OLLY #1 ①

SM & Beyond
Timeline

1820 Faraday, Field
1875 Light Waves
Maxwell, Hertz } Electricity plus Magnetism = 1^{st} Unification

1890s (a) Unit of Charge
Thompson & Millikan CET] e^-
(b) X-rays (c) Radioactivity

Planck 1900
Einstein 1905
Rutherford
} Qtm of action $\hbar$: Plank
STR
Discovery of nucleus : { Bulk of atomic mass, +ve charge

1911 Bohr Atom

$\boxed{1917 \quad GTR}$

$(\not{\partial} - m)\psi = 0$

Jordan / Fermi

Dirac, Heisenberg, Schrodinger, Chadwick

1920 QM : QFT

1930 : Neutron + Proton = Nucleus $m_n - m_p = 1.36$ MeV $\mathcal{L}_F = G_F^A(\bar{\psi}\Gamma_A\psi)(\bar{\psi}\Gamma_A\psi)$

1932 : Fermi Theory for Beta decay $c = \hbar = 1$ $n \to p + e^- + \bar{\nu}_e$

1937 : Yukawa ⇒ Massive "Meson" mediate inter nuclear force

$m_\gamma = 0, \Leftrightarrow R_{EM} = \infty$

$m \neq 0 \Leftrightarrow R_{Nuc} \sim 10^{-13}$ cm] How?

⇓

100 MeV

→ Muon 105 MeV → Heavy electron : I. Rabi "Who ordered the Muon?"
Not strongly interacting with p, n.

Feynman, Schwinger, Tomonaga, Dyson

→ Pion ~ 135 MeV

1945-1950 QED = QFT of $(\bar{e}, \gamma)$ → 1000 MeV?

1950-1960 → ZOO $T \sim 10^{-23}$ sec ~ time light takes to travel distance of nucleus.

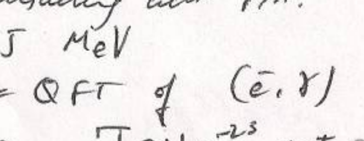
1957-60 { SU(3) classification → Quark
{ Gell Mann, Neeman, Zweig } of Hadronic Particle → Ace

composites of fundamental rep

$\pi^\pm \ \pi^0$ $\binom{u}{d} \sim 2$ of $SU(2)$
$\underline{2} \times \underline{2} = 3 + 1$
p, n
$2 \times 2 \times 2$

1957 — MSGF Model of Weak interaction V-A Theory (Vector, Axial Vector)
1961

Lee & Yang (Parity)



$\mathcal{L}_F = G_F \bar{\psi}_1 (1-\gamma_5)\gamma^\mu \psi_2$
$\bar{\psi}_3 (1-\gamma_5) \psi_3$
$+ G_F^2 \bar{\psi}_1 \sigma^{\mu\nu} \psi_2$
$\bar{\psi}_3 \sigma_{\mu\nu} \psi_3$

spin 1
Intermediate Vector Boson

1950s : Yang-Mills (-Shaw)

Revise : QED

QED : gauge theory i.e. with Local $U(1)$ rephasing sym

$U(1) = \{e^{i\theta} ; \theta \in [0, 2\pi]\}$

$\psi \to e^{i\theta(x)}\psi$

$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \theta(x)$

Noether (Revise)
Continuous sym
$\partial_\mu j^\mu = 0$
$Q = \int dx^3 j^0(x,t)$; $\frac{dQ}{dt} = 0$

?

$U(1) \to SU(n)$ or any compact gp } Non-Abelian $g_1 g_2 \neq g_2 g_1$

$SU(2) = T^1, T^2, T^3 \longrightarrow W^1, W^2, W^3$; $R_{weak} \sim \sqrt{G_F}$

$\frac{W^1 \pm W^2}{\sqrt{2}} = W^{\pm}$

$= 10\,eV$

$\sim (100\,GeV)^{-1}$

1961 : Goldstone, Anderson, Nambu

If $Q|vac\rangle \neq 0 \Rightarrow m_{Gold} = 0 \Rightarrow R_{ange} = \infty$ — $m_g \leq 10^{-2}\,eV$

$\left[\begin{matrix} WW \\ NG \end{matrix} : \right\} [Q, H] = 0$, $Q|vac\rangle_{WW} = 0$

$Q|vac\rangle_{WW} \neq 0$

$\left(A_\mu^{m=0} + (N.G)^{m=0}\right) =$ Massive Photon

If $|vac\rangle$ is invari- then $U^{iQ}|vac\rangle = \mathbb{1}|vac\rangle$
under the action Q, $\Rightarrow Q|vac\rangle = 0$

~~If~~ If not in $\Leftrightarrow Q|vac\rangle \neq 0$

Ques How is $Q|vac\rangle = 0$ or not $\neq 0$ related to mass of mediators?

Ques: Whats a ~~could be~~ mediator really?

2

1950s QED
1954/5 { YMS (Yang Mill Shaw)
Parity Violation {Yang ~~Mills~~ Lee; Wu}

2-component theory of Solons {Landau, Pauli} ?

$\begin{pmatrix}\psi_1\\ \psi_4\end{pmatrix}$ → $\gamma_5 = 1 \begin{pmatrix}x\\x\end{pmatrix}$, → $\gamma_5 = -1 \begin{pmatrix}x\\x\end{pmatrix}$

Dirac γ^μ

Quarks Aces

ZOO 1957-60

$T_{strong} \sim 10^{-23}$ sec (time for light to pass through the nucleus)

$T_{strange} \simeq 10^{-10} - 10^{-16}$ sec

[Gellmann – Nee'man] Zweig

SU(2) Isospin (Heisenberg 1940)

$\begin{pmatrix}p\\n\end{pmatrix}$ $\Delta m_N = 1.36$ MeV, $m_N = 936$ MeV

SU(3) – octets, decuplets

1961: Glashow IVB? / WI (Weak Interaction) $\Leftarrow W^\pm$, $\gamma \overset{?}{=} W^3$?] SU(2)

$M_{W^\pm} = R^{-1}_{weak} \propto 100$ GeV] ?

→ what did we know first? Range or mass (for weak interaction)

Hadrons
Meson: $Q\bar{Q}'$
Baryon: $QQ'Q''$

1961-62: Goldstone thm

$\left.\begin{array}{l}\partial_\mu J^\mu = 0\\ \dot{Q}(t) = 0\end{array}\right\} \Rightarrow [Q, H] = 0$ $\quad Q|vac\rangle \neq 0$? How?

⇓

$m_{Goldstone} = 0 \Rightarrow R_{Gold} = \infty$??

→ verify: is it the mass of the particle?

Higgs, Englert, Brout, Kibble, Guralnik, Anderson,

(Goldstone + A_μ $m_\gamma = 0$) ⇒ Massive spin ? 1 (3 polarization)

QED: polarized?

$\mathcal{L} = \gamma (D_\mu \phi)^\dagger (D^\mu \phi) - \underline{V(\phi^\dagger\phi)} - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$

Ask: Chanchha

$\qquad\qquad = \frac{\lambda}{2}(\phi^\dagger\phi - v^2)^2$

assignments back

Weinberg: PRL: 1967

A model for ~~WI~~ of Leptons

$\boxed{SU(2)_L \times U_Y(1)}$ → $W^\pm, Z^0$; → A Photon

$(2, -1) = \begin{pmatrix}\nu\\ e\end{pmatrix}_L$ $\quad e_R(1, Y = -2)$

$\Phi = \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}$

1973 Gargamelle CERN } Weak Neutral Currents

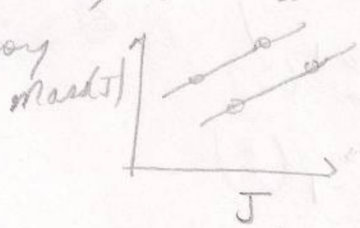
~~$\partial\phi$~~ $\partial_\phi V = 0 \Rightarrow \langle\phi^0\rangle \neq 0$

$(2, \frac{1}{3}) = \begin{pmatrix} u \\ d \end{pmatrix}_L \quad u_R \quad d_R$

$\qquad\qquad\qquad (1, 4/3) \quad (1, -2/3)$

$Q = T_{3L} + \frac{Y}{2}$

Strong interact theory



$\longrightarrow$ Motivated the string model

Regge Trajectory

String theory

1967 : SLAC DIS $\rightarrow$ Deep Inelastic Scattering

$R \sim \frac{1}{E}$ $\quad n^0 \rightarrow udd$

$\quad p \rightarrow uud$

1975 : $SU(3)_c \times SU(2)_L \times U(1)$ $\quad R \sim \frac{1}{E_\gamma} \quad 10^{-13} cm$

?

$$\left.\begin{matrix} Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L & u_R & d_R & \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L & e_R \\ \begin{pmatrix} c \\ s \end{pmatrix}_L & c_R & d_R & \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L & \mu_R \\ \begin{pmatrix} t \\ b \end{pmatrix}_L & t_R & b_R & \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L & \tau_R \end{matrix}\right\}$$

1975 - 2012

$\boxed{\text{1998 : Neutrino Oscillation} \quad m_\nu \neq 0}$

$$\Phi = \begin{pmatrix} \phi^+ \\ \frac{Re\,\phi^0 + Im\,\phi^0}{\sqrt{2}} \end{pmatrix}$$

H^0

Olly

#3

Group Theory

$G = \{ \{g_1 g_2 \cdots g \cdots\} ; g_1 \odot g_2 = g_3 ;$

$g \cdot e = e \cdot g = g \quad \forall g \in G$

$g^{-1} \cdot g = g \cdot g^{-1} = e \}$

Discrete: $\quad \mathbb{Z}_2 = \{ e, o \}$ $\quad 1, -1$

P, C, T

R-parity?

$e^2 = e \quad e \cdot o = o \cdot e = o$

$o^2 = e$

$\cancel{g \cdot e = e \cdot g = g \quad \forall g \in G}$

$\cancel{g^{-1} \cdot g = g \cdot g^{-1} = e}$

$\mathbb{Z}_n = \{ \omega^n = 1 \}$

$\omega_m = \exp\left(\frac{2\pi i m}{n}\right)$

$m = 0, 1, \ldots, n-1$

Similarity transform $\quad U \phi_i U^\dagger(\phi, \pi) = R_{ij} \phi_j$

$U|i\rangle = R_{ij} |j\rangle$

NB: We won't worry about the 2nd quantized operator derivation in implementing our symmetry.

$O(2)$

2×2 orthogonal matrices

$O^T = O^{-1} ; \quad O^T O = O O^T = \mathbb{1}_2 = e$

$(\det O)^2 = 1 \qquad O : \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R, R \right\}$

$\det O = \pm 1$

$O(2) \simeq SO(2) \times \mathbb{Z}_2$

$\left((\;), (\;), (\;) \right) \times \left\{ I, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

$O = \mathbb{1} + A + \mathcal{O}(A^2)$ → generator × parameter

$O^T = O^{-1}$

$\Rightarrow A^T = -A \quad ; \quad (\cancel{1 + A^T + \mathcal{O}(A^2)})$

$$(1 + A^T + \dots)(1 + A + \dots) = \mathbb{1} + \mathcal{O}(A^2)$$

$$\Rightarrow \quad A^T = -A$$

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad R = e^{\ominus \theta \epsilon}$$

$$\epsilon^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad = \mathbb{1} + \theta\epsilon + \mathcal{O}(\theta^2)$$

$$e = \sum_{n=0}^{\infty} \frac{\theta^{2n}(-\mathbb{1})^n}{(2n)!} + \sum_{n=0}^{\infty} \frac{\theta^{2n+1}}{(2n+1)!}(-\mathbb{1})^n \epsilon$$

$$= \cos\theta + \epsilon \sin\theta$$

using $\ominus$ sign

$$\begin{pmatrix} V_1' \\ V_2' \end{pmatrix} = R \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} c_\theta V_1 - s_\theta V_2 \\ s_\theta V_1 + c_\theta V_2 \end{pmatrix}$$

$V_i' = R_{ij} V_j \longleftarrow$ Vector/Fund (ir)reducible rep of $O(2)$

$$T'_{i_1 i_2 \dots} = R_{i_1 i_1'} R_{i_2 i_2'} \dots R_{i_n i_n'} T_{i_1' \dots i_n'}$$

2^n elements $\qquad R^T \mathbb{1}_2 R = \mathbb{1}_2$

Invariant Tensors, eg. δ_{ij}

$$\Delta_{ij} = \delta_{ij}$$

$$\Delta'_{ij} = R_{ii'} R_{jj'} \delta_{i'j'}$$

$$= R_{ii'} R^T_{i'j}$$

$$= \delta_{ij}$$

$$e^{-\theta\epsilon}\epsilon e^{\theta\epsilon} = \epsilon$$

$$E_{ij} = \epsilon_{ij}$$

$$E'_{ij} = R_{ii'} R_{jj'} \epsilon_{i'j'}$$

So then ~~using~~

$$= \cancel{(\det R)}^{1} \epsilon_{ij}$$

$$= \epsilon_{ij} = E_{ij}$$

Recall: $M_{i_1 i_1'} \dots M_{i_n i_n'} \epsilon_{i_1 \dots i_n}$

$$= \det M \; \epsilon_{i_1' \dots i_n'}$$

☺

$T_{ij} = A_{ij} + S_{ij}$

$\quad\quad\ \ \parallel \quad\quad \parallel$

$\quad\ -A_{ji} \quad +S_{ji}$

$\quad\quad\ \ \parallel$

$\quad\quad a\epsilon$

(obvious)

2

1

(for a 2×2 only)

\# of indep. components ⟶ 1   3

$\operatorname{Tr} S' = \operatorname{tr} RSR^T$

$S'_{ij} = R_{ii'} R_{jj'} S_{i'j'}$

$= (RSR^T)_{ij}$

$$S_{ij} = \hat{S}_{ij} + \frac{1}{2}\delta_{ij}(S_{kk})$$

$\hat{S}_{ii} = 0$

1

2

$3 = 2 + 1$

$\hat{S}_{ijk}$ $\hat{S}_{iik} = 0$

$S_{11k} + S_{22k} = 0$] 2 each (8-2 = 4?)

8

~~2 — 8 —~~

$\left.\begin{matrix} S_{111} \\ S_{112} \\ S_{221} \\ S_{222} \end{matrix}\right\} - 2 = 2$

$U(1) \approx \operatorname{spin}(2)$

$O(2n)$ ⎫ Clifford

$O(2n+1)$ ⎭ $\underline{\text{Algebra}}$

$\gamma^1 \ldots \gamma^{2n}$

$\underline{\text{related}}$ to $\operatorname{spin}(2n)$

their gen. satisfy same relation. commutation

$\operatorname{sp}(2n+1)$

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij}\, \mathbb{1}_{2^n}$$

$2^n \times 2^n$ matrix

$i = 1, \ldots 2n$

$O(2)$ ⊢ γ^1, γ^2

$\quad\quad\ \ 2\times2 \quad 2\times2$

$n = 1$, $\{\gamma^i, \gamma^j\} = 2\delta^{ij}\, \mathbb{1}_2$

$\gamma^1 = \sigma^1$, $\gamma^2 = \sigma^2$

σ_{ij}

$$J^{(12)} = \frac{-i}{4}[\gamma^1, \gamma^2]$$

$$= \frac{-i}{4}[\sigma^1, \sigma^2] = \frac{\sigma^3}{2}$$

Need an anti-Hermitian Matrix

$\therefore\ U = e^{\theta A}$ $U^\dagger = U^T = U^{-1} = e^{-\theta A}$

$\theta \in [0, 2\pi)$

$e^{\theta\epsilon} e^{\theta'\epsilon} = e^{(\theta+\theta' \bmod 2\pi)\epsilon}$

$$\mathrm{spin}(2)\ \{ S = e^{i\theta J_{12}},\ \theta \in [0, 4\pi)\ ;\ S^\dagger = S^{-1} \}$$

$$e^{i\theta J_{12}} e^{i\theta' J_{12}} = e^{i(\theta + \theta' \bmod 4\pi) J_{12}}$$

$$e^{\frac{i\theta\sigma_3}{2}} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \qquad \gamma_F \sim \gamma_1 \gamma_2$$

$$[\sigma_1, \sigma_2] = 2i\sigma_3$$

$$\text{def}^n:\ \gamma_F \equiv -i\gamma_1\gamma_2 = \sigma_3$$

NB: $\gamma_F^2 = \mathbb{1}$

$\Rightarrow$ eigenvalue $\gamma_F = \pm 1$

$$\sigma_3 \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix} = \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix}$$

Def^n: $P_\pm = \frac{(1 \pm \gamma_F)}{2}$

$$\sigma_3 \begin{pmatrix} 0 \\ \psi_- \end{pmatrix} = -\begin{pmatrix} 0 \\ \psi_- \end{pmatrix}$$

NB: $[\gamma_F, \cancel{\gamma} J_{12}] = 0$

$$U(1) = \{ e^{i\theta};\ \theta \in [0, 2\pi) \}$$

$$(e^{i\theta})^{-1} = e^{-i\theta}$$

$$e^{i\theta} e^{i\theta'} = e^{i((\theta + \theta') \bmod 2\pi)}$$

$$O(2) \qquad V_\pm = \frac{V_1 \pm i V_2}{\sqrt{2}} \qquad \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & +i \\ 1 & -i \end{pmatrix}}_{U} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$$

check: $\cancel{\cos}$ $\begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}$ $\qquad V_\pm' = e^{\pm i\theta} V_\pm$

$$\phi_q' = e^{iq\theta} \phi_q$$

1-O preserves norm, $\quad V'^T V' = V^T V$

$$\psi^{\dagger\prime} \psi' = \psi^\dagger \psi \qquad \phi_q' \psi_r' = e^{i(q+r)\theta}$$

charge q rep.

$$q = \frac{m}{n}$$

#4

$O(2) = e^{\theta\epsilon}$ 9 Jan

$O^T = O^{-1}$

$V^T V = V_1^2 + V_2^2$

$V^T \underbrace{\mathbb{1}}_{\text{Euclidean Metric}} V = V_1^2 + V_2^2$

$V^T \underbrace{\eta}_{\text{Minkowski}} V = V_1^2 - V_2^2$ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

So we want $L^T \eta L = \eta$ | $R^T \mathbb{1}_2 R = \mathbb{1}_2$

$\cancel{L = e^{\beta\sigma_1}}$ $L = e^{\alpha\sigma_1} = e^{-\alpha\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$

σ_1 is symmetric

$L^T = L, \; L = e^{-\alpha\sigma_1}\sigma_3 = \sigma_3 e^{\alpha\sigma_1}$

$\sigma_3 = \eta^{(2)}$

(copy from prashansa)

$= \begin{pmatrix} ch\,\alpha & sh\,\alpha \\ sh\,\alpha & ch\,\alpha \end{pmatrix}$

---

spin $(1,1) \longrightarrow \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

$= e^{i\omega_{12}J^{(12)}} = e^{i\omega_{12}\frac{\sigma_3}{2}}$] unitary

$S_L = e^{i\omega_{01}J^{(01)}}$] no longer unitary

$= e^{-\frac{\omega_{01}\sigma_3}{2}}$

$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$

$\psi_+ \to e^{-\frac{\omega_{01}}{2}}$

$\psi_- \to e^{+\frac{\omega_{01}}{2}}$

$\gamma^0 = \sigma^1$

$\gamma^1 = \pm i\sigma^2 = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$J^{01} = -\frac{i}{4}[\gamma^0, \gamma^1]$

$= -\frac{i}{4}[\sigma^1, \pm i\sigma^2]$

$= \frac{\pm}{2} i\sigma^3$

$i\sigma^3$

$\gamma^1\gamma^2 = \pm i\sigma^1\sigma^2$

$O(2)$

$\det R = 1$

$e^{\theta\epsilon}(-(R_1^{-1}R_2))$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e$ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R_2$

1

#4

$O(2) = e^{\theta\epsilon}$ 9 Jan

$O^T = O^{-1}$

$V^T V = V_1^2 + V_2^2$

Euclidean Metric

$V^T \mathbb{1} V = V_1^2 + V_2^2$

$V^T \eta V = V_1^2 - V_2^2$

Minkowski $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

So we want $L^T \eta L = \eta$ | $R^T \mathbb{1}_2 R = \mathbb{1}_2$

~~$L = e^{\beta\sigma_1}$~~ $L = e^{-\alpha\sigma_1} = e^{-\alpha\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$

σ_1 is symmetric

$L^T = L, L = e^{-\alpha\sigma_1}\sigma_3 = \sigma_3 e^{\alpha\sigma_1}$

$\sigma_3 = \eta^{(2)}$

(copy from prashansa)

$\Sigma = \begin{pmatrix} ch\alpha & sh\alpha \\ sh\alpha & ch\alpha \end{pmatrix}$

---

spin $(1,1) \longrightarrow \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

$= e^{i\omega_{12} J^{(12)}} = e^{i\omega_{12}\frac{\sigma_3}{2}}$] unitary

inor

$S_L = e^{i\omega_{01} J^{(01)}}$] No longer unitary

$= e^{-\frac{\omega_{01}\sigma_3}{2}}$

$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$

$\psi_+ \to e^{-\frac{\omega_{01}}{2}}$

$\psi_- \to e^{+\frac{\omega_{01}}{2}}$

$\gamma^0 = \sigma^1$

$\gamma^1 = \pm i\sigma^2 = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$J^{01} = -\frac{i}{4}[\gamma^0, \gamma^1]$

$= -\frac{i}{4}[\sigma^1, \pm i\sigma^2]$

$= \frac{\pm}{2} i\sigma^3$

$i\sigma^3$

$\gamma^1\gamma^2 = \pm i\sigma^1\sigma^2$

$O(2)$ det $R = 1$

$e^{\theta\epsilon}(-(R_1^{-1}R_2))$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = R_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R_2$

$O(3) = \left\{ \hat{R}^{+} = \hat{R}^{-1} \right\}$

$= V_1^2 + V_2^2 + V_3^2 = V^T V = V'^T V'$

$\left(\det \hat{R}\right)^2 = 1$

$\det \hat{R} = \pm 1$

$O(3) = SO(3) \times (\text{reflection})$

$\uparrow$

$\det R = 1$

$R = e^A$

$A^T = -A$

$\begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}$

$\Downarrow$

There're 3 real parameters.

$A = \frac{1}{2} a_{[ij]} A^{[ij]} = a_{12} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_{31} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + a_{23} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

& $-A^{[21]} = A^{[12]}$ & so on.

$\boxed{\left(A^{[ij]}\right)_{kl} \equiv \delta^i_{[k}\delta^j_{l]}}$

Rotation in 3-d, don't commute

$t_i = \frac{1}{2} \epsilon_{ijk} A^{[jk]}$

$[A^{ij}, A^{kl}]_q \sim \pm \delta^{[ik} A^{j]l]}$

$t_1 = \frac{1}{2}(\epsilon_{123} A^{23} + \epsilon_{132} A^{32}) = A^{[23]}$

$t_i = \frac{1}{2} \epsilon_{ijk} A^{[jk]}$

$[A, B]^T = (AB - BA)^T = (BA - AB) = -[A, B]$

$[t_i, t_j] \sim \epsilon_{ijk} t_k$

$t_i \equiv -iT_i \Rightarrow [T_i, T_j] \sim \epsilon_{ijk} (iT_k)$

$R = e^{\sum_i \theta_i t_i}$ ← Canonical / Exponential Parameters

$= e^{\vec{\theta} \cdot \vec{t}} = e^{\theta_1 t_1 + \theta_2 t_2 + \theta_3 t_3}$

$R = e^{\theta_1 t_1} e^{\theta_2 t_2} e^{\theta_3 t_3} \simeq \mathbb{1}_3 + \vec{\theta} \cdot \vec{t}$

$R^T = e^{-\theta_3 t_3} e^{-\theta_2 t_2} e^{-\theta_1 t_1}$

$RR^T = R^T R = \mathbb{1}_3$

BCH Thm: $e^A e^B = \exp\left(A + B + \frac{1}{2}[A, B] + \frac{1}{12}\left\{[A, [A, B]] + [B, [B, A]]\right\} \ldots\right)$

$e^{\vec{\theta} \cdot \vec{t}} e^{\vec{\theta}' \cdot \vec{t}} = e^{\vec{\theta}'' \cdot \vec{t}}$

Claim: $\vec{\theta}'' = \vec{\theta} + \vec{\theta}' + \frac{1}{2}(\vec{\theta} \times \vec{\theta}') + \theta^2\theta' + \theta'^2\theta + \ldots$

NB: Generators can yield multiplication rules

SO(3), Mat rep

$3 \times 3 \longrightarrow [t_i, t_j] = \epsilon_{ijk} t_k$ then $\exists$ infinite ds

$d \times d \longrightarrow [T_i, T_j] = \epsilon_{ijk} T_k$

NB: The multiplication rules remain same
$\therefore$ they depend on the commutators only

$V_i' = R_{ij} V_j$

$V' = RV \qquad i, j = 1, 2, 3$

$T_{i_1 \ldots i_n} = R_{i_1 i_1'} \ldots R_{i_n i_n'} T_{i_1' \ldots i_n'}$

is an n index Tensor

Invariant Tensors of SO(3)

$\rightarrow \Delta_{ij} = \delta_{ij} \rightarrow \Delta'_{ij} = \delta_{ij} = \Delta_{ij}$

$\rightarrow \quad \epsilon_{ijk} = \epsilon_{ijk} \rightarrow \epsilon'_{ijk} \cancel{(\det)} = \epsilon_{ijk} = \epsilon_{ijk}$

in SO(3) NB: $\epsilon_{i'j'} R_{ii'} \cancel{\left(\frac{R_{jj'}}{R_{jk'}}\right)} = \epsilon_{ij} R_{jk'} \Rightarrow \epsilon_{i'k'} R_{ii'} = \epsilon_{ij} R_{jk'}$

Similarly $so(3)$:

$A_{ij} = -A_{ji}$

$A'_{ij} = R_{ii'} R_{jj'} A_{i'j'}$

$B_i' = \frac{1}{2} \epsilon_{ijk} A'_{jk} = \frac{1}{2} \epsilon_{ijk} R_{jj'} R_{kk'} A_{j'k'}$

$\boxed{B_i' = R_{ij} B_j}$

$S_{ij} = \hat{S}_{ij} + \frac{\delta_{ij}}{3} (S)$

You can trade an anti-symmetric tensor for a vector

$T_{[i_1 i_2]} \ldots i_n$ $\quad 5 = 2 \cdot 2 + 1$

$N^2 - \frac{N}{2}$

$T_{i_1 \ldots i_n} \rightarrow S_{(j_1 \ldots j_m)}$

$\downarrow$

$\sum \hat{S}_{iij_3 \ldots j_m} = 0$

m index symmetric tensor on N valued indices

$$\frac{N(N+1) \ldots \cancel{N}(N+M-1)}{M!}$$

$$= {}^{N+M-1}C_M = \frac{(N+M-1)!}{M!\,(N-1)!}$$

12-Jan '14

#Olly 2.1

$\# O(3)$ $T_{[i_1 i_2]}\cdots \to \hat{S}_{i_1 \cdots i_n}\cdots$

$\binom{N+M-1}{M} = \frac{N(N+1)\cdots(N+M-1)}{M!\,(N-1)!} - \frac{N(N+1)(N+M-3)}{(M-2)!}$

$-\binom{N+M-3}{M-2}$

$d[\hat{S}_{i}\cdots i_M] = 2M+1$

$[t_i, t_j] = \pm \epsilon_{ijk} t_k$ where $t = iT$

$t^T = -t$, $t^* = t$

$[T_i, T_j] = i\epsilon_{ijk}T_k$

for $N=3$, $\binom{M+2}{M} - \binom{M}{M-2} = \frac{(M+2)(M+1)}{2} - M(M$

$=$ calc $= 2M+1$

Spin(3) γ^i $i=1,2,3$

$S\{\gamma^i, \gamma^j\}S^{-1} = 2\delta^{ij}\, S\mathbb{1}S^{-1}$

$(\gamma^i)' = S\gamma^i S^{-1}$ We use $\gamma^i = \sigma^i$ (could use $S\sigma^i S^{-1}$)

Recall: $\left.\begin{matrix}O(2)\\O(3)\end{matrix}\right\}$ 2×2 γ

$\left.\begin{matrix}O(2M)\\O(2M+1)\end{matrix}\right]$ both use $2^M \times 2^M$

$J_{ij} = -\frac{i}{4}[\gamma_i, \gamma_j]$

$\Rightarrow J_{12} = -\frac{i}{4}\underbrace{[\sigma_1, \sigma_2]}_{2i\sigma_3} = \frac{\sigma_3}{2}$

$J_{31} = \frac{\sigma_2}{2}$; $J_{23} = \frac{\sigma_1}{2}$

$T^a \equiv \frac{\sigma^a}{2}$ generates spin(3)

$[T^a, T^b] = i\epsilon^{abc}T^c$

$\parallel\mid$ so(3) commutation relation

so(3) $\simeq$ spin(3) $\to$ algebra

is isomorphic

$1 + i\vec{\theta}\cdot\vec{T} = U$
$+ \mathcal{O}(\theta^2)$

since $\vec{T}^+ = \vec{T}$,

$U = e^{i\vec{\theta}\cdot\vec{T}}$; thus $U^+ = U^{-1} = U(-\theta)$

$\chi^*_\alpha \psi_\alpha = \bar{\chi}^\alpha \psi_\alpha = \bar{\chi}^\beta \delta^\alpha_\beta \psi_\alpha$

$T'_{\alpha_1 \cdots \alpha_M}$

$T'^{\alpha_1 \cdots}_{\ \ \alpha_L}$

$T'^{\alpha_1\alpha_2}_{\ \ \beta_1\beta_2}\cdots$

$(\delta^\alpha_\beta)' = \bar{U}^\alpha_{\ \alpha'} U_\beta^{\ \beta'}\, \delta^{\alpha'}_{\beta'}$

$= \bar{U}^\alpha_{\ \beta'} U_\beta^{\ \beta'}$

old rotation

$= (U^*_{\alpha\beta'})(U_{\beta\beta'})$

$= (UU^+)_{\alpha\beta} = \mathbb{1}_{\alpha\beta}$

new rotation $= \delta^\alpha_\beta$

$\psi'_\alpha = U_\alpha^{\ \beta}\psi_\beta$

$\alpha = 1, 2$

$\chi^+\psi = \chi'^+\psi'$

$\psi' = U\psi$

$\chi' = U\chi$ $\longrightarrow$ $\boxed{\chi'_\alpha = U_\alpha^{\ \beta}\chi_\beta}$

$\chi'^+ = \chi^+ U^+$

$\chi'^* = U^*\chi^*$ $\to \chi'^*_\alpha = (U_\alpha^{\ \beta})^*\chi^*_\beta$

$\bar{\chi}^{\alpha\prime} = \bar{U}^\alpha_{\ \beta}\bar{\chi}^\beta$

$\chi'_\alpha = U_\alpha^{\ \beta}\chi_\beta$

$U^+ = U^{-1}$

$\bar{U}^\alpha_{\ \beta} = (U_\alpha^{\ \beta})^*$

$\bar{U}^\alpha_{\ \beta}$

$=0$ $\alpha=\beta$

TAS $\epsilon_{12} = 1$

$\Rightarrow \left.\begin{matrix}\epsilon_{\alpha\beta}\\ \epsilon^{\alpha\beta}\end{matrix}\right\}$ are also invariant tensor

could be $\epsilon^{12} = 1$ or $\epsilon^{12} = -1$

$U = e^{i\vec{\theta}\cdot\vec{T}}$

$= e^{i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}$

$\operatorname{tr}\vec{\sigma} = 0$

$\boxed{\begin{matrix}U^+ = U^{-1}\\ \det U = 1\end{matrix}} \equiv SU(N)$

$\det M = \exp \operatorname{tr}\ln M$

$\det U = \exp\operatorname{tr}(i\vec{\theta}\cdot\vec{T}) = e^0 = 1$

$\epsilon_{\alpha\beta} = \epsilon_{\alpha\beta}$

$\epsilon'_{\alpha\beta} = U_\alpha^{\ \alpha'} U_\beta^{\ \beta'}\epsilon_{\alpha'\beta'} = \epsilon_{\alpha\beta}$

$U_{\alpha\alpha'}U_{\beta\beta'}\epsilon_{\alpha'\beta'} = \det U\,\epsilon_{\alpha\beta}$

$U_\alpha^{\ \beta} = \left(e^{i\frac{\vec{\theta}\cdot\vec{\sigma}}{2}}\right)_\alpha^{\ \beta} \equiv (U_{\alpha\beta})$

removed

$U^T = e^{i\vec{\tilde{\theta}}\cdot\frac{\vec{\sigma}}{2}}$; $\vec{\tilde{\theta}} = (\theta_1, -\theta_2, \theta_3)$

$= \epsilon\, e^{\frac{i}{2}(\theta_1\sigma_1 - \theta_2\sigma_2 + \theta_3\sigma_3)} = e^{-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}}\epsilon$

$\epsilon = i\sigma_2 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$

$\Rightarrow \epsilon U^T = U^+\epsilon$ (yes this is correct; $\sigma_1^T = \sigma_1$, $\sigma_3^T = \sigma_3$

$\begin{pmatrix}0&1\\1&0\end{pmatrix}^T = \begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\begin{pmatrix}1&0\\0&-1\end{pmatrix}^T = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$)

$U^T\epsilon = \epsilon U^+$

$\Rightarrow \epsilon U = U^*\epsilon$

$\chi'^T\epsilon\psi' = \chi^T U^T\epsilon U\psi = \chi^T\epsilon\underbrace{U^+U}_{\mathbb{1}}\psi = \chi^T\epsilon\psi$

$\begin{pmatrix}\chi' = U\chi\\ \psi' = U\psi\end{pmatrix}$

$\tilde{\chi} \equiv \epsilon\chi$

$\begin{pmatrix}\tilde{\chi}_1\\ \tilde{\chi}_2\end{pmatrix} = \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = \begin{pmatrix}\chi_2\\ -\chi_1\end{pmatrix}$

$\tilde{\chi}' = \epsilon\chi'$
$= \epsilon U\chi$
$= U^*(\epsilon\chi)$

$\begin{pmatrix}\chi_2\\ -\chi_1\end{pmatrix}' = U^*\begin{pmatrix}\chi_2\\ -\chi_1\end{pmatrix}$

$\boxed{2^* \sim 2}$

$\psi'_\alpha = U_\alpha{}^{\alpha'}\psi_{\alpha'}$ $\quad S^{\alpha\beta\,\prime} = U^\alpha{}_{\alpha'}U^\beta{}_{\beta'}S^{\alpha'\beta'}$

$T_{[\alpha_1\alpha_2]\cdots\alpha_M} \Rightarrow \underbrace{S_{\alpha_1\cdots\alpha_M}}$ | dim: Totally anti-symmetric, is the most general $SU(2)$ tensor (can't take trace) :-

$T_{\alpha\beta} = S_{\alpha\beta} + A_{\alpha\beta}$ $\quad D(S^{(M)}) = \binom{N+M-1}{M} = \binom{M+1}{M} = \cdots = M+1$ $\quad 2j+1 = \boxed{M+1}$

M	$d(S^{(M)})$
1	$2 = 2(\frac{1}{2})+1$
2	$3 = 2(1)+1$
3	$4 = 2(\frac{3}{2})+1$

$O(2) \to \text{spin}(2) \to O(1,1)$, $Sp(1,1)$ $\to O(3) \to \text{spin}(3) = SU(2)$

$\text{spin}(2) \to U(1)$

unitary gps $U(N)$

$N\times N \quad U^\dagger = U^{-1} \quad \hat{U}^\dagger\hat{U} = \hat{U}\hat{U}^\dagger = \mathbb{1}_N$

$|\det\hat{U}|^2 = 1 \Rightarrow \det\hat{U} = e^{i\theta}$

$\hat{U} = Ue^{i\theta/N}$

$\det\hat{U} = (\det U)\,e^{i\theta}$

$U(N) = SU(N)\times U(1)$

($N\times N$, unitary mat, $\det U = 1$)

{traceless Hermitian Matrices}

$SU(2) = U = e^{i\vec{\theta}\cdot\vec{\sigma}/2}$

$U = e^{iH} \quad H^\dagger = H$

$U^\dagger = e^{-iH'} = U^{-1}$

$\det U = 1 = \exp \ln e^{iH} = e^0 = 1$

(if tr $H = 0$)

#2.2

Diagonal generators (Cartan Subalgebra)

Hermitian $U^\dagger H U = \Lambda = \text{Diag}(\lambda_1 \ldots \lambda_n)$

$(E_{(1)}, E_{(2)} \ldots E_{(n)})$

columns = Eigenvectors of H

$n\times n$ $H = H^\dagger \Rightarrow n$ eigenvectors $\perp$ wrt Hermitian inner products $E^\dagger_{(i)} E_j = \delta_{ij}$

$(2j+1) = n\times n$ $[\mathcal{T}_a, \mathcal{T}_b] = i\epsilon_{abc}\mathcal{T}_c$

$SU(2) \to T_1\ T_2\ T_3$ 2×2 $[T_a, T_b] = i\epsilon_{abc}T_c$

$\text{tr}\, T_a = 0$ $\frac{\sigma_3}{2} = \frac{1}{2}\text{Diag}(1,-1)$

Spin j representation in Vector space supporting action of $SU(2)$

I $|j, j\rangle$, $|j, j-1\rangle$, $\vdots$, $|j, -j+1\rangle$, $|j, -j\rangle$

$\mathcal{T}_3 = \text{Diag}(j, j-1, \ldots -j)$

$j = \frac{1}{2}$ $\mathcal{T}_3 = T_3 = \frac{1}{2}\text{Diag}(1,-1)$

$\langle j, m'|\mathcal{T}_\pm|j, m\rangle$

$V_\pm = \frac{1}{\sqrt{2}}(v_1 \pm i v_2)$

$O(2)$ $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ $V^T V = V'^T V'$ $O^T = O^{-1}$

$\begin{pmatrix}v_1\\v_2\end{pmatrix} \xrightarrow{U} \begin{pmatrix}V+\\V-\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&i\\1&-i\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}$ $\overset{\shortparallel}{U}$

$V^T V = V(+ V_-) = 2V_+V_-$

$V = U^\dagger \hat{V}$ $V^T = \hat{V}^T U^*$

$V^T V = \hat{V}^T \underbrace{(U^* U^\dagger)}_{\begin{pmatrix}0&1\\1&0\end{pmatrix}} \hat{V}$

$= (V_+\ V_-)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}V+\\V-\end{pmatrix}$

Similarly, let's see what $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ transforms to

$(\hat{V}_+, \hat{V}_-)\underbrace{U^*\begin{pmatrix}0&1\\-1&0\end{pmatrix}U^\dagger}_{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}\begin{pmatrix}\hat{V}_+\\V_-\end{pmatrix}$

$SU(2)$

$S(\alpha_1 \ldots \alpha_N)$

$S_{\alpha\beta} = S_{\beta\alpha} \to$

			T_3
S_{11}	—	S_{++}	1
$S_{12}+S_{21}$	—	S_0	0
S_{22}	—	S_{--}	-1

$\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} \leftarrow$ vector of $SO(3)$ same thing as $SO(2)$ for $SO(3)$

Unitary Gps

$U(N)\{\hat{U}^\dagger = \hat{U}^{-1};\ N\times N \text{ unitary}\}$

$\hat{U}^\dagger \hat{U} = 1$ $\hat{U} = \left(\exp\frac{i\theta}{N}\mathbb{1}_N\right)U$

$|\det\hat{U}|^2 = 1$

$\overset{\shortparallel}{e^{i\theta}}$ $\det\hat{U} = \det U e^{i\theta}$

$U(N) = SU(N)\times U(1)$

$SU(N) = \{U^\dagger = U^{-1};\ \det U = 1\}$

$(e^{iH})^\dagger = e^{-iH} = (e^{iH})^{-1}$ $\Rightarrow H_{ii} = H_{ii}^*$, $H_{ij} = H_{ji}^*$ $i<j$

$\uparrow$ $H^\dagger = H$ (no sum, diagonal elements)

N real (diagonal)

$\frac{N(N-1)}{2}\times 2$ (below diagonal) (complex)

$\overset{\shortparallel}{N^2}$

$\to$ Now also $\det U = 1 \Rightarrow \text{tr}\, H = 0$

$SU(N)\begin{cases}H = \sum_a \theta_a T_a\\ T_a\ \ N^2-1\end{cases}$

So there're N^2-1 independent Hermitian traceless $N\times N$ linearly independent matrices

:((some T_a part missing)

Gell-Mann Basis for $SU(N)$

$N=2$ Pauli $\sigma_1 = \begin{pmatrix}0&+1\\+1&0\end{pmatrix}$ $\sigma_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$ $\sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

Normalization is $\text{Tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$

$N=2$ $\sigma^{(12)} = \begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\sigma^{[12]} = i\begin{pmatrix}0&+1\\-1&0\end{pmatrix}$

$N=3$ $\lambda_1 = \left(\begin{array}{c|c}\sigma_1&0\\\hline 0&0\end{array}\right) = \lambda^{(12)}$ $\lambda_2 = \left(\begin{array}{c|c}\sigma_2&\\\hline &0\end{array}\right)$

$\lambda^{(13)} = \lambda_4$ $\lambda^{[13]} = \lambda_5$

$\lambda^{(23)} = \lambda_6$ $\lambda^{[23]} = \lambda_7$

$\lambda_3 = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$ $\lambda_8 = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}\frac{1}{\sqrt{3}} \to$ square trace = same in all

$\to$ traceless

Q. How many diagonal generators?

A. You found $N-1$ real independent #s $\Rightarrow$ $N-1$ diagonal generators

what're left are off diagonal matrices

:(MISS!

$T^A = \frac{\lambda^A}{2} \qquad A = 1, \dots 8$

$\mathrm{tr}\, T^A T^B = \frac{1}{2} \delta^{AB}$

$$\begin{array}{ccc} & T_3 & T_8 \, 2\sqrt{3} \\ \psi_1 & \frac{1}{2} & 1 \\ \psi_2 & -\frac{1}{2} & 1 \\ \psi_3 & 0 & -2 \end{array}$$

$T_8 = \frac{\lambda_8}{2} = \frac{1}{2\sqrt{3}} \, \mathrm{Diag}(1, 1, -2)$

$\psi^+ \psi = \psi'^+ \psi'$ where $\psi' = U\psi$

$\psi'_\alpha = U_\alpha{}^\beta \psi_\beta$

2.3

$SU(N)$: $N\times N$ Hermitian traceless
$a = 1, \ldots, N^2-1$ dim
$\mathrm{tr}\, T^a T^b = S^{ab}$ ← symmetric real; orthogonal $R^{ab'}T^{a'}$

$$R_{aa'}R_{bb'}\,\mathrm{Tr}(T^{a'}T^{b'}) = \mathrm{Tr}(T'^{a}T'^{b})$$
$$= (RSR^T)_{ab} = [\mathrm{Diag}(s_1,\ldots,s_N)]_{ab}$$
$$T''^{a} = \frac{T'^{a}}{\sqrt{s_a}} \quad \mathrm{tr}(T''^a T''^b) = \frac{1}{2}\delta^{ab} \Rightarrow \mathrm{tr}((T^a)^2) = \frac{1}{2}$$
DOUBT: Why O?

U, $U = e^{i\theta\cdot T}$, $\psi' = U\psi$, $V^{(\bar{3})}\otimes V^{(3)}\ldots$

$$\begin{pmatrix}\psi_1\\ \vdots\\ \psi_N\end{pmatrix} = U\begin{pmatrix}\psi_1\\ \vdots\\ \psi_N\end{pmatrix} \to \psi'_i = U_{ij}\psi_j \xrightarrow{\text{Notation}} \psi'_i = U_i^{\ j}\psi_j$$
$$U = \exp i\frac{\lambda^a}{2}\theta^a \;;\; a = 1,\ldots,N^2-1$$
$$\psi_i'^{*} = (U_i^{\ j})^*\psi_j^* \quad \bar{U}^i_{\ j} \equiv (U_i^{\ j})^*$$
$$\chi^{i\prime} = \bar{U}^i_{\ j}\chi^j$$

M index tensors

$\psi\in\mathbb{C}^N$ $\quad \psi^\dagger\psi = \psi'^\dagger\psi'$

$$T'_{i_1\ldots i_M} = U_{i_1}^{\ i_1'}\ldots U_{i_M}^{\ i_M'}T_{i_1'\ldots i_M'}$$

Aim: Find irreducible

$SU(3)$: $M=2$ $\quad 9\to 3+6$
$$T_{ij} = A_{ij} + S_{ij}$$
$$T' = UTU^T \quad [(T-T^T)']^T = [UTU^T - UT^TU^T]^T$$
DOUBT SEEMS WRONG $A'^T = -A$; $S'^T = S$ $= -A'$

$E_{i_1\ldots i_N} \equiv \epsilon_{i_1\ldots i_N}$ $\quad \left(\epsilon_{12\ldots N} = +1,\ \text{TAS}\right)$

$$E'_{i_1\ldots i_N} = U_{i_1}^{\ i_1'}\ldots U_{i_N}^{\ i_N'}\epsilon_{i_1'\ldots i_N'}$$
$$= (\cancel{\det U})\,\epsilon_{i_1\ldots i_N}$$
$$= E_{i_1\ldots i_N}$$

(missed fill) $\underbrace{(U_{i_1}^{\ j_1})^* U_{i_1}^{\ i_1'}}_{3\times 3} U_{i_2}^{\ i_2'} U_{i_3}^{\ i_3'}$
$\epsilon_{j_1 i_2' i_3'} = \epsilon_{i i_2 i_3}$ $(U_{i_1}^{\ j_1})^*$
$\vdots$
$U_{i_3}^{\ i_3'}\epsilon_{j_1 i_2 i_3'} =$
$\epsilon_{i_1 i_2 i_3}(U_{i_1}^{\ j_1})^*$ $(U_{i_2}^{\ j_2})^*$

$\chi^i\psi_i = \chi'^i\psi'_i$

A_{ij}
$$\bar{\chi}^i = \frac{1}{2}\epsilon^{ijk}A_{jk}$$
Eg: $\bar{\chi}^1 = A_{23}$, $\bar{\chi}^2 = A_{31}$, $\bar{\chi}^3 = A_{12}$

Ex: $\Rightarrow \bar{\chi}'^i = \bar{U}^i_{\ j}\bar{\chi}^j$

More general Tensor

$$T'^{\,j_1\ldots j_L}_{i_1\ldots i_M} = \bar{U}^{j_1}_{\ j_1'}\ldots\bar{U}^{j_L}_{\ j_L'}\ldots U_{i_1}^{\ i_1'}\ldots U_{i_M}^{\ i_M'}T^{j_1'\ldots j_L'}_{i_1'\ldots i_M'}$$
$$\epsilon'_{i_1\ldots i_N} = \epsilon_{i_1\ldots i_N};\ \epsilon'^{i_1\ldots i_N} = \epsilon^{i_1\ldots i_N}$$
$$\delta'^i_{\ j} = \delta^i_{\ j}$$

$T_{ij} = A_{ij} + S_{ij}$
$\quad\quad\ 3 \quad\ \ 6$
$\quad\quad \bar{3}$ (c.c. rep)

$$T_i^{\ j} = \hat{T}_i^{\ j} + \frac{1}{N}\delta_i^{\ j}(T_k^{\ k})$$
$N^2 \quad N^2-1 \quad 1$

SU(N) algebra

$$\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$$
$$T^{a\dagger} = T^a$$
$$\mathrm{tr}(T^a) = 0$$

$$(i[T_a,T_b])^\dagger = -i(T_b^\dagger T_a^\dagger - T_a^\dagger T_b^\dagger) = +i[T_a,T_b]$$

$\mathrm{tr}([T_a,T_b]) = 0$; $i[T_a,T_b] = -f_{abc}T_c$
$[T_a,T_b] = if_{abc}T_c$ real

$\cancel{i}[T_a,T_b] \neq if_{abc}T_c$
$\because$ Hermitian

NB: (a) $T'^a = UT^aU^\dagger$
$$[T'^a, T'^b] = if_{abc}T'^c$$
(b) $T'^a = R^{ab}T^b$
s.t. $R^T = R^{-1}$; $\mathrm{Tr}(T'^aT'^b) = \mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$

$$[T_a', T_b'] = R_{aa'}R_{bb'}[T_{a'},T_{b'}]$$
$$= R_{aa'}R_{bb'}if_{a'b'c'}T_{c'}$$
$$\quad\quad\quad R^{-1}_{c'c}T'_c$$
$$= iR_{aa'}R_{bb'}f_{a'b'c'}R_{c'c}T'_c$$
$$\Rightarrow f_{abc} = iR_{aa'}R_{bb'}R_{cc'}f_{a'b'c'}$$

$$\mathrm{tr}([T_a,T_b]T_d) = if_{abc}\,\underbrace{\mathrm{tr}(T_cT_d)}_{\frac{1}{2}\delta_{cd}} \Rightarrow \boxed{f_{abc} = -2i\,\mathrm{Tr}[T_a,T_b]T_c}$$

Ex. (Later)
① calculate f_{abc} for $N=3$ $\quad T^a = \frac{\lambda^a}{2}$

② Show f_{abc} is anti-symmetric on transposition of any two indices.

N^2-1
$d\times d$ matrices $\{\mathcal{T}^a\}$ are said to furnish a d dimension d dimensional rep of $SU(N)$
if $[\mathcal{T}^a,\mathcal{T}^b] = if^{abc}\mathcal{T}^c$

Show by BCH $\left(U = e^{i\theta^a\mathcal{T}^a}\right.$ has the same mult rule as $U = e^{i\theta_aT_a}$
$\quad\quad\downarrow$
lie in fundamental domain

$$(\mathcal{T}^a)_{bc} = if_{bac}$$
$$e^{i\theta_a\mathcal{T}_a} = e^{-\theta_a\mathcal{F}_a}$$
$$F_a^T = -F_a \quad (F_a)_{bc} = f_{bac}$$
$$R^T = R^{-1}$$
$$R^* = R$$
$$R_{ab} = e^{-\theta^a F^a}$$

Representations of SU(N)

Tensors of fundamental N : $\psi_i = \begin{pmatrix}\psi_1\\ \vdots \\ \psi_N\end{pmatrix}$ $\psi_i' = U_i{}^j \psi_j$

$T'_{i_1 \ldots i_M} = U_{i_1}{}^{i_1'} \cdots U_{i_M}{}^{i_M'} T_{i_1' \ldots i_M'}$ $\psi_i^{*\prime} = (U_i{}^j)^* \psi_j^*$

$\epsilon_{i_1 \ldots i_N}$ is invariant

$U = e^{i\theta^a T_a}$

$U^* = e^{i\theta^a(-T^{a*})}$, $-T^{a*} = -T^{aT}$

$\left[\begin{matrix} T^a = T^{a\dagger} \\ T^{a*} = T^{aT}\end{matrix}\right]$

su(N) $[T^a, T^b] = i f^{abc} T^c$

$T_{ij} = S_{ij} + A_{ij}$

su(3)	9	6	3 — 3^*	
SU(N)	N^2	$\frac{N(N+1)}{2}$	$\frac{N(N-1)}{2}$	

N^2-1 mat $[(-T^a)^*, (-T^a)^*] \quad (-T^{c*})\, if^{abc}$

$(X^a)_{bc} = if_{bac} = i(F^a)_{bc} \rightarrow N^2-1$ dimensional ~~real~~ irreducible rep of SU(N) adjoint representation. $U = e^{iX^a\theta^a} = e^{-F^a\theta^a} = R_A(\theta)$

ASK JD

$\Rightarrow (F^a)^T = -F^a$

Ex: Show $(X^a)_{bc} = if_{abc}$ define an N^2-1 dimensional irreducible of SU(N)

Jacobi Identity: $[T^a,[T^b,T^c]] + \text{cyclic} = 0 \Rightarrow if^{bcd}\, if^{ade} T^e + \text{cyclic} = 0$

$\square$ ψ_i N, $\bar{N}$ $3^* \sim A_{ij}$ SU(3)

$A'_{i_1 \ldots i_{N-1}} = U_{i_1}{}^{i_1'} \cdots U_{i_{N-1}}{}^{i_{N-1}'} A_{i_1' \ldots i_{N-1}'}$

$T_{[i_1 \ldots i_N]} \sim 1$ (This is why you have trace as invariant)

$\bar{A}^i = \frac{1}{(N-1)!} \epsilon_{i i_1 \ldots i_{N-1}} A_{i_1 \ldots i_{N-1}} = \frac{1}{(N-1)!}\epsilon^{i i_1 \ldots i_{N-1}} A_{i_1 \ldots i_{N-1}}$ $\epsilon' = \epsilon \Rightarrow \boxed{\bar{A}^{i\prime} = U^i{}_j \bar{A}^j}$ almost done VERIFY

$\square$ ψ_i N

$N-1$ $A_{[i_1 \ldots i_{N-1}]}$ $\bar{N}$ $\sim \bar{A}^i$

$T_{[i_1 \ldots i_{N-1}]j} \sim T^i{}_j = \hat{T}^i{}_j + \frac{\delta^i_j}{N}(T^k_k)$

$S(i_1 i_2 i_3 \ldots i_M) = d(S_{i_1 \ldots i_M}) = \frac{N(N+1)\cdots(N+M-1)}{M!}$ $T_{[i_1 \ldots (i_{N-1}]j)}$

SU(N) adjoint $N-1$

Ex: Show this is equivalent to extracting the totally anti-symmetric piece

$U T^a U^\dagger$

$U = e^{i\theta\cdot T}$

$= e^{i\theta\cdot T} T^a e^{-i\theta\cdot T}$

$= T^a + \underbrace{i[\theta\cdot T, T^a]}_{i\theta^b[T^b,T^a] = i\, f_{bac}^{\ldots}{}^c} + \frac{i^2}{2!}[\theta\cdot T,[\theta\cdot T, T^a]] + \ldots$

$= T^a + i[\theta\cdot T, T^a] + \frac{i^2}{2!}[\theta\cdot T,[\theta\cdot T,T^a]] \ldots$

$T_{(R)bc} = (e^{i\theta\cdot X})_{bc} = \delta_{bc} + i\theta^a \underbrace{(X^a)_{bc}}_{if_{bac}} + \ldots = \delta_{bc} - \theta^a f_{bac} + O(\theta^2)$

Ex: SU(2) $V^a \frac{\sigma^a}{\sqrt{2}} = V$! $V' = UVU^\dagger$ show $\sim S_{\alpha\beta}$ & find the connection.

$e^{i\theta\cdot T} T^a e^{i\theta\cdot T} = R^{-1}(\theta)_{ab} T^b$

any representation of SU(N)

$A'^a = R^{ab} A^b$ $\quad e^{i\theta\cdot X}$

$\psi_i' = U_i{}^j \psi_j$

$A = A^a(T^a\sqrt{2})$ $\quad \mathrm{Tr}\, A^2 = (\sum_a A_a^2)$

$(A^a)^2$

$\mathrm{tr}(A^2) = 2(\mathrm{Tr}\, T^a T^b) A^a A^b = \sum_a (A^a)^2$

$T^a A'^a = R^{ab} A^b T^a$

$= A^b (R^{-1})_{ba} T^a$ $\quad$ adjoints

$\quad (U T^b U^\dagger)$ $\quad \boxed{A' = UAU^\dagger}$