Lattice Dynamics

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1 Introduction

We explore the dynamics of a 1D spring mass lattice, using an equivalent electronic/electrical circuit. We consider two configurations which correspond to

1. an array of point masses (identical) connected by springs of equal spring constant
2. an array of point masses (identical) connected by springs with alternate springs having the same spring constant

We show rigorously which quantities correspond to which in both setups. We then evaluate the dispersion relation, viz the relation between the frequency ω and (TODO: Name of k) k theoretically and plot it. In the same graph, we plot the applied frequency ω and measured value of k.

There are certain curious theoretical questions which the experiment has raised, some of which we have addressed and the remaining have been stated, in the last section.

2 Background Theory

2.1 Spring Mass System

Consider the following spring mass system. We write the forces on the n^{th} oscillator as

$$m\ddot{x}_n = -\gamma(x_n - x_{n-1}) + \gamma(x_{n+1} - x_n)$$

and we know that $x_n = Ae^{i(kna-\omega t)}$ can be used to find the dispersion relation, that is ω as a function of k. We skip the detailed derivation here and come back to it in the LC circuit equivalent of the same.

Next, consider next the spring mass system as given below. We again write two sets of equations for the forces on both types of n^{th} oscillator as

$$m_1\ddot{x}_n = -\gamma_2(x_n - x'_n) + \gamma_1(x'_{n-1} - x_n)$$

$$m_2\ddot{x}'_n = -\gamma_1(x'_n - x_{n+1}) + \gamma_2(x_n - x'_n)$$

where we take $m_1 = m_2 = m$ which can be solved by using $x_n = Ue^{i(kna-\omega t)}$ and $x'_n = Ve^{i(kna-\omega t)}$. Again the detailed solution is discussed in the next section.

2.2 L, C circuit equivalent

Consider the following LC circuit. We can write down using Kirchoff's law the following:

$$-L\ddot{q}_n - C^{-1}(q_n - q_{n-1}) + C^{-1}(q_{n+1} - q_n) = 0$$

The equivalence then is that $L \leftrightarrow m$, $1/C \leftrightarrow \gamma$ and $x \leftrightarrow q$. It is important to note that we may differentiate the equation again with respect to time to obtain $x \leftrightarrow \dot{q} = i$. Note however that we'll consider this equivalence only when we deal with observables in the laboratory. In the rest of the analysis, we use the former. Further, observe that we could've easily used $C \leftrightarrow m$, $1/L \leftrightarrow \gamma$ in this case, but this will not work in general for obvious reasons. Again, we may use $q_n = Ae^{i(kna-\omega t)}$ to obtain

$$\begin{aligned} \dot{q} &= -i\omega q \\ \ddot{q} &= \omega^2 q \\ \implies \quad L\omega^2 &= C^{-1}e^{ikna} + C^{-1}e^{-ikna} - 2C^{-1} \\ \implies \quad \omega &= \sqrt{\tfrac{2}{LC}(1 - \cos(ka)} \end{aligned}$$

Note here that kna is reminiscent of the solution we used in the spring mass system where a had some significance (interatomic distance), however now there's none. Further, we only use the product ka.

Now in the usual case, we impose a periodic boundary condition, which (claim) results in giving N distinct solutions for ω by quantizing k as $\frac{2\pi m}{Na}$(where N is the number of entities). [can be proven] This is also expected since for N oscillators, there should be N normal modes.

However, in the given experimental setup, viz. our case, we do not have a periodic boundary. Infact, the differential equations don't fully capture the physical system because the end points of the 1D chain, don't satisfy the equations. To overcome the issue, we imagine the chain to be infinite in length and consider only a part of it. We therefore hypothesize that we are giving an input charge $q_0(t) = Ae^{-i\omega t}$ on the zeroth capacitor (just a label) and read the charge at the last capacitor (due to the inductor on the left), viz. $q_{N-1} = Ae^{-i(ka(N-1)-\omega t)}$ (although experimentally we're measuring current which is equivalent as was explained earlier). Since we're taking the real part of these quantities, we find that if we plot q_0 on the X-axis and q_{N-1} on the Y-axis using the XY mode on the oscillator, we get a graph of $\cos(\omega t)$ vs $\cos(\omega t - \phi)$, where $\phi = ka(N-1)$. Obviously, when $\phi = 0$, we'll get a straight line and for $\phi = \pi/2$ we'll get a circle. This information can be used to find the value of ka for various values of ω, which is to say we find only those ωs for which ϕ is of the form $m\pi/2$.

Also note that there's an upper limit to $\omega = 2/\sqrt{LC}$ because, well, $\cos(ka)$ is bounded.

Next consider the following LC circuit. We can write down using Kirchoff's law the following:

$$-L_1\ddot{q}_n - C_2^{-1}(q_n - q_n') + C_1^{-1}(q_{n-1}' - q_n) = 0$$

$$-L_2\ddot{q}_n' - C_1^{-1}(q_n' - q_{n+1}) + C_2^{-1}(q_n - q_n') = 0$$

(claim) It is an interesting fact that regardless of which of the following cases we take

- $L_1 = L_2 = L$ and capacitors different
- $C_1 = C_2 = C$ and inductors different

the expression for ω is identical, given we interchange in one of the expressions, Ls with Cs.

If we consider the physical case, viz. Cs different, and $L_1 = L_2 = L$, and use $q_n = Ue^{i(kna-\omega t)}$, $q_n' = Ve^{i(kna-\omega t)}$ we obtain by equating the determinant of the algebraic equations to zero,

$$\left(\omega^2 - \frac{1}{LC_1} - \frac{1}{LC_2}\right)^2 = \left(\frac{1}{LC_2} + \frac{e^{-ika}}{LC_1}\right)\left(\frac{1}{LC_2} + \frac{e^{ika}}{LC_1}\right)$$

which finally yields

$$\omega = \frac{1}{LC_1} + \frac{1}{LC_2} \pm \sqrt{\left[\frac{1}{LC_1}\right]^2 + \left[\frac{1}{LC_2}\right]^2 + \frac{2}{L^2C_1C_2}\cos ka}$$

So using this we can plot $\omega(k)$ against k; obviously for each value of k we'll get two values of ω. Now the smallest separation between two values of $\omega(k)$ will be

$$\Delta\omega \equiv 2\sqrt{\left[\frac{1}{LC_1}\right]^2 + \left[\frac{1}{LC_2}\right]^2 - \frac{2}{L^2C_1C_2}} = 2\left(\frac{1}{LC_1} - \frac{1}{LC_2}\right)$$

. If instead we plot k vs ω, there'll be a region of ω, namely

$$\omega \in \left(\frac{1}{LC_1} + \frac{1}{LC_2} - \frac{\Delta\omega}{2}, \frac{1}{LC_1} + \frac{1}{LC_2} + \frac{\Delta\omega}{2}\right)$$

for which there'll not exist any value of k. Thus we expect a bandgap. Also note that k is defined upto a multiple of $2\pi/a$. One final point that needs to be mentioned in addition is that even though $\omega(k)$ is multi-valued (viz. for a given k there're two ωs), when we restrict k to $(0, \pi/a)$, then the inverse function $k(\omega)$ is single-valued.

The one important caveat in this case is that for determining the value of ka from the observation we must realize that $n \in \{0, 1, 2, 3, 4\}$ because we used q_n and q_n' which together repeat only 5 times. Thus we must have $\phi = ka(\frac{N}{2} - 1)$.

3 Procedure

1. Connect the outputs to the Oscilloscope and set to the XY mode.

2. Put both the switches in the direction of the mono-atomic label.
3. Set to the low frequency mode (the apparatus)
4. Set the amplitude to be roughly the maximum
5. Using the knobs R_1 and R_2 and by adjusting the frequency, we can adjust the X and Y amplitude such that for $\phi = \pi/2$ we roughly get a circle (definition of ϕ is given in the previous section)
   (a) It is expected that changing the frequency will change the axis of the ellipse (of which circle will be a special case) and finally into a line and so on.
   (b) Changing R_1 and R_2 shouldn't change the axis, but infact it was observed that changing R_1 does change the frequency
6. Change the frequency to change between consecutive figures of the form \ O / O and so on.
7. This must be repeated till the frequency abruptly drops to zero for a reasonable range of the frequency.
8. Repeat the procedure with both switches in the direction of the di-atomic (although it is misleading strictly speaking) label.

4 Observations

4.1 Time Summary

5 Result

Both the mono-atmoic and 'di-atomic' (read the next section) yielded results that matched theoretical predictions.

6 Explorable Questions

1. What is the spring mass analogue when we replace $L \leftrightarrow C$
2. How is it that $\omega(k)$ found in the more general case, is identical if in one of the following situations, we apply $L_x \leftrightarrow C_x$
   (a) $L_1 \neq L_2$ and $C = C_1 = C_2$
   (b) $L = L_1 = L_2$ and $C_1 \neq C_2$

These questions relate the physical case of di-atomic interactions[1] with the physical situation created in the apparatus. They are mathematically distinct in terms of dynamics viz. you can't relate L and Cs with ms and γ (spring constant); but for the quantity ω we can find such a relation.

7 Critique

The manual provided was found to be mostly misleading. However it did motivate some of the problems listed. The procedure however was found to be mostly accurate.

The most important theoretical improvement that can be made is the realization that our physical system is not properly modeled by the differential equations we wrote down. We have accordingly motivated the use modified formulae for obtaining ϕ as stated which gives much more accurate results experimentally than quoted in the manual. We were also able to account for the various ranges where ω has no corresponding k which was confirmed experimentally.

No experimental improvements could be made/suggested as the apparatus is essentially an inaccessible black box.

[1] See for instance Classical Harmonic Theory, Solid State Physics, Ashcroft and Mermin