Measuring Geometric Phase using 3 Pin Holes :.

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We describe a method a of measuring the geometric phase optically, using 3 pin holes. This method is in stark contrast with conventional methods where states are evolved and the relative phase measured after completion of a cycle.

1 Introduction

1.1 Geometric Phase

Concept of geometric phase was discovered as a relative phase that is introduced in an adiabatic and cyclic evolution of a state, between the final and initial state. This relative phase has both the geometric phase and what is known as the dynamic phase, which is the time integral of the Hamiltonian. The concept since then, has been generalized to work in cases of non-adiabatic evolution and then even to non-cyclic evolution.

1.2 Conventional Measurement Scheme

Experimentally, conventional measurement of the geometric phase is associated with directly measuring the total phase difference between the evolved and unevolved reference states. This requires phase calibration, which is an experimental complication. Further, one must eliminate the dynamic phase to obtain the geometric phase.

1.3 Kinematic Geometric Phase and the Proposed Scheme

Direct observation of geometric phases using a three-pinhole interferometer **PHYSICAL REVIEW A 81, 012104 (2010)**, *Kobayashi et. al.* propose a method of measurement of geometric phase, which is built on the kinematic approach to geometric phase. In this approach, the geometric phase is attributed to the structure of the Hilbert space itself and doesn't require dynamics (evolution of states). One can show that Geometric Phase is a ray space object and more explicitly, it has been shown that for n points in the ray space, there exists an entity known as the Bargmann invariant, from which the geometric phase can be recovered. This geometric phase is that for a closed curve obtained by connecting the n points with geodesics.

The proposed scheme involves use of 3 quantum states and letting them interfere directly, viz. without evolving the system. From the intereferogram thus obtained, using the 3 vertex Bargmann invariant and some more details specific to the setup, it has been shown that the geometric phase can be recovered. Further, since the initial 3 states are known, one can theoretically evaluate the Geometric phase. Theory and experimental observations have been shown to agree. Since this method doesn't require evolution, the need for phase calibration and removal of dynamic phase has been eliminated.

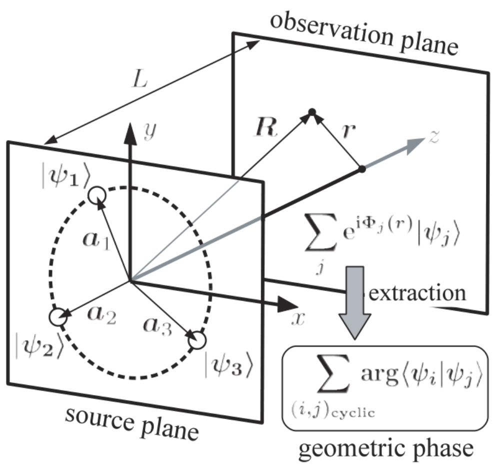

Figure 1: Schematic of the setup

2 Geometric Phase and Bargmann Invariant

As Figure 1 shows, three states $|\psi_j\rangle$ $(j = 1, 2, 3)$ represent the internal states of the photon from three pinholes. The resultant interferogram should and does contain three distinct interference fringes due to each pinhole pair. To start with, superimpostion of $|\psi_1\rangle$ and a $U(1) = e^{i\phi}$ shifted $|\psi_2\rangle$ leads to an intensity of

$$I \propto 1 + |\langle\psi_1|\psi_2\rangle| \cos\left(\phi + \arg\langle\psi_1|\psi_2\rangle\right)$$

which can be derived easily as

$$\begin{aligned}
|\psi\rangle &= e^{i\phi}|\psi_1\rangle + |\psi_2\rangle \\
\langle\psi|\psi\rangle &= \left(\langle\psi_1| e^{-i\phi} + \langle\psi_2|\right)\left(e^{i\phi}|\psi_1\rangle + |\psi_2\rangle\right) \\
&= \langle\psi_1|\psi_1\rangle + \langle\psi_2|\psi_2\rangle + \left(e^{-i\phi}e^{i\arg\langle\psi_1|\psi_2\rangle}|\langle\psi_1|\psi_2\rangle| + \text{hc}\right) \\
&\propto 1 + |\langle\psi_1|\psi_2\rangle| \cos\left(\phi + \arg\langle\psi_1|\psi_2\rangle\right)
\end{aligned}$$

Thus the interference fringes are shifted by the relative phase between the states. In case of constructive interference, this phase is zero and the states are called *in phase*. Note however that this property of the states is *non transitive*. To make this explicit, consider the following expression, which is manifestly invariant under a local change of phases (gauge invariance).

$$\begin{aligned}
\Delta_3(\psi_1, \psi_2, \psi_3) &\equiv \langle\psi_1|\psi_2\rangle\langle\psi_2|\psi_3\rangle\langle\psi_3|\psi_1\rangle \\
&= \text{tr}\left(|\psi_2\rangle\langle\psi_2|\psi_3\rangle\langle\psi_3|\psi_1\rangle\langle\psi_1|\right) \\
\implies \arg\left(\Delta_3(\psi_1, \psi_2, \psi_3)\right) &= \arg\left(\langle\psi_1|\psi_2\rangle\langle\psi_2|\psi_3\rangle\langle\psi_3|\psi_1\rangle\right) \\
&= \sum_{(i,j)\text{cycle}} \arg\langle\psi_j|\psi_j\rangle
\end{aligned}$$

Here, if $\arg\langle\psi_1|\psi_2\rangle = 0$ and $\arg\langle\psi_2|\psi_3\rangle = 0$, that is if the states 1 and 2 are in phase, 2 and 3 are in phase, then $\arg\left(\Delta_3(\psi_1, \psi_2, \psi_3)\right) = \arg\langle\psi_1|\psi_3\rangle \neq 0$ in general.

Δ_3 is defined to be a 3 point **Bargmann Invariant** and $\arg\Delta_3$, the **Pancharatnam Phase**. We recall/claim that the Pancharatnam phase can be related to the geometric phase corresponding to that of a triangle in the B space (or if you like, that of a system whose trajectory is given by the triangle).[1] This triangle has as vertices, the points corresponding to the 3 states. Its sides are formed by joining these points by geodesics.[2] The relation is

$$\phi_{\text{goem}} = -\arg\Delta_3\left(\psi_1, \psi_2, \psi_3\right)$$

It can be further shown, that for a two state system (such as polarization of photon, the case here), $\arg\Delta_3$ is proportional to the solid angle (Ω) of the spherical triangle on the Bloch sphere with the three states as the vertices.

$$\phi_{\text{goem}} = -\frac{\Omega}{2}$$

3 Geometric Phase and Ridge Lines

Consider again, the three pinholes irradiated with monochromatic light of wavenumber k. Without any loss of generality, the location of the three pinholes can be parametrized by $\mathbf{a}_j (j = 1, 2, 3)$ such that $|\mathbf{a}_j| = a\,\forall\, j$, by simply choosing the origin to be the circumcenter of the triangle. The source plane is labeled to be at $z = 0$. It is known that the state of a photon can be expressed as a combination of a spatial part represented by the spherical wave and the internal, polarization state part. The state of photon from the j^{th} pin-hole[3], on the observation plane $z = L$ is

$$|\psi(\mathbf{r})\rangle = C\sum_{j=1}^{3}\frac{\exp\left[i\left(k\left|\mathbf{R}-\mathbf{a}_j\right| + \phi_j\right)\right]}{\left|\mathbf{R}-\mathbf{a}_j\right|}\left|\psi_j\right\rangle$$

where $\mathbf{R}$ is the position vector on the observation plane $(z = L)$; $\mathbf{r} \equiv \mathbf{R} - (\mathbf{R}.\mathbf{z})\,\mathbf{z}$ (component of $\mathbf{R}$ perpendicular to $\mathbf{z}$, the unit vector along z-axis); C is dimensionless normalization; ϕ_j is the phase of $|\psi_j\rangle$ (the polarization state of the j^{th} source). If we make the para-axial approximation, we get

$$\begin{aligned}
\mathbf{R}-\mathbf{a}_j &= (L\mathbf{z}+\mathbf{r}) - \mathbf{a}_j \\
\left|\mathbf{R}-\mathbf{a}_j\right| &= \left[\left((L\mathbf{z}+\mathbf{r})-\mathbf{a}_j\right)\left((L\mathbf{z}+\mathbf{r})-\mathbf{a}_j\right)\right]^{1/2} \\
&= \left[(L\mathbf{z}+\mathbf{r})(L\mathbf{z}+\mathbf{r}) - 2(L\mathbf{z}+\mathbf{r})\,\mathbf{a}_j + \mathbf{a}_j^2\right]^{1/2} \\
&= \left[L^2 + 2\cancelto{0}{L\mathbf{z}.\mathbf{r}} + \mathbf{r}^2 - 2(\cancelto{0}{L\mathbf{z}.\mathbf{r}} + \mathbf{r}.\mathbf{a}_j) + \mathbf{a}_j^2\right]^{1/2} \\
&= \left[L^2 + 2\frac{r^2+a^2}{2} - 2(\mathbf{r}.\mathbf{a}_j)\right]^{1/2} \\
&= L\left[1 + 2\left(\frac{r^2+a^2}{2L^2} - \frac{(\mathbf{r}.\mathbf{a}_j)}{L^2}\right)\right]^{1/2} \\
&\approx L\left[1 + \frac{r^2+a^2}{2L^2} - \frac{(\mathbf{r}.\mathbf{a}_j)}{L^2}\right] \\
&= L + \frac{r^2+a^2}{2L} - \frac{(\mathbf{r}.\mathbf{a}_j)}{L}
\end{aligned}$$

[1]B 'space' is a *subset* of H (Hilbert Space) with $\langle\psi|\psi\rangle = 1$. Ray space is the subspace formed by $\rho_\psi = |\psi\rangle\langle\psi|$ (idea is to remove the phase freedom)

[2]Assuming an appropriately defined metric, this we had proved in the course. It hinges on the fact that for a closed loop in B space, the total phase is zero. In general $\phi_{\text{total}} = \phi_{\text{dyn}} + \phi_{\text{geom}}$ where for a geodesic between 2 points, $\phi_{\text{dyn}} = \arg\langle\psi_1|\psi_2\rangle$. With these assumptions and the fact that dynamic phase is additive, the result stated should follow.

[3]If we assume that all the pinholes have the same probability of transmission

$$|\psi(\mathbf{r})\rangle \approx C\sum_{j=1}^{3}\frac{\exp\left[i\left(k\left(L+\frac{r^2+a^2}{2L}-\frac{(\mathbf{r}.\mathbf{a}_j)}{L}\right)+\phi_j\right)\right]}{L}|\psi_j\rangle$$

where we've used the fact that $\mathbf{a}_j^2 = a^2$. In the last step, in writing the denominator approximately as L, we have also used a stronger condition, that is $|\mathbf{R}-\mathbf{a}_j| \ll \left(L^3/k\right)^{1/4} \ll L$.

The intensity distribution can then be written as

$$\begin{aligned}
p(x,y) &= \langle\psi(\mathbf{r})|\psi(\mathbf{r})\rangle \\
&= \frac{C^2}{L^2}\sum_{j,k=1}^{3}\exp\left[-i\left(k\left(\cancel{L+\frac{r^2+a^2}{2L}}-\frac{(\mathbf{r}.\mathbf{a}_j)}{L}\right)+\phi_j\right)\right]\exp\left[i\left(k\left(\cancel{L+\frac{r^2+a^2}{2L}}-\frac{(\mathbf{r}.\mathbf{a}_k)}{L}\right)+\phi_k\right)\right]\langle\psi_j|\psi_k\rangle \\
&= \frac{C^2}{L^2}\left\{3+\sum_{j\neq k=1}^{3}\exp\left[i\left(k\left(\frac{(\mathbf{r}.\mathbf{a}_j)}{L}-\frac{(\mathbf{r}.\mathbf{a}_k)}{L}\right)+\phi_k-\phi_j+\arg\langle\psi_j|\psi_k\rangle\right)\right]|\langle\psi_j|\psi_k\rangle|\right\} \\
&= \frac{C^2}{L^2}\left\{3+\sum_{(j,k)\mathrm{cycle}}\cos\left[k\left(\frac{(\mathbf{r}.\mathbf{a}_j)}{L}-\frac{(\mathbf{r}.\mathbf{a}_k)}{L}\right)+\phi_k-\phi_j+\arg\langle\psi_j|\psi_k\rangle\right]|\langle\psi_j|\psi_k\rangle|\right\} \\
&= \frac{C^2}{L^2}\left\{-3+\sum_{(i,j)\mathrm{cycle}}P_{ij}(x,y)\right\}
\end{aligned}$$

which finally yields

$$p(x,y) = \frac{C^2}{L^2}\left\{-3+\sum_{(i,j)\mathrm{cycle}}P_{ij}(x,y)\right\}$$

where

$$P_{ij}(x,y) \equiv 2\left(1+\cos\left[(\mathbf{k}_{ij}.\mathbf{r})-\phi_{ij}+\arg\langle\psi_i|\psi_j\rangle\right]|\langle\psi_i|\psi_j\rangle|\right)$$

with $k_{ij} \equiv k(\mathbf{a}_i-\mathbf{a}_j)/L$ and $\phi_{ij} \equiv \phi_i - \phi_j$. This equation infact corresponds to that of the double slit interference fringes between two states $|\psi_i\rangle$ and $|\psi_j\rangle$. Thus, the three pin hole interferogram consists of sets of interference fringes with distinct directions, given by $\mathbf{k}_{ij}$. Evidently, the geometric phase information is contained in P_{ij}. To extract this, we look at the condition for maximum intensity

$$(\mathbf{k}_{ij}.\mathbf{r})-\phi_{ij}+\arg\langle\psi_i|\psi_j\rangle = 2n_{ij}\pi$$

which is of the from $\mathbf{r}.\mathbf{k} = c$ (in 3d, this is a plane, in 2d, its a line obviously)

$$\begin{aligned}
(x\mathbf{x}+y\mathbf{y}).(k_x\mathbf{x}+k_y\mathbf{y}+k_z\mathbf{z}) &= c \\
&= xk_x + yk_y
\end{aligned}$$

Here n_{ij} are integers. These equations define sets of three distinct parallel lines which are called the **ridge lines** on the observation plane. The area S of the triangle formed by these, called the **ridge triangle**. Using the formula for finding the area of a triangle, given equations of the lines making the edges as

$$\begin{aligned}
a_1x+b_1y+c_1 &= 0 \\
a_2x+b_2y+c_2 &= 0 \\
a_3x+b_3y+c_3 &= 0
\end{aligned}$$

is

$$\frac{\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}^2}{2C_1C_2C_3}$$

where C_i = cofactor of c_i, we finally obtain

$$S = \frac{L^2}{4k^2 S_0} \{\arg [\Delta_3 (\psi_1, \psi_2, \psi_3)] - 2n\pi\}^2$$

where $n = n_{12} + n_{23} + n_{31}$ and S_o is the area of the triangle of the three pinholes.

Some points are wroth noting here.

- The area of ridge triangle is related to $\arg \Delta_3$. We define an **elemental ridge triangle** to be a ridge triangle that encloses no ridge lines. For example, for $0 \leq \arg \Delta_3 < 2\pi$ the ridge triangles with $n = 0$ and 1 are elemental. Clearly, the S with $n = 0$ is proportional to square of geometric phase $\arg \Delta_3$; for $n = 1$ it is proportional to $2\pi - \arg \Delta_3$.

- Secondly, the gauge invariance of Δ_3 carries over to the area since by introducing a phase shift to one of pinholes, two sets of ridge lines are displaced but the area is conserved.

- Third, the equation is proportional to the geometric phases regardless of the vectors $\mathbf{a}_i$, thus essentially any geometry of the three pinholes can form the ridge triangle.

4 Extraction of Ridge Lines

In order to determine the ridge lines, rather than the straightforward way of observing individual interference fringes $P_{ij}(x, y)$ by closing one of the holes, we extract all the ridge lines in a single shot from the combined interferogram $p(x, y)$. To start with, note that for arbitrary $\mathbf{b}$, $\mathbf{k}$ we have

$$\begin{aligned} \mathbf{b}.\nabla (\mathbf{k}.\mathbf{r}) &= b_i \frac{\partial}{\partial x_i} k_j r_j \\ &= b_i k_j \left(\frac{\partial}{\partial x_i} r_j \right) \\ &= b_i k_j \delta_{ij} \\ &= \mathbf{b}.\mathbf{k} \end{aligned}$$

Now we define

$$\mathbf{b}_i \equiv \mathbf{e}_z \times (\mathbf{a}_j - \mathbf{a}_k) = \frac{L}{k} \mathbf{e}_z \times \mathbf{k}_{jk}$$

where $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$ NB: $\mathbf{b}_i \mathbf{k}_{jk} = 0$. Further note that the vector b_i in $z = 0$ is determined only by the geometry of the three pinholes. If we consider the directional derivative along b_i, since b_i is orthogonal to k_{jk} we can[4] now eliminate one of the interference fringes $P_{ij}(x, y)$ from the total interferogram $p(x, y)$ while the other fringes remain sinusoidal functions. Another such directional derivative isolates the desired oscillation term.

Let's explicitly derive this. Recall: $P_{ij}(x, y) \equiv 2 (1 + \cos [(\mathbf{k}_{ij}.\mathbf{r}) - \phi_{ij} + \arg \langle\psi_i|\psi_j\rangle] |\langle\psi_i|\psi_j\rangle|)$. Let's start with performing the first directional derivative.

[4]The derivation is given below. This is just to motivate the approach.

$$
\begin{aligned}
(\mathbf{b}_1.\nabla)\, p(x,y) &= (\mathbf{b}_1.\nabla)\, \frac{C^2}{L^2} \left\{ -3 + \sum_{(i,j)\text{cycle}} P_{ij}(x,y) \right\} \\
&\propto \sum_{(i,k)\text{cycle}} (\mathbf{b}_1.\nabla)\, P_{ik}(x,y) \\
&\propto \sin\left[(\mathbf{k}_{12}.\mathbf{r}) - \phi_{12} + \arg\langle\psi_1|\psi_2\rangle\right] |\langle\psi_1|\psi_2\rangle|\, \cancelto{\mathbf{b}_1.\mathbf{k}_{12}}{(\mathbf{b}_1.\nabla)\,(\mathbf{k}_{12}.\mathbf{r})} \\
&+ \sin\left[(\mathbf{k}_{23}.\mathbf{r}) - \phi_{23} + \arg\langle\psi_2|\psi_3\rangle\right] |\langle\psi_2|\psi_3\rangle|\, \cancelto{0}{\mathbf{b}_1.\mathbf{k}_{23}} \\
&+ \sin\left[(\mathbf{k}_{31}.\mathbf{r}) - \phi_{31} + \arg\langle\psi_3|\psi_1\rangle\right] |\langle\psi_3|\psi_1\rangle|\, \mathbf{b}_1.\mathbf{k}_{31}
\end{aligned}
$$

Now when it is applied again along $\mathbf{b}_2$, we get

$$
\begin{aligned}
(\mathbf{b}_2.\nabla)\,(\mathbf{b}_1.\nabla)\, p(x,y) &\propto -\cos\left[(\mathbf{k}_{12}.\mathbf{r}) - \phi_{12} + \arg\langle\psi_1|\psi_2\rangle\right] |\langle\psi_1|\psi_2\rangle|\, \mathbf{b}_1.\mathbf{k}_{12}\mathbf{b}_2.\mathbf{k}_{12} \\
&\quad - \cos\left[(\mathbf{k}_{31}.\mathbf{r}) - \phi_{31} + \arg\langle\psi_3|\psi_1\rangle\right] |\langle\psi_3|\psi_1\rangle|\, \mathbf{b}_1.\mathbf{k}_{31}\cancelto{0}{\mathbf{b}_2.\mathbf{k}_{31}} \\
&\quad \propto \cos\left[(\mathbf{k}_{12}.\mathbf{r}) - \phi_{12} + \arg\langle\psi_1|\psi_2\rangle\right] |\langle\psi_1|\psi_2\rangle|
\end{aligned}
$$

which is precisely what we'd claimed to start with. NB: $(\mathbf{b}_2.\nabla)\,(\mathbf{b}_1.\nabla)\, p(x,y) = (\mathbf{b}_1.\nabla)\,(\mathbf{b}_2.\nabla)\, p(x,y)$

So in general then, I can write

$$
(\mathbf{b}_i.\nabla)\,(\mathbf{b}_j.\nabla)\, p(x,y) \propto \cos\left[(\mathbf{k}_{ij}.\mathbf{r}) - \phi_{ij} + \arg\langle\psi_i|\psi_j\rangle\right] |\langle\psi_i|\psi_j\rangle|
$$

Interferograms for the three pinholes are shown in Figure 3(a) and the three sets of ridge lines thus extracted are shown in Figure 3(b). Thus it is easy to determine the pure geometric phase instantaneously as the square root of the area of the ridge triangle extracted directly from the three pinhole interferogram for three arbitrary states.

5 Experiment

For the experiment (Figure 2), a $532nm$ green laser source was used for illuminating a thin copper foil that is perforated with three $0.1mm$ radius pinholes. These pinholes lie on an equilateral triangle of side $1.5mm$. The observation plane was placed at a distance of approximately $2m$ from the pinholes, where the interfering patterns were captured using a charged coupled device (CCD) camera. This camera had a resolution of 640×480 pixels with each pixel being $9\mu m \times 8\mu m$.

The polarization states from the left, right and upper pinholes are $|\psi_1\rangle = (\sqrt{3}\,|H\rangle + i\,|V\rangle)/2$ and $|\psi_2\rangle = (i\sqrt{3}\,|H\rangle + |V\rangle)/2$ and $|\psi_3\rangle = \cos|H\rangle + \sin|V\rangle$ where $|H\rangle$ and $|V\rangle$ are the horizontal and vertical polarization states respectively. On the Poincare sphere $|\psi_1\rangle$ and $|\psi_2\rangle$ are both located at a latitude of $\pm 60^o$ on the prime meridian and $|\psi_3\rangle$ is located on the equator at a longitude of 2θ which is can be varied according to linear polarizer LP1. The geometric phase is proportional to the solid angle Ω of the spherical triangle formed by $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$ on the Poincare sphere as $\arg\Delta_3 = -\Omega/2$.

$$
\arg\Delta_3(\psi_1\psi_2\psi_3) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\theta\right)
$$

The geometric phase moves between 0 and 2π with respect to θ. Figure 3(a) shows experimentally obtained interferograms for various values of θ and Figure 3(b) shows the extracted ridge lines where the shaded triangles depict the elemental ridge triangles ($n = 0$). The area of ridge triangle varies with the spherical triangle on the Poincare sphere (Figure 3(c)). The relationship between the elemental ridge triangle and geometric phase is analyzed in Figure 3(d) where area of the triangle normalized by

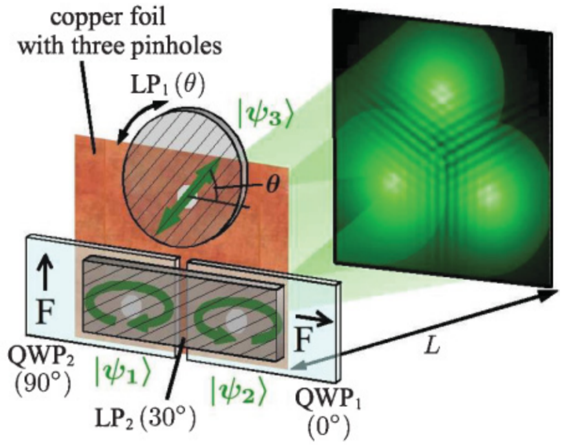

Figure 2: Experimental Setup Schematic

the maximum area is plotted as a function of the geometric phase. The solid line is the theoretical curve while the dots represent the experimental results and they seem to agree well. The other elemental ridge triangle ($n = 1$) related to the complementary area on the Poincare sphere $4\pi - \Omega$ is visible in Figure 3(b).

A local phase shift was introduced by inserting a thin ($0.15mm$ thick) glass plate in front of each hole, which gave variations in ridge triangles, as shown in Figure 4. Figure 4(a) shows the ridge lines without a phase shift for reference. When a phase shift was introduced at pinhole 1, as shown in Figure 4(b), two interference fringes $P_{12}(x, y)$ and $P_{31}(x, y)$ suffer the same phase shift and are simultaneously displaced towards pinhole 1. Thus, the ridge triangle is only parallely displaced along the ridge line of fringe $P_{23}(x, y)$ but is not deformed. Again, a phase shift applied to pinhole 2 and 3 has no influence on the size of the ridge triangle as shown in Figure 4(c) and (d) respectively.

6 References

http://math.stackexchange.com/questions/901819/direct-formula-for-area-of-a-triangle-formed-by-three

http://en.wikipedia.org/wiki/Bloch_sphere

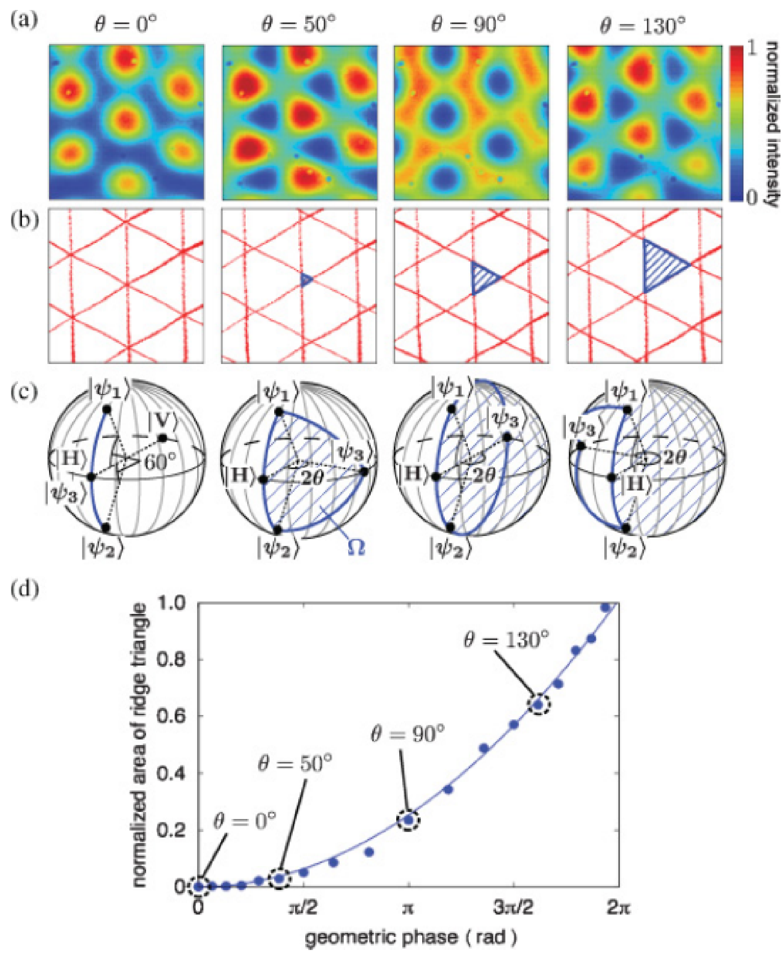

Figure 3: Interferogram, data extraction, physical states, prediction and data