SUDESHNA

1

S.H. STROGATZ | 20% Term Paper

CHAOS | FRACTALS | COMPLEXITY

DYNAMICAL SYSTEMS | DETERMINISTIC → UNPREDICTABLE

SOLVABLE

MAPS

DISCRETE

$$\underline{x}^{n+1} = \underline{f}(\underline{x}^n)$$

DIFF. EQNs

$$\dot{x}_1 = f_1(\underline{x})$$
$$\dot{x}_2 = f_2(\underline{x})$$
$$\vdots$$
$$\dot{x}_n = f_n(\underline{x})$$

~~CHAOS F~~

POINCARE

→ you're asking the wrong question

worry about qualitative soln.

ROBERT MAY

Not do | CLASSICAL MECH

Ch-10, 11

Experiments

MAPS 1-DIM

ODE : 3

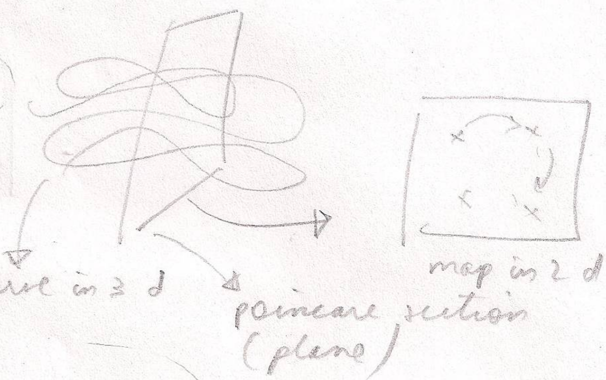
MAPS : INVERTIBLE - 2

x_n $x_0, x_1, \ldots$ ORBIT or TRAJECTORY

$n = 0, 1, \ldots$

NON-LINEAR : PIECEWISE LINEAR

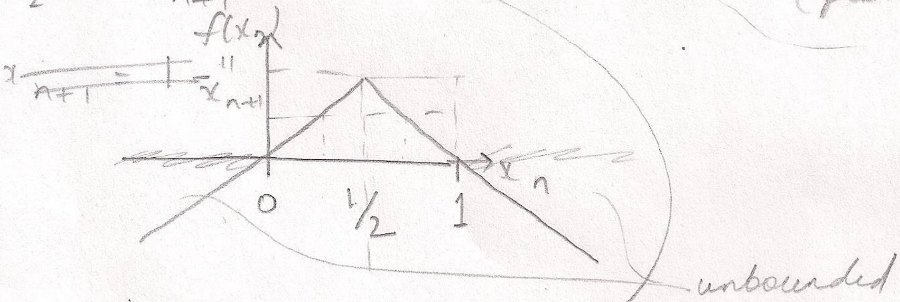
$$\cancel{x_n}\ x_{n+1} = 1 - 2\left|x_n - \tfrac{1}{2}\right|$$



for $x_n < \frac{1}{2}$, $x_{n+1} = 2x_n$

$x_n > 1/2$ $x_{n+1} = 2(1 - x_n)$

So now, $x_{n+1} = x_{n+1}$



$x < 0$
$\to -\infty$
$x > 1$
$\to -\infty$

look at,

$[0, 1]$

$$0 \le 1 - 2\left|x_n - \tfrac{1}{2}\right| \le 1$$

if $0 \le x_0 \le 1$: ORBIT'S BOUNDED

1) STRETCH $\to$ EXPONENTIAL DIVERGENCE OF NEARBY ORBITS

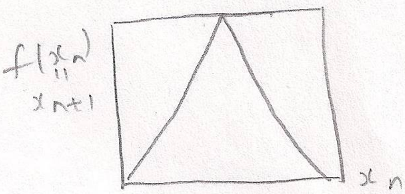
SENSITIVE DEPENDENCE ON INITIAL CONDITIONS

2) FOLD

COMPOSE MAP m times

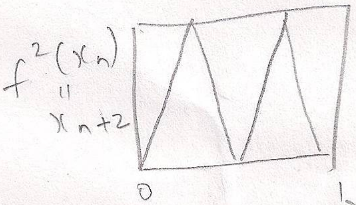
$$F^m(x) = F(F^{m-1}(x))$$
$$= F(F(F \ldots F(x)))$$

$$x_{n+m} = F^m(x_n)$$



Trick ; plot zeros, rest is linear

$x_0 \pm \frac{1}{2^m}$ $\to$ to see the effect of an error



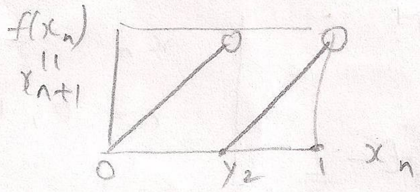
well, you could get 0 or 1 !

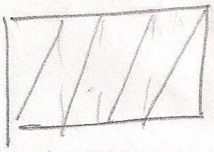
+ if you increase res by 3 digits & 3 then iterate will again destroy.

Extreme sensitivity.

$f(x) = 2x \mod 1$

$x_{n+1} = 2x_n \mod 1$





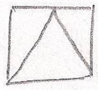
$\frac{1}{2^m}$

#3

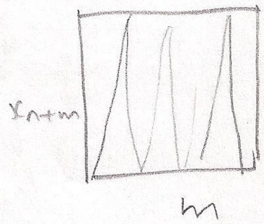
ITERATED MAPS

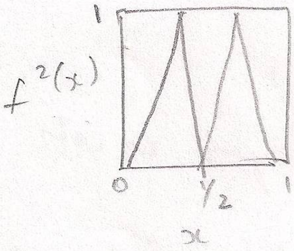
STRETCH } NON-INVERTIBLE
FOLD } PIECEWISE

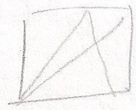
$$\begin{cases} f(x) = 1 - 2\left|x - \frac{1}{2}\right| \\ f(x) = 2x \mod 1 \end{cases}$$




$f^m \equiv x_m$







$$x = 0.a_1 a_2 \ldots a_j \ldots$$
$$= \sum_{j=1}^{\infty} 2^{-j} a_j$$

alg. eg.

$0.a_1 a_2 \ldots \quad \times 2$

$a_1 . a_2 \ldots$

$0.010\ldots$

$\frac{1}{3} \times 2 = \frac{2}{3} < 1 \quad 0$

$\frac{2}{3} \times 2 = \frac{4}{3} > 1 \quad 1$

$\frac{1}{3} \times 2 = \frac{2}{3} < 1 \quad 0$

$$x_{n+1} = f(x_n) = 2x_n \mod 1$$

$x_0 = 0.a_1 a_2 \ldots$

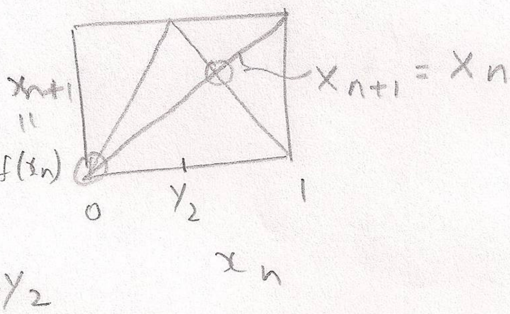
$x_1 = 0.a_2 a_3 \ldots$

$x_2 = 0.a_3 a_4 \ldots$

PERIODIC POINTS (ORBITS) OF A MAP

→ FIXED PT $\quad x_{n+1} = x_n$

eg.



$$f(x^*) = x^*$$

$x > 1/2$

$2 - 2x^* = x^*$

$x^* = 2/3$

$x < 1/2$

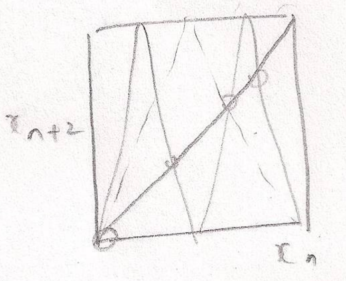
$x_{n+1} = 2x_n \quad x^* = 0$

→ P-CYCLE $\quad x_i \neq x_j$

$x_0, x_1 \ldots x_{p-1} \quad x_0 \ldots x_{p-1}$

$x_{n+p} \equiv x_n$

$$x_j = f^p(x_j)$$



For shift map

$0.a_1 a_2 a_3 \dots a_p a_1 \dots a_p \dots$

$0.00\dots$

$0.11\dots$

$0.01\dots \equiv 1/3$

$0.10\dots \equiv 2/3$

NB: $\frac{1}{3} \times 2 = 2/3$

$\frac{2}{3} \times 2$

$1/3$

Period 2 — 1 cycle exists

How many period p cycles are there (including trivials)?

(p is prime then non-trivial is $2^p - 1$)

& $\sim \frac{2^p}{p} = N_p$

$p = 11$ $\quad N_{11} = 186$

$= 13$ $\quad N_{13} = 630$

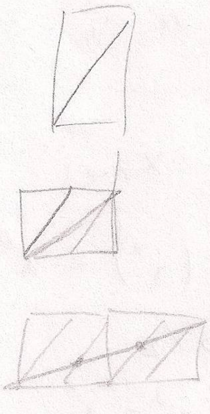
$= 17$ $\quad = 7710$

$p = 4$

$2^4 = 16$

1 : 0, 1

2 : $1/3$, $2/3$



$x = f_x b^{q-E}$

$\frac{1}{b} < f < 1$

Let precision of f be t bit

~~Max val. of q of q be~~

Then $\frac{\delta x}{x} = \left(\frac{1}{2}\right) b^{-t}$

$\left(x_1(1+\epsilon_1) + x_2(1+\epsilon_2)\right)(1+\epsilon)$

$\sim x_1\epsilon_1 + x_2\epsilon_2 + (x_1+x_2)\epsilon + O(\epsilon_x^2)$

$\leq \cancel{x_1+x_2)} \; \epsilon \left(|x_1| + |x_2| + |x_1+x_2|\right)$

$17^{1/2}$

checking prime smaller than this.

7^m

$\log_7(7^m \cancel{2^n} 2^n)$

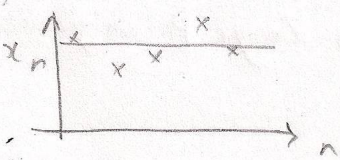
$m' \leq m + \log_7(2^n)$

~~3~~ 4

PERIODIC ORBIT

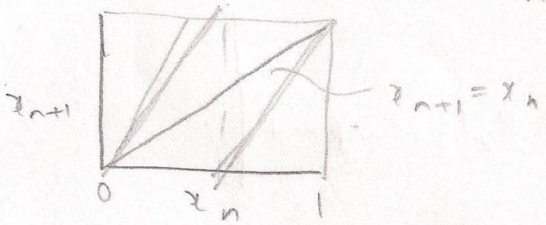
$x_{n+p} = x_n$

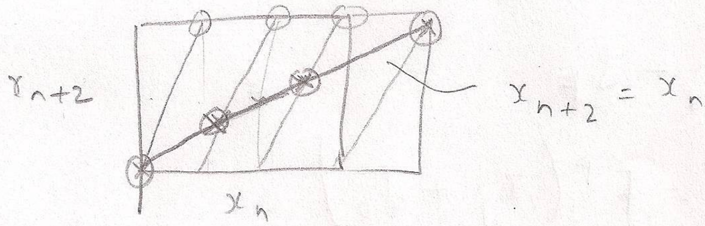
p-cycle

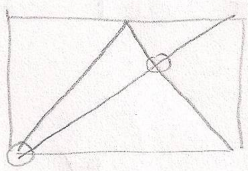


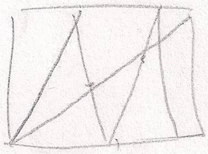
intersections

$x^* = f^p(x^*)$









$0 . a_1 a_2 \ldots$

PERIODIC ~~POTENTIAL~~ ORBIT

$x_{n+1} = 2x_n \mod 1$

2^p

$p = 1$ $\quad 0.00 \ldots \quad 0$

$\quad\quad\quad 0.11 \ldots \quad 1$

$p = 2$ $\quad 2^2 = 4$

$0.0101 \ldots = 1/3$

$0.1010 \ldots = 2/3$

$$\sum_n \frac{1}{2^{2n}} = \sum_n \frac{1}{4^n} = \frac{1}{4-1} = \frac{1}{3}$$

$p = 3$ $\quad 2^3 = 8$

$\left.\begin{array}{l} 0.010 \\ .100 \\ .001 \end{array}\right]$ $\quad \frac{6}{2}$

$\left.\begin{array}{l} .110 \\ .011 \\ .101 \end{array}\right]$

$x_{n+2} = x_n = f(f(x_n))$

$= 2(2x_n)$

$x^* = 0$

$x_{n+2} = x_n = 2(2 - 2x_n)$

$\Rightarrow x^* = \frac{4}{5}$

$(2 - 2(2-2x)) = x$

$p = 4$

$2^4 = \cancel{8}\ 16$

$p = 1 : 2$

$p = 2 : 2$

$(16 - 4) = 12$; $\frac{12}{4} = 3$ Distinct Period 4

$$N_p = \frac{2^p - 2}{p}$$

p : PRIME

$$N_p \sim \frac{2^p}{p}$$

for large p in general

$$X_p = X_0$$

$$X_j = F^p(X_j) \quad j = 0, 1 \ldots p-1$$

$$X_j + \delta_0$$

$$X_p = X_j + \delta_p$$

$$X_j + \delta_p = F^p(x_j + \delta_0)$$

$$= F^p(x_j) + \delta_0 F^{p'}(x)\Big|_{x_j} + O(\delta_0^2)$$

$$\Rightarrow \quad \delta_p = \lambda_p \delta_0$$

$$\lambda_p = \frac{d}{dX} F^p(x)\Big|_{x = x_j}$$

$$= \frac{d}{dX_n} X_{n+p}\Big|_{X_j}$$

Notation: $X = x$

$$\frac{dx_{n+p}}{dx_n} = \frac{dx_{n+1}}{dx} \cdot \frac{dx_{n+2}}{dx_{n+1}} \cdot \ldots \frac{dx_{n+p}}{dx_{n+p-1}}$$

$$= f'(x_n) f'(x_{n+1}) \ldots f'(x_{n+p})$$

This doesn't depend on where you are in the orbit. It'll cycle!

$$x_j + \delta_p = x_j + \lambda_p \delta_0$$

$$x_j + \delta_{2p} = x_j + \lambda_p \delta_p$$

$$= x_j + \lambda_p \lambda_p \delta_0 = x_j + (\lambda_p)^2 \delta_0$$

$$\boxed{\delta_{np} = \lambda_p^n \delta_0}$$

$|f'(x)| = 2$

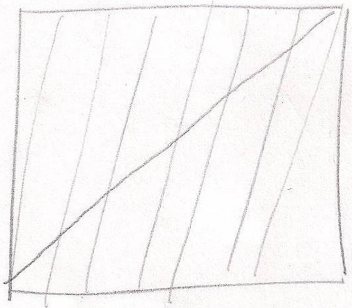
$|\lambda_p| = 2^p > 1$

<u>All orbits are unstable.</u>

Recall: Countable
Uncountable

NB: Countable maybe dense

Recall: Dense $\Rightarrow$ $\exists$ atleast one pt in your set ~~at~~ in $[x-\varepsilon, x+\varepsilon]$ $\forall\ \varepsilon > 0$.

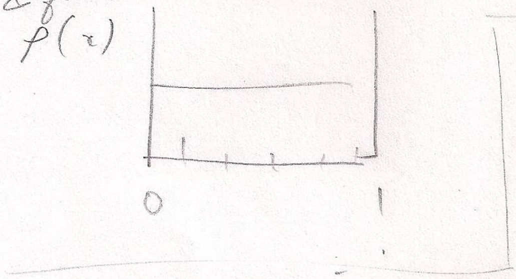


$2^{-p} < \varepsilon$

$p > \dfrac{\ln\left(\frac{1}{\varepsilon}\right)}{\ln 2}$

$\left[\dfrac{1}{2^p}(m-1),\ \dfrac{1}{2^p} m\right]$ $\quad m = 1, 2 \ldots 2^p$

Exp: Run a point. Consider some area & plot how often that area was visited
& you'll see



N ---

1 2 3 ---

p/q

1/1 2/1 3/1 4/1 - - -

1/2 2/2 3/2 4/2 - - -

#2·1

chaos

$x_{n+1} = \lambda x_n (1 - x_n)$ → Model 1: Explosion/Extinction

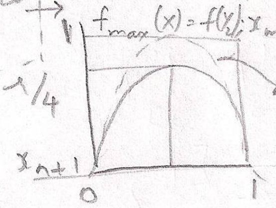
$x_{n+1} = \lambda x_n \quad \lambda \geq 1$ explosion

$0 < \lambda \leq 1$ extinction

Robert May (1976) Nature

Population model

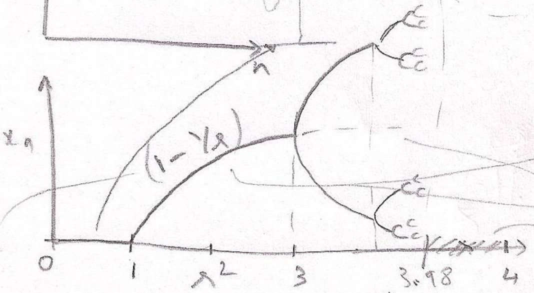
$f_{max}(x) = f(x_{max}) = \frac{\lambda}{4}$

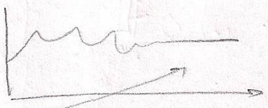


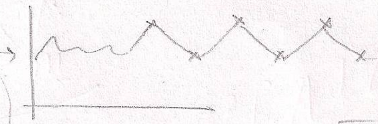
at $\lambda = 4$

if $\lambda > 4$, you'll go $-\infty$!

$x_0 =$







Period doubles

unstable branch

Def^n: Bifurcation ≡ You get qualitatively different behaviour, ~~every~~ when a parameter is changed.

People: Thomae & grossman

Feigenbaum

$2^n \left\{ \lim_{n \to \infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} = \delta = 4.669\ldots \right.$

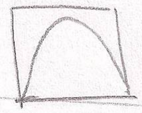
① CHAOTIC BANDS

② ODD CYCLES

③ at $\lambda = 4$, CHAOTIC ATTRACTOR

\# UNIMODAL: Maps qualitatively with one hump

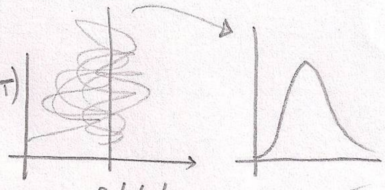
\# unimodals, δ was same.



Fluid Experiments → Period Doubling observed.

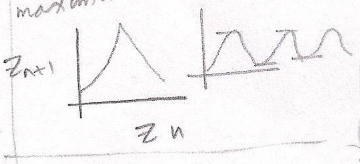
(even the # was matching.

B-Z experiment: complicated set of chemicals mixed & produced



$T = 53s$ $\quad B(t)$ → bromine conc.

3 ODES consecutive maxima



$x^* = f(x^*) = \lambda x^* (1 - x^*)$

$x^* = 0$

$x^* = 1 - \frac{1}{\lambda}$ for $\lambda > 1$

$f'(x)\big|_{x^*} = \lambda - 2\lambda x^*$

$f'(0) = \lambda$

$\lambda < 1$ → stable

$\lambda > 1$ → unstable

$f'(1 - \frac{1}{\lambda}) = \cancel{\lambda} \lambda - 2\lambda + 2$

$= 2 - \lambda$

$-1 < 2 - \lambda < 1$ for stable

$1 < \lambda < 3$

2·2

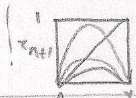
Chaos

$N_{n+1} = \lambda N_n$
$N_{n+1} = \lambda^n N_0$
$\lambda > 1$; Explosion

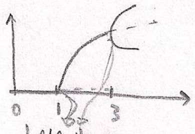
$\lambda' \to \lambda\left[1 - \frac{N_n}{N_c}\right]$

$\frac{N}{N_c} = x$ $\quad N_{n+1} = \lambda' N_n$

$\to x_{n+1} = \lambda x_n (1 - x_n)$ $\quad [0,1]$



$\lambda \leq 4$



$|f'(x)| = 1$

$\lambda = 3$

$\text{abs}(f'(x)|_{x^*}) < 1$

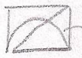
$f'(x^*) = \lambda - 2\lambda x^*$

$\Rightarrow -1 < 2 - \lambda < 1$
$\Rightarrow 1 < \lambda < 3$

1 period
Stable → unstable

NB: Slope tells you stability; $< 45°$ is stable.
NB: There only one soln. when $0 < \lambda < 1$

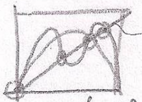
$0 < \lambda < 1$



$\lambda > 1$

PITCHFORK

$x_{n+2} = ff(x_n))$

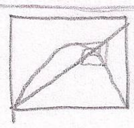


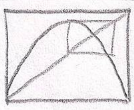
$x_{n+2} = x_n$

put $f^2(x) = x$ & you'll get

$x = \lambda\{\lambda x(1-x)\}\{1 - \lambda x(1-x)\}$
two factors are $x^* = 0$, $x^* = 1 - \frac{1}{\lambda}$
long division then quadratic
& the roots then are $p, q = \frac{\lambda + 1 \pm \sqrt{(\lambda-3)(\lambda+1)}}{2\lambda}$

COBWEB

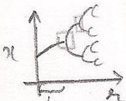




$3 < \lambda < 1 + \sqrt{6}$

$$\lambda = \frac{d}{dx}\left(ff(x)\right)\Big|_{x=p} = f'(f(p))f'(p) = f'(q)f'(p)$$
$$= \lambda^2[1 - 2(p+q) + 4pq]$$
$$= 4 + 2\lambda - \lambda^2$$
$$|\lambda| = |4 + 2\lambda - \lambda^2| < 1$$

Claim: All bifurcations happen like this | Next Week/Class: Tangent Bifurcation; odd cycles.



$|f'(x^*)| > 1$ Repeller

$|f'(x^*)| < 1$ attractor

control parameter

window for period 1

my slope must go from

$+1 \rightarrow -1 \rightarrow$ unstable

$f' = 0$ will happen somewhere.

2.3 Thursday

r_n : 2^n cycle first appears

$r_1 = 3 \rightarrow 2$

$r_2 = 3.449 \rightarrow 4$

$r_3 = 3.544 \rightarrow 8$

$r_4 = 3.5644 \rightarrow 16$

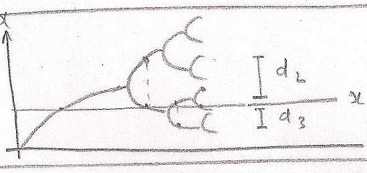
$r_n \rightarrow r_\infty$ (accumulation 2^∞)

$$\delta = \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\ldots$$

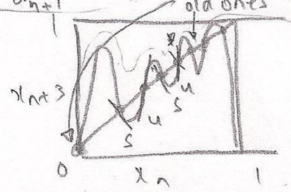
$\rightarrow$ superstable $\rightarrow x_m \rightarrow f(x_m) \rightarrow$ Max

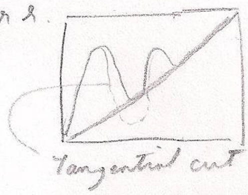
$\lambda = f'(x_1) f'(x_2) \ldots f'(x_p)$

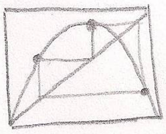
For $\lambda = 0$, we just need $f'(x) = 0$ for some $x \in x_p$



$\frac{d_n}{d_{n+1}} = \alpha = -2.5029$ Quantitative Universality







SARKOUSKII ORDERING

LI & YORKE → first used the word chaos.