Bag #3.1

Linear Algebra

System of Linear Eq^{n}s:

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2$$

~ represent ~ lines.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; \quad A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$A\vec{X} = \vec{B}$$

Round off errors: If the determinant is ~ 0, i.e. when the lines are almost parallel

Methods of evaluation (baby stuff)

(a) $$(a_{21} * a_{11} - a_{21} * a_{11})x + (a_{22} * a_{11} - a_{12} * a_{21})y = b_2 * a_{11} - b_1 * a_{21}$$

(b) $$\left(a_{21} - a_{11}\left(\frac{a_{21}}{a_{11}}\right)\right)x + \left(a_{22} - a_{12}\left(\frac{a_{21}}{a_{11}}\right)\right)y = b_2 - b_1\left(\frac{a_{21}}{a_{11}}\right)$$

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} - a_{12}\left(\frac{a_{21}}{a_{11}}\right) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 - b_1 \frac{a_{21}}{a_{11}} \end{pmatrix}$$

The second approach will take ~ 10 steps, for 2 equations.
This can cause problems if $a_{11} \sim 0$. So instead iterate through the rows & put the largest pivot row first (called pivot)

Gauss Elimination: Upper Triangular form
Gauss-Seidel Elimin^n: Diagonal form

In general the operations needed

$$\left.\begin{matrix} n-1 \\ n-2 \\ \vdots \\ 1 \end{matrix}\right\} = 1 + 2 + \dots + (n-1)$$

$$2 \times \Big[\underbrace{\frac{n(n-1)}{2}}_{\text{ASK Prashansa}} (S + M) + (n-1)D \Big]$$

$$\simeq 4\, O(n^2)$$

$\to$ for diagonal final

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Case: a_{22} & $a_{21} \sim \epsilon$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \to \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} - \underbrace{a_{12}a_{21}}_{\text{This } \sim \epsilon^2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 - \underbrace{b_1 \frac{a_{21}}{a_{11}}}_{\text{this may!}} \end{pmatrix}$$

maybe the round off error of a_{22} & not affect it

Eg:

$$\left.\begin{matrix} x + \epsilon y = 1 \\ \epsilon x + y = 1 \end{matrix}\right] \begin{matrix} y = -\frac{x}{\epsilon} + \frac{1}{\epsilon} \\ y = 1 - \epsilon x \end{matrix}$$

$$\left.\begin{matrix} x + \epsilon y = 1 \\ 0 + y = 1 - \epsilon \end{matrix}\right\} \Rightarrow \begin{aligned} x &= 1 - \epsilon y \\ &= 1 - \epsilon(1 - \epsilon) \\ &= 1 - \epsilon + O(\epsilon^2) \end{aligned}$$

(neglecting ϵ^2)

These things can come back & hit you.

The issue is that you have no handle on the errors.
The solution is SVD

GK: Amdahl's Law: If s: seq. fraction, p: parallel fraction $\Big\}$ if $s + p = 1$, then the speedup $= 1/s$

Bagla # 3.2

Gauss-Jordan
Gauss Elimination $O(n^2)$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

$A\vec{x} = \vec{B}$

Row equilibriation
$|a_{ij}| \leq 1 \; \forall i$
divide an entire eq^n by the largest.

Column eq.
$|a_{ij}| \leq 1$, $\max |a_{ij}| = 1$
for each j

These help to avoid comparisons/operation of largely different #s.

Thm: $A = LU$ (condition on A?)
- upper triangular
- lower triangular.

Demand L to be of the form

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}$$

can verify that

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l^{-1}_{21} & 1 & 0 & 0 \\ l^{-1}_{31} & l^{-1}_{32} & 1 & 0 \\ l^{-1}_{41} & l^{-1}_{42} & & 1 \end{bmatrix}$$

where l^{-1}_{ij} is the mat element of L^{-1} & can be obtained by

Methods
$A\vec{x} = \vec{b}$
$U\vec{x} = \_\_\,\vec{b}$
↳ our old form
we do
$\underbrace{L U}_{A}\vec{x} = \vec{b}$
$U\vec{x} = L^{-1}\vec{b}$

1) Unit Diag L
Doolittle
2) Unit Diag U
Crout's algorithm

$$\overset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{a_{21}}{a_{11}} & 1 & 0 & 0 \\ \frac{a_{31}}{a_{11}} & & 1 & 0 \\ \frac{a_{41}}{a_{11}} & & & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

So for a given A

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & & \cdots & \\ & & & \end{bmatrix}$$

I'll have $a_{22} = \frac{a_{21}}{a_{11}} \cdot a_{12} + x \Rightarrow x = a_{22} - \frac{a_{21}}{a_{11}} \cdot a_{12}$

In general,

$$l_{i1} = \frac{a_{i1}}{a_{11}} \; ; \quad u_{kj} = a_{kj} - \sum_{i=1}^{k-1} l_{ki} u_{ij}$$

Other algorithm is (Doolittle's)

$$l_{ik} = \frac{a_{ik} - \sum_{i=1}^{k-1} l_{ij} u_{jk}}{u_{kk}}$$

the issue is that you may have $u_{kk} = 0$

$A = U \Sigma V^T$
$U^T U = \mathbb{1}$
$V V^T = \mathbb{1}$
$\Sigma = \text{diag}\{\sigma_1 \ldots \sigma_n\}$
$U \equiv$ eigen-vector of $A^T A$ (put in a column)
$V \equiv$ $A^T A$ (put in a row)
$\sigma_i = \sqrt{\lambda_i (A^T A)}$
↳ not multiplication
λ_i is the eigenvalue of $A^T A$

$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_n \geq 0$