Lattice Dynamics

Vivek Sagar

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1 Aim of the experiment

The aim of this experiment is to study the dispersion relations in the mono-atomic and diatomic lattices and measure the cutoff frequency in the former and the band gap (between the acoustic and optical modes) in the latter.

2 Theory

2.1 Mono-atomic lattice

Consider a chain of inductors (of inductance) L. Imagine a wire parallel to this chain, connected to the chain after each inductor through a capacitor with a capacitance C. Let I_n be the current through the n-th inductor. The equation governing the dynamics of such a system can be written using the Kirchoff's voltage rule.

$$-L\frac{dI_n}{dt} - \frac{q_n - q_{n+1}}{C} + \frac{q_{n-1} - q_n}{C} = 0 \tag{1}$$

Here q_n is the charge on the n-th capacitor due to current I_n. Clearly,

$$\frac{d^2q_n}{dt^2} = \frac{dI_n}{dt} \tag{2}$$

The dynamics are similar to a spring mass system of n point masses (of mass L) connected to each other with springs of force constants $\frac{1}{C}$. The q_n is then analogous to the displacement of the nth particle. Such a system of ODEs can be solved by putting the following guess solution.

$$q_n = Ae^{i(knQ-wt)} \tag{3}$$

Which yields,

$$Lw^2Ae^{i(knQ-wt)} = \frac{1}{C}Ae^{i(knQ-wt)}e^{ikQ} + \frac{1}{C}Ae^{i(knQ-wt)}e^{-ikQ} - 2\frac{1}{C}Ae^{i(knQ-wt)} \tag{4}$$

Solving the above equation gives,

$$w(k) = \sqrt{\frac{2(1-\cos kQ)}{LC}} \tag{5}$$

The frequency can now be obtained by the following relation:

$$\nu = \frac{2\pi}{w} \tag{6}$$

In this experiment, we have a series of 10 LC components. One supplies an alternating current and a single LC component introduces a phase lag. Naively thinking (as we did!), we can put the input current as I_1 and every time the current crosses the nth LC component, it picks up a phase lag of kQ. So, $I_2 = I_1 e^{i(-wt+kQ)}$ and so on. Finally, we measure I_{10} and find how much phase lag has been introduced in total. Clearly, we expect a phase difference of $(n-1)kQ$. This would give us the value of kQ. One might ask, how are we representing a system of infinite no. of LC components by a system of 10 components. And, that's a good question, but I don't know the answer! Anyhow, we increase ν, so that we can make the phase difference equal to an integral multiple of $\frac{\pi}{2}$. We then plot I_{10} vs I_1, and obtain an ellipse (assume the amplitude won't remain same) for the odd multiples and straight lines (with slopes ± 1). At a certain stage, (see equation 5), coskQ would become equal to 1. Consequently, there's an upper bound on the value of w, which one can call the cutoff frequency.

2.2 Diatomic Lattice

What happens if the adjacent atoms in a spring-mass system are not identical? We can analyze the spring mass system where the adjacent masses are different but the spring constants are identical. Our analogical LC circuit would then consist of different inductors (with inductance L_1 and L_2) and identical capacitors (C_0). In turns out, in our experimental circuit we have the same inductor (L_0) throughout the chain, but different capacitors ($C_1,\ C_2$). Surprisingly, the resulting dispersion relation is the same, if one makes the transformation $L_i \to C_i$. Thus, we proceed with the analysis for the case when the spring constants are different.

The resulting equations are:

$$-L\frac{dI_n}{dt} - \frac{q_n - q_n^{'}}{C_2} + \frac{q_{n-1}^{'} - q_n}{C_1} = 0 \tag{7}$$

$$-L\frac{dI_n^{'}}{dt} - \frac{q_n^{'} - q_{n+1}}{C_1} + \frac{q_n - q_n^{'}}{C_2} = 0 \tag{8}$$

Consider now the case when Cs different and inductance is equal to L. We can then use $q_n = Ue^{i(knQ-wt)}$, $q_n' = Ve^{i(knQ-wt)}$. One obtains two equations with constant coefficients in U and V, so the determinant of the coefficients should be zero for a non-trivial solution, which yields the following relation.

$$\left(w^2 - \frac{1}{LC_1} - \frac{1}{LC_2}\right)^2 = \left(\frac{1}{LC_2} + \frac{e^{-ikQ}}{LC_1}\right)\left(\frac{1}{LC_2} + \frac{e^{ikQ}}{LC_1}\right)$$

which implies,

$$w(k) = \frac{1}{LC_1} + \frac{1}{LC_2} \pm \sqrt{\left[\frac{1}{LC_2}\right]^2 + \left[\frac{1}{LC_1}\right]^2 + \frac{2}{L^2C_1C_2} coskQ}$$

This gives rise to two solutions, depending upon the sign of the term in the square root. The positive root corresponds to the optical band of the dispersion relation while the negative band corresponds to the acoustic band. The band gap can be calculated by finding the value of k for which the difference between the two solutions is minimum. Clearly, such a solution exists for $k = (2m+1)\pi,\ m \in \mathbb{N}$.

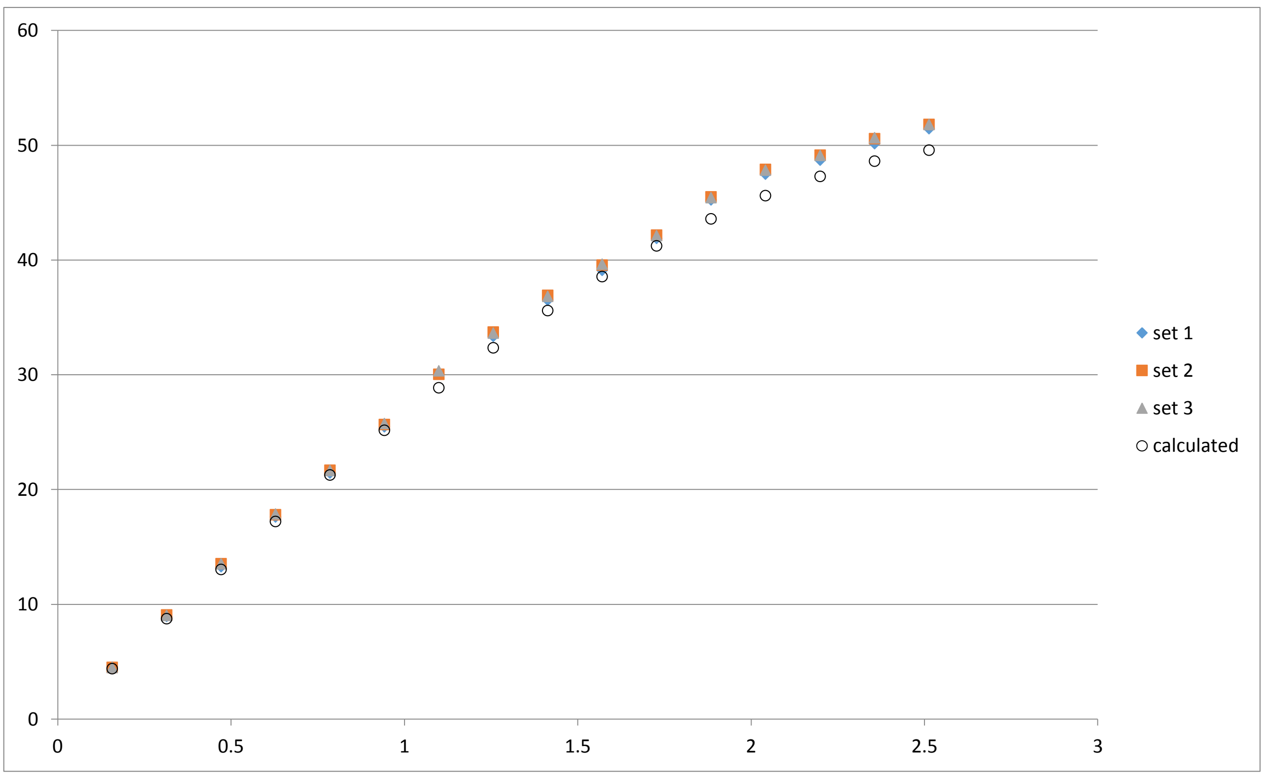
In this experiment, the index n would vary from 1 to 5. So, like in the previous case, the phase difference between the input and outout current is $4kQ$.

3 Procedure

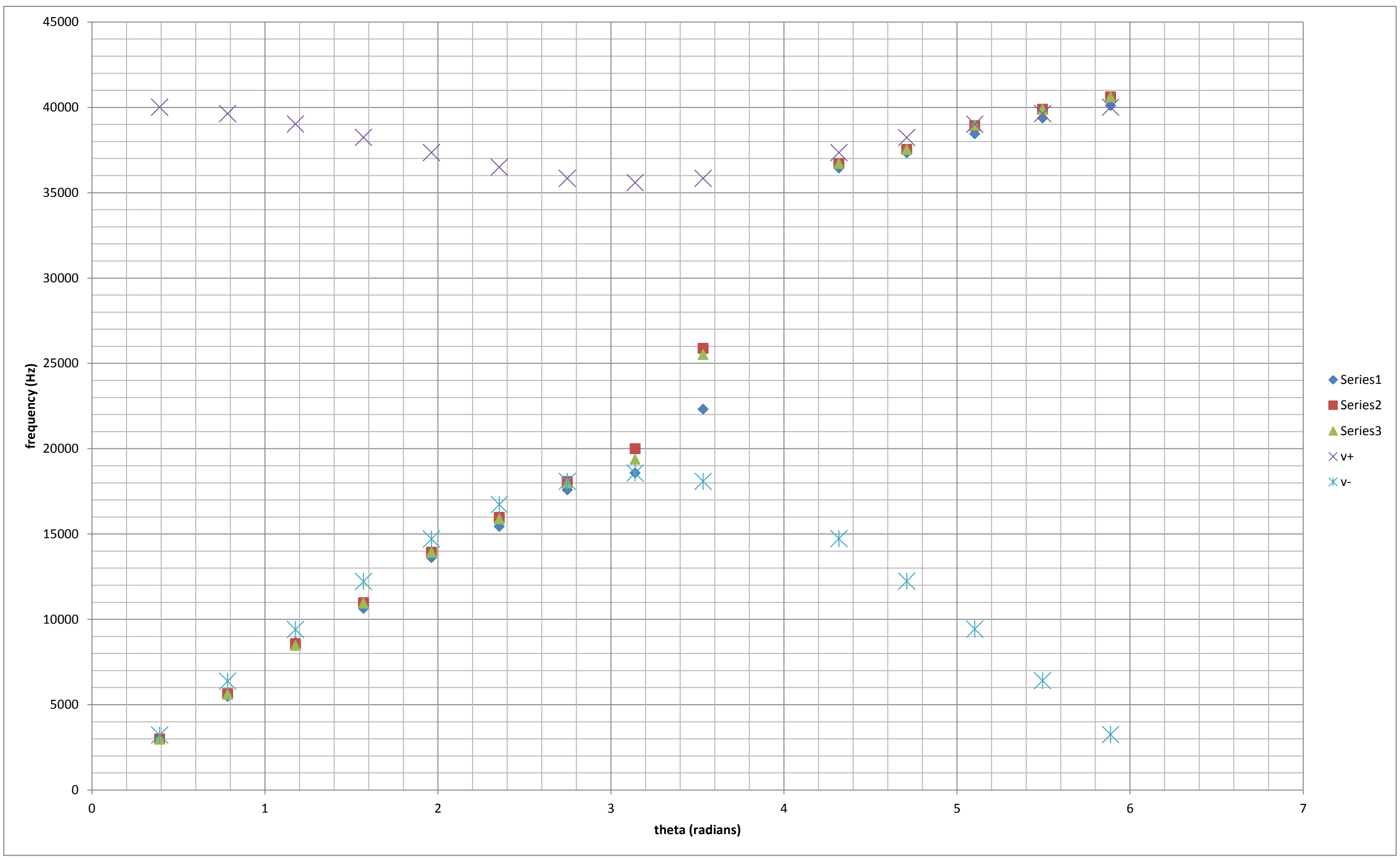
The experimental setup is shown in the figure 1. For both types of lattices, one notes the value of ν for which the total phase difference between the input(X) and output current(Y) is $m\frac{\pi}{2},\ m \in \mathbb{N}$. For the odd multiples, one obtains elliptical figures on the X-Y mode of oscilloscope(the amplitude of the signal need not remain the same) which can be transformed into circles by modifying the knob R_2. Ideally speaking, R_2 should control the ratio of amplitudes of the input and output signals without modifying the w. But that doesn't happen in this setup, consequently the knob should be kept constant throughout the experiment, especially during the diatomic case. R_1 can be thought of as a resistor between I_1 and the first L.

4 Observations and result

CRO pattern	Ne	Degree	Radians	Freq(set1)	Freq(set2)	Freq(set3)	2/LC*(1-cos)	sqrt	
/	0	0	0						
O	90	9	0.1570796	4.517	4.508	4.49	759.6123494	27.56106582	4.386479863
\	180	18	0.3141593	8.931	9.068	9.025	3015.368961	54.9123753	8.739575965
O	270	27	0.4712389	13.275	13.543	13.543	6698.729811	81.84576844	13.02615862
/	360	36	0.6283185	17.601	17.799	17.861	11697.77784	108.1562659	17.21360434
O	450	45	0.7853982	21.438	21.685	21.648	17860.61952	133.6436288	21.27004413
\	540	54	0.9424778	25.475	25.666	25.751	25000	158.113883	25.16460605
O	630	63	1.0995574	30.083	30.035	30.345	32898.99283	181.3807951	28.86765012
/	720	72	1.2566371	33.342	33.711	33.634	41317.59112	203.2672898	32.35099395
O	810	81	1.4137167	36.503	36.911	36.83	50000	223.6067977	35.58812717
\	900	90	1.5707963	39.104	39.552	39.639	58682.40888	242.2445229	38.55441326
O	990	99	1.727876	41.887	42.17	42.135	67101.00717	259.038621	41.22727698
/	1080	108	1.8849556	45.247	45.506	45.442	75000	273.8612788	43.58637623
O	1170	117	2.0420352	47.482	47.897	47.835	82139.38048	286.5996868	45.61375685
\	1260	126	2.1991149	48.699	49.138	49.116	88302.22216	297.1568982	47.29398922
O	1350	135	2.3561945	50.144	50.581	50.649	93301.27019	305.4525662	48.61428579
/	1440	144	2.5132741	51.445	51.816	52	96984.63104	311.4235557	49.56459828



theta	set 1	set2	set 3	w+	w-	v+(hz)	v-
0.3925	2968	2989	2978				
0.785	5484	5639	5614	251388.5	20230.85994	40009.71499	3219.841363
1.1775	8657	8572	8473	248998.3	40065.98268	39629.31212	6376.699194
1.57	10634	10974	10940	245185.2	59073.39257	39022.43466	9401.822432
1.9625	13615	13936	13921	240243.5	76736.43423	38235.94804	12212.98282
2.355	15440	15981	15879	234672.7	92380.57411	37349.31871	14702.82502
2.7475	17585	18068	17992	229269.5	105075.8302	36489.37716	16723.33778
3.14	18582	20002	19372	225172.9	113589.6825	35837.37702	18078.35945
3.5325	22312	25881	25512	223606.8	116642.3179	35588.13139	18564.20147
4.3175	36443	36711	36695	225148.5	113637.9445	35833.50119	18086.04058
4.71	37338	37533	37521	234626.8	92496.94089	37342.02274	14721.34536
5.1025	38446	38922	38952	240200	76872.57675	38229.02035	12234.65058
5.495	39368	39882	39879	245149.1	59222.92854	39016.6929	9425.621822
5.8875	40096	40614	40615	248972.8	40224.19336	39625.25223	6401.879205
		41487	41530	251375.3	20393.97242	40007.61733	3245.801519
				252201.2	0	40139.06665	0



45000
40000
35000
30000
25000
20000
15000
10000
5000
0
frequency (Hz)
0
1
2
3
4
5
6
7
theta (radians)
Series1
Series2
Series3
v+
v-

5 Comments

I couldn't understand the physical reason behind the spring mass system with different masses and same spring constants and that with different springs constants and identical masses giving rise to the same dispersion relation. Moreover, if we transform the $L_i \to C_i$ and vice versa, we do get the same equations. However, we could not think of the spring-mass analogue of the system where the positions of the capacitor and inductor has been interchanged. Furthermore, why are we able to predict the behavior of a chain of finite no. of LC elements using the analysis of an infinite chain of these elements? A rigorous explanation of why the phase difference between the output and input current is $(N-1)kQ$ and not NkQ where N is the number of LC elements in the circuit is also missing.