Topics
(a) Calculation of Indices
(b) Decomposition of a tensor product
(c) Spin(4)
Spin(1,3) ← Lorentz ⊕ Fermions (Spinors)
Dirac
Weyl
Maj

Olly #3.2

$\begin{Bmatrix} S_{ij} \\ A_{ij} \end{Bmatrix}$; $-A_{ji}$

$A_{ij} = A_{\bar i \bar j}, A_{\bar i 3}, A_{33}$

$S_3 = 0 + (1/2)^2 + (1/2)^2 + 0 = 1/2$

$S_{ij} = \{ S_{\bar i \bar j}, S_{\bar i 3}, S_{33} \}$

$\mathrm{tr}(7^a 7^b) = S_2(D)\,\delta^{ab}$

\# of indexes

Choose: $7^a = 7$

Diagonal generator of any SU(2) subgp

$SU(N): T^a = \frac{\lambda^a}{2}; \frac{\lambda^1}{2}, \frac{\lambda^2}{2}, \frac{\lambda^3}{2}$

$\frac{1}{2}\begin{pmatrix} \sigma^3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Eg: $SU(3): T_3 = \frac{1}{2}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$

$\mathrm{tr}((\lambda^a)^2) = 2$

$S_2(3) = \frac{1}{2}$

$a = 1,2,3 = \bar a$, $(3,4,5,6,7,8)$

$[T^{\bar a}, T^{\bar b}] = i\epsilon^{\bar a \bar b \bar c} T^{\bar c}$

$\frac{1}{2}\begin{pmatrix} \sigma^a & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} = \\ x \end{pmatrix}$

$(1)^2 + (1/2)^2 + 0 = 5/2$

Adjoint: $T_{[ij]k}$; $T_{[ij\,k]} = 0$

$T^i{}_k$; $T^i{}_i = 0$

$T_i{}^j$; $T_i{}^i = 0 = T_1{}^1 + T_2{}^2 + T_3{}^3 = 0$

9 components − 1 constraint = 8

SU(3) also has $N^2 - 1 = 8$ components

$T_{\bar i}{}^{\bar j}, T_{\bar i}{}^3, T_3{}^{\bar i}, T_3{}^3$

$(\hat T_{\bar i}{}^{\bar j}, (T_1{}^1 + T_2{}^2))$

S_2 2, 0, 0, $1/2 + 1/2 + 0 = 3$

$SU(N): i = \bar i \oplus \hat i = 3, \ldots N$

$N = 2 + (N-2)$ singlets of SU(2)

$S_2(N) = \frac{1}{2} + 0 = \frac{1}{2}$

Adjoint $A_i{}^j$ $A_i{}^i = 0$

$A_{\bar i}{}^{\bar j}$ $A_{\bar i}{}^{\hat i}$, $A_{\hat i}{}^{\bar i}$ $A_{\hat i}{}^{\hat j}$

$\hat A_{\bar i}{}^{\bar j} + (A_{\bar i}{}^{\bar i})$ 0

2 $\frac{(N-2)}{2} \times \frac{1}{2} \times 2$

for the caps; value for SU(2); for the second piece

$S_2(N) = N$

In general then,

S_{ijk}	
$S_{\bar i \bar j \bar k}$	5
$S_{\bar i \bar j \hat k}$	$(N-2)\,2$
$S_{\bar i \hat j \hat k}$	$\frac{(N-2)(N-1)}{2} \times \frac{1}{2}$
$S_{\hat i \hat j \hat k}$	0

Casimirs

$\sum_a 7^a 7^a \equiv 7^a 7^a$, $D \times D$

Schur's Lemma

$\boxed{7^a 7^a = C_2(D)\,\mathbb{1}_D}$

second casimir

$[7^a 7^a, 7^b] = 0$ (by TAS of f_{abc})

$\mathrm{tr}(7^a 7^b) = \delta^{ab} S_2(D)$

We want to relate S_2 & C_2.

Now R: Representation

D(R): Dimension / # of generators

use this as

$\mathrm{tr} \sum_a 7^a 7^a = C_2(R)\,\mathrm{tr}\,\mathbb{1}_D$

$\Rightarrow \boxed{d(R) S_2(R) = C_2(R) D(R)}$

$7^a = T^a \otimes \mathbb{1} + \mathbb{1} \otimes T^a$

$\mathrm{tr}(7^a 7^b) = \mathrm{tr}[(T^a \otimes \mathbb{1} + \mathbb{1} \otimes T^a)(T^b \otimes \mathbb{1} + \mathbb{1} \otimes T^b)]$

complete this

$N = 2 + (N-2)$

i $\bar i$ $\hat i = 2, \ldots, N$

$SU(N) \sim SU(2) \times SU(N-2)$

$U(1)$ (N^2-1)

$N-1$ 1 $N-3$

$2\sqrt{3}T_8$ 1 −2] ASK

3 = 2 + 1

$SU(2)$ $\begin{pmatrix} x \\ x \\ v \end{pmatrix}$; $\frac{1}{2}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} = T_3$

$T_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$

$\psi_i = \psi_{\bar i} + \psi_{\hat i}$

$SU(N) = SU(2) \times SU(N-2) \times U_{N^2-1}(1)$

$N = (2, 1, 1) + (1, N-2, \frac{-2}{N-2})$

calculated as

$Q = \begin{pmatrix} a\mathbb{1}_2 & \\ & b\mathbb{1}_{N-2} \end{pmatrix}$

Take $a = 1$, & $\mathrm{tr}\,Q = 0$ to get

$b = \frac{2}{2-N} = \frac{-2}{N-2}$;

S_{ij}	$\frac{N(N+1)}{2}$
$S_{\bar i \bar j}$	3
$S_{\bar i \hat i}$	$(N-2)$ doublets
$S_{\hat i \hat j}$	$\frac{(N-2)(N-1)}{2}$

\# symmetric index ij of 2 indices

& then $\frac{N(N+1)}{2} = (3, 1, 2) + (2, N-2, 1 + \frac{2}{2-N}) + (1, \frac{(N-2)(N+1)}{2}, -\frac{4}{N-2})$

SU(N) Tensor Products

$\psi_i \chi_j \equiv T_{ij} = A_{ij} + S_{ij}$

$N_A A_{ij} = \frac{\psi_{[i}\psi_{j]}}{2}$

$N_B S_{ij} = \frac{\psi_{(i}\psi_{j)}}{2}$

SU(3)

$A = \psi_i x^i$

$3 \times 3 = \bar{3} + 6$

$3 \times \bar{3} = \psi_i x^j = 8 + 1 \quad \hat{A}_i^{\ j} = \psi_i x^j + \#(\psi_k x^k)\delta^i_j ; \hat{A}^i_{\ i} = 0$

$3 \times 6 = x_i S_{jk} = x_{(i} S_{jk)} + x_{[i} S_{j]k}\, \epsilon^{ijl} = A^l_{\ k}$

$3 \times 8 = x_i \psi^i_{\ k} + \underset{6}{\overset{10}{x_{[i}\psi^j_{\ e]}}} + x_{(i}\psi^l_{\ k)}$

$x_i \quad 4$

$x_{(ij)} \quad \frac{4 \cdot 5}{2} = 10$

$x_{[ij]} \quad \frac{4 \cdot 3}{2} = 6$

$\hat{x}_i^{\ j} \sim \hat{x}_{i[lmn]} \sim 4^2 - 1 = 15$

$\hat{x}^{\ k}_{[ij]} \quad : \quad 20$ ☺

$6 \times 4 = 24$

$\hat{x}^{[kl]}_{[jm]} \quad \hat{x}^{ik}_{il} = 0 \cdot 20' ?$ ☺

$S_{(ijkl)} : \frac{4 \cdot 5 \cdot 6 \cdot 7}{4} = 35$

$M^{[kl]}_{[ij]} \quad 60 - \cancel{16}\,15 = 4 \times 5$

$M^{il}_{ij} = 0 - 16$

but $M^{[ij]}_{(ij)} = 0$

so you overcounted one term

$[T_a, T_b] = i f_{abc} T_c$

$f_{abc} = \cancel{\#}2(-i\, \mathrm{Tr}([T_a, T_b] T_c))$

$\{\lambda^a, \lambda^b\} = d^{abc}\lambda^c + \dots \delta^{ab} \mathbb{1}_N$

$\mathbb{1}_N, \ \lambda^a \to 1, \dots N^2 - 1$: N Herm. Matrices

$\{\lambda^a, \lambda^b\} = d^{abc}\lambda^c + 4\frac{\delta^{ab}}{N}\mathbb{1}_N$

$* \quad d^{abc} = \frac{1}{2}\mathrm{Tr}(\{\lambda^a, \lambda^b\}\lambda^c)$

$X'^{abc} = R^{*}{}^{aa'} R^{bb'} e^{cc'} \underbrace{X^{a'b'c'}}$

$U(\theta)\lambda^a U^+(\theta)$

$\overset{\shortparallel}{e^{-1}(\theta)^{ab}\lambda^b}$

$\Rightarrow U^+ \lambda^a U = R^{ab}\lambda^b$

Recall: $e^{i\theta \cdot 7}\, 7\, e^{-i\theta \cdot 7}$

$\overset{\shortparallel}{R^{ab}(-\theta) 7^b}$

$\mathrm{Tr}(\lambda^{a'}\lambda^{b'}\lambda^{c'})$

$\downarrow$ could be

$[\quad]$ or

$\{\quad \}$

NB: $\mathrm{Tr}(\lambda^{a'}\lambda^{b'}\lambda^{c'})$

goes to

$\mathrm{Tr}(U\lambda^a U^+ U \lambda^b U^+ U \lambda^c U^+)$

$X'^{abc} = X^{abc}$

$(X^{abc})(\lambda^a)_{i'}^{\ i}(\lambda^b)_{j'}^{\ j}(\lambda^c)_{k'}^{\ k} = X^{ijk}_{i'j'k'}$

$\shortparallel$ could be f^{abc} or d^{abc}

$\hat{U} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad |\alpha|^2 + |\beta|^2 = 1$

$\sum_{i=1}^{4} x_i^2 = 1 \quad S^3$

NB: compactness ☺

Diag Generator of SU(2)

$H_1 = T_3$

$E_1 = \frac{T_1 + iT_2}{\sqrt{2}}$

$E_{-1} = \frac{T_1 - iT_2}{\sqrt{2}}$

$[H_1, E_{\pm 1}] = \pm E_{\pm 1}$

$[E_+, E_-] = T_3 = H_1$

$\alpha = \pm 1$ Roots of SU(2)

Same we'll try for SU(3)

SU(3)

$E^{(12)} = E_1 = \frac{T_1 + iT_2}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$E^{(13)} = E_2 = \frac{T_4 + iT_5}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$E^{(23)} = E_3 = \frac{T_6 + iT_7}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$(H_1, H_2) = (T_3, T_8)$

$[\vec{H}, E_i] = \vec{\beta}_i E_i$

$E_i = -\vec{\beta}_i E_i$

$E_i \to E_{\vec{\beta}}$

$E_i \to E_{-\vec{\beta}}$

$(e_i)_k = \delta_{ik} \quad T_3 = \frac{1}{2}\mathrm{diag}(1, -1, 0)$

$T_8 = \frac{1}{2\sqrt{3}}\mathrm{diag}(1, 1, -2)$

$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} \overset{T_3 \quad T_8}{\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)}$ $\vec{\mu}_1$

$e_2 = \begin{pmatrix} 0 \\ \vdots \\ \vdots \end{pmatrix} \left(\frac{-1}{2}, \frac{1}{2\sqrt{3}}\right)$ $\vec{\mu}_2$

$e_3 = \begin{pmatrix} 0 \\ \vdots \\ \vdots \end{pmatrix} \left(0, -\frac{1}{\sqrt{3}}\right)$ μ_3

Weight vectors of fund SU(3)

$e_1 \to |\vec{\mu}_1\rangle$

$e_2 \to |\vec{\mu}_2\rangle$

$e_3 \to |\vec{\mu}_3\rangle$

$e_i \sim |i\rangle \leftrightarrow |\mu_i\rangle$

$H_j|\vec{\mu}_i\rangle = (\vec{\mu}_i)_j |\vec{\mu}_i\rangle = (H_j)_{ii}|\vec{\mu}_i\rangle$ → no sum! diagonal

eg. $H_1|\vec{\mu}_1\rangle = (H_1)_{11}|\vec{\mu}_1\rangle$

$= \frac{1}{2}|\vec{\mu}_1\rangle$

$[E_{\vec{\beta}}, E_{-\vec{\beta}}] = \beta \cdot H$ ☺

$[E_{\vec{\beta}}, E_{\vec{\beta}'}] = 0$ if $\vec{\beta} + \vec{\beta}'$ is not a root.

$\vec{\beta}_1 = (1, 0)$

$\vec{\alpha}_1 = \vec{\beta}_2 = \frac{1}{2}(1, \sqrt{3})$

$\vec{\alpha}_2 = \vec{\beta}_3 = \frac{1}{2}(-1, \sqrt{3})$

(Any Lie group) Cartan Sub-Algebra

$[H_i, H_j] = 0 ; [\vec{H}, E_{\vec{\gamma}}] = \vec{\gamma} E_{\vec{\gamma}}$

Positive Root: First non-zero component +ve

$\vec{\beta}_1 = (1, 0)$ ← Non simple

$\vec{\beta}_2 = \frac{1}{2}(1, \sqrt{3}) = \alpha_1$

$-\vec{\beta}_3 = \frac{1}{2}(1, -\sqrt{3}) = \alpha_2$

Simple (+ve) roots

$\vec{\beta}_1 = \vec{\beta}_2 + (-\vec{\beta}_3)$

$[E_{\vec{\beta}_1}, E_{-\vec{\beta}_2}] = \frac{-1}{\sqrt{2}} E_{-\vec{\beta}_3}$

$[E_{\vec{\beta}_1}, E_{\vec{\beta}_3}] = \frac{1}{\sqrt{2}} E_{\vec{\beta}_2}$

$[E_{\vec{\beta}_2}, E_{\vec{\beta}_3}] = \frac{1}{\sqrt{2}} E_{\vec{\beta}_1}$

$[E_{\vec{\beta}}, E_{\vec{\beta}'}] = N(\beta, \beta') E_{\vec{\beta} + \vec{\beta}'}$

$[\vec{H}, E_{\vec{\beta}}] = \vec{\beta} \cdot E_{\vec{\beta}}$

$[H_i, H_j] = 0$

Cartan Chivalie Form

$$\vec{H} E_{\vec{\beta}} |\vec{\mu}\rangle = ([\vec{H}, E_{\vec{\beta}}] + E_{\vec{\beta}} \vec{H}) |\vec{\mu}\rangle$$

$$\vec{H} |\vec{\mu}\rangle = \vec{\mu} |\vec{\mu}\rangle$$

$$= (\vec{\beta} + \vec{\mu}) E_{\vec{\beta}} |\vec{\mu}\rangle$$

$$\sim |\vec{\mu} + \vec{\beta}\rangle$$

$E^{(ij)}$ $\bar{E}^{(ij)}$ $T_3^{(ij)}$

$$E_{\pm} = \frac{E_{\pm\vec{\beta}}}{\|\vec{\beta}\|} \; ; \; E_3 = \frac{\vec{\beta}\cdot\vec{H}}{\|\vec{\beta}\|^2}$$ form $SU(2)$ algebra $\forall \vec{\beta}$

For $SU(2)$; $\dfrac{\alpha_1 \cdot \alpha_2}{\|\alpha_1\|\|\alpha_2\|} = \dfrac{-1}{2} = \cos\theta_{12} = 120°$

represented by 

For $SU(N)$ we'll have 

Olly # 3.4

$(\vec{\mu}_i) = (\mu_1, \mu_2, \dots \mu_{N-1})_{ii}$

Weights of N states of fundamental of SU(N)

$e_1 = \begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix} \leftrightarrow i \leftrightarrow |\vec{\mu}_1\rangle = (\frac{1}{2}, \frac{1}{2\sqrt{3}}, \dots \frac{1}{\sqrt{2N(N-1)}}) = (\vec{\mu})_{ii}$

$|\vec{\mu}_2\rangle = (-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \dots$

$|\vec{\mu}_3\rangle = (0, -\frac{1}{2\sqrt{3}}, \dots$

$-\vec{\mu}_i$

$A^i_{\ j} \longleftrightarrow$ generators of SU(N) transform in the adjoint $\Rightarrow$ roots

$\vec{\mu}_j$

$SU(2) = \mu_1 = \frac{1}{2}, \ \mu_2 = -\frac{1}{2} \Big\} \alpha = \pm 1$

$SU(3): |\mu_1\rangle = |\frac{1}{2}, \frac{1}{2\sqrt{3}}\rangle$

$|\mu_2\rangle = |-\frac{1}{2}, \frac{1}{2\sqrt{3}}\rangle$

$|\mu_3\rangle = |0, \frac{-2}{2\sqrt{3}}\rangle$

$\mu_1 - \mu_2, \ \mu_1 - \mu_3, \ \mu_3 - \mu_2$

$|(\frac{1}{2}, \frac{\sqrt{3}}{2})\rangle = \beta_2$

$|(1, 0)\rangle = \beta_1 = \mu_1 - \mu_2 = \alpha_1$

$\boxed{\mu_3 - \mu_2 = |(\frac{1}{2}, -\frac{\sqrt{3}}{2})\rangle = \beta_3 = \alpha_2}$

General scheme for decomposition

$SU(N) \to SU(N_1) \times SU(N - N_1) \times U(1)$

$Q = \begin{pmatrix} N_2 \mathbb{1}_{N_1} & \\ & -N_1 \mathbb{1}_{N_2}\end{pmatrix}$

$T_{i_1 \dots i_l}^{j_1 \dots j_m}$

$SO(N) \quad V^T W = V'^T W'$

$\det O = 1; \quad V' = OV$

$O^T = O^{-1}$

Example

$SO(4) \quad R = e^A = e^{\frac{w_{ij}}{2} A^{(ij)}}$

$(A^{ij})_{kl} = \delta_k^{[i}\delta_l^{j]} = -[A^{ji}]_{kl}$

$\Rightarrow \frac{N(N-1)}{2}$ independent

$J^{ij} = i A^{ij}$

$[J^{ij}, J^{kl}] = i\delta^{k[i} J^{j]l}$

$H \longrightarrow [J^{12}, J^{34}, \dots]$

$SO(4) \approx O(3) \times O(3)$

$\mathrm{spin}(4) \simeq SU(2) \times SU(2)$

$SO(2N)$ — Dual of N index antisymmetric tensor

$\tilde{F}_{i_1 \dots i_n} \equiv \frac{-i^n}{n!} \epsilon_{i_1 \dots i_{2n}} F_{i_{n+1} \dots i_{2n}}$

NB: A.Sym

$so(4): F_{ij} = -F$

$\tilde{F} = -\frac{(i)^2}{2!} \epsilon F$

$\frac{4 \cdot 3}{2} = 6 \longrightarrow 3 + 3$

$\text{g. } \tilde{F}_{\underset{12}{ij}} = \frac{1}{2} \epsilon_{\underset{1234}{ijkl}} F_{\underset{43}{kl}} = F_{34}$

Vector of SO(4):

$V_1 \ V_2 \ V_3 \ V_4$

$\frac{V_1 + iV_2}{\sqrt{2}} \equiv V_{2\dot{2}}$

$A_{ij} = -A_{ji}$

$\tilde{A}^{+}_{ij} = \pm A_{ij}$

$\alpha \dot{\alpha}$

$A^{+}_{(\alpha\dot{\alpha}),(\beta\dot{\beta})} = \epsilon_{\alpha\beta}\delta_{\dot{\alpha}\dot{\beta}}$

$A^{-} \qquad\quad = \epsilon_{\dot{\alpha}\dot{\beta}}\delta_{\alpha\beta}$

$J^{\pm}_{ij} \equiv \frac{1}{2}(J_{ij} \pm \tilde{J}_{ij}) \quad i, j = (1, 2, 3)$

$J^{\pm}_{12} = \frac{1}{2}(J_{12} \pm J_{34})$

$T^{\pm}_{k} = \frac{1}{2} \epsilon_{kij} J^{\pm}_{ij}$

$[T^{\pm}_k, T^{\pm}_l] = i\epsilon_{kjl} T^{\pm}_l$ Exercise

$[T^{\pm}_k, T^{\mp}_l] = 0$

$V_{1\dot{1}} = \frac{-V_1 + iV_2}{\sqrt{2}}; \quad V_{1\dot{2}} = \frac{V_3 + iV_4}{\sqrt{2}}$

$V_{2\dot{1}} = \frac{V_3 - iV_4}{\sqrt{2}}$; so that we

have $\underset{\psi_\alpha}{\underline{SU(2)_L}} \times \underset{\psi_{\dot{\alpha}}}{\underline{SU(2)_R}}$ (just label) via $\psi'_\alpha = U_\alpha^{\ \alpha'} \psi_{\alpha'}$ & $\psi'_{\dot{\alpha}} = U_{\dot{\alpha}}^{\ \dot{\alpha}'} \psi_{\dot{\alpha}'}$

claim to verify: $\begin{pmatrix}V_1\\\vdots\\V_4\end{pmatrix}' = \underset{SO(4)}{R} \begin{pmatrix}V_1\\\vdots\\V_4\end{pmatrix}$

Lorentz grp SO(1,3)

$V^\mu W_\mu = V_\nu \eta^{\mu\nu} W_\mu$

$= V^\nu \eta_{\mu\nu} W_\mu$

The definition is

$V^T \eta W = V'^T \eta W'$

$\eta = \mathrm{Diag}(-1, \mathbb{1}_3)$

$V' = e^L V \quad ; \Rightarrow \Lambda^T \eta \Lambda = \eta$

$= \Lambda V$

$L^T \eta = -\eta L$

$\Lambda^T = e^{L^T}; \quad \Lambda^T \eta = \eta \Lambda^{-L} \dots$

$\Rightarrow \Lambda^T \eta \Lambda = \eta \Lambda^{-1} \Lambda = \eta$

$A^{ab}_{cd} = \delta^{[a}_{k} \delta^{b]}_{d}$

$(L_{\mu\nu})_{kl} = \delta k [\mu \eta \nu] l$

$\text{g. } L_{01} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0\end{pmatrix}$ NB: It's symmetric!

$\mathcal{L} \equiv -i L_{\mu\nu}$ so that

$\Lambda = e^{i \frac{w}{2}^{\mu\nu} \mathcal{L}_{\mu\nu}}$

& $[\mathcal{L}_{\mu\nu}, \mathcal{L}_{\rho\theta}] = i\eta_{[\rho}^{[\mu} \mathcal{L}^{\nu]}_{\theta]}$

same as SO(1,3)

$\mathcal{L}_{\bar{a}\bar{b}} \quad K_{\bar{b}4} = i \mathcal{L}_{\bar{b}0}$

$\underset{1,2,3}{\uparrow}$

$T^{\pm}_{\bar{a}} = \frac{1}{2} \epsilon_{\bar{a}\bar{b}\bar{c}} \mathcal{L}_{\bar{b}\bar{c}} \pm K_{\bar{b}4}$

$\Downarrow$

$\underbrace{\text{Satisfies } SU(2)_L \times SU(2)_R}$

$\Downarrow$

Lorentz irrep (j_L, j_R)

$j_X = 0, \frac{1}{2}, 1, \dots$

γ_i for SO(2N) defined iteratively

$\gamma_i^{(m+1)} = \gamma_i^{(m)} \otimes \sigma_3 \quad m = 1, \dots N-1$

$\gamma^{(m+1)}_{2m+1} = \mathbb{1}_{2m} \otimes \sigma_1$

$\gamma^{(m+1)}_{2m} = \mathbb{1}_{2m} \otimes \sigma_2$

$\gamma_1^{(2)} = \sigma_1 \otimes \sigma_3$

$\gamma_2^{(2)} = \mathbb{1}_2 \otimes \sigma_1$

$\gamma_4^{(2)} = \mathbb{1}_2 \otimes \sigma_2$