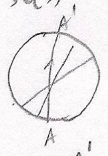
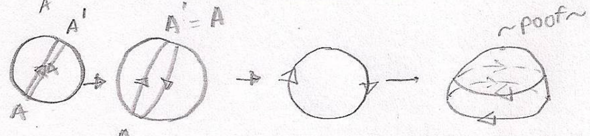
$so(3)$



G : "Topological space"

(a) G is a Lie gp of order "r" if its also a "smooth manifold"



If you twice, you can show like so the paths can be made shrinkable.

(b) Smooth manifold union of connected open sets with each of the sets homeomorphic to a connected subset of Euclidian space of r dim.

Spirit of topology: look at concepts of continuity etc in the simplest possible way;
Lie gps are special & support such

a description.

If the corresponding space is compact then the Lie gp is called compact.
★ Closed + Bounded

For every cover of compact Lie gp, ∃ a finite subcover.

Eg: R in G near identity
$a \in G$ a, R
$G \equiv$ Union of $a_1 R, a_2 R \ldots$

$a_1 R$ $a_2 R$

Simply Connected: All paths can be shrunk to zero continuously | eg. $O(2)$ No continous transformation

det -1 det 1

(1) Defining representation of gp

V real n dimensional

$x' = Rx$

$x'^T x' \overset{dem}{=} x^T x$

$x R^T I R x = x^T x$

$R^T R = R R^T = I$

$x^T y$: inner product

$x^T x$ is $(\text{length})^2$

$x^T I x$

$O(n_1, n_2)$ $n_1 + n_2 = n$

Inner Product

$$g = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{matrix} n_1 \\ n_2 \end{matrix}$$

Let $x\, g\, y$ be the inner product

Linear transformation Λ

Then $x'^T g x' \overset{demand}{=} x^T g x$

$\Rightarrow \Lambda^T g \Lambda = \Lambda$

$n_1 = n_2 = 1$; find representation.

$Sp(2n, R)$

Mechanics phase space

n dim, n coordinates, q_i

n momenta, p_i

ξ_a $a = 1 \ldots | \ldots 2n$

q_s p_s

$\{\xi_a, \xi_b\} = \beta_{ab}$

Linear Transformation $\xi' = S\xi$

$\{\xi_a', \xi_b'\} \overset{dem}{=} \{\xi_a, \xi_b\}$

NB:

$$\beta_{ab} = \left(\begin{array}{c|c} 0 & \begin{matrix} 1 & & \\ & 1 & \\ & & 1 \end{matrix} \\ \hline \begin{matrix} -1 & & \\ & -1 & \\ & & -1 \end{matrix} & 0 \end{array}\right)$$

$S^T \beta S = \beta$; Form a gp

$sp(2n, R)$: Canonical Transformation

Arvind 3.2

Symmetries and Conservation Laws

(i) Particle $\vec{p}$ conserved no forces

Newton's First Law

space homogeneous $\quad \vec{p}$ is conserved

(ii) Two particles $\vec{p}_1 = m_1\vec{v}_1$; $\vec{p}_2 = m_2\vec{v}_2$

$\vec{F}_{21} = -\vec{F}_{12}$ $\quad \boxed{p_1 + p_2 = \text{const}}$

$\dot{\vec{p}}_1 = \vec{F}_{12}$ $\quad \dot{\vec{p}}_1 = \vec{F}_{21}$

$F_{12} = -\nabla_1 V(\vec{x}_1 - \vec{x}_2)$; $F_{21} = -\nabla_2 V(\vec{x}_1 - \vec{x}_2)$

translational invariance of V

Lagrangian Mechanics

n degrees of freedom

$q_r \; (r = 1, 2, \ldots n)$

$L(q, \dot{q})$

$p_r = \dfrac{\partial L}{\partial \dot{q}_r}$ $\quad q_1 \ldots q_n$, $\; p_1 \ldots p_n$

Hamiltonian $H(q, p)$

$\dot{q}_r p_r - L(q, \dot{q})$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0$$

$\dfrac{d}{dt} f(q, \dot{q}, t) = 0$ along a soln.

L doesn't depend upon q_k

$\dfrac{d(p_k)}{dt} = 0 \quad \dfrac{\partial L}{\partial q_k} = 0$

p_k is const. of motion

q_k cyclic co-ordinate

$\dfrac{\partial L}{\partial t} = 0$ corresponding quantity is Hamiltonian

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_r}\dot{q}_r + \frac{\partial L}{\partial \dot{q}_r}\ddot{q}_r$$

$0 = \dfrac{d}{dt}(\dot{q}_r p_r - L)$; NB: $\dfrac{d}{dt}(\dot{q}_r p_r)$

$L(q, \dot{q})$ is invariant under infinitesimal translation q_k, which are time independent.

If $\delta q_r = \epsilon \delta_{rk}$

$\Downarrow$

$\delta L = 0$

then p_k is a const. of motion

Symmetry statement

Infinitesimal Symmetries / Conservation Laws

If $\delta q_r = \epsilon \phi_r(q) \Rightarrow \delta L = 0$

ϵ small

Ex: Damped H.O.

$m\ddot{x} = -kx - \Gamma\dot{x}$

(a) Euler Lagrange $L(x, \dot{x}, t)$?

(b) Hamiltonian

(c) Is H = total energy

Point transformation $\phi_r(q)$ function in configuration space.

Lagrangian is invariant if I make an infinitesimal pt. transformation in configuration space

Q: Is there a const. of motion in this transformation?

$\delta q_r = \epsilon \phi_r(q_r)$

$$\delta \dot{q}_r = \delta \frac{d}{dt} q_r = \frac{d}{dt}(\delta q_r)$$

$$= \epsilon \frac{d\phi_r}{dt} = \epsilon \frac{\partial \phi_r}{\partial q_k} \cdot \dot{q}_k$$

$$\delta L = 0 = \frac{\partial L}{\partial q_r}\delta q_r + \frac{\partial L}{\partial \dot{q}_r}\delta \dot{q}_r$$

"Euler-Lagrange"

$$= \left(\frac{d}{dt} p_r\right)\left(\epsilon \phi_r(q)\right) + p_r\left(\epsilon \frac{\partial \phi}{\partial q_k} \cdot \dot{q}_k\right)$$

$$= \frac{d}{dt}\left(\epsilon p_k \phi_k(q)\right) = 0$$

$\Rightarrow p_k \phi_k(q)$ is constant of motion

Ex: Local Derivation use EOM.

Global derivation requires only

$\displaystyle\int_{t_1}^{t_2} L\,dt = A$; $\delta A \overset{\text{claim}}{=} 0$

Arvind # 3.3

References
Symmetries & conservation laws in classical & Q. Mechanics
K.S. Mallesh, S Chaturvedi, V Balkrishna, R Simon & N Mukunda
1. Resonance Feb 2011
2. Resonance March 2011

Lagrangian System

$L(q,\dot q) \quad L(q,\dot q,t)$

$\delta q_r = \epsilon \varphi_r(q)$ &

$\delta L = 0 \Rightarrow \sum_r p_r \varphi_r(q)$ is a const. of motion

Phase Space, Hamiltonian Desc.

PB $\{f,g\} = \frac{\partial f}{\partial q_r}\frac{\partial g}{\partial p_r} - \frac{\partial g}{\partial q_r}\cdot\frac{\partial f}{\partial p_r}$

$\{q_r, q_s\} = \{p_r, p_s\} = 0$

$\{q_r, p_s\} = \delta_{rs}$

Not a pt. transformation:

$\delta q = \epsilon \phi_r(q,\dot q)$

$\delta \dot q = \frac{d}{dt}\delta q = \epsilon \frac{d}{dt}\phi_r(q,\dot q)$

$\delta L = 0 \qquad L' = L + \frac{d}{dt}(F)$

$= \epsilon \frac{d}{dt} F(q,\dot q)$

EOM; solns are invariant

$G(q,p) = p_r\phi_r(q,\dot q) - F(q,\dot q)$

$\delta q_r = \epsilon\{q_r, G(q,p)\}$

$\delta p_r = \epsilon\{p_r, G(q,p)\}$

Dynamical Symmetry

Canonical Transformation

$(q,p) \to (Q(q,p), P(q,p))$

Q, P have same PB structure canonical variables

Infinitesimal Canonical Transf.

$Q_r = q_r + \delta q_r;\ P_r = p_r + \delta p_r$

$\delta q = \epsilon\{q_r, G(q,p)\}$

$\delta p = \epsilon\{p_r, G(q,p)\}$

Eg. If $\delta q_r = \epsilon\phi_r(q)$

$= \epsilon\{q_r, p_l\phi_l(q)\}$

(easy & obvious to check)

Now $\delta p_r = \delta\left(\frac{\partial L}{\partial \dot q_r}\right)$

$= \frac{\partial^2 L}{\partial q_l \partial \dot q_r}\delta q_l + \frac{\partial^2 L}{\partial \dot q_l \partial \dot q_r}\delta \dot q_l$

$\delta q_l = \epsilon\varphi_l(q)$

$\delta \dot q_l = \epsilon\frac{\partial\varphi_l(q)}{\partial q_m}\dot q_m$

$= \frac{\partial^2 L}{\partial q_l\partial \dot q_r}\epsilon\varphi_l(q) + \frac{\partial^2 L}{\partial \dot q_l\partial \dot q_r}\cdot\epsilon\frac{\partial\varphi_l(q)}{\partial q_m}\cdot\dot q_m$

$\delta L = 0 = \frac{\partial L}{\partial q_l}\epsilon\varphi_l(q) + \frac{\partial L}{\partial \dot q_l}\epsilon\frac{\partial\varphi_l}{\partial q_m}\cdot\dot q_m$

$\Rightarrow \frac{\partial^2 L}{\partial q_l\partial \dot q_r}\epsilon\varphi_l(q) + \frac{\partial^2 L}{\partial \dot q_l\partial \dot q_r}\left(\epsilon\frac{\partial\varphi_l}{\partial q_m}\cdot\dot q_m\right)$

$+ \frac{\partial L}{\partial \dot q_l}\epsilon\frac{\partial\varphi_l}{\partial q_r} = 0 \qquad \because \frac{\partial \dot q_m}{\partial \dot q_r} = \delta_{mr}$

$= -\frac{\partial L}{\partial \dot q_l}\epsilon\frac{\partial\varphi_l}{\partial q_r}$

claim $= \epsilon\{p_r, p_l\varphi_l(q)\}$

verification

$= \underbrace{\frac{\partial p_r}{\partial q_k}}_{0}\cdot\frac{\partial}{\partial p_k}(p_l\varphi_l(q))$

$- \underbrace{\frac{\partial p_r}{\partial p_k}}_{\delta_{rk}}\cdot\frac{\partial}{\partial q_k}(p_l\varphi_l(q))$

$= \frac{\partial}{\partial q_r} \cdots = -\frac{\partial L}{\partial \dot q_l}\frac{\partial\varphi_l(q)}{\partial q_r}$

Two Dimensional Harmonic Oscillator

"Symmetries"

$L = \frac{1}{2} m \dot q_\alpha \dot q_\alpha - \frac{1}{2} m w^2 q_\alpha q_\alpha$

Consider $U(2)$ 2x2 Matrices $UU^+ = U^+U = \mathbb{1}$

$U = I + i\epsilon h$

$U^+U = I \quad h^+ = h$ Hermitian Matrices

$h_{\alpha\beta} = \frac{h_{\alpha\beta} + h^*_{\alpha\beta}}{2} + i\frac{(h_{\alpha\beta} - h^*_{\alpha\beta})}{2i}$

$= \underset{\text{sym}}{S_{\alpha\beta}} + i\underset{\text{anti-sym}}{a_{\alpha\beta}}$

$\delta q_\alpha = \epsilon(a_{\alpha\beta}q_\beta - w^{-1}s_{\alpha\beta}\dot q_\beta)$

$q_1: \delta q_1 = \epsilon(\overset{0}{a_{11}}q_1 + a_{12}q_2 - w^{-1}(s_{11}\dot q_1 + s_{12}\dot q_2))$

Therefore now, exercise

$\delta L = \epsilon\frac{dF}{dt};$

$F = \frac{m}{2}(w\, q_\alpha q_\beta - w^2 q_\alpha \dot q_\beta) S_{\alpha\beta}$

$G(q,p) = p_\alpha\delta q_\alpha - \epsilon F$

$G = p_\alpha(a_{\alpha\beta}q_\beta - w^{-1}s_{\alpha\beta}\dot q_\beta) - \frac{m}{2}(\ldots)$

(explicitly)

also, $a_{12} = -a_{21} \neq 0$

$s_{11} =$ { handwriting }

$c_1 = (mw^{-1})p_1^2 + mwq_1^2$

$c_2 = (mw^{-1})p_2^2 + mwq_2^2$

$c_3 = p_1q_2 - p_2q_1$

$c_4 = (mw^{-1})p_1p_2 + mwq_1q_2$

$q_1 \quad q_2 \quad p_1 \quad p_2$

$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}; \quad a^+ = (a^*, b^*)$

$a_\alpha \equiv \frac{p_\alpha - imwq_\alpha}{\sqrt{mw}}$

$S_0 = c_1 + c_2 = a^+a$

$S_1 = c_1 - c_2 = a^+\sigma_3 a$

$S_2 = 2c_4 = a^+\sigma_1 a$

$S_3 = -2c_3 = a^+\sigma_2 a$