Lattice Dynamics

Jan, February 2015

1 Introduction

We explore the dynamics of a 1D spring mass lattice, using an equivalent electronic/electrical circuit. We consider two configurations which correspond to

1. an array of point masses (identical) connected by springs of equal spring constant
2. an array of point masses (identical) connected by springs with alternate springs having the same spring constant

We show rigorously which quantities correspond to which in both setups. We then evaluate the dispersion relation, viz the relation between the frequency ω and (TODO: Name of k) k theoretically and plot it. In the same graph, we plot the applied frequency ω and measured value of k.

There are certain curious theoretical questions which the experiment has raised, some of which we have addressed and the remaining have been stated, in the last section.

2 Background Theory

2.1 Spring Mass System

Consider an infinite point masses connected to each other by identical springs. We write the forces on the n^{th} oscillator as

$$m\ddot{x}_n = -\gamma(x_n - x_{n-1}) + \gamma(x_{n+1} - x_n)$$

and we know that $x_n = Ae^{i(kna-\omega t)}$ can be used to find the dispersion relation, that is ω as a function of k. We skip the detailed derivation here and come back to it in the LC circuit equivalent of the same.

Next, consider next the case in which the masses alternate between m_1 and m_2, while the spring constants go γ_1 and γ_2. We again write two sets of equations for the forces on both types of n^{th} oscillator as

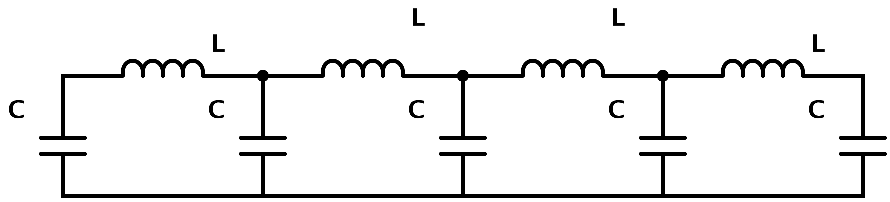
$$m_1\ddot{x}_n = -\gamma_2(x_n - x'_n) + \gamma_1(x'_{n-1} - x_n)$$

$$m_2\ddot{x}'_n = -\gamma_1(x'_n - x_{n+1}) + \gamma_2(x_n - x'_n)$$

where we take $m_1 = m_2 = m$ which can be solved by using $x_n = Ue^{i(kna-\omega t)}$ and $x'_n = Ve^{i(kna-\omega t)}$. Again the detailed solution is discussed in the next section.

2.2 L, C circuit equivalent

Consider an infinite generalization of the following LC circuit.



We can write down using Kirchoff's law the following:

$$-L\ddot{q}_n - C^{-1}(q_n - q_{n-1}) + C^{-1}(q_{n+1} - q_n) = 0$$

The equivalence then is that $L \leftrightarrow m$, $1/C \leftrightarrow \gamma$ and $x \leftrightarrow q$. It is important to note that we may differentiate the equation again with respect to time to obtain $x \leftrightarrow \dot{q} = i$. Note however that we'll consider this equivalence only when we deal with observables in the laboratory. In the rest of the analysis, we use the former. Further, observe that we could've easily used $C \leftrightarrow m$, $1/L \leftrightarrow \gamma$ in this case, but this will not work in general for obvious reasons. Again, we may use $q_n = Ae^{i(kna-\omega t)}$ to obtain

$$\begin{aligned}
\dot{q} &= -i\omega q \\
\ddot{q} &= \omega^2 q \\
\implies \quad L\omega^2 &= C^{-1}e^{ikna} + C^{-1}e^{-ikna} - 2C^{-1} \\
\implies \quad \omega &= \sqrt{\tfrac{2}{LC}(1-\cos(ka)}
\end{aligned}$$

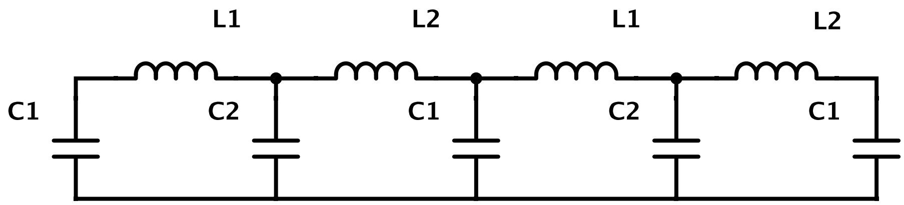
Note here that kna is reminiscent of the solution we used in the spring mass system where a had some significance (interatomic distance), however now there's none. Further, we only use the product ka.

Now in the usual case, we impose a periodic boundary condition, which (claim) results in giving N distinct solutions for ω by quantizing k as $\frac{2\pi m}{Na}$(where N is the number of entities). [can be proven] This is also expected since for N oscillators, there should be N normal modes.

However, in the given experimental setup, viz. our case, we do not have a periodic boundary. Infact, the differential equations don't fully capture the physical system because the end points of the 1D chain, don't satisfy the equations. To overcome the issue, we imagine the chain to be infinite in length and consider only a part of it. We therefore hypothesize that we are giving an input charge $q_0(t) = Ae^{-i\omega t}$ on the zeroth capacitor (just a label) and read the charge at the last capacitor (due to the inductor on the left), viz. $q_{N-1} = Ae^{-i(ka(N-1)-\omega t)}$ (although experimentally we're measuring current which is equivalent as was explained earlier). Since we're taking the real part of these quantities, we find that if we plot q_0 on the X-axis and q_{N-1} on the Y-axis using the XY mode on the oscillator, we get a graph of $\cos(\omega t)$ vs $\cos(\omega t - \phi)$, where $\phi = ka(N-1)$. Obviously, when $\phi = 0$, we'll get a straight line and for $\phi = \pi/2$ we'll get a circle. This information can be used to find the value of ka for various values of ω, which is to say we find only those ωs for which ϕ is of the form $m\pi/2$.

Also note that there's an upper limit to $\omega = 2/\sqrt{LC}$ because, well, $\cos(ka)$ is bounded.

Next consider again an infinite generalization of the following LC circuit.



We can write down using Kirchoff's law the following:

$$-L_1\ddot{q}_n - C_2^{-1}(q_n - q_n') + C_1^{-1}(q_{n-1}' - q_n) = 0$$

$$-L_2\ddot{q}_n' - C_1^{-1}(q_n' - q_{n+1}) + C_2^{-1}(q_n - q_n') = 0$$

(claim) It is an interesting fact that regardless of which of the following cases we take

- $L_1 = L_2 = L$ and capacitors different
- $C_1 = C_2 = C$ and inductors different

the expression for ω is identical, given we interchange in one of the expressions, Ls with Cs.

If we consider the physical case, viz. Cs different, and $L_1 = L_2 = L$, and use $q_n = Ue^{i(kna-\omega t)}$, $q_n' = Ve^{i(kna-\omega t)}$ we obtain by equating the determinant of the algebraic equations to zero,

$$\left(\omega^2 - \frac{1}{LC_1} - \frac{1}{LC_2}\right)^2 = \left(\frac{1}{LC_2} + \frac{e^{-ika}}{LC_1}\right)\left(\frac{1}{LC_2} + \frac{e^{ika}}{LC_1}\right)$$

which finally yields

$$\omega = \frac{1}{LC_1} + \frac{1}{LC_2} \pm \sqrt{\left[\frac{1}{LC_1}\right]^2 + \left[\frac{1}{LC_2}\right]^2 + \frac{2}{L^2C_1C_2}\cos ka}$$

So using this we can plot $\omega(k)$ against k; obviously for each value of k we'll get two values of ω. Now the smallest separation between two values of $\omega(k)$ will be

$$\Delta\omega \equiv 2\sqrt{\left[\frac{1}{LC_1}\right]^2 + \left[\frac{1}{LC_2}\right]^2 - \frac{2}{L^2C_1C_2}} = 2\left(\frac{1}{LC_1} - \frac{1}{LC_2}\right)$$

. If instead we plot k vs ω, there'll be a region of ω, namely

$$\omega \in \left(\frac{1}{LC_1} + \frac{1}{LC_2} - \frac{\Delta\omega}{2}, \frac{1}{LC_1} + \frac{1}{LC_2} + \frac{\Delta\omega}{2}\right)$$

for which there'll not exist any value of k. Thus we expect a bandgap. Also note that k is defined upto a multiple of $2\pi/a$. One final point that needs to be mentioned in addition is that even though $\omega(k)$ is multi-valued (viz. for a given k there're two ωs), when we restrict k to $(0, \pi/a)$, then the inverse function $k(\omega)$ is single-valued.

The one important caveat in this case is that for determining the value of ka from the observation we must realize that $n \in \{0, 1, 2, 3, 4\}$ because we used q_n and q'_n which together repeat only 5 times. Thus we must have $\phi = ka(\frac{N}{2} - 1)$.

3 Procedure

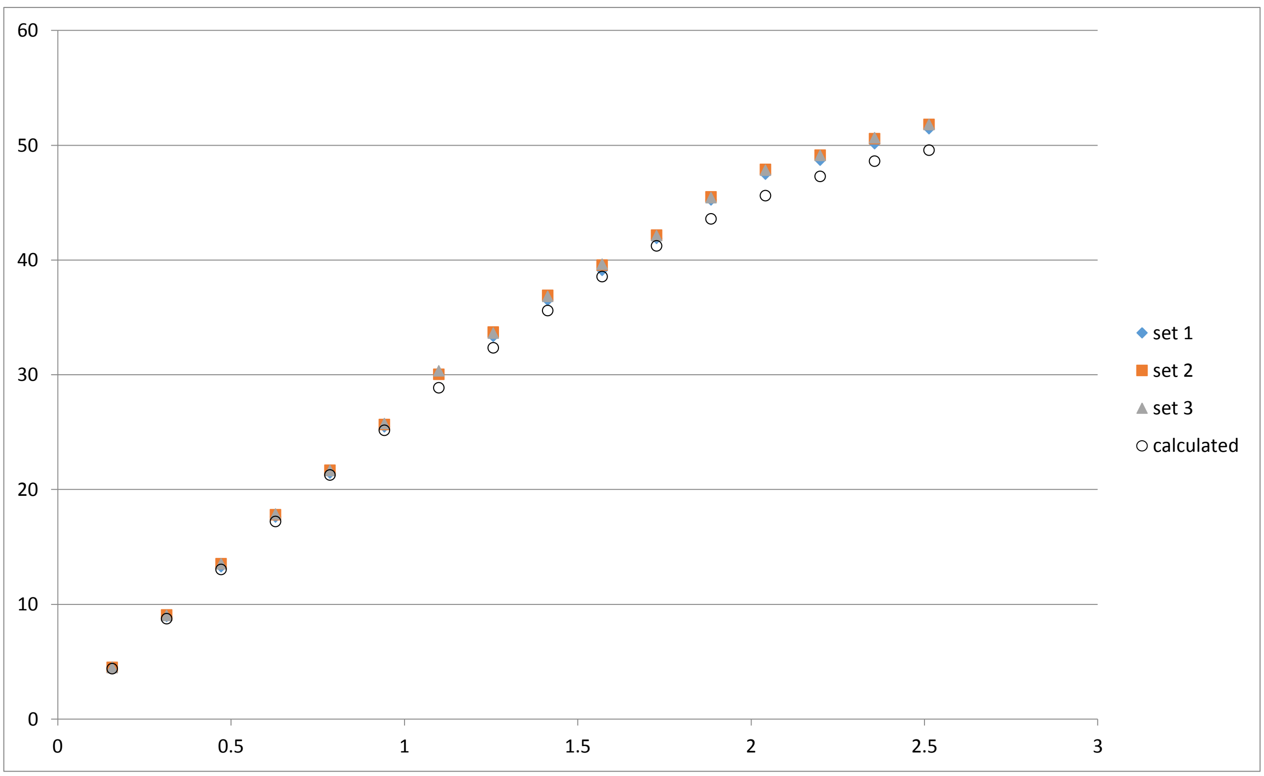
1. Connect the outputs to the Oscilloscope and set to the XY mode.
2. Put both the switches in the direction of the mono-atomic label.
3. Set to the low frequency mode (the apparatus)
4. Set the amplitude to be roughly the maximum
5. Using the knobs R_1 and R_2 and by adjusting the frequency, we can adjust the X and Y amplitude such that for $\phi = \pi/2$ we roughly get a circle (definition of ϕ is given in the previous section)
   (a) It is expected that changing the frequency will change the axis of the ellipse (of which circle will be a special case) and finally into a line and so on.
   (b) Changing R_1 and R_2 shouldn't change the axis, but infact it was observed that changing R_1 does change the frequency
6. Change the frequency to change between consecutive figures of the form \ O / O and so on.
7. This must be repeated till the frequency abruptly drops to zero for a reasonable range of the frequency.
8. Repeat the procedure with both switches in the direction of the di-atomic (although it is misleading strictly speaking) label.

4 Observations

4.1 Mono-atomic

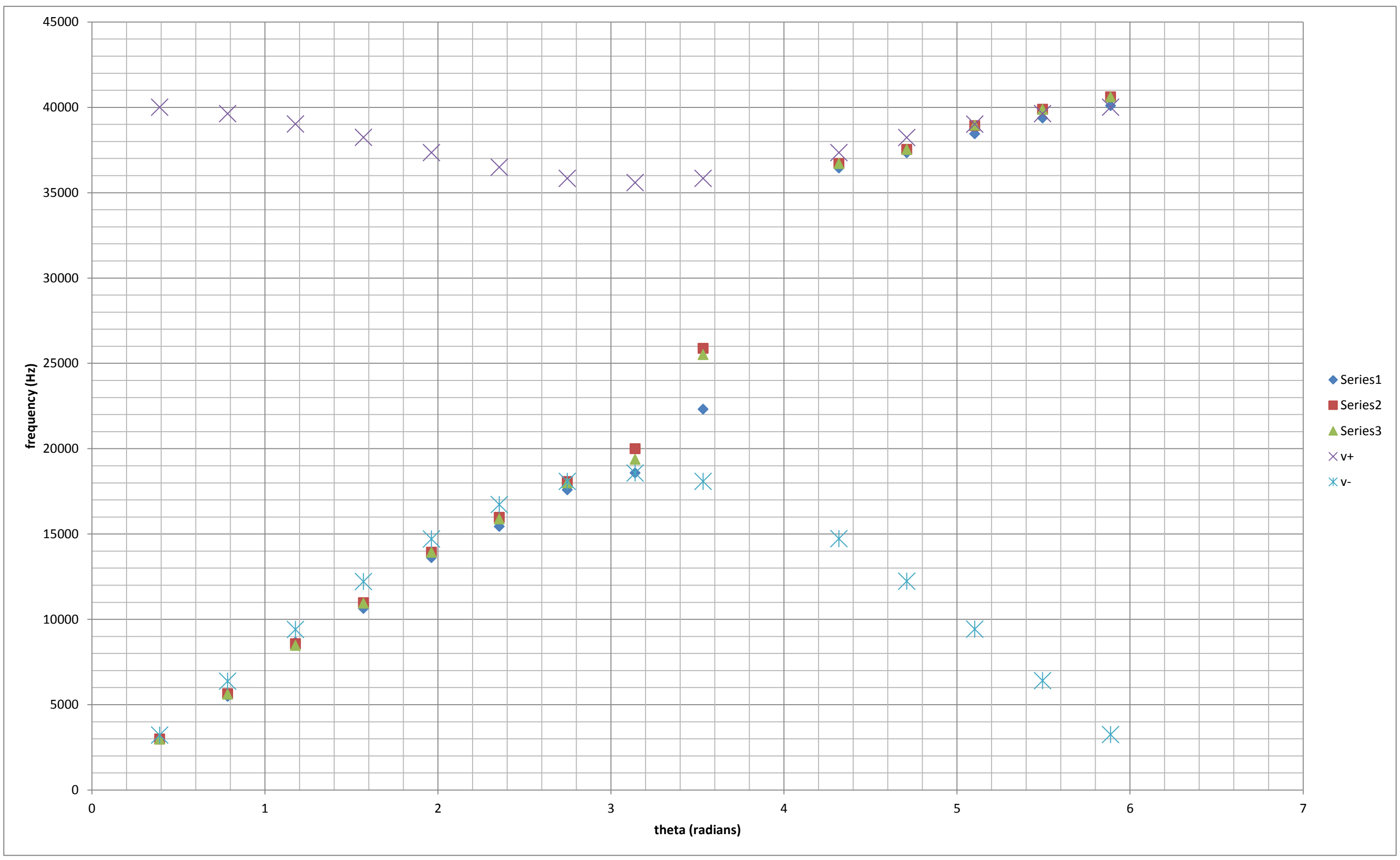
The graph has been plotted for frequency (kHz) vs. ϕ (radians)

CRO pattern	Ne	Degree	Radians	Freq(set1)	Freq(set2)	Freq(set3)	2/LC*(1-cos)	sqrt	
/	0	0	0						
O	90	9	0.1570796	4.517	4.508	4.49	759.6123494	27.56106582	4.386479863
\	180	18	0.3141593	8.931	9.068	9.025	3015.368961	54.9123753	8.739575965
O	270	27	0.4712389	13.275	13.543	13.543	6698.729811	81.84576844	13.02615862
/	360	36	0.6283185	17.601	17.799	17.861	11697.77784	108.1562659	17.21360434
O	450	45	0.7853982	21.438	21.685	21.648	17860.61952	133.6436288	21.27004413
\	540	54	0.9424778	25.475	25.666	25.751	25000	158.113883	25.16460605
O	630	63	1.0995574	30.083	30.035	30.345	32898.99283	181.3807951	28.86765012
/	720	72	1.2566371	33.342	33.711	33.634	41317.59112	203.2672898	32.35099395
O	810	81	1.4137167	36.503	36.911	36.83	50000	223.6067977	35.58812717
\	900	90	1.5707963	39.104	39.552	39.639	58682.40888	242.2445229	38.55441326
O	990	99	1.727876	41.887	42.17	42.135	67101.00717	259.038621	41.22727698
/	1080	108	1.8849556	45.247	45.506	45.442	75000	273.8612788	43.58637623
O	1170	117	2.0420352	47.482	47.897	47.835	82139.38048	286.5996868	45.61375685
\	1260	126	2.1991149	48.699	49.138	49.116	88302.22216	297.1568982	47.29398922
O	1350	135	2.3561945	50.144	50.581	50.649	93301.27019	305.4525662	48.61428579
/	1440	144	2.5132741	51.445	51.816	52	96984.63104	311.4235557	49.56459828



4.2 'Di-atomic'

The graph has been plotted for frequency (Hz) vs. ϕ (radians) [although at that point, we were calling it θ]



45000
40000
35000
30000
25000
20000
15000
10000
5000
0
frequency (Hz)
0
1
2
3
4
5
6
7
theta (radians)
Series1
Series2
Series3
v+
v-

theta	c1	C2	2*one by c1*one byc2*costheta	second term
0.3925	0.000000147	0.00000004		
0.785	one by c1	one by c2	3.14271E+14	31393433394
1.1775	6802721.088	25000000	2.40608E+14	30197438120
1.57	one by c1 square	one by c2 square	1.30352E+14	28313055379
1.9625	4.6277E+13	6.25E+14	2.70859E+11	25914240750
2.355	L	one by L	-1.29852E+14	23268550616
2.7475	0.001	1000	-2.40225E+14	20761790986
3.14	first term		-3.14063E+14	18900105128
3.5325	31802721088		-3.40136E+14	18197290765
4.3175			-3.14477E+14	18889138667
4.71			-1.30852E+14	23247037014
5.1025			-8.12578E+11	25893328033
5.495			1.29351E+14	28295365823
5.8875			2.39841E+14	30184735357
			3.13855E+14	31386806978
			3.40136E+14	31802721088

theta	set 1	set2	set 3	w+	w-	v+(hz)	v-
0.3925	2968	2989	2978				
0.785	5484	5639	5614	251388.5	20230.85994	40009.71499	3219.841363
1.1775	8657	8572	8473	248998.3	40065.98268	39629.31212	6376.699194
1.57	10634	10974	10940	245185.2	59073.39257	39022.43466	9401.822432
1.9625	13615	13936	13921	240243.5	76736.43423	38235.94804	12212.98282
2.355	15440	15981	15879	234672.7	92380.57411	37349.31871	14702.82502
2.7475	17585	18068	17992	229269.5	105075.8302	36489.37716	16723.33778
3.14	18582	20002	19372	225172.9	113589.6825	35837.37702	18078.35945
3.5325	22312	25881	25512	223606.8	116642.3179	35588.13139	18564.20147
4.3175	36443	36711	36695	225148.5	113637.9445	35833.50119	18086.04058
4.71	37338	37533	37521	234626.8	92496.94089	37342.02274	14721.34536
5.1025	38446	38922	38952	240200	76872.57675	38229.02035	12234.65058
5.495	39368	39882	39879	245149.1	59222.92854	39016.6929	9425.621822
5.8875	40096	40614	40615	248972.8	40224.19336	39625.25223	6401.879205
		41487	41530	251375.3	20393.97242	40007.61733	3245.801519
				252201.2	0	40139.06665	0

4.3 Time Summary

Jan 12	Monday	Finished the record for the previous experiment | started reading this experiment's theory
Jan 13	Tuesday	Working on the mono-atomic part | worked out the theory and performed the experiment
Jan 16	Friday	Started the di-atomic case observations | trying to understand the cause of upper limit, attempted decaying oscillator solution (Landau) | figured division by 9 instead of 10 must be done | started work on the di-atomic
Jan 19	Monday	[unwell]
Jan 20	Tuesday	Working on the di-atomic part | figured why the solution must be bounded in mono-atomic, started analyzing the theory for di-atomic
Jan 23	Friday	Had an embarrasing moment | signs for capacitor discharge were messed up | confirmed the theoretical solution by plotting it against the observations | results didn't match though | figured that the division had to be done by 4
Jan 26	Monday	[Republic Day]
Jan 27	Tuesday	[Prashansa fell | Hospital]
Jan 30	Friday	Tried to answer some theoretical questions raised by comparing results of the manual and our analysis | took two more sets of readings for both the mono-atomic and di-atomic case
Feb 2	Monday	Started writing the record in all its glory!
Feb 3	Tuesday	[absent | accidentally overslept]
Feb 6	Friday	[exams]
Feb 9	Monday	[half lab: Helping Kishor with his interview in Europe] attempted reading the next experiment
Feb 10	Tuesday	Completed the record | started performing the next experiment | red LED observations taken (by Sagar)

5 Result

Both the mono-atmoic and 'di-atomic' (read the next section) yielded results that matched theoretical predictions.

6 Explorable Questions

1. What is the spring mass analogue when we replace $L \leftrightarrow C$
2. How is it that $\omega(k)$ found in the more general case, is identical if in one of the following situations, we apply $L_x \leftrightarrow C_x$
   (a) $L_1 \neq L_2$ and $C = C_1 = C_2$
   (b) $L = L_1 = L_2$ and $C_1 \neq C_2$

These questions relate the physical case of di-atomic interactions[1] with the physical situation created in the apparatus. They are mathematically distinct in terms of dynamics viz. you can't relate L and Cs with ms and γ (spring constant); but for the quantity ω we can find such a relation.

7 Critique

The manual provided was found to be mostly misleading. However it did motivate some of the problems listed. The procedure however was found to be mostly accurate.

The most important theoretical improvement that can be made is the realization that our physical system is not properly modeled by the differential equations we wrote down. We have accordingly motivated the use modified formulae for obtaining ϕ as stated which gives much more accurate results experimentally than quoted in the manual. We were also able to account for the various ranges where ω has no corresponding k which was confirmed experimentally.

No experimental improvements could be made/suggested as the apparatus is essentially an inaccessible black box.

8 References and Acknowledgements

We used

- http://www.digikey.com/schemeit# for creating the schematics
- Aschroft and Mermin's book on Solid State Physics for looking at some results
- Landau for looking at damped oscillators

I acknowledge the contribution of my team members, Vivek and Prashansa who worked diligently and contributed in a fundamental way to the understanding of the theory and performance of the experiment.

[1] See for instance Classical Harmonic Theory, Solid State Physics, Ashcroft and Mermin