Mukunda's Lectures [train notes]

March 10, 2015

1 Berry's Phase (original)

$$R \in \text{multidimensional parameter space}$$

$$R \to R(t) \implies H(R) \to H(R(t))$$

We assume that the time evolution is cyclic. Let $R(t)$ run over a closed loop

$$C = \{R(t)|0 \leq t \leq T\} = \text{closed loop in parameter space}$$

$$R(T) = R(0)$$

For each R (in the domain of interest), there're eigenvectors and eigenvalues of $H(R)$ and we can write the usual

$$H(R)\,|n; R\rangle = E_n(R)\,|n; R\rangle$$

$$\langle n'; R|\,|n; R\rangle = \delta_{n'n} \qquad \sum_n |n; R\rangle\,\langle n; R| = 1$$

We assume nondegenercy and no level crossing in the domain of interest in the R space. Now there's a remark in the paper that (I can't understand) says that $|n; R\rangle$ is assumed continuous and single valued in R, in contrast to the previous case (where we assumed $(\psi_n(t), \dot{\psi}_n(t)) = 0$. Don't know what relevance it has though. Fine, it clarifies it in the next step. It says each $|n; R\rangle$ is free upto a phase dependent on n and R but is otherwise assumed to be well-defined (single-valued) in the domain of interest.

The schrodinger equation is

$$i\hbar\dot{\psi}(t) = H(R(t))\psi(t)$$

We know from the adiabatic theorem (which I read from wikipedia, there in the scanned notes) that

$$\psi(0) = |n; R(0)\rangle \implies \psi(t) \approx exp\left([i\theta_n(t)] + [i\gamma_n(t)]\right)|n; R(t)\rangle$$

where $i\theta_n(t) = -i/\hbar \int_0^t dt' E_n(R(t'))$, $\gamma_n(t)$ in those notes had been explicitly derived. Here, we ignore that and simply plug this as the guess solution in the schrodinger equation. Lets do just that.

$$i\hbar\left[i\left(\dot{\theta_n} + \dot{\gamma_n}\right)e^{i(\theta_n+\gamma_n)}\,|n; R(t)\rangle + e^{i(\theta_n+\gamma_n)}\frac{d}{dt}\,|n; R(t)\rangle\right]$$

Okay so now i'm getting confused with differentiation. Lets check.

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}f(g(x)) = f'(g(x)) * g'(x)$$

Never mind that. Sorry. Back to the task.

$$i\hbar\left[i\left(\dot{\theta_n}+\dot{\gamma_n}\right)e^{i(\theta_n+\gamma_n)}|n;R\rangle+e^{i(\theta_n+\gamma_n)}\frac{d}{dt}|n;R\rangle\right]\approx H(R)|\psi(t)\rangle$$

$$i\hbar\left[\left(\cancel{-\frac{i}{\hbar}E_n(R)}+i\dot{\gamma_n}\right)e^{i(\theta_n+\gamma_n)}|n;R\rangle+e^{i(\theta_n+\gamma_n)}\frac{d}{dt}|n;R\rangle\right]\approx\cancel{E_n(R)e^{i(\theta_n+\gamma_n)}}|n;R\rangle$$

$$i\dot{\gamma_n}e^{i(\theta_n+\gamma_n)}|n;R\rangle+e^{i(\theta_n+\gamma_n)}\frac{d}{dt}|n;R\rangle\approx 0$$

DOUBT: I would now want to understand what difference would it have made if I had written $i\theta_n(t)=-i/\hbar\,[E_n(R(t))t]$, because even then $i\dot{\theta_n}=-i/\hbar\,[E_n(R)]$ and would've happily cancelled out.

NB: This is essentially the statement

$$\frac{d}{dt}\left(e^{i\theta_n}|n;R\rangle\right)\approx 0$$

Continuing, I want to find γ_n, so I go back to the previous step and write

$$i\dot{\gamma_n}|n;R\rangle+\frac{d}{dt}|n;R\rangle\approx 0$$

$$\begin{aligned}i\dot{\gamma_n}&\approx-\langle n;R|\frac{d}{dt}|n;R\rangle\\ \dot{\gamma_n}&\approx i\langle n;R|\frac{d}{dt}|n;R\rangle\\ &\approx i\langle n;R|\nabla|n;R\rangle.\dot{\mathbf{R}}\end{aligned}$$

NB:

- We now assume that the paratmeter space is 3 dimensional.
- ∇ is in the R space, not the physical 3d space.

Further, I can integrate to get an expression for

$$\begin{aligned}\gamma_n&\approx i\int_0^t\langle n;R|\nabla|n;R\rangle.\frac{d\mathbf{R}}{dt}dt\\ &\approx i\int_{\mathbf{R}(0)}^{\mathbf{R}(t)}\langle n;R|\nabla|n;R\rangle.d\mathbf{R}\end{aligned}$$

NB:

- For convenience, I haven't written $|n;\mathbf{R}\rangle$ but it's implied.
- I'll just show that γ_n is real, viz. the integral is purely imaginary

$$\begin{aligned}\nabla\langle n;R|n;R\rangle=0&=\langle n;R|\left(\nabla|n;R\rangle\right)+\left(\nabla\langle n;R|\right)|n;R\rangle\\ &=2\mathrm{Re}\left(\langle n;R|\nabla|n;R\rangle\right)\end{aligned}$$

  that tells us $\langle n;R|\left(\nabla|n;R\rangle\right)$ has no real part, viz. it is purely imaginary.

Now recall we're moving along a closed curve so we get

$$
\begin{aligned}
\gamma_n &\approx i \oint_C \langle n; R| \nabla |n; R\rangle .dR \\
&\approx \mathrm{Re}\left(i \oint_C \langle n; R| \nabla |n; R\rangle .dR \right) && [\text{we proved } \gamma_n \text{ is purely Real}] \\
&\approx i \oint_C \alpha.dR; && [\alpha \equiv \langle n; R| \nabla |n; R\rangle] \\
&\approx i \int_S (\nabla \times \alpha).da; && [\text{using stokes thm, implicit}] \\
\gamma_n &\approx i \int_S (\nabla \times \alpha).da \\
\gamma_n &\approx i \int_S \epsilon_{ijk} \frac{\partial}{\partial x_i} \left(\langle n; R| \frac{\partial}{\partial x_j} |n; R\rangle \right) da_k \\
&\approx i \int_S \epsilon_{ijk} \left(\frac{\partial}{\partial x_i} \langle n; R| \frac{\partial}{\partial x_j} |n; R\rangle + \cancel{\langle n; R| \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} |n; R\rangle} \right) da_k && [\text{using anti-symmetry in i j}] \\
&\approx i \int_S (\nabla \langle n; R|) \times (\nabla |n; R\rangle).d\mathbf{a} \\
&\approx \sum_m i \int_S (\nabla \langle n; R|) |m; R\rangle \times \langle m; R| (\nabla |n; R\rangle).d\mathbf{a} \\
&\approx \sum_m i \int_S i\mathrm{Im}\left((\nabla \langle n; R|) |m; R\rangle \times \langle m; R| (\nabla |n; R\rangle) \right).d\mathbf{a} && [\gamma_n \text{ is purely Re}] \\
&\approx - \sum_{m \neq n} \int_S \mathrm{Im}\left((\nabla \langle n; R|) |m; R\rangle \times \langle m; R| (\nabla |n; R\rangle) \right).d\mathbf{a} && [\langle n; R| (\nabla |n; R\rangle) \text{ is real}]
\end{aligned}
$$