Proposition: Let $x_1, \dots x_n$ be n random variables (possibly correlated; for us, x_i denotes winning the i^{th} game).
taking values in $\{0,1\}$

$\Pi \in S_n$ be a random variable, uniformly distributed.

Then

$$\Pr\left\{\left|\frac{x_{\Pi(1)} + x_{\Pi(2)} + \dots x_{\Pi(n-1)}}{n-1} - \underbrace{\sum_{\pi \in S_n} \frac{E(x_{\pi(n)})}{|S_n|}}_{= E(x_{\Pi(n)})}\right| > s\right\} \leq 2e^{-cns^2}$$

for some $c > 0$.

Lemma 1:

Let x_is & Π be as before.

Then

$$\Pr\left\{\left|x_{\Pi(1)} + x_{\Pi(2)} + \dots x_{\Pi(n-1)} - \frac{1}{|S_n|}\sum_{\pi \in S_n}(x_{\pi(1)} + \dots x_{\pi(n-1)})\right| \geq s\right\} \leq 2e^{-\frac{cs^2}{n}}$$

for some $c > 0$.

Lemma 2: Let x_is be as before.
Then

$$\Pr\left\{\left|\frac{1}{|S_n|}\sum_{\pi \in S_n}\left[\frac{n}{n-1}(x_{\pi(1)} + \dots + x_{\pi(n-1)}) - E(x_{\pi(1)} + \dots + x_{\pi(n)})\right]\right| \geq s\right\} \leq 2e^{-\frac{cs^2}{n}}$$

for some $c > 0$

[Proof of Lemma 1:

Let: $Q_i := E(x_{\Pi(1)} + x_{\Pi(2)} + \dots + x_{\Pi(n-1)} \mid F^{i-1})$

where $\{F^i\}_{i=0}^n$ is a filteration which specifies $\Pi(i)$

& F^0 specifies $x_1, x_2 \dots x_n$.

NB1: We saw that Q_i forms a martingale.

NB2: $|Q_i - Q_{i-1}| \leq d$ for some $d > 0$.

Recursion: We can therefore apply the Azuma-Hoeffding inequality

to get $\Pr\{|Q_1 - Q_n| > s\} \leq e^{-cs^2}$.

to get $\Pr\{|Q_1 - Q_n| > \varepsilon\} \leq e^{-c\varepsilon^2}$.

NB 3: $Q_1 = \frac{1}{|S_n|} \sum_{\pi \in S_n} (X_{\pi(1)} + X_{\pi(2)} + \dots X_{\pi(n-1)})$

$Q_n = X_{\Pi(1)} + X_{\Pi(2)} + \dots X_{\Pi(n-1)}$

$\square$

Proof of Lemma 2:

Let $Y_i = \frac{1}{|S_n|} \sum_{\pi \in S_n} X_{\pi(i)}$, for $i \in \{1, 2, \dots n\}$

NB: $Y_1 = Y_2 \dots = Y_n$; Def^n: $Y := \frac{1}{n} \sum_{i=1}^{n} Y_i$

Narrator: We want to estimate $\frac{1}{|S_n|} \sum_{\pi \in S_n} \mathbb{E}(X_{\pi(n)}) = \mathbb{E}\left(\sum_{\pi \in S_n} \frac{X_{\pi(n)}}{|S_n|}\right)$

$= \mathbb{E}(Y)$.

This motivates the following def^n.

Def^n: $Z_i := \mathbb{E}(Y \mid F^{i-1})$

fixes $\Pi = \pi$ and $X_{\pi(1)}, X_{\pi(2)} \dots X_{\pi(i-1)}$
& $F^0 = \{\phi, \Omega\}$

NB: $\{Z_i, F^i\}$ form a martingale ($\because$ as in the e.g., F^i is a filtr^n)

& $|Z_{i+1} - Z_i| <$ bounded by some const.

Narrator: From Azuma-Hoeffding, we have that

$$\Pr\{|Z_0 - Z_n| > \varepsilon\} \leq 2e^{-c\varepsilon^2} \text{ for some } c.$$

NB: $Z_n = Y$ & $Z_0 = \underbrace{\mathbb{E}(Y)}$

$$= \frac{1}{|S_n|} \sum_{\pi \in S_n} \mathbb{E}(X_{\pi(1)} + \dots + X_{\pi(n)})$$

$$= \frac{\left(\sum_{\pi \in S_n} X_{\pi(1)} + X_{\pi(2)} + \dots X_{\pi(n-1)}\right) + \sum_{\pi \in S_n} X_{\pi(n)}}{n \cdot |S_n|}$$

$$= \frac{\cancel{n}}{n-1} \frac{\left(\sum_{\pi \in S_n} X_{\pi(1)} + \dots X_{\pi(n-1)}\right)}{\cancel{n} \cdot |S_n|}$$

$\frac{1}{n}\left(\sum_{i=1}^{n} X\right) = \frac{nX}{n} = S$

$\frac{(n-1)X}{n} = S'$

$\frac{n}{n-1}$