ASTRONOMY & ASTROPHYSICS

IDC 201

ASSIGNMENT

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MS11021

1. Find the comoving distance r as a function of redshift z for Einstein - de Sitter Universe ($k=0$, $\Omega_m = 1$)

$$\int_r^0 dr = \int \frac{c\,dt}{a(t)} = \int c \frac{dt}{da} \frac{da}{a(t)} = \int c \frac{da}{\dot{a}\,a} = \int c \frac{da}{a^2(\dot{a}/a)}$$

$$\Rightarrow r = -\int_a^{a_0} \frac{c\,da}{H\,a^2} = -cH_0^{-1} \int_a^{a_0} \frac{da}{(H/H_0)\,a^2}$$

Also $1+z = \frac{a_0}{a} \Rightarrow dz = -\frac{a_0}{a^2} da$

$$= -\frac{a_0^2}{a^2} \frac{da}{a_0}$$

$$\Rightarrow \boxed{a_0 r = cH_0^{-1} \int_z^0 \frac{dz}{H(z)/H_0}}$$

This expression gives distance of object at any given red shifts for the Einstein - de Sitter model.

2. find out the energy density in CMBR, if the radiation has a temperature 2.726 K. Assume a Black Body spectrum for the radiation. Use the answer to find out density parameter Ω_{CMBR}. Also find out the number density of photons in CMBR.

given $T = 2.726\ K$

$$\text{energy density} = \rho_{rad} = aT^4$$
$$= 7.6 \times 10^{-15}\ \text{erg cm}^{-3} K^{-4} \times (2.726)^4$$
$$= 4.2 \times 10^{-13}\ \text{erg cm}^{-3}$$
$$= \frac{4.2 \times 10^{-13} \times 10^{-7}\ J}{10^{-6}\ m^3}$$

$$\Omega_{CMBR} = \frac{\rho_{rad}}{c^2 \rho_c}$$
$$= \frac{4.2 \times 10^{-14}\ J/m^3}{(3 \times 10^8)^2_{m^2/s^2} \times 10^{-26}\ kg/m^3}$$
$$= \frac{4.2 \times 10^{-14}}{9 \times 10^{16} \times 10^{-26}}$$
$$= 0.00042$$

3. Use the Virial Theorem and show that the average pressure

$$\bar{P} = \frac{-1}{3}\frac{\langle E_{gr}\rangle}{V}$$

where V is the volume of the star. Use values of $M_\odot$ & $R_\odot$ and assume that the sun is made up purely of ionised hydrogen & Estimate ~~Egr~~ $\bar{P}$ for sun.

Ans- According to Virial theorem, $2\langle E_{kn}\rangle + \langle E_{gr}\rangle = 0$

$$\Rightarrow 2\left(\frac{3}{2}NKT\right) = -\langle E_{gr}\rangle$$

$$\Rightarrow NKT = -\frac{1}{3}\langle E_{gr}\rangle$$

$N \equiv$ no. of particles $= nN_A$ (n = no. of moles)

$$\Rightarrow nN_AkT = \frac{-\langle E_{gr}\rangle}{3}$$

$$\Rightarrow nRT = -\frac{\langle E_{gr}\rangle}{3}$$

$$\Rightarrow \bar{P}V = \frac{-\langle E_{gr}\rangle}{3}$$

$$\Rightarrow \bar{P} = -\frac{1}{3}\frac{\langle E_{gr}\rangle}{V}$$

For sun $M_\odot = 2\times10^{30}$ kg , $R_\odot = 6.96\times10^{8}$ m

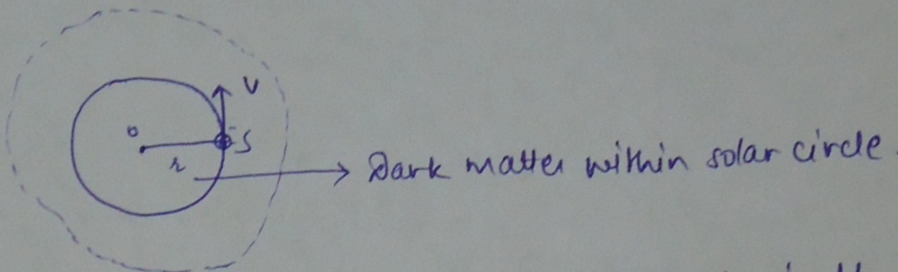
$$\vec{P}_\odot = -\frac{1}{3}\frac{\left(-\frac{3}{5}\frac{GM_\odot^2}{R_\odot}\right)^2}{\left(\frac{4}{3}\pi R_\odot^3\right)} = \frac{3}{20\pi}\cdot G\frac{M_\odot^2}{R_\odot^4}$$

$$\Rightarrow \vec{P}_\odot = \frac{3\times6.67\times10^{-11}\times(2\times10^{30})^2}{20\times3.14\times(6.96\times10^{8})^4}$$

$$= \frac{3\times6.67\times4}{20\times3.14\times(6.96)^4}\frac{\times10^{-11+60}}{\times10^{32}}$$

$$\bar{P}_\odot = 5.4\times10^{13}\ N/m^2.$$

4. Given that the Sun is at a distance of 8.5 kpc from the Galactic centre and its circular speed is 240 km/s estimate the mass of dark matter within the solar circle (assume that the dark matter distribution is spherically symmetric).



Assuming the dark matter distribution to be spherically symmetric, we can assume the total mass of dark matter at the galactic centre 'O'.

$$\Rightarrow \frac{G M_\odot M_{DM}}{r^2} = \frac{M_\odot v^2}{r}$$

$$\Rightarrow M_{DM} = \frac{r v^2}{G}$$

Now, $r = 8.5\,kpc = 8.5 \times 3.1 \times 10^{18}\,m = 2.62 \times 10^{19}\,m$

$\Rightarrow v = 240\,km/s$

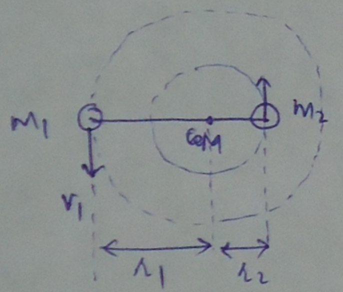
$$\Rightarrow M_{DM} = \frac{2.62 \times 10^{17}\,m \times 240 \times 10^{3}\,m/s}{6.67 \times 10^{-11}}$$

$$\Rightarrow M_{DM} = 2.26 \times 10^{38}\,kg.$$

$$\Rightarrow M_{DM} = 1.13 \times 10^{8}\,M_\odot$$

5. For an eclipsing binary the observed maximum radial velocities for the two stars are 20 km/s and 5 km/s respectively. The period is 5 years. After the eclipse starts, it takes 0.3 days for intensity to fall to its minimum. The duration of the eclipse is 1.3 days. Assume that orbits are circular and also that the orbit is seen edge on

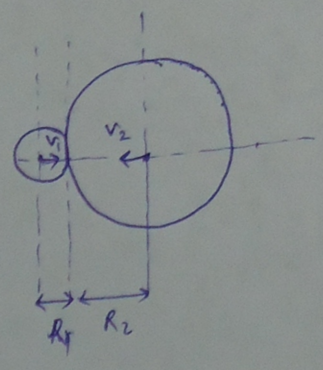
- Find the mass of each star.
- Find the radius of each star.



$m_1 r_1 = m_2 r_2$

$m_1 v_1 = m_2 v_2$

$\Rightarrow \dfrac{m_1}{m_2} = \dfrac{5}{20} = \dfrac{1}{4}$ and $r_1 = \dfrac{v_1}{\omega}$, $r_2 = \dfrac{v_2}{\omega}$



Reference frame on m_2,

$|v_{(m_1/m_2)}| = v_1 + v_2$

Time taken by m_1 to completely enter $m_2 = \dfrac{2R_1}{|v_{m_1/m_2}|}$

$\Rightarrow 0.3 \times 24 \times 3600 = \dfrac{2R_1}{25 \times 10^3}$

$R_1 = 3.24 \times 10^8$ m

$\sim 0.5\, R_\odot$

Duration of eclipse $= \dfrac{2(R_1 + R_2)}{|v_{m_1/m_2}|}$

$\dfrac{1.3 \times 24 \times 3600 \times 25 \times 10^3}{2} = R_1 + R_2$

$R_2 = 1.08 \times 10^9$ m

$\simeq 1.55\, R_\odot$

for Binary systems, $\omega = \sqrt{\dfrac{(m_1+m_2)\,G}{(r_1+r_2)^3}}$

$\Rightarrow \quad \omega = \sqrt{\dfrac{(m_1+m_2)\,G\,\omega^3}{(v_1+v_2)^3}}$

$\Rightarrow \quad \omega = \omega \sqrt{\dfrac{G(m_1+m_2)}{(v_1+v_2)^3}\,\omega}$

$\Rightarrow \quad \dfrac{(25\times10^3)^3}{\omega} = 6.67\times10^{-11}(m_1+m_2)$

$\Rightarrow \quad m_1+m_2 = 5.87\times10^{30}\,\text{kg}$

Now $\quad 4m_1 + m_1 = 5.87\times10^{30}\,\text{kg}$

$m_1 = 1.18\times10^{30}\,\text{kg} \approx 0.6\,M_\odot$

and $m_2 = 4m_1$

$m_2 = 4.7\times10^{30}\,\text{kg} \approx 2.4\,M_\odot$

6. Density parameter of Baryons is $\Omega_B = 0.045$. If the universe is made up of Hydrogen (76% by mass) and Helium (24%) then find out the average number density of electrons. Compute electron to photon ratio for the universe.

$$\Omega_B = 0.045$$

$$\text{No. of } e^- = \frac{0.76\, M}{m_p} + \frac{0.24\, M}{4 m_p} \times 2$$

where M is the total mass.

$$\therefore\ n_e = \Omega_B \rho_c \left(\frac{0.76}{m_p} + \frac{0.24}{2 m_p} \right)$$

$$= \frac{0.045 \times 10^{-26}}{1.67 \times 10^{-27}} \left(0.76 + 0.12 \right)$$

$$= 0.0237 \times 10^{1}\ m^{-3}.$$

$$= 0.24\ m^{-3}$$

7. Consider a radial material arm extending from a galactic radius of 4 kpc to 10 kpc at some initial time. Due to differential rotation, this hypothetical radial line winds up into a material spiral arm. Assuming a flat rotation curve, estimate the <u>pitch angle</u> of the spiral after 10^{10} years.

Taking speed of particles to be v, we can write trajectory of particle as
$$r(r,t) = (r\cos\omega_r t,\ r\sin\omega_r t) \quad \text{where } \omega_r = \frac{v}{r}$$
where r is constant and t is varying.

Tangent of circle in which a particular object is rotating is
$$\dot{r}(r,t) = (-r\sin\omega_r t,\ r\omega_r\cos\omega_r t)$$
$$\Rightarrow \|\dot{r}(r,t)\| = \sqrt{r^2\omega_r^2\sin^2(\omega_r t) + r^2\omega_r^2\cos^2\omega_r t} = r\omega_r$$

If we keep t fixed now, then the curve describes the position of the objects as a function of 'r' at some instant 't'.
$$r(r,t) = (r\cos\omega_r t,\ r\sin\omega_r t)$$
$$r(r,t) = \left[r\cos\left(\frac{vt}{r}\right),\ r\sin\left(\frac{vt}{r}\right)\right]$$

The tangent of the curve joining all the objects can be given as
$$r'(r,t) = \left[\left\{r\frac{vt}{r^2}\sin\left(\frac{vt}{r}\right) + \cos\left(\frac{vt}{r}\right)\right\},\ \left\{-r\frac{vt}{r^2}\cos\left(\frac{vt}{r}\right) + \sin\left(\frac{vt}{r}\right)\right\}\right]$$

$$\Rightarrow r'(r,t) = \left[\omega_r t\sin\omega_r t + \cos\omega_r t,\ -\omega_r t\cos\omega_r t + \sin\omega_r t\right]$$
$$\|r'(r,t)\| = \sqrt{\omega_r^2t^2(\sin^2\omega_r t + \cos^2\omega_r t) + (\cos^2\omega_r t + \sin^2\omega_r t)}$$

$$\Rightarrow \|r'(\lambda,t)\| = \sqrt{\omega_\lambda^2 t^2 + 1}$$

Now $\dot{r}(\lambda,t) \cdot r'(\lambda,t) = \|\dot{r}(\lambda,t)\| \, \|r'(\lambda,t)\| \cos\phi$

$$\cos\phi = \left[\frac{-r\omega_\lambda^2 t \sin^2\omega_\lambda t - r\omega_\lambda^2 t\cos^2\omega_\lambda t}{r\omega_\lambda\sqrt{\omega_\lambda^2 t^2 + 1}}\right]$$

$$\Rightarrow \cos\phi = \frac{-r\omega_\lambda^2 t}{r\omega_\lambda\sqrt{\omega_\lambda^2 t^2 + 1}} = \frac{-\omega_\lambda t}{\omega_\lambda\sqrt{t^2 + \frac{1}{\omega_\lambda^2}}}$$

$$\Rightarrow \cos\phi = \frac{-t}{\sqrt{t^2 + \frac{r^2}{v^2}}}$$

$$= \frac{-vt}{\sqrt{v^2t^2 + r^2}}$$

$$\phi = \cos^{-1}\left(-\frac{vt}{\sqrt{v^2t^2 + r^2}}\right)$$

Here $t = 10^{10}$ years

$v = 250$ km/s $= 250 \times 10^3$ m/s

$r = 7$ kpc $= 7 \times 10^3 \times 3.1 \times 10^{16} = 21.1 \times 10^{19}$ m

$$\phi = \cos^{-1}\left(\frac{10^{10} \times 3.15 \times 10^7 \times 250 \times 10^3}{\sqrt{10^{20} \times (250 \times 10^3)^2 \times (3.15 \times 10^7)^2 + (21.1 \times 10^{19})^2}}\right)$$

$$= \cos^{-1}\left(\frac{787.5 \times 10^{20}}{\sqrt{196875 \times 10^{40} + 44521 \times 10^{40}}}\right)$$

$$= \cos^{-1}\left(\frac{787.5 \times 10^{20}}{491.32 \times 10^{20}}\right)$$

$$= \cos^{-1}($$

8. We believe that the main source of energy in stars is nuclear fusion, In main sequence stars; this is due to conversion of H to He. Conversion of four Hydrogen nuclei into a He nucleus in the p-p chain in low mass stars results in the release of 26.2 MeV into components other than neutrinos. In stars more massive than $4 M_\odot$, the primary channel, ~~for~~ conversion is the C-N-O cycle and here around 25 MeV is released into components other than neutrinos.

- Assuming that each star converts a fixed fraction say 15% of its mass from H to He, write an expression for lifetime of stars as a function of mass and luminosity. You may assume that the luminosity does not change with time during the Hydrogen burning phase.

$$L \propto M^3$$

$$\Rightarrow \frac{L_*}{L_\odot} = \left(\frac{M_*}{M_\odot}\right)^3 \Rightarrow L_* = \frac{M_*^3 L_\odot}{M_\odot^3}$$

$$\text{Energy released per proton} = \frac{26.2 \text{ MeV}}{4}$$

$$\Rightarrow \text{No. of protons in star converted to He} = \frac{M_* \times 0.15}{m_p}$$

$$\Rightarrow \text{Total energy released} = \left(\frac{26.2 \text{ MeV}}{4}\right)\left(\frac{M_* \times 0.15}{m_p}\right)$$

$$\Rightarrow \text{Lifetime} = \frac{\text{Energy}}{\text{Luminosity}} = \cancel{\frac{(26.2 \times 10^6 \times 1.6 \times 10^{-19}) \times 0.15 \times M_\odot^3}{4 \times 1.67 \times 10^{-27} \times M_*^3 \times L_\odot}}$$

$$= \frac{(26.2 \times 1.6 \times 10^{-19} \times 10^6) \times M_* \times 0.15 \times M_\odot^3}{4 \times 1.67 \times 10^{-27} \times M_*^3 \times L_\odot}$$

$$= \frac{26.2 \times 1.6 \times 0.15 \times 8 \times 10^{103}}{4 \times 1.67 \times 4 \times M_*^2 \times 10^{-1}}$$

$$= 1.88 \times 10^{104} \; M_*^{-2}$$

- Use the known parameters of the Sun to estimate its lifetime. You may assume that He fraction in the sun is 0.26 and the rest of its is in the form of Hydrogen.

No. of protons in Sun converted to He $= \frac{(0.74 M_\odot)(0.15)}{m_p}$

Energy released $= \left[\frac{0.74 \times M_\odot \times 0.15}{m_p}\right] \times \frac{26.2 \text{ MeV}}{4}$

$$T_\odot = \frac{\text{Energy}}{\text{Luminosity}}$$

$$= \frac{0.74 \times (2 \times 10^{30}) \times 0.15 \times (26.2 \times 10^6 \times 1.6 \times 10^{-19})}{(1.67 \times 10^{-27}) \times 4 \times (4 \times 10^{26})}$$

$$= \frac{0.74 \times 2 \times 0.15 \times 26.2 \times 1.6 \times 10^{17}}{1.67 \times 16 \times 10^{-1}}$$

$$= 3.48 \times 10^{17} \text{ seconds}$$

$$= \frac{3.48 \times 10^{17}}{3.15 \times 10^{7}} \text{ years}$$

$$= 1.1 \times 10^{10} \text{ years} \approx 11 \text{ billion yrs.}$$

- If the luminosity of stars scales in proportion with the mass as $M^{3.5}$ then find out the dependence of the life time of stars on the mass.

$$\frac{L_*}{L_\odot} = \left(\frac{M_*}{M_\odot}\right)^{7/2}$$

$$\Rightarrow \tau = \frac{(26.2 \times 10^6 \times 1.6 \times 10^{-19}) \times (M_* \times 0.15)}{4 \times (1.67 \times 10^{-27}) \times L_*}$$

$$= \frac{(26.2 \times 10^{6} \times 1.6 \times 10^{-19})\, M_* \; 0.15 \, (M_\odot)^{7/2}}{4 \times 1.67 \times 10^{-27} \times L_\odot \times M_*^{7/2}}$$

$$= \frac{26.2 \times 1.6 \times 0.15 \times 10^{-13} \times (2 \times 10^{30})^{7/2}}{4 \times 1.67 \times 10^{-27} \times 4 \times 10^{26} \times M_*^{5/2}}$$

$$= \frac{71.13 \times 10^{92}}{26.72 \times 10^{-1}} M_*^{-5/2}$$

$$= 2.66 \times 10^{93} \, M_*^{-2.5}$$

9. Consider a situation where radiation pressure is very large inside a star. Then

$$\bar{P} = \bar{P}_{gas} + \bar{P}_{rad} = -\frac{1}{3} \frac{E_{gr}}{V}$$

- Write down the average pressure in terms of the average density, mass of star and some constants.

Solution: $E_{gr} = \frac{3}{5} \frac{G M_*^2}{R_*}$

$\bar{P}_{gas} = \frac{N}{V} k_B T = \frac{\rho k_B T}{\mu}$ where ρ is the mass density and μ is the avg molecular mass.

$\bar{P}_{rad} = \frac{1}{3} a T^4$

$$\Rightarrow \bar{P} = \frac{\rho k_B T}{\mu} + \frac{1}{3} a T^4 = -\frac{G M_*^2}{20 \pi R_*^4}$$

- If Radiation and Gas pressures are equal, then show that

$$\bar{P}_{rad} = \left(\frac{3}{a}\right)^{1/3} \left(\frac{k \rho}{\mu}\right)^{4/3}$$

$$\bar{P}_{gas} = \bar{P}_{rad}$$

$$\Rightarrow \frac{\rho k_B T}{\mu} = \frac{1}{3} a T^4$$

$$\Rightarrow T = \left(\frac{3\rho k_B}{a\mu}\right)^{1/3}$$

$$\Rightarrow \bar{P}_{rad} = \frac{1}{3} a \left(\frac{3\rho k_B}{a\mu}\right)^{4/3} = \left(\frac{3}{a}\right)^{1/3} \left(\frac{\rho k_B}{\mu}\right)^{4/3}$$

- If radiation pressure becomes stronger than the gas pressure then the star approaches instability. Find out the expression for maximum mass of a star. Estimate the numerical value in units of the solar mass.

If $\bar{P}_{rad} > \bar{P}_{gas}$ star cannot remain stable. ∴ The total pressure can be at the maximum twice radiation pressure, when radiation pressure equals gas pressure.

$$\bar{P} = 2\left(\frac{3}{a}\right)^{1/3} \left(\frac{\rho k_B}{\mu}\right)^{4/3} = \frac{3\, G M_*^2}{20\pi R_*^4}$$

$$\Rightarrow \frac{40}{3} \frac{\pi\, \cancel{R_*^4}}{G} \left(\frac{3}{a}\right)^{1/3} \left(\frac{M_*\, k_B}{\frac{4}{3}\pi \cancel{R_*^3}/\mu}\right)^{4/3} = M_*^2$$

$$\Rightarrow \frac{40}{3} \pi \frac{\cancel{\ }}{G} \left(\frac{3}{a}\right)^{1/3} \left(\frac{3\, k_B}{4\pi\mu}\right)^{4/3} = M_*^{2-\frac{4}{3}}$$

$$\Rightarrow \frac{40 \times 3.14}{3 \times 6.67 \times 10^{-11}} \left(\frac{3 \times 10^{-6}}{7.6 \times 10^{-15} \times 10^{-7}}\right)^{1/3} \left[\frac{6 \times 1.4 \times 10^{-16} \times 10^{-7}}{4 \times 3.14 \times 1.6 \times 10^{-27}}\right]^{4/3} = M_*^{2/3}$$

$$\Rightarrow \left[6.277 \times 10^{+11} \left(0.395 \times 10^{16}\right)^{1/3} \left(0.418 \times 10^{4}\right)^{4/3}\right]^{2/3} = M_*$$

$$\Rightarrow \left[6.277 \times 10^{11} \times \left(\frac{\cancel{0.734}}{1.58} \times 10^{5}\right) \times \left(1.611 \times 10\right)^{4}\right]^{2/3} = M_*$$

$$\Rightarrow \left[66.8023 \times 10^{20}\right]^{2/3} = 3.5 \times 10^{14}\,\text{kg} = M_*$$

Absurd !!

10. Molecular wieght μ is defined as an average mass of a molecule when multiple species are present. If the fraction of Hydrogen by mass is given by X and that of He is denoted by Y, then write an expression for μ assuming that both the species are fully ionised. How does this change when He^- is singly ionised instead?

When both H and He are fully ionised, we have two ions (H^+ and He^{2+}) and 3 electrons – total 5 particles.

Let total Mass = M

Mass of Hydrogen = XM ; Mass of Helium = YM

No. of particles = 5

$$\mu = \frac{XM + YM}{2\left(\frac{XM}{m_p}\right) + 3\left(\frac{YM}{4m_p}\right)}$$

Since Hydrogen gives 2 particles $\therefore \frac{2XM}{m_p}$

and Helium gives 3 particles $\therefore \frac{3YM}{4m_p}$

$$\Rightarrow \mu = \frac{(X+Y)(4m_p)}{8X + 3Y}$$

Similarly, when He is singly ionised

$$\mu = \frac{XM + YM}{2\left(\frac{XM}{m_p}\right) + 2\left(\frac{YM}{4m_p}\right)} = \frac{(X+Y)(2m_p)}{4X + Y}$$

since Hydrogen gives 2 particles $\therefore 2\left(\frac{XM}{m_p}\right)$

and Helium again gives 2 particles $\therefore 2\left(\frac{YM}{4m_p}\right)$