underbrace{\epsilon^{2k-1}}_{(2k+1)-1-1}\,\dfrac{(k-i)!\,(k+i)!\,(-1)^{k+i}}{i}} = \sum_{\substack{i=-k\\ i\neq 0}}^{k} \frac{(-1)^{k+i}(i)\,f(x+i\epsilon)\,[x+i\epsilon]}{\epsilon^{2k-1}(k+i)!\,(k-i)!}$$

$\because \prod_{j=-k}^{k}$ $j\neq i$ $j\neq 0$

Thus a complete, symmetric ladder can be written as (put $x = j\epsilon$ & add the y component)

$$h_{ead} = \sum_{j=j_0}^{\Gamma} \sum_{\substack{i=-k \\ i\neq 0}}^{k} \frac{(-1)^{k+i}\,(i)\, f((j+i)\epsilon, j\epsilon)}{\epsilon^{2k-1}\,(k+i)!\,(k-i)!} [(j+1)\epsilon, j\epsilon]$$

where the ladder was terminated at $\Gamma = \frac{y}{\epsilon}$ which can be enforced by demanding

$$f(x,y) = g(x,y)\left(\prod_{i=1}^{k}\left((\Gamma+i)\epsilon - x\right)\right)\underbrace{\left(\prod_{i=1}^{k}\left((\Gamma+i)\epsilon - y\right)\right)}_{\text{symmetry}}$$

(validity)
NB: $f(-\lambda,y)$ would have a degree $\leq \underbrace{(2k)}_{n} - 2$. in λ.

since k are already taken hence

$g(-\lambda, y)$ can have degs. $\leq k-2$ in λ

Also $g(-\lambda,y)$ should be ≥ 0 for $\lambda > 0$ & $\Gamma\epsilon \geq y > 0$ (until bottom cropping)

(truncating further)
NB: We can't fully crop the bottom as we have only $k-2$ zeros to play with. That's OK $\because$ we intend to join this to the WCF protocol below.

$\therefore$ It is still useful to truncate as many points as possible.

Truncating at $y = j_0\epsilon$

$$g(x,y) = C\,(-1)^k\left(\prod_{i=1}^{k-2}\left((j_0-i)\epsilon - x\right)\right)\left(\prod_{i=1}^{k-2}\left((j_0-i)\epsilon - y\right)\right)$$

(sign)
NB: The overall sign is chosen so that

$g(-\lambda,y) > 0$ for $\lambda > 0$ & $\Gamma\epsilon \geq y > j_0\epsilon$.

Figure 8. A ladder with $2k = 8$ points

(left over points)
NB: head has exactly three points left of the first truncation band; $\underset{i=k-2}{(j_0 - k + 2)\epsilon} \leq x \leq \underset{i=1}{(j_0 - 1)\epsilon}$.

There would be three points at

$[(j_0 - k)\epsilon, j_0\epsilon]$ $\quad$ $[(j_0 - k + 1)\epsilon, j_0\epsilon]$

$[(j_0 - k + 1)\epsilon, (j_0 + 1)\epsilon]$

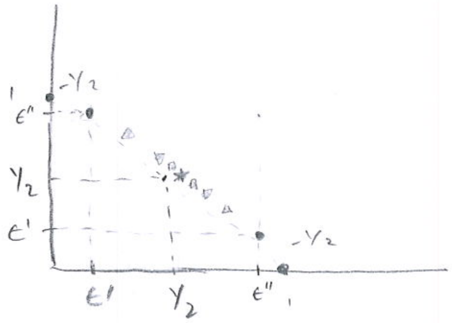
(see figure 8)

Similarly for the bottom.

(last split)
NB: If we add a split to the ● as a move, we get the game of type figure 8

Claim: With some fine tuning, one can obtain

$$h + v = -\frac{1}{2}[1, 0] - \frac{1}{2}[0, 1] + \frac{1}{2}[1 - \epsilon'', \epsilon'] + \frac{1}{2}[\epsilon', 1 - \epsilon'']$$



- • First step (game)
- △ Final point of next game
- ▽ " " third
- □ " " fourth
- ☆ Final point

for some some $0 < \epsilon'' < \epsilon'$.
For $k \to \infty$, $\epsilon' \to \epsilon''$ which means one can use a sequence of such steps (see illustration) to obtain a protocol with arbitrarily small bias. ☺

Approach: We will not pursue this specific construction. We'll find a procedure that works in one big step.

§ 5.1.4 Mixing ladders with points on the axes

Goal: Describe a family of protocols (in the TIPG framework) that converges to zero bias.

Approach: We will mix the ladder with points on the diagonal.

NB: The 1/6 protocol will become a special case!

Overview: The complete protocol has 3 basic steps.

$$\frac{1}{2}[1,0] + \frac{1}{2}[0,1] \to \frac{1}{2}\left(\sum_{j=3^*/\epsilon}^{\Gamma} p(j\epsilon)\left([j\epsilon,0] + [0,j\epsilon]\right)\right)$$

$$\rightarrow \frac{1}{2}\left([z^*, z^*-k\epsilon] + [z^*-k\epsilon, z^*]\right) \rightarrow 1\,[z^*, z^*]$$

where (as usual) k would denote "# " points on a ladder rung

ϵ is small (lattice size)

Γ would denote "point of truncation"

$1/2 < z^* < 1$,

$p(z)$ to be determined.

NB : There are some obvious constraints such as

$k\epsilon < z^*$ &

$\frac{z^*}{\epsilon} \in \mathbb{Z}$ — (basically that z^* is on the lattice).

which will be resolved for $\epsilon \to 0$ & $\Gamma \to \infty$ for a given k.

Approach: Prove the aforesaid process is valid for some z^* & then find the smallest $z^* = p^*_A = p^*_B$ for a given k.

Intuition: (1) The first transition, is splits along the axes.

(2) second transition, hard step with ladders.

(3) Third transition, trivially valid.

Step 1

NB: The first transition is valid if we can enforce

$$1 = \sum_{j=z^*/\epsilon}^{\Gamma} p(j\epsilon) \quad \& \quad 1 \geq \sum_{j=z^*/\epsilon}^{\Gamma} \frac{p(j\epsilon)}{z(j\epsilon)}$$

prob. conservation

validity of splits (i.e. $\frac{1}{z}$ on avg.)

step 2

Remark: will be accomplished by a ladder that collects weight from the axes & deposits it near $x = z^*$ $y = z^*$.

: We need as usual a f^n $h(x, y)$ s.t. $v(x,y) = h(y,x)$ &

$$h + v = -\frac{1}{2}\left(\sum_{j=z^*/\epsilon}^{\Gamma} p(j\epsilon)\left([j\epsilon, 0] + [0, j\epsilon]\right)\right) + \frac{1}{2}\left([z^*, z^* - k\epsilon] + [z^* - k\epsilon, z^*]\right)$$

Intuition: h will have terms of the form $-\frac{1}{2} p(j\epsilon)\,[0, j\epsilon]$ which suggests that our ladder would have terms of the form $\dfrac{-f(0, j\epsilon)}{\prod_{j \neq i}(x_j - x_i)}$ where in the denominator, $x_j = 0$ can no longer be excluded ($\because$ its present in the ladder!). Thus even though we will deal with $\dfrac{-f(x_i, y_i)}{\prod_{j \neq i}(x_j - x_i)}$ like terms as we did before, the main difference is inclusion of the $x_j = 0$ term.

So what?: So far we exploited the fact that product of distances of points horizontally or vertically (upto the sign) was the same.

NB: These terms $p(x_i, y_i)$ & $-p(y_i\, x_i)$ would differ by a factor of $(0-x_i)$ or $(0-y_i)$.

To fix this, $f(x,y)$ must add this "missing" factor in V & h., ~~However~~, then $f(x,y)$ ceases to be symmetric. Thus we pull out a factor of "Y_y" (for $h(x,y) = -h(y,x)$) from $f(x,y)$. This is allowed ∵ we care for the degree in λ of $f(-\lambda, y)$.

& Y_y is +ve.

We therefore define

$$h = \sum_{j=z^*/\epsilon}^{\Gamma} \left(-\frac{p(j\epsilon)}{2}[0, j\epsilon] + \sum_{\substack{i=-k \\ i\neq 0}}^{k} -\frac{f(\overbrace{(j+i)\epsilon}^{x_i}, j\epsilon)}{j\epsilon \prod_{x_\ell \neq x_i} (x_\ell - x_i)} [(j+i)\epsilon, j\epsilon] \right)$$

NB: has the $(0-x_i) = -(j+i)\epsilon$ term

To use lemma 31, we must express the first term as a f^n of f in the right form.

$$\frac{p(j\epsilon)}{2} = \frac{f(0, j\epsilon)}{j\epsilon \prod_{x_\ell \neq 0} \underbrace{(x_\ell - 0)}_{=(\ell+j)\epsilon}} = \frac{f(0, j\epsilon)}{\epsilon^{2k+1}(-k+j)(-k+1+j)\dots j \dots (k-1+j)(k+j)}$$

$$= \frac{f(0, j\epsilon)}{\epsilon^{2k+1} \prod_{l=j-k}^{j+k} l}$$

Now we use the freedom in choosing f to fully truncate the top of the ladder, and as much of the bottom as permitted.

$$f(x,y) = c\,(-1)^{k-1} \underbrace{\left(\prod_{i=1}^{k-1} (\zeta^* - i\epsilon - x)\right)\left(\prod_{i=1}^{k-1} (\zeta^* - i\epsilon - y)\right)}_{\text{(Bottom)}}$$

$$\cdot \underbrace{\left(\prod_{i=1}^{k} (\Gamma_\epsilon + i\epsilon - x)\right)\left(\prod_{i=1}^{k} (\Gamma_\epsilon + i\epsilon - y)\right)}_{\text{Top.}}$$

NB: $f(x,y) = f(y,x)$

$f(-\lambda, y) \geq 0 \quad \forall\ \lambda > 0\ \&\ \zeta^* \leq y \leq \Gamma_\epsilon$

(for ϵ small enough; $k\epsilon < \zeta^*$)

& $f(-\lambda, y)$ has deg. $2k-1$ in λ which is valid $\because\ \exists\ 2k+1$ points (including the one on the diagonal) & 1 point at the border (axis).

NB 2: 

$\zeta^* - (k-1)\epsilon$

$[\zeta^* - k\epsilon, \zeta^* \epsilon]$ $\quad$ $\zeta^* - \epsilon$ (trunc^n bottom)

$\because$ we used $(k-1)$ zeroes, only one point remains at the left bottom (for h & bottom bottom for v).

By prob. conservation (which holds for valid transitions, which these are by lemma 31)

at $[\zeta^* - k\epsilon, \zeta^*]$ & $[\zeta^*, \zeta^* - k\epsilon]$ these two points must carry the same prob. that was present on the axis

Nearly done. We have shown the second step can be accomplished through the ladder of $f(x,y)$ as defined.

We return to the first step to find the smallest z^* achievable. We do this in the $\epsilon \to 0$ & $\Gamma \to \infty$ limit (in the formal proof, which I won't discuss, the finite analysis is also given).

We start with

$$f(0,z) = (\epsilon)\, C' \prod_{i=1}^{k-1} (z^* - i\epsilon - z) \underbrace{\prod_{i=1}^{k} (\Gamma\epsilon + i\epsilon - z)}$$

$$\downarrow$$

$$(\epsilon)\, \underbrace{C' \prod_{i=1}^{k-1} (z^* - z)\, (\Gamma\epsilon)^k}_{= C''} \quad (\to 0)$$

where C' & C'' are k dependent constants & we assumed $\frac{\Gamma - z/\epsilon}{\Gamma} \to 1$.

$$p(z) = \frac{2 f(0, j\epsilon)}{z_{(j\epsilon)} \prod_{l \neq 0} x_l} \to (\epsilon)\, \frac{C''' (z - z^*)^{k-1}}{z^{2k+1}} \quad (\to 0)$$

The two constraints of step 1 can now be expressed as

$$1 = \int_{z^*}^{\infty} p(z)\, dz \quad \& \quad 1 = \int_{z^*}^{\infty} \frac{p(z)}{z}\, dz$$

(we saturated)

Simplif^n:
We can use $w = \frac{z^*}{z}$ to obtain a Beta function reps.

with (I didn't check)

$$\int_{z^*}^{\infty} \frac{(z-z^*)^j}{z^l}\,dz = (z^*)^{j-l+1} \int_0^1 w^{l-j-2}(1-w)^j\,dw$$

$$\left(B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}\right)$$

$$= (z^*)^{j-l+1}\, B(l-j-1,\ j+1)$$

$$= (z^*)^{j-l+1}\, \frac{(l-j-2)!\,(j)!}{(l-1)!}$$

Using this & equating the constraints (& thereby cancelling c'') we obtain

$$\cancel{(z^*)}^{\cancel{(k-1)-(2k+1)+1}} \frac{\cancel{(2k+1-(k-1)-2)!}\ \cancel{(k-1)!}}{\cancel{(2k+1-1)!}}$$

$$= z^{*\ \cancel{(k-1)-(2k+2)+1}\ -1} \frac{(2k+2-(k-1)-2)\cancel{!}\ \cancel{(k-1)!}}{(2k+2-1)\cancel{!}}$$

$$\Rightarrow \quad z^* = \frac{k+1}{2k+1}$$

& that's it! We've created a protocol with

$$p^*_A = p^*_B = \frac{k+1}{2k+1} \quad \left(= \frac{1}{6} + \frac{1}{2} \text{ for } k=1\right)$$