PHY659: GROUP THEORY ASSIGNMENT PROBLEMS

1. Calculate the Poisson bracket for $L_k = \epsilon_{ijk} x_i p_j$ where $(x_i, p_i), i = 1, 2, 3$ are canonical coordinates and momenta. What are the corresponding quantum relations? When will the Poisson brackets of the L_k with a function of coordinates and momenta vanish ?

2. A system has Lagrangian $L = \dot{z}^* \dot{z} - V(z^* z)$ where z is a complex coordinate. L is evidently invariant under re-phasing of z. Find the corresponding constant of motion.

3. Find the eigenvalues and normalized eigenvectors of

   a)Pauli matrices $\sigma_a \quad , a = 1, 2, 3$

   b) $\sum_a \sigma_a$

   c)$U = Exp(i\theta^a \sigma_a) \quad (\theta_a \quad real)$

   d)$\begin{pmatrix} 1-i & 3-5i \\ 3-5i & 2 \end{pmatrix}$

4. 4) Find the Diagonalizing unitary transformation(s) and spectral decomposition of

   a)$\begin{pmatrix} 2 & 4-7i \\ 4+7i & -3 \end{pmatrix}$

   b) $\begin{pmatrix} a & c \\ c^* & b \end{pmatrix}$, a,b real, c complex.

   c)$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a,b, c,d complex.

   Show that in case (c) the phases of the eigenvectors can be chosen so that the diagonal form is semi-positive definite. When are the eigenvalues degenerate in cases b) and c) ?

5. For N fermionic fields $\psi_i, i = 1..N$ with Lagrangian density

$$\mathcal{L} = \overline{\psi} i \gamma^\mu \partial_\mu \psi_i - \mathcal{L}_{int}((\bar{\psi}_i \psi_i) \tag{1}$$

   show that currents

$$j^\mu[H] = \overline{\psi}_i \gamma^\mu H_{ij} \psi_j \tag{2}$$

   (where H is an arbitrary N x N Hermitian matrix) are conserved. Find the commutation algebra $[Q[H], Q[H']]$ obeyed by the second quantized charges obtained from these currents.

6. For N real scalar fields $\phi_i, i = 1..N$ with Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i - V(\phi_i \phi_i) \tag{3}$$

show that the currents

$$j^\mu_{ij} = \partial^\mu \phi_{[i}\phi_{j]} \tag{4}$$

are conserved and the corresponding second quantized charges Q_{ij} obey the algebra of the O(N) group.

7. 7) The abelian groups Z_n are defined as the sets of n-th roots of unity closed under multiplication. Construct the multiplication table and regular representation of Z_n for $n = 2, 3, 4$.

8. 8) The non-abelian groups S_n are defined as the groups of permutation on n objects. Construct the multiplication table and regular representation of S_n for $n = 2, 3$.

9. a) If $[A, B] = A$ calculate $exp(\alpha A)\ B\ exp(-\alpha A)$. Repeat if $[A, B] = B$

   b) Calculate the terms up to second order in θ_a and ϕ_a in the product $exp(i\theta^a T_a)\ exp(i\phi^b T_b)$ in terms of the structure constants of the algebra obeyed by the Hermitian generators T_a.

10. Calculate the expectation value of J_x in the states

    a) $exp(-i\theta J_y/\hbar)|j, m>$

    b)$exp(-i(\theta J_x + \theta' J_y)/\hbar)|j, m>$

    where $J_{x,y}$ obey the angular momentum algebra and $|j, m>$ are the states of the spin j representation.

11. An algebra $\mathcal{A}$ is a vector space equipped with an additional 'vector' product($\diamond$) that maps two vectors $V, W \in \mathcal{A}$ in to a vector $X \in \mathcal{A} : V \diamond W = X$. Choosing a basis $e_k, k = 1...dim(\mathcal{A})$ in the vector space and using completeness of the basis implies that

    $$e_i \diamond e_j = C^k_{ij} e_k \tag{5}$$

    where the C^k_{ij} are called the structure constants of the algebra.

    Find the structure constants of the algebras of $N \times N$ matrices defined by

    a) $[e_{ij}]_{kl} = \delta_{ik}\delta_{jl} \qquad i, j, k, l = 1....N$ and matrix multiplication as the vector product.

    b)$[e^S_{ij}] = e_{ij} + e_{ji}$ as basis elements and anti commutation as the vector product.

    c) $[e^A_{ij}] = e_{ij} - e_{ji}$ as basis elements and commutation as the vector product. Show that this vector product is a derivation i.e

    $$A \diamond (B \diamond C) = (A \diamond B) \diamond C) + B \diamond (A \diamond C) \tag{6}$$

    Such an algebra is called a **Lie** algebra.

12. Consider the algebra generated by $S_1 = (T_1 + T_2)/(\sqrt{3}), S_2 = (T_1 - T_2)/(\sqrt{5}), S_3 = (T_3 + T_2)/(\sqrt{7})$, where $T_a = \sigma_a/2$. Find the commutation algebra obeyed by the $S_a, a = 1, 2, 3$ and find the transformation $S'_a = L_{ab}S_b$ such that $TrS'_aS'_b = \delta_{ab}/2$

13. For $SU(3)$, consider the diagonal generators $H'_1 = Diag(1, -1, 0)/2$, $H'_2 = Diag(1, 0, -1)/2, H'_3 = Diag(0, 1, -1)/2$. Let $\{H'_1, H'_2\} = \vec{H}', E_1 = W^1_2/(\sqrt{2})$, $E_2 = W^1_3/(\sqrt{2}), E_3 = W^2_3/(\sqrt{2})$ where $(W^k_l)^j_i = \delta^k_i\delta^j_l$. Write $[E_i, E^\dagger_i] = \alpha'^{(i)}_j H'_j$ and determine $\alpha'^{(i)}$. Calculate c_{ij} such that $H'_i = c_{ij}H_j$ where $\vec{H} = (\lambda_3, \lambda_8)/2$ are the Gell Mann diagonal generators and determine therefrom the roots of the SU(3) algebra.

14. Write the weights in the fundamental(defining representation) of SU(N) for $N = 2, 3, 4, 5$ with respect to the GellMann Diagonal generators $\{\lambda^{i^2-1}/2, i = 2, 3, ...N\}$. Determine the norm of the weights. Show that the positive roots can be written as differences of these weight vectors and choose $N - 1$ of these as simple positive roots which cannot be written as linear combinations of the other positive roots. Determine the lengths and inner products of the simple roots chosen and detrmine the SU(N) Dynkin diagram therefrom.

15. Find the fundamental weights $\vec{\mu}_j (j = 1, 2...N - 1)$ such that

$$2\vec{\alpha}_i \cdot \vec{\mu}_j/\vec{\alpha}_i^2 = \delta_{ij} \tag{7}$$

for the simple roots $\vec{\alpha}_i$ of SU(N). Determine the integers that characterize the weights within the $N, \overline{N}$, Adjoint, symmetric and antisymmetric 2 index tensors for $N = 3, 4, 5$.

16. Find the SU(N) invariants one can form from contractions of up to 3 irreducible SU(N) tensors with up to 3 indices each for N=2,3,4,5.

17. Compare the expansions of $R = exp(i\theta_a\mathcal{F}_a)$ and $X'_a = exp(i\theta \cdot X)X_a exp(-i\theta \cdot X) = \tilde{R}_{ab}X_b$ to establish the relation between R and $\tilde{R}$. here $\mathcal{F}_a = -if_{abc}$ are the adjoint irrep generators in terms of the structure constants of the algebra f_{abc}.

18. Enumerate the SU(N) irreps of dimension up to $2(N^2-1)$ for $N = 2, 3, 4, 5$ by considering irreducible tensors with p upper and q lower indices. Find their decomposition under $SU(2) \times SU(N - 2)$ for $N \geq 3$ (make the obvious required change for the special case N=3). Calculate their Dynkin indices and Anomaly numbers.

19. Show that an $SO(4)$ vector 4-plet transforms as a bi-doublet w.r.t $\simeq SU(2)_- \times SU(2)_+$

$$\Phi' = U_-\Phi U_+^\dagger \tag{8}$$

where $U_\pm = exp(i\vec{\theta}_\pm \cdot \vec{T}_\pm)$ and

$$\Phi = \begin{pmatrix} V_{\hat{2}} & V_{\bar{\hat{1}}} \\ V_{\hat{1}} & -V_{\bar{\hat{2}}} \end{pmatrix} \tag{9}$$

where

$$\begin{pmatrix} V_{\hat{1}} \\ V_{\bar{\hat{1}}} \end{pmatrix} = U_2 \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \tag{10}$$

(and similarly for $V_{\hat{2}}, V_{\bar{\hat{2}}}$) are the complexified components of the 4-plet.

20. Show that the tensors of SO(3) are all obtainable as completely symmetric traceless tensors $S_{i_1...i_N}; S_{i_1 i_1 i_3...i_N} = 0$. For N=1..6 check the counting of the components matches with the dimension $2j+1$ of a integer spin j representation of $SU(2)$ as expected from $SU(2) \simeq Spin(3)$.

21. Consider the SO(N) tensors $V_i, \{\hat{S}_{(ij)}; \hat{S}_{ii} = 0\}, A_{[ij]}, \{\hat{S}_{(ijk)}; \hat{S}_{iij} = 0\}; (i,j = 1,2...N)$. Calculate the irrep dimensions and Dynkin indices. For $N = 4$ determine how they transform under $\simeq SU(2)_- \times SU(2)_+$.

22. Consider completely antisymmetric tensors of $SO(2n)$ and $SO(2n+1)$ with up to n indices for $n = 1...5$. Determine the dimensions of the corresponding irreps and their Dynkin indices. Decompose them into representations of the (canonically embedded) $SO(n+1) \times SO(N-n-1)$.

23. Same as the previous problem for completely symmetric traceless representations.

24. The two dimensional spinor of SO(4) splits into chiral spinors $2^{\pm}$ with $\gamma_F = \pm 1$. If we define

$$\begin{aligned} 2^- &= |\psi^- >= \psi_1| + - > + \psi_2| - + > \\ 2^+ &= |\psi^+ >= \psi_{\dot{1}}| + + > - \psi_{\dot{2}}| - - > \end{aligned} \tag{11}$$

Derive the expressions for $\psi^T B \chi$ and $\psi^T B \gamma^i \chi V_i$ in terms of the chiral spinors and the bidoublet components introduced in Problem 1.

25. Write out the generators of $Spin(4) \simeq SU(2)_- \times SU(2)_+$ explicitly. Show they do not mix spinors with even and odd γ_F and that $T_a^{\pm}$ annihilate spinors with $\gamma_F = \mp 1$, while $T_a^{\pm}$ act on spinors with $\gamma_F = \pm$ as on a spinor of $SU(2)_{\pm}$.

26. The 8 spinor of $SO(6) \simeq SU(4)$ decomposes into chiral spinors $4^{\pm} = \Psi_a^{\pm}, a = (\pm\mp\mp), (\mp\pm\mp), (\mp\mp\pm), (\pm\pm\pm)$. These chiral spinors can be identified with the fundamental$(4 = [\psi_\mu], \mu = 1,2,3,4)$ and anti-fundamental $(\bar{4} == [\hat{\psi}^\mu], \mu = 1,2,3,4)$ of $SU(4)$. Find the map of the spinor components Ψ to SU(4) components $\psi, \hat{\psi}$ such that $\Psi^T B_2 \Phi = \psi \cdot \hat{\phi} + \hat{\psi} \cdot \phi$. Find also $\Psi^T B_1 \Phi$ as well as $\sum_{i=1}^{6} \Psi^T B_2 \gamma_i \Phi V^i$.

27. A common form for the gamma matrices of SO(4) is

$$\gamma'_{\bar{i}} = \begin{pmatrix} 0 & \tau^{\bar{i}} \\ \tau^{\bar{i}} & 0 \end{pmatrix} \qquad \bar{i} = 1,2,3 \qquad \gamma'_4 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \tag{12}$$

While the Wilczek-Zee iterative construction gives

$$\gamma_{1,2} = \tau_{1,2} \times \tau_3 = \begin{pmatrix} \tau_{1,2} & 0_2 \\ 0_2 & -\tau_{1,2} \end{pmatrix}$$
$$\gamma_{3,4} = 1_2 \times \tau_{1,2} = \begin{pmatrix} 0_2 & (1,-i)1_2 \\ (1,i)1_2 & 0_2 \end{pmatrix} \tag{13}$$

Find the Unitary transformation that connects ($\gamma_i' = U\gamma_i U^\dagger$)these representations of the SO(4) Clifford algebra.

28. Prove

$$[AB, CD] = A\{B, C\}D - AC\{B, D\} - C\{A, D\}B + \{A, C\}DB \tag{14}$$

and use it to show that the Spin(N) and Spin(1,N-1) generators $J_{ij} = -i\gamma_i\gamma_j/2; \quad (i \neq j = 1..N)$ and $J_{\mu\nu} = -i\gamma_\mu\gamma_\nu/2; \quad (\mu \neq \nu = 0..N-1)$ satisfy the SO(N) and SO(1,N-1) commutation relations.

29. Check that $\mathcal{L}_{\mu\nu} = -i\delta_{\kappa[\mu}\eta_{\nu]\rho}, \quad (\mu \neq \nu = 0..N-1)$ satisfy the $SO(1, N-1)$ commutation relations and that for $N = 4$ the *Hermitian* generators

$$T_a^{\pm} = \frac{1}{4}\epsilon_{abc}\mathcal{L}_{bc} \pm \frac{i}{2}\mathcal{L}_{a0} \tag{15}$$

satisfy the commutation relations of $\simeq SU(2)_+ \times SU(2)_-$.

Find the relation between the parameters associated with the Hermitian generators $T_a^{\pm}$ and the 6 Lorentz parameters $(\omega_{\mu\nu}, \mu, \nu = 0, 1, 2, 3)$. If the generators are hermitian why then are the group elements in general non unitary ?