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LECTURE SERIES

Classical Theory of Constrained Systems

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Contents

Chapter 1

Introduction and Overview - from Lagrangian to Hamiltonian Dynamics - regular versus singular cases

The basics of classical analytical dynamics in Lagrangian and Hamiltonian forms, and the passage between them, will be assumed known. Now we recall the main features to establish notations and terminology. As we proceed, many statements, etc. will be described carefully leaving the proofs as exercises.

Given an N-dimensional configuration space Q, with (local) generalized coordinates q^j, j = 1,2,...,N for some system. For Lagrangian mechanics the complete set of 2N variables are q^j and $\dot{q^j} = \frac{d}{dt}q^j$. These are (local) coordinates of the 2n-dimensional space TQ, the tangent bundle on Q (Q$\rightarrow$TQ). Given a Lagrangian L$(q, \dot{q})$, omitting explicit time dependence just for simplicity, the Hessian matrix is defined as a real symmetric NXN matrix with elements:

$$H_{jk}(q,\dot{q}) = \frac{\partial^2 L(q,\dot{q})}{\partial\dot{q}^j\partial\dot{q}^k}, j,k = 1,2,...,N \tag{1.1}$$

Then the regular or standard case is defined as

$$det(H_{jk}(q,\dot{q})) \neq 0 \tag{1.2}$$

$(H_{jk}(q,\dot{q}))$ is non-singular.

This is motivated by the form of the kinetic energy for non-relativistic mechanical systems. First immediate consequence: from the Lagrangian equations of motion (EOM)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^j} - \frac{\partial L}{\partial q^j} = 0, j = 1, 2, ..., N, \tag{1.3}$$

we can solve for all N accelerations:

$$\ddot{q}^j = f^j(q, \dot{q}), j = 1, 2, ..., N \tag{1.4}$$

forming a set of N second order ODE's in time. Second consequence: the usual way in which canonical momenta p_j are introduced is by a system ϕ_L of defining equations

$$\phi_L : p_j = \frac{\partial L}{\partial \dot{q}^j}, j = 1, 2, ..., N \tag{1.5}$$

Then, when (1.2) is valid, we can 'invert' (1.5) and express each $\dot{q}^j$ as some function of q's and p's, (Implicit function theorem) A better way to view the system (1.5) is (Figure 1.1):

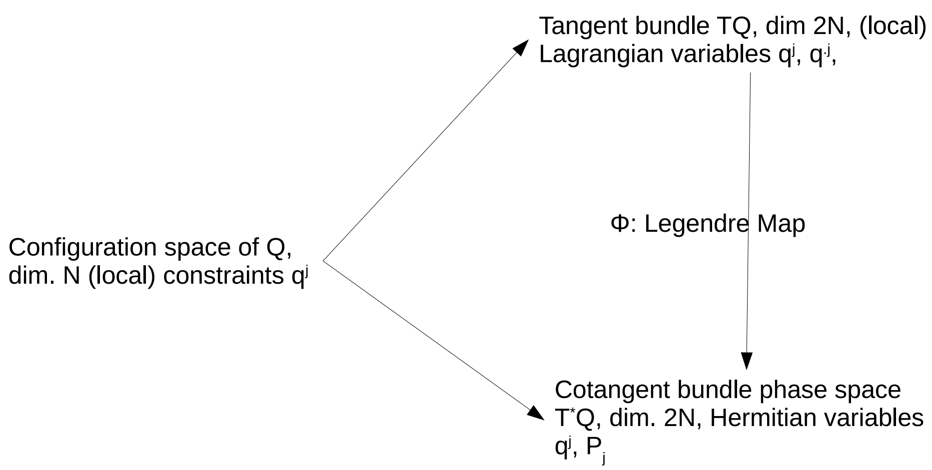

Figure 1.1

Then in the standard case:

${\phi_L}^{-1}$ exists; ϕ_L = one-to-one onto map TQ $\rightarrow T^*Q$; each $\dot{q}^j$ = some function of q's and p's. So any

$$f(q,\dot{q}) = Somef'(q,p) \tag{1.6}$$

The Hamiltonian, a function in phase space T^*Q, is defined by the Legendre transformation:

$$H(q,p) = p_j\dot{q}_j - L(q,\dot{q}), \tag{1.7}$$

and then the Lagrangian EOM(1.3) appears in Hamiltonian form as 2N first order ODE's in time:

$$\dot{q}^j = \{q^j, H\} = \frac{\partial H(q,p)}{\partial p_j}, j = 1,2,...,N; \tag{1.8}$$

$$\dot{p}_j = \{p_j, H\} = -\frac{\partial H(q,p)}{\partial q^j}, j = 1,2,...,N; \tag{1.9}$$

So,

$$\frac{df(q,p,t)}{dt} = \{f(q,p,t), H(q,p)\} + \frac{\partial f(q,p,t)}{\partial t} \tag{1.10}$$

in general.

Here the Poisson Bracket, PB, among phase space functions is used:

$$\{f(q,p), g(q,p)\} = \frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q^j} \tag{1.11}$$

This has the properties of linearity, anti-symmetry, and Jacobi Identity.

In principle, in the two version of the EOM, the solutions are:

$$q(0), \dot{q(0)} \rightarrow q(t)\forall t, \tag{1.12}$$

trajectory in Q, so also in TQ;

$$q(0), p(0) \rightarrow q(t), p(t)\forall t, \tag{1.13}$$

trajectory in T^*Q.

Now we define the singular case as corresponding to

$$det(H_{jk}(q,\dot{q})) = 0, \tag{1.14}$$

$(H_{jk}(q,\dot{q}))$ = singular matrix.

Many important physical cases come in this category. The first systematic study of such systems was presented by Dirac in lectures at "First Canadian Mathematical Congress" in 1949 at the UBC in Vancouver, then published in the CJM in 1950. He later gave the famous Yeshiva Lectures in 1964. The subject is called Generalized Hamiltonian Dynamics, sometimes Constrained Hamiltonian Dynamics. There had been some other attempts, notably by P.G. Bergmann, but it was Dirac who set up a comprehensive framework.

Compared to the standard case, many modifications are needed. Before going into the details, an overview is useful, so that we have an idea of what lies ahead:

1. The Lagrangian EOM (1.3) are unable to yield expressions for all the accelerations $\ddot{q}^j$. Instead, they lead in general to some 'constraint' relations among q^j, $\dot{q}^j$ which have to be analysed.

2. For the same reason, the Legendre map ϕ_L: TQ $\rightarrow T^*$Q becomes non-invertible, it is a many to one map. Its range is part of T^*Q, rather than all of it. This part is determined by a set of (primary) constraints, relations among the q^j and p_j, so these are determined by ϕ_L.

3. Based on a study of the properties of ϕ_L, the Lagrangian EOM (1.3) can be cast into an initial Hamiltonian form. This involves a starting Hamiltonian, the (primary) constraints of (point 2) above, and some non phase space variables.

4. In contrast to the regular case, now one has to go through a step by step consistency analysis of the initial Hamiltonian EOM and initial constraints. The final form of the general EOM is reached only at the end of this analysis.

In principle one can give a purely Lagrangian treatment of singular systems. The Dirac theory however works with the Hamiltonian or phase space framework from the beginning. The entire consistency analysis is done in phase space. The main factors are:

1. It permits a systematic analysis of the process of building up of constraints on the q's and p's, while always maintaining the PB form of the EOM.

2. It motivates the separation of constraints into two different types, the first class and the second class, and brings out their differences.

3. The ultimate reduction in the number of independent variables or phase space degrees of freedom is achieved by replacing PB's by Dirac Brackets or DBs.

All these new features in the handling of the EOM have repercussions on the description of symmetry transformations, and their associated constants of motion (COM) which act as their phase space generators.

Now we go on to describing Dirac's analysis of the dynamics, i.e., the EOM. Then we look at the consequences of symmetries in the sense of Noether's theorem. As we proceed we will make many simplifying assumptions so that a clear overall picture emerges. In practice each singular system is singular in its own way, with its own specific features, they are best handled on a case-by-case basis.

Chapter 2

Singular Case dynamics in phase space form - Consistency Analysis

This is a long chapter with several parts.

2.1 Structure of Legendre map ϕ_L in singular case

The rank of the Hessian matrix in (1.1) is less than N, so let:

$$rank(H_{jk}(q,\dot{q}) = \frac{\partial L^2(q,\dot{q})}{\partial \dot{q}^j \partial \dot{q}^k} = K, 0 < K < N \tag{2.1}$$

Then we find on the basis of equation (1.5) defining the map ϕ_L:

1. Only a subset of K canonical momenta, along with the N q's, from (N+K) independent variables 'in phase space'.
2. The remaining (N-K) momenta are functions of the above (N+K) independent q's and p's, obtained by eliminating the $\dot{q}^j$ in equation (1.5).
3. In trying to 'invent' equation (1.5), we find that we can express only K of the $\dot{q}$'s in terms of the q's , p's and the remaining (N-K) $\dot{q}$'s.

To grasp these features better, let us depict the situation in this way (Table 2.1).

Clearly, number of independent(dependent) momenta = number of solved(unsolved) velocities. Entries (iii) and (iv) are appearing in the singular case. We can also say,

TABLE 2.1

Content of $\phi_L : p_j = \frac{\partial L}{\partial \dot{q}^j}$	**Properties of momenta**	**Properties of Velocities**
	(i) K independent ones	(ii) K solved ones.
	(iii) (N-K) dependent functions of K independent ones are N q's	(iv) (N-K) unsolved

the map ϕ_L: TQ $\rightarrow T^*$Q is many to one, while the domain TQ is of dimension 2N, the range within T^*Q is of lower dimension (N+K), a region Σ contained within T^*Q:

$$\phi_L : (q, \dot{q})\epsilon TQ \rightarrow (q, p = \frac{\partial L}{\partial \dot{q}})\epsilon\Sigma \subset T^*Q, \tag{2.2}$$

dim TQ = dim T^*Q = 2N, dim Σ = N+K $<$ 2N.

This 'sub-manifold' $\Sigma \subset T^*$Q is defined by the relations (iii) of Table 2.1, written in a flexible notation as follows:

$$\Sigma = \{(q, p)\epsilon T^*Q|\phi_m(q, p) = 0, m = 1, 2, ..., (N - K)\} \subset T^*Q \tag{2.3}$$

Points in T^*Q, outside Σ are <u>not</u> obtained as images v of any (q, $\dot{q}$) in TQ; given any (q,p) $\epsilon\Sigma$, the 'pre-image' ${\phi_L}^{-1}$(q,p), all (q,$\dot{q}$) mapped by ϕ_L into (q,p), is a region in TQ of dimension (N-K); this region is parametrized by the unsolved velocities (iv) of (Table 2.1). Of course, the functions ϕ_m(q,p) in equation (2.3) are algebraically independent.

To get a feeling for these features of the singular case, it may help to look at the simple case of, L(q,$\dot{q}$) at most quadratic in velocities:

$$L(q, \dot{q}) = \frac{1}{2}A_{jk}\dot{q}^j\dot{q}^k + \text{lower order terms}, \tag{2.4}$$

$$A = (A_{jk}) = \text{NXN symmetric numerical matrix} \tag{2.5}$$

So, the Hessian matrix is A.

So, referring to the (Table 2.1), for those degrees of freedom for which the momenta are independent, the velocities are solved; for those whose velocities are dependent, the velocities are unsolved.

2.2 The EOM in phase space - initial version

Let f(q,$\dot{q}$) be a function of the 2N independent Lagrangian variables. We are in general unable to express it entirely in terms of phase space variables. We can at best express it as a function of K independent momenta, N-K unsolved velocities, and all the q's. There is some freedom here in the way we choose independent momenta and unsolved velocities.

For the Legendre transform of L(q,$\dot{q}$), however, we find simplifications. As in the standard case, we form the expression

$$p_j\dot{q}^j - L(q,\dot{q}) \tag{2.6}$$

We find that on passing to T^*Q, this can be expressed as a function of q's and independent p's alone. That is, for each point (q,p) $\epsilon\Sigma$, the expression (2.6) has the same value at all points in ${\phi_L}^{-1}$(q,p) $\subset$ TQ. Thus we define the (initial) Hamiltonian as:

$$H_o(q,p) = p_j\dot{q}^j - L(q,\dot{q}), \tag{2.7}$$

realizing that due to the freedom in the choice of the independent momenta, the functional form of H_0 is not unique. In addition, we need to work with an expression for H_o valid not only on Σ but also in an immediate neighbourhood of it. This will allow us to form PB's of H_o with other phase space quantities, treating the q^j and p_j as all independent, and only in the end limit them to Σ. Dirac introduces the 'weak equality' sign $\approx$ for an equation valid on Σ but not away from it. A strong equality $=$ is valid even away from Σ. In this sense, the freedom in the form of H_o(q,p) amounts to being allowed to add any linear combination of the primary constraints ϕ_m(q,p) to it, with coefficients being functions of q's and p's. These are exemplified by these statements:

Definition of Σ: ϕ_m(q,p) ≈ 0;

General initial Hamiltonian $=$ H_o(q,p) $+$ (q,p)-dependent linear combination of ϕ_m(q,p).

Now we develop the phase space form of the Lagrangian EOM (1.3) for a singular system. In the regular case, the 'genuine' EOM for $\dot{p}_j$ in (1.10) arose from the Lagrangian EOM by using the relation:

$$\frac{\partial L(q,\dot{q})}{\partial q^j} = -\frac{\partial H(q,p)}{\partial q^j} \tag{2.8}$$

which is a consequence of the Legendre transformation (1.7). Now the replacements for this relation and the $\dot{q}^j$ EOM in (1.10) are obtained by varying the q's and p's independently in (2.7), while taking account of the primary constraints by the use of Lagrange multipliers. This gives:

$$\dot{q}^j = \frac{\partial H_o}{\partial p^j} + V_m\frac{\partial \phi_m}{\partial p_j} \approx \{q^j, H_o + V_m\phi_m\}, \tag{2.9}$$

$$\frac{\partial L}{\partial q^j} = -\frac{\partial H_o}{\partial q^j} - V_m \frac{\partial \phi_m}{\partial q^j} \approx \{p_j, H_o + V_m \phi_m\}, j = 1, 2, ..., N \tag{2.10}$$

The Lagrange multipliers are intentionally written as V_m - they are essentially the unsolved velocities, so to that extent equations (2.9) are 'empty'. In any case this part of (2.13) shows the extent to which the Legendre map ϕ_L has been inverted. Further, since the V_m are not (as yet) phase space expressions, their PB's are undefined, but the $\approx$ sign saves us from trouble!

The full set of (initial) Hamiltonian EOM for a singular system, plus the primary constraints, is:

$$\dot{q}^j \approx \{q^j, H_o + V_m \phi_m\}, \tag{2.11}$$

$$\dot{p}_j = \{p_j, H_o + V_m \phi_m\}, \phi_m \approx 0, \tag{2.12}$$

We have also the general EOM for any f(q,p,t):

$$\frac{df(q,p,t)}{dt} \approx \{f, H_o + V_m \phi_m\} + \frac{\partial f}{\partial t} \tag{2.13}$$

2.3 Consistency analysis

If a point (q,p) $\epsilon\Sigma$ is chosen at time t, then at time t + δt, the changes δq^j, δp_j are given by equations (2.11) and (2.12):

$$\delta q^j \approx \{q^j, H_o + V_m \phi_m\}\delta t \tag{2.14}$$

$$\delta p_j \approx \{p_j, H_o + V_m \phi_m\}\delta t \tag{2.15}$$

For consistency, we must demand that (q + δq, p + δp) $\epsilon\Sigma$ as well, is that ϕ_m(q + δq, p + δp) must vanish. From equation (2.13) this means:

$$\delta \phi_m \approx \{\phi_m, H_o\}\delta t + \{\phi_m, \phi_{m'}\} V_m, \delta t \approx 0, \forall m \tag{2.16}$$

The $\approx$ sign means that these must be obeyed modulo ϕ_m(q,p) ≈ 0, $\forall$ m. These conditions can lead to more constraints on the q's and p's , and or some restrictions on the V_m, to be obeyed at any time t.

To orient ourselves, we first look at two extreme and contrasting cases in each of which the consistency analysis ends already at this stage. The reasons will be very different. Later we look at the general case which lies between these extremes.

2.3.1 Case(i) - First Class Case

This is when H_o and ϕ_m are such that all the PB's in (2.16) vanish over Σ:

$$\{\phi_m, H_o\} \approx \{\phi_m, \phi_{m'}\} \approx 0, \text{modulo } \phi_m \approx 0 \tag{2.17}$$

Then the conditions on (2.16) are satisfied. Notice that this is consistent with the known fact that H_o is determined only upto a linear combination of the ϕ_m. Equations (2.11) and (2.12) are already the final form of the dynamics, the V_m remain undetermined and must be chosen in some way if we wish to look for definite solutions of the EOM. Phase space trajectories of the system stay always within Σ.

In this extreme case, there are no new constraints, and both ϕ_m and H_o are called first class functions.

2.3.2 Case(ii) - Second Class Case

In this opposite case, the matrix $(\{\phi_m, \phi_{m'}\})$ is non-singular over Σ. Then, (N-K) must be even, and the consistency condition (2.16) determines all the V_m as phase space functions. Once again the consistency analysis has ended at equation (2.16), and the solutions for V_m can be put into the EOM (2.11,12,13) to give their final forms. There are no new constraints beyond the primary ones, which form a self-perpetuating system, and no unknown velocities either. The constraints $\phi_m \approx$ are called a second class set, and (as we see later) the EOM (2.11,12,13) lead to the Dirac Bracket (DB).

The general situation lies between these extremes. Eq.(2.16) can lead to two kinds of consequences:

1. New constraints independent of the primary ones, written as:

   $$\chi_{..}(q,p) \approx 0 \tag{2.18}$$

   These are called secondary constraints, they limit the physically accessible region of T^*Q to some Σ' smaller than Σ:

   $$\Sigma' = \{(q,p)\epsilon T^*Q | \phi_m(q,p) \approx \chi_{..}(q,p) \approx 0\} \subset \Sigma \subset T^*Q \tag{2.19}$$

   Here onwards, weak equations refer to Σ' rather than to Σ.

2. Some V_m may be determined in terms of q, p and the other V_m, as linear inhomogeneous expressions. The number so determined is the rank of $(\{\phi_m, \phi_{m'}\})$ on

Σ'. For simplicity we may suppose that rank($\{\phi_m, \phi_{m'}\}$) is constant over Σ, and remains the same under further restriction to $\Sigma' \subset \Sigma$.

Once secondary constraints have appeared, the next step is to add the requirements:

$$\{\chi.., H_o\} + \{\chi.., \phi_{m'}\}V_{m'} \approx 0 \text{modulo } \phi_m \approx \chi.. \approx 0 \tag{2.20}$$

These can again lead to two kind of consequences - more (tertiary) constraints, further determination of the V_m - and so on.

When and how do we recognize that the consistency analysis has come to an end? Clearly when no new constraints are generated and no new determinations of the V_m arise. We decide their next.

2.4 Conclusion of analysis - final form of EOM and constraints

Denote the secondary, tertiary,... constraints generically by χ..(q,p) ≈ 0. Let the finally undetermined (unknown) velocities be V_A, some subset of the original V_m. Let the partial determinations of the V_m be written as:

$$V_m = {V_m}^{(0)} + V_A C_{Am}, \tag{2.21}$$

where ${V_m}^{(0)}$, C_{Am} are phase space functions. (So some of these equations are empty). Then the initial combination $H_o + V_m\phi_m$ becomes:

$$H_o + V_m\phi_m = H_o + V_{m(0)}\phi_m + V_A C_{Am}\phi_m = H + V_A\phi_A, \tag{2.22}$$

$$H = H_o + {V_m}^{(0)}\phi_m, \phi_A = C_{AM}\phi_m \tag{2.23}$$

The EOM and constraints are now:

$$\frac{df(q,p,t)}{dt} \approx \frac{\partial f}{\partial t} + \{f, H\} + V_A\{f, \phi_A\}, \tag{2.24}$$

$$\phi_m(q,p) \approx 0, \chi..(q,p) \approx 0 \tag{2.25}$$

The corresponding sub-manifold in phase space is:

$$\Sigma_f = \{(q,p)\epsilon T^*Q | \phi_m(q,p) \approx \chi..(q,p) \approx 0\} \subset ... \subset \Sigma' \subset \Sigma \subset T^*Q, \tag{2.26}$$

and weak equality now refers to Σ_f.

The signs that the analysis has concluded are that the conditions:

$$\{\phi_m, H\} + V_A\{\phi_m, \phi_A\} \approx 0, \tag{2.27}$$

$$\{\chi.., H\} + V_A\{\chi.., \phi_A\} \approx 0, \tag{2.28}$$

are obeyed over Σ_f. This means that we have:

$$\{\phi_m \text{or } \chi.., H \text{or } \phi_A\} \approx 0 \text{modulo } \phi_m \approx \chi.. \approx 0 \tag{2.29}$$

A final Hamiltonian H, and special linear combinations ϕ_A of ϕ_m, have appeared. Any f(q,p) is said to be first class if its PB's with all constraints vanishes over Σ_f:

$$f(q,p) \text{first class } <=> \{f, \phi_m \text{or } \chi..\} \approx 0 \text{modulo } \phi_m \approx \chi.. \approx 0 \tag{2.30}$$

Otherwise it is second class. So both H and ϕ_A are first class - the latter are a maximal number of primary first class constraints, and each one comes with one finally undetermined V_A in the final EOM.

How do the two extreme cases (i), (ii) fit into the generalized picture? In both, the only constraints are the primary ones, there are no χ's, so $\Sigma_f = \Sigma$.

In Case (i), we find:

$$H_o, \phi_m \text{first class; } H = H_o; V_A = \text{all } V_m \tag{2.31}$$

The general solution to the EOM always lies in Σ, and involves (N-K) arbitrary (gauge like) functions of time.

In Case (ii) on the other hand, the ϕ_m form a second class set. So (N-K) is even; there are no ϕ_A and no V_A. We have:

$$C_{mm'}\{\phi_{m'}, \phi_{m''}\} \approx \delta_{mm''}; \tag{2.32}$$

$$V_m \approx -C_{mm'}\{\phi_{m'}, H_o\}; \tag{2.33}$$

$$H = H_o - \phi_m C_{mm'}\{\phi_{m'}, H_o\} \tag{2.34}$$

The final EOM is:

$$\frac{df}{dt} \approx \frac{\partial f}{\partial t} + \{f, H_o\} - \{f, \phi_m\} C_{mm'}\{\phi_{m'}, H_o\}, \tag{2.35}$$

with no arbitrary terms, We will see that we have an instance of the DB.

2.5 The Dirac Bracket

The difference between first class functions and second class functions can be brought out nicely. The infinitesimal canonical transformation (CT) generated by any f(q,p) is given by:

$$\delta q^j = \varepsilon\{q^j, f\}, \delta p_j = \varepsilon\{p_j, f\}, |\varepsilon| << 1; \tag{2.36}$$

$$\delta g(q,p) = \varepsilon\{g, f\}, \text{any } g(q,p) \tag{2.37}$$

So if f is first class, this CT preserves the region $\Sigma_f \subset T^*$Q in eq. (2.26):

$$\delta\phi_m(q,p) \approx \delta\chi_{..}(q,p) \approx 0; \tag{2.38}$$

$$q, p\epsilon\Sigma_f \to q + \delta q, p + \delta p\epsilon\Sigma_f \tag{2.39}$$

But for second class f, points on Σ_f generally move out of Σ_f.

Now we are free to replace the ϕ_m by non-singular linear combinations of themselves; while $\chi_{..}$ can be replaced by non-singular linear combinations of themselves plus linear combinations of ϕ_m; in all these, functions of q,p can be used as coefficients. Using these freedoms, we extract maximum numbers of first class constraints in each set:

Primary $\phi_m \to$ primary first class ϕ_A, remaining second class ϕ's;

Secondary $\chi_{..} \to$ first class $\chi_{..}$'s, remaining second class χ's.

Only the ϕ_A appear in the final EOM (2.24,25) with undetermined coefficients V_A.

Let us combine all the independent second class ϕ's and χ's into one set $\{\theta_a(q,p)\}$. It is easy to see that over Σ_f:

$$det(\{\theta_a, \theta_b\}) \neq 0, \tag{2.40}$$

so the number of θ's must be even. With this rearrangement of the constraints we express the final EOM (2.24, 25) and constraints in this way:

$$\frac{df}{dt} \approx \frac{\partial f}{\partial t} + \{f, H\} + V_A\{f, \phi_A\}; \tag{2.41}$$

$\phi_A \approx 0$, independent first class $\chi_{..}$'s ≈ 0; $\theta_a \approx 0$. As H and ϕ_A are first class, we have in particular:

$$\{H \text{or } \phi_A, \theta_a\} \approx 0 \tag{2.42}$$

We now replace the PB's in (2.41) by a new structure, the DB: it allows us to convert the constraints $\theta_a \approx 0$ into algebraic equations and eliminate variables. For any two f(q,p), g(q,p) the DB is defined thus:

$$\{f, g\}^* = \{f, g\} - \{f, \theta_a\}C^{ab}\{\theta_b, g\}, \tag{2.43}$$

$$C^{ab}\{\theta_b, \theta_c\} = \delta_c{}^a \tag{2.44}$$

(No weak equations here!) We recognise now that in eq. (2.35) we have the DB of f and H_o with respect to the second class ϕ_m in that situation. Apart from linearity in f and in g, and antisymmetry upon interchange (as for PB's) we have the key properties:

$$\{f, gh\}^* = g\{f, h\}^* + \{f, g\}^*h; \tag{2.45}$$

$$\{f, \{g, h\}^*\}^* + \{g, \{h, f\}^*\}^* + \{h, \{f, g\}^*\}^* = 0; \tag{2.46}$$

$$\{f, \theta_a\}^* = 0 \tag{2.47}$$

These are all 'strong equations'. Then the EOM (2.41) become translated to (2.42):

$$\frac{df}{dt} \approx \frac{\partial f}{\partial t} + \{f, H\}^* + V_A\{f, \phi_A\}^*, \tag{2.48}$$

$\phi_A \approx 0$, independent first class χ..'s ≈ 0.

2.6 Gauge constraints

Two general points remain to be made.

1. In the final EOM (2.24, 25) only the primary first class constraints ϕ_A appear, each multiplied by one unknown velocity. Dirac has suggested adding more terms, one for each secondary... first class constraint accompanied by one 'unknown' coefficient like the V_A. This restores symmetry between primary and secondary constraints but certainly goes beyond the originally given singular Lagrangian dynamics.

2. Returning to eq.(2.24,25), one way to determine the V_A is to impose an equal number of 'external' gauge fixing constraints. ξ_A(q,p) ≈ 0 such that their being maintained in time fixes the V_A. Thus,

$$\xi_A \approx 0, \frac{d\xi_A}{dt} \approx 0 \Rightarrow \{\xi_A, H\} + \{\xi_A, \phi_B\}V_B \approx 0, \tag{2.49}$$

and our aim will be realized if

$$det(\{\xi_A, \phi_B\}) \neq 0 \tag{2.50}$$

These constraints are external to the Lagrangian, and the total set (ϕ_A, ξ_A) form a second class set in the sense of eq. (2.35). Then in a recursive manner one can pass to the DB with respect to this set, after having set up the DB with respect to the θ_a.

Chapter 3

Symmetry transformations and their generators

3.1 Standard Lagrangian - Noether and Non-Noether Cases

In the case of a standard Lagrangian, we recall that we have two kinds of symmetry transformations: the Noether (N) type, and the Non-Noether (Non-N) type. The significant results are in summary:

3.1.1 N-type symmetry

$$\delta q^j = \varepsilon \phi^j(q) \Rightarrow \delta L(q, \dot{q}) = \varepsilon \frac{dF(q)}{dt}, |\varepsilon| << 1; \tag{3.1}$$

$$G(q,p) = F(q) - p_j \phi^j(q) = \text{linear in p, Constant of. motion(COM)}, \tag{3.2}$$

$$\frac{dG}{dt} = \{G, H\} = 0; \tag{3.3}$$

$$\delta q^j = \varepsilon \{G(q,p), q^j\} \text{at } \phi_L \text{level, no use of EOM}, \tag{3.4}$$

$$\delta p_j = \varepsilon \{G(q,p), p_j\} \text{at } \phi_L \text{level, no use of EOM}, \tag{3.5}$$

$$\delta f(q,p) = \varepsilon \{G(q,p), f(q,p)\} \text{in general, no use of EOM}, \tag{3.6}$$

The functions ϕ^j(q) give the action of an infinitesimal point transformation on configuration space Q, which is a symmetry if the change in L is as given above.

3.1.2 Non-N type symmetries

$$\delta q^j = \varepsilon \phi^j(q, \dot{q}) \Rightarrow \delta L = \varepsilon \frac{dF(q, \dot{q})}{dt}, |\varepsilon| << 1; \tag{3.7}$$

$$G(q,p) = F(q,\dot{q}) - p_j\phi^j(q,\dot{q}) = \text{non-linear in p, COM,} \tag{3.8}$$

$$\frac{dG}{dt} = \{G, H\} = 0; \tag{3.9}$$

$$\delta q^j = \varepsilon\{G(q,p), q^j\}\text{at } \phi_L\text{level, no use of EOM;} \tag{3.10}$$

$$\delta p_j = \varepsilon\{G(q,p), p_j\}\text{defined using EOM;} \tag{3.11}$$

$$\partial f(q,p) = \varepsilon\{G(q,p), f(q,p)\}\text{in general, uses EOM} \tag{3.12}$$

The ϕ^j(q,$\dot{q}$) describe a dynamical symmetry, not a point transformation.

3.2 Singular Lagrangian - symmetries

Now we turn to the Noether and Non-Noether symmetries of a singular Lagrangian system. Some general points may be made right away. The action and step-by-step analysis of an (infinitesimal) symmetry transformation does not lead to any constraints additional to those already encountered in the analysis of the EOM. In that sense, this is not in the nature of a consistency analysis but only a drawing out of the consequences of being a symmetry transformation. Every symmetry transformation respects or preserves the constraints produced by the dynamics. The successive stages of analysis of a symmetry transformation and its associated COM stay one step behind the corresponding stages in the analysis of the dynamics in Section II.

For simplicity consider directly a non-Noether type infinitesimal symmetry, as the Noether type is a special case (with simplifying features). The definition of such a symmetry is the same as before, except that L(q,$\dot{q}$) is singular:

$$\delta q^j = \varepsilon\phi^j(q,\dot{q}) \Rightarrow \delta L(q,\dot{q}) = \varepsilon\frac{dF(q,\dot{q})}{dt} \tag{3.13}$$

We first find that

$$F(q,\dot{q}) - p_j\phi^j(q,\dot{q}) = G_o(q,p)\text{over } \Sigma, \text{i.e., modulo } \phi_m \approx 0; \tag{3.14}$$

similar to the case of H_o(q,p), there is no dependence on the unsolved velocities. Next we find that δq^j has the form expected of an infinitesimal canonical transformation:

$$\delta q^j = \varepsilon\{G_o(q,p) - u_m\phi_m(q,p), q^j\}\text{modulo } \phi_m \approx 0 \tag{3.15}$$

The u_m are analogous to the unsolved velocities V_m in the dynamics, they are in fact some of the δq^j themselves, and to that extent these equations are 'empty'. The definition and evaluation of δp_j is more subtle; they do turn out to have the forms expected for a

CT but require the validity of the primary constraints as well as their time derivatives. Similarly, the result that G_o(q,p) is a COM requires both these conditions. Collecting results we find that:

$$\delta p_j \approx \varepsilon\{G_o - u_m\phi_m, p_j\}, \tag{3.16}$$

$$\{G_o, H_o + V_m\phi_m\} \approx 0, \tag{3.17}$$

$$\{G_o - u_m\phi_m, \phi_{m'}\} \approx 0, \tag{3.18}$$

$$\text{all modulo } \phi_m \approx 0, \{\phi_m, H_o + V_m, \phi_{m'}\} \approx 0 \tag{3.19}$$

The meaning of equation (3.18) is that the symmetry transformation automatically preserves the primary constraints provided the dynamics does so. Recall that eq. (3.19) is the source of secondary constraints and partial determination of V_m. Similarly eq. (3.18) leads to a partial 'determination' of the u_m, so some terms proportional to the ϕ_m get added to G_o.

As the process continues this phase space description of the symmetry transformation gets more and more sharply defined. At the end the generator of the transformation as a CT on phase space reaches a form similar to the final Hamiltonian in eq. (2.22,23):

$$G_o - u_m\phi_m = G - u_A\phi_A, \tag{3.20}$$

G = first class, COM.

Like the V_A in the final EOM (2.24), the u_A are those parts of the symmetry transformation which remain untranslatable fully to phase space. Since both G and ϕ_A are first class, this generator 'survives' the passage to DB's:

$$\delta f(q,p) \approx v\varepsilon\{G - u_A\phi_A, f\} \approx \varepsilon\{G - u_A\phi_A, f\}^*, \text{any f.} \tag{3.21}$$

3.3 Gauge type symmetries

Finally we look at the gauge type symmetries for a singular Lagrangian system. As for arbitrary functions of time remain in the final form of the EOM, such symmetries are natural. We see first class constraints playing a role. Only the results of a detailed analysis will be given here.

We consider two types of gauge symmetries:

$$\text{Type I} : \delta q^j = \varepsilon(f(t)\phi^j(q,\dot{q})), |\varepsilon| << 1, f(t) \text{ arbitrary;} \tag{3.22}$$

$$\text{Type II} : \delta q^j = \varepsilon(f(t)\phi^{(1)j}(q,\dot{q}) + \dot{f}(t)\phi^{(2)j}(q,\dot{q})), |\varepsilon| << 1, f(t) arbitrary. \tag{3.23}$$

Suppose the changes produced in L(q,$\dot{q}$) are of these types:

$$\text{Type I} : \delta L(q,\dot{q}) = \varepsilon \frac{d(f(t)F(q,\dot{q}))}{dt}, \tag{3.24}$$

$$\text{Type II} : \delta L(q,\dot{q}) = \varepsilon \frac{d(f(t)F^{(1)}(q,\dot{q}) + \dot{f}(t)F^{(2)}(q,\dot{q}))}{dt} \tag{3.25}$$

Then we have corresponding types of Non-Noether symmetries. We can now go through the analysis sketched in the previous paper, and at the end we find the generators in the sense of eq. (3.20) to have these forms:

Type I : G - $u_A\phi_A$ = f(t) (linear combination of primary first class constraints ϕ_A);

TypeII : G - $u_A\phi_A$ = f(t) (linear combination of ϕ_A and secondary first class constraints χ...) + $\dot{f}$(t) (linear combination of ϕ_A).

These results bring out the role of the first class constraints, both primary and secondary. We also see how the two features of the generator being a COM and of f(t) being an arbitrary function of time get reconciled - the generator is a (first class) constraint and is limited to the value zero!

Chapter 4

Simple examples to illustrate the theory

Two toy examples:

4.1 Case of N = 2, K = 0; Q = R^2, T^*Q = R^4

$$L = \frac{1}{2}(q_2\dot{q_1} - q_1\dot{q_2}) - \frac{1}{2}({q_1}^2 + {q_2}^2) : \tag{4.1}$$

Two primary constraints: $\phi_1 = p_1 - \frac{1}{2}q_2, \phi_2 = p_2 + \frac{1}{2}q_1; \{\phi_1, \phi_2\} = -1$

$$H_o = \frac{1}{2}({q_1}^2 + {q_2}^2), \text{initial Hamiltonian:} H_o + V_1\phi_1 + V_2\phi_2 \tag{4.2}$$

$$\dot{\phi_1} = \{\phi_1, H_o\} + V_1\{\phi_1, \phi_1\} + V_2\{\phi_1, \phi_2\} \approx 0 \Rightarrow V_2 = -q_1; \tag{4.3}$$

$$\dot{\phi_2} = \{\phi_2, H_o\} + V_1\{\phi_2, \phi_1\} \approx 0 \Rightarrow V_1 = q_2; \tag{4.4}$$

No χ's; ϕ_m are a second class set, go to their DB:

$$\{f, g\}^* = \{f, g\} - \{f, \phi_m\}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1}\{\phi_{m'}, g\} = \{f, g\} - \{f, \phi_1\}\{\phi_2, g\} + \{f, \phi_2\}\{\phi_1, g\} \tag{4.5}$$

Then $\{q_1, q_2\}^* = 1 \rightarrow$ one degree of freedom, one canonical pair, just the SHO!

4.2 Case of N = 2, K = 1; Q = R^2

$$L = \frac{1}{2}(\dot{q_1} + \dot{q_2})^2 - V(q_1, q_2); P_1 = P_2 = \dot{q_1} + \dot{q^2}; \tag{4.6}$$

One primary constraint : $\phi = p_1 - p_2$

H_o can be chosen as $\frac{1}{2}{p_1}^2 + V(q_1, q_2)$, initial Hamiltonian $H_o + V\phi$

$$\frac{d\phi}{dt} \approx \{\phi, H_o\} = \frac{\partial V}{\partial q_2} - \frac{\partial V}{\partial q_1} \approx 0 \text{for consistency.} \tag{4.7}$$

Choice (a) $V(q_1, q_2) = \frac{k}{2}(q_1 + q_2)^2$: no secondary χ, ϕ is first class, $V = V_A =$ free.

per Gauge type symmetry $\delta q_1 = \varepsilon f(t), \delta q_2 = -\varepsilon f(t); \delta L = 0, G = -f(t)\phi$

G generates this gauge symmetry: $\delta q_1 = \varepsilon\{G, q_1\}, \delta q_2 = \varepsilon\{G, q_2\}, \delta p_1 = \delta p_2 = 0$

Choice (b) $V(q_1, q_2) = \frac{k}{2}(q_1 - q_2)^2; \{\phi, H_o\} = -2k(q_1 - q_2) \approx 0 \Rightarrow$

Secondary constraints $\chi = q_1 - q_2$

$\frac{d\chi}{dt} = \{\chi, H_o\} + V\{\chi, \phi\} \approx 0 \Rightarrow V = -\frac{1}{2}p_1$ ϕ and χ are second class:

$\theta_1 = \phi = p_1 - p_2, \theta_2 = \chi = q_1 - q_2; (\{\theta_a, \theta_b\})^{-1} = \left(\begin{smallmatrix} 0 & -2 \\ 2 & 0 \end{smallmatrix}\right)^{-1} = \frac{1}{2}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right);$

DB: $\{f, g\}^* = \{f, g\} - \frac{1}{2}\{f, \theta_1\}\{\theta_2, g\} + \frac{1}{2}\{f, \theta^2\}\{\theta_1, g\}$

$\{q_1, p_1\}^* = \frac{1}{2}; q_2 = q_1 \text{and } p_1 = p_2 \text{strongly.}$

Final $H = \frac{1}{2}{p_1}^2 =$ free particle in one dimension.

4.3 Geodesic motion in Riemannian space

Q = Riemann space of dim n, local coordinate q^j. Metric tensor $g_{jk}(q)$.

'Free particle in Q': L $= (\dot{q^2})^{\frac{1}{2}}, \dot{q^2} = g_{jk}(q)\dot{q^j}\dot{q^k}, q^j = q^j(s), \dot{q^j} = \frac{dq^k(s)}{ds}$, s any 'evolution parameter'.

$\phi_L : q^j, \dot{q^j} \to q^j, p_j = \frac{g_{jk}(q)\dot{q^k}}{(\dot{q^2})^{\frac{1}{2}}}$.

One primary constraint: $\phi(q, p) = g^{jk}(q)p_j p_k - 1, \text{also } H_o = 0$

Initial EOM: $\dot{q^j} \approx \{q^j, V\phi\} = 2Vg^{jk}(q)p_k,$

$\dot{p_j} \approx \{p_j, V\phi\} = -Vg^{kl}(q)p_k p_l.$

We can 'evaluate' V: $p_j = \frac{1}{2V}g_{jk}(q)\dot{q^k}; \phi \approx 0 \Rightarrow V = \frac{1}{2}(\dot{q^2})^{\frac{1}{2}}$

Now 'by hand' choose s = arc length, then $\dot{q^2} = 1$ and v $= \frac{1}{2}$. Use this in EOM.

$\dot{q}^j = g^{jk}(q)p_k, \dot{p}_j = \frac{-1}{2}g^{kl}, j^{(q)}p_k p_l = \frac{1}{2}g_{kl}, j^{(q)}\dot{q}^k\dot{q}^l$

$\therefore \frac{d}{ds}(g_{jk}(q)\dot{q^k}) = \frac{1}{2}g_{kl,j}(q)\dot{q^k}\dot{q^l} or g_{jk}(q)\ddot{q^k} = \frac{1}{2}g_{kl,j}(q)\dot{q^k}\dot{q^l} - g_{jk,l}(q)\dot{q^k}\dot{q^l}$

i.e. $\ddot{q^j} + \Gamma_{kl}{}^j(q)\dot{q^k}\dot{q^l} = 0,$

$\Gamma_{kl}{}^j(q) = \frac{1}{2}g^{jk}(q)(g_{mk,l}(q) + g_{ml,k}(q) - g_{kl,m}(q)) =$ Christoffel symbol.

4.4 Relativistic free particle

Use x^μ and not q^μ. Metric $\eta_{\mu\nu} = (+1,-1,-1,-1), \mu v = 0,1,2,3$. Evolution of parameter τ arbitrary, Lorentz invariant. N =4, K=3 case.

$L = -m(\dot{x})^{\frac{1}{2}} = -m(\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -m(\dot{x^o}^2 - \dot{x^j}\dot{x^j})^{\frac{1}{2}}$

$\phi_L : x^\mu, \dot{x^\mu} \to x^\mu, p_\mu = -m\frac{\dot{x_\mu}}{(\dot{x^2})^{\frac{1}{2}}}$

One primary constraint $\phi(x,p) = p^\mu p_\mu - m^2$; initial $H_o = 0$, so Hamiltonian is Vϕ.

PB's: $\{x^\mu, p_\nu\} = \delta_\nu{}^\mu, \{x^\mu, x^\nu\} = \{p_\mu, p_\nu\} = 0.$

Initial EOM:

$\dot{x^\mu} \approx \{x^\mu, V\phi\} = 2Vp^\mu, \dot{p}_\mu \approx 0.$

No secondary constraints as $\{\phi, V\phi\} \approx 0 : \phi$ is first class ϕ_A; V is free, V_A.

Gauge type symmetry generated by ϕ: reparameterization

$\delta x^\mu \approx \varepsilon f(t)\dot{x^\mu}, \delta\dot{x^\mu} = \varepsilon(f(t)\ddot{x^\mu} + \dot{f(t)}\dot{x^\mu}) \Rightarrow \delta L = \frac{d(\varepsilon f L)}{dt}$

Corresponding COM is

$G_o = \varepsilon f(t)(L - \dot{x^\mu}p_\mu) = 0$

so this symmetry is generated by a multiple of primary first class ϕ:

$\delta x^\mu = \{-\varepsilon\mu\phi, x^\mu\} = 2\varepsilon u p^\mu, \delta p_\mu = 0,$

So $u = -\frac{1}{2m}f(t)(\dot{x^2})^{\frac{1}{2}}$

Two choices of gauge constraints to fix τ

(a) Laboratory time choice

$\xi(x,p,\tau) = x^o - \tau \approx 0.$

$\frac{d\xi}{d\tau} = \frac{\partial\xi}{\partial\tau} + \{\xi, V\phi\} \approx 0 \Rightarrow V = \frac{1}{2p^o}$

ϕ and ξ are second class set: $\theta_1 = \phi, \theta_2 = \xi$:

$(\{\theta_a, \theta_b\})^{-1} = \begin{pmatrix} 0 & -2p^o \\ 2p^o & 0 \end{pmatrix}^{-1} = \frac{1}{2p^o}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$

$\{f, g\}^* = \{f, g\} - \frac{1}{2p^o}(\{f, \phi\}\{\xi, g\} - \{f, \xi\}\{\phi, g\})$

Basic DB are:

$\{x^j, p_k\}^* = \delta_k{}^j, \{x^j, x^k\}^* = \{p_j, p_k\}^* = 0, Canonical;$

$\phi = \xi = 0$ are identities.

(b) Proper time choice

$\xi(x, p, \tau) = x, p - m\tau \approx 0$

$\frac{d\xi}{d\tau} \approx 0 \Rightarrow V = \frac{1}{2m}$, simpler than above

$\begin{pmatrix} \{\phi,\phi\} & \{\phi,\xi\} \\ \{\xi,\phi\} & \{\xi,\xi\} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -2m^2 \\ 2m^2 & 0 \end{pmatrix}^{-1} = \frac{1}{2m^2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$

$\{f, g\}^* = \{f, g\} - \frac{1}{2m^2}(\{f, p^2\}\{x.p, g\} - \{f, x.p\}\{p^2, g\})$

which leads to a small surprise:

$\{\chi^j, \chi^k\}^* = \frac{-1}{m^2}(\chi^j P^k - \chi^K P^j), \{\chi^j, P_k\}^* = \delta_k{}^j - \frac{1}{m^2}P^j P_k,$

$\{P_j, P_k\}^* = 0$

With either choice, (a) or (b), we can work out the generators of Lorentz transformations which are symmetries of L.

Chapter 5

Concluding comments

Dirac theory originally developed as a preparation for quantisation. It works for Maxwell and Yang Mill's theories as well, etc. General quantisation procedure is rather elaborate.

For finite numbers of degrees of freedom, there are several 'beautiful' papers. Hanson and Regge 1974. Relativistic Spherical Top - Q = $\Re^4 XSO(3,1)$, etc. Very beautiful piece of work but upon quantization leads to (2j + 1) -fold degeneracy of each spin j.

This theory as a whole a 'completion' of classical mechanical methods - good to have and use when needed - can handle any L(q,$\dot{q}$) if physically consistent.