Is contextuality a necessary feature of quantum mechanics?

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We show that one can violate the KCBS inequality, with a non-contextual deterministic toy-model. Investigation of this results in identification of a new property, we call 'non-multiplicativity', which is sufficient to arrive at a violation. Infact, it is suggested that even contextual models are 'non-multiplicative'. We then delineate a general 'non-multiplicative' construction, which is non-contextual and still manages to explain all the results of quantum mechanics (including the violation), by restricting a known and successful hidden variable formulation of QM. Effectively thus, we show that contextuality is not a necessary feature of quantum mechanics, viz. the violation of the KCBS inequality doesn't entail contextuality.

I. INTRODUCTION

There are two tests of hidden variable theories, which have gained popularity in the physics community, Bell's Locality test and Kochen & Specker's Contextuality test. Both were designed to rule out certain classes of theories, and with them, their associated properties. In case of the former, locality is given up, although one still preserves no-signalling. In the case of the latter, it is believed that non-contextuality must be given up. Since the class of theories which are ruled out, satisfy our intuitive understanding of nature (classical properties), their exclusion entails that quantum mechanics has these non-classical features, which can be harnessed as a resource to perform tasks that were otherwise impossible.

The paper is organized as follows. In section §II we describe a non-contextual toy-model that violates the KCBS inequality, a representative contextual model and finally explore the implications of the aforesaid, including arriving at a definition of the term 'non-multiplicative'. In the following section, we generalize the result by using restricted ideas from Bohmian Mechanics and completing QM appropriately. The toy-model we show is infact a specific case of the generalization.

II. CONTEXTUALITY AND MULTIPLICATIVITY

We start with stating the famous construction due to Peres and Mermin, where for $\hat{C}_j := \prod_i \hat{A}_{ij}$, $\hat{R}_i := \prod_j \hat{A}_{ij}\, \forall\, i, j \in \{1, 2, 3\}$ and

$$\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} \\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} \\ \hat{A}_{31} & \hat{A}_{32} & \hat{A}_{33} \end{bmatrix} = \begin{bmatrix} \hat{\mathbb{I}} \otimes \hat{\sigma}_x & \hat{\sigma}_x \otimes \hat{\mathbb{I}} & \hat{\sigma}_x \otimes \hat{\sigma}_x \\ \hat{\sigma}_y \otimes \hat{\mathbb{I}} & \hat{\mathbb{I}} \otimes \hat{\sigma}_y & \hat{\sigma}_y \otimes \hat{\sigma}_y \\ \hat{\sigma}_y \otimes \hat{\sigma}_x & \hat{\sigma}_x \otimes \hat{\sigma}_y & \hat{\sigma}_z \otimes \hat{\sigma}_z \end{bmatrix},$$

one can immediately check (a) $\hat{C}_3 = -\hat{\mathbb{I}}$, $\hat{C}_j = \hat{\mathbb{I}}\,(j \neq 3)$, $\hat{R}_i = \hat{\mathbb{I}}, \forall\, j, i$ and that (b) operators along a given row (also along a column) commute. Further,

$$\hat{\chi}_{\text{KCBS}} = \hat{R}_1 + \hat{R}_2 + \hat{R}_3 + \hat{C}_1 + \hat{C}_2 - \hat{C}_3 = 6.\hat{\mathbb{I}}$$

which facilitates experimental verification by evaluation of $\langle \hat{\chi}_{\text{KCBS}} \rangle$ by measuring $\left\langle \hat{R}_i \right\rangle$ and $\left\langle \hat{C}_j \right\rangle$. Now imagine a theory that assigns values to these observables, as given in Mat. (1), representing a given state, which satisfies all conditions listed in (a), except $\hat{C}_3 \to 1$ instead of -1.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{1}$$

One can check that it is not easy to modify the assigned values, s.t. all constraints are satisfied. In fact it is impossible [put citation]. It clearly follows that such simplistic 'deterministic' theories, can't explain all predictions of Quantum Mechanics (QM). Explicitly, for the aforesaid

$$\langle \hat{\chi}_{\text{KCBS}} \rangle \leq 4, \tag{2}$$

and this holds quite generally, for such theories, where the ensemble is taken to be identical copies of the given state.

In the aforesaid simple 'deterministic' model, we have made two tacit assumptions. First, we assumed that upon 'measuring' an observable, the assignment (to the other operators) remains invariant, and therefore the state remains invariant. Second, we assumed, while calculating $\langle \hat{\chi} \rangle$, that the value assigned to $\hat{R}_i$, $\hat{C}_i$ is obtained by multiplying the values assigned to it's defining constituent operators.

Let us view the validity of these from the point of view of quantum mechanics. It is obvious that the first assumption is wrong in general; according to the postulates of QM, the state after measurement (taken to be projective), must be an eigenstate of the observable. The second assumption must be stated more precisely before proceeding. For any n compatible operators, $\{\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_n\}$, consider $\hat{C} = f(\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_n)$. Let $m_i(*)$ refer to the value assigned to the operator, or equivalently, the value observed upon measurement of an operator $\hat{A}$, where i parametrizes the sequence of measurements. The second assumption then, is that

$$m_1(\hat{C}) = f(m_{k_1}(\hat{B}_1), m_{k_2}(\hat{B}_2), \ldots, m_{k_n}(\hat{B}_n)), \tag{3}$$

where $\mathbf{k} := \{k_1, k_2, \ldots, k_n\} \in \{\{1, 2, \ldots, n\} +$ all possible permutations$\}$. To see if it is valid in QM, note that one can construct eigenkets

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$|\mathbf{b} = \{b_1, b_2, \ldots, b_n\}\rangle$, s.t. $B_i |\mathbf{b}\rangle = b_i |\mathbf{b}\rangle$. If the state of the system is given by one of these, then it is trivial to see that the second assumption holds. It can be proven [see appendix] that this assumption holds in general also. It must however be emphasized that, QM doesn't enforce

$$m_1(\hat{C}) = f(m_1(\hat{B}_1), m_1(\hat{B}_2), \ldots, m_1(\hat{B}_n)) \quad (4)$$

in general, since for instance, for the state $|11\rangle := |1\rangle \otimes |1\rangle$ written in the computational basis, $\hat{B}_1 = \hat{\sigma}_x \otimes \hat{\sigma}_x$, $\hat{B}_2 = \hat{\sigma}_y \otimes \hat{\sigma}_y$ and $\hat{C} = \hat{B}_1 \hat{B}_2 = -\hat{\sigma}_z \otimes \hat{\sigma}_z$, we have $m_1(\hat{C}) = -1$ while $m_1(\hat{B}_1) = \pm 1$ and $m_1(\hat{B}_2) = \pm 1$, independently, according to QM, with probability half. Henceforth, a model/theory that satisfies equation (3) will be referred to as being *sequentially multiplicative* and one that satisfies equation (4) will be termed *multiplicative*. Note that for the simple 'deterministic' model, both sequential multiplicativity and multiplicativity become equivalent, since the state doesn't change upon being measured.

We construct a 'deterministic' toy-model, which yields a consistent assignment and also a violation of the KCBS inequality. The assignments are made by a three step process.
1. Initial State: Choose an appropriate initial state $|\psi\rangle$ (say $|00\rangle$).
2. Hidden Variable (HV): Toss a coin and assign $c = +1$ for heads, else assign $c = -1$.
3. Predictions/Assignments: For an operator $\hat{p}' \in \{\hat{A}_{ij}, \hat{R}_i, \hat{C}_j\, (\forall\, i, j)\}$ check if $\exists$ a λ, s.t. $\hat{p}' |\psi\rangle = \lambda |\psi\rangle$. If $\exists$ a λ, then assign λ as the value. Else, assign c.
So far the model has only predicted the outcomes of measurements. If however, a measurement is made on the system, then although we know the result from the predictions, we must update the state $|\psi\rangle$ of the system, depending on which observable is measured and arrive at new predictions, using the aforesaid steps. The following final step fills precisely this gap.
4. Update: Say $\hat{p}$ was observed. If $\hat{p}$ is s.t. $\hat{p} |\psi\rangle = \lambda |\psi\rangle$, then leave the state unchanged. Else, find $|p_\pm\rangle$ (eigenkets of $\hat{p}$), s.t. $\hat{p} |p_\pm\rangle = \pm |p_\pm\rangle$ and update the state $|\psi\rangle \to |p_c\rangle$.

Let us explicitly apply the aforesaid algorithm, to the state $|\psi\rangle = |00\rangle$. Say we obtained tails, and thus $c = -1$. To arrive at the assignments, note that $|00\rangle$ is an eigenket of only $\hat{R}_i, \hat{C}_j$ and $\hat{A}_{33} = \hat{\sigma}_z \otimes \hat{\sigma}_z$. Thus, in the first iteration, all these should be assigned their respective eigenvalues. The remaining operators must be assigned c (see equation (5)). Two remarks are in order. First, this model is *non*-multiplicative, for $m_1(\hat{C}_3) = 1 \neq m_1(\hat{A}_{13}) m_1(\hat{A}_{23}) m_1(\hat{A}_{33}) = -1$. Second, we must impose sequential multiplicativity as a consistency check of the model, which in particular entails that $m_1(\hat{C}_3) = m_1(\hat{A}_{33}) m_2(\hat{A}_{23}) m_3(\hat{A}_{13})$. To illustrate this, we must choose to measure $\hat{A}_{33}$. According to step 4, since $|00\rangle$ is an eigenstate of $\hat{A}_{33}$, the final state remains $|00\rangle$.

Iteration	$i = 1$	$i = 2$	$i = 3$
$\lvert\psi_{\text{init}}\rangle$	$\lvert 00\rangle$	$\lvert 00\rangle$	$\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}}$
HV/Toss	$c = -1$	$c = -1$	$c = +1$
Predictions (Assignments)	$m_1(\hat{A}_{ij}) \doteq \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & +1 \end{bmatrix}$	$m_2(\hat{A}_{ij}) \doteq \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & +1 \end{bmatrix}$	$m_3(\hat{A}_{ij}) \doteq \begin{bmatrix} +1 & +1 & +1 \\ +1 & +1 & -1 \\ +1 & +1 & +1 \end{bmatrix}$
	$m_1(\hat{R}_i), m_1(\hat{C}_j) = +1\, (j \neq 3)$; $m_1(\hat{C}_3) = -1$	$m_2(\hat{R}_i), m_2(\hat{C}_j) = +1\, (j \neq 3)$; $m_2(\hat{C}_3) = -1$	$m_3(\hat{R}_i), m_3(\hat{C}_j) = +1\, (j \neq 3)$; $m_3(\hat{C}_3) = -1$
Operator Measured	$\hat{A}_{13} = \hat{\sigma}_z \otimes \hat{\sigma}_z; m_1(\hat{A}_{13}) = +1$	$\hat{A}_{23} = \hat{\sigma}_y \otimes \hat{\sigma}_y; m_2(\hat{A}_{23}) = -1$	$\hat{A}_{33} = \hat{\sigma}_x \otimes \hat{\sigma}_x; m_3(\hat{A}_{33}) = +1$
$\lvert\psi_{\text{final}}\rangle$	$\lvert 00\rangle$	$\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}}$	$\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}}$

(5)

For the next iteration, $i = 2$, say we again yield $c = -1$. Since $|\psi\rangle$ is also unchanged, the assignment remains invariant. For the final step, we choose to measure $\hat{p} = \hat{A}_{23} (= \hat{\sigma}_y \otimes \hat{\sigma}_y)$, to proceed with sequentially measuring $\hat{C}_3$. To simplify calculations, we note

$$|00\rangle = \frac{(|\tilde{+}\tilde{-}\rangle + |\tilde{-}\tilde{+}\rangle)/\sqrt{2} + (|\tilde{+}\tilde{+}\rangle + |\tilde{-}\tilde{-}\rangle)/\sqrt{2}}{\sqrt{2}},$$

where $|\tilde{\pm}\rangle = |0\rangle \pm i\,|1\rangle$ (eigenkets of $\hat{\sigma}_y$). Since $|00\rangle$ is manifestly not an eigenket of $\hat{p}$, we must find $|p_-\rangle$, since $c = -1$. It is immediate that $|p_-\rangle = \left(|\tilde{+}\tilde{-}\rangle + |\tilde{-}\tilde{+}\rangle\right)/\sqrt{2} = \left(|00\rangle + |11\rangle\right)/\sqrt{2}$, which becomes the final state.

For the final iteration, $i = 3$, say we yield $c = 1$. So far, we have $m_1(\hat{A}_{33}) = 1$ and $m_2(\hat{A}_{23}) = -1$. We must obtain $m_3(\hat{A}_{13}) = 1$, independent of the value of c, to be consistent. Let's check that. According to step 3, since $\hat{\sigma}_x \otimes \hat{\sigma}_x \left(|00\rangle + |11\rangle\right)/\sqrt{2} = 1\left(|00\rangle + |11\rangle\right)/\sqrt{2}$, $m_3(\hat{A}_{13}) = 1$ indeed. As a remark, it maybe emphasised that the $m_2(\hat{A}_{33}) = m_3(\hat{A}_{33})$ and $m_2(\hat{A}_{23}) = m_3(\hat{A}_{23})$, which essentially expresses compatibility of these observables, viz. measurement of $\hat{A}_{13}$ doesn't affect the result one would obtain by measuring operators compatible to it (granted they have been measured once before).

III. GENERALIZATIONS

IV. APPENDIX

A. Proof of Sequential Multiplicativity

Let $m_i(\hat{O})$ refer to the outcome of measuring an operator $\hat{O}$, where i parametrizes the sequence of the measurement. Sequential multiplicativity is the statement that $m(\hat{P}) = m_1(\hat{P}_1)m_2(\hat{P}_2) = m_1(\hat{P}_2)m_2(\hat{P}_1)$. The proof is simple. Consider a set of compatible operators, which are also complete. For simplicity, let's say the set is given by $S = \{\hat{P}_1, \hat{P}_2\}$ (not anti-commutation, its a set). Now consider the operator $\hat{P} = \hat{P}_1 + \hat{P}_2$. Let $|\psi\rangle$ be s.t. $\hat{P}|\psi\rangle = p|\psi\rangle$ but $\nexists$ any λ s.t. $\hat{P}_i|\psi\rangle = \lambda_i|\psi\rangle$ (else the result we're trying to prove will be trivial; as $|\psi\rangle$ is a simultaneous eigenket of P_1, P_2 and P, with eigenvalues p_1, p_2 and p_1p_2). If you're uncomfortable with this 'claim' or its relevance, one can use $P_1 = \sigma_x \otimes \sigma_y$, $P_2 = \sigma_y \otimes \sigma_x$ so that $P = \sigma_z \otimes \sigma_z$. Now, the state $|00\rangle$ makes for a non-trivial $|\psi\rangle$. Coming to the proof, first we define $\hat{P}_i|p_1, p_2\rangle = p_i|p_1, p_2\rangle$ note that it suffices for me to show that $|\psi\rangle$ must be a superposition of only those simultaneous eigenkets $|p_1, p_2\rangle$ s.t. $p_1p_2 = p$. This can be proved as follows. Consider

$$\begin{aligned} \hat{P}|\psi\rangle &= p|\psi\rangle \\ \implies \langle p_1, p_2|\hat{P}|\psi\rangle = p_1p_2\langle p_1, p_2|\psi\rangle &= p\langle p_1, p_2|\psi\rangle . \end{aligned}$$

It follows immediately, that if $\langle p_1, p_2|\psi\rangle \neq 0$, then it entails that $|\psi\rangle$ contains $|p_1, p_2\rangle$ as an eigenket in its superposition, and $p_1p_2 = p$. However, if $\langle p_1, p_2|\psi\rangle = 0$, then we won't ever see p_1 or p_2 as outcomes, so it is irrelevant. That then completes the proof.

This can be readily generalized to any number of operators, even if they're not complete (in which case the remaining operators maybe added to complete the set, but the eigenvalues corresponding to them, don't play any role in the proof, and thus making its extension trivial).