Effect of Peculiar Velocities on Observing Density Contrast

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Abstract—**It is known that for large distances and negligible peculiar velocities, the observed red-shift is related directly to the distance of the source, viz. $Z = HR/c$. Using this as an approximation, one can find the density contrast by observing representative objects in the sky, viz. $\delta^S = \delta^S(\theta, \phi, R = cZ/H)$. An improvement can be made by accounting for non-zero peculiar velocities. Let $Z = HS/c$, where $S = R$ if we assume zero peculiar velocity. Our objective here, is to find a relation between the observed quantity $\delta^S = \delta^S(\theta, \phi, S)$ and the relevant quantity $\delta^R = \delta^R(\theta, \phi, R)$. The main importance of this, is that even in the linear theory, where peculiar velocities are small, δ^S and δ^R are significantly different, although in this discussion, we will not prove this.**

I. Background

The notation and known results used in what follows, have been briefly summarized here. The position of an object, is given by $\vec{R} = a\vec{r}$, where a quantifies scaling/expansion of the universe and $\vec{r}$ represents the co-moving coordinate. Consequently, the velocity is given by $\vec{V} = \dot{a}\vec{r} + a\dot{\vec{r}}$. Recalling, $H \equiv \dot{a}/a$ and $\vec{u} \equiv \dot{\vec{r}}$ (peculiar velocity), we obtain

$$\vec{V} = H\vec{R} + a\vec{u}. \tag{1}$$

Recall that the red-shift $Z \equiv (\lambda_{\text{obs}} - \lambda_{\text{em}})/\lambda_{\text{em}}$. Using the Doppler effect for light, we have $\lambda_{\text{em}} = \left(\frac{1-\beta}{1+\beta}\right)\lambda_{\text{obs}}$, which entails $Z \approx \beta$, where $\beta = V_{\text{los}}/c$, and V_{los} is the line of sight speed. Further, it can be shown, that for electromagnetic radiation, that was emitted at t_{em}, and observed at t_{obs}, $\lambda_{\text{obs}}/\lambda_{\text{em}} = a(t_{\text{obs}})/a(t_{\text{em}})$ which entails

$$\frac{a_{\text{obs}}}{a_{\text{em}}} = 1 + Z.$$

We will use some results from the linear theory for density contrast, which is defined implicitly as $\rho(\vec{r}, t) = \overline{\rho}(t)(1 + \delta(\vec{r}, t))$, where $\overline{\rho}$ is the background/averaged mass density, ρ is the mass density. In the Newtonian limit, using the fluid approach, it is known that

$$\frac{\partial \delta}{\partial t} + \vec{\nabla}.[(1+\delta)\vec{u}] = 0,$$
$$\frac{\partial u}{\partial t} + \frac{2\dot{a}}{a}\vec{u} + (u.\nabla)u = -\frac{1}{a^2}\vec{\nabla}\phi,$$
$$\nabla^2\phi = 4\pi G a^2 \overline{\rho}\delta,$$

hold ($\vec{\nabla} \equiv \partial/\partial r_x \hat{x} + \partial/\partial r_y \hat{y} + \partial/\partial r_z \hat{z}$), while for small u, in the linear limit, it follows from these that,

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho\delta = 0.$$

This can be solved to obtain two independent solutions, $D_{\pm}(t)$. The following results will be useful.

(i) It is found that $D_+ = a$ is a growing solution (grows with time), for an Einstien De Sitter (EDS) universe. In general also, it has been shown that

$$d\ln D_+/d\ln a = f. \tag{2}$$

where f depends on cosmological parameters Ωs.

(ii) The most general solution can be written as

$$\delta(\vec{r}, t) = \delta_+(\vec{r})\frac{D_+(t)}{D_+(t_i)} + \delta_-(\vec{r})\frac{D_-(t)}{D_-(t_i)}.$$

It follows after some analysis, that if we start with

$$\vec{v} = -\vec{\nabla}\psi, \tag{3}$$

where $\vec{v} \equiv d\vec{r}/dD_+$ and $\psi \equiv 2a\phi/3H_0^2\Omega_{\text{nr}}D_+$, then $\delta_- = 0$. Also, from the definition of ψ, it follows

$$\nabla^2\psi = \frac{\delta}{D_+} \tag{4}$$

II. The Relation between δ^S and δ^R

We start with $Z \approx \frac{\vec{V}.\hat{r}}{c}$ and precisely define $\vec{S} \equiv Z\hat{r}cH^{-1}$ to obtain $\vec{S} = (R + aH^{-1}\vec{u}.\hat{r})\hat{r}$ (using equation (1)). $\vec{u}$ can be expressed as $\frac{d\vec{r}}{dt} = \frac{dr}{dD_+}\frac{dD_+}{da}\frac{da}{dt}$, which entails $\vec{S} = (R + (H^{-1}\dot{a})D_+\vec{v}f(\Omega_{\text{nr}}).\hat{r})\hat{r}$ (using equation (2)). We are interested in the present time, in which case, if we

assume $D_+(t_0) = 1$, we have

$$\vec{S} = \vec{R} + f_0(v.\hat{r})\hat{r}$$

where $f_0 \equiv f(\Omega_{\text{nr0}}, \Lambda_0)$. From conservation of mass, we must have $\left(1 + \delta^S(\vec{S})\right) d^3\vec{S} = \left(1 + \delta^R(\vec{R})\right) d^3\vec{R}$, where $\vec{S}$ is related to $\vec{R}$ as stated earlier. Note that we must conserve $(1 + \delta)d\tau$ and not $\delta d\tau$ (where $d\tau$ is the volume element). Since, $d^3\vec{S} = \frac{\partial(S_x, S_y, S_z)}{\partial(R_x, R_y, R_z)} d^3\vec{R}$, effectively we are only required to evaluate the Jacobian to find an explicit relation. Evaluating the Jacobian directly is tedious. Instead, one may note that one can write $S\hat{r} = R(1 + U/R)\hat{r}$, where $U = f_0(\vec{v}.\hat{r})$, which entails that in spherical coordinates, θ and ϕ remain unchanged. Consequently $d^3\vec{S} = S^2 dS \sin\theta d\theta d\phi$ can be written as $(1 + U/R)^2(1 + dU/dR)R^2 dR \sin\theta d\theta d\phi$, which entails $J = \left(1 + \frac{U}{R}\right)^2 \left(1 + \frac{dU}{dR}\right)$. The required relation then, is

$$1 + \delta^R(\vec{R}) = \left(1 + \delta^S(\vec{S})\right)\left(1 + \frac{U}{R}\right)^2\left(1 + \frac{dU}{dR}\right).$$

where the second term in the Right Hand Side (RHS), maybe dropped for distant objects. However, U is still unknown and to resolve that, for the approximation, we note that we only require dU/dR. Let us work in the fourier space, with a single mode to simplify calculations and later sum the modes. We start with $\vec{v} = \vec{v}_k e^{-i\vec{k}.\vec{R}}$, $\psi = \psi_k e^{-i\vec{k}.\vec{R}}$ and substitute them in equation (3) to get $\vec{v}_k = i\vec{k}\psi_k$. Further, for $\delta = \delta_k e^{-i\vec{k}.\vec{R}}$, using equation (4), we get $\vec{v}_k = -i\vec{k}\delta_k/k^2$. Also, substituting for $\vec{v}$ in U, we have

$$U = f_0\vec{v}_k.\hat{r}e^{-i\vec{k}.\vec{R}} = -\frac{if_0\delta_k\vec{k}.\hat{r}e^{-i\vec{k}.\vec{R}}}{k^2} = -\frac{if_0\delta_k\mu e^{-i\vec{k}.\vec{R}}}{k}$$

$$\frac{dU}{dR} = f_0\mu^2\delta_k e^{-i\vec{k}.\vec{R}} = -f_0\mu^2\delta,$$

where $\mu = \hat{k}.\hat{r}$, the cosine of the angle between $\vec{k}$ and line of sight. Consequently we have $(1 + \delta^S) \approx (1 + \delta^R)\left(1 - f_0\delta^R\mu^2\right)^{-1}$, where we suppressed arguments of δ. Now keeping to first order in δ^R, we have $\delta^S \approx \delta^R(1 - f_0\mu^2)$. Substituting $\delta^S = \delta_k^S e^{-i\vec{k}.\vec{R}}$ and $\delta^R = -\delta_k^R e^{-i\vec{k}.\vec{R}}$, we have the final relation

$$\delta_k^S(\vec{k}) \approx \delta_k^R(\vec{k})\left(1 + f_0\mu^2\right).$$

The power spectrum is given by the square of δ_k, which is

$$\begin{aligned}
P^S(\vec{k}, \mu) &\approx \left(1 + f_0\mu^2\right)^2 P^R(\vec{k}) \\
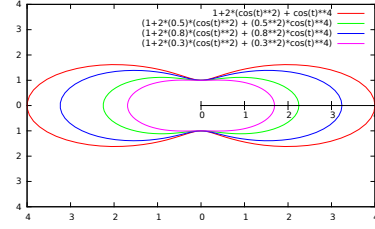&= (1 + 2f_0\mu^2 + f_0^2\mu^4)P^R(\vec{k}). \quad (5)
\end{aligned}$$

Figure 1. Representative polar plot of $P^S(\vec{k}, \mu)$, for a given $\left|\vec{k}\right|$, and $f = 0.3, 0.5, 0.8$ and 1.0. LOS is along the x-axis.
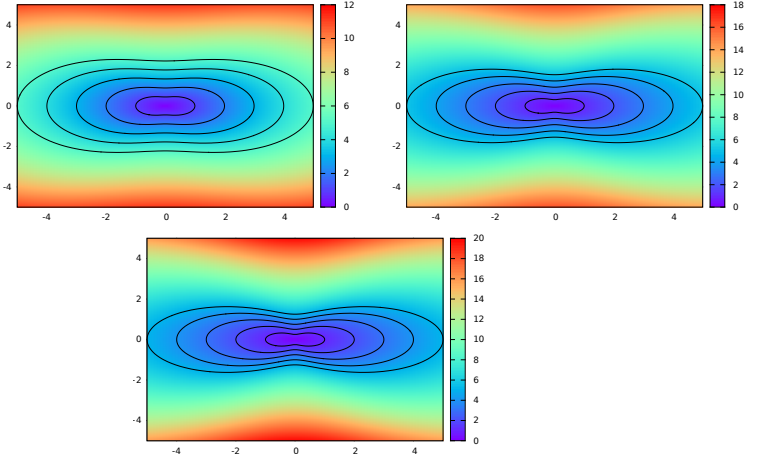

Figure 2. Representative contour plots for $f = 0.5, 0.8$ and 1.0. LOS is along the y-axis.

III. IMPLICATIONS

The aforesaid relation is of particular interest, for now, given the power spectrum (of the density contrast), one can fit a curve to find the value of f_0 using equation (5). For instance, for a given value of $\left|\vec{k}\right|$, one can plot P^S as a function of the angle between $\vec{k}$ and the line of sight. This dependence will only come from the μ dependent part and is represented in figure 1, since δ_k^R is expected to be isotropic. Another possibility is to plot contours of constant P^S, representative plots of which are given in figure 2 while an observed plot is given in figure 3, which is calculated using co-relations which we will not discuss here.

IV. CONCLUSION

We have shown how to correct for the errors caused by neglecting the peculiar velocity, using linear theory. We have also shown that this correction, while at first sight seems to require the knowledge of the potential, to be relevant, it can be converted into a form that allows us to estimate the value of cosmological parameters. This was first pointed out by N. Kaiser [1].
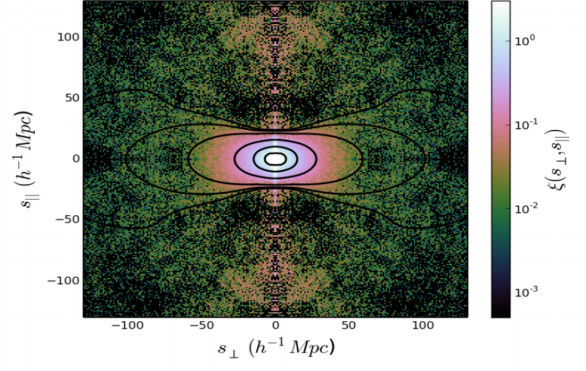
Figure 3. An observed contour plot of δ^S, with LOS along the y-axis. [3]

V. ACKNOWLEDGEMENTS

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