Sudeshna # 3.3

"Renormalization"

$f(x,r)$ $\quad x_m$: max of f

r_n : 2^n-cycle

R_n : 2^n-cycle superstable

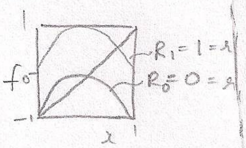
$(r_n < R_n < r_{n+1})$

Explicit: $f(x,r) = r - x^2$

R_0

$x^* = R_0 - x^{*2}$ $\quad \Big| \quad \frac{df}{dx} = -2x \Rightarrow x^* = 0$

$\Rightarrow \quad R_0 = 0$

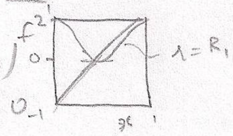


$(-2p)(-2q) = 0$

Let $p = 0$ (could've chosen $q = 0$)

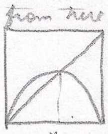
$f(0) = r \Rightarrow f^2(0) = r - r^2 = 0$

$R_1 - R_1^2 = 0 \Rightarrow R_1 = 1$

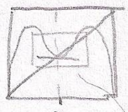


NB: f from here is assumed $r x(1-x)$

$f(x,R_0)$



$f^2(x,R_1)$



$x \to x - x_m$ $\quad$ do this in $x_{n+1} = r x_n (1 - x_n)$

1) $x \to x - x_m$

2) scale by alpha $\quad \alpha f^2\left(\frac{x}{\alpha}, R_1\right)$

rescale

$\alpha = -2.5$

$f^2(x,R_1) \to \alpha f^2\left(\frac{x}{\alpha}, R_1\right) \approx f(x,R_0)$

$f^2 \to f^4$ $\qquad f^2\left(\frac{x}{\alpha}, R_1\right) \approx \alpha f^4\left(\frac{x}{\alpha^2}, R_2\right)$

$R_1 \to R_2$ $\qquad f(x,R_0) \approx \alpha^2 f^4\left(\frac{x}{\alpha^2}, R_2\right)$

$f(x,R_0) \approx \alpha^n f^{(2^n)}\left(\frac{x}{\alpha^n}, R_n\right)$

$\lim_{n\to\infty} \alpha^n f^{(2^n)}\left(\frac{x}{\alpha^n}, R_n\right) \to g_0(x)$

"Universal"

$g_i(x) \quad f(x,R_i)$

$g_i(x) \approx \alpha^n f^{(2^n)}\left(\frac{x}{\alpha^n}, R_{n+i}\right)$

As $R_i \to R_\infty$

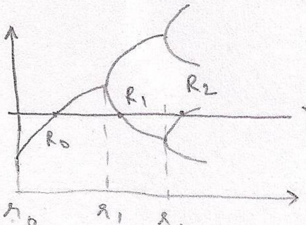
we get $f(x,R_\infty) \approx \alpha f^2\left(\frac{x}{\alpha}, R_\infty\right)$

Without too much loss, we have

$g'(0) = 0 \quad \because x \to x - x_m$

$g(0) = 1$

Sudeshna #3.4

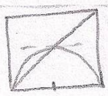


$X_m = 1/2$

R_0 ? $f'(x^*)\big|_{x^* = 1 - \frac{1}{r}} = 0 \Rightarrow 2 - r = 0$

$\Rightarrow R_1 = 2$

$\Rightarrow x^* = 1 - \frac{1}{R_1} = \frac{1}{2}$

$f(x, R_0)$ 

$f^2(x, R_1)$ 

$\alpha = -2.5$

$\alpha f^2\left(\frac{x}{\alpha}, R_1\right) \simeq f(x, R_0)$

$\Rightarrow f(x, R_0) \approx \alpha^n f^{2n}\left(\frac{x}{\alpha^n}, R_n\right)$

$g_0(x)$

$g_i(x) \approx \alpha^n f^{2n}\left(\frac{x}{\alpha^n}, R_{n+1}\right)$

$\Rightarrow g_\infty(x) \equiv g(x)$

$g(x) = \alpha\, g^2\left(\frac{x}{\alpha}\right)$; $g'(0) = 0 \because x \to x - x_m$

& $g(0) = 1 \because$

If g is a soln., so is $\mu\, g\left(\frac{x}{\mu}\right)$

Now $g(0) = \alpha\left(g(g'(0))\right)$

$\Rightarrow 1 = \alpha\, g(1)$

$\Rightarrow \alpha = 1/g(1)$

$x_{n+1} = -(1+\mu)x_n + a x_n^2$

$x \to x/a$ is the same as putting $a = 1$

Thus,

$x_{n+1} = -(1+\mu)x_n + x_n^2$

NB: When $x \approx 0$, $\mu \approx 0$

$f'(x) = -(1)$

2-cycle.

f^2 near p.

Let $p \to q \to p$

$p = f(q)$

$q = f(p)$

$f(p) = -(1+\mu)p + p^2 = q$

$f(q) = -(1+\mu)q + q^2 = p$

Subtract

$\Rightarrow p + q = \mu$

Multiply

$\Rightarrow pq = -\mu$

$p = \dfrac{\mu + \sqrt{\mu^2 + 4\mu}}{2}$

$q = \dfrac{\mu - \sqrt{\mu^2 + 4\mu}}{2}$

You want to see what happens around p (near)

$f^2(p + \eta_n) = p + \eta_{n+1}$

algebra

for $c = 4\mu + \mu^2 - 3\sqrt{\mu^2 + 4\mu}$

$\eta_{n+1} = (1 - 4\mu - \mu^2)\eta_n + c\eta_n^2$

$\tilde{x}_n = c\eta_n$

$\tilde{x}_{n+1} = (1 - 4\mu - \mu^2)\tilde{x}_n + \tilde{x}_n^2$

(of) $= -(1 + \tilde{\mu})\tilde{x}_n + \tilde{x}_n^2$

then $\tilde{\mu} = \mu^2 + 4\mu - 2$

for birth of the ~~next~~ 2 cycle, we have $\tilde{\mu} = 0$ ($\because$ slope ≈ -1 at change of stability)

$\Rightarrow \mu = -2 + \sqrt{6}$

old result: $r = 1 + \sqrt{6}$

this is $\because$ we shifted μ.

$r_2 = 3 + (-2 + \sqrt{6}) = 1 + \sqrt{6}$

$\mu_{k-1} = \mu_k^2 + 4\mu_k - 2$

$\mu_k = -2 + \sqrt{6 + \mu_{k-1}}$

$\mu_k \to \mu^* \Rightarrow \mu^* = \mu^{*2} + 4\mu^* - 2$

$\Rightarrow \mu^* = \frac{1}{2}(-3 + \sqrt{17}) \approx 0.56$

$\Rightarrow r_\infty (= 3 + \mu^*) = 3.56$

Now what is α!

$\tilde{x}_n = c\eta_n$

$c = 4\mu + \mu^2 - 3\sqrt{\mu^2 - 4\mu} \approx -2.24 \,(= \alpha)$ for $\mu = \mu^*$

about 10% off.

$\delta \approx \dfrac{(\mu_{k-1} - \mu^*)}{(\mu_k - \mu^*)}$

use L'Hopital's rule to get and use

$d\mu_{k-1} = \mu_k^2 + 4\mu_k - 2$

$\Rightarrow d\mu_{k-1} = 2\mu_k d\mu_k + 4 d\mu_k$

to get $\delta = \dfrac{d\mu_{k-1}}{d\mu_k}\Big|_{\mu^*} = 2\mu_k + 4 \approx 5.12$