Sudeshna 11·1
(No 1Q. x series)
exams

2-d Systems

F.P. ✓

Limitcycles

Linear

$\dot{x} = ax + by$

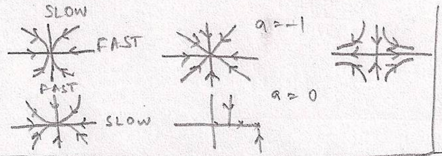
$\dot{y} = cx + dy$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$; $\dot{\vec{x}} = A\vec{x}$

$\vec{x}^* = 0$

seperable

$A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$



$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\tau = \lambda_1 + \lambda_2$

$\Delta = \lambda_1 \lambda_2$

$\lambda_{1,2} = \dfrac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$

$\lambda_1 \neq \lambda_2$ distinct real

$e^{\lambda_1 t}$

$e^{\lambda_2 t}$

COMPLEX CONJUGATES

$\lambda_{1,2} = \alpha + i\omega$

$\alpha = \tau/2$, $\omega = \dfrac{\sqrt{4\Delta - \tau^2}}{2}$

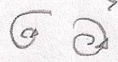
$\alpha = 0$

$\lambda_{1,2} = \pm i\omega$

elliptic fixed point.

α : $Re(\lambda) < 0$ stable

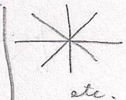
> 0 unstable



$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2$

$A\vec{v} = \lambda \vec{v}$

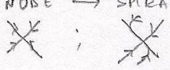


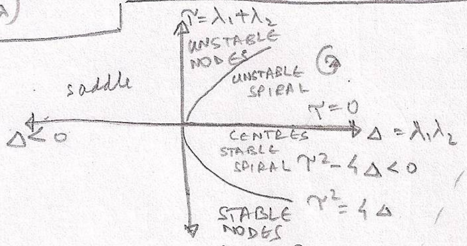
etc.

$A = \begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix}$; $A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$

its a borderline b/w

NODE $\longrightarrow$ SPIRAL





Linear 2d $\to$ Non-linear

fixed pt : classify ?
closed orbits : stable ? $\Big\}$ flow

$\vec{x}(t+T) = \vec{x}(t)$

$\dot{x}_1 = f_1(x_1, x_2)$
$\dot{x}_2 = f_2(x_1, x_2)$

$f_1(x_1^*, x_2^*) = 0$
$f_2(x_1^*, x_2^*) = 0$

chaos 11.4
(no 11.3, 11.2 was lab)

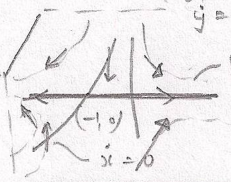
eg.
$\dot{x} = x + e^{-y}$
$\dot{y} = -y$
$y(t) = y_0 e^{-t}$
$\dot{x} \approx x + 1$

$\dot{x} = x + e^{-y}$
$\dot{y} = -y$
$y(t) = y_0 e^{-t}$
$\dot{x} \approx x + 1$

$\dot{y} = 0 \Rightarrow y^* = 0$
$\dot{x} = 0 \Rightarrow$
$x + e^{-0} = 0$
$x^* = -1$

$(-1, 0)$

Nullcline
$\dot{x} = 0$
or
$\dot{y} = 0$



Saddles
Spirals
Nodes $\Big]$ Structurally stable [Robust]

[Centres; $Re(\lambda) = 0$

Thm:
Hartman-Borgman

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Remember: at x^*, y^*

$\dot{u} = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}$

$(\because f(x^*, y^*) = 0)$

similarly

$\dot{v} = u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y}$

stability (local)

Non-linear $\to$ linearize

$\dot{x} = f(x, y) \quad x^*, y^*$
$\dot{y} = g(x, y)$

$f(x^*, y^*) = 0$
$g(x^*, y^*) = 0$

$u \equiv x - x^*$
$v \equiv y - y^*$

$\dot{x} = -x + x^3 \equiv f$
$\dot{y} = -2y \equiv g$
$\dot{y} = 0$

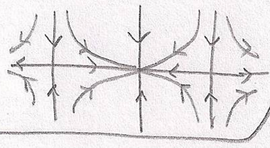
$x = 0$
$x = +1, -1$

$(0,0) \quad (+1,0) \quad (-1,0)$

$$J = \begin{pmatrix} -1+3x^2 & 0 \\ 0 & -2 \end{pmatrix}$$

$(0,0)$ $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ STABLE NODE

$(\pm 1, 0)$ $A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ SADDLES



Sudeshna 12.1

$\dot{x}_1 = f_1(x_1, \dots x_n)$

$\vdots$

$\dot{x}_n = f_n(x_1, \dots x_n)$

$\frac{\partial f_i}{\partial x_j}$

(IV)

1) Fixed Points; solve n simultaneous eq^{n}s.

2) Stability of fixed points

Jacobian.

$\downarrow$

eigenvalue/eigenvector

(a) $Re(\lambda_i) \neq 0$; Robust

3) Nullclines $\dot{x}_i = 0 \rightarrow$ flow accross nullclines

(IV)

(b) Poincare Bendixson

2d < FP, closed orbit.

LOTKA-VOLTERA MODEL

2 species – competition

$\dot{x} = x(3-x-2y)$

$\dot{y} = y(2-x-y)$

$\dot{x} = 0$

$\dot{y} = 0$

1) $x^* = 0$, $y^* = 0$

2) $x^* = 0$, $y^* = 2$

3) $y^* = 0$, $x^* = 3$

4) Simultaneous $x^* = 1$, $y^* = 1$

$$J = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{pmatrix}$$

For $(0,0)$, $(0,2)$, $(3,0)$ & $(1,1)$ we find J

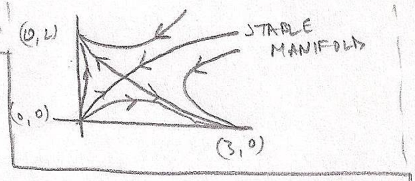
$(0,0) \rightarrow J = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

$\lambda_{1,2} = 3, 2$

x-ax is v fast

(II)

Basin of Attraction



$(1,1)$ $J = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$

$\tau = -2$

$\Delta = -1$

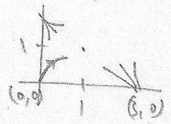
saddle

$\lambda_{1,2} = -1$

$(3,0)$

$J = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$

eigenvector (3,1)



$(0,2) \rightarrow J = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$

$\lambda_{1,2} = -1, -2$

$\downarrow$ eigenvector

$(1,-2)$

Conservative System

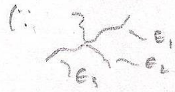
$F = m\ddot{x} \rightarrow 2d$

$F(x) = -\frac{dV}{dx}$

$m\dot{x}\ddot{x} + \frac{dV}{dx} \cdot \dot{x} = 0 \rightarrow \frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 + V(x)\right] = 0$

Energy

$\Leftrightarrow$ Can't have attracting points



$E_1 = E_2 = E_3 \dots$

so E must be trivial, contrary to assumption)

$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$

$F(x) = x - x^3$

$\dot{x} = y$

$\dot{y} = x - x^3$

$x^* = 0$, $y^* = 0$

$y^* = 0$, $x^* = \pm 1$

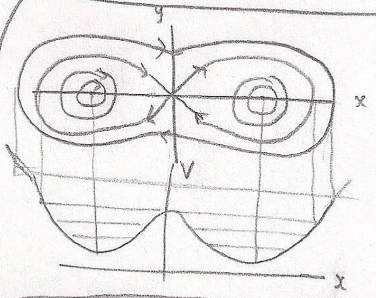
$(0,0)$ $(1,0)$ $(-1,0)$

$J = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{pmatrix}$ $\tau = 0$

$\Delta = 3x^2 - 1$

$(0,0)$ $\Delta = -1$: (Saddle)

$(\pm 1, 0)$ $\Delta = 2$: centre



(IV) The text has various examples

Existence & Uniqueness Thm:

Consider the initial value problem $\dot{\vec{x}} = \vec{f}(\vec{x})$, $\vec{x}(0) = \vec{x}_0$. Suppose $\vec{f}$ is continuous & all its partial derivatives are also continuous in some open connected set $D \subset \mathbb{R}^n$. Then for $\vec{x}_0 \in D$, the initial value problem has a soln. $\vec{x}(t)$ on time interval $(-\tau, \tau)$ about $t = 0$, & the soln. is unique.

$\Downarrow$

Two trajectories mustn't intersect.

For vector fields in a plane

(IV) Poincare Bendixson Thm: If a trajectory is confined to a closed, bounded region & there're no fixed points in the region, then the trajectory must approach a closed orbit.

$\dot{x} = f(x, y)$

$\dot{y} = g(x, y)$

$u = x - x^*$

$v = y - y^*$

$\dot{u} = \dot{x}$

$= f(x^* + u, y^* + v)$

$= \cancel{f(x^*, y^*)} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv)$

$= u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}$

$\dot{v} = u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y}$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \dots$$

claim:

The linearized system gives a qualitatively correct picture of the phase portrait so long as its not a borderline case.

Def^n: Robust case: Repellers, Attractors, Saddles

Marginal case: Centres, (+1 more)

Criterion: Atleast one λ st. $Re(\lambda) = 0$

Def^n: Hyperbolic fixed points $\equiv Re(\lambda) \neq 0$ for both (all) λ.

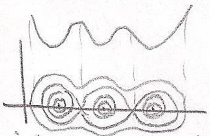
Hartman-Grobman Thm: For hyperbolic fixpts, linearization faithfully captures the phase potrait.

(II)

Given an attracting fixed point x^*, we define its basin of attraction as the set of init conditions x_0, st. $x(t) \rightarrow x^*$ as $t \rightarrow \infty$.

Basin Boundary = Stable manifold of saddle.

Conservative
↳ centre
↳ saddle



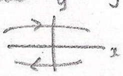
TODO: Homoclinic Orbit
(क) Hetroclinic
Reversible

Time Reversible
$t \to -t$?

$\frac{dx}{dt}$: reverses sign $\quad \dot{x} = y$

$\frac{d^2x}{dt^2}$: invariant $\quad \dot{y} = \frac{F(x)}{m}$

$\left.\begin{array}{l} t \to -t \\ y \to -y \end{array}\right]$



$\left.\begin{array}{l} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{array}\right\}$ $\quad f(x,-y) = -f(x,y)$ odd
$\quad g(x,-y) = g(x,y)$ even

You want under leaves the system invariant.

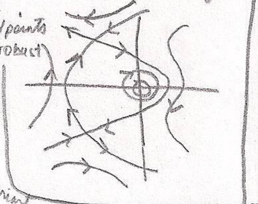
$\dot{x} = y - y^3 \quad \dot{x} = 0 \quad (0,0)$ its reversible
$\dot{y} = -x - y^2 \quad \dot{y} = 0 \quad \because t \to -t$
$y \to -y$ leaves it unchanged

$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\left.\begin{array}{l} T = 0 \\ \Delta = 1 \end{array}\right\}$ centre

$\left.\begin{array}{l} (-1,1) \\ (-1,-1) \end{array}\right\}$ more fixed points

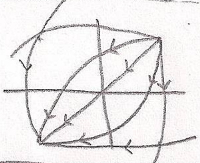
claim: reversible ⇒ robust

$R(x,y)$
$R^2(x,y) = (x,y)$
$\left.\begin{array}{l} t \to -t \\ \vec{x} \to R(\vec{x}) \end{array}\right\}$ invariant



$\dot{x} = -2\cos x - \cos y$
$\dot{y} = -2\cos y - \cos x$

$\left.\begin{array}{l} t \to -t \\ x \to -x \\ y \to -y \end{array}\right\}$ leaves invariant the eq's.



Pendulum

$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$

L, θ, m, g

$\tau = \omega t \quad \omega = \sqrt{\frac{g}{L}}$

$\frac{d^2\theta}{d\tau^2} + \sin\theta = 0$

$\dot{\theta} = \nu$

$\dot{\nu} = -\sin\theta$

$(\theta^*, \nu^*) = (k\pi, 0)$

$(0,0)$; $(\pi, 0) \longrightarrow J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\tau < 0, \Delta = -1$
$\lambda_{1,2} = -1, 1$
Saddle.

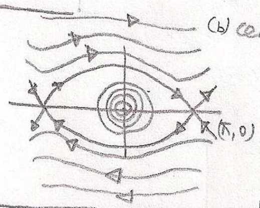
Invariant under $t \to -t$
$\nu \to -\nu$

$E = \frac{1}{2}\nu^2 - \underbrace{\cos\theta}_{1 - \frac{1}{2}\theta^2}$

$E \sim \frac{1}{2}(\nu^2 + \theta^2) - 1$: circle

(a) Energy of seperatrix: put $(\pi, 0)$ in E.

(b) centre → spiral when damping is introduced



Index

Thm: Consider $\dot{\vec{x}} = \vec{f}(\vec{x})$ with $\vec{x} = (x,y) \in \mathbb{R}^2$, f is continuously differentiable. Suppose ∃ a conserved quantity $E(\vec{x})$ & that $\vec{x}^*$ is an isolated fixed pt. (no fixed pt. in a small neighbourhood) If $\vec{x}^*$ is also a local minima (or maxima) of E, then all trajectories sufficiently close to $\vec{x}^*$ are closed.

Thm: Suppose that origin $\vec{x}^* = 0$ is a linear centre for the continuously differentiable system
$\dot{x} = f(x,y)$
$\dot{y} = g(x,y)$
& suppose that the system is reversible then sufficiently close to the origin, all trajectories are closed curves.

(क) Homoclinic orbits: Trajectories that start & end at the same fixed point

NB: Even though centres are predicted by linearization, they're indeed reliable ∵ the system is conservative.

Heteroclinic trajectories
or saddle connections ≡ Trajectories joining two saddle points

Index Theory

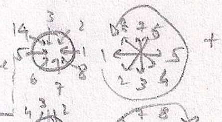
$\dot{\vec{x}} = \underline{f}(\vec{x})$

$\phi = \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right)$

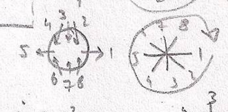
$[\phi_c] \Rightarrow I_c = \frac{1}{2\pi}[\phi]_c$

1) Shouldn't sit on a fixed pt.
2) Shouldn't intersect itself.

2d plane

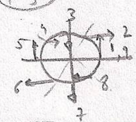


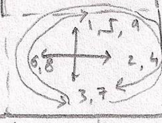
+1



−1

$\dot{x} = x^2 y$
$\dot{y} = x^2 - y^2$
$x^2 + y^2 = 1$





(a) $I_c = I_{c'}$
$c' \to c$
No f.p. crossed
Proof: Plumps ∴ value is int & continuity

$-\pi + 2\pi - \pi = 0$

(b) If c doesn't enclose a f.p.
$\leftarrow I_c = 0$ ∵ then you shrink to zero.

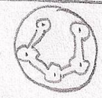
(c) Reverse arrows − Index is invariant
$\phi \to \phi + \pi \Rightarrow$ Index doesn't change

(d) If c is a traj, $I_c = 1$

stable & unstable node
$I_c = 1$

Saddle

$I_c = -1$



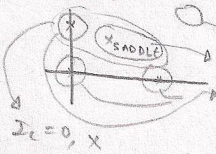
Thm: 'n' isolated fixed pt.
$\underline{x}_1^*, \underline{x}_2^*, \ldots$

$\frac{1}{2\pi}[\phi]_c = I_{c'}$
$= \frac{1}{2\pi}\sum[\phi]_{f.p.}$

$\dot{x} = x e^{-x}$
$\dot{y} = 1 + x + y^2$
No fixed points
⇓
No closed orbit.

$\dot{x} = x(3 - x - 2y)$
$\dot{y} = y(2 - x - y)$

If closed curve
$\Rightarrow I_c = 1 \Rightarrow$ Fixed Points exist



→ No fixed pt
→ saddle, $I_c = -1$
→ Index Theory will OK, but $x = 0$ & $y = 0$ are fixed pts, you can't cross 'em.

$I_c = 0$, ✗

Next: Limit Cycles
Relaxation Oscillators