OLLY #1 (1)

SM & Beyond 1820 Faraday, Field

Timeline 1875 Light Waves
Maxwell, Hertz } Electricity plus Magnetism = 1st Unification

1890s (a) Unit of Charge
Thompson & Millikan CET] e^-
(b) X-ray (c) Radioactivity

Planck 1900 } Qtm of action $\hbar$: Plank
Einstein 1905 } STR
Rutherford Discovery of nucleus : { Bulk of atomic mass, +ve charge

1911 Bohr Atom

$\boxed{1917 \quad GTR}$ $(\not\partial - m)\psi = 0$

Dirac, Heisenberg, Schrodinger, Chadwick

1920 QM : QFT / Jordan Fermi

1930 : Neutron + Proton = Nucleus $m_n - m_p = 1.36$ MeV $\mathcal{L}_F = G_F^A(\bar\psi \Gamma_A \psi)(\bar\psi \Gamma_A \psi)$

1932 : Fermi Theory for Beta decay $c = \hbar = 1$ $n \to p + e^- + \bar\nu_e$

1937 : Yukawa ⇒ Massive "Meson" mediate inter nuclear force
$m_\gamma = 0, \Leftrightarrow R_{EM} = \infty$
$m \neq 0 \Leftrightarrow R_{Nuc} \sim 10^{-13}$ cm] How?
⇓
100 MeV
→ Muon 105 MeV → Heavy electron : J. Rabi "Who ordered the Muon?"
Not strongly interacting with p, n.

Feynman, Schwinger, Tomonaga, Dyson } → Pion ~ 135 MeV

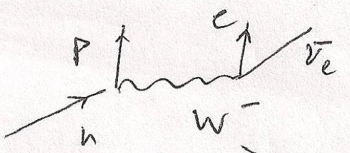
1945-1950 QED = QFT of $(\bar e, \gamma)$ → 1000 MeV ?

1950-1960 → Zoo $T \sim 10^{-23}$ sec ~ time light takes to travel distance of nucleus.

1957-60 { SU(3) classification → Quark
{ Gell Mann, Neeman, Zweig } of Hadrons
Particle → Ace
→ composites of fundamental rep
$\pi^\pm \quad \pi^0$ } $\binom{u}{d} \sim 2$ of SU(2)
$\underline{2} \times \underline{2} = 3 + 1$
$\left.\begin{matrix} p \\ n \end{matrix}\right\}$ $2 \times 2 \times 2$

1957
1961 — MSGF Model of Weak interaction V-A Theory → Vector → Axial Vector

Lehi Pal (Parity)



spin 1 } Intermediate Vector Boson

$\mathcal{L}_F = G_F \bar\psi_1 (1 - \gamma_5)\gamma^\mu \psi_2 \; \bar\psi_3 (1 - \gamma_5) \psi_3 + G_F^2 \bar\psi_1 \sigma^{\mu\nu} \psi_2 \; \bar\psi_3 \sigma_{\mu\nu} \psi_3$

O

1950s : Yang-Mills (-Shaw)

Revise : QED

QED : Gauge Theory i.e. with Local U(1) rephasing Symm

$U(1) = \{e^{i\theta}; \theta \in [0, 2\pi]\}$

$\psi \to e^{i\theta(x)}\psi$

$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \theta(x)$

?

$U(1) \to SU(n)$ etc any compact gp } Non-Abelian $g_1 g_2 \neq g_2 g_1$

Noether (Revise)
Continuous Symm
$\partial_\mu j^\mu = 0$
$Q = \int dx^3 j^0(x,t)$, $\frac{dQ}{dt} = 0$

$SU(2) = T^1, T^2, T^3 \rightarrow W^1, W^2, W^3$?

$\frac{W^1 \pm W^2}{\sqrt{2}} = W^\pm$

$R_{weak} \sim \sqrt{G_F}$ $\parallel$ $10 eV$ $\sim (100 GeV)^{-1}$

1961 : Goldstone, Anderson, Nambu

If $Q|vac\rangle \neq 0 \Rightarrow m_{Gold} = 0 \Rightarrow R_{ange} = \infty$ $\quad m_g \leq 10^{-2} eV$

$\left.\begin{matrix} WW \\ NG \end{matrix}\right\} [Q, H] = 0, \quad Q|vac\rangle_{WW} = 0$

$Q|vac\rangle_{WW} \neq 0$

$\left(A_\mu^{m=0} + (N.G)^{m=0}\right) =$ Massive Photon

If $|vac\rangle$ is invari. then $U^{iQ}|vac\rangle = 1|vac\rangle$ under the action Q, $\Rightarrow Q|vac\rangle = 0$

~~If~~ If not in $\Leftrightarrow Q|vac\rangle \neq 0$

Ques How is $Q|vac\rangle = $ or not $\neq 0$ related to mass of mediator?

Ques: ~~could be~~ Whats a mediator really?

1950s QED
1954/5 { YMS (Yang Mill Shaw) Parity Violation { Yang ~~Mills~~ Lee / Wu } Dirac Eq^n

2-component theory of Solons { Landau, Pauli } ?

$\begin{pmatrix} \psi_1 \\ \psi_4 \end{pmatrix}$ $\gamma_5 = 1 \begin{pmatrix} x \\ x \end{pmatrix}$, $\gamma_5 = -1 \begin{pmatrix} x \\ x \end{pmatrix}$

ZOO 1957-60

Quarks Aces

Gellmann – Nee'man, Zweig

$T_{strong} \sim 10^{-23}$ sec (time for light to pass through the nucleus)

$T_{strange} \simeq 10^{-10} - 10^{-16}$ sec

SU(2) Isospin (Heisenberg 1940s)

$\begin{pmatrix} p \\ n \end{pmatrix}$ $\Delta m_N = 1.36$ MeV, $m_N = 936$ MeV

SU(3) – octets, decuplets

Hadrons: Meson: $Q\bar{Q}'$, Baryon: $QQ'Q''$

1961: Glashow IV(B)? / WI (Weak Interaction) $\Leftarrow W^\pm$, $\gamma \stackrel{?}{=} W^3$?] SU(2)

$M_{W^\pm} = R^{-1}_{weak} \simeq 100$ GeV] ?

what did we know first? Range or mass (for weak interaction)

1961-62: Goldstone thm

$\left.\begin{aligned}\partial_\mu J^\mu = 0 \\ \dot{Q}(t) = 0\end{aligned}\right\} \Rightarrow [Q, H] = 0$ $Q|vac\rangle \neq 0$ ¿ How?

$\Downarrow$

$m_{Goldstone} = 0 \Rightarrow R_{Gold} = \infty$??

verify: is it the mass of the particle?

Higgs, Englert, Brout, Kibble, Guralnik ~~Anderson~~,

QED: polarized

(Goldstone + A_μ $m_\gamma = 0$) $\Rightarrow$ Massive spin 1 (3 polarization) ?

$\mathcal{L} = \gamma (D_\mu \phi)^\dagger (D^\mu \phi) - \underline{V(\phi^\dagger \phi)} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ Ask: Chanchha

$\qquad = \frac{\lambda}{2} (\phi^\dagger \phi - v^2)^2$

assignments back

Weinberg: PRL: 1967

A model for WI of Leptons

$\boxed{SU(2)_L \times U_Y(1)} \rightarrow W^\pm, Z^0$; A photon

$(2, -1) = \begin{pmatrix} \nu \\ e \end{pmatrix}_L$ $e_R(1, Y = -2)$

1975 Gargamelle CERN } Weak Neutral Currents

$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$

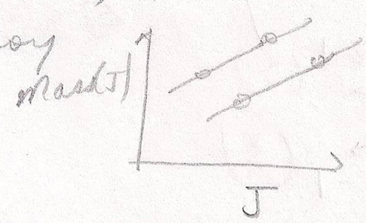
$\cancel{\partial\phi}$ $\partial_\phi V = 0 \Rightarrow \langle \phi^0 \rangle \neq 0$

$\left(2, \frac{1}{3}\right) = \begin{pmatrix} u \\ d \end{pmatrix}_L \quad u_R \quad d_R$

$\qquad\qquad (1, 4/3) \quad (1, -2/3)$

$Q = T_{3L} + \frac{Y}{2}$

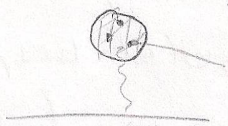
Strong interact theory



Regge Trajectory $\longrightarrow$ Motivated the string model

String theory

1967 : SLAC DIS $\rightarrow$ Deep Inelastic Scattering



$R \sim \frac{1}{E}$

$n^0 \rightarrow udd$

$p \rightarrow uud$

$R \sim \frac{1}{E_\gamma} \quad 10^{-13}\,cm$

1975 : $SU(3)_c \times SU(2)_L \times U(1)$

?

$$\begin{array}{llllll} Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L & u_R & d_R & \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L & e_R \\ \begin{pmatrix} c \\ s \end{pmatrix}_L & c_R & d_R & \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L & \mu_R \\ \begin{pmatrix} t \\ b \end{pmatrix}_L & t_R & b_R & \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L & \tau_R \end{array}$$

1975 - 2012

1998 : Neutrino Oscillation

$m_\nu \neq 0$

$$\Phi = \begin{pmatrix} \phi^+ \\ \frac{\mathrm{Re}\,\phi^0 + \mathrm{Im}\,\phi^0}{\sqrt{2}} \end{pmatrix}$$

H^0

Olly

#3

Group Theory

$$G = \{ \{g_1 g_2 \cdots g \cdots\} ;\ g_1 \odot g_2 = g_3 ;$$

$$g \cdot e = e \cdot g = g \quad \forall\, g \in G$$

$$g^{-1} \cdot g = g \cdot g^{-1} = e \}$$

Discrete: $\qquad \mathbb{Z}_2 = \{ \overset{1}{e}, \overset{-1}{o} \}$

P, C, T
R-parity?

$$e^2 = e \qquad e \cdot o = o \cdot e = o$$
$$o^2 = e$$

$$\cancel{g \cdot e = e \cdot g = g \quad \forall\, g \in G}$$
$$\cancel{g^{-1} \cdot g = g \cdot g^{-1} = e}$$

$$\mathbb{Z}_n = \{ w^n = 1 \}$$

$$w_m = \exp\left(\frac{2\pi i m}{n}\right)$$

$$m = 0, 1, \ldots, n-1$$

<u>Similarity transform</u> $\quad i U \phi_i U_\Sigma^+ (\phi, \pi) = R_{ij} \phi_j$

$$U|i\rangle = R_{ij} |j\rangle$$

NB: We won't worry about the 2nd quantized operator derivation in implementing our symmetry.

<u>$O(2)$</u>

2×2 orthogonal matrices

$$O^T = O^{-1} ;\quad O^T O = O O^T = \mathbb{1}_2 = e$$

$$(\det O)^2 = 1 \qquad O = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R ,\ R \right\}$$

$$\det O = \pm 1$$

$$O(2) \simeq SO(2) \times \mathbb{Z}_2$$

$$\left(\begin{pmatrix} \ \end{pmatrix}, \begin{pmatrix} \ \end{pmatrix}, \begin{pmatrix} \ \end{pmatrix} \right) \times \left\{ I, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

$$O = \mathbb{1} + A + \mathcal{O}(A^2) \qquad \text{generator} \times \text{parameter}$$

$$O^T = O^{-1}$$

$$\Rightarrow A^T = -A \ ; \qquad (\cancel{1 + A^T + \mathcal{O}(A^2)})$$

$$(1 + A^T + \dots)(1 + A + \dots) = \mathbb{1} + \mathcal{O}(A^2)$$

$$\rightarrow \quad A^T = -A$$

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad R = e^{\ominus\theta\epsilon}$$

$$\epsilon^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad = \mathbb{1} + \theta\epsilon + O(\theta^2)$$

$$e = \sum_{n=0}^{\infty} \frac{\theta^{2n}(-\mathbb{1})^n}{(2n)!} + \sum_{n=0}^{\infty} \frac{\theta^{2n+1}}{(2n+1)!}(-\mathbb{1})^n \epsilon$$

$$= \cos\theta + \epsilon \sin\theta$$

using $\ominus$ sign

$$\begin{pmatrix} V_1' \\ V_2' \end{pmatrix} = R\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} c_\theta V_1 - s_\theta V_2 \\ s_\theta V_1 + c_\theta V_2 \end{pmatrix}$$

$V_i' = R_{ij} V_j \longleftarrow$ Vector/Fund (ir)reducible rep of $O(2)$

$$T'_{i_1 i_2 \dots} = R_{i_1 i_1'} R_{i_2 i_2'} \dots R_{i_n i_n'} T_{i_1' \dots i_n'}$$

2^n elements

$$R^T \mathbb{1}_2 R = \mathbb{1}_2$$

Invariant Tensors, eg. δ_{ij}

$$\Delta_{ij} = \delta_{ij}$$

$$\Delta'_{ij} = R_{ii'} R_{jj'} \delta_{i'j'}$$

$$= R_{ii'} R^T_{i'j}$$

$$= \delta_{ij}$$

$$e^{-\theta\epsilon} \epsilon e^{\theta\epsilon} = \epsilon$$

$$E_{ij} = \epsilon_{ij}$$

$$E'_{ij} = R_{ii'} R_{jj'} \epsilon_{i'j'}$$

So then ~~using~~

$$= (\det R)^{1} \epsilon_{ij}$$

$$= \epsilon_{ij} = E_{ij}$$

Recall: $M_{i_1 i_1'} \dots M_{i_n i_n'} \epsilon_{i_1 \dots i_n}$

$$= \det M \; \epsilon_{i_1' \dots i_n'}$$

$$T_{ij} = A_{ij} + S_{ij}$$

$$A_{ij} = -A_{ji}, \quad S_{ij} = +S_{ji}$$

(for a 2×2 only)

$A_{ij} = a\,\epsilon$ (obvious)

$Tr\,S = tr\,RSR^T$

$S'_{ij} = R_{ii'}R_{jj'}S_{i'j'} = (RSR^{-1})_{ij}$

$$S_{ij} = \hat{S}_{ij} + \frac{1}{2}\delta_{ij}(S_{kk})$$

\# of indep components: 1 | 3 | 2 | 1

$\hat{S}_{ii} = 0$

$3 = 2 + 1$

$\hat{S}_{ijk}$

$\hat{S}_{iik} = 0$

$S_{11k} + S_{22k} = 0$

2 each (is it?)

8

~~2 8~~

$$\left.\begin{matrix} S_{111} \\ S_{112} \\ S_{221} \\ S_{222} \end{matrix}\right\} - 2 = 2$$

$U(1) \approx spin(2)$

$$\left.\begin{matrix} O(2n) \\ O(2n+1) \end{matrix}\right\} \text{Clifford Algebra}$$

$\gamma^1 \ldots \gamma^{2n}$

<u>related</u> to $spin(2n)$

their gen satisfy same commutation relation

$sp(2n+1)$

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij}\,\mathbb{1}_{2^n}$$

$2^n \times 2^n$ matrix

$i = 1, \ldots 2n$

$O(2) \vdash \gamma^1, \gamma^2$

$2\times 2 \quad 2\times 2$

$n = 1, \quad \{\gamma^i, \gamma^j\} = 2\delta^{ij}\,\mathbb{1}_2$

$\gamma^1 = \sigma^1, \quad \gamma^2 = \sigma^2$

σ_{ij}

$$J^{(12)} = \frac{-i}{4}[\gamma^1, \gamma^2] = \frac{-i}{4}[\sigma^1, \sigma^2] = \frac{\sigma^3}{2}$$

Need an anti-Hermitian Matrix

$U = e^{\theta A} \qquad U^\dagger = U^T = U^{-1} = e^{-\theta A}$

$\theta \in [0, 2\pi]$

$e^{\theta\epsilon} e^{\theta'\epsilon} = e^{(\theta+\theta' \bmod 2\pi)\epsilon}$

spin(2) $\{ S = e^{i\theta J_{12}} ,\ \theta \in [0, 4\pi) \ ; S^\dagger = S^{-1} \}$

$$e^{i\theta J_{12}} e^{i\theta' J_{12}} = e^{i(\theta + \theta' \bmod 4\pi) J_{12}}$$

$$e^{\frac{i\theta\sigma_3}{2}} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$$

$\gamma_F \sim \gamma_1 \gamma_2$

$[\sigma_1, \sigma_2] = 2i\sigma_3$

def^n: $\gamma_F \equiv -i\gamma_1\gamma_2$

$= \sigma_3$

NB: $\gamma_F^2 = \mathbb{1}$

& eigenvalue $\gamma_F = \pm 1$

def^n: $P_\pm = \dfrac{(1 \pm \gamma_F)}{2}$

NB: $[\gamma_F, \cancel{\gamma} J_{12}] = 0$

$$\sigma_3 \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix} = \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix}$$

$$\sigma_3 \begin{pmatrix} 0 \\ \psi_- \end{pmatrix} = -\begin{pmatrix} 0 \\ \psi_- \end{pmatrix}$$

$U(1) = \{ e^{i\theta} ; \ \theta \in [0, 2\pi) \}$

$$(e^{i\theta})^{-1} = e^{-i\theta}$$

$$e^{i\theta} e^{i\theta'} = e^{i(\theta + \theta') \bmod 2\pi}$$

$O(2)$ $\quad V_\pm = \dfrac{V_1 \pm i V_2}{\sqrt{2}}$

$$\underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & +i \\ 1 & -i \end{pmatrix}}_{U} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$$

Check: ~~$\cos$~~ $\begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}$ — $V_\pm' = e^{\pm i\theta} V_\pm$

O preserves norm $\quad V'^T V' = V^T V$

$\psi'^\dagger \psi' = \psi^\dagger \psi$

$\phi_q' \psi_r' = e^{i(q+r)\theta}$

$\phi_q' = e^{iq\theta} \phi_q$

charge q rep

$q = \frac{m}{n}$

#4

9 Jan

$O(2) = e^{\theta\epsilon}$

$O^T = O^{-1}$

$V^T V = V_1^2 + V_2^2$

Euclidean Metric

$V^T \mathbb{1} V = V_1^2 + V_2^2$

$V^T \eta V = V_1^2 - V_2^2$

Minkowski $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

So we want $L^T \eta L = \eta$ | $R^T \mathbb{1}_2 R = \mathbb{1}_2$

$\cancel{L = e^{\beta\sigma_1}}$ $\quad L = e^{\alpha\sigma_1} = e^{-\alpha\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$

σ_1 is symmetric

$L^T = L,\ L = e^{-\alpha\sigma_1}\sigma_3 = \sigma_3 e^{\alpha\sigma_1}$

$\sigma_3 = \eta^{(2)}$

(copy from prashansa)

Σ

$\Sigma = \begin{pmatrix} ch\,\alpha & sh\,\alpha \\ sh\,\alpha & ch\,\alpha \end{pmatrix}$

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spin $(1,1)$ $\longrightarrow$ $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

$S_{\text{spinor}} = e^{i\omega_{12}\sigma^{(12)}} = e^{i\omega_{12}\frac{\sigma_3}{2}}$] unitary

$S_L = e^{i\omega_{01}J^{(01)}}$] no longer unitary

$= e^{-\frac{\omega_{01}\sigma_3}{2}}$

$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$

$\psi_+ \to e^{-\frac{\omega_{01}}{2}}$

$\psi_- \to e^{+\frac{\omega_{01}}{2}}$

$\gamma^0 = \sigma^1$

$\gamma^1 = \pm i\sigma^2 = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$J^{01} = -\frac{i}{4}[\gamma^0, \gamma^1]$

$= -\frac{i}{4}[\sigma^1, \pm i\sigma^2]$

$= \frac{\pm}{2} i\sigma^3$

$i\sigma^3$

$\gamma^1\gamma^2 = \pm i\sigma^1\sigma^2$

$O(2)$

$\det R = 1$

$e^{\theta\epsilon}\left(-(R_1^{-1}R_2)\right)$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = R_1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R_2$

4

1

$O(2) = e^{\theta\epsilon}$ 9 Jan

$O^T = O^{-1}$

$V^T V = V_1^2 + V_2^2$

$V^T \mathbb{1} V = V_1^2 + V_2^2$ Euclidean metric

$V^T \eta V = V_1^2 - V_2^2$

η Minkowski $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

so we want $L^T \eta L = \eta$ | $R^T \mathbb{1}_2 R = \mathbb{1}_2$

$\cancel{L = e^{\beta\sigma_1}}$ $L = e^{\alpha\sigma_1} = e^{-\alpha\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$

σ_1 is symmetric

$L^T = L, L = e^{-\alpha\sigma_1} \sigma_3 = \sigma_3 e^{\alpha\sigma_1}$

$\sigma_3 = \eta^{(2)}$

(copy from prashansa)

$= \begin{pmatrix} ch\,\alpha & sh\,\alpha \\ sh\,\alpha & ch\,\alpha \end{pmatrix}$

---

spin $(1,1)$ $\longrightarrow$ $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

$= e^{i\omega_{12} J^{(12)}}$ spinor $= e^{i\omega_{12}\frac{\sigma_3}{2}}$] unitary

$S_L = e^{i\omega_{01} J^{(01)}}$] no longer unitary

$= e^{-\frac{\omega_{01}\sigma_3}{2}}$

$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$

$\psi_+ \to e^{-\frac{\omega_{01}}{2}}$

$\psi_- \to e^{+\frac{\omega_{01}}{2}}$

$\gamma^0 = \sigma^1$

$\gamma^1 = \pm i\sigma^2 = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$J^{01} = -\frac{i}{4}[\gamma^0, \gamma^1]$

$= -\frac{i}{4}[\sigma^1, \pm i\sigma^2]$

$= \frac{\pm}{2} i\sigma^3$

$i\sigma^3$

$\gamma^1\gamma^2 = \pm i\sigma^1\sigma^2$

$O(2)$ $\det R = 1$

$e^{\theta\epsilon}(-(R_1^{-1}R_2))$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e$ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R_2$

$O(3) = \{ \hat{R}^{T} = \hat{R}^{-1} \}$

$= V_1^2 + V_2^2 + V_3^2 = V^T V = V'^T V'$

$(\det \hat{R})^2 = 1$

$\det \hat{R} = \pm 1$

$O(3) = SO(3) \times (\text{reflection})$

$\det R = 1$

$R = e^A$

$A^T = -A$

$$\begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}$$

$\Downarrow$

There're 3 real parameters.

$A = \frac{1}{2} a_{[ij]} A^{[ij]} = a_{12} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_{31} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + a_{23} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

& $-A^{[21]} = A^{[12]}$ & so on.

$\boxed{(A^{[ij]})_{kl} \equiv \delta^{i}_{[k} \delta^{j}_{l]}}$

Rotations in 3-d, don't commute

$t_i = \frac{1}{2} \epsilon_{ijk} A^{[jk]}$

$[A^{ij}, A^{kl}]_{pq} \sim \pm \delta^{[i[k} A^{j]l]}$

$t_1 = \frac{1}{2} (\epsilon_{123} A^{23} + \epsilon_{132} A^{32}) = A^{[23]}$

$t_i = \frac{1}{2} \epsilon_{ijk} A^{[jk]}$

$[t_i, t_j] \sim \epsilon_{ijk} t_k$

$[A, B]^T = (AB - BA)^T = (BA - AB) = -[A, B]$

$t_i \equiv -i T_i \Rightarrow [T_i, T_j] \sim \epsilon_{ijk} (i T_k)$

$R = e^{\sum_i \theta_i t_i}$ ← Canonical / Exponential Parameters

$= e^{\vec{\theta} \cdot \vec{t}} = e^{\theta_1 t_1 + \theta_2 t_2 + \theta_3 t_3}$

$R = e^{\theta_1 t_1} e^{\theta_2 t_2} e^{\theta_3 t_3} \simeq \mathbb{1}_3 + \vec{\theta} \cdot \vec{t}$

$R^T = e^{-\theta_3 t_3} e^{-\theta_2 t_2} e^{-\theta_1 t_1}$

$R R^T = R^T R = \mathbb{1}_3$

BCH Thm: $e^A e^B = \exp(A + B + \frac{1}{2}[A,B] + \frac{1}{12} \{ [A,[A,B]] + [B,[B,A]] \} \ldots)$

$e^{\vec{\theta} \cdot \vec{t}} e^{\vec{\theta}' \cdot \vec{t}} = e^{\vec{\theta}'' \cdot \vec{t}}$

Claim: $\vec{\theta}'' = \vec{\theta} + \vec{\theta}' + \frac{1}{2} (\vec{\theta} \times \vec{\theta}') + \theta^2 \theta' + \theta'^2 \theta + \ldots$

NB: Generators can yield multiplication rules

SO(3), Mat rep

$3 \times 3 \longrightarrow [t_i, t_j] = \epsilon_{ijk} t_k$ then $\exists$ infinite ds

$d \times d \longrightarrow [T_i, T_j] = \epsilon_{ijk} T_k$

$d \times d$

NB: The multiplication rules remain same $\because$ they depend on the commutators only

$V_i' = R_{ij} V_j$

$V' = R V \qquad i, j = 1, 2, 3$

$T_{i_1 \ldots i_n} = R_{i_1 i_1'} \ldots R_{i_n i_n'} T_{i_1' \ldots i_n'}$

is an n index Tensor

Invariant Tensors of SO(3)

$\rightarrow \Delta_{ij} = \delta_{ij} \rightarrow \Delta'_{ij} = \delta_{ij} = \Delta_{ij}$

$\rightarrow \quad \epsilon_{ijk} = \epsilon_{ijk} \rightarrow \epsilon'_{ijk} \cancel{(\det R)} = \epsilon_{ijk} = \epsilon_{ijk}$

NB: $\epsilon_{i'j'} R_{ii'} \cancel{R_{jj'} R_{jk'}} = \epsilon_{ij} R_{jk'} \Rightarrow \epsilon_{i'k'} R_{ii'} = \epsilon_{ij} R_{jk'}$

Similarly $so(3)$:

$A_{ij} = -A_{ji}$

$A'_{ij} = R_{ii'} R_{jj'} A_{i'j'}$

$B_i' = \frac{1}{2} \epsilon_{ijk} A'_{jk} = \frac{1}{2} \epsilon_{ijk} R_{jj'} R_{kk'} A_{j'k'}$

$\boxed{B_i' = R_{ij} B_j}$

You can trade an anti-symmetric tensor for a vector.

$T[i_1 i_2] \ldots i_n \qquad 5 = 2 \cdot 2 + 1$

$\cancel{N^2} - \frac{N}{2}$

$S_{ij} = \hat{S}_{ij} + \frac{\delta_{ij}}{3}(S)$

$T_{i_1 \ldots i_n} \to S(j_1 \ldots j_m)$

$\downarrow$

$\sum \hat{S}_{iij_3 \ldots j_m} = 0$

m index symmetric tensor on N valued indices

$$\frac{N(N+1) \ldots \cancel{N}(N+M-1)}{M!}$$

$$= {}^{N+M-1}C_M = \frac{(N+M-1)!}{M!\,(N-1)!}$$

12-Jan '14

#Olly 2.1

$O(3)$ $\quad T_{[i_1 i_2]} \cdots$

$\downarrow$

$\hat{S}_{i_1 \cdots i_n \cdots}$

$$\binom{N+M-1}{M} = \frac{N(N+1)\cdots(N+M-1)}{M!\,(N-1)!} - \frac{N(N+1)(N+M-3)}{(M-2)!}$$

$$-\binom{N+M-3}{M-2}$$

$d[\hat{S}_{i \cdots i_M}] = 2M+1$

$[t_i, t_j] = \pm \epsilon_{ijk} t_k$ where $t = iT$

$t^T = -t$, $t^* = t$

$[T_i, T_j] = i\epsilon_{ijk} T_k$

for $N=3$, $\binom{M+2}{M} - \binom{M}{M-2} = \frac{(M+2)(M+1)}{2} - M(M$

$=$ calc $= 2M+1$

spin(3) $\quad \gamma^i \quad i = 1,2,3$

$S\{\gamma^i, \gamma^j\}S^{-1} = 2\delta^{ij}\, S\mathbb{1}S^{-1}$

$(\gamma^i)' = S\gamma^i S^{-1}$ We use $\gamma^i = \sigma^i$ (could use $S\sigma^i S^{-1}$)

Recall: $\left.\begin{matrix} O(2) \\ O(3)\end{matrix}\right\}$ $2\times 2\ \gamma$

$\left.\begin{matrix} O(2M) \\ O(2M+1)\end{matrix}\right]$ both use $2^M \times 2^M$

$J_{ij} = \frac{-i}{4}[\gamma_i, \gamma_j]$

$\Rightarrow J_{12} = -\frac{i}{4}\underbrace{[\sigma_1 \sigma_2]}_{2i\sigma_3} = \frac{\sigma_3}{2}$ $\quad J_{31} = \frac{\sigma_2}{2}$; $J_{23} = \frac{\sigma_1}{2}$

$T^a \equiv \frac{\sigma^a}{2}$ generates spin(3)

$[T^a, T^b] = i\epsilon^{abc} T^c$

$|||$

so(3) commutation relation

so(3) $\underbrace{\simeq}_{\text{is isomorphic}}$ spin(3) $\rightarrow$ algebra

$1 + i\vec{\theta}\cdot\vec{T} = u + \mathcal{O}(\theta^2)$

since $\vec{T}^+ = \vec{T}$,

$u = e^{i\vec{\theta}\cdot\vec{T}}$; thus $U^+ = U^{-1} = U(-\theta)$

$\chi^{*}{}_\alpha \psi_\alpha = \bar{\chi}^\alpha \psi_\alpha = \bar{\chi}^\beta \delta^\alpha{}_\beta \psi_\alpha$

$T'_{\alpha_1 \cdots \alpha_M}$

$T'^{\alpha_1 \cdots \alpha_L}$

$T'^{\alpha_1\alpha_2 \cdots}{}_{\beta_1\beta_2 \cdots}$

$(\delta^\alpha{}_\beta)' = \bar{U}^\alpha{}_{\alpha'} U_\beta{}^{\beta'} \delta^{\alpha'}{}_{\beta'}$

$= \bar{U}^\alpha{}_{\beta'} U_\beta{}^{\beta'}$

old notation $= (U^*{}_{\alpha\beta'})(U_{\beta\beta'})$

$= (UU^+)_{\alpha\beta} = \mathbb{1}_{\alpha\beta}$

new notation $= \delta^\alpha{}_\beta$

$\psi'_\alpha = U_\alpha{}^\beta \psi_\beta$

$\alpha = 1, 2$

$\bar{\chi}^{\alpha'} = \bar{U}^\alpha{}_\beta \bar{\chi}^\beta$

$\chi'_\alpha = U_\alpha{}^\beta \chi_\beta$

$\chi^+\psi = \chi'^+\psi'$

$\psi' = U\psi$

$\chi' = U\chi \longrightarrow \boxed{\chi'_\alpha = U_\alpha{}^\beta \chi_\beta}$

$\chi'^+ = \chi^+ U^+$

$\chi'^* = U^* \chi^* \quad \rightarrow \chi'^*_\alpha = (U_\alpha{}^\beta)^* \chi^*_\beta$

$\qquad\qquad |||$

$\qquad\qquad \bar{U}^\alpha{}_\beta$

$U^+ = U^{-1}$

$\bar{U}^\alpha{}_\beta = (U_\alpha{}^\beta)^*$

$=0$ $\alpha = \beta$

TAS $\epsilon_{12} = 1$

$\Rightarrow \left.\begin{matrix}\epsilon_{\alpha\beta} \\ \epsilon^{\alpha\beta}\end{matrix}\right\}$ are also invariant tensor

could be $\epsilon^{12} = 1$ or $\epsilon^{12} = -1$

$U = e^{i\vec{\theta}\cdot\vec{T}}$

$= e^{i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}$

$\boxed{\begin{matrix} U^+ = U^{-1} \\ \det U = 1\end{matrix}} \equiv SU(N)$

$\mathrm{tr}\,\vec{\sigma} = 0$

$\det M = \exp \mathrm{tr} \ln M$

$\det U = \exp \mathrm{tr}(i\vec{\theta}\cdot\vec{T}) = e^0 = 1$

$\epsilon_{\alpha\beta} = E_{\alpha\beta}$

$E'_{\alpha\beta} = U_\alpha{}^{\alpha'} U_\beta{}^{\beta'} \epsilon_{\alpha'\beta'} = \epsilon_{\alpha\beta}$

$\qquad |||$

$U_{\alpha\alpha'} U_{\beta\beta'} \epsilon_{\alpha'\beta'} = \det U\, \epsilon_{\alpha\beta}$

$U_\alpha{}^\beta = \left(e^{i\frac{\vec{\theta}\cdot\vec{\sigma}}{2}}\right)_\alpha{}^\beta \equiv (U_{\alpha\beta})$

removed

$U^T = e^{i\vec{\theta}'\cdot\frac{\vec{\sigma}}{2}}$; $\vec{\theta}' = (\theta_1, -\theta_2, \theta_3)$

$= \epsilon\, e^{\frac{i}{2}(\theta_1\sigma_1 - \theta_2\sigma_2 + \theta_3\sigma_3)} = e^{-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}}\epsilon$

$\epsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$

$\Rightarrow \epsilon U^T = U^+\epsilon$ (yes this is correct; $\sigma_1{}^T = \sigma_1$, $\sigma_3{}^T = \sigma_3$)

$U^T\epsilon = \epsilon U^+$ $\quad \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}^T = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$

$\Rightarrow \epsilon U = U^*\epsilon$

$\chi'^T \epsilon \psi' = \chi^T U^T \epsilon U \psi = \chi^T \epsilon \underbrace{U^+ U}_{I} \psi = \chi^T \epsilon \psi$

$\begin{pmatrix}\chi' = U\chi \\ \psi' = U\psi\end{pmatrix}$

$\tilde{\chi} \equiv \epsilon\chi$

$\begin{pmatrix}\tilde{\chi}_1 \\ \tilde{\chi}_2\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix}\chi_1 \\ \chi_2\end{pmatrix} = \begin{pmatrix}\chi_2 \\ -\chi_1\end{pmatrix}$

$\tilde{\chi}' = \epsilon\chi'$

$= \epsilon U \chi$

$= U^*(\epsilon\chi)$

$\begin{pmatrix}\chi_2 \\ -\chi_1\end{pmatrix} = U^*\begin{pmatrix}\chi_2 \\ -\chi_1\end{pmatrix}$

$\boxed{2^* \sim 2}$

$\psi'_\alpha = U_\alpha{}^{\alpha'}\psi_{\alpha'}$ $\qquad S^{\alpha\beta\,'} = U^\alpha{}_{\alpha'}U^\beta{}_{\beta'}S^{\alpha'\beta'}$

$T_{[\alpha_1\alpha_2]\cdots\alpha_M} \Rightarrow \underbrace{S_{\alpha_1\cdots\alpha_M}}$ | dim: Totally anti-symmetric, is the most general $SU(2)$ tensor (Cant take trace) :-

$T_{\alpha\beta} = S_{\alpha\beta} + A_{\alpha\beta}$ $\qquad D(S^{(M)}) = \binom{N+M-1}{M} = \binom{M+1}{M} = \cdots = M+1$

$2j+1 = \boxed{M+1}$

M	$d(S^{(M)})$
1	$2 = 2(\frac{1}{2})+1$
2	$3 = 2(1)+1$
3	$4 = 2(\frac{3}{2})+1$

$O(2) \longrightarrow Spin(2) \longrightarrow O(1,1)$, $Sp(1,1)$ $\to O(3) \to Spin(3) = SU(2)$

$Spin(2) \to U(1)$

Unitary gps $U(N)$

$N\times N \quad U^+ = U^{-1} \quad \hat{U}^+\hat{U} = \hat{U}\hat{U}^+ = \mathbb{1}_N$

$|\det \hat{U}|^2 = 1 \Rightarrow \det \hat{U} = e^{i\theta}$

$\hat{U} = U e^{i\theta/N}$

$\det \hat{U} = (\det U)\, e^{i\theta}$

$U(N) = SU(N)\times U(1)$

$SU(N)$ ($N\times N$, unitary mat, $\det U = 1$)

{traceless Hermitian Matrices}

$SU(2) = U = e^{i\vec{\theta}\cdot\vec{\sigma}/2}$

$U = e^{iH} \quad H^+ = H$

$U^+ = e^{-iH'} = U^{-1}$

$\det U = 1 = \exp \ln e^{iH} = e^0 = 1$

$(\Rightarrow \mathrm{tr}\, H = 0)$

#2.2

Diagonal Generators (Cartan Subalgebra)

Hermitian $U^+ H U = \Lambda = \text{Diag}(\lambda_1 \dots \lambda_n)$

$U = (E_{(1)}, E_{(2)} \dots E_{(n)})$

columns = Eigenvectors of H

$n\times n$ $H = H^+ \Rightarrow$ n eigenvectors $\perp$ wrt Hermitian inner products $E^+_{(i)} E_j = \delta_{ij}$

$(2j+1) = n\times n$ $[\mathcal{T}_a, \mathcal{T}_b] = i\epsilon_{abc}\mathcal{T}_c$

$\overline{SU(2)} \rightarrow T_1\ T_2\ T_3$ | $2\times 2\ [T_a, T_b] = i\epsilon_{abc}T_c$

$\text{Tr}\, T_a = 0$ $\quad \frac{\sigma_3}{2} = \frac{1}{2}\text{Diag}(1,-1)$

Spin; representation in Vector space supporting action of SU(2)

I $\quad |j, j\rangle$ $\quad \mathcal{T}_3 = \text{Diag}(j, j-1, \dots -j)$

$|j, j-1\rangle$ $\quad j = \frac{1}{2}$ $\quad \mathcal{T}_3 = T_3 = \frac{1}{2}\text{Diag}(1,-1)$

$\langle j, m' | \mathcal{T}_\pm | j, m\rangle$

$\vdots$

$|j, -j+1\rangle$ $\quad V_\pm = \frac{1}{\sqrt{2}}(v_1 \pm i v_2)$

$|j, -j\rangle$

$O(2)$ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad V^T V = V'^T V'$ $\quad O^T = O^{-1}$

$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \xrightarrow{U} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}}_{= U} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$V^T V = V_{(+}, V_{-)} = 2V_+V_-$

$V = U^+\hat{V}$ $\quad V^T = \hat{V}^T U^*$

$V^T V = \hat{V}^T \underbrace{(U^* U^+)}_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \hat{V}$

$= (V_+\ V_-)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} V_+ \\ V_- \end{pmatrix}$

Similarly, lets see what $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ transforms to

$(\hat{V}_+, \hat{V}_-)\ \underbrace{U^*\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}U^+}_{= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}\begin{pmatrix} \hat{V}_+ \\ \hat{V}_- \end{pmatrix}$

SU(2)

$S(\alpha_1 \dots \alpha_N)$

			T_3
$S_{\alpha\beta} = S_{\beta\alpha} \rightarrow$	S_{11} —	S_{++}	1
	$S_{12}+S_{21}$ —	S_0	0
	S_{22} —	S_{--}	−1

$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \leftarrow$ vector of SO(3) same thing as SO(2) for SO(3)

Unitary Grps

$U(N)\{\hat{U}^+ = \hat{U}^{-1};\ N\times N$ unitary$\}$

$\hat{U}^+\hat{U} = 1$ $\quad \hat{U} = (\exp \frac{i\theta}{N}\mathbb{1}_N)\, U$

$|\det \hat{U}|^2 = 1$

$\overset{\shortparallel}{e^{i\theta}}$ $\quad \det\hat{U} = \det U e^{i\theta}$

$U(N) = SU(N)\times U(1)$

$SU(N) = \{U^+ = U^{-1};\ \det U = 1\}$

$(e^{iH})^+ = e^{-iH} = (e^{iH})^{-1}$ $\Rightarrow H_{ii} = H_{ii}^*$

$H_{ij} = H_{ji}^*\ i<j$

$H^+ = H$ $\quad$ (no sum, diagonal elements)

N real (diagonal)

$\frac{N(N-1)}{2}\times 2$ (below diagonal) (complex)

$\overset{\shortparallel}{N^2}$

$\rightarrow$ Now also $\det U = 1 \Rightarrow \text{tr}\, H = 0$

$SU(N)\begin{cases} H = \sum_a \theta_a T_a \\ T_a : N^2 - 1 \end{cases}$

So there're N^2-1 independent Hermitian traceless $N\times N$ linearly independent matrices

Gell-Mann Basis for SU(N)

(some T_a not missing)

$N=2$ Pauli $\sigma_1 = \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}$

$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Normalization is $\quad \text{Tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$

$N=2$ $\quad \sigma^{(12)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \sigma^{[12]} = i\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$

$N=3$ $\quad \lambda_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix} = \lambda^{(2)}$ $\quad \lambda_2 = \begin{pmatrix} \sigma_2 & \\ & 0 \end{pmatrix}$

$\lambda^{(13)} = \lambda_4$ $\quad \lambda^{[13]} = \lambda_5$

$\lambda^{(23)} = \lambda_6$ $\quad \lambda^{[23]} = \lambda_7$

$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\frac{1}{\sqrt{3}} \rightarrow$ square trace = same in all

traceless

Q. How many diagonal generators?

A. You find N−1 real independent #s $\Rightarrow$ N−1 diagonal generators. What're left are off diagonal matrices

MISSI

$T^A = \frac{\lambda^A}{2} \qquad A = 1, \ldots 8$

$\mathrm{Tr}\; T^A T^B = \frac{1}{2}\delta^{AB}$

$$\begin{array}{ccc} & T_3 & T_8\, 2\sqrt{3} \\ \psi_1 & 1/2 & 1 \\ \psi_2 & -1/2 & 1 \\ \psi_3 & 0 & -2 \end{array}$$

$T_8 = \frac{\lambda_8}{2} = \frac{1}{2\sqrt{3}} \operatorname{Diag}(1, 1, -2)$

$\psi^+ \psi = \psi'^+ \psi'$ where $\psi' = U\psi$

$\psi'_\alpha = U_\alpha{}^\beta \psi_\beta$

\# 2.3

SU(N) : $N \times N$ Hermitian traceless

$a = 1, \ldots, N^2-1$ dim

$\mathrm{tr}\, T^a T^b = S^{ab}$ ← symmetric real, orthogonal $R^{ab'} T^{a'}$

$$R_{aa'} R_{bb'} \mathrm{Tr}(T^{a'} T^{b'}) = \mathrm{Tr}(T'^a T'^b)$$

$$= (R S R^T)_{ab} = [\mathrm{Diag}(s_1, \ldots, s_N)]_{ab}$$

$$T''^a = \frac{T'^a}{\sqrt{s_a}} \quad \mathrm{tr}(T''^a T''^b) = \frac{1}{2}\delta^{ab} \Rightarrow \mathrm{tr}((T^a)^2) = \frac{1}{2}$$

DOUBT: Why 0?

U, $a = e^{i\theta \cdot T}$, $\psi' = U\psi$, $V^{(1)} \otimes V^{(1)} \ldots$

$$\begin{pmatrix}\psi_1' \\ \vdots \\ \psi_N'\end{pmatrix} = U \begin{pmatrix}\psi_1 \\ \vdots \\ \psi_N\end{pmatrix} \rightarrow \psi_i' = U_{ij}\psi_j \xrightarrow{\text{Notation}} \psi_i' = U_i{}^j \psi_j$$

$$U = \exp i \frac{\lambda^a}{2}\theta^a \;;\; a = 1, \ldots, N^2-1$$

$$\psi_i'^{*} = (U_i{}^j)^* \psi_j^* \quad \bar{U}^i{}_j \equiv (U_i{}^j)^*$$

$$\chi'^i = \bar{U}^i{}_j \chi^j$$

M index tensors

$\psi \in \mathbb{C}^N \quad \psi^\dagger\psi = \psi'^\dagger\psi'$

$$T'_{i_1 \ldots i_M} = U_{i_1}{}^{i_1'} \cdots U_{i_M}{}^{i_M'} T_{i_1' \ldots i_M'}$$

$E_{i_1 \ldots i_N} \equiv \epsilon_{i_1 \ldots i_N}$ $\left(\epsilon_{12\ldots N} = +1 \atop \text{TAS}\right)$

Aim: Find irreducible

SU(3) : $M = 2$ $\quad 9 \rightarrow 3 + 6$

$T_{ij} = A_{ij} + S_{ij}$

$T' = U T U^T$ $\quad [(A - A^T)']^T = [U T U^T - U T^T U^T]^T$

DOUBT SEEMS WRONG $A'^T = -A$; $S'^T = S$ $= -A'$

$$E'_{i_1 \ldots i_N} = U_{i_1}{}^{i_1'} \cdots U_{i_N}{}^{i_N'} \epsilon_{i_1' \ldots i_N'}$$

$$= (\cancel{\det U})\, \epsilon_{i_1 \ldots i_N}$$

$$= E_{i_1 \ldots i_N}$$

(missed fill) $\underbrace{(U_{i_1}{}^{j_1})^* U_{i_1}{}^{i_1'}}_{3\times 3} U_{i_2}{}^{i_2'} U_{i_3}{}^{i_3'} \epsilon_{j_1 i_2' i_3'} = \epsilon_{i i_2 i_3} (U_{i_1}{}^{j_1})^*$

$\vdots$

$U_{i_3}{}^{i_3'} \epsilon_{j_1 i_2 i_3'} = \epsilon_{i_1 i_2 i_3} (U_{i_1}{}^{j_1})^* (U_{i_2}{}^{j_2})^*$

More general tensors

$$T'^{j_1 \ldots j_L}_{i_1 \ldots i_M} = \bar{U}^{j_1}{}_{j_1'} \cdots \bar{U}^{j_L}{}_{j_L'} \cdots U_{i_1}{}^{i_1'} \cdots U_{i_M}{}^{i_M'} T^{j_1' \ldots j_L'}_{i_1' \ldots i_M'}$$

$\epsilon'_{i_1 \ldots i_N} = \epsilon_{i_1 \ldots i_N}$; $\epsilon'^{i_1 \ldots i_N} = \epsilon^{i_1 \ldots i_N}$

$\delta'^i{}_j = \delta^i{}_j$

$\chi^i \psi_i = \chi'^i \psi_i'$

A_{ij}

$\bar{\chi}^i = \frac{1}{2}\epsilon^{ijk} A_{jk}$

Eg: $\bar{\chi}^1 = A_{23}$

$\bar{\chi}^2 = A_{31}$

$\bar{\chi}^3 = A_{12}$

Ex: $\Rightarrow \bar{\chi}'^i = \bar{U}^i{}_j \bar{\chi}^j$

$T_i{}^j = \hat{T}_i{}^j + \frac{1}{N}\delta_i{}^j (T_k{}^k)$

$N^2 \quad N^2-1 \quad 1$

$T_{ij} = A_{ij} + S_{ij}$

$3 \quad 6$

$\bar{3}$ (c.c. rep)

SU(N) algebra

$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$

$T^{a\dagger} = T^a$

$\mathrm{tr}(T^a) = 0$

$\mathrm{tr}([T_a, T_b]) = 0$; $i[T_a, T_b] = -f_{abc} T_c$

$[T_a, T_b] = i f_{abc} T_c$ real

$(i[T_a, T_b])^\dagger = -i(T_b^\dagger T_a^\dagger - T_a^\dagger T_b^\dagger)$

$= +i[T_a, T_b]$

$\cancel{i}[T_a, T_b] \neq i f_{abc} T_c$

NB: (a) $T'^a = U T^a U^\dagger$

$[T'^a, T'^b] = i f_{abc} T'^c$

(b) $T'^a = R^{ab} T^b$

s.t. $R^T = R^{-1}$; then $\mathrm{tr}(T'^a T'^b) = \mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$

$\because$ Hermiticity $[T_a', T_b'] = R_{aa'} R_{bb'} [T_{a'}, T_{b'}]$

$= R_{aa'} R_{bb'} i f_{a'b'c'} T_{c'}$

$T_{c'} = R^{-1}_{c'c} T'_c$

$= i R_{aa'} R_{bb'} f_{a'b'c'} R_{c'c'} T'_c$

$\Rightarrow f_{abc} = i R_{aa'} R_{bb'} R_{cc'} f_{a'b'c'}$

$$\mathrm{tr}([T_a, T_b] T_d) = i f_{abc} \underbrace{\mathrm{tr}(T_c T_d)}_{\frac{1}{2}\delta_{cd}} \Rightarrow \boxed{f_{abc} = -2i \,\mathrm{Tr}[T_a, T_b] T_c}$$

Ex. (later)

① calculate f_{abc} for $N = 3$ $\quad T^a = \frac{\lambda^a}{2}$

② Show f_{abc} is anti-symmetric on transposition of any two indices.

N^2-1

$d \times d$ matrices $\{\mathcal{T}^a\}$ are said to furnish a d dimension · d dimensional rep of SU(N)

if $[\mathcal{T}^a, \mathcal{T}^b] = i f^{abc} \mathcal{T}^c$

Show by BCH $\left(U = e^{i\theta^a \mathcal{T}^a}\right.$ ↓ lie in fundamental domain $\left.\right)$ has the same mult rule as $U = e^{i\theta_a T_a}$

$(\mathcal{T}^a)_{bc} = i f_{bac}$

$e^{i\theta_a \mathcal{T}_a} = e^{-\theta_a \mathcal{F}_a}$

$F_a^T = -F_a$; $(F_a)_{bc} = f_{bac}$; $R^T = R^{-1}$

$R^* = R$

$R_{ab} = e^{-\theta^a F^a}$

Representations of SU(N)
Tensors of fundamental N : $\psi_i = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}$ $\psi_i' = U_i^{\ j}\psi_j$

$T'_{i_1 \dots i_M} = U_{i_1}^{\ i_1'} \cdots U_{i_M}^{\ i_M'} T_{i_1' \dots i_M'}$

$\epsilon_{i_1 \dots i_N}$ is invariant

$T_{ij} = S_{ij} + A_{ij}$

SU(3)	9	6	3 ——	3^*
SU(N)	N^2	$\frac{N(N+1)}{2}$	$\frac{N(N-1)}{2}$	

$\psi_i^{*\prime} = (U_i^{\ j})^* \psi_j^*$

$U = e^{i\theta^a T_a}$

$U^* = e^{i\theta^a(-T^{a*})}$, $-T^{a*} = -T^{aT}$

$T^a = T^{a\dagger}$, $T^{a*} = T^{aT}$

su(N)
$[T^a, T^b] = i f^{abc} T^c$

N^2-1 mat $[(-T^a)^*, (-T^a)^*] = (-T^{c*})\, i f^{abc}$

$(X^a)_{bc} = i f_{bac} = i(F^a)_{bc} \rightarrow N^2-1$ dimensional real irreducible rep of SU(N) adjoint representation. ASK $U = e^{iX^a\theta^a} = e^{-F^a\theta^a} = R_A(\theta)$

$\Rightarrow (F^a)^T = -F^a$

Ex: Show $(X^a)_{bc} = i f_{abc}$ define an N^2-1 dimensional irreducible of SU(N)

Jacobi Identity: $[T^a,[T^b,T^c]] + \text{cyclic} = 0 \Rightarrow i f^{bcd}\, i f^{adc} T^e + \text{cyclic} = 0$

$\square$ ψ_i N, $\bar{N}$

$T^{[i_1 \dots i_N]} \sim 1$ (This is why you have trace as invariant)

$3^* \sim A_{ij}$ SU(3)

$A'_{i_1 \dots i_{N-1}} = U_{i_1}^{\ i_1'} \cdots U_{i_{N-1}}^{\ i_{N-1}'} A_{i_1' \dots i_{N-1}'}$

$\bar{A}^i \equiv \frac{1}{(N-1)!} \epsilon^{i i_1 \dots i_{N-1}} A_{i_1 \dots i_{N-1}}$
$= \frac{1}{(N-1)!} \epsilon^{i i_1 \dots i_{N-1}} A_{i_1 \dots i_{N-1}}$

$\epsilon' = \epsilon \Rightarrow \boxed{\bar{A}^{i\prime} = U^i_{\ j} \bar{A}^j}$ almost done VERIFY

$\square$ ψ_i N

N-1 $A_{[i_1 \dots i_{N-1}]}$ $\bar{N}$ $\sim \bar{A}^i$

$T_{[i_1 \dots i_{N-1}]j} \sim T^i_{\ j} = \hat{T}^i_{\ j} + \frac{\delta^i_j}{N}(T^a_a)$

$S(i_1 i_2 i_3 \dots i_M) = d(S_{i_1 \dots i_N}) = \frac{N(N+1)\cdots(N+M-1)}{M!}$ $T_{[i_1 \dots (i_{N-1}]j)}$

SU(N) adjoint N-1

Ex: Show this is equivalent to extracting the totally anti-symmetric piece

$U T^a U^\dagger$
$U = e^{i\theta\cdot T}$
$= e^{i\theta\cdot T} T^a e^{-i\theta\cdot T}$
$= T^a + i[\theta\cdot T, T^a] + \frac{i^2}{2!}[\theta\cdot T,[\theta\cdot T, T^a]] + \dots$
$i[\theta\cdot T, T^a] = i\theta^b[T^b, T^a] = i\, i f_{bac} T^c$

$\hat{T} = T^a + i[\theta\cdot T, T^a] + \frac{i^2}{2!}[\theta\cdot T,[\theta\cdot T, T^a]] \dots$

$T_{(R)bc} = (e^{i\theta\cdot X})_{bc} = \delta_{bc} + i\theta^a (X^a)_{bc} + \dots$, $(X^a)_{bc} = i f_{bac}$
$= \delta_{bc} - \theta^a f_{bac} + O(\theta^2)$

$e^{i\theta\cdot T} T^a e^{-i\theta\cdot T} = R^{-1}(\theta)_{ab} T^b$
adj representation of SU(N)

$A'^a = R^{ab} A^b$, $R = e^{i\theta\cdot X}$

$\psi_i' = U_i^{\ j} \psi_j$

$A = A^a (T^a \sqrt{2})$ $\operatorname{Tr} A^2 = (\sum_a A_a^2)$
$(A^a)^2$

$\operatorname{tr}(A^2) = 2(\operatorname{Tr} T^a T^b) A^a A^b$
$= \sum (A^a)^2$

$T^a A'^a = R^{ab} A^b T^a$
$= A^b (R^{-1})_{ba} T^a$
$= A^b (U T^b U^\dagger)$

adjoints $\boxed{A' = U A U^\dagger}$

Ex: SU(2) $\frac{V^a \sigma^a}{\sqrt{2}} = V$! $V' = U V U^\dagger$ shows $S_{\alpha\beta}$ & find the connection