Is Quantum Mechanics truly Contextual?

Atul Singh Arora[1] and Arvind[1]

[1] *Indian Institute of Science Education & Research (IISER), Mohali*[*]

Violation of the KCBS inequality doesn't imply contextuality, since we explicitly show how a non-contextual hidden variable (NCHV) theory can yield a violation. The class of theories which can't violate the inequality, are thus refined to 'non-contextual multiplicative hidden variable' (NCMHV) theories. The said NCHV theory, which violates the KCBS inequality, infact reproduces all predictions of Quantum Mechanics (QM). We present a test of multiplicativity, which each multiplicative theory, including NCMHV must and does satisfy. QM itself fails it, ruling out a new class of theories, including NCMHV. Effectively we show that we can describe quantum phenomena, without contextuality entering the picture.

I. INTRODUCTION

There are two tests of hidden variable theories, which have gained popularity in the physics community, Bell's Locality test and Kochen & Specker's Contextuality test. Both were designed to rule out certain classes of theories, and with them, their associated properties. In case of the former, locality is given up, although one still preserves no-signalling. In the case of the latter, it is believed that non-contextuality must be given up. Since the class of theories which are ruled out, satisfy our intuitive understanding of nature (classical properties), their exclusion entails that quantum mechanics has these non-classical features, which can be harnessed as a resource to perform tasks that were otherwise impossible.

II. CONTEXTUALITY AND MULTIPLICATIVITY

Consider a set of 9 observables, as listed in Mat. (1), which take the value ± 1, where $\hat{C}_j := \prod_i \hat{A}_{ij}$ and $\hat{R}_i := \prod_j \hat{A}_{ij}$ $(i, j \in \{1, 2, 3\})$.

$$\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} \\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} \\ \hat{A}_{31} & \hat{A}_{32} & \hat{A}_{33} \end{bmatrix} \tag{1}$$

This is motivated by the Peres Mermin (PM) construction, where observables along a row (and also column) commute. For an appropriate choice of $\hat{A}_{ij}$, they showed that

$$\hat{C}_3 = -\hat{1},\ \hat{C}_j = \hat{1}\,(j \neq 3),\ \hat{R}_i = \hat{1},\ \forall\, j, i. \tag{2}$$

where $\hat{1}$ is identity. Consequently,

$$\hat{\chi}_{\text{KCBS}} = \hat{R}_1 + \hat{R}_2 + \hat{R}_3 + \hat{C}_1 + \hat{C}_2 - \hat{C}_3 = 6.\hat{1} \tag{3}$$

for QM, given the PM situation. Experimentally, one evaluates $\langle \hat{\chi}_{\text{KCBS}} \rangle$ by measuring $\left\langle \hat{R}_i \right\rangle$ and $\left\langle \hat{C}_j \right\rangle$.

Now imagine a theory that assigns values to these observables, as given in Mat. (4), representing a given state, which satisfies all conditions listed in Eqns. (2), except $\hat{C}_3 \to 1$ instead of -1.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{4}$$

One can check that it is not easy to modify the assigned values, s.t. all constraints are satisfied. In fact it is impossible [put citation]. It clearly follows that such simplistic 'deterministic' theories, can't explain all predictions of Quantum Mechanics (QM). Explicitly, for the aforesaid

$$\langle \hat{\chi}_{\text{KCBS}} \rangle \leq 4, \tag{5}$$

and this holds quite generally, for such theories, where the ensemble is taken to be identical copies of the given state. To modify the nature of the theory itself, consider $\hat{A} = \hat{\sigma}_x \otimes \hat{\sigma}_y$ and $\hat{B} = \hat{\sigma}_y \otimes \hat{\sigma}_x$, which are s.t. $[\hat{A}, \hat{B}] = 0$. Simultaneous eigenstates $|a, b\rangle$ can be constructed s.t. $\hat{A}\,|a, b\rangle = a\,|a, b\rangle$ and $\hat{B}\,|a, b\rangle = b\,|a, b\rangle$. Consequently, $f(\hat{A}, \hat{B})\,|a, b\rangle = f(a, b)\,|a, b\rangle$, where f is an arbitrary function. It appears reasonable therefore, that if a theory assigns $m(\hat{A}) = a$ and $m(\hat{B}) = b$, the value assigned to $\hat{C} := \hat{A}\hat{B} = -\hat{\sigma}_z \otimes \hat{\sigma}_z$ in particular, must be $m(\hat{C}) = ab$, where $m(\hat{*})$ represents the value a measurement will yield, corresponding to the observable. However this is not a requirement, since for the state $|11\rangle := |1\rangle \otimes |1\rangle$ written in the computational basis, $m(\hat{C}) = -1$ while $m(\hat{A}) = \pm 1$ and $m(\hat{B}) = \pm 1$, independently, according to QM, with probability half. Thus, even if according to some theory, $m(\hat{A}) = 1$ and $m(\hat{B}) = 1$, it can still assign $m(\hat{A}\hat{B}) = -1$. Accordingly, we define a new property, *multiplicativity,* which a theory is said to have, if the value assigned to commuting observables, entails that the value assigned to an arbitrary function of these observables, equals the value obtained by replacing the observables with their corresponding values, viz. if $m(f(\hat{A}, \hat{B})) = f(m(\hat{A}), m(\hat{B}))$, $\forall\, f$, $\hat{A}$ and $\hat{B}$ s.t. $[\hat{A}, \hat{B}] = 0$. Specifically it requires $m(\hat{A}\hat{B}) = m(\hat{A})m(\hat{B})$.

If we now consider theories that are not multiplicative (henceforth termed as non-multiplicative theories), then it is almost obvious that an assignment can be made that violates inequality (5). For instance, imagine a theory that assigns values to $\hat{A}_{ij}$ as in Eqn. (4), and $m(\hat{R}_i) = 1$, $m(\hat{C}_1) = m(\hat{C}_2) = 1$,

[*] ms11003@iisermohali.ac.in, to.AtulArora@gmail.com

$m(C_3) = -1$ to trivially and precisely satisfy Eqn. (2). Consequently, for this theory, $\langle\chi_{\mathrm{KCBS}}\rangle = 6 > 4$, where the average again, is done over identical ensembles.

Non-multiplicative theories are interesting for atleast two reasons. First, contrary to the apparently artificial construction exemplified and its limitless freedom in making assignments, there is already a quantum theory, which is non-multiplicative. It is completely deterministic, in principle, and all its predictions match with those of QM. The second reason, we postpone to the end of this section.

Note that we could've taken multiplicativity for granted, and tried to check if some other aspect of the theory can be changed, to still allow for a violation of inequality (5). This approach is already known and it results in the notion of *contextuality*, which requires that the values assigned to an (or strictly, at least one) observable, depends on which set of commuting observables, it is measured with. Here we construct a scheme for such a contextual assignment, as an algorithm, which will engender a deterministic assignment and will explicitly yield a violation. The aim is to see if a test can be conceived that distinguishes between non-multiplicativity and contextuality.

Imagine that initially the assignment is as given in Mat. (4). The system is assumed to be capable of remembering the last three observables that were measured. The general scheme is that upon measurement of an observable, yield the value as saved in the matrix, append the observable in the 3 element memory and update the matrix, once the context (set of commuting observables) is known, to satisfy the requirements of Eqns. (2).

If for example, we start with measuring $\hat{A}_{33}$. The system will yield $m(\hat{A}_{33}) = 1$, in accordance with the values in the matrix. The memory would read $[*, *, A_{33}]$. Since the context is not known yet, the matrix is left unchanged. Next, say $\hat{A}_{23}$ is measured. The system yields $m(\hat{A}_{23}) = 1$ and the memory becomes $[*, A_{33}, A_{23}]$. The context at this stage, is known. To satisfy $m(\hat{C}_1) = -1$, the matrix is updated to Mat. (6). Now if $\hat{A}_{13}$ is measured, the result is $m(\hat{A}_{13}) = -1$.

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{6}$$

The reader can convince him(her)self that the algorithm facilitates satisfaction of all constraints given in (2) and consequently, violates (5). As a remark, we must add that the given contextual construction is multiplicative, since we assumed $m(\hat{C}_3) = m(\hat{A}_{33}\hat{A}_{23}\hat{A}_{13}) = m(\hat{A}_{33})m(\hat{A}_{23})m(\hat{A}_{13})$ and the former non-multiplicative construction was non-contextual, as the assignment was independent of which observable is measured.

The fact that non-multiplicative theories can violate the KCBS inequality, without being contextual, is the second reason why they are interesting. It shows that a violation of the KCBS inequality, doesn't imply that the theory must be contextual. Henceforth, the terms 'Contextual Multiplicative Hidden Variable' (CMHV) and 'Non-Contextual Non-Multiplicative Hidden Variable' (nCnMHV) will be used to refer to these theories succinctly.

III. CAN CONTEXTUALITY BE DISTINGUISHED FROM NON-MULTIPLICATIVITY EXPERIMENTALLY

Before constructing a general test, we focus our attention to the two specific constructions, delineated in the previous section, that violate (5). So far, to evaluate $\langle\hat{\chi}_{\mathrm{KCBS}}\rangle$, we have been evaluating $\left\langle \hat{R}_i \right\rangle$ and $\left\langle \hat{C}_j \right\rangle$, which in turn are evaluated as, $\left\langle m(\hat{C}_j) \right\rangle = m(\hat{C}_j)$ for example, where the equality follows from the identical ensemble assumption. Note that for the aforesaid CMHV construction, although strictly we had $m(\hat{C}_j) = m_3(\hat{A}_{13})m_2(\hat{A}_{23})m_1(\hat{A}_{33})$, where $m_l(\hat{*})$ denotes measuring an observable, at time a step parametrized by l, re-ordering the measurement sequence leaves the result invariant. Thus we were justified at writing $m(\hat{C}_j) = m(\hat{A}_{33})m(\hat{A}_{23})m(\hat{A}_{13})$. Accordingly, we modify the KCBS test by replacing measurement of $\hat{C}_j$ (for example) by a sequential measurement of $\hat{A}_{3j}$, $\hat{A}_{2j}$, $\hat{A}_{1j}$, viz. we define

$$\begin{aligned}\langle\chi_2\rangle := \left\langle m^{R(1)} \right\rangle + \left\langle m^{R(2)} \right\rangle + \left\langle m^{R(3)} \right\rangle + \\ \left\langle m^{C(1)} \right\rangle + \left\langle m^{C(2)} \right\rangle - \left\langle m^{C(3)} \right\rangle\end{aligned}$$

where $m^{R(i)} = m_1(\hat{A}_{i1})m_2(\hat{A}_{i2})m_3(\hat{A}_{i3})$ and $m^{C(j)} = m_1(\hat{A}_{1j})m_2(\hat{A}_{2j})m_3(\hat{A}_{3j})$. It immediately follows that a simplistic 'deterministic' theory (as described earlier), must satisfy

$$\langle\chi_2\rangle \leq 4, \tag{7}$$

as they are multiplicative, which makes $\langle\chi_2\rangle = \langle\chi_{\mathrm{KCBS}}\rangle$. Thus a violation must signal non-classicality. Now a multiplicative theory that violates (5), will also violate (7), including our CMHV construction. However, one can violate (5) without violating (7), as is the case for the nCnMHV construction presented here.

- Try to arrive at general conclusions for the following cases:

	$\langle\chi_2\rangle > 4$	$\langle\chi_2\rangle \leq 4$
$\langle\chi_{\mathrm{KCBS}}\rangle > 4$	(nCnM+CM)?	nCnM?
$\langle\chi_{\mathrm{KCBS}}\rangle \leq 4$	?	classical

IV. QUANTUM MECHANICS AS A NON-CONTEXTUAL HIDDEN VARIABLE THEORY

- Generate the table for all observables, assuming some hidden variables and show the violation of the KCBS inequality
- Generalize the test to phase space, where the competition is more fierce (perhaps) & Discuss the RS Theory (perhaps)