Symmetry
Finite Subgroups of the Rotation Group
SP Status

Atul Singh Arora   June 8, 2012

Sir,

Writing Mathematics using the conventional windows tools like Word, Outlook etc. was becoming rather impossible and it had been resulting in a break of flow of my thoughts. Consequently I switched to LATEX and Sublime Text. Also I realized that when I start writing to you about a problem systematically, I usually end up solving it myself.

The document henceforth would consist of the work I've done and understood. Things I haven't been able to understand, I would mark with a **Doubt**, so that you can spot it easily and don't have to go through the entire text, while at the same time, providing you with the context so that you can reply accordingly.

POLE OF GROUP ELEMENT: Let G be a finite subgroup of SO_3, of order $N > 1$. For now, consider only the rotation elements (of $\mathbb{R}^3$) $\in G$. Poles for such an element $h(\neq 1) \in SO_3$ are defined as the intersection points of the axis of rotation with the unit sphere $\mathbb{S}^2$. h can't be identity since the definition of pole requires existence of an axis. Clearly, for each such h, there are 2 poles.

POLE OF A GROUP: Similarly in general, pole is defined for an element $g(\neq 1) \in G$. This pole can also be referred to as pole of the group.

Thus, a pole of G, is a point, fixed by a group element $g \neq 1$.

G-ORBITS AND POLES: According to Artin, and I quote "The set $\mathcal{P}$ of poles of G, is a union of G-orbits. So G operates on $\mathcal{P}$."

Here's the analysis of the second part of this statement. The set $\mathcal{P}$ of poles is basically a set of points. Each of these points can be "operated" upon by an element of G. In accordance with the definition of the word "operate" in terms of group action, we must verify that the point obtained after multiplication with an element $g(\neq)1 \in G$ is also a pole. The other properties of group action are easy to verify.

Let us prove it before continuing.
Consider a pole $p \in \mathcal{P}$.
Now $\exists\, h(\neq 1)$ s.t. $hp = p$ [from the definition of pole]
Let g be an element of G. We have to show that $gp = q(say)$ is also a pole,
that is, $\exists\, k(\neq 1)$ s.t. $kq = q$
Lets replace q with gp in both sides, and p with hp in the RHS in the equation above.
So we need to solve for k in the equation
$\Rightarrow kgp = ghp$

$\Rightarrow k = ghg^{-1}$

Since G is a group, ghg^{-1} exists and thus k exists, proving gp to also be a pole of the group (specifically of k). Hence, we have shown G operates on $\mathcal{P}$.

Now for the first part of the statement. First, orbits are defined for a particular element of a set. However, if the set is the same as the orbit, then it needn't be specified.

Immediate Context before the doubt | In this case, let's fix a pole. The orbit of this pole under the group action would be the set of all poles obtained by operation of all $g \in G$. We just proved that each element of the group, when operates on a pole, creates another pole (of an element present in the group). Union of all the orbits must therefore be equal to $\mathcal{P}$, as $\mathcal{P}$ was defined to be the set of all poles of G.

Doubt | However, Artin has presented this in the other way as can be seen from the quote. What am I missing here?

SPIN: For a given $g \in G$, spin is the number of unordered pairs (g, p), where p is a pole of the group G.

Since each rotation (excluding identity) has two poles, there are two unordered pairs associated, and thus the spin of each such rotation is two.

DERIVING A FAMOUS FORMULA: The first objective here is to find a relation between the order of stabilizers for different p, and the order of the group, G.

Now, for a given pole p, all elements $g \in G$, that leave p unchanged, form a set G_H. G_H is a cyclic subgroup and also a stabilizer by definition. The fact that its a subgroup can be verified easily as was done for the kernel. It is cyclic because the group G is finite. It then follows that G_H is generated by rotation by the smallest angle $\theta(>0)$ present in the group. If the order of the group G_H is given by r_p, then $\theta = 2\pi/r_p$.

Doubt | *(Supplement explanation)* Is it correct to note here that the stabilizers G_H for different poles p, will either form the same subgroup, or be disjoint (can be readily verified). The case would be former if and only if the poles are a result of intersection of the same axis. The latter would happen for all other cases. Also, if such subgroups were created for every pole p of G, their union would exhaust the group G.
Now if all such stabilizers, minus their identity element, are made into a union, they would consequently contain twice as many elements as there are in (G minus the Identity element). i.e.

$$\sum_{p \in \mathcal{P}} r_p - 1 = 2 \times (|G| - 1) \tag{1}$$

It took me a while to get here. This relation is exactly the same as that given in Artin, but I would want to confirm if the reasoning is correct.

(Textbook method explained) Now since p is a pole, the stabilizer G_H will contain at least one element other than identity, and thus $r_p > 1$. Consequently $r_p - 1$ elements (since we're

excluding identity), stabilize p. Each of these elements thus has a spin two.

So every group element except the identity has two poles, implying its spin is 2. Taking $|G| = N$, there are $2(N-1)$ spins. Recalling that spin means the number of unordered pairs (g, p) for a given p, total spins would be the total number of unordered pairs (g, p). Now if we sum over all $r_p - 1$ (for excluding identity), then also we are counting all such pairs. So the same relation (equation 1) follows from this school of thought.

So lets move forward. To simplify the LHS of equation 1, we will use the counting formula.

We already know from the counting formula (Size of a Group $G=$ Order of coset of $H \times$ Number of Cosets) that

$$|G| = |G_H| \times |\text{Orbit of } G_H| \qquad (2)$$

since there is a bijective map between the orbit of G_H and the cosets of G_H.

Let n_p denote $|\text{Orbit of } G_H|$. Rewriting both equations in terms of N, r_p and n_p, we get

$$N = r_p \times n_p \qquad (3)$$

Now we can see that if two poles p and p' are in the same orbit, then the order of their orbits is the same, i.e. $n_p = n_{p'}$. Equation 3 demands the order of their Stabilizers to also be the same, i.e. $r_p = r_{p'}$.

Let's us arbitrarily denote different orbits by $O_1, O_2, ...O_k$. Now we note that if $n_i = n_p$ (note that the p is the same as it was in the previous context, a particular pole in the summing, although this result is not dependant on it) so by equation 3 we have $r_i = r_p$.

It is right here that I got stuck, which caused me to initiate writing like this in the first place. Now the trick here is to realize this very essential fact which is as follows.

We are summing over all poles p in equation 1. Now if $\exists p$ s.t. $p \in O_i$, then $\exists n_i$ poles in the orbit, each with the same number of stabilizers, i.e. $(r_p - 1) = (r_i - 1)$ since $r_p = r_i$.

Now we can, from the left side of the equation, take out the contribution of all such poles to the total spin, and express it as $n_i \times (r_i - 1)$. Also, each pole must belong to some orbit, therfore the entire sum (the LHS) may be written as

$$\sum_{i=1}^{k} n_i \times (r_i - 1) = 2 \times (N - 1) \qquad (4)$$

Take r_i common from LHS and divide by N on both sides, to obtain

$$\sum_{i=1}^{k} (1 - \frac{1}{r_i}) = 2 \times (1 - \frac{1}{N}) \qquad (5)$$

And that was the 'famous' formula I'd never even seen before! As the book says, this formula might look small, but its a very strong tool.

The power of this function will be explored tomorrow.