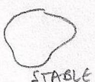
(क)

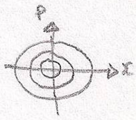
Sudeshna 13.1

Limit Cycles: Isolated — Nonlinear

2d < F.P. / Closed Orbits



STABLE



Self-sustained osc.

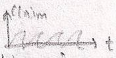
Van Der Pol Oscillator

$\ddot{x} + \mu(x^2-1)\dot{x} + x = 0$

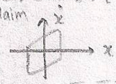
$|x|>1$: +ve

$|x|<1$: -ve

? claim



claim



$\dot{x} = F_x$

$\dot{y} = F_y$

$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$

$\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x}$

$\dot{x} = \sin y = -\frac{\partial V}{\partial x}$

$\dot{y} = x\cos y = -\frac{\partial V}{\partial y}$

$V(x,y) = -x\sin y$

Gradient System

Liapunov f^n

1) $V(\vec{x}) > 0 \quad \forall\ \vec{x} \neq \vec{x}^*$

$\vec{x}^*$: f.p.

2) $\frac{dV}{dt} < 0 \quad \forall\ \vec{x} \neq \vec{x}^*$

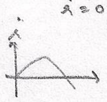
Poincare Oscillator

$\dot{r} = r(1-r^2)$

$\dot{\theta} = 1$

$\dot{r} = 0$ $\quad r^* = 0$: U

$r^* = 1$: S



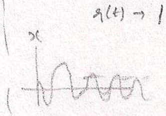
$f'(r) = 1-3r^2$

(only $r(1-r^2)$ → $r^2(1-r)$ would work)

$x(t) = r(t)\cos(\theta(t))$

⇓

$r(t) \to 1 \quad t+\theta_0$



Gradient System

$\dot{\vec{x}} = -\vec{\nabla}V$

$\Delta V = \int_0^T \frac{dV}{dt}\,dt = \int_0^T \nabla V \cdot \dot{\vec{x}}\,dt$

$= -\int_0^T |\dot{x}|^2\,dt < 0$

$\dot{\vec{x}} = 0 \Rightarrow$ f.p.

Energy like → Liapunov f^n

$\ddot{x} + (\dot{x})^3 + x = 0$

$E(x,\dot{x}) = \frac{1}{2}(\dot{x}^2 + x^2)$

$\Delta E = \int_0^T \dot{E}\,dt$

$\frac{dE}{dt} = \dot{x}(x + \ddot{x})$

$= \dot{x}(-\dot{x}^3)$

$= -\dot{x}^4 \leq 0$

⇒ No closed Orbits

Dulac's Criterion

(ग)

$\dot{\vec{x}} = \vec{f}(\vec{x})$ $\quad \nabla\cdot(\underbrace{g\dot{\vec{x}}}_{=\vec{F}})$

$\iint \nabla\cdot\vec{F} = \oint \vec{f}\cdot\hat{n}$

doesn't change sign

⇓

$\neq 0$

Trapping Region

2d → No chaos

3d

Non-invertible 1d map → chaos

(ख) Defn: Limit Cycle ≡ an isolated closed trajectory where isolated means that the neighboring trajectories are not closed.

(घ) Dulac's

Let $\dot{\vec{x}} = \vec{f}(\vec{x})$ be a continuously differentiable vector field defined on a simply connected subset R of the plane. If ∃ a continuously differentiable real valued function $g(\vec{x})$, s.t. $\nabla\cdot(g(\vec{x})\dot{\vec{x}})$ has the same sign throughout R, then there are no closed orbits lying entirely in R.

Proof: Let there be a closed orbit C entirely in R & A be area inside C. Then

$$\iint_A \nabla\cdot(g\dot{\vec{x}})\,dA = \oint_C g\dot{\vec{x}}\cdot\hat{n}\,dl$$

LHS $\neq 0$ ∵ of sign assumption; RHS = 0 ∵ C is a trajectory

Sudeshna 13.2
(13.3 missing | wedding)

Limit Cycle - nonlinear

No closed orbits

① Gradient Systems
$\vec{\dot{x}} = -\nabla V$

② Liapunov Function

③ $\nabla \cdot (g \vec{\dot{x}})$

$\dot{x} - (-\dot{y})$
$= -x + ay + x^2y + b - ay - x^2y$
$= -x + b$
for $x > b$, $\Rightarrow (\dot{x}) < (-\dot{y})$

Existence of closed orbits

Poincare - Bendixson Thm.

① R

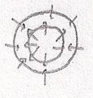
② $\vec{\dot{x}} = \vec{f}(\vec{x})$

③ No F.P. in R

④ Trajectory confined in R, trapping region

$$J = \begin{pmatrix} -1 + 2xy & a + x^2 \\ -2xy & -(a+x^2) \end{pmatrix}$$

$\dot{r} = r(1-r^2) + \mu r \cos\theta$



$\dot{r} > 0$

$r(1-r^2) + \mu r \cos\theta > 0$

$1 - r^2 - \mu > 0$

$r_{min} < \sqrt{1-\mu}$

$r_{max} > \sqrt{1+\mu}$

So then $\sqrt{1-\mu} < r < \sqrt{1+\mu}$

closed orbit

$\Delta > 0$

$\tau > 0$, U → closed orbits

$\tau < 0$, stable F.P., no orbit.

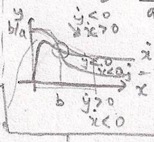
Nullclines

$\dot{x} = -x + ay + x^2y$

$\dot{y} = b - ay - x^2y$

$\dot{x} = 0 \Rightarrow y = \frac{x}{a+x^2}$

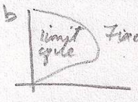
$\dot{y} = 0 \Rightarrow y = \frac{b}{a+x^2}$



$\dot{x} = 0$; $y = \frac{x}{a+x^2}$

$\dot{y} = 0$; $y = \frac{b}{a+x^2}$

Magn



$\dot{x} \approx x^2y$

$\dot{y} \approx -x^2y$

$\frac{dy}{dx} \sim -1$

$x = 0$

$\dot{x} = ay > 0$

$\dot{y} = b - ay > 0$

$y = 0$

$\dot{x} = -x < 0$

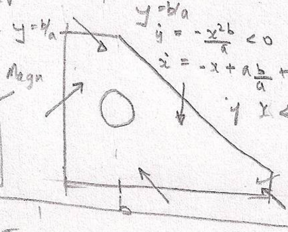
$\dot{y} = b > 0$

$y = b/a$

$\dot{y} = -\frac{x^2 b}{a} < 0$

$\dot{x} = -x + a\frac{b}{a} + x^2\frac{b}{a}$

if $x < b$, then



Sudeshna

14.3 (14.2 was off)
(14 | missing | wedding)

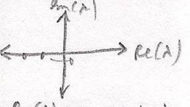
CRITICAL modes $\rightarrow$ 1d
$\downarrow$
Enslaved
2d
Fixed Points / Closed Orbits

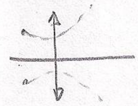
$\dot{r} = \mu r - r^3$
$\dot{\theta} = \omega + b r^2$

$\mu < 0$; $r = 0$: stable
$\mu > 0$, $r = \sqrt{\mu}$: stable
$\mu_c = 0$

Hopf Bifurcation

Supercritical
Subcritical





$Re(\lambda) < 0$, $\Delta = \lambda_1 \lambda_2$

$x = r\cos\theta$, $y = r\sin\theta$
$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$
$= \mu x - \omega y$ + cubic terms

Similarly,
$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$
$= \omega x + \mu y$ + higher order.

So,

$\dot{x} = \mu x - \omega y$
$\dot{y} = \omega x + \mu y$

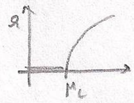
$J = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$

$(\mu - \lambda)^2 + \omega^2 = 0$

$Re(\lambda)$ gives stability (ক)

F.P. $\rightarrow$ Limit Cycle

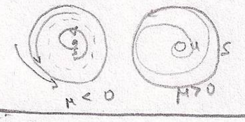
(a) $r \sim \sqrt{\mu}$



(b) Freq $\omega \sim Im(\lambda)$

Subcritical

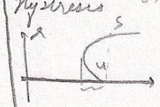
$\dot{r} = \mu r + r^3 - r^5$
$\dot{\theta} = \omega + b r^2$



(ক) Generically, for supercritical Hopf bifurcations,

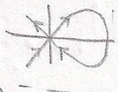
1. The size of the limit cycle increases from zero proportional to $\sqrt{\mu - \mu_c}$, for μ close to μ_c.
2. The freq. of the limit cycle $\approx \omega = Im\,\lambda$ for $\mu = \mu_c$.
   It's correct to $O(\mu - \mu_c)$; period is $\therefore T = \frac{2\pi}{Im\,\lambda} + O(\mu - \mu_c)$

Hystresis



Homoclinic Bifurcation

Homoclinic Orbit



Global Bifurcations

Chemical Reactions B-Z:

2 ODES

$\frac{d[\,]}{dt}$

	Amplitude		Period
Supercritical Hopf	$O(\sqrt{\mu})$	small	$O(1)$
Saddle	$O(1)$	large	$O(1)$
∞ period	$O(1)$		$O(\frac{1}{\sqrt{\mu}})$
Homoclinic	$O(1)$		$\sim O(\ln \mu)$

claimed in general

$\therefore$ Bottleneck like

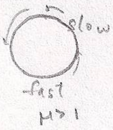
it's an exercise

Saddle-Node Bifurcation of cycles

$\dot{r} = \mu r + r^3 - r^5$; $\mu = -1/4$

Infinite Period Bifurcation

$\dot{r} = r(1 - r^2)$
$\dot{\theta} = \mu - \sin\theta$



amp $\sim O(1)$ &

freq $\sim \frac{1}{\sqrt{\mu - \mu_c}}$