Notes

02 April 2021 3:56 PM

sample space — all possible outcomes

σ-algebra — space of events

Def^n (Discrete-time Martingale). Let $(\Omega, F, \mathbb{P})$ be a prob. space.

prob. f^n — assigns prob. to each event

$\{X_i, F_i\}_{i=1}^{n}$ $n \in \mathbb{N}$ is a Martingale if

random variable — σ-algebras

set of events, closed under unions and intersections

1. F_i's form a "filtration" i.e. $F_0 \subseteq F_1 \subseteq F_2 \dots \subseteq F_n \subseteq F$

trivial σ-algebra $\{\emptyset, \Omega\}$ — full σ-algebra

2. $X_i \in \mathbb{L}^1(\Omega, F_i, \mathbb{P})$ $\forall i \in \{0, \dots n\}$

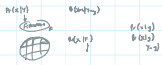
i.e. (a) each X_i is defined on the same sample space Ω

(b) $\mathbb{E}(|X_i|) = \int_\Omega |X_i(\omega)| \, \mathbb{P}(d\omega) < \infty$.

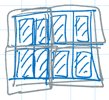
3. $\forall i \in \{1, \dots n\}$

$$\boxed{X_{i-1} = \mathbb{E}(X_i \mid F_{i-1})}$$

holds almost surely.



NB1: $X_j = \mathbb{E}[X_i \mid F_j]$ $\forall i > j$



F_{i-2} F_{i-1}

$$X_{i-1} = \mathbb{E}[X_i \mid F_{i-1}]$$

$$X_{i-2} = \mathbb{E}[X_{i-1} \mid F_{i-2}] = \mathbb{E}[\underline{\mathbb{E}[X_i \mid F_{i-1}]} \mid F_{i-2}]$$

high res — low res

$$= \mathbb{E}[X_i \mid F_{i-2}]$$

$$\vdots$$

$$X_j = \mathbb{E}[X_i \mid F_j]$$

high res — low-res

E.g. Arbitrary filtration $\{F_i\}_{i=0}^{n}$

$$X_i := E(X \mid F_i) \quad \forall i \in \{0, 1, \dots n\}$$

Then, $\{X_i, F_i\}_{i=0}^{n}$ is a martingale.

proof:

$$\underline{X_{i-1}} \overset{\text{assert}}{=} \mathbb{E}[X_i \mid F_{i-1}] = \mathbb{E}[\mathbb{E}[X \mid F_i] \mid F_{i-1}]$$

$$= \mathbb{E}[X \mid F_{i-1}] = X_{i-1}$$

NB: $F_0 = \{\emptyset, \Omega\}$, $F_n = F$

$$X_0 = \mathbb{E}[X \mid F_0] = \mathbb{E}[X]$$

$$X_n = \mathbb{E}[X \mid F_n] = X$$

Theorem 2.2.1 (The Azuma–Hoeffding inequality).

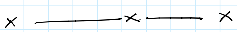
Let $\{X_k, F_k\}_{k=0}^{n}$ (real valued) martingale seq.

Assume $\cdot$ $d_1, \dots d_n$ s.t. $|X_k - X_{k-1}| \leq d_k$ $\forall k \in \{1, \dots n\}$

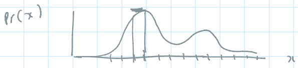
zero excluded.

Then $\forall \lambda > 0$,

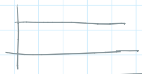
$$\mathbb{P}(|X_n - X_0| \geq \lambda) \leq 2 \exp\left(\frac{-\lambda^2}{2 \sum_{k=1}^{n} d_k^2}\right).$$



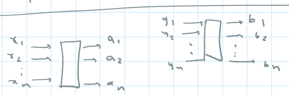
stefano et.al. 2010



$\rightarrow$ guess x:



$H_{min} = -\log \max_x P(x)$



$s = (x^n, y^n) \qquad \ell = (a^n, b^n)$

$$H_\infty(R|S=s) = -\log \max_\ell P(\ell|s)$$

$$= \min_{a^n b^n} \left(-\log P(a^n b^n | x^n y^n)\right)$$

Aim: lower bound $\quad -\log P(a^n b^n | x^n y^n) \quad \forall\ a^n b^n \ \&\ x^n y^n$

Assert: $\quad -\log_2 P(a^n b^n | x^n y^n) \underset{\text{assert}}{=} \sum_{i=1}^{n} -\log P(a_i b_i | x_i y_i w^i)$

$$w^i := (a^{i-1}\, b^{i-1}\, x^{i-1}\, y^{i-1})$$

$a^j = (a_1 \dots a_j)$
$b^j = (b_1 \dots b_j)$
$x^j = (x_1 \dots x_j)$
$y^j = (y_1 \dots y_j)$

Narrator: Device can be characterised by

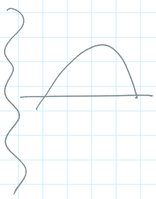
$$P(a_i b_i | x_i y_i w^i)$$

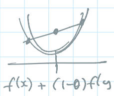
$$I(w^i) = \sum_{abxy} c_{abxy} \dots$$

"Thm": $\exists f$ s.t. $\quad -\log_2 P(a^n b^n | x^n y^n) \geq \sum_{i=1}^{n} f(I(w^i))$

a convex f^n

$$\geq n f\Big(\underbrace{\frac{1}{n}\sum_{i=1}^{n} I(w^i)}\Big)$$





$\theta f(x) + (1-\theta) f(y) \geq f(\theta x + (1-\theta) y)$

Def^n: $\quad \hat{I}_i := \sum_{abxy} c_{abxy} \dfrac{\overset{\text{indicator}}{\chi}(a_i = a, b_i = b, x_i = x, y_i = y)}{pr(x,y)}$

Claim: $\quad I(w^i) = \mathbb{E}(\hat{I}_i | w^i)$

Def^n: $\quad \hat{I} = \frac{1}{n}\sum_{i=1}^{n} \hat{I}_i$

Narrator: $\hat{I}$ is an estimator for $\frac{1}{n}\sum_{i=1}^{n} I(w^i)$

Def^n: $\min_{xy} p(x,y) = q$;

Assum^n: $q > 0$.

Def^n: $\quad Z^k := \sum_{i=1}^{k} \Big(\hat{I}_i - \underbrace{I(w^i)}_{\mathbb{E}(\hat{I}_i | w^i)}\Big)$

Claim: $\{Z^k, W^k\}$ is a martingale seq.
understood to be the σ-algebra generated by W^k.

Proof: $\quad Z^0 := 0$

$Z^{i-1} = \mathbb{E}(Z^i | w^i) \qquad$ — $\qquad Z^0 = \mathbb{E}(Z^1 | \cancel{w^1}) = \mathbb{E}(Z^1) = \mathbb{E}(\hat{I}_1) - \mathbb{E}(\hat{I}_1 | \cancel{w^1}) = 0$

$w^i = (a^{i-1}, b^{i-1}, x^{i-1}, \dots)$

(1) $\mathbb{E}(|Z^k|) < \infty \qquad \forall k$

(2) . . . .

(3) $\mathbb{E}(Z^k \mid w^1 \dots w^j) = \mathbb{E}(Z^k \mid \underline{w^j}) = Z^{j-1} \quad \forall \; j < k.$

F^{j-1}

$\mathbb{E}(Z^{3} \mid w^3) = Z^{3}$

$\mathbb{E}(Z^3 \mid F^2) = Z^2$

$$Z^j = \overbrace{\sum_{i=1}^{j-1} \hat{I}_i - \mathbb{E}(\hat{I}_i \mid w^i)}^{Z^{j-1}} + \hat{I}_j - \mathbb{E}(\hat{I}_j \mid w^j)$$

$$Z^{j-1} \overset{\text{assert}}{=} \mathbb{E}(Z^j \mid w^j) = \sum_{i=1}^{j-1} \underbrace{\mathbb{E}(\hat{I}_i \mid w^j)} - \underbrace{\mathbb{E}(\mathbb{E}(\hat{I}_i \mid w^i) \mid w^j)}$$

$$+ \cancel{\mathbb{E}(\hat{I}_j \mid w^j)} - \cancel{\mathbb{E}(\hat{I}_j \mid w^j)} \qquad \mathbb{E}(\hat{I}_i \mid w^i) \quad \because i \le j$$

$\because i \le j-1$

$\mathbb{E}(\hat{I}_i \mid w^j) = \hat{I}_i$

involves $a_i b_i x_i y_i$ for $i < j$

$$= \sum_i \hat{I}_i - \mathbb{E}(\hat{I}_i \mid w^i) = Z^{j-1}$$

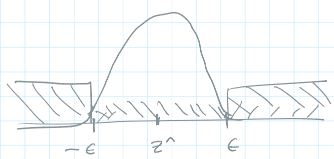
NB: $|Z^i - Z^{i-1}| \le |\hat{I}_i - I(w^i)| \le \frac{1}{q} + I_q$ ↳ max. Bell violation for QM.

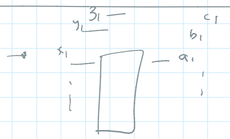
$\because$ x would be 1 for exactly one term in any experiment

c's are bounded by 1.

Using Azuma-Hoeffding,

$$\frac{1}{n}\sum_{i=1}^{n} I(w^i) \approx_{\exp} \frac{1}{n}\sum_{i=1}^{n} \hat{I}_i - \epsilon$$





$(x_1, x_2 \dots x_n) \sim \mathcal{D}(\dots)$

$x_i = \begin{cases} 1 & \text{if the } i^{th} \text{ game is won.} \\ 0 & \text{else.} \end{cases}$

Suppose: p is chosen randomly.

is a permutation on $\{1, 2, \dots n\}$.

Given: $x_{p(1)} \cdot x_{p(2)} \cdot \dots \cdot x_{p(n-1)} = 1$,

estimate $\mathbb{E}(x_{p(n)} \mid x_{p(1)} \dots x_{p(n-1)} = 1)$

Claim: $\sum_{p \in_R S_n} \mathbb{E}(x_{p(n)} \mid x_{p(1)} \dots x_{p(n-1)} = 1) \approx_{\exp} 1$

$Z_i := x_{p(1)} \dots x_{p(i-1)} - \mathbb{E}(x_{p(i)} \mid x_{p(1)} \dots x_{p(i-1)})$

$\vdots$

$Z_0 = 0$

$Y_i := 1 - x_{p(i)}$;

$$z_0 = 0$$

$$Y_i := 1 - X_{p(i)} \ ;$$

$$Z_\ell := \sum_{i=1}^{\ell-1} Y_i - \mathbb{E}(Y_\ell \mid F^{\ell-1})$$

$$\sum_{i=1}^{\ell} Y_i - \sum_{i=1}^{\ell} \mathbb{E}(Y_i \mid F^{i-1})$$

$$Z_{\ell-1} \overset{\text{required}}{=} \mathbb{E}(Z_\ell \mid F^{\ell-1}) = \sum_{i=1}^{\ell-1} \underbrace{(Y_i \mid F^{\ell-1})} - \mathbb{E}(Y_\ell \mid F^{\ell-1}) = Z_\ell$$

$$\mathbb{E}_{i=1}(Y_i \mid F^{i-1})$$

$$X_i$$

$$\sum_{i=1}^{n-1} I_i \approx_{\varepsilon, p} \sum_{i=1}^{n-1} \hat{I}_i$$

$$\sum_{i=1}^{n} I_i \approx_{\varepsilon, p} \sum_{i=1}^{n} \hat{I}_i \Leftarrow$$

$\hat{I}_n$ — don't want to measure

diff

$$I_n$$

$$I_n \approx \hat{I}_n$$

$$\Pr\left[\frac{1}{\ell}\left[\sum_{k=1}^{\ell} E(X_k \mid F^{k-1}) - \sum_{k=1}^{\ell} X_k\right] < \varepsilon\right]$$

$$\sum_{k=1}^{m} X_k + \underbrace{\sum_{k=m+1}^{\ell} X_k}_{\ell - m}$$

$$\Pr\left(\frac{1}{\ell}\sum_{k=1}^{\ell} E(X_k \mid F^{k-1}) - \frac{\sum_{k=1}^{m} X_k}{\ell} < \varepsilon + \frac{\sum_{k=m+1}^{\ell} X_k}{\ell}\right)$$