Bohmian Mechanics and Contextuality in (q,p)

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Introduction

Abstract

There are atleast two known theories that describe the same physical world; Quantum Mechanics and Bohmian Mechanics (also uses the Schrödinger Equation). While the latter is not popularly known, it provides exceptional clarity about certain aspects of reality. So far, no tests are known that can distinguish these two. One fundamental difference between them is that that the latter is deterministic (in the sense that (q,p) are well defined). The purpose of this thesis is to get these theories head on; we aim to construct a theoretical situation where this type of determinism is refuted by Quantum Mechanics (using generalization of the GHZ test, contextuality etc.) and analyze it using Bohmian Mechanics. This is a step towards understanding the relation between contextuality and non-locality.

Content

I will first discuss Bohmian Mechanics and then go on to discuss the known standard tests of determinism (GHZ and contextuality). Thereafter I will describe how the GHZ test is explained from the Bohmian perspective and also discuss a generalization of the GHZ test to phase space (q,p). Finally, I'll show some simulation results which will be generalized in the future to perform the generalized GHZ test using BM.

Bohmian Mechanics [5, 4, 2]

Background

The Quantum Mechanics that is taught, is usually the one which uses the 'Copenhagen interpretation'. This interpretation asserts that the most complete possible specification of an individual system, is in terms of ψ which yields only probabilistic results. While it can be shown to be consistent, it is worth exploring the reasons for believing this assertion. David Bohm[a] in an attempt to investigate the truth behind this, constructed a theory with 'hidden variables' (positions and momenta (q,p) of particles) that in principle completely specified the system but in practice get averaged over. He was able to show that his theory yields the same results as Quantum Mechanics in all the physical situations he considered. Such a theory is worth studying because the following are at stake.

(1) Clarity: First, the widely held notion that at the atomic level, we must give up any concievable precise description of nature, is plain false because there exists a heretic counter example. Second, deriving Classical Mechanics from QM (in it's usual interpretation) isn't possible due to the arbitrary distinction between the classical and quantum worlds. Within BM, classical mechanics can be recovered clearly.

(2) Accuracy of conclusions: The Bell test showed that there can't be hidden variable theories consistent with predictions of QM. Yet Bohmian Mechanics (Bohm's hidden variable theory) is consistent with QM; it allows the violation of Bell's inequality. The point is that we must be extremely careful about the conclusions we draw from our equations/experiments. The Bell test excludes *local* hidden variable theories, and Bohmian mechanics is explicitly *non-local*.

There are a host of interesting questions which can be raised. For instance, one could ask why position and momentum aren't simultanesouly determinable if in princple they're well defined? In the double slit then, the particle goes through one of the slits? Can one observe these trajectories? If particles have trajectories, what happens to identical particles? What happens to spins? Does the explicit non-locality entail we can communicate faster than light? Can one distinguish between Bohmian Mechanics and the usual Quantum Mechanics expeirmentally? All these questions, except the last, have been solved or atleast addressed.

Formalism

According to Bohm's original formulation of Bohmian Mechanics, a particle is associated with (1) a position and momentum (q,p), precisely and continuously defined & (2) a wave (ψ). For their description, the following are assumed:

- The ψ-field satisfies the Schrödinger equation.
- The particle momentum is restricted to $mv = p = \nabla S = \hbar\,\mathrm{Im}(\nabla\psi/\psi)$, where $\psi = Re^{iS/\hbar}$ and Im is the imaginary part.
- In practice, we don't control/predict precise locations of the particle; instead we have a stistical ensemble with probability densities $\rho(q) = |\psi(q)|^2$.

Comments:

(1) Note that the observers play no fundamental role in the formalism. If $\hbar = 0$ then we recover the classical Hamilton-Jacobi equation. Unlike QM, BM has a clear classical limit.

(2) These are readily generalized for N particles. Non locality in that case becomes explicit; $p_i = \nabla_i S(q_1, q_2, \ldots, q_N)$ viz. momentum of the i^{th} particle depends on the instantaneous positions of all particles.

(3) Extension to spins: In BM, the particle only has (q,p). The spin is associated only with the wavefunction. For a spinor, say $\Psi \equiv (\psi_+, \psi_-)^T$, the generalization is that $mv = \hbar\,\mathrm{Im}((\Psi, \nabla\Psi)/(\Psi,\Psi))$ where $(.,.)$ represents inner product in the spin space $\mathbb{C}^2$.

[a] Historically, de Broglie had formulated a similar theory and then gave it up until Bohm independently re-discovered it

Determinism

The GHZ test [7]

Objective: To show that realism is incompatible with QM.

Assume: Three particles are allowed to interact and three observers are given one particle each. The interaction is such that the following holds. There are two properties of these particles one can measure, X or Y. The outcome of the measurement is either 1 or -1.

Construction: Interestingly, for a specific initial state of these particles, if they measure $A = X \otimes Y \otimes Y$ then the outcome is guarenteed by QM to be $+1$. This also holds for $B = Y \otimes X \otimes Y$ and $C = Y \otimes Y \otimes X$. However, if $D = X \otimes X \otimes X$ is measured, then the result is -1.[a] Explicitly, this can be achieved with 3 spin half particles for example, with $|\psi\rangle = (|000\rangle - |111\rangle)/\sqrt{2}$ (where $\sigma_z |0/1\rangle = \pm|0/1\rangle$) and X, Y, Z as pauli spin operators.

Hypothesis: Assume that the world is deterministic, viz. the properties had predefined values. Then if we evaluate ABC, then we know by construction that it must be $= 1$. However, it is also true that $ABC = D$ (because $Y^2 = 1$). By construction we also know that $D = -1$. Thus we arrive at $+1 = -1$.

Conclusion: This entails that our hypothesis must be wrong. More precisely, this implies that we can't have non-contextual determinism where the qualification "non-contextual" is subtle but necessary.

Contextuality [7, 8]

Two observables A and B are mutually compatible if the result of measuring A doesn't depend on whether B is measured (before, after, simultanesouly or not measured at all). If we restrict ourselves to hidden variable models that assert that A and B have predefined values, irrespective of which compatible observable is measured, then such a theory would be termed "non-contextual" and deterministic. Kochen-Specker proved that such theories, viz. non-contextual deterministic theories are inconsistent with QM. Mermin and Peres showed this for a four-level system. Consider the following operators.

$$\begin{array}{ccc} A_{11} = \sigma_z \otimes \mathbb{I} & A_{12} = \mathbb{I} \otimes \sigma_z & A_{13} = \sigma_z \otimes \sigma_z \\ A_{21} = \mathbb{I} \otimes \sigma_x & A_{22} = \sigma_x \otimes \mathbb{I} & A_{23} = \sigma_x \otimes \sigma_x \\ A_{31} = \sigma_z \otimes \sigma_x & A_{32} = \sigma_x \otimes \sigma_z & A_{33} = \sigma_y \otimes \sigma_y \end{array}$$

Note that operators along a given row commute. This also holds for a given column and thus these are compatible. Also note that the measurement product along any row (R_K) or column (C_k) is 1, except for column three; $C_3 = -1$. Thus, QM predicts $\prod_{k=1,2,3} R_k C_k = -1$, in contrast to non-contextual models.[b] Since no experiment yields ideal results, an inequality must be constructed. It has been shown that all non-contextual theories must satisfy $\langle \chi_{KS} \rangle = \langle R_1 \rangle + \langle R_2 \rangle + \langle R_3 \rangle + \langle C_1 \rangle + \langle C_2 \rangle - \langle C_3 \rangle \leq 4$. QM yields $\langle \chi_{KS} \rangle = 6$.

Remarks:

(1) Note that Mermin's test is state independent, unlike the GHZ test.

(2) While there's a subtle connection between the Bell test and Contextuality, the latter is more suited for testing determinism (non-contextual) because the locality assumption is not required.

[a] The tensor has been ommited henceforth. The details have been

[b] TODO: Verify this

GHZ, Spins, Phase Space and BM

In this section we'll first sketch the analysis of the GHZ situation using BM. That BM is able to explain this is not very surprising since spins are not assigned predefined values in BM, unlike (q,p). A generalization of the GHZ test to phase space is thus desired. Such a generalization has been discussed in the literature and is briefly summarized.

Spin GHZ | BM Analysis [3]

Assuming that Stern Gerlach type apparatus are used to measure the spins of the different particles, the initial state of the system maybe described as $|\Psi(r_1, r_2, r_3, t=0)\rangle = (\psi_{+++}|000\rangle - \psi_{---}|111\rangle)/\sqrt{2}$, where r_i represents the position vector in the frame of the i^{th} observer. If the particles are assumed to be gaussian initially and propogating (with speed v_0 for instance) along the axes of their respective SG apparatus, then one can further define $\psi_{\pm\pm\pm}$. It can be shown that[a] the further evolution of $\psi_{\pm\pm\pm}$ can be written as products of 3 single particle solutions of SG setup, which was analyzed by Bohm himself. Once $|\Psi(r_i, t)\rangle$ is known, one can evaluate the equation of motion for the three particles using the formalism of BM. If the SG apparatus are setup to measure say XYY, then it is found that (in the directioni relevent to measurement), four attractors basins form: $(+++)$, $(+--)$, $(--+)$ and $(-+-)$. The product is always $+1$ as was predicted by QM. However, when the SG apparatus are setup to measure XXX, the trajectories are found to obey equations which posses four attractive basins: $(---)$, $(-++)$, $(+-+)$ and $(++-)$. In this case, we get -1 as the product, again in agreement with QM.

Remarks:

Non locality enters from the fact that the attractor basins depend on the settings of *all* SG apparatus. Contextuality from this perspective is essentially the statement that the results of an experiment, depend on the experiment being performed. [4]

[a] details have been skipped for brevity and clarity

Phase space GHZ [6]

Consider unitary operators X, Y and the following redefinitions; $A = X^\dagger Y Y^\dagger$, $B = Y^\dagger X^\dagger Y$, $C = Y Y^\dagger X^\dagger$ and $D = XXX$. If the following anti-commutations hold, then we'll arrive at a GHZ like situation; $\{X, Y\} = 0$ and $\{X, Y^\dagger\}$. Given this, it follows that (1) A, B, C, D all commute and (2) $ABCD = -\mathbb{I}$. Thus any simultaneous eigenstate of A, B, C, D will result in the GHZ situation. Explicitly, in phase space, for some length scale L, $X \equiv \exp(i\sqrt{\pi}x/L)$ and $Y \equiv \exp(i\sqrt{\pi}pL)$ ($\hbar$ is chosen to be 1 in this section) satisfies the aforesaid conditions. To construct the simultaneous eigenstates, observe that for

$$|\uparrow\rangle_{x_0,p_0} \equiv \frac{1}{\sqrt{2}}\Big(\sum_{k=-\infty}^{\infty} e^{i\pi 2kp_0}|x = x_0 + 2k\rangle + i\sum_{k=-\infty}^{\infty} e^{i\pi(2k+1)p_0}|x = x_0 + 2k+1\rangle\Big),$$

$$|\downarrow\rangle_{x_0,p_0} \equiv \frac{1}{\sqrt{2}}\Big(\sum_{k=-\infty}^{\infty} e^{i\pi 2kp_0}|x = x_0 + 2k\rangle - i\sum_{k=-\infty}^{\infty} e^{i\pi(2k+1)p_0}|x = x_0 + 2k+1\rangle\Big),$$

we yield[a] $X|\uparrow\rangle = |\downarrow\rangle$, $Y|\uparrow\rangle = i|\downarrow\rangle$ and $Z|\uparrow\rangle = |\uparrow\rangle$ and similarly for $|\downarrow\rangle$. From this, the generalization of the GHZ state is found to be $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle)$.

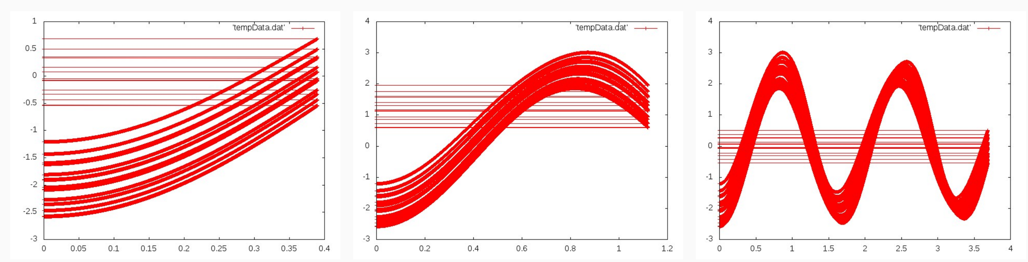
With the states and the operators defined in phase space, the GHZ test has been extended.

[a] here $x_0, p_0 \in [0, 1[$ and are numbers. Strictly one must write in place of x, $\sqrt{\pi}Lx$ and for p, $\sqrt{p}/L$.

Achievements and Outlook

Results

In addition to narrowing the problem and surveying the literature, the first few stages of writing a BM simulator have been achieved. This is of special interest since analytic solutions to Bohmian trajectories are rarely simple. Trajectories for free evolution, squeezed state evolution under harmonic potential (shown in the figure) and a one dimensional analogue of the double slit experiment have been simulated and found to be qualitatively satisfactory. The simulator was written in Fortran, uses RK-4 and spline interpolation for evaluation.



Immediate Goals

Numerically, extension to many particles and ability to handle spins are the essential next steps. These are required to validate the spin GHZ test as described. Theoretically, improvement of the phase space GHZ test is desired so that normalizable states can yield the paradox. The analogue of the SG apparatus for measuring p is essential for a Bohmian analysis.

Future Scope

A more ambitious goal would be to explore phase space contextuality [1] using BM to understand its relation with non locality more directly. A puzzling question is that while formally in QM, spins and (q,p) are handled rather similarly, why can't we extend BM in a manner such that spins are as 'deterministic' as (q,p)? It is worth attempting to find such a formulation or to show that it doesn't exist. This is of great interest for this answer must depend on the fundamental difference between spins and (q,p) as properties. The thesis problem is a step in this direction.

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