Arrind # 9.1

Classical

Phase space; Points in phase space, q, p, Poisson Bracket, $H(q,p)$

Physical observable: Hamiltonian

Canonical transformation CM

Const. of motion is observable

PB

State, Hilbert space

Complex linear vector space

Hamiltonian Operator

Hermitian Operators

Unitary transformations

thus there's a corresponding Hermitian Operator

Commutator

If $\hat{G}$ is a COM, then $\hat{G}$ is Hermitian

$$U(\alpha) = e^{-i\alpha\hat{G}/\hbar} \qquad \alpha \text{ constant, real}$$

$$U^\dagger(\alpha) = U^{-1}(\alpha)$$

Quantum state if initially is $|\Psi\rangle$, then

$$|\Psi'\rangle = U|\Psi\rangle$$

Schrodinger's Eqn: $i\hbar \frac{d}{dt}|\Psi\rangle = \hat{H}(t)|\Psi\rangle$

$$0 = \frac{d\hat{G}(t)}{dt} = \frac{\partial \hat{G}}{\partial t} - \frac{i}{\hbar}\underbrace{[\hat{G}(t), \hat{H}(t)]}_{\text{PB}}$$

$\because$ COM

$$\Rightarrow \frac{\partial \hat{G}}{\partial t} = \frac{i}{\hbar}[\hat{G}(t), \hat{H}(t)]$$ using which

Claim: we have

$$|\Psi'\rangle = \overbrace{e^{-i\alpha\hat{G}(t)/\hbar}}^{\text{unitary}} \underbrace{|\Psi\rangle}_{\text{state}}$$

$$\Rightarrow i\hbar \frac{d}{dt}|\Psi'\rangle = \hat{H}(t)|\Psi'\rangle$$

Prove it.

Arvind 4.3

Symm QM

$\hat{G}$ observable COM

$U(\alpha) = e^{-\frac{i\alpha\hat{G}}{\hbar}}$

If $|\psi(t)\rangle$ is a soln of SE,

then $|\psi'(t)\rangle = U(\alpha)|\psi(t)\rangle$

is also a soln.

This is not available in CM

$q_r \to q_r + \epsilon\{q_r, G(q,p)\}$

$p_r \to p_r + \epsilon\{p_r, G(q,p)\}$

Finite transformation:

Apply it n times with

$n \to \infty$ $\epsilon \to 0$

Let $C(\alpha)$ be a finite transformation. Then we have

$f(q,p) \xrightarrow{C(\alpha)} f(q,p) + \frac{\alpha}{1!}\{G(q,p), f(q,p)\} + \frac{\alpha^2}{2!}\{G,\{G,f\}\} + \dots$

$\bar{G}(q,p) \equiv \frac{\partial G}{\partial q_r}\frac{\partial}{\partial p_r} - \frac{\partial G}{\partial p_r}\frac{\partial}{\partial q_r}$

then $f'(q,p) = f(q,p) + \frac{\alpha}{1!}\bar{G}f + \frac{\alpha^2}{2!}\bar{G}(\bar{G}f) + \dots$

so that $f'(q,p) = e^{\alpha\bar{G}(q,p)} f(q,p)$

now $C(\alpha)f(q,p) = e^{\alpha\bar{G}(q,p)} f(q,p)$

$\hat{A} \xrightarrow{U(\alpha)} \hat{A}' = U(\alpha)\hat{A}U^{-1}(\alpha)$

Recall: $e^{-\lambda\hat{B}}\hat{A}e^{\lambda\hat{B}} = \hat{A} + \lambda[\hat{B},\hat{A}] + \frac{\lambda^2}{2!}[\hat{B},[\hat{B},\hat{A}]] + \frac{\lambda^3}{3!}[\hat{B}, \dots$

Defn: $\hat{\bar{G}}(\cdot) = (i\hbar)^{-1}[\hat{G}, \cdot]$

superoperators (operate on operators)

$\rho' = \sum_j E_j^\dagger \hat{\rho} E_j$; $\sum_j E_j^\dagger E_j = I$

ρ $n\times n$ matrix, Hermitian

$\begin{pmatrix} \text{super} \\ \text{op} \end{pmatrix} \begin{pmatrix} \rho_{11} \\ \rho_{12} \\ \vdots \\ \rho_{nn} \end{pmatrix} = \begin{pmatrix} \rho_{11}' \\ \vdots \\ \rho_{nn}' \end{pmatrix}$

So now

$\hat{A}' = \hat{A} + \alpha\hat{\bar{G}}\hat{A} + \frac{\alpha^2}{2!}\hat{\bar{G}}\hat{\bar{G}}\hat{A} + \dots$

$= e^{\alpha\hat{\bar{G}}}\hat{A}$

Lie Group $\mathcal{G}$ — Symmetry group of physical systems classical / Quantum Mech.

order "r real parameters" $\alpha \equiv \{\alpha_1, \alpha_2, \dots, \alpha_r\}$

$\mathcal{G}$: $g(\alpha) = \exp(\alpha_i e_i)$

$g(\alpha)g(\beta) = g(f(\alpha,\beta))$

$[e_i, e_k] = c_{jk}^l e_l$

Defn: $e_i \equiv$ Canonical coordinates in the gp space

group commutation relations, Lie algebra correspond to $\mathcal{G}$.

Defn: c_{ik}^l structure constant of $\mathcal{G}$.

CM

$g(\alpha)$ is a CT, $C(\alpha) = e^{\alpha_i\bar{G}_i}$

$C(\alpha)C(\beta) = C(f(\alpha,\beta))$

$[\bar{G}_i, \bar{G}_k] = c_{jk}^l \bar{G}_l$

$[G_j, G_k] = c_{jk}^l G_l + d_{jk} + \dots$

QM

$g(\alpha)$ $U T U = e^{-i\alpha_j G_j/\hbar}$

$U(\alpha)U(\beta) = e^{iw(\alpha,\beta)} U(f(\alpha,\beta))$

$[\hat{\bar{G}}_j, \hat{\bar{G}}_k] = c_{jk}^l \hat{\bar{G}}_l$

$(i\hbar)^{-1}[\hat{G}_j, \hat{G}_k] = c_{jk}^l \hat{G}_l + d_{jk} + \dots$

Classical Theory of Constraint Mechanics

Configuration Space Q, dimension N

$q^j \quad j = 1, 2, \ldots N$

Lagr. Mech. $q^j = \frac{d}{dt} q^j = \dot{q}^j$

local coordinates

TQ ≡ Tangent Bundle over Q

$\mathcal{L}(q, \dot{q}) \rightarrow$ Hessian Matrix $N \times N$

$$H_{jk}(q, \dot{q}) = \frac{\partial^2 L(q, \dot{q})}{\partial \dot{q}^j \partial \dot{q}^k}, \quad j k = 1, \ldots N \quad (1.1)$$

Standard case ≡ $\det(H_{jk}(q, \dot{q})) \neq 0$,

$H_{jk}(q, \dot{q})$ nonsingular (1.2)

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^j} - \frac{\partial \mathcal{L}}{\partial q^j} = 0 \quad (1.3) \xrightarrow{\Leftrightarrow} \ddot{q}^j = f^j(q, \dot{q}), \; j = 1, \ldots N \quad (1.4)$$

(given H_{jk} is non-sing)

$$\varphi_{\mathcal{L}} : p_j = \frac{\partial L}{\partial \dot{q}^j} \quad j = 1, 2, \ldots N \quad (1.5)$$

collection of the set of eq^ns

Claim: Consequence of H being non-singular is that we can invert these eq^ns to obtain $\dot{q}^j$ from $p_j \; \forall j$.

Hint: Implicit function thm.

Next: What are the major changes that'll come in the non-singular case

Mukunda §1.1

Consider a composition space "unscheduled" $Q = \{q^j\}$ dim N

Tangent Bundle TQ, dim $2N$
Lagrangian variables $q^j, \dot{q}^j$ (1.6)

Co-Tangent Bundle or phase space T^*Q, dim $2N$, Hamiltonian variables q^j, p_j

$\varphi_{\mathcal{L}}$ = Legendre Map

Abraham & Marsden – "Foundations of Mechanics"

Standard case: φ_L^{-1} exists $\Leftrightarrow \varphi_L$ = one-one, onto $TQ \rightarrow T^*Q$

each $\dot{q}^j$ = some f^n of q, p &

$$f(q, \dot{q}) = f'(q, p) \quad (1.7)$$

$$H(q, p) = p_j \dot{q}^j - \mathcal{L}(q, \dot{q}) \quad (1.8)$$

$$\dot{q}^j = \{q^j, H\} = \frac{\partial H(q, p)}{\partial p_j}$$

$$\dot{p}_j = \{p, H\} = -\frac{\partial H(q, p)}{\partial q^j}$$

$$\frac{df}{dt}(q, p, t) = \{f, H\} + \frac{\partial f}{\partial t}(q, p, t) \quad (1.9)$$

NB: $H = H(q, p)$ time independent

$$\{f(q, p), g(q, p)\} \equiv \frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q^j} \quad (1.10)$$

- anti-symmetric
- linear in f & $g \rightarrow$? (anti?)
- Jacobi Identity

$q(0), \dot{q}(0) \longrightarrow q(t) \; \forall t$, trajectory in Q, also in TQ

$q(0), p(0) \longrightarrow q(t), p(t) \; \forall t$, trajectory in T^*Q (1.11)

Singular Case: $\det(H_{jk}(q, \dot{q})) = 0$

$\Updownarrow$

H_{jk} is a singular matrix

Motivation:

More fundamental cases fall in Singular case

: EM, GTR, YM – singular matrix

: Proceedings of the Cambridge ~~Phy~~ Philosop. Society – PAM Dirac

: L. Rosenfeld (was with Neils Bohr)

: 1949 – First Canadian Mathematical Congress, UBC in Vancouver
Canadian Journal of Math
vol. 2, page 129 (1950)

: 1964 – Yeshiva University
"Lectures on QM"

: PG Bergmann (in the context of GTR)
"Generalized Hamiltonian Dynamics"
"Constrained Hamiltonian Dynamics"

(i) Lagrangian EOM do not determine all $\ddot{q}^j$'s
This is same as saying we only have constraints among q^j & $\dot{q}^j$

(ii) $\varphi_{\mathcal{L}} : TQ \to T^*Q$ becomes non-invertible, many to one map. viz. $\varphi_{\mathcal{L}}$ maps to a part of T^*Q.

(iii) $\mathcal{L}$ EOM $\to$ Hamiltonian EOM in pleminary way (some $\dot{q}^j$ will be there)

(iv) Step by step consistency analysis

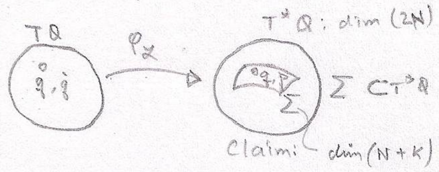
$$\text{rank}\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^j \partial \dot{q}^k}\right) = K < N \quad (2.1)$$

~~Constant~~ Content of
$$\varphi_{\mathcal{L}} =: p_j = \frac{\partial L}{\partial \dot{q}^j}$$

Claim

Properties of momentum
(i) K independent
(ii) N−K dependent

Properties of velocities
(iii) K ~~independent~~ solved
(iv) N−K ~~dependent~~ unsolved (2.2)



$$\varphi_{\mathcal{L}} : (q, \dot{q}) \in TQ \longrightarrow \left(q, p = \frac{\partial L}{\partial \dot{q}}\right) \in T^*Q \quad (2.3)$$

$\because 2N - (N+K) = N-K$

$$\Sigma = \{ (q,p) \in T^*Q \mid \phi_m(q,p) = 0,\ m = 1, 2 \dots, N-K \} \quad (2.4)$$

Another way to see this is that (N−K) moments are dependent.

Eg. $\mathcal{L} = \frac{1}{2} A_{ij} \dot{q}_i \dot{q}_j \dots (2.5)$ where $A_{ij} = A_{ji} \in \mathbb{R}$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \cancel{2} \times \frac{1}{\cancel{2}} A_{ij} \dot{q}_i \delta_{jk} = A_{ik} \dot{q}_i$$

$\Rightarrow p_j = A_{jk} \dot{q}^k \quad (2.6)$

If $\det(A) = 0$

$\Rightarrow \exists\ c^j$ s.t. $A_{ij} c^j = 0$

For $\det(A) = 0$ we thus have for some c^j
$c^j p_j = 0$ which give the constraint

The # of such constraints will be (N−K) $\because$ # of ~~variables~~ null vectors for rank k matrices

Claim: If a p is independent, then $\dot{q}$ corresponding to it can be solved.

If a p is ~~not~~ dependent, then the corresponding $\dot{q}$ can't be solved.

We note in general that $f(q, \dot{q})$ can't be written in q, p only.

However, in case of the following

$$p_j \dot{q}^j - \mathcal{L}(q, \dot{q}) = H_0(q, p)$$

is always expressable in ~~the form~~ ~~$p_j \dot{q}^j$~~ terms of H_0.

Mukunda 6.3x

$$p_j \dot{q}^j - L(q, \dot{q})$$

Regular H | singular $H_0(q,p)$, $\phi_m(q,\phi) = 0$

$$\dot{q}^j = \frac{\partial H}{\partial p_j} \dashrightarrow \dot{q}^j = \frac{\partial H_0}{\partial p_j} + v_m \frac{\partial \phi_m}{\partial p_j}$$

v_m → Lagrange multiplier

$$\frac{\partial L}{\partial q^j} = -\frac{\partial H}{\partial q^j} \qquad \frac{\partial L}{\partial q^j} = -\frac{\partial H_0}{\partial q^j} - v_m \frac{\partial \phi_m}{\partial q^j} \quad (2.13)$$

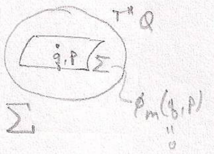
$$\dot{q}^j = \frac{\partial H_0}{\partial p_j} + v_m \frac{\partial \phi_m}{\partial p_j} = \{q^j, H_0\} + v_m\{q^j, \phi_m\}$$
$$\approx \{q^j, H_0 + v_m \phi_m\}$$

$$\dot{p}_j = -\frac{\partial H_0}{\partial q^j} - v_m \frac{\partial \phi_m}{\partial q^j} = \{p, H_0\} + v_m\{p_j, \phi_m\}$$
$$\approx \{p_j, H_0 + v_m \phi_m\}$$
$$\phi_m(q,p) \approx 0$$

$$\frac{df}{dt} \approx \frac{\partial f}{\partial t} + \{f, H_0 + v_m \phi_m\} \quad (2.15)$$

$$\delta q^j \approx \{q^j, H_0 + v_m \phi_m\} \delta t$$
$$\delta p_j \approx \{p_j, H_0 + v_m \phi_m\} \delta t \quad (2.16)$$
$$\delta \phi_m \approx \{\phi_m, H_0\} \delta t + \{\phi_m, v_{m'} \phi_{m'}\} \delta t \approx 0 \quad (2.17)$$

$\approx$ Weak equation
$=$ Strong equation



$$q, p \in \Sigma \longrightarrow q + \delta q, p + \delta p \in \Sigma$$

$$\{\phi_m, H_0\} + \{\phi_m, \phi_{m'}\} v_{m'} \approx 0$$

case(1):
$\{\phi_m, H_0\} = 0$
$\{\phi_m, \phi_{m'}\} = 0$
⇒ You could choose any arbitrary v_m as f^n of time

case(2):
$(\{\phi_m, \phi_{m'}\})$ is non-singular
⇒ $v_{m'}$ are known
∵ I can invert $\{\phi_m, \phi_{m'}\}$
NB: $\{\phi_m, \phi_{m'}\}$ is anti symmetric
∴ Must be even dimension else det = 0

Claim: For any anti-symmetric matrix, if the dimension's even, the det is 0.

$$v_m \approx -C_{mm'}\{\phi_{m'}, H_0\}$$

$$\Rightarrow \frac{df}{dt} \approx \{f, H_0\} + v_m\{f, \phi_m\}$$

$$\frac{df}{dt} \approx \{f, H_0\} - \{f, \phi_m\} C_{mm'} \{\phi_{m'}, H_0\}$$

also called the Dirac Bracket.

$$\{\phi_m, H_0\} + \{\phi_m, \phi_{m'}\} v_{m'} \approx 0$$

$\chi(q,p) \approx 0$ — Secondary constraints | Partial determination of some v_m's in terms of others

$$\Sigma = \{(q,p) \in T^*Q \mid \phi_m(q,p) \approx 0\} \subset T^*Q$$
$$\to \Sigma' = \{(q,p) \in T^*Q \mid \phi_m(q,p) = 0, \chi\ldots(q,p) \approx 0\}$$
$$\bigcap \Sigma \subset T^*Q$$
$$\to \Sigma'' = \ldots \subset \Sigma' \subset \Sigma \subset T^*Q$$

Now you want χ to hold at all time. Then

$$\{\chi, H_0\} + \{\chi, \phi_m\} v_m \approx 0$$

This will produce tertiary condition & so that must be preserved over time & so on.

$$\frac{df}{dt} \approx \frac{\partial f}{\partial t} + \{f, H_0\} + \{f, \phi_m\} v_m$$

indetermined till the end.

$$\phi_m \approx 0, \ \chi \approx 0 \ ; \ v_m \approx v_m^{(0)}(q,p) + v_A C_{Am}(q,p)$$

Claim: Since all constraints are of the form linear in v_m with an inhomogeneous part v_m is the most general soln.

The conditions we demanded should be automatically satisfied (assuming you've reached the end)

$$\{\phi_m \text{ or } \chi, H_0\} + \{\phi_m \text{ or } \chi, \phi_{m'}\}\left(v_{m'}^{(0)}(q,p) + v_A C_{Am'}(q,p)\right) \approx 0$$

$$\underbrace{\{\phi_m \text{ or } \chi, H_0 + v_{m'}^{(0)} \phi_{m'}\}}_{\overset{??}{0}} + \underbrace{\{\phi_m \text{ or } \chi, C_{Am'} \phi_{m'}\}}_{\overset{??}{0}} v_A \approx 0$$

Initial EOM

$$\frac{df}{dt}(q,p) = \frac{\partial f}{\partial t} + \{f, H_0\} + \{f, \phi_m\} v_m \quad , v_m \text{ free}$$

$\phi_m(q,p) = 0$ – primary constraint (arises from the struct. of the Lagrangian)

$\{\phi_m, H_0\} + \{\phi_m, \phi_{m'}\} v_{m'} \approx 0$ – consequence of EOM, $\frac{d}{dt}\phi_m = 0$; ϕ_m must remain fixed.

more constraints $\chi_{\dots}(q,p) \approx 0$ | some restriction on v_m

Then again,

$$\{\chi_{\dots}, H_0\} + \{\chi_{\dots}, \phi_{m'}\} v_{m'} \approx 0$$

& so on.

$\phi_m \approx 0, \chi_{\dots} \approx 0$; since some of the vs will eventually be (maybe) still be free, we have to have

$$v_m = v_m^{(0)}(q,p) + c_{Am}(q,p) v_A \quad , v_A \equiv \text{subset of } v_m \text{ that're free.}$$

So now, $\frac{df}{dt} \approx \frac{\partial f}{\partial t} + \{f, H_0\} + \{f, \phi_m\}\left(v_m^{(0)}(q,p) + c_{Am}(q,p) v_A\right)$

$$\approx \frac{\partial f}{\partial t} + \{f, H\} + \{f, \phi_A\} v_A \quad \text{where}$$

$H = H_0 + \phi_m v_m^{(0)}$

& $\phi_A = \phi_m c_{Am}$ $\quad \because \phi_m$ is set zero in the end (in a weak eq^n)

$\Rightarrow \{\phi_m, H\} \approx 0, \ \{\phi_m, \phi_A\} \approx 0$
$\{\chi_{\dots}, H\} \approx 0, \ \{\chi_{\dots}, \phi_A\} \approx 0$ (2.26) ; $\because$ the analysis is over and v_As were free, can't possibly have more constraints.

$$\Sigma = \{(q,p) \mid \phi_m(q,p) = 0\} \longrightarrow \Sigma_f = \{(q,p) \mid \phi_m = 0, \chi_{\dots} = 0\}$$

$$\cap \ \Sigma \subset T^*Q$$

$\{\phi_m \text{ or } \chi_{\dots}, H \text{ or } \phi_A\} \approx 0$ on Σ_f, modulo $\phi_m \approx 0, \chi_{\dots} \approx 0$ (2.27)

Def^n: $f(q,p)$ is 1st class $\equiv \{f, \phi_m \,\&\, \chi_{\dots}\} \approx 0$
2nd class $\equiv$! 1st class

Mukanda 6.4

The Dirac Bracket

$\phi_m \to$ free to form independent linear combinations (over (p,q) variables) — form as many 1st class ϕ_A as possible — remaining must be second class

$\chi_{\dots} \to$ free to form independent linear combinations + can add pieces of ϕ_m — form as many 1st class χ as possible — remaining are 2nd class. $\longrightarrow \Theta_a(q,p)$

Dirac: Primary & Secondary difference not so important $\because$ primary depends on $\mathcal{L}$ & various $\mathcal{L}$s may give

Primary constraints $\phi_m \approx 0$
Secondary constraints $\chi_{\dots} \approx 0$

$$\Sigma_f = \{(q,p) \in T^*Q \mid \phi_m \approx \chi_{\dots} \approx 0\} \subset T^*Q$$

$f(q,p)$ is first class $\equiv \{f, \phi_m \text{ or } \chi_{\dots}\} \approx 0$ on Σ_f

Infinitesimal canonical transf

$\delta q = \varepsilon\{q, f\}$, $\delta p = \varepsilon\{p, f\}$ some $f(q,p)$ $\varepsilon \ll 1$

$\delta g(q,p) = \varepsilon\{g, f\}$

So a first class f^n when used to perform a canonical transformation, leave the constraints invariant within Σ_f.

	1st class	2nd class
Primary constraint	ϕ_A	.. $\to \theta_a(q,p)$
Secondary constraint	..	..

claim:

$\det(\{\theta_a, \theta_b\}) \neq 0$ dim must be even

$$C^{ab}(q,p)\{\theta_b(q,p), \theta_c(q,p)\} = \delta^a_c$$

$$\{f,g\}^* = \{f,g\} - \{f,\theta_a\} \underset{\text{anti-symmetric}}{C^{ab}} \{\theta_b, g\}$$

Dirac bracket

- linearity & anti-symmetry (obvious)
- Derivation law $\{f, gh\}^* = g\{f,h\}^* + \{f,g\}^* h$
- $\{f, \{g,h\}^*\}^* + \dots + \dots = 0$ Jacobi (apparently not so simple.
- $\{f, \theta_c\}^* = 0$ (use $C^{ab}\{\theta_b, \theta_c\} = \delta^a_c$)

In the eq^n of motion, we can replace the bracket with dirac's bracket $\because$ H, & ϕ_A being first class $\Rightarrow \{H \text{ or } \phi_A, \theta\} \approx 0$.

We thus have

$$\frac{df}{dt} \approx \frac{\partial f}{\partial t} + \{f, H\}^* + \{f, \phi_A\}^* v_A$$

Now still we have v_A s as unknown

We thus impose some conditions by defining $\xi_A(q,p) \approx 0$ where

$$\frac{d}{dt}\xi_A(q,p) \approx 0 \Rightarrow v_A \text{ is determined}$$

Gauge-constraints

Mukunda 6.5.

Symmetry transformation & their generator

Noether type symmetry

$\delta q^j = \varepsilon \phi^j\{q\}$ → infinitesimal point transf where $|\varepsilon| \ll 1$

$q \in Q \to q + \delta q \in Q$

$\delta L(q, \dot q) = \varepsilon \frac{d}{dt} F(q,p)$

$G(q,p) = F(q) - p_j \phi^j(q) = \text{COM}$

$\frac{dG}{dt} = \{G, H\} = 0$

$\delta q^j = \varepsilon\{G, q^j\}$; $\delta p_j = \varepsilon\{G, p_j\}$

sign

Symm $\longleftrightarrow$ COM $G(q,p)$

CT

Mukunda 6·6

$\delta q^j = \varepsilon \phi^j(q,\dot{q}) \Rightarrow \delta \mathcal{L}(q,\dot{q}) = \varepsilon \frac{d}{dt} F(q,\dot{q})$ (3·3)

$F(q,\dot{q}) - p_j \phi^j(q,\dot{q}) = G_0(q,p)$ modulo $\phi_m \approx 0$ (3·4)

$\delta q^j \approx \varepsilon \{ G_0(q,p) - u_m \phi_m(q,p), q^j \}$ modulo $\phi_m \approx 0$ (3·5)

(algebraic modifications)

$\Rightarrow \delta p_j \approx \varepsilon \{ G_0 - u_m \phi_m, p_j \}$ (a) $\rightarrow$ proves that it's a canonical transformation

$\{ G_0, H_0 + v_m \phi_m \} \approx 0$ (b) $\rightarrow \frac{d}{dt} G_0 \approx 0$ Symmetry generator

$\{ G_0 - u_m \phi_m, \phi_{m'} \} \approx 0$ (c) $\rightarrow$ preserves constraints

must satisfy $\{ \phi_m, H_0 + v_{m'} \phi_{m'} \} \approx 0$ (3·6)

$\delta q^j = \varepsilon f(t)\, \phi^j(q,\dot{q}) \Rightarrow \delta \mathcal{L}(q,\dot{q}) = \varepsilon \frac{d}{dt}\left(f(t) F(q,\dot{q}) \right)$

Type I

$G - u_A \phi_A = f(t)$ (primary 1st class constraints)

ϕ_A of the gauge type

Type II

$\delta q^j = \varepsilon \left(f(t) \phi^{(1)j}(q,\dot{q}) + \dot{f}(t) \phi^{(2)j}(q,\dot{q}) \right)$

$\delta \mathcal{L} = \varepsilon \frac{d}{dt} \left(f(t) F^{(1)}(q,\dot{q}) + \dot{f}(t) F^{(2)}(q,\dot{q}) \right)$

$G - u_A \phi_A = f(t)$ (linear combin of ϕ_A + secondary 1st class χ)

$+ \dot{f}(t)$ (" " ϕ_A)

Ann of Physics 99, 408 (1976)

Phys Scripta 21, 783 (1980)

A. J. Hansan & T. Ragge Ann of Phys 87, 498 (1974) "The relativistic top"

Hansan, Regge & Teitelboim 1976

Constrained Hamiltonian System

Lincei