Symmetry

Finite Subgroups of the Rotation Group

SP Status

Atul Singh Arora June 9, 11 & 12, 2012

Dear Sir,

June 11, 2012

Today the opposite of what usually happens happened. I initially did get stuck, but I was able to resolve that issue without having to write it down (I have marked it). However it was when I started to put it on record for future reference that I realize, it was just an illusion, for I hadn't in fact understood the concept after all! Fortunately I did eventually.
Also, I haven't been able to finish the analysis for this section still. The case of symmetries of an octahedron and the most interesting case of icosahedral groups has still not been done. I would most likely be able to finish that part by tomorrow.

June 12, 2012

Strange as it may sound, I now seem to feel that I can BEGIN learning group theory, as though all the work I've done so far, was like learning words. I glanced through the next chapter and that seems to interest me too! There is also another mathematical problem that might be related and one that I would like to study. Could we please meet once before Friday to discuss this? Also, I need to send an official document, certifying that I'm working here, for KVPY. I will be in Delhi for a week, from Saturday (June 16) to Sunday (June 24) and would resume work only once I return.

Application of the Famous Formula: The famous formula is:

$$\sum_{i=1}^{k}(1-\frac{1}{r_i}) = 2\times(1-\frac{1}{N}) \tag{1}$$

Recalling that N is the order of the group which is not trivial, hence $N > 1$. Also, N is an whole number, and therefore the smallest value it can have is 2. So the RHS ≥ 1. Also, as $N \to \infty$, the RHS $\to 2$, but remember N is finite. So effectively the $1 \leq$ RHS < 2. Also, each term in the LHS $\geq \frac{1}{2}$, since $r_i \geq 1$.

Now since the LHS must equal the RHS, there can't be more than 3 terms of LHS, else the sum would become ≥ 2, which the RHS can't reach for any value of N.

Dividing this into 3 and classifying, we get
One orbit:
So for a single orbit, $k = 1$. So, the LHS becomes

$$1-\frac{1}{r} < 1 \tag{2}$$

while the RHS

$$2\times(1-\frac{1}{N}) \geq 1 \tag{3}$$

So this case is impossible.
Two orbits:
For two orbits, we would have

$$(1-\frac{1}{r_1})+(1-\frac{1}{r_2})=2-\frac{2}{N} \tag{4}$$

which is the same as

$$\frac{1}{r_1}+\frac{1}{r_2}=\frac{2}{N} \tag{5}$$

Doubt | From this itself, Artin concludes that since r_i divides N, the equation will hold only when $r_1 = r_2 = N$. I was unable to see why this was so. However, a little manipulation got me to the same result, but it still doesn't seem obvious to me. What am I missing? Here's what I'd done.
Replaced N once with r_1n_1 and one with r_2n_2 to get

$$\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{r_1n_1}+\frac{1}{r_2n_2} \tag{6}$$

rearranged

$$\frac{1}{r_1}(1-\frac{1}{n_1})=\frac{1}{r_2}(\frac{1}{n_2}-1) \tag{7}$$

simplified

$$\frac{1}{r_1n_1}(n_1-1)=\frac{1}{r_2n_2}(1-n_2) \tag{8}$$

since $r_1n_1 = r_2n_2$

$$n_1+n_2=2 \tag{9}$$

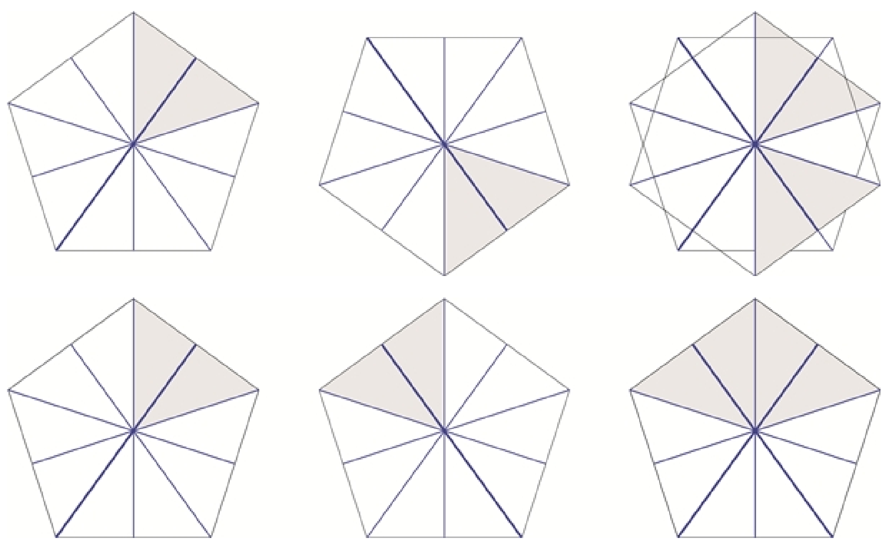
And since each orbit must contain atleast one element, $n_i \geq 1$. So the only possible solution is
$n_1 = n_2 = 1$
$\Rightarrow r_1 = r_2 = 1$.
So since there are only two poles, both fixed by all elements in G hence, the only possibility (of the type of elements in the group) is rotation about a single axis, passing through both these poles (read points!).

Doubt Context | Now as Artin says, is the most interesting case.
Three orbits: What the text says till Case 1: $r_1 = r_2 = 2$ and $r_3 = k$ s.t. $N = 2k$, is clear. For further clarity its given as
$r_i = 2, 2, k;\quad n_i = k, k, 2;\quad N = 2K$
It goes on to then say that there's one pair of poles p, p' making the orbit O_3. So far so good as it readily follows from the value of n_3.
Doubt | This is where I'm stuck.
It asserts, *Half* of the elements of G fix p, and the other *half* interchange p and p'.
Elephant in the room is, why Half?
This is what I had in mind, but I'm not sure.

My Analysis:
Now we know that O_3, contains 2 elements since n_3 is 2. For a pole in this orbit, say p as

used above, $r_p = r_3$ [terms have the meaning as per their prior definition]. This means that the stabilizer of the pole, has order k and these are rotations about the axis passing through the origin and the pole (read point) p. Since there are only two poles, the other pole p' must lie on this very axis. Thus, the same K stabilizers, stabilize it. However the group has $2K$ elements. The other elements are NOT stabilizers and hence MUST interchange p and p'. So they are 'reflections' which in $\mathbb{R}^3$ become rotations by π about a line perpendicular to the line containing the poles. So half of them are fix p, other half interchange p and p'.

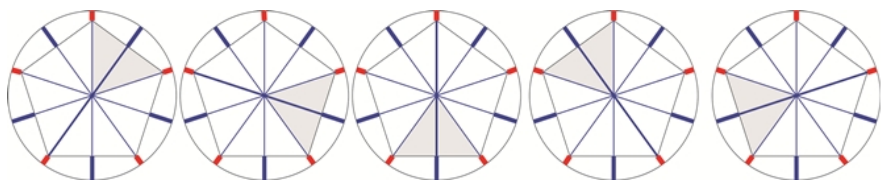
So effectively, there are K rotations about the axis pp'. The rest of the rotations have their axis contained in a plane perpendicular to the pp' axis and passing through its mid point. This is so that each such rotation does infact swap the poles p and p'. **Doubt** | This is where the story gets even more interesting. I initially thought like so. I imagined a point in the said plane. Then I pictured it getting rotated by an angle $\theta = 2\pi/K$ (why this, the rigorous proof is given in the text, basically its because the group [since they're stabilizers] of rotations is finite) along the pp' axis. The orbit of the point makes the vertices of a regular K-gon. Then I went on to imagined "reflecting" the K-gon, for the orbit is obtained by operating the point with all group elements. However, here's the mistake I made. I ended up reflecting the pentagon (that's what I'd imagined for simplicity) along an axis which doesn't exist in the group! This resulted in expansion of the orbit and that kept me startled for a while, until I realized that the pentagon must be "reflected", and by that I mean, rotate by π along one of the axis contained in the plane. The result in that case of the "reflection" is again the same pentagon. So the orbit obtained in general will be K-gon.



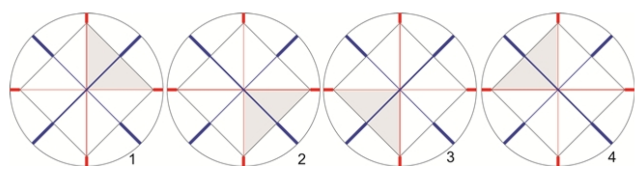
The trouble however is that there are supposed to be two distinct orbits, each with K distinct elements, with 2 stabilizers. Now one stabilizer is identity, and so the other must be rotation by *some* angle (the angle must actually be π, since p and p' are the only points in their orbit) along an axis containing the element.
As has been shown, one of the orbits consists of the poles corresponding to the vertices of the k-gon. (I've come back to poles since poles are what we derived the "famous formula" using.)

The text says that the vertices & the centres of the faces of the K-gon *correspond* to the remaining poles.

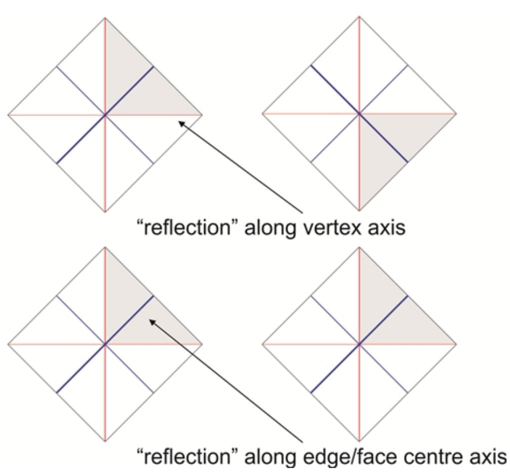
So essentially, the centre of faces, which in this case may even be taken to be the centre of the edges, also form an orbit such that each element has 2 stabilizers (one is identity, the other rotation by π [effectively a reflection along the chosen axis]).
Now two interesting cases arise. When K is odd, say 5 for a pentagon, the poles made by opposite face/edge centre and vertex, correspond to the same axis. Yet, when the pentagon is rotated, the vertices go to the vertices, and the face/edge centres go to themselves, preserving 2 different orbits. Take a moment to realize that there aren't any other poles such that their stabilizer is of order two and they restrict the orbit of p and p' to p, p'.
In this case, the reflections as was shown earlier, map the pentagon to a pentagon so the orbits are preserved.



And when K is even, say 4, the poles made by opposite faces/edge centres form an axis, and those made by opposite vertices, form a different axis. And again, the poles made by the faces goto faces (under rotation by $2\pi/K$) & correspondingly the poles made by the vertices, goto vertices. So the orbits continue to be separate!



NOTE: I did not include reflection in the analysis above since again, the reflections along both kind of axis, map the square to a square, and the orbits are thereby preserved.



With that done, lets move on to the next case.
The arguments in the book are easy to follow. The conclusion is given as follows:

(a) $r_i = 2, 3, 4;\ n_i = 6, 4, 4;\ N = 12$

    The poles in the orbit O_3 are the vertices of a regular tetrahedron, and G is the tetrahedral group T of its 12 rotational symmetries.

(b) $r_i = 2, 3, 4;\ n_i = 12, 8, 6;\ N = 24$

    The poles in the orbit O_3 are the vertices of a regular octahedron, and G is the octahedral group O of its 24 rotational symmetries.

(c) $r_i = 2, 3, 5;\ n_i = 30, 20, 12;\ N = 60$

    The poles in the orbit O_3 are the vertices of a regular icosahedron, and G is the icosahedral group I of its 60 rotational symmetries.

    Take a note and verify the fact, that r_i represents the number of edges, the number of faces and the number of vertices respectively. Why that's happening (aside from the ordering) is partially answered if one thinks of them as caused by the rotation of different stabilizers (of varied order) to form different orbits.

In Artin the explanation for the number of symmetries for the first two shapes is probably left as an exercise.
However an attempt to understand the concept of "truncated polyhedron" seems futile without first deriving these simpler, more intuitive results.

(a) So for a tetrahedron, we can begin the analysis by considering the rotational symmetry about its vertices. Note that the symmetrical axis about a vertex, passes through the centre of the face on the opposite side. So we will not count the rotational symmetry for the faces. So we have a 3 fold symmetry, of which one element in the group will be identity, which will be common, so there are 2 distinct group elements corresponding to each vertex. Also, there are 4 vertices. So there are $2 \times 4 = 8$ non-identity elements in the group, because of symmetry of vertices (and faces). The remaining elements come from the rotation along an axis through the centre of edges, but we must be careful here as well, for each axis of symmetry passes through the centre of 2 edges. Also, the symmetry is 2 fold, and only 1 element is therefore non-identity. There are 6 edges, 3 of which share an axis, so we have $1 \times 3 = 3$ more non-identity elements in the group.
    The total then becomes $8 + 3 + 1$ (the identity element) $= 12$ and that's precisely what was expected.

(b) Now the next shape is an octahedron. So first let's resolve the simplest rotational symmetries. Consider rotations about the axis joining opposite vertices, which have a 4 fold symmetry. There are 6 vertices and therefore 3 such axis. So total contribution from this rotational symmetry will be $3 \times (4 - 1) = 9$. Next consider axis created by joining opposite edge centres. There are 12 edges & thus 6 edge centres. The rotational symmetry is 2 fold. Thus their contribution is $6 \times (2-1) = 6$. The last symmetry is easier to visualize if we picture a cube, and its centres of faces forming the octahedron. Now consider the body diagonal of the cube. There are 4 body diagonals and around each there is a 3 fold symmetry. So the final contribution will be $4 \times (3-1) = 8$. So the total number of symmetries becomes $9+6+8+1$ (identity) $= 24$, as was expected and is given in the text.

Doubt | Interesting Observation: As was shown in the discussion for the k-gon, poles corresponding to vertices (or edge centres) do NOT span the entire set of poles. Similarly in a cube, the edge centres don't span the poles of a cube, but the figure they span is called a *truncated polyhedron*, as claimed by the text.

(c) To justify this assertion, Artin uses an entire page. So this won't be quite as straight forward. So let's start. Let V be the orbit O_3 of order 12. Thus, the order of stabilizer of each pole in the orbit will be 5. Now we choose any pole p, present in V and "declare it" to be the north pole of the unit sphere. Now that is sufficient to derive an equator from it (a unit circle with centre at origin, that lies on a plane perpendicular the line containing p and the origin) and also a south pole (diametrically opposite point to p). Now we let H be the stabilizer of p which as stated earlier, must have order 5. Thus H is a cyclic group, generated by a rotation (say x) about p with an angle $2\pi/5$. **Doubt** | At this stage Artin asserts there must be 2 H-orbits of order 1 and then goes on to identifying them as north & south poles.
Continuing with my analysis, since H consists of orthogonal operations, thus if p is stabilized, so will its diametrically opposite point be, i.e. the south pole. So H has 1 orbit with 2 elements.
Now since H has order 5, the remaining poles (i.e. poles that have a stabilizer of order 5), form two H-orbits of order 5. **Doubt** | Now on what rigorous mathematical grounds should I back that argument? I have to somehow show that all poles that have a stabilizer of order 5, under the action of H, form an orbit of order 5.
Back to Artin, by symmetry, either, one of the H-orbits is in the northern hemisphere and one is in the southern hemisphere, or else, both are on the equator. Let us name the orbits as $\{q_0, .., q_4\}$ and $\{q'_0, .., q'_4\}$, where $q_i = x^i q_0$ and $q'_i = x^i q'_0$. (recall x is the generator of H)
Let $|x, y|$ denote the spherical distance between the points x and y on the unit sphere. We note that $d = |p, q_i|$ is independent of $i = 0, ..., 4$.
Doubt | Now the explanation that Artin provides here, is that $\exists\, h \in H$ s.t. $hq_0 = q_i$. I initially couldn't see how the result follows from this, but since H is just rotations along the polar axis, the spherical distance of a point from the pole remains unchanged, under the action of H.
The same argument can be used to prove $d' = |p, q'_i|$ will also be independent of i. Now the only two values $d = |p, p_i|$ can take are: $0,$ (say) d. Similarly the only two values $d' = |p, q'_i|$ will take would be: (say) d', π. So the values $|p, p'|$ where $p' \in V$ (the orbit), are $0, d, d', \pi$.
By the definition of d (that is the fact that the elements q_i were defined to be the ones between the equator and the north pole, or on the equator), $d \leq \pi/2$. Similarly we can say $d' \geq \pi/2$.

Now let's show that the orbit with 5 elements does NOT lie on the equator. For this, let's first note that the operation of G on V is transitive (simply because V was one of the orbits, so the conclusion follows from the definition of transitivity). This essentially means that we could've picked any p as the north, and $|p, p'|$ would've had the same 4 possible values. Now this implies, if we choose q_i as our north pole (but following the old notation), then $|q_i, q_{i+1}| = d$ (try visualizing to help your intuition). However, there are 5 poles in the orbit q_i so their angular separation can NOT be $= \pi/2$, for the sum of angles between them MUST add up to 2π (and not $5\pi/2$!). So this implies their angular separation, which we just showed equals d, must be $< \pi/2$ and hence, the poles do NOT lie on the equator.
Since $|p, q_i| = d = |q_i, q_{i+1}|$, the north pole p and points of the orbit q_i, q_{i+1} form equilateral triangles! Since all the triangles share a point p, their are five congruent triangles with a common vertex, forming the face of an Icosahedron.
Doubt | Artin concludes from the above that poles of the group, correspond to the vertices of an Icosahedron. However to me it seems incomplete since the relation between q and q' hasn't explicitly been derived, although calculation of d and d' (which can be easily computed) should be sufficient to show that the rest of the faces are also built off of the same equilateral triangles.

The images of the polygons used in this document were specifically created for this purpose.