Symmetry

Finite Subgroups of the Rotation Group

SP Status

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Application of the Famous Formula: The famous formula is:

$$\sum_{i=1}^{k}(1-\frac{1}{r_i}) = 2\times(1-\frac{1}{N}) \tag{1}$$

Recalling that N is the order of the group which is not trivial, hence $N > 1$. Also, N is an whole number, and therefore the smallest value it can have is 2. So the RHS ≥ 1. Also, as $N \to \infty$, the RHS $\to 2$, but remember N is finite. So effectively the $1 \leq$ RHS < 2. Also, each term in the LHS $\geq \frac{1}{2}$, since $r_i \geq 1$.

Now since the LHS must equal the RHS, there can't be more than 3 terms of LHS, else the sum would become ≥ 2, which the RHS can't reach for any value of N.

Dividing this into 3 and classifying, we get
One orbit:
So for a single orbit, $k = 1$. So, the LHS becomes

$$1-\frac{1}{r} < 1 \tag{2}$$

while the RHS

$$2\times(1-\frac{1}{N}) \geq 1 \tag{3}$$

So this case is impossible.
Two orbits:
For two orbits, we would have

$$(1-\frac{1}{r_1}) + (1-\frac{1}{r_2}) = 2-\frac{2}{N} \tag{4}$$

which is the same as

$$\frac{1}{r_1}+\frac{1}{r_2} = \frac{2}{N} \tag{5}$$

Doubt | From this itself, Artin concludes that since r_i divides N, the equation will hold only when $r_1 = r_2 = N$. I was unable to see why this was so. However, a little manipulation got me to the same result, but it still doesn't seem obvious to me. What am I missing?
Here's what I'd done.
Replaced N once with $r_1 n_1$ and one with $r_2 n_2$ to get

$$\frac{1}{r_1}+\frac{1}{r_2} = \frac{1}{r_1 n_1}+\frac{1}{r_2 n_2} \tag{6}$$

rearranged

$$\frac{1}{r_1}(1-\frac{1}{n_1}) = \frac{1}{r_2}(\frac{1}{n_2}-1) \tag{7}$$

simplified

$$\frac{1}{r_1 n_1}(n_1-1) = \frac{1}{r_2 n_2}(1-n_2) \tag{8}$$

since $r_1 n_1 = r_2 n_2$

$$n_1+n_2 = 2 \tag{9}$$

And since each orbit must contain atleast one element, $n_i \geq 1$. So the only possible solution is
$n_1 = n_2 = 1$
$\Rightarrow r_1 = r_2 = 1$.
So since there are only two poles, both fixed by all elements in G hence, the only possibility (of the type of elements in the group) is rotation about a single axis, passing through both these poles (read points!).

Doubt Context | Now as Artin says, is the most interesting case.
Three orbits: What the text says till Case 1: $r_1 = r_2 = 2$ and $r_3 = k$ s.t. $N = 2k$, is clear. For further clarity its given as
$r_i = 2, 2, k; \quad n_i = k, k, 2; \quad N = 2K$
It goes on to then say that there's one pair of poles p, p' making the orbit O_3. So far so good as it readily follows from the value of n_3.
Doubt | This is where I'm stuck.
It asserts, *Half* of the elements of G fix p, and the other *half* interchange p and p'.
Elephant in the room is, why Half?
This is what I had in mind, but I'm not sure.

My Analysis:
Now we know that O_3, contains 2 elements since n_3 is 2. For a pole in this orbit, say p as used above, $r_p = r_3$ [terms have the meaning as per their prior definition]. This means that the stabilizer of the pole, has order k and these are rotations about the axis passing through the origin and the pole (read point) p. Since there are only two poles, the other pole p' must lie on this very axis. Thus, the same K stabilizers, stabilize it. However the group has $2K$ elements. The other elements are NOT stabilizers and hence MUST interchange p and p'. So they are 'reflections' which in $\mathbb{R}^3$ become rotations by π about a line perpendicular to the line containing the poles. So half of them are fix p, other half interchange p and p'.