# Theory of Wigner Distributions in Quantum Mechanics

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## Contents



## 1 Introduction: Kinematics of Cartesian Quantum Systems

The basic ideas of this subject go back to Weyl and Wigner, and slightly later Moyal. Our discussion will be largely at the kinematic level. We will use Dirac's notation extensively.

To begin with, we discuss "Cartesian" systems in Quantum Mechanics, or "Cartesian Quantum Mechanics". This means systems whose basic dynamical variables are Cartesian positions and conjugate Cartesian momenta. So spin is excluded. Later we will look at spin and other finite dimensional quantum systems.

For one degree of freedom (dof), we have one canonical pair of hermitian operators obeying the Heisenberg canonical commutation relation (CCR).

$$\hat{q}^\dagger = \hat{q}; \quad \hat{p}^\dagger = \hat{p}; \quad [\hat{q}, \hat{p}] = i\hbar \tag{1.1}$$

In addition, they are assumed to be irreducible:

$$[\hat{q} \,\mathrm{or}\, \hat{p}, \hat{A}] = 0 \Leftrightarrow \hat{A} = aI, \quad \mathrm{multiple\,of\,identity} \tag{1.2}$$

Informally this also means that every operator $\hat{A}$ pertaining to the system is expressible as a function of $\hat{q}$ and $\hat{p}$. The meaning and spirit of this will be clarified as we proceed.

Such operators $\hat{q}$ and $\hat{p}$ are 'unbounded', they can not be applied to all state vectors or wave functions. They have definite domains on which alone they can act. To avoid such domain problems, these CCR can be represented in the unitary Weyl form:

$$e^{ia\hat{q}}e^{ib\hat{p}} = e^{-i\hbar ab}e^{ib\hat{p}}e^{ia\hat{q}} \tag{1.3}$$

where $a$ and $b$ are real. Equivalently for all real $q$, $p$, $q'$, $p'$:

$$e^{\frac{i}{\hbar}(p'\hat{q}-q'\hat{p})}e^{\frac{i}{\hbar}(p\hat{q}-q\hat{p})} = e^{\frac{i}{2\hbar}(p'q-q'p)}e^{\frac{i}{\hbar}[(p'+p)\hat{q}-(q'+q)\hat{p}]} \tag{1.4}$$

Here all these operators are unitary; i.e. they have no domain problems. The most familiar representations of $\hat{q}$ and $\hat{p}$ are in terms of Schrodinger wave functions in position space, or wave functions in momentum space; in a Hilbert space:

$$\begin{aligned}
\mathcal{H} &= \left\{\psi(q) \in \left|\int_{-\infty}^{\infty} dq \mid \psi(q)\right|^2 < \infty\right\} \\
&= \left\{\phi(p) \in \left|\int_{-\infty}^{\infty} dp \mid \phi(p)\right|^2 < \infty\right\}, \\
\phi(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dq e^{-ipq/\hbar}\psi(q); \\
(\hat{q}\psi)(q) &= q\psi(q), (\hat{p}\psi)(q) = -i\hbar\frac{d\psi(q)}{dq}; \\
(\hat{p}\phi)(p) &= p\phi(p), (\hat{q}\phi)(p) = i\hbar\frac{d\phi(p)}{dp} \qquad (1.5)
\end{aligned}$$

Each of $\hat{q}$ and $\hat{p}$ can be applied only to wave functions in its domain. Formally eqs. (1) can be verified.

We use Dirac notation and idealized eigen functions of $\hat{q}$ and $\hat{p}$. Each of them has all real numbers as possible eigenvalues:

$$\begin{aligned}
\hat{q}|q\rangle &= q|q\rangle; \quad \langle q'|q\rangle = \delta(q'-q), \quad -\infty < q', q < \infty, \\
\int_{-\infty}^{\infty} dq|q\rangle\langle q| &= 1 \quad \text{on}\,\mathcal{H}; \\
\hat{p}|p\rangle &= p|p\rangle; \quad \langle p'|p\rangle = \delta(p'-p), \quad -\infty < p', p < \infty, \\
\int_{-\infty}^{\infty} dp|p\rangle\langle p| &= 1 \quad \text{on}\,\mathcal{H}; \\
\langle q|p\rangle &= \frac{1}{\sqrt{2\pi\hbar}} e^{iqp/\hbar} \qquad (1.6)
\end{aligned}$$

The wave functions $\psi(q)$, $\phi(p)$ in eqs.(1.5) are the following overlaps or inner products.

$$|\psi\rangle \in \mathcal{H}: \quad \psi(q) = \langle q|\psi\rangle, \quad \phi(p) = \langle p|\psi\rangle \qquad (1.7)$$

Since both $\hat{q}$ and $\hat{p}$ have all real numbers as eigenvalues, it is meaningful to displace or shift them by any c-numbers. For any real $q$ and $p$, clearly $\hat{q} - q$ and $\hat{p} - p$ obey eqs. (1.1), so they must be unitarily related to $\hat{q}$ and $\hat{p}$ respectively. This is achieved by so called displacement operators which are just what appear in eqs. (1.4). So we define:

$$\begin{aligned}
D(q,p) &= e^{i(p\hat{q}-q\hat{p})}/\hbar, \quad -\infty < q, p < \infty, \\
D(q,p)^\dagger D(q,p) &= 1 \qquad (1.8)
\end{aligned}$$

Their actions on operators and ket vectors are obtained quite easily:

$$\begin{aligned}
D(q',p')D(q,p) &= e^{\frac{i}{2\hbar}(p'q-q'p)} D(q'+q, p'+p); \qquad (a) \\
D(q,p)(\hat{q}\,\text{or}\,\hat{p})D(q,p)^{-1} &= \hat{q} - q \quad \text{or} \quad \hat{p} - p; \qquad (b) \\
D(q,p)|q'\rangle &= e^{\frac{i}{\hbar}(q'+\frac{q}{2})p}|q'+q\rangle, \\
D(q,p)|p'\rangle &= e^{\frac{-i}{\hbar}(p'+\frac{p}{2})q}|p'+p\rangle \qquad (c) \qquad (1.9)
\end{aligned}$$

## 2 The Weyl map and the Moyal Bracket

For a classical system with one Cartesian dof, dynamical variables are (real) functions $f(q,p)$; i.e., functions over the two dimensional phase space plane. For the 'corresponding' quantum system, dynamical variables are (hermitian)operators $\hat{F}$ acting on the Hilbert space $\mathcal{H}$ in eqs. (1.5). As $\hat{q}$

and $\hat{p}$ are an irreducible pair, every $\hat{F}$ is in principle some 'function' of $\hat{q}$ and $\hat{p}$. The Weyl map is a rule or a convention to set up a one-to-one correspondence between classical (real) functions $f(q,p)$ on the one hand, and (hermitian) operators $\hat{F}$ on the other.

To begin, consider finite degree polynomials in $q$ and $p$. For linear expressions, we of course set

$$q \to \hat{q}, \quad p \to \hat{p}, \quad \sigma q - \tau p \to \sigma\hat{q} - \tau\hat{p} \tag{2.1}$$

For higher degree expressions, we use the idea of 'symmetrised polynomials' to allow for the fact that $\hat{q}$ and $\hat{p}$ do not commute. The Weyl rule can be stated in several equivalent ways. For any non negative integers $m$ and $n$, we define the correspondence:

$$\begin{aligned} q^m p^n &= \text{coefficient of } \frac{(m+n)!}{m!n!}\sigma^m(-\tau)^n \text{ in } (\sigma q - \tau p)^{m+n} \to \\ \widehat{(q^m p^n)} &= \text{coefficient of } \frac{(m+n)!}{m!n!}\sigma^m(-\tau)^n \text{ in } (\sigma\hat{q} - \tau\hat{p})^{m+n} \end{aligned} \tag{2.2}$$

As examples, we have:

$$\begin{aligned} \widehat{(q^m)} &= (\hat{q})^m; \\ \widehat{(q^m p)} &= \frac{1}{(m+1)}(\hat{q}^m\hat{p} + \hat{q}^{m-1}\hat{p}\hat{q} + \hat{q}^{m-2}\hat{p}\hat{q}^2 + \ldots\ldots + \hat{p}\hat{q}^m); \ldots\ldots; \\ \widehat{(p^n)} &= (\hat{p})^n \end{aligned} \tag{2.3}$$

This is formal as all these are unbounded operators. Nevertheless we can see that this rule does map classical real $f(q,p)$ to hermitian quantum $\hat{F}$.

There are other ways to set up rules to go from classical $f(q,p)$ to quantum $\hat{F}$ used in other contexts, but for the purpose of the Wigner distribution, the Weyl rule is the appropriate one. Incidentally the same rule eq. (2.2) can be expressed as follows: for any real $\sigma$ and $\tau$,

$$e^{i(\sigma q - \tau p)} \to e^{i(\sigma\hat{q} - \tau\hat{p})} \equiv D(\hbar\tau, \hbar\sigma) \tag{2.4}$$

Based on this, we can use the methods of Fourier analysis to express the Weyl rule in yet another way. Suppose a classical $f(q,p)$ has the Fourier integral representation:

$$f(q,p) = \frac{1}{2\pi}\int\int d\tau d\sigma \tilde{f}(\sigma,\tau)e^{i(\sigma q - \tau p)} \tag{2.5}$$

Then by eq. (2.4) and linearity, we have:

$$\begin{aligned} f(q,p) \to \hat{F} &= \frac{1}{2\pi}\int\int d\tau d\sigma \tilde{f}(\sigma,\tau)e^{i(\sigma\hat{q} - \tau\hat{p})} \\ &= \frac{1}{2\pi\hbar^2}\int\int dq dp \tilde{f}(\frac{p}{\hbar}, \frac{q}{\hbar})D(q,p) \end{aligned} \tag{2.6}$$

Now we can see that, using many nice properties of the $D(q,p)$, this can be inverted. Using first eq. (1.9 c) and then eq. (1.9 a):

$$\begin{aligned} Tr[D(q,p)] &= \int_{-\infty}^{\infty} dq' \langle q'|D(q,p)|q'\rangle \\ &= \int_{-\infty}^{\infty} dq' \langle q'|q'+q\rangle e^{\frac{i}{\hbar}(q'+\frac{q}{2})p} \\ &= 2\pi\hbar\delta(q)\delta(p); \qquad (a) \end{aligned}$$

$$\begin{aligned} Tr[D(q',p')^\dagger D(q,p)] &= Tr[D(-q',-p')D(q,p)] \\ &= e^{\frac{i}{2\hbar}(q'p - p'q)}Tr[D(q-q', p-p')] \\ &= 2\pi\hbar\delta(q-q')\delta(p-p') \qquad (b) \end{aligned} \tag{2.7}$$

This means that in the space of operators on $\mathcal{H}$ and with respect to the 'Hilbert Schmidt inner product', the $D(q,p)$ form an orthonormal basis. Incidentally, we can say eq. (2.6) is the quantum operator form of the Fourier integral representation (2.5). Combining eqs. (2.6) and (2.7), we get:

$$Tr[D(q',p')^{\dagger}\hat{F}] = \frac{1}{\hbar}\tilde{f}\left(\frac{p'}{\hbar},\frac{q'}{\hbar}\right) \tag{2.8}$$

So via eq. (2.5), we arrive at the inverse to the Weyl map (2.6):

$$f(q,p) = \frac{1}{2\pi\hbar}\int\int dq'dp' e^{i(p'q-q'p)/\hbar} Tr[D(q',p')^{\dagger}\hat{F}] \tag{2.9}$$

Let now $g(q,p)$ be another classical function with Fourier transform $\tilde{g}(\sigma,\tau)$, mapped by the Weyl rule to the operator $\hat{G}$. Since by Fourier inversion we know how to pass from $\tilde{f}$ to $f$ and $\tilde{g}$ to $g$ and eq (2.9) shows how to pass $\tilde{F}\to f$, $\tilde{G}\to g$, we can put it all together and can obtain the relation

$$Tr[\tilde{G}^{\dagger}F] = \frac{1}{2\pi\hbar}\int\int dqdp f(q,p)g(q,p)^{*} \tag{2.10}$$

Thus, square integrable functions over the classical phase plane are taken by the Weyl rule to operators with finite Hilbert-Schmidt norm.

A somewhat surprising instance of the $f(q,p)\leftrightarrow\hat{F}$ correspondence will be found useful later on. The choice

$$f(q,p) = \delta(q)\delta(p), i.e, \tilde{f}(\sigma,\tau) = \frac{1}{2\pi} \tag{2.11}$$

leads to

$$\hat{F} = \frac{1}{(2\pi\hbar)^2}\int\int dqdp D(q,p) \tag{2.12}$$

What is this operator? If we apply it to $|q'>$ and use eq(1.9c) we get

$$\begin{aligned}
\hat{F}|q'> &= \frac{1}{(2\pi\hbar)^2}\int\int dqdp e^{i(q'+\frac{q}{2})p/\hbar}|q'+q> \\
&= \frac{1}{2\pi\hbar}\int dq\delta(q'+\frac{q}{2})|q'+q> = \frac{1}{\pi\hbar}|-q'>, \\
\text{i.e. } \hat{F} &= \frac{1}{\pi\hbar}\hat{P}, \quad \hat{P} = \text{Parity operator}
\end{aligned} \tag{2.13}$$

Thus

$$f(q,p) = \delta(q)\delta(p) \leftrightarrow \hat{F} = \frac{\hat{P}}{\pi\hbar} \tag{2.14}$$

where of course

$$\hat{P}(\hat{q}\,\text{or}\,\hat{p})\hat{P}^{-1} = -\hat{q}\text{ or }-\hat{p}, \quad \hat{P}^2 = I \tag{2.15}$$

**The Moyal Product and Bracket**

Since the Weyl rule gives a one-to-one correspondence between classical phase space functions and quantum operators, it follows that the associative law of non-commutative multiplication of operators can be expressed in terms of the classical functions themselves. Introduce the notation:

$$f(q,p) \xrightarrow{Weyl} \hat{F} = (f(q,p))_w \tag{2.16}$$

Then we want to find the structure of the * product in

$$(f(q,p))_w\,(g(q,p))_w = (f(q,p)*g(q,p))_w \tag{2.17}$$

For exponential functions, after relabelling variables, eq(1.4) says:

$$\begin{aligned}
\left(e^{i(p''q-q''p)/\hbar}\right)_w\left(e^{i(p'q-q'p)/\hbar}\right)_w &= \left(e^{i(p''q'-q''p')/2\hbar} \quad e^{i(p''q-q''p)/\hbar} \quad e^{i(p'q-q'p)/\hbar}\right)_w \\
&= \left(e^{i(p''q-q''p)/\hbar} * e^{i(p'q-q'p)/\hbar}\right)_w, \\
e^{i(p''q-q''p)/\hbar} * e^{i(p'q-q'p)/\hbar} &= e^{\frac{i}{2\hbar}(p''q'-q''p')} e^{\frac{i}{\hbar}(p''q-q''p)} e^{\frac{i}{\hbar}(p'q-q'p)} \\
&= e^{\frac{i}{2\hbar}\left(\frac{\hbar}{i}\frac{\partial}{\partial q_2} i\hbar\frac{\partial}{\partial p_1} - i\hbar\frac{\partial}{\partial p_2}\frac{\hbar}{i}\frac{\partial}{\partial q_1}\right)} e^{i(p''q_2-q''p_2)/\hbar} e^{i(p'q_1-q'p_1)/\hbar} \qquad (2.18)
\end{aligned}$$

By linearity we can generalize to any $f(q,p)$ and $g(q,p)$:

$$f(q,p) * g(q,p) = e^{\frac{i\hbar}{2}\left(\frac{\partial}{\partial q_2}\frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2}\frac{\partial}{\partial q_1}\right)} f(q_2,p_2) g(q_1,p_1)\Big|_{q_1=q_2=q, p_1=p_2=p} \qquad (2.19)$$

This is the Moyal product or multiplication rule for classical phase space functions, an exact rendering of multiplication of quantum operators. For the commutator, we have:

$$[(f(q,p))_w, (g(q,p))_w] = (f(q,p) * g(q,p) - g(q,p) * f(q,p))_w,$$

$$\begin{aligned}
& f(q,p) * g(q,p) - g(q,p) * f(q,p) \\
&= \left[\exp\left\{\frac{i\hbar}{2}\left(\frac{\partial}{\partial q_2}\frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2}\frac{\partial}{\partial q_1}\right)\right\}\left(f(q_2,p_2)g(q_1,p_1) - g(q_2,p_2)f(q_1,p_1)\right)\right]_{q_1=q_2=q,\, p_1=p_2=p} \\
&= 2i\,\sin\left(\frac{\hbar}{2}\left(\frac{\partial}{\partial q_2}\frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2}\frac{\partial}{\partial q_1}\right)\right) f(q_2,p_2)g(q_1,p_1)\Big|_{q_1=q_2=q,\, p_1=p_2=p} \\
&= i\hbar\left\{f(q,p), g(q,p)\right\} + O(\hbar^3) \qquad (2.20)
\end{aligned}$$

The leading term is essentially the classical PB $\{f, g\}$, discovered by Dirac in September 1925!

## 3 The Wigner distribution and its properties

The Weyl map $f(q,p) \to \hat{F} = (f(q,p)_w)$ is such that the physical dimensions of $f$ and $\hat{F}$ are the same. This is clear from eqs. (2.5, 2.6). Now let $\hat{\rho}$ be the density matrix of a (pure or mixed) state of the quantum system, obeying

$$\hat{\rho}^\dagger = \hat{\rho} \geq 0, \quad Tr(\hat{\rho}) = 1 \qquad (3.1)$$

In the pure case,

$$\hat{\rho} = |\psi><\psi|, \quad <\psi|\psi>= 1 \qquad (3.2)$$

In the state $\hat{\rho}$ the expectation value of any observable $\hat{F}$ is given by:

$$<\hat{F}>_{\hat{\rho}} = Tr(\hat{\rho}\hat{F}) \qquad (3.3)$$

The idea of the Wigner distribution is to express this in the classical looking form familiar from classical statistical mechanics. For any $\hat{F} = (f(q,p))_w$, we want a function $W(q,p)$ to represent $\hat{\rho}$ so that

$$Tr(\hat{\rho}\hat{F}) = \int\int dq dp W(q,p) f(q,p) \qquad (3.4)$$

The general result (2.10) tells us how to define $W(q,p)$: setting $\hat{G} = \hat{\rho}$ there, which makes $g(q,p)$ real, $W(q,p)$ must be such that

$$(W(q,p))_w = \frac{1}{2\pi\hbar}\hat{\rho} \qquad (3.5)$$

Thus the rule to pass from $\hat{\rho}$ to $W(q,p)$ has this additional factor compared to the Weyl map rule. From eq (2.9), and taking account of this extra factor, we find:

$$\begin{aligned} W(q,p) &= \frac{1}{(2\pi\hbar)^2}\int\int dq'dp' e^{i(p'q-q'p)/\hbar} Tr(\hat{\rho}D(q',p')^\dagger) \\ &= \frac{1}{(2\pi\hbar)^2}\int\int dq'dp' e^{i(q'p-p'q)/\hbar} Tr(\hat{\rho}D(q',p')) \\ &= \frac{1}{(2\pi\hbar)^2}\int\int dq'dp' \int dq'' e^{i(q'p-p'q)/\hbar} < q''|\hat{\rho}D(q',p')|q'' > \\ &= \frac{1}{(2\pi\hbar)^2}\int\int dq'dp' \int dq'' e^{i(q'p-p'q)/\hbar + i(q''+\frac{q'}{2})p'/\hbar} < q''|\hat{\rho}|q''+q' > \\ &= \frac{1}{2\pi\hbar}\int\int dq'dq'' e^{ipq'/\hbar}\delta(q''+\frac{q'}{2}-q) < q''|\hat{\rho}|q''+q' > \\ &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dq' e^{ipq'/\hbar} < q-q'/2|\hat{\rho}|q+q'/2 > \end{aligned} \tag{3.6}$$

For a pure state $\hat{\rho} = |\psi><\psi|$ we get:

$$W_\psi(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dq' e^{ipq'/\hbar}\psi(q-q'/2)\psi(q+q'/2)^* \tag{3.7}$$

The basic properties of the Wigner distribution follow from those of $\hat{\rho}$:

$$\hat{\rho}^\dagger = \hat{\rho} \implies W(q,p) \text{ real}; \quad Tr\hat{\rho} = 1 \implies \int\int dqdpW(q,p) = 1 \tag{3.8}$$

However, $\hat{\rho} \geq 0$ does not mean $W(q,p) \geq 0$. In general, taking $\hat{F} = \hat{\rho}'$ in eq (3.4) we find :

$$\begin{aligned} f(q,p) &= 2\pi\hbar W_{\hat{\rho}'}(q,p), \\ Tr(\hat{\rho}'\hat{\rho}) &= 2\pi\hbar\int\int dqdpW_{\hat{\rho}'}(q,p)W_{\hat{\rho}}(q,p) \end{aligned} \tag{3.9}$$

Since the left hand side can certainly vanish, for example mutually orthogonal pure states, in principle $W(q,p)$ can become negative in some parts of the phase plane.

On the other hand we find that point wise the value of $W(q,p)$ is bounded in magnitude. From eqs (2.14, 3.4), taking $\hat{F} = \frac{1}{\pi\hbar}\hat{\rho}$ we see that

$$Tr(\hat{\rho}\hat{P}) = \pi\hbar W(0,0), \quad |W(0,0)| \leq \frac{1}{\pi\hbar} \tag{3.10}$$

This is true all over the phase plane since from eqs (1.9c, 3.6):

$$\hat{\rho}' = D(q_0,p_0)^{-1}\hat{\rho}D(q_0,p_0) \Leftrightarrow W'(q,p) = W(q+q_0,p+p_0) \tag{3.11}$$

So we can extend eq.(3.10):

$$|W(q,p)| \leq \frac{1}{\pi\hbar} \tag{3.12}$$

We can also combine eqs (3.10, 3.11) to see that

$$W(q,p) = \frac{1}{\pi\hbar}Tr(\hat{\rho}D(q,p)\hat{P}D(q,p)^{-1}) \tag{3.13}$$

So the Wigner distribution at each $(q,p)$ is an expectation value. This formula was 'first' found by me in 1978. Later Wootters (1987) gave the name 'phase point operators' for the expression in eq (3.13), for the finite dimensional case.

As examples, we calculate the Wigner distributions for the ground state and first excited state of the simple harmonic oscillator. For simplicity, we set $m = \omega = \hbar = 1$, so the Hamiltonian is:

$$\hat{H} = \frac{1}{2}\left(\hat{q}^2 + \hat{p}^2\right) \tag{3.14}$$

The normalized wave functions are:

$$\psi_0(q) = \frac{1}{\pi^{1/4}} e^{-q^2/2}, \quad \psi_1(q) = \frac{\sqrt{2}}{\pi^{1/4}} q e^{-q^2/2} \tag{3.15}$$

Then using the standard integrals

$$\int_{-\infty}^{\infty} dq e^{-q^2}(1, q^2) = \sqrt{\pi}(1, 1/2) \tag{3.16}$$

from eq. (3.7), we find

$$W_0(q,p) = \frac{1}{\pi} e^{-\left(q^2+p^2\right)}, \quad W_1(q,p) = \frac{2}{\pi}\left(q^2 + p^2 - 1/2\right) e^{-\left(q^2+p^2\right)} \tag{3.17}$$

Since

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq dp W_0(q,p) W_1(q,p) = 0, \tag{3.18}$$

$W_1(q,p)$ does indeed have to become negative sometimes!

There are however limits to how negative $W(q,p)$ can become. If we go back to eq. (3.4) and consider functions of $q$ alone, $f(q)$,then clearly $\hat{F} = f(\hat{q})$. And for these we do have quantum mechanically determined non-negative probability distributions. Similarly also for $f(p)$, when $\hat{F} = f(\hat{p})$. For a general state $\hat{\rho}$ and then in the pure case $\hat{\rho} = |\psi\rangle\langle\psi|$, 'we recover the marginals':

$$\begin{aligned}\int_{-\infty}^{\infty} dp\, W(q,p) &= \langle q|\hat{\rho}|q\rangle \quad \text{or} \quad |\psi(q)|^2; \\ \int_{-\infty}^{\infty} dq\, W(q,p) &= \langle p|\hat{\rho}|p\rangle \quad \text{or} \quad |\phi(p)|^2 \end{aligned}\tag{3.19}$$

So this is how the Wigner distribution makes contact with genuine probabilities and why it is called a quasi-probability distribution.

## 4 Extension to many degrees of freedom

All the formulae in Sections II and III generalize easily, we present only some of them and develop useful notations as well.

For $n$ Cartesian canonical pairs, we have $2n$ basic hermitian operators $\hat{q}_j$, $\hat{p}_j$, $j = 1, 2, ......, n$ obeying the CCR's

$$[\hat{q}_j, \hat{p}_k] = i\hbar\delta_{jk}, \quad [\hat{q}_j, \hat{q}_k] = [\hat{p}_j, \hat{p}_k] = 0 \tag{4.1}$$

A compact way of expressing them is to define a $2n$ component column vector with operator entries:

$$\begin{aligned}\hat{\xi} &= \begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \\ \hat{\xi}_3 \\ . \\ . \\ \hat{\xi}_{2n} \end{pmatrix} = \left(\hat{\xi}_a\right), \quad a = 1, 2, ..., 2n; \quad \hat{\xi}_a^\dagger = \hat{\xi}_a; \\ \hat{\xi}_j &= \hat{q}_j, \quad \hat{\xi}_{j+n} = \hat{p}_j \end{aligned}\tag{4.2}$$

Then the CCR's (4.1) can be written as

$$[\hat{\xi}_a, \hat{\xi}_b] = i\hbar\beta_{ab}, \quad \beta = \begin{pmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & 0 \end{pmatrix} \tag{4.3}$$

This $2n \times 2n$ matrix $\beta$ is very important and will play a role in the discussion of the symplectic groups $Sp(2n, R)$. Its basic properties, apart from reality and anti-symmetry, are

$$\beta^\dagger = \beta^{-1} = -\beta, \quad \beta^2 = -I_{2n\times 2n} \tag{4.4}$$

We can use both the split $q - p$ notation and the $\xi$ notation as convenient.

The Weyl form of the CCR's (4.1, 4.3) is directly expressed in terms of the displacement operators which are unitary.

$$\begin{aligned} D(\underline{q}, \underline{p}) = D(\xi) = e^{i(\underline{p}.\hat{\underline{q}} - \underline{q}.\hat{\underline{p}})/\hbar} = e^{-i\xi^T\beta\hat{\xi}/\hbar}, \\ \xi = (q_1...q_n, p_1...p_n)^T \in R^{2n}, \quad \underline{q} \in R^n, \quad \underline{p} \in R^n \end{aligned} \tag{4.5}$$

Then the CCR's appear as the composition law for the $D(\xi)$:

$$D(\xi')D(\xi) = e^{\frac{-i}{2\hbar}\xi'^T\beta\xi} D(\xi' + \xi) \tag{4.6}$$

The standard representations in terms of wave functions are:

$$\begin{aligned} \mathcal{H} &= \{\psi(\underline{q}) \in C | \int_{R^n} d^n q |\psi(\underline{q})|^2 < \infty\} \\ &= \{\phi(\underline{p}) \in C | \int_{R^n} d^n p |\phi(\underline{p})|^2 < \infty\}, \\ \phi(\underline{p}) &= \frac{1}{(2\pi\hbar)^{n/2}} \int_{R^n} d^n q e^{-i\underline{p}.\underline{q}/\hbar} \psi(\underline{q}); \\ (\hat{q}_j\psi)(\underline{q}) &= q_j\psi(\underline{q}), \quad (\hat{p}_j\psi)(\underline{q}) = -i\hbar\frac{\partial}{\partial q_j}\psi(\underline{q}); \\ (\hat{q}_j\phi)(\underline{p}) &= i\hbar\frac{\partial}{\partial p_j}\phi(\underline{p}), \quad (\hat{p}_j\phi)(\underline{p}) = p_j\phi(\underline{p}) \end{aligned} \tag{4.7}$$

The main properties of the $D(\xi)$, in addition to (4.6), are:

$$\begin{aligned} D(\xi)\hat{\xi}_a D(\xi)^{-1} &= \hat{\xi}_a - \xi_a; \\ D(\underline{q}, \underline{p})|\underline{q}'\rangle &= e^{\frac{i}{\hbar}\underline{p}\cdot(\underline{q}' + \frac{\underline{q}}{2})}|\underline{q}' + \underline{q}\rangle, \\ D(\underline{q}, \underline{p})|\underline{p}'\rangle &= e^{\frac{-i}{\hbar}\underline{q}\cdot(\underline{p}' + \frac{\underline{p}}{2})}|\underline{p}' + \underline{p}\rangle \end{aligned} \tag{4.8}$$

Of course we supplement (4.7) with

$$\begin{aligned} \psi(\underline{q}) &= \langle\underline{q}|\psi\rangle, \quad \phi(\underline{p}) = \langle\underline{p}|\psi\rangle; \\ \langle\underline{q}|\underline{p}\rangle &= \frac{1}{(2\pi\hbar)^{n/2}} e^{i\underline{q}.\underline{p}/\hbar} \end{aligned} \tag{4.9}$$

The Weyl map or rule to go from classical (real) phase space functions to Hermitian quantum operators on $\mathcal{H}$ can be given at several levels:

$$\begin{aligned} (\underline{\sigma}.\underline{q} - \underline{\tau}.\underline{p})^N &\rightarrow (\underline{\sigma}.\hat{\underline{q}} - \underline{\tau}.\hat{\underline{p}})^N, \quad N = 1, 2, ....; \\ e^{i.(\underline{\sigma}.\underline{q} - \underline{\tau}.\underline{p})} &\rightarrow e^{i(\underline{\sigma}.\hat{\underline{q}} - \underline{\tau}.\hat{\underline{p}})} \end{aligned} \tag{4.10}$$

For general classical functions on the phase space $R^{2n}$, we can use the Fourier route generalizing (2.5, 2.6):

$$\begin{aligned} f(\underline{q},\underline{p}) &= \frac{1}{(2\pi)^n}\int\int d^n\tau d^n\sigma \hat{f}(\underline{\sigma},\underline{\tau})e^{i.(\underline{\sigma}.\underline{q}-\underline{\tau}.\underline{p})} \rightarrow \\ \hat{F} &= \frac{1}{(2\pi)^n}\int\int d^n\tau d^n\sigma \hat{f}(\underline{\sigma},\underline{\tau})e^{i(\underline{\sigma}.\hat{\underline{q}}-\underline{\tau}.\hat{\underline{p}})} \end{aligned} \quad (4.11)$$

Then generalizing eqns (2.7 , 2.10), we have

$$\begin{aligned} Tr(D(\xi')^\dagger D(\xi)) &= (2\pi\hbar)^n\delta^{(2n)}(\xi'-\xi); \\ Tr(\hat{G}^\dagger\hat{F}) &= \frac{1}{(2\pi\hbar)^n}\int_{R^{2n}} d^{2n}\xi g(\xi)^* f(\xi) \end{aligned} \quad (4.12)$$

The basic equations for the Wigner distribution are straight forward generalisations of Sections II, III:

$$\begin{aligned} W(\xi) = W(\underline{q},\underline{p}) &= \frac{1}{(2\pi\hbar)^n}\int_{R^n} d^nq' e^{i\underline{p}.\underline{q}'/\hbar}\langle\underline{q}-\frac{1}{2}\underline{q}'|\hat{\rho}|\underline{q}+\frac{1}{2}\underline{q}'\rangle \\ &= \frac{1}{(2\pi\hbar)^n}\int_{R^n} d^nq' e^{i\underline{p}.\underline{q}'/\hbar}\psi(\underline{q}-\frac{1}{2}\underline{q}')\psi(\underline{q}+\frac{1}{2}\underline{q}')^* \quad \text{in pure case;} \quad (a); \\ Tr(\hat{\rho}\hat{F}) &= \int_{R^{2n}} d^{2n}\xi W(\xi)f(\xi), \qquad (b) \\ Tr(\hat{\rho}'\hat{\rho}) &= (2\pi\hbar)^n\int_{R^{2n}} d^{2n}\xi W'(\xi)W(\xi); \qquad (c) \\ \int_{R^n} d^npW(\underline{q},\underline{p}) &= \langle\underline{q}|\hat{\rho}|\underline{q}\rangle \text{ or } |\psi(\underline{q})|^2, \\ \int_{R^n} d^nqW(\underline{q},\underline{p}) &= \langle\underline{p}|\hat{\rho}|\underline{p}\rangle \text{ or } |\phi(\underline{p})|^2; \qquad (d) \\ \hat{\rho}' = D(\xi_0)^{-1}\hat{\rho}D(\xi_0) &\Leftrightarrow W'(\xi) = W(\xi+\xi_0) \qquad (e) \end{aligned} \quad (4.13)$$

## 5 Wigner Distributions in finite dimensions- The spin $1/2$ case

The preceding Sections depend crucially on the structure and consequences of the CCR's (1.1, 4.1, 4.3) for Cartesian systems. For finite dimensional quantum systems, when $dim\mathcal{H} < \infty$, new features are to be expected though the Cartesian case can be a guide. We now describe a treatment by Feynman for two dimensional quantum systems - particle with spin $\frac{1}{2}$, two-level atom, photon polarization states etc.

As motivation recall the following features of the one - dimensional Cartesian case:

**(i)** $W(q,p)$ is a real function of two independent real continuous c- number arguments $q$ and $p$; we regard them as possible eigen values of the non commuting operators $\hat{q}$ and $\hat{p}$ respectively.

**(ii)** The (marginal) integrals of $W(q,p)$ over $p$ (respectively over $q$) give the quantum mechanical probability distributions for position $\hat{q}$ (respectively momentum $\hat{p}$) in the concerned quantum state.

**(iii)** We have the relation (3.4) re-expressing the operator form of a general quantum mechanical expectation value in completely c- number terms.

These feature are carried over to the spin $\frac{1}{2}$ case as follows. The three basic operators (apart from the unit operator) are the hermitian $2\times 2$ Pauli matrices $\sigma_x, \sigma_y, \sigma_z$, or $\sigma_j$, $j = 1,2,3$ obeying the familiar commutation relations.

$$[\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l. \quad (5.1)$$

We change the Cartesian scheme in this way:

$$\begin{aligned} &\hat{q}\rightarrow\sigma_z, \quad q\rightarrow\epsilon = \pm 1 = \text{eigen values of } \sigma_z; \\ &\hat{p}\rightarrow\sigma_x, \quad p\rightarrow\epsilon' = \pm 1 = \text{eigen values of } \sigma_x; \\ &\text{phase space point } (q,p)\rightarrow \text{four } '\text{points}' \, (\epsilon,\epsilon') \end{aligned} \quad (5.2)$$

The most general state is a $2 \times 2$ density matrix $\hat{\rho}$ obeying

$$\hat{\rho}^\dagger = \hat{\rho} \geq 0, \quad Tr\hat{\rho} = 1 \tag{5.3}$$

As is well known, it can be expressed in the form

$$\begin{aligned} \hat{\rho} &= \frac{1}{2}(I + \underline{n}.\underline{\sigma}), \quad \underline{n} = \text{real } 3 - \text{dimensional vector}, \ |\underline{n}| \leq 1; \\ |\underline{n}| &= 1 \Leftrightarrow \text{pure state}, \quad |\underline{n}| < 1 \Leftrightarrow \text{mixed state}. \end{aligned} \tag{5.4}$$

Given $\hat{\rho}$ , the probabilities for $\sigma_z = \epsilon = \pm 1$, and for $\sigma_x = \epsilon' = \pm 1$, are given by standard quantum mechanics as

$$\begin{aligned} \text{Prob.}(\hat{\rho}|\sigma_z = \epsilon) &= Tr(\hat{\rho}.\frac{1}{2}(1 + \epsilon\sigma_z)) = \frac{1}{2}(1 + \epsilon n_z); \\ \text{Prob.}(\hat{\rho}|\sigma_x = \epsilon') &= Tr(\hat{\rho}.\frac{1}{2}(1 + \epsilon'\sigma_x)) = \frac{1}{2}(1 + \epsilon' n_x) \end{aligned} \tag{5.5}$$

We now need two things to put together the Weyl-Wigner picture: each state $\hat{\rho}$ should be represented by a corresponding c- number Wigner distribution $W(\epsilon, \epsilon')$ ; and each of the operators $\sigma_j$ should be represented by corresponding c- number functions $f_j(\epsilon, \epsilon')$ in such a way that the analogue of eqn (3.4) is obeyed. Here are the Feynman prescriptions which achieve all this:

$$\begin{aligned} \hat{\rho} &\longrightarrow W(\epsilon, \epsilon') = \frac{1}{4}Tr\{\hat{\rho}(1 + \epsilon\sigma_z + \epsilon'\sigma_x + \epsilon\epsilon'\sigma_y)\}; && (a) \\ \sigma_x, \sigma_y, \sigma_z &\longrightarrow f_x(\epsilon, \epsilon'),\ f_y(\epsilon, \epsilon'),\ f_z(\epsilon, \epsilon') = \epsilon', \epsilon\epsilon', \epsilon && (b) \end{aligned} \tag{5.6}$$

Then indeed we find:

$$\begin{aligned} Tr(\hat{\rho}\sigma_j) &= \sum_{\epsilon, \epsilon' = \pm 1} W(\epsilon, \epsilon') f_j(\epsilon, \epsilon'); \\ \sum_{\epsilon'} W(\epsilon, \epsilon') &= \text{Prob.}(\hat{\rho}|\sigma_z = \epsilon), \\ \sum_{\epsilon} W(\epsilon, \epsilon') &= \text{Prob.}(\hat{\rho}|\sigma_x = \epsilon'); \\ \sum_{\epsilon, \epsilon'} W(\epsilon, \epsilon') &= 1 \end{aligned} \tag{5.7}$$

Then general expectation values, marginals and overall normalisation all work out satisfactorily. Using eqn (5.4) in (5.6) we find:

$$W(\epsilon, \epsilon') = \frac{1}{4}(1 + \epsilon' n_1 + \epsilon'\epsilon n_2 + \epsilon n_3) \tag{5.8}$$

This is just a collection of four real numbers, normalized and with non negative marginals. How now is the 'Wigner quality' recognized or expressed? If any four real numbers $W(\pm 1, \pm)$ are given, do they make up a physically acceptable Wigner distribution corresponding to a quantum state? The necessary and sufficient condition for this is that they be expressible in the form of the right hand side of eq. (5.8) for some $\underline{n}$ with $|\underline{n}| \leq 1$.

In the same year, 1987, as Feynman's work, Wootters also published an important paper on finite dimensional Wigner distributions, in which the phrase 'phase point operators' appeared. Since 2005-2006, Chaturvedi, Simon, myself and some others initiated an approach based on early ideas of Dirac. A fair amount of work is done, but it is still work in progress. An important point is that the problems in odd and in even dimensions are very different, the former being considerably simpler.