Effect of Peculiar Velocities on Density Contrast

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***Abstract*—It is known that for large distances and negligible peculiar velocities, the observed red-shift is related directly to the distance of the source, viz. $Z = HR/c$. Using this as an approximation, one can find the density contrast by observing representative objects in the sky, viz. $\delta^S = \delta^S(\theta, \phi, R = cZ/H)$. An improvement can be made by accounting for non-zero peculiar velocities. Let $Z = HS/c$, where $S = R$ if we assume zero peculiar velocity. Our objective here, is to find a relation between the observed quantity $\delta^S = \delta^S(\theta, \phi, S)$ and the relevant quantity $\delta^R = \delta^R(\theta, \phi, R)$. The main importance of this, is that even in the linear theory, where peculiar velocities are small, δ^S and δ^R are significantly different, although in this discussion, we will not prove this.**

I. Background

The notation and known results used in what follows, have been briefly summarized here. The position of an object, is given by $\vec{R} = a\vec{r}$, where a quantifies scaling/expansion of the universe and $\vec{r}$ represents the co-moving coordinate. Consequently, the velocity is given by $\vec{V} = \dot{a}\vec{r} + a\dot{\vec{r}}$. Recalling, $H \equiv \dot{a}/a$ and $\vec{u} \equiv \dot{\vec{r}}$ (peculiar velocity), we obtain

$$\vec{V} = H\vec{R} + a\vec{u}. \tag{1}$$

Recall that the red-shift $Z \equiv (\lambda_{\text{obs}} - \lambda_{\text{em}})/\lambda_{\text{em}}$. Using the Doppler effect for light, we have $\lambda_{\text{em}} = \left(\frac{1-\beta}{1+\beta}\right)\lambda_{\text{obs}}$, which entails $Z \approx \beta$, where $\beta = V_{\text{los}}/c$, and V_{los} is the line of sight speed. Further, it can be shown, that for electromagnetic radiation, that was emitted at t_{em}, and observed at t_{obs}, $\lambda_{\text{obs}}/\lambda_{\text{em}} = a(t_{\text{obs}})/a(t_{\text{em}})$ which entails

$$\frac{a_{\text{obs}}}{a_{\text{em}}} = 1 + Z.$$

We will use some results from the linear theory for density contrast, which is defined implicitly as $\rho(\vec{r}, t) = \overline{\rho}(t)(1 + \delta(\vec{r}, t))$, where $\overline{\rho}$ is the background/averaged mass density, ρ is the mass density. In the Newtonian limit, using the fluid approach, it is known that

$$\begin{aligned}
\frac{\partial \delta}{\partial t} + \vec{\nabla}.[(1+\delta)\vec{u}] &= 0, \\
\frac{\partial u}{\partial t} + \frac{2\dot{a}}{a}\vec{u} + (u.\nabla)u &= -\frac{1}{a^2}\vec{\nabla}\phi, \\
\nabla^2\phi &= 4\pi G a^2 \overline{\rho}\delta,
\end{aligned}$$

hold ($\vec{\nabla} \equiv \partial/\partial r_x \hat{x} + \partial/\partial r_y \hat{y} + \partial/\partial r_z \hat{z}$), while for small u, in the linear limit, it follows from these that,

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho\delta = 0.$$

This can be solved to obtain two independent solutions, $D_\pm(t)$. The following results will be useful.
(i) It is found that $D_+ = a$ is a growing solution (grows with time), for an Einstien De Sitter (EDS) universe. In general also, it has been shown that (TODO: figure the assumption, and if Ω_{nr} is the initial one)

$$d\ln D_+/d\ln a = f(\Omega_{\text{nr}}). \tag{2}$$

(ii) The most general solution can be written as

$$\delta(\vec{r}, t) = \delta_+(\vec{r})\frac{D_+(t)}{D_+(t_i)} + \delta_-(\vec{r})\frac{D_-(t)}{D_-(t_i)}.$$

It follows after some analysis, (TODO: state the assumption, which cosmology) that if we start with

$$\vec{v} = -\vec{\nabla}\psi, \tag{3}$$

where $\vec{v} \equiv d\vec{r}/dD_+$ and $\psi \equiv 2a\phi/3H_0^2\Omega_{\text{nr}}D_+$, then $\delta_- = 0$. Also, from the definition of ψ, it follows

$$\nabla^2\psi = \frac{\delta}{D_+} \tag{4}$$

II. The Relation between δ^S and δ^R

We start with $Z \approx \frac{\vec{V}.\hat{r}}{c}$ and precisely define $\vec{S} \equiv Z\hat{r}cH^{-1}$ to obtain $\vec{S} = (R + aH^{-1}\vec{u}.\hat{r})\hat{r}$ (using equation (1)). $\vec{u}$ can be expressed as $\frac{d\vec{r}}{dt} = \frac{dr}{dD_+}\frac{dD_+}{da}\frac{da}{dt}$, which entails $\vec{S} = (R + (H^{-1}\dot{a})D_+\vec{v}f(\Omega_{\text{nr}}).\hat{r})\hat{r}$ (using equation (2)). We are interested in the present time, in which case, if we assume $D_+(t_0) = 1$, we have

$$\vec{S} = \vec{R} + f_0(v.\hat{r})\hat{r}$$

where $f_0 \equiv f(\Omega_{\text{nr}\,0})$ and we used equation (3). From conservation of mass, we must have $\left(1 + \delta^S(\vec{S})\right)d^3\vec{S} = \left(1 + \delta^R(\vec{R})\right)d^3\vec{R}$, where $\vec{S}$ is related to $\vec{R}$ as stated earlier. Note that we must conserve $(1+\delta)d\tau$ and not $\delta d\tau$ (where $d\tau$ is the volume element). Since, $d^3\vec{S} = \frac{\partial(S_x, S_y, S_z)}{\partial(R_x, R_y, R_z)}d^3\vec{R}$, effectively we are only required to evaluate the Jacobian to find an explicit relation. Evaluating the Jacobian directly is tedious. Instead, one may note that one can write $S\hat{r} = R(1 + U/R)\hat{r}$, where $U = f_0(\vec{v}.\hat{r})$, which entails that in spherical coordinates, θ and ϕ remain unchanged. Consequently $d^3\vec{S} = S^2 dS \sin\theta d\theta d\phi$ can be written as $(1 + U/R)^2(1 + dU/dR)R^2 dR \sin\theta d\theta d\phi$, which entails $J = \left(1 + \frac{U}{R}\right)^2 \left(1 + \frac{dU}{dR}\right)$. The required relation then, is

$$1 + \delta^R(\vec{R}) = \left(1 + \delta^S(\vec{S})\right)\left(1 + \frac{U}{R}\right)^2\left(1 + \frac{dU}{dR}\right).$$

where the second term in the Right Hand Side (RHS), maybe dropped for distant objects. However, U is still unknown and to resolve that, for the approximation, we note that we only require dU/dR. Let us work in the fourier space, with a single mode to simplify calculations and later sum the modes. We start with $\vec{v} = \vec{v}_k e^{-i\vec{k}.\vec{R}}$, $\psi = \psi_k e^{-i\vec{k}.\vec{R}}$ and substitute them in equation (3) to get $\vec{v}_k = i\vec{k}\psi_k$. Further, for $\delta = \delta_k e^{-i\vec{k}.\vec{R}}$, using equation (4), we get $\vec{v}_k = i\vec{k}\delta_k/k^2$. Also, substituting for $\vec{v}$ in U, we have

$$U = f_0\vec{v}_k.\hat{r}e^{-i\vec{k}.\vec{R}} = \frac{if_0\delta_k\vec{k}.\hat{r}e^{-i\vec{k}.\vec{R}}}{k^2} = \frac{if_0\delta_k\mu e^{-i\vec{k}.\vec{R}}}{k}$$
$$\frac{dU}{dR} = f_0\mu^2\delta_k e^{-i\vec{k}.\vec{R}} = f_0\mu^2\delta,$$

where $\mu = \hat{k}.\hat{r}$, the cosine of the angle between $\vec{k}$ and line of site. Consequently we have $(1 + \delta^S) \approx (1 + \delta^R)\left(1 + f_0\delta^R\mu^2\right)^{-1}$, where we suppressed arguments of δ. Now keeping to first order in δ^R, we have $\delta^S \approx \delta^R(1 - f_0\mu^2)$. Substituting $\delta^S = \delta^S_k e^{-i\vec{k}.\vec{R}}$ and $\delta^R = \delta^R_k e^{-i\vec{k}.\vec{R}}$, we have the final relation

$$\delta^S_k(\vec{k}) \approx \delta^R_k(\vec{k})\left(1 - f_0\mu^2\right).$$

The power spectrum is given by the square of δ_k, which is

$$P^S(\vec{k}, \mu) = \left(1 - f_0\mu^2\right)P^R(\vec{k}).$$