The real Symplectic groups and their uses in physics; Uncertainty Relations

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Contents

1 The group $Sp(2, \mathcal{R})$

We return to the case of Cartesian coordinates and momenta in the QM, obeying the Heisenberg CCR's. Many useful and important results will be carefully described and proofs often left as good exercises. There will be important connections to the theory of Wigner distributions. Sometimes we will set $\hbar = 1$.

For one dof,we have a canonical pair of hermitian irreducible operators $\hat{q}$, $\hat{p}$ obeying

$$[\hat{q}, \hat{p}] = iI \tag{1.1}$$

As we have seen,translations in $\hat{q}$, $\hat{p}$ are unitarily induced by the displacement operators:

$$D(q_0, p_0) = e^{i(p_0\hat{q} - q_0\hat{p})},$$

$$D(q_0, p_0)^{-1} \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} D(q_0, p_0) = \begin{pmatrix} \hat{q} + q_0 \\ \hat{p} + p_0 \end{pmatrix} \tag{1.2}$$

Next we consider linear homogeneous transformations preserving eq. (1.1):

$$\begin{pmatrix} \hat{q'} \\ \hat{p'} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix}, \qquad a,\, b,\, c,\, d \,\text{real},$$

$$[\hat{q'}, \hat{p'}] = i \quad \Leftrightarrow ad - bc = 1 \tag{1.3}$$

This leads us to define the two-dimensional real symplectic group

$$Sp(2, \mathcal{R}) = \left\{ S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \text{real} \,\middle|\, det\, S = ad - bc = 1 \right\} \tag{1.4}$$

The symplectic aspect is not obvious in this form, but for many dof, it becomes clearer. This group turns out to be isomorphic to two other three-dimensional groups:

$$Sp(2, \mathcal{R}) = SL(2, \mathcal{R}) = SU(1, 1) = \text{double cover of}\, SO(2, 1) \tag{1.5}$$

The matrices $S \in Sp(2, \mathcal{R})$ are easily seen to obey:

$$S^T \in Sp(2, \mathcal{R}); \qquad S^T \beta S = \beta = i\sigma_z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

$$S^{-1} = \beta^{-1} S^T \beta \tag{1.6}$$

Three useful subgroups and their conventional names are the following:

$$K = SO(2) = \left\{ r(\phi) = \begin{pmatrix} \cos\phi/2 & -\sin\phi/2 \\ \sin\phi/2 & \cos\phi/2 \end{pmatrix} \mid 0 \le \phi \le 4\pi \right\} \tag{1.7}$$

These are phase space rotations and form a maximal compact subgroup. Both Fourier transformation $\mathcal{F}$ and parity P are in K:

$$\mathcal{F} = r(\pi) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\beta \quad : \quad \hat{q}' = -\hat{p},\, \hat{p}' = \hat{q};$$

$$P = r(2\pi) = \mathcal{F}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad : \quad \hat{q}' = -\hat{q},\, \hat{p}' = -\hat{p} \tag{1.8}$$

Next we have a one-dimensional abelian subgroup:

$$A = \left\{ m(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} \mid -\infty < \eta < \infty \right\} \tag{1.9}$$

These are reciprocal scalings of $\hat{q}$ and $\hat{p}$. Lastly we have the 'lens' subgroup, a so-called nilpotent subgroup:

$$N = \left\{ l(\xi) = \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix} \mid -\infty < \xi < \infty \right\} \quad : \quad \hat{q}' = \hat{q},\, \hat{p}' = \hat{p} + \xi\hat{q} \tag{1.10}$$

An important result is the 'Iwasawa decomposition' : any $S \in Sp(2,\mathcal{R})$ can be uniquely written as a product of factors from these three subgroups:

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = l(\xi)m(\eta)r(\phi),$$

$$\xi = (ac+bd)/(a^2+b^2), \quad \eta = \ln(a^2+b^2), \quad \phi = 2\arg(a-ib) \tag{1.11}$$

2 The metaplectic unitary representation of $Sp(2,\mathcal{R})$

The Stone-von Neumann theorem says that apart from unitary equivalence, there is a unique irreducible hermitian representation of the CCR (1.1). Therefore $\hat{q}'$ and $\hat{p}'$ must be related to $\hat{q}$ and $\hat{p}$ by some unitary transformation:

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad : \quad \begin{pmatrix} \hat{q}' \\ \hat{p}' \end{pmatrix} = S \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} = \overline{U}(S)^{-1} \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} \overline{U}(S) \tag{2.1}$$

Each $\overline{U}(S)$ is unique upto an S-dependent phase. Then one can easily prove the composition law

$$\overline{U}(S')\overline{U}(S) = (\text{phase dependent on} S' \text{and} S)\overline{U}(S'S) \tag{2.2}$$

We can try and simplify this phase to the maximum extent possible by using the freedom of phase in each factor $\overline{U}$. When we do this, it turns out that there is a residual 'two-valuedness' which cannot be eliminated:

$$\overline{U}(S')\overline{U}(S) = \pm\overline{U}(S', S) \tag{2.3}$$

So we say the operators $\overline{U}(S)$ give a double valued unitary representation of $Sp(2,\mathcal{R})$. It is called the metaplectic representation of $Sp(2,R)$, it is reducible and is the direct sum of two irreducible parts.

The action of the $\overline{U}(S)$ on the displacement operators in (1.2) is very simple:

$$\overline{U}(S)^{-1}D(q_0,p_0)\overline{U}(S) = D(q_0',p_0'), \quad \begin{pmatrix} q_0' \\ p_0' \end{pmatrix} = S^{-1}\begin{pmatrix} q_0 \\ p_0 \end{pmatrix} \tag{2.4}$$

It is good to check that this is consistent with the composition law (2.3).

Now we examine the explicit form of $\overline{U}(S)$. Since upon conjugation by $\overline{U}(S)$, $\hat{q}$ and $\hat{p}$ transform linearly, $\overline{U}(S)$ should be exponentials of (anti hermitian) quadratic expressions in $\hat{q}$, $\hat{p}$. These will be the generators of $\overline{U}(S)$. To start with, let us find the matrix generators of the defining representation(1.4) of $Sp(2, \mathcal{R})$ and their commutation relations (CCR's). Using conventional notations:

$$\begin{aligned} r(\phi) &= e^{-i\phi J_0}, \quad J_0 = \frac{1}{2}\sigma_2; \\ m(\eta) &= e^{-i\eta K_1}, \quad K_1 = \frac{i}{2}\sigma_3; \\ l(\xi) &= e^{-i\xi(J_0+K_2)}, \quad J_0 + K_2 = \frac{1}{2}\sigma_2 + \frac{i}{2}\sigma_1. \end{aligned} \tag{2.5}$$

Therefore

$$\begin{gathered} J_0 = \frac{1}{2}\sigma_2, \quad K_1 = \frac{i}{2}\sigma_3, \quad K_2 = \frac{i}{2}\sigma_1; \\ [J_0, K_1] = iK_2, \quad [J_0, K_2] = -iK_1, \quad [K_1, K_2] = -iJ_0 \end{gathered} \tag{2.6}$$

So at the level of $\overline{U}(S)$, we should have:

$$\overline{U}(r(\phi)) = e^{-i\phi\hat{J}_0} \quad : \quad e^{i\phi\hat{J}_0}\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}e^{-i\phi\hat{J}_0} = r(\phi)\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix} \Rightarrow$$

$$i\left[\hat{J}_0, \begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}\right] = -\frac{i}{2}\sigma_2\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-\hat{p}\\ \hat{q}\end{pmatrix} \Rightarrow \hat{J}_0 = -\frac{1}{4}(\hat{q}^2 + \hat{p}^2);$$

$$\overline{U}(m(\eta)) = e^{-i\eta\hat{K}_1} \quad : \quad e^{i\eta\hat{K}_1}\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}e^{-i\eta\hat{K}_1} = m(\eta)\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix} \Rightarrow$$

$$i\left[\hat{K}_1, \begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}\right] = \frac{1}{2}\sigma_3\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\hat{q}\\ -\hat{p}\end{pmatrix} \Rightarrow \hat{K}_1 = \frac{1}{4}(\hat{q}\hat{p} + \hat{p}\hat{q});$$

$$\overline{U}(l(\xi)) = e^{-i\xi(\hat{J}_0+\hat{K}_2)} \quad : \quad e^{i\xi(\hat{J}_0+\hat{K}_2)}\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}e^{-i\xi(\hat{J}_0+\hat{K}_2)} = l(\xi)\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix} \Rightarrow$$

$$\begin{aligned} i\left[\hat{J}_0 + \hat{K}_2, \begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}\right] &= -\frac{i}{2}(\sigma_2 + i\sigma_1)\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix} \Rightarrow i\left[\hat{K}_2, \begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix}\right] = \frac{1}{2}\sigma_1\begin{pmatrix}\hat{q}\\ \hat{p}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\hat{p}\\ \hat{q}\end{pmatrix} \Rightarrow \\ \hat{K}_2 &= \frac{1}{4}(\hat{p}^2 - \hat{q}^2) \end{aligned} \tag{2.7}$$

Actually the phase freedom in $\overline{U}(S)$ leads to an additive c-number freedom in each generator. We have chosen the generators so that they obey the same CR's (2.6) as in the defining representation:

$$[\hat{J}_0, \hat{K}_1] = i\hat{K}_2, \quad [\hat{J}_0, \hat{K}_2] = -i\hat{K}_1, \quad [\hat{K}_1, \hat{K}_2] = -i\hat{J}_0 \tag{2.8}$$

After this, there is no more freedom left.

Now that the operators $\overline{U}(S)$ are explicitly constructed, we can ask what their 'matrix elements' in the coordinate basis look like. After some careful algebra, the results turn out to be as follows(some delicate points here):

$$S = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in Sp(2, \mathcal{R}) :$$

$$\begin{aligned} \langle q'|\overline{U}(S)|q\rangle &= \frac{e^{-i\pi/4}}{\sqrt{|b|}}\exp\left\{i(dq'^2 - 2q'q + aq^2)/2b\right\}, b \neq 0; \\ & e^{iacq^2/2}\delta\left(\frac{q'}{a} - q\right), b = 0 \end{aligned} \tag{2.9}$$

This is called the one dof generalized Huyghens kernel. It is useful in classical optics as well as in quantum mechanics for quadratic hamiltonians.

3 Metaplectic group action on Wigner Distributions

This turns out to be very simple and elegant. We combine eq. (2.4) above, an earlier result expressing the Wigner distributions in terms of displacement operators, and a new fact: since the generators of $\overline{U}(S)$ are quadratic in $\hat{q}$, $\hat{p}$, they and $\overline{U}(S)$ commute with parity. Let us express all these in the ξ notation:

$$S \in Sp(2,\mathcal{R}) \;:\; \overline{U}(S)^{-1}D(\xi)\overline{U}(S) = D(S^{-1}\xi);$$
$$W(\xi) = \frac{1}{2\pi\hbar}Tr\left[\hat{\rho}D(\xi)^{-1}\hat{\rho}D(\xi)\right];$$
$$\overline{U}(S)\hat{P} = \hat{P}\overline{U}(S) \tag{3.1}$$

Then we get as consequences:

$$\begin{aligned} \hat{\rho} \to \hat{\rho}' &= \overline{U}(S)^{-1}\hat{\rho}\,\overline{U}(S) \Rightarrow \\ W(\xi) &\to W'(\xi) = W(S\xi). \end{aligned} \tag{3.2}$$

It is instructive to check that this behaviour is consistent with the composition law (2.3) for $\overline{U}(S)$:

$$\begin{aligned} \hat{\rho''} &= \overline{U}(S')^{-1}\hat{\rho'}\overline{U}(S') = \overline{U}(SS')^{-1}\hat{\rho}\overline{U}(SS') \Rightarrow \\ W''(\xi) &= W'(S'\xi) = W(SS'\xi) \end{aligned} \tag{3.3}$$

We can combine this behaviour of $W(\xi)$ with its behaviour under phase space displacements derived earlier,

$$\hat{\rho'} = D(\xi_0)^{-1}\hat{\rho}D(\xi_0) \Leftrightarrow W'(\xi) = W(\xi+\xi_0), \tag{3.4}$$

to conclude that for any $S \in Sp(2,\mathcal{R})$ and any ξ_0,

$$W(\xi) \text{ is a Wigner Distribution} \Leftrightarrow W(S\xi+\xi_0) \text{ is a Wigner Distribution} \tag{3.5}$$

This combines with the marginals properties of $W(\xi)$ with respect to q and p,

$$\int_{-\infty}^{\infty}(dp \text{ or } dq)W(q,p) = \langle q|\hat{\rho}|q\rangle \text{or} \langle p|\hat{\rho}|p\rangle \geq 0, \tag{3.6}$$

to give a more general 'marginals' type result: the integral of $W(\xi)$ along any straight line in the phase plane is non negative, essentially a quantum mechanical probability distribution for some combination of $\hat{q}$ and $\hat{p}$:

$$\int\int dqdpW(q,p)\delta(lq+mp-n) \geq 0, \quad \text{any real} l,m,n, \quad l^2+m^2>0 \tag{3.7}$$

4 One dof Uncertainty Principles and their invariances

For any pure or mixed state, $|\psi\rangle$ or $\hat{\rho}$, the uncertainties or spreads in position and in momentum are defined by:

$$(\Delta q)^2 = \left\langle(\hat{q}-\langle\hat{q}\rangle)^2\right\rangle = \left\langle\hat{q}^2\right\rangle - \langle\hat{q}\rangle^2,$$
$$\langle\hat{q}\rangle = \langle\psi|\hat{q}|\psi\rangle \,\text{or}\, Tr(\hat{\rho}\hat{q}), \quad \langle\hat{q}^2\rangle = \langle\psi|\hat{q}^2|\psi\rangle \quad \text{or} \quad Tr(\hat{\rho}\hat{q}^2);$$
$$(\Delta p)^2 = \left\langle(\hat{p}-\langle\hat{p}\rangle)^2\right\rangle = \left\langle\hat{p}^2\right\rangle - \langle\hat{p}\rangle^2,$$
$$\langle\hat{p}\rangle = \langle\psi|\hat{p}|\psi\rangle \,\text{or}\, Tr(\hat{\rho}\hat{p}), \quad \langle\hat{p}^2\rangle = \langle\psi|\hat{p}^2|\psi\rangle \quad \text{or} \quad Tr(\hat{\rho}\hat{p}^2) \tag{4.1}$$

Then by using the Cauchy Schwartz inequality, we easily obtain the Heisenberg form of the Uncertainty Principle

$$\Delta q\Delta p \geq \hbar/2 \tag{4.2}$$

For a pure state, it is well known that for this inequality to be saturated and become an equality, $\Delta q\Delta p = \hbar/2$, apart from a phase space displacement the wave function must be a real Gaussian. Without loss of generally we can assume that $\langle\hat{q}\rangle = \langle\hat{p}\rangle = 0$. This can be achieved by action by $D(\langle\hat{q}\rangle, \langle\hat{p}\rangle)$. Then

$$\begin{aligned} \Delta q\Delta p = \frac{\hbar}{2} \Leftrightarrow \psi_a(q) &= \frac{1}{(\pi a)^{1/4}} e^{-q^2/2a}, \quad \Delta q = (\frac{a}{2})^{1/2} \\ \phi_a(p) &= \frac{1}{\hbar}(\frac{a}{\pi})^{1/4} e^{-ap^2/2\hbar^2}, \Delta p = \frac{\hbar}{(2a)^{1/2}}; \quad a > 0 \end{aligned} \tag{4.3}$$

We can next ask for the subgroup of $Sp(2,\mathcal{R})$ which preserves the form of the Heisenberg Uncertainty Principle (4.2). It is easy to see that this consists of

$$\begin{aligned} \text{Reciprocal Scale changes, (1.9)}: &\quad \hat{q} \to e^{\eta/2}\hat{q}, \quad \hat{p} \to e^{-\eta/2}\hat{p}; \\ \text{Fourier transformation}\,\mathcal{F},\ (1.8): &\quad \hat{q} \to -\hat{p}, \quad \hat{p} \to \hat{q}; \end{aligned} \tag{4.4}$$

and their products. So parity $P = \mathcal{F}^2$ is included. In explicit form this is the subgroup of $Sp(2,\mathcal{R})$ matrices made up of

$$S_H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -a \\ a^{-1} & 0 \end{pmatrix} \,\Big|\, a \text{ real}, \neq 0 \right\} \subset Sp(2,\mathcal{R}) \tag{4.5}$$

There is a stronger form of the U P called the Schrodinger U P, which is form invariant under all of $Sp(2,\mathcal{R})$.It involves one more quantity of the nature of an uncertainty or spread, a cross term:

$$\Delta(q,p) = \frac{1}{2}\langle\{\hat{q} - \langle\hat{q}\rangle, \hat{p} - \langle\hat{p}\rangle\}\rangle = \frac{1}{2}\langle\{\hat{q},\hat{p}\}\rangle - \langle\hat{q}\rangle\langle\hat{p}\rangle \tag{4.6}$$

Then by an argument similar to that which gives (4.2), we get the stronger statement:

$$(\Delta q)^2(\Delta p)^2 - (\Delta(q,p))^2 \geq \hbar^2/4 \tag{4.7}$$

In fact this Schrodinger U P implies the earlier one. Every state $\hat{\rho}$ necessarily obeys both (4.2) and (4.7), the latter being stronger.

This U P can be expressed in matrix form, which we see later generalizes to any number of dof. From the operator column vector $\hat{\xi} = \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix}$ we form in any state $\hat{\rho}$:

$$\hat{\xi} - \langle\hat{\xi}\rangle = \begin{pmatrix} \hat{q} - \langle\hat{q}\rangle \\ \hat{p} - \langle\hat{p}\rangle \end{pmatrix}, \quad \langle\hat{\xi}\rangle = Tr(\hat{\rho}\hat{\xi});$$

$$\hat{\Omega} = (\hat{\xi} - \langle\hat{\xi}\rangle)(\hat{\xi} - \langle\hat{\xi}\rangle)^\dagger = \begin{pmatrix} \hat{q} - \langle\hat{q}\rangle \\ \hat{p} - \langle\hat{p}\rangle \end{pmatrix} (\hat{q} - \langle\hat{q}\rangle \quad \hat{p} - \langle\hat{p}\rangle);$$

$$\Omega = Tr(\hat{\rho}\hat{\Omega}) = \begin{pmatrix} (\Delta q)^2 & \langle(\hat{q} - \langle\hat{q}\rangle)(\hat{p} - \langle\hat{p}\rangle)\rangle \\ \langle(\hat{p} - \langle\hat{p}\rangle)(\hat{q} - \langle\hat{q}\rangle)\rangle & (\Delta p)^2 \end{pmatrix} \tag{4.8}$$

Clearly Ω is hermitian non-negative. By rewriting the off-diagonal terms using anti commutators and commutators, we find:

$$\Omega = V + \frac{i}{2}\hbar\beta, \quad V = \begin{pmatrix} (\Delta q)^2 & \Delta(q,p) \\ \Delta(q,p) & (\Delta p)^2 \end{pmatrix}, \quad \beta = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.9}$$

The 'variance matrix' V is real symmetric, and we have the U P

$$V + \frac{i}{2}\hbar\beta \geq 0 \tag{4.10}$$

This is just the Schrodinger UP (4.7) as

$$det(V + \frac{i}{2}\hbar\beta) = (\Delta q)^2(\Delta p)^2 - (\Delta(q,p))^2 - \frac{\hbar^2}{4} \tag{4.11}$$

It is now easy to show that for any $S \in Sp(2,\mathcal{R})$:

$$\hat{\rho} \to \overline{U}(S)\hat{\rho}\overline{U}(S)^{-1} \Rightarrow \quad V \to SVS^T, \tag{4.12}$$

and since from eq. (1.6), $S\beta S^T = \beta$, the U P (4.10) is $Sp(2,\mathcal{R})$ covariant.

Pure states saturating the Schrodinger U P (4.7) are complex Gaussians which generalized the real ones in eq. (4.3). After making $\langle\hat{q}\rangle = \langle\hat{p}\rangle = 0$, we have:

$$(\Delta q)^2(\Delta p)^2 - (\Delta(q,p))^2 = \frac{\hbar^2}{4} \;\Leftrightarrow$$

$$\Psi_{a,b}(q) = \left(\frac{a}{\pi(a^2+b^2)}\right)^{1/4} e^{-q^2/2(a+ib)},$$

$$(\Delta q)^2 = (a^2+b^2)/2a, (\Delta p)^2 = \hbar^2/2a, \quad \Delta(q,p) = \hbar b/2a, \quad a > 0 \tag{4.13}$$

In terms of the Wigner distribution, the variance matrix V is especially simple. Indeed from the Weyl map we know that

$$Tr\left(\hat{\rho}\{\hat{q},\, \hat{p},\, \hat{q}^2,\, \hat{q}\hat{p}+\hat{p}\hat{q},\, \hat{p}^2\}\right) = \int\int dqdpW(q,p)\left(q,p,q^2,2qp,p^2\right) \tag{4.14}$$

so the elements of V are formed out of the moments of $W(q,p)$ upto the second order.

5 Definition of Squeezed states, invariances, examples

Let us now set $\hbar = 1$ for simplicity, and also assume $\hat{q}$ and $\hat{p}$ are dimensionless. So they obey $[\hat{q},\hat{p}] = i$. The Heisenberg UP (4.2) is

$$\Delta q\Delta p \geq \frac{1}{2} \tag{5.1}$$

This is saturated by the ground state of the oscillator, and in a symmetric manner:

$$\hat{H} = \frac{1}{2}(\hat{q}^2+\hat{p}^2), \psi_0(q) = \frac{1}{\pi^{1/4}}e^{-q^2/2}$$

$$\hat{H}\psi_0(q) = \frac{1}{2}\psi_0(q), \Delta q = \Delta p = \frac{1}{\sqrt{2}}; \quad \Delta(q,p) = 0 \tag{5.2}$$

On the basis of this UP, a state is said to be squeezed if

$$\begin{aligned} \text{either}\,\Delta\text{q} \;&<\; \frac{1}{\sqrt{2}}, \text{then necessarily}\,\Delta p > \frac{1}{\sqrt{2}},\ \text{squeezing in}\,q, \\ \text{or}\,\Delta p \;&<\; \frac{1}{\sqrt{2}}, \text{then necessarily}\,\Delta q > \frac{1}{\sqrt{2}},\ \text{squeezing in}\,p \end{aligned} \tag{5.3}$$

We can find examples among the complex Gaussian pure states $\Psi_{a,b}(q)$ in eq. (4.13):

$$\begin{aligned} \Psi_{a,b}(q)\,\text{is} \quad &\text{squeezed} \quad \text{in } q \text{ if } 0 < a < 1,\ b^2 < a(1-a); \\ &\text{squeezed} \quad \text{in } p \text{ if } a > 1,\ any\ b \end{aligned} \tag{5.4}$$

(The real centred Gaussians $\psi_a(q)$ of eq. (4.3) are a subset of these for $b = 0$)

This definition of squeezing is not preserved under any continuous subgroup of $Sp(2,\mathcal{R})$. (it is of course preserved under Fourier transformation $\mathcal{F}$). Another definition of squeezing is in terms of the variance matrix V. Under $Sp(2,\mathcal{R})$, V transforms as in eq. (4.12), $V' = SVS^T$, so in general V' and V have different eigenvalue spectra. But if $S \in SO(2) \subset Sp(2,\mathcal{R})$, eq. (1.7), then V' and V have the same eigenvalues since $S^T = S^{-1}$. So another interesting definition is to say that a state $\hat{\rho}$ is squeezed if and only if

$$l(V) = \text{lesser eigenvalue of}\, V < \frac{1}{2} \tag{5.5}$$

Then a squeezed state remains squeezed under all $S \in SO(2)$, but of course not under the whole of $Sp(2,\mathcal{R})$.

These two alternative definitions of squeezing are related in the sense that squeezing as in (5.3) implies squeezing as in (5.5) but not conversely:

$$\Delta q \text{ or } \Delta p < \frac{1}{\sqrt{2}} \Rightarrow l(V) < \frac{1}{2} \tag{5.6}$$

So the definition (5.3) is more restrictive. With either definition, a squeezed state is non classical in a well defined technical sense.

For the centred complex Gaussians (4.13), we have

$$V = \frac{1}{2a}\begin{pmatrix} a^2+b^2 & b \\ b & 1 \end{pmatrix}, \quad det\, V = \frac{1}{4} \tag{5.7}$$

Therefore the product of the eigenvalues of V is $\frac{1}{4}$. If then V is not a multiple of the unit matrix, ie $V \neq \frac{1}{2}I$, its lesser eigenvalue must obey $l(V) < \frac{1}{2}$. The states $\Psi_{a,b}(q)$ are therefore squeezed in the sense of the definition (5.5) if either $b \neq 0$, any a; or $b = 0$, $a \neq 1$. This includes the cases (5.4) but others too, in agreement with the one way implication (5.6).

6 Many dof, the groups $Sp(2n,\mathcal{R})$

The CCR's for n dof, and a suitable notation, have been introduced earlier in the discussion of Wigner distributions. We have $2n$ hermitian operators $\hat{q}_j,\ \hat{p}_j,\ j = 1,2,3,...,n$ arranged into a $2n$-component column of operators

$$\hat{\xi} = \left(\hat{\xi}_a\right) = (\hat{q}_1 \quad \hat{q}_2.. \quad \hat{q}_n \quad \hat{p}_1 \quad \hat{p}_2.. \quad \hat{p}_n)^T, \quad a = 1,2,...,2n \tag{6.1}$$

(Alternatively we could arrange them in successive canonical pairs $\hat{q}_1\, \hat{p}_1\, \hat{q}_2\, \hat{p}_2 .... \ \hat{q}_n\, \hat{p}_n$). The CCR's are

$$\begin{aligned} &[\hat{\xi}_a, \hat{\xi}_b] = i\hbar\beta_{ab}, \beta = \begin{pmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & 0 \end{pmatrix}, \\ &\beta^T = \beta^{-1} = -\beta \end{aligned} \tag{6.2}$$

A real linear transformation, real to maintain hermiticity,

$$\hat{\xi}'_a = S_{ab}\hat{\xi}_b, \quad \hat{\xi}' = S\hat{\xi}, \tag{6.3}$$

will preserve the CCR's (6.2) only if the $2n \times 2n$ real matrix S obeys a suitable condition:

$$\begin{aligned} &[\hat{\xi}'_a, \hat{\xi}'_b] = i\hbar\beta_{ab} \Leftrightarrow S_{ac}S_{bd}\beta_{cd} = \beta_{ab}, \\ \text{i.e.,} \quad &S\beta S^T = \beta \end{aligned} \tag{6.4}$$

This leads to the definition of the real symplectic group in $2n$ dimensions:

$$Sp(2n,\mathcal{R}) = \left\{ S = 2n \times 2n \text{ real matrix} \ \middle|\ S\beta S^T = \beta \right\} \tag{6.5}$$

So this is a multi dimensional generalization of $Sp(2,\mathcal{R})$ in eq. (1.4) (in which however β does not appear explicitly though as eq. (1.6) shows it is 'really present'). This is a (non compact) Lie group of dimension $n(2n+1)$ as we see later(So for $n=1$ we have $Sp(2,\mathcal{R})$ of dimension 3). While the groups $SO(n,\mathcal{R})$ and $SU(n)$ are defined in every (real or complex) dimension n, the family $Sp(2n,\mathcal{R})$ is defined only in even dimensions.

A utilitarian review of these groups is given in Arvind et al., Pramana 45, 471-497(1995). A more detailed treatment has been given by M. de Gosson, in a book and papers.

It is natural to describe these matrices S in block form, made up of four $n\times n$ real matrices:

$$\begin{aligned} S &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n,\mathcal{R}) \Leftrightarrow \\ & AB^T,\, CD^T \text{ symmetric}, \quad AD^T - BC^T = I_{n\times n} \end{aligned} \tag{6.6}$$

Some useful properties are these:

$$\begin{aligned} &\text{i)} \quad \beta \in Sp(2n,\mathcal{R}) \\ &\text{ii)} \quad S \in Sp(2n,\mathcal{R}) \Rightarrow -S,\, S^T,\, S^{-1},\, , \in Sp(2n,\mathcal{R}) \\ &\text{iii)} \quad S^{-1} = \beta S^T \beta^{-1} \\ &\text{iv)} \quad det\, S = +1 \end{aligned} \tag{6.7}$$

The last property, unimodularity, is 'delicate' and requires some effort to establish; the definition (6.5) permits $det\, S = \pm 1$, but in fact $det\, S = -1$ is not possible.

There are several useful subgroups of $Sp(2n,\mathcal{R})$, of which we mention only three:

(a) $GL(n,r)$ is the n^2- dimensional general linear group in n dimensions, contained in $Sp(2n,\mathcal{R})$ by the identification, using the block form (6.6) :

$$A \in GL(n,\mathcal{R}), B = C = 0, \quad D = (A^{-1})^T \tag{6.8}$$

(b) $O(n,\mathcal{R})$ is the $\frac{1}{2}n(n-1)$ - dimensional real orthogonal group in n real dimensions, contained in $GL(n,\mathcal{R})$ and so in $Sp(2n,\mathcal{R})$:

$$A = D \in O(n,\mathcal{R}), \quad B = C = 0 \tag{6.9}$$

(c) $U(n)$ is the n^2 - dimensional unitary group in n complex dimensions contained in $Sp(2n,\mathcal{R})$ by the identification:

$$\begin{aligned} U &= X - iY \in U(n), \quad U^\dagger U = UU^\dagger = I_{n\times n} \leftrightarrow \\ S(X,Y) &= \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \quad \in Sp(2n,\mathcal{R}) \end{aligned} \tag{6.10}$$

Here X and Y are the real and imaginary parts of U, and the conditions in eq. (6.6) can be verified. This $U(n)$ is the maximal compact sub group of $Sp(2n,\mathcal{R})$, and it is the interesting of two intersecting real matrix groups in $2n$ dimensions:

$$O(2n,\mathcal{R}) \cap Sp(2n,\mathcal{R}) = \left\{ S(X,Y) \in Sp(2n,\mathcal{R}) \,\middle|\, X - iY \in U(n) \right\}, \tag{6.11}$$

So, the matrices $S(X,Y)$ in eq. (6.10) are real orthogonal.

In the Pramana review quoted above, some useful global ways of expressing each $S \in Sp(2n,\mathcal{R})$ as a product of simpler factors, so called decomposition theorems, are described.

Compared to the Lie algebras of the groups $U(n)$ and $O(n)$, the Lie algebra of $Sp(2n,\mathcal{R})$ and the basic commutation relations are rather elaborate. We give a brief overview with more details available in the Pramana review.

Even though the defining representation of $Sp(2n,\mathcal{R})$ consists of real matrices, keeping in mind the uses in the quantum mechanical context we retain the ' quantum mechanical i' in the identification of generators. Then for elements close to the identity in the defining representation we find:

$$\begin{aligned} & S \cong I - i\epsilon J \in Sp(2n,\mathcal{R}), \quad |\epsilon| << 1 \Leftrightarrow J\, \text{pure imaginary}, \\ & J\beta + \beta J^T = 0, \;\; i.e, \;\; J\beta \text{ and } \beta J \text{ symmetric} \end{aligned} \tag{6.12}$$

Therefore a complete independent set of J's is obtained by starting with a complete independent set of $2n \times 2n$ real symmetric matrices, and either pre or post multiplying them by $i\beta$:

$$\begin{aligned} J &= i\beta \times (\text{real symmetric}\, 2n \times 2n\, \text{matrices}) \\ &= (\text{real symmetric}\, 2n \times 2n\, \text{matrices}) \times i\beta \end{aligned} \tag{6.13}$$

The number of such independent J's is clearly $n(2n+1)$, which is thus the dimension of $Sp(2n,\mathcal{R})$. Taking the first alternative in eq (6.13), we get a basis for the generators in the defining representation:

$$\begin{aligned} J^{(0)}_{ab} &= J^{(0)}_{ba}\,, \quad a,b = 1,2,...,2n; \\ (J^{(0)}_{ab})_{cd} &= i\,(\delta_{cd}\beta_{cb} + \delta_{ad}\beta_{ca})\,. \end{aligned} \tag{6.14}$$

Their commutation relations are:

$$[J^{(0)}_{ab}, J^{(0)}_{cd}] = i(\beta_{ac}J^{(0)}_{bd} + \beta_{bc}J^{(0)}_{ad} + \beta_{ad}J^{(0)}_{cb} + \beta_{bd}J^{(0)}_{ca}) \tag{6.15}$$

Therefore in a general (unitary) representation of $Sp(2n,\mathcal{R})$, we have (hermitian)generators $J_{ab} = J_{ba}$ obeying

$$[J_{ab}, J_{cd}] = i(\beta_{ac}J_{bd} + \beta_{bc}J_{ad} + \beta_{ad}J_{cb} + \beta_{bd}J_{ca}) \tag{6.16}$$

With this choice of basis for the Lie algebra of $Sp(2n,\mathcal{R})$, the generator of the subgroups listed in eqs (6.8 - 6.10) can be identified. It helps to use a split index notation:

$$a,b = 1,2,....,2n \to j\alpha,\ k\beta\ :\ j,k = 1,2,...,n; \quad \alpha,\beta = 1,2. \tag{6.17}$$

Indices j, k label the n dof, while $\alpha = 1$ and 2 correspond to the $\hat{q}$ and the $\hat{p}$ in a canonical pair:

$$\begin{aligned} \hat{\xi}_1 \equiv \hat{\xi}_{11} = \hat{q}_1, ...\hat{\xi}_n \equiv \hat{\xi}_{n1} = \hat{q}_n; \\ \hat{\xi}_{n+1} \equiv \hat{\xi}_{12} = \hat{p}_1, ...\hat{\xi}_{2n} \equiv \hat{\xi}_{n2} = \hat{p}_n. \end{aligned} \tag{6.18}$$

Then the generators of the subgroups in eqs. (6.8 - 6.10) are:

$$\begin{aligned} GL(n,\mathcal{R}) &: \ J_{j_1,k_2}\,, \quad j_1,k_2 = 1,2,...,n; \\ O(n,\mathcal{R}) &: \ J_{j_1,k_2} - J_{k_1,j_2}\,, \quad k_1,j_2 = 1,2,...n; \\ U(n) &: \ J_{j_1,k_2} - J_{k_1,j_2}, \quad J_{j_1,k_1} + J_{j_2,k_2}\,, \quad j_2,k_2 = 1,2,...n; \end{aligned} \tag{6.19}$$

7 Metaplectic unitary representation of $Sp(2n,\mathcal{R})$, effect on Wigner distribution

This is a generalization of the one dof case in Sections II and III. For any(finite) number of dof, the Stone-von Neumann theorem says that there is,apart from unitary equivalence, only one hermitian irreducible representation of the CCR's (6.2). In the Schrodinger form, this is set up as follows:

$$\begin{aligned} \mathcal{H} &= \left\{\psi(\underline{q}) \in C,\, \underline{q} \in R^n \mid \|\psi\|^2 = \int_{R^n} d^n q |\psi(\underline{q})|^2 < \infty\right\}; \\ (\hat{q}_j\psi)(\underline{q}) &= q_j\psi(\underline{q}), \quad (\hat{p}_j\psi)(\underline{q}) = -i\hbar\frac{\partial}{\partial q^j}\psi(\underline{q}) \end{aligned} \tag{7.1}$$

Therefore since for any $S \in Sp(2n,\mathcal{R})$ the transformation (6.3) preserves (6.2), there must be a unitary transformation $\overline{U}(S)$, unique upto an S-dependent phase, implementing (6.3):

$$S \in Sp(2n,\mathcal{R})\ :\ \hat{\xi}' = S\hat{\xi} = \overline{U}(S)^{-1}\hat{\xi}\,\overline{U}(S), \quad \overline{U}(S)^{\dagger}\overline{U}(S) = I \,\text{on}\, \mathcal{H} \tag{7.2}$$

As with one dof, the general composition law similar to (2.2) follows:

$$S', S \in Sp(2n, \mathcal{R}) \ : \ \overline{U}(S')\overline{U}(S) = (\text{phase dependent on}\, S',\ S)\overline{U}(S'S), \tag{7.3}$$

and using the phase freedom available in each factor $\overline{U}$, this can be maximally simplified to read

$$\overline{U}(S')\overline{U}(S) = \pm\overline{U}(S'S) \tag{7.4}$$

This is the two-valued metaplectic unitary representation of $Sp(2n, \mathcal{R})$, a true unitary representation of a group $Mp(2n)$ which covers $Sp(2n, \mathcal{R})$ twice. It is the direct sum of two unitary irreducible representations, for any n.

The hermitian generators of this representation are all quadratic expressions in $\hat{q}_j$ and $\hat{p}_j$. In the notation of eqs. (6.16, 6.17) we find:

$$\hat{J}_{j_1,k_1} = \hat{q}_j\hat{q}_k, \quad \hat{J}_{j_1,k_2} = \frac{1}{2}\{\hat{q}_j, \hat{p}_k\}, \quad \hat{J}_{j_2,k_2} = \hat{p}_j\hat{p}_k, \quad j,k = 1,2.....n \tag{7.5}$$

The effect of this unitary representation on Wigner distributions also generalizes eq. (3.2) :

$$\begin{aligned} W(\xi) &= \frac{1}{(2\pi\hbar)^n}\int_{R^n} d^n q' \langle \underline{q} - \frac{1}{2}\underline{q}' \mid \hat{\rho} | \underline{q} + \frac{1}{2}\underline{q}'\rangle e^{i\underline{p}.\underline{q}'/\hbar}; \\ \hat{\rho}' &= \overline{U}(S)^{-1}\hat{\rho}\,\overline{U}(S) \ \Leftrightarrow \ W'(\xi) = W(S\xi) \end{aligned} \tag{7.6}$$

8 The UP for many dof, $Sp(2n, \mathcal{R})$ covariance

This is in the spirit of the matrix form of the Schrodinger UP for one dof in Section IV. We follow the route of eq (4.8) :

$$\begin{aligned} &\text{State}\, \hat{\rho} \longrightarrow \langle\hat{\xi}\rangle = Tr(\hat{\rho}\hat{\xi}) \rightarrow \hat{\Omega} = (\hat{\xi} - \langle\hat{\xi}\rangle)(\hat{\xi} - \langle\hat{\xi}\rangle)^T \rightarrow \\ &\Omega = Tr(\hat{\rho}\hat{\Omega}), \quad \Omega_{ab} = Tr(\hat{\rho}\hat{\xi}_a\hat{\xi}_b) - \langle\hat{\xi}_a\rangle\langle\hat{\xi}_b\rangle \end{aligned} \tag{8.1}$$

It is obvious that

$$\Omega^\dagger = \Omega \geq 0 \tag{8.2}$$

Bringing in anti commutators and commutators, we find:

$$\begin{aligned} \Omega &= V + \frac{i}{2}\hbar\beta, \\ V_{ab} &= \frac{1}{2}Tr\left(\hat{\rho}\{\hat{\xi}_a, \hat{\xi}_b\}\right) - \langle\hat{\xi}_a\rangle\langle\hat{\xi}_b\rangle \end{aligned} \tag{8.3}$$

The variance matrix V is $2n$ dimensional real symmetric, and the n dof UP is the matrix statement

$$V + \frac{i}{2}\hbar\beta \geq 0 \tag{8.4}$$

This is manifestly covariant under $Sp(2n, \mathcal{R})$ since

$$\begin{aligned} \hat{\rho}' &= \overline{U}(S)\hat{\rho}\overline{U}^{-1} \Rightarrow V' = SVS^T, \\ \text{and}\ \beta &= S\beta S^T \ \text{as well} \end{aligned} \tag{8.5}$$

In terms of $W(\xi)$,

$$\begin{aligned} \langle\hat{\xi}_a\rangle = \langle\xi_a\rangle &= \int_{R^{2n}} d^{2n}\xi \ \xi_a W(\xi), \\ V_{ab} &= \int_{R^{2n}} d^{2n}\xi \ \xi_a\xi_b W(\xi) - \langle\xi_a\rangle\langle\xi_b\rangle \end{aligned} \tag{8.6}$$

just the first and second moments of the Wigner distribution.

Looking at the behaviour of V in (8.5), since V is real symmetric, we could make V' diagonal if S could be chosen from the group $SO(2n,\mathcal{R})$. However here $S \in Sp(2n,\mathcal{R})$, and by eq. (6.11) $Sp(2n,\mathcal{R}) \cap SO(2n,\mathcal{R}) = U(n)$ is a 'very small part', the maximal compact subgroup, of $Sp(2n,\mathcal{R})$. Nevertheless, thanks to a remarkable theorem of Williamson, it turns out that since V is positive definite, a consequence of the UP (8.4), it can be 'brought to diagonal form':

$$\begin{aligned} &\text{V is real symmetric positive definite} \Rightarrow V' = SVS^T \\ = \quad & diag(k_1,k_2,...,k_n,k_1,k_2,...,k_n), k_j > 0, \text{ some } S \in Sp(2n,\mathcal{R}) \end{aligned} \tag{8.7}$$

This is actually just one of the many consequences of Williamson's theorem; for our purposes a very economical proof has been given by Chaturvedi, Simon and Srinivasan in Journal of Math Phys. By the use of scaling transformations, we have put V' into a particularly convenient form. The content of the matrix UP is now the statement

$$V_{ij} \geq \hbar/2, \qquad j = 1,2,...,n \tag{8.8}$$

In general , the k_j in eq. (8.7) are not the eigenvalues of V, since S may not be real orthogonal. A useful squeezing criterion works however with the eigenvalues of V, and it can be stated as follows:

$$\begin{aligned} &\hat{\rho}\,\text{is a squeezed state} \Leftrightarrow \\ &\qquad \left(S(X,Y)VS(X,Y)^T\right)_{aa} < \frac{1}{2}, \text{some } \ X - iY \in U(n), \text{ some } a \\ \Leftrightarrow \quad & l(V) = \text{least eigenvalue of } V < \frac{1}{2} \end{aligned} \tag{8.9}$$

To be clear, it is good to emphasize that while the UP (8.4) is covariant under $Sp(2n,\mathcal{R})$, the squeezing criterion (8.9) is only preserved under the subgroup $U(n) \subset Sp(2n,\mathcal{R})$.

9 The 'Wigner Quality' of a phase space function, Gaussian Wigner distributions

We have seen that any density matrix $\hat{\rho}$ for an n dof Cartesian quantum system can be (completely) described by its Wigner distribution eq (7.6). Hermiticity and unit trace of $\hat{\rho}$ translate easily in terms of $W(\xi)$:

$$\begin{aligned} \hat{\rho}^\dagger = \hat{\rho} &\Leftrightarrow W(\xi) \text{ real}, \\ Tr\hat{\rho} = 1 &\Leftrightarrow \int_{R^{2n}} d^{2n}\xi W(\xi) = 1 \end{aligned} \tag{9.1}$$

The remaining condition $\hat{\rho} \geq 0$ is more delicate. It certainly does not imply $W(\xi) \geq 0$ pointwise over the classical phase space, but is expressed more subtly by a combination of earlier formulae we have developed. For any unit vector $|\psi\rangle \in \mathcal{H}$,a pure state, denote the corresponding Wigner distribution by $W_{|\psi\rangle}(\xi)$:

$$W_{|\psi\rangle}(\xi) = \frac{1}{(2\pi\hbar)^n} \int_{R^n} d^n q'\, \psi(\underline{q} - \frac{1}{2}\underline{q}')\psi(\underline{q} + \frac{1}{2}\underline{q}')^* e^{i\underline{p}\cdot\underline{q}'/\hbar} \tag{9.2}$$

Then we have:

$$\begin{aligned} &\hat{\rho} \geq 0 \Leftrightarrow \langle\psi|\hat{\rho}|\psi \geq 0 \text{ for all } |\psi\rangle \in \mathcal{H} \Leftrightarrow \\ &\int_{R^{2n}} d^{2n}\xi W(\xi) W_\psi(\xi) \geq 0 \text{ for all normalized } |\psi\rangle \in \mathcal{H} \end{aligned} \tag{9.3}$$

This is the expression of the 'Wigner quality' of $W(\xi)$. We can say: any given function $W(\xi)$ which is real and normalized as in eq (9.1) is a Wigner distribution corresponding to some quantum state $\hat{\rho}$ if and only if it also obeys (9.3). In general, this statement cannot be made significantly simpler.

However for a real normalized Gaussian phase space function, we can go much further and bring out the 'Wigner quality' more explicitly. For simplicity let us assume we are dealing with a centred Gaussian. Such a function is completely determined by a $2n \times 2n$ real symmetric positive definite matrix G:

$$W_G(\xi) = \frac{1}{\pi^n}\sqrt{detG}e^{-\xi^T G\xi}, \quad G^T = G > 0,$$
$$\int_{R^{2n}} d^{2n}\,\xi\, W_G(\xi) = 1 \tag{9.4}$$

We easily find that its 'variance matrix' is

$$V_{ab} = \int_{R^{2n}} d^{2n}\xi\, \xi_a\xi_b W_G(\xi) = \frac{1}{2}(G^{-1})_{ab} \tag{9.5}$$

Then by an elementary analysis we can show that the UP (8.4) for V guarantees the 'Wigner quality' of $W_G(\xi)$:

$$W_G(\xi) \text{ is a Wigner distribution } \Leftrightarrow G^{-1} + i\hbar\beta \geq 0 \tag{9.6}$$

If now using Williamson's theorem we transform G to diagonal form via some $S \in Sp(2n, \mathcal{R})$ and attain

$$SGS^T = diag(r_1, r_2, ..., r_n, r_1, r_2, ..., r_n), \tag{9.7}$$

the condition (9.6) becomes

$$0 < r_j \leq 1/\hbar, \qquad j = 1, 2, ..., n \tag{9.8}$$