Arvind | Classical / QM Topics

- * Gp Theory & Sym in Physics
- * Symplectic Gps & their use in Physics (CM & QM)
- * Geometric Phases
- * Classical Theory of Constrained Dynamics
- * Quantum Theory of Angular Momentum
- * Theory of Wigner Distributions
- * ~~The Wigner thm of UIR's of Poincare Gp~~
  (U → unitary rep)

□ Arvind
○ NM

* Dissipative QM.

Book 1: Lectures on adv. math methods — NM

Midsem 1 - same slot — 25%
Midsem 2 - study/lectures instead — 25%
Presentation
Final — 50%

Arvind

8 Jan '15

#2

Gp : has a binary operation (also called gp multiplication)

1. Closure : if $a, b \in G$, $c = ab \in G$
2. Associativity : $a(bc) = (ab)c$
3. Identity : $e \in G$ s.t. $ea = ae = a$ ~~about~~ $\forall\, a \in G$
4. Inverse : $\exists\, a^{-1} \in G$ s.t. $a^{-1}a = e$ $\forall\, a \in G$.

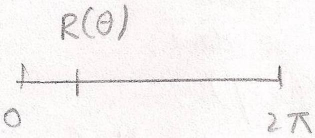
Needn't be commutative viz. $ab \neq ba$ in general.

Eg. $\{1\}$, $\{1, -1\}$, $\{1, -1, i, -i\}$

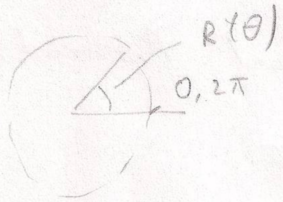
Consider : $\{a, a^2, a^3 \ldots a^{n-1}, a^n = 1\}$ — Cyclic gp $\equiv$ Implicit Def^n

Consider : Gp of rotations in 2-D.

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} ; \quad 0 \leq \theta < 2\pi$$



representation



Gp Test : (1) $R(\theta_1) R(\theta_2) = R(\theta_3)$

use trigo & you'll get $\theta_3 = \theta_1 + \theta_2$

(2) Matrix Multiplication is associative

(3) $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\theta = 0$

(4) $R^{-1}(\theta) = R(-\theta)$ use $(-\theta) \bmod 2\pi$ ← will it work?

Def^n $\equiv$

Conjugate : If $a, b \in G$, & b is s.t. $b = cac^{-1}$ where $c \in G$, then a, b are conjugate

Defining properties:

Equivalence Relation : $a \sim a$, $a \sim b \Rightarrow b \sim a$

$a \sim b$, $b \sim c \Rightarrow a \sim c$

Claim : Conjugation is an equivalence

Def^n: Equivalence Class $\equiv$ subset S s.t. if $k_2, k_1 \in S$

$k_1 \sim k_2$

Eg: Find equivalent classes $\{1, -1, i, -i\}$

$\{1\}$ $\{-1\}$ $\{i\}$ $\{-i\}$

Eg: $\{a, a^2, a^3, a^4, a^5, a^6 = 1\}$

Def^n: $H \equiv$ Subgp $\equiv$ If G is a gp, $H \subset G$ & H is a gp.

$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ Def^n: Left Coset $\equiv$

$aH \equiv \{ah\}; a \in G, h \in H\}$

$\Rightarrow aH$ is the left coset of H in G.

Thm: The cosets are either disjoint or identical.

Proof: If $a_1 h_1 = a_2 h_2$, & $a_1, a_2 \in G$ but $a_1, a_2 \notin H$

$a_1 h_1 = a_2 h_2$

$a_1 = a_2 h_2 h_1^{-1}$

Now the first is $a_1 H = a_2 h_2 h_1^{-1} H = a_2 H$

so done.

(half)

NB: Since $e \in H, G$; $a_1 H, a_2 H \ldots$ aren't subgps.

Def^n: Invariant (Normal) Subgps $\equiv$ for $H \subset G$, $g \in G$,

$H_g = \{ g h g^{-1} \mid h \in H$ for a given $g\}$

If H is special, $H_g = H$

Def^n: H is a normal subgp. if Left Coset $\equiv$ right coset

Recall: Factor gp G, H $\{a_j H\}$

$\frac{G}{H}$

Eg: (1) $G = \{1, -1, i, -i\}$ $H = \{1, -1\}$

$H = \{1, -1\}$ $\{\{1,-1\}, \{i, -i\}\}$

$iH = \{i, -i\}$

Def^n: Orthogonal $R^T R = R R^T = I$

NB: $\det(R^T R) = (\det(R))^2 = 1 \Rightarrow \det R = \pm 1$

: No. of indep param in $SO(2)$, $O(2)$

Def^n: $SO(2)$ is $O(2)$ with $\det = 1$

special/proper

Remarks: Can't relate

Def^n: $U(n) \equiv U$ s.t. $U^\dagger U = U U^\dagger = I$

NB: $\det(U^\dagger U) = (\det U)^* (\det U) = |\det U|^2 = 1$

$\Rightarrow |\det U| = 1 \Rightarrow \det U = e^{i\theta}$

#2.1

Abelian + Non-abelian gps

G abelian $ab = ba$ for all $a, b \in G$; commutative gp.

If $ab \neq ba$ for any pair $a, b \in G$; non abelian or non-commutative gp.

Commutator $\equiv g(a,b) = aba^{-1}b^{-1}$

- If $g(a,b) = e$ $\forall a, b \in G$, then commutative
- Products of commutators form a gp
- Larger $g(G,G)$, more non-abelian G
- ★ Normal subgp is $g(G,G)$: $g(\ \ )g^{-1}$
- ☆ $G/g(G,G)$ Factor gp "Abelian" $g \in G$

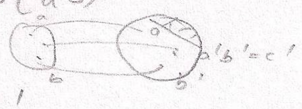
Example: SO(2) U(1) (& not SU(1))

Relationships / Homomorphism

$\varphi\ G \rightarrow G'$

$a \in G$ $\varphi(a) = a' \in G'$

$\varphi(a)\varphi(b) = \varphi(a \cdot b)$



Many $\rightarrow$ One

$\varphi(e) = e'$

one-one, onto : Isomorphism G, G' "same"

★ $\varphi(G)$ is a subgp of G' ★ If $G' = G$, then its an automorphism

Kernel of Homomorphism $K = \{a \in G\ \varphi(a) = e'\}$

K is a subgp of G.

Normal / Invariant subgp — left coset = right coset "onto"

G/K Factor gp

$\varphi : (G/K) \longrightarrow G'$ is isomorphism

Direct Product of Gps

$G_1\ G_2$ $G \equiv SO(2) \times SO(2)$

$G_1 \times G_2$ Direct Prod

(a_1, a_2) $a_1 \in G_1$, $a_2 \in G_2$

$(a_1, a_2)(b_1, b_2) = (a_1 b_1, a_2 b_2)$

$(a_1, a_2)^{-1} = (a_1^{-1}\ a_2^{-1})$

$I = (e_1, e_2)$

🏠: Topological Spaces:

6 Feb - Noclass

Arvind #2.2

$SO(2)$ "Vector space"
V set of "vectors" ; Field
if $v_1 \in V$
$v_2 \in V$
then $f_1 v_1 + f_2 v_2 \in V$
* Linear Transform
* Basis in "n" Dimension ; # max. independent (linearly) vectors
Basis $\{e_i\}$

Linear Transformation
if $O v_1 = v_2 \quad \forall v_1 \in V$
then O is known

Group Representation

"Linear reps" on finite dim vector spaces real & complex
Group G linear vector space V
(i) $D(g') D(g) = D(g'g)$ — apply them successively on vectors — multiply gp elements
(ii) $D(e) = 1$
(iii) $D(g)^{-1} = D(g^{-1})$

Basis in V $\{e_i\}$ $i = 1, \ldots n$
$D(g) e_j$ Determines $D(g)$
missing

G $\quad g \to D(g)$ Representation
V_1 is a subspace of V, which is closed under linear combination
$x \in V_1$ vector in V_1
if $g \to D(g) x = y \in V_1 \; \forall x \in V_1$
then V_1 is invariant of space under the actions of V_1
& if $D(\cdot)$ rep is reducible
if V_1 doesn't exist $D(\cdot)$ is irreducible

It maybe
$z \notin V_1$
$D(g) z \in V$

$V = V_1 \oplus V_2$
$\begin{vmatrix} (\,) \\ (\,) \end{vmatrix}$
If I can find V_2 where V_2 is also invariant, then V is "decomposable"

missing

Irreducible rep.
$D(\cdot)$ V_1 invariant
Reducible
Reducible decomposable
Reducible non-decomposable
V_2 is not an invariant subspace
$\begin{pmatrix} D_1(g) & \\ & D_2(g) \end{pmatrix}$ ← decomposable

Unitary + Orthogonal Representation

$x, y \in V$ dual space
$(x, y) \quad \langle x | y \rangle$
linear in y, antilinear in x.
if $(x, y) = 0$, x is orthogonal to y
$(x, c_1 y_1 + c_2 y_2) = c_1 (x, \ldots$

Rep G $D(\cdot)$
$(D(g) x, D(g) y) = (x, y)$

$U|x\rangle = |x'\rangle$
$U|y\rangle = |y'\rangle$
$\langle x|y\rangle \stackrel{C/a}{=} \langle x'|y'\rangle$
$= \langle x| U^\dagger U |y\rangle$

S_3 $\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$ $\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$ $\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$ $\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$ $\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$

3 Dimension vector space "complex"

I $\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$, $\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$, $\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$, $\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$, $\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$

$D(g')D(g) = D(gg')$

$D(e) = I$

$D^{-1}(g) = D(g)$

G $D(\cdot)$ V rep1

$D'(\cdot)$ V' rep2

(i) equiv

$D'(g) = T\,D(g)\,T^{-1}$ $\forall g \in G$

$\text{Dim } V = \text{Dim } V'$ | "change of basis

Intrinsic

"Explicit Basis"

G $v_i \in V_1$

$D_1(\cdot)$

$D_2(\cdot)$

☹ missed $n_1 = \text{Dim}\{e_j'\}$

(i) Space "Tensor product"

spanned by $n_1 \cdot n_2$ vectors

$e_i \otimes e_j^{(2)}$ "$V_1 \times V_2$" "span"

$v \in V_1 \otimes V_2$

$v = \sum_{ij} v_{ij}\, e_i^{(1)} \otimes e_j^{(2)}$

<Truncated> (copy from prashant)

Shur's Lemma: Let there be 2 representations

$D(\cdot)$ IR V $\quad D'(\cdot)$ IR V' & if $\exists$ a T s.t.

$D'(g)T = T D(g)$ $\quad \forall g \in G$, then

(i) $D(\cdot)$, $D'(\cdot)$ are inequivalent & $T = 0$

(ii) $D(\cdot)$, $D'(\cdot)$ are equivalent & $T \neq 0$ & T^{-1} exists.

Given $D(g)$

$g \to D(g)$ IR (Matrix)

$g \to (D(g)^T)^{-1}$ Contragradient to D

$g \to (D(g)^{\dagger})^{-1}$ Adjoint to D

$g \to D(g)^*$ conjugate rep

"Need not be equivalent"

If D is unitary $\Leftrightarrow$ Self Adjoint

Contragredient $\Leftrightarrow$ complex conjugate

Rotation 3; Euler angles

2 axes

"3 real parameters" "Rigid"

"axis angle"