First Mid-Sem (Session 2012-13) ①

September 4^{th}, 2012 MTH201 (Curves & Surfaces)

Solutions

Soln. 1 :

$$D_{f(x,y)} = \begin{pmatrix} \frac{\partial}{\partial x}(\sin x \cos y) & \frac{\partial}{\partial y}(\sin x \cos y) \\ \frac{\partial}{\partial x}(x^2 - y) & \frac{\partial}{\partial y}(x^2 - y) \end{pmatrix}$$

$$= \begin{pmatrix} \cos x \cos y & -\sin x \sin y \\ 2x & -1 \end{pmatrix}$$

$\Rightarrow D_{f(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which is invertible as

$\det(D_{f(0,0)}) = -1 \neq 0$.

Now at $x = 0, y = \frac{\pi}{2}$

$D_{f(0,\frac{\pi}{2})} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ & this matrix is not invertible.

Sol^n. 2:

We compute

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$$

Take $x = r\cos\theta$, $y = r\sin\theta$

then $(x,y)\to(0,0)$ corresponds to $r\to 0$

and

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \frac{r^3\cos^3\theta}{r^2}$$

$$= \lim_{r\to 0} r\cos^3\theta$$

$$= 0$$

$$= f(0,0).$$

Thus f is continuous at $(0,0)$.

Sol.^{n} 3:

(a) Let at $t = t_0$ the curve γ passes through $(0,0,0)$. Then

$$\left(\frac{1}{3}(1+t_0)^{3/2},\ \frac{1}{3}(1-t_0)^{3/2},\ \frac{t_0}{\sqrt{2}} \right) = (0,0,0)$$

$$\Rightarrow \quad \frac{t_0}{\sqrt{2}} = 0 \quad \text{and} \quad \frac{1}{3}(1+t_0)^{3/2} = 0$$

which is a contradiction.

Thus the curve does not pass through $(0,0,0)$.

(b) We compute:

$$\dot{\gamma}(t) = \left(\frac{1}{3} \cdot \frac{3}{2} \cdot (1+t)^{1/2},\ \frac{1}{3} \cdot \frac{3}{2} \cdot (-1) \cdot (1-t)^{1/2},\ \frac{1}{\sqrt{2}} \right)$$

$$= \left(\frac{1}{2}(1+t)^{1/2},\ \frac{-1}{2}(1-t)^{1/2},\ \frac{1}{\sqrt{2}} \right)$$

$$\Rightarrow \|\dot{\gamma}(t)\|^2 = \frac{1}{4}(1+t) + \frac{1}{4}(1-t) + \frac{1}{2}$$

$$= 1$$

$$\Rightarrow \|\dot{\gamma}(t)\| = 1.$$

Therefore γ is a unit speed curve.

Sol^n 4: We compute

$$\dot{r}(t) = \Big(a\big(e^{bt}(-\sin t + b\cos t)\big),\ a\big(e^{bt}(\cos t + b\sin t)\big)\Big)$$

$$\Rightarrow \|\dot{r}(t)\|^2 = a^2 e^{2bt}\Big(\sin^2 t + b^2\cos^2 t - \cancel{2b\sin t\cos t} + \cos^2 t + b^2\sin^2 t + \cancel{2b\sin t\cos t}\Big)$$

$$\Rightarrow \|\dot{r}(t)\|^2 = a^2 e^{2bt}(1+b^2)$$

$$\Rightarrow \|\dot{r}(t)\| = \left(a\sqrt{1+b^2}\right)e^{bt}$$

And the arc length function will be:

$$s(t) = \int_0^t \|\dot{r}(u)\|\,du = \int_0^t a\sqrt{1+b^2}\ e^{bu}\,du$$

$$= \left(a\sqrt{1+b^2}\right)\frac{e^{bu}}{b}\Bigg|_{u=0}^{u=t}$$

$$= \frac{a\sqrt{1+b^2}}{b}\left(e^{bt}-1\right).$$

$$\Rightarrow \frac{s(2t)}{s(t)} = \frac{e^{2bt}-1}{e^{bt}-1} = e^{bt}+1$$

$$\Rightarrow \frac{s(2)}{s(1)} = e^b + 1 \Rightarrow b = \log_e\left(\frac{s(2)}{s(1)} - 1\right).$$

Sol^n. 5: Since r is a unit speed curve,

we have

$$\|\dot{r}(t)\| = 1 ;$$

i.e $\|\dot{r}(t)\|^2 = 1$

but $\|\dot{r}(t)\|^2 = \dot{r}(t) \cdot \dot{r}(t)$

Therefore $\frac{d}{dt}\left(\dot{r}(t) \cdot \dot{r}(t)\right) = \frac{d}{dt}(1) = 0$

$$\Rightarrow \dot{r}(t) \cdot \ddot{r}(t) + \ddot{r}(t) \cdot \dot{r}(t) = 0$$

$$\Rightarrow 2\dot{r}(t) \cdot \ddot{r}(t) = 0$$

$$\Rightarrow \dot{r}(t) \cdot \ddot{r}(t) = 0$$

Therefore $\dot{r}(t)$ and $\ddot{r}(t)$ are orthogonal to each other.