Steps and Motivation

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Abstract

I was told once by one of my mathematic professors, Kapil H. Paranjape, that understanding a proof is not about understanding how one step follows from the previous. I could intuitively understand the meaning, but couldn't quite feel it. It was in a philosophy class that it occured to me that there are times in Quantum Mechanics, when I can understand the steps, but I can't grasp the idea. This was made apparent by our Physics Optics Laboratory instructor, Dr. Mandip Singh, when he asked us about how we would get the classical harmonic oscillator's oscillations in a quantum harmonic oscillator. So here, I have charted out the motivation behind the large spread of steps, so that I can understand the subject, and not only the steps of the derivations. Perhaps this is what *they* mean by developing intuition.

Part I
The Theory of Angular Momentum

1 Rotations and Angular Momentum

In this section, we aim at finding what we really mean by angular momentum and what we don't want it to mean.

1. We first note that rotations R are not commutative (recall all its group properties too).
   (a) We find the commutation relation till second order.
   (b) But we can show that infinitesimal rotations (upto first order) are in fact commutative.
2. Then you make the first leap. You recall the relation between infinitesimal translation and the linear momentum. Thereby define angular momentum J_k to be the operator that generates infinitesimal rotation. (CAVEAT: it is

not always the same as $r \times p$, but it follows to be so in classical mechanics) It is important to pause here and think about what we're doing. We just created an operator (actually three since $k \in \{x, y, z\}$, but fix one of it in your head) that acts on some ket $|\alpha\rangle$ in a space which doesn't necessarily have dimension 3. It is the operators that are three, each corresponding to one of the physically observable dimensions of space. Very important to note.

3. We now define the finite rotation operator $\mathscr{D}$ using the definition of exponential as usual.

4. So far awesome. Now we want some commutation relation between the *angular momentum.* It seems reasonable to think that we can use the commutation of R.

   (a) For this it is *postulated* that $\mathscr{D}_k$ and R have the same group properties

   (b) Now we use the commutation relation of R to get a commutation relation between $\mathscr{D}_k$ which in turn gives a commutation relation between J_k, called the fundamental commutation relation of angular momentum.

2 Spin $\frac{1}{2}$ Systems and Finite Rotations

In this section, we look at two approaches.

1. In one, we construct a rotated ket, but keep the operator the same.

2. In the other, we construct a Heisenberg like operator (by sandwitching the rotation operator), but keep the ket the same.
   In both cases we calculate the expectation value, and as expected (pun intended), it turns out to be the same[1]. For clarity, for a spin half system, with rotation about the z-axis, I'll state the result

$$\langle S_y \rangle \rightarrow \langle S_y \rangle \, cos\phi + \langle S_x \rangle \, sin\phi$$

   and as expected (both intuitively (which intuition?) and from the commutation $\mathscr{D}_z$ with S_z),

$$\langle S_z \rangle \rightarrow \langle S_z \rangle$$

3. As it turns out, it is clear that the expectation value of the spin operator does behave like components of a 3D vector being transformed. And the second method shows that we haven't used any property of a spin half system to get the result; the result is true for all spin operators. So that's the intuition!

[1] Use the Baker-Hausdorff lemma for evaluating using the second method

4. Here's an un-intuitive result. The state ket (note it is any arbitrary state) picks up a minus sign upon a 360^0 rotation ($|\alpha\rangle \to -|\alpha\rangle$). Another 360^0 rotation makes it plus again. However, as far as the expectation value is concerned, we don't see this!

5. Here's an amazing application. The energy associated with a magnetic dipole is $\mu.B$. Since μ is related to the spin, the hamiltonian for such a system (with B fixed) is a constant times $S_z$say. So then the time evolution operator becomes a rotation operator!

6. With this, we actually can see the change in sign of a ket. The idea is simple. Let a beam of particles split (so that the state is the same) and make them go through two paths, say A and B. In one of the paths, add a magnetic field. This will change the state of the ket. Now make both interfere and observe the interference as a function of the rotation caused (can be done either by changing the time or the magnetic field). You get a verifyable result!

7. The other two subsections in the text are almost like solved example. Although somethings to think about include why pauli matrices have a -1 determinant and a zero trace (explicitly it is obvious, but so is the fact that it is hermitian; however hermiticity follows from the spin operators being observable)

3 SO(3), SU(2) and Euler Rotations

Since these notes are for you; I expect some familiarity with groups.

1. Intuitively (you can look at the book for rigour) there are 3 independent variables for specifying a rotation in 3D space. The group of such transformations (length and parity preserving) is called SO(3).

2. We've seen how unitary operators preserve the norm of kets (think of time evolution), viz the 'length'. It follows from unitarianism (hehe) that for a general 2x2 unitary operator's matrix representation, you can have at most 3 independent variables (4 if you include a scalar $e^{i\gamma}$, but you can get rid of that by demanding the determinant to be unity [unimodularity]). Do you smell the relation yet?

   (a) Now the cool part; We can always relate a unimodular unitary operator to a rotation (we saw that explicitly)

   (b) Can we do the other way, associate a rotation to every unimodular unitary operator? Think. Think about the 2π rotation. We (hopefully) clearly can't.

3. Euler Rotations are well (you know them intuitively very well), skipped. Just look at how the rotation is transformed from the axis of the rigid body, to the rotations about fixed axis.

4 Density Operators and Pure vs Mixed Ensambles

This has been skipped for now

5 Eigenvalues and Eigenstates of Angular Momentum[2]

We do precisely that, find eigenvalues and eigenstates of the angular momentum operators (well indirectly anyway) for a general dimension of the ket space.

1. We start with finding out commutation relations between J^2 and J_k. They commute. We label their simultaneous eigenket as $|a, b\rangle$.

2. We then define ladder operators called $J_\pm = J_x \pm iJ_y$. We also find commutation relations between $J_\pm$ and J_z. We observe that $J_\pm$ and J^2 commute.

3. Now we find the effect of the operator $J_z J_\pm$ on $|a, b\rangle$. Here we find something interesting. $J_\pm$ acts on a ket to increase b to $b + \hbar$, leaving a unchanged! Observe the cleverness of the trick. Note the similarity with the harmonic oscillator.

4. Lets see if we can get some constraints on a and b. It is now shown that $J^2 - J_z^2$ results in a hermitian operator by expressing it in terms of $J_\pm$! So now since its expectation values are real. From here itself we get $a \geq b^2$.

5. Let's do better, let's find the maximum and minimum b. That said, one can find $b_{\text{max}}$by getting $J_- J_+$ to act on a maxed out ket $|a, b_{\text{max}}\rangle$ and expressing the expression in terms of J_z and J^2. You should get $a - b_{\text{max}}^2 - b_{\text{max}}\hbar = 0$. Similarly you can get the minimum as plus $\hbar$. So on comparison of the equations, we now have $b_{\text{max}} = -b_{\text{min}}$.

6. Let's now make things convenient. Logically it now follows $b_{\text{max}} = b_{\text{min}} + n\hbar$. So we must have $b_{\text{max}} = \frac{n\hbar}{2}$. Wait here to note that the n (its an integer) that we've found, is more like a constant at this stage than an index as you may think. What does it depend on? We'll answer shortly, but before that we finally, for convenience, define $j = b_{\text{max}}/\hbar$.

7. From the relation we found between a and b, we have $a = \hbar^2 j(j+1)$, where j obviously is either an integer or a half integer. So our j is actually dependent on the value of a, the eigenvalue of J^2. This determines the range of b, the eigenvalues of J_z, a result we've been blindly learning from class X!

[2]This is not what it should've been, but it is important as an algorithm anyway.

8. Also for convenience, we define m such that $b = m\hbar$. Now if j is a half integer, so will m be. Else they're both happy integers.

9. Bravo, you've done it. Here's the final piece

$$J^2 \left|j, m\right\rangle = j(j+1)\hbar^2 \left|j, m\right\rangle$$

and

$$J_z \left|j, m\right\rangle = m\hbar \left|j, m\right\rangle$$