Indian Institute of Science Education and Research Mohali

Curves and Surfaces (MTH201)
Academic Session 2012-13
Exercise Sheet 1

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1. For what functions f the curve $\gamma(t) = (\cos t, \sin t, f(t))$ is planar?
2. A curve is given in the polar form $\gamma(\theta) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$. Show that its curvature is given by
$$\kappa(\theta) = \frac{2r'(\theta)^2 - r(\theta)r''(\theta) + r(\theta)^2}{(r(\theta)^2 + r'(\theta)^2)^{3/2}}.$$
3. Find all curves for which $\kappa = 1 = \tau$.
4. Calculate curvature and torsion for the curve $\gamma(t) = \left(t, \frac{1+t}{t}, \frac{1-t^2}{t}\right)$.
5. Calculate the curvature, torsion and Fernet frame for the curve $\gamma(t) = (\cos 2t, \sin 2t, 2\sin t)$.
6. Compute the Frenet frame for the curve $\gamma(t) = \left(t, \frac{t^2}{2}, \frac{t^3}{6}\right)$. Show that as t approaches ∞ the curve γ becomes a straight line, and the Frenet frame becomes: $\mathbb{T} = (0, 0, 1), \mathbb{N} = (0, -1, 0), \mathbb{B} = (1, 0, 0)$.
7. For a unit speed curve γ show that: $\dddot{\gamma} = -\kappa^2\,\mathbf{T} + \dot{\kappa}\,\mathbf{N} + \kappa\tau\,\mathbf{B}$.
8. Find a unit speed plane curve γ for which $\gamma(0) = (0, 0)$ and the curvature function is $\kappa(s) = \frac{1}{1+s^2}$.
9. Determine if the curve $\gamma(t) = (ae^t\cos t, ae^t\sin t, be^t)$ is a cylindrical helix.
10. (a) Show that the curve $\gamma(t) = (t - \sqrt{3}\sin t, 2\cos t, \sqrt{3}t + \sin t)$ is a cylindrical helix.
    (b) Find a helix $\theta(t) = (a\cos t, a\sin t, bt)$ and a rigid motion that M such that M takes γ to θ.
11. Find the equation of the osculating plane of the curve $\gamma(t) = (a\sin t + b\cos t, a\cos t + b\sin t, c\sin 2t)$ at $t = 0$.
12. Prove that curvature and torsion of the curve $\gamma(t) = (a(3t - t^3), 3at^2, a(3t + t^3))$ are equal.
13. For what values of a and b the curvature and torsion of the curve $\gamma(t) = (ae^t\cos t, ae^t\sin t, be^t)$ are equal.
14. Let γ be a unit speed curve with nowhere vanishing curvature and torsion. Show that the trace of γ is contained on the surface of a sphere of radius R if and only if
$$\frac{1}{\kappa(s)^2} + \frac{\dot{\kappa}(s)^2}{\kappa(s)^4\tau(s)^2} = R^2.$$
15. Let γ be a unit speed curve with nowhere vanishing curvature and torsion. Show that the trace of γ is contained on the surface of a sphere if and only if
$$\frac{\tau(s)}{\kappa(s)} + \frac{d}{ds}\left(\frac{1}{\tau(s)}\frac{d}{ds}\left(\frac{1}{\kappa(s)}\right)\right) = 0.$$
From this conclude that a non planar curve with constant curvature can not lie on a sphere.
16. Let γ be a smooth unit speed nowhere vanishing curvature space curve whose torsion is a constant $\tau = c$ and whose trace is contained is a sphere. Prove that the curvature function of γ looks like:
$$\kappa(s) = \frac{1}{a\cos cs + b\sin cs}$$
for some constant $a, b \in \mathbb{R}$.
17. Show that the evolute of the ellipse $\gamma(t) = (a\cos t, b\sin t)$ is the curve $\alpha(t) = \left(\frac{a^2 - b^2}{a}\cos^3 t, \frac{b^2 - a^2}{b}\sin^3 t\right)$.
18. Show that the evolute of the parabola $y = ax^2$ is the curve whose trace is given by $27x^2 = 16a\left(y - \frac{1}{2a}\right)^3$.
19. Prove that the trace of the curve $\gamma(t) = (t\cos t, t\sin t, t)$ lies on a cone. Find the curvature and torsion of this curve.