An Alternative to Contextuality

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I. Background

- Einstein: 'locality' $\implies$ Quantum Mechanics (QM) is incomplete [3].
- Bell: 'locality' $\implies \langle \hat{B} \rangle \leq 2$; For some $|\psi\rangle$, QM $\implies \langle \hat{B} \rangle = 2\sqrt{2}$ [1]. Verified experimentally (without loop holes in 2015)
- Comment: At roughly the same time, various physicists had produced proofs of the claim that one can't complete QM satisfactorily, that a sensible complete 'hidden variable' (HV) description of nature was impossible.
- Bohmian Mechanics (BM): a HV description, that (i) 'completes' QM in a simple, clear, precise but non-local manner, and (ii) is deterministic [2].
- Defn: *Deterministic* $\equiv$ If in principle, the outcome of measuring each observable is predictable, given the HVs.
- Comment: Bell's inequality requires entanglement in some form, to prove Einstein's notion of locality incorrect. Recently, another peculiar feature of QM has been identified, namely contextuality.
- Impl Defn: *Context* $\equiv$ If $[\hat{A}, \hat{B}] = 0$ and $[\hat{A}, \hat{C}] = 0$ but $[\hat{B}, \hat{C}] \neq 0$, then possible contexts are $\hat{A}$, $\hat{A}$ and $\hat{B}$ or $\hat{A}$ and $\hat{C}$ [5].
- Defn: *Non-contextual* $\equiv$ Value an operator takes, depends only on the state (including 'hidden variables') and the choice of the operator A (not it's context) [5].
- Defn: *Contextual* $\equiv$ Value an operator takes, depends on it's context [5].
- Comment: This notion arises, atleast in certain explicit constructions, where one is unable to assign values to operators, consistent with predictions of QM.
- Aim: Understand how a deterministic theory can be consistent with the notion of contextuality.

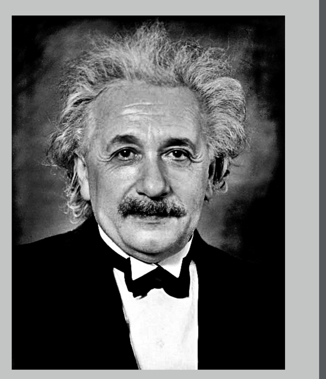

Figure 1: A. Einstein

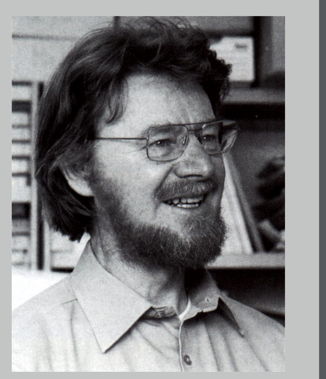

Figure 2: J. Bell

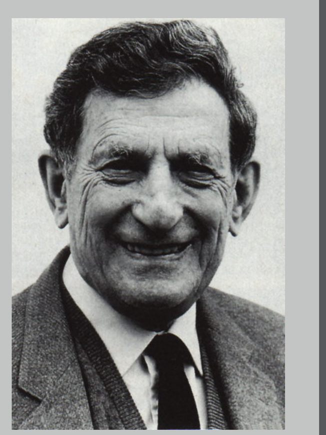

Figure 3: D. Bohm

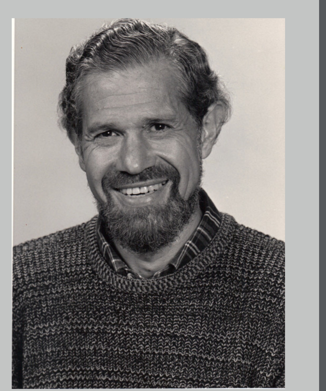

Figure 4: S. B. Kochen

II. Overview

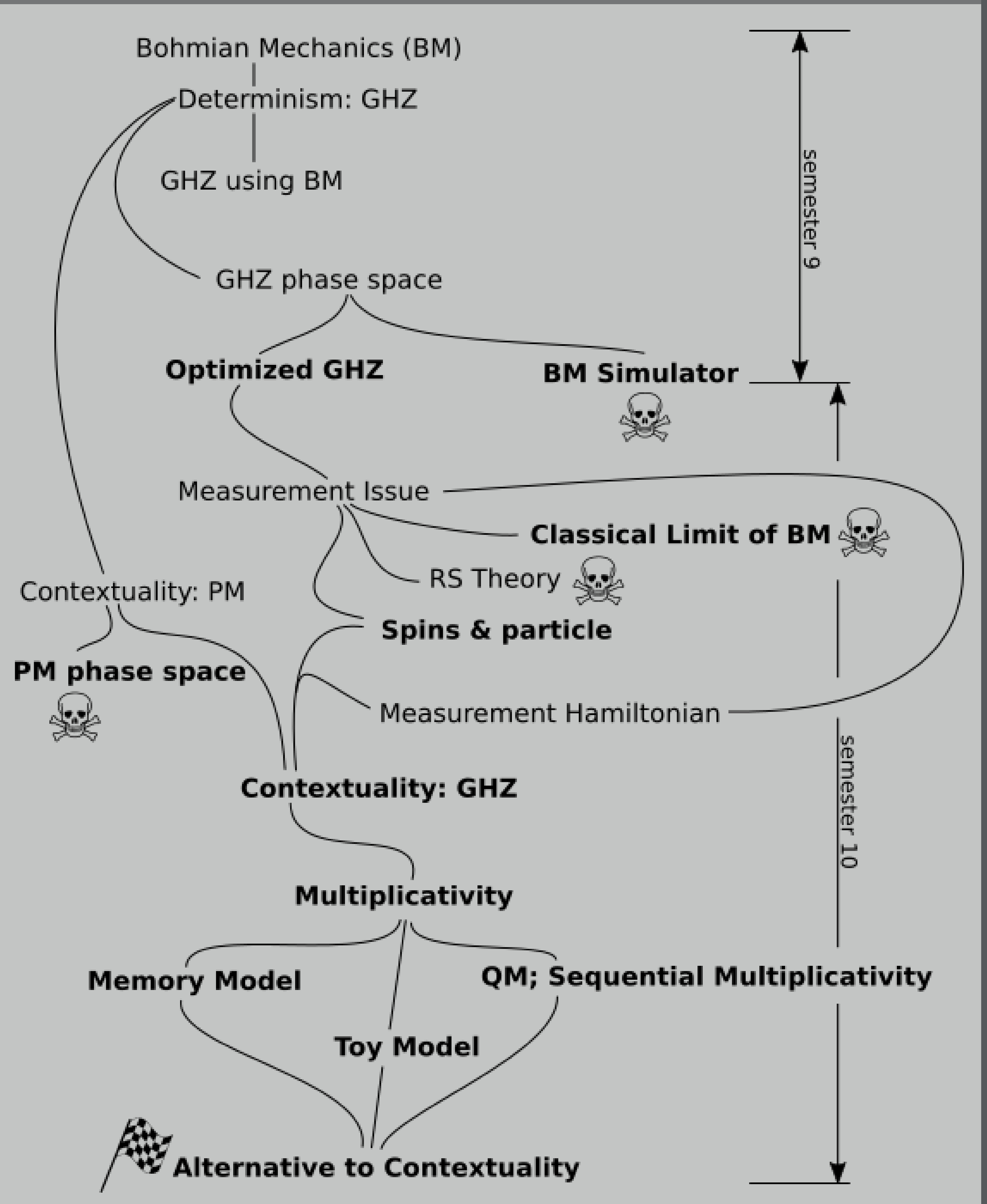

Figure 5: Exploration flow: Boldface titles represent new results

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III. Multiplicativity

- Defn: *Compatible operators* $\equiv$ Two observables $\hat{A}$ and $\hat{B}$ are mutually compatible if, given that the system is prepared in a state s.t. measurement $\hat{A}$ yields repeatable results, measurement of $\hat{B}$ doesn't change the result of measuring $\hat{A}$. For projective measurements, its equivalent to $[\hat{A}, \hat{B}] = 0$.
- Defn: *Multiplicativity* $\equiv$ For compatible operators $\hat{B}_i$, a model is multiplicative iff

$$f(m_1(\hat{B}_1), m_1(\hat{B}_2), \dots, m_1(\hat{B}_n)) = \tag{1}$$
$$m_1(f(\hat{B}_1, \hat{B}_2, \dots, \hat{B}_n)), \tag{2}$$

where $m_j(\circledast)$ represents the assigned value of the operator, and j encodes the sequence of measurement. Note that this is an ontological statement and can't be experimentally tested.

- Defn: *Sequential Multiplicativity* $\equiv$ For compatible operators $\hat{B}_i$, a model is sequentially multiplicative iff

$$f(m_{k_1}(\hat{B}_1), m_{k_2}(\hat{B}_2), \dots, m_{k_n}(\hat{B}_n)) = \tag{3}$$
$$m_1(f(\hat{B}_1, \hat{B}_2, \dots, \hat{B}_n), \tag{4}$$

where $\mathbf{k} \equiv (k_1, k_2, \dots, k_n) \in ((1, 2, \dots, n) +$ all possible permutations $)$, $m_j(\circledast)$ represents the assigned value of the operator, and j encodes the sequence of measurement.

- Example: $\hat{B}_1 = \hat{\sigma}_x \otimes \hat{\sigma}_y$, $\hat{B}_2 = \hat{\sigma}_y \otimes \hat{\sigma}_x$ so that $\hat{C} = \hat{B}_1\hat{B}_2 = \hat{\sigma}_z \otimes \hat{\sigma}_z$. $|\psi\rangle = |00\rangle$, so that $m_1(\hat{C}) = 1$, while $m_1(\hat{B}_1) = \pm 1$. If say $m_1(\hat{B}_1) = -1$, then $\psi \to$ (figure this) so that entails $m_2(\hat{B}_2) = -1$ as well, consistent with $m_1(\hat{C}) = m_1(\hat{B}_1)m_2(\hat{B}_2)$.
- Claim: Quantum Mechanics is sequentially multiplicative.

IV. Contextuality - PM Test

Kochen-Specker proved that non-contextual theories, are inconsistent with QM [6]. Mermin and Peres showed this for a four-level system [4].

- Simplified Proof: Consider the following operators.

$$\hat{A}_{ij} \doteq \begin{bmatrix} \hat{\sigma}_z \otimes \hat{\mathbb{I}} & \hat{\mathbb{I}} \otimes \hat{\sigma}_z & \hat{\sigma}_z \otimes \hat{\sigma}_z \\ \hat{\mathbb{I}} \otimes \hat{\sigma}_x & \hat{\sigma}_x \otimes \hat{\mathbb{I}} & \hat{\sigma}_x \otimes \hat{\sigma}_x \\ \hat{\sigma}_z \otimes \hat{\sigma}_x & \hat{\sigma}_x \otimes \hat{\sigma}_z & \hat{\sigma}_y \otimes \hat{\sigma}_y \end{bmatrix}$$

Note that operators along a given row (column) commute.

$$\hat{R}_i \equiv \prod_j \hat{A}_{ij} = \mathbb{I} \tag{5}$$

$$\hat{C}_j \equiv \prod_i \hat{A}_{ij} = \begin{cases} +\hat{\mathbb{I}} & (j \neq 3) \\ -\hat{\mathbb{I}} & (j = 3) \end{cases} \tag{6}$$

It entails that $\prod_{k=1,2,3} \hat{R}_k \hat{C}_k = -\hat{\mathbb{I}}$, whereas non-contextual models would yield $+1$.
NB: We also assumed multiplicativity.
To facilitate experimental validation, it has been shown that non-contextual models satisfy Eq. 7, while QM yields $\langle \hat{\chi}_{PM} \rangle = \mathbf{6}$.

$$\langle \hat{\chi}_{PM} \rangle = \langle \hat{R}_1 \rangle + \langle \hat{R}_2 \rangle + \langle \hat{R}_3 \rangle + \langle \hat{C}_1 \rangle + \langle \hat{C}_2 \rangle - \langle \hat{C}_3 \rangle \leq \mathbf{4} \tag{7}$$

- Conclusion: Deterministic theories, that satisfy both (a) non-contextuality and (b) multiplicativity, are inconsistent with QM.

The Toy Model — Example

Iteration	$i = 1$	$i = 2$	$i = 3$
$\lvert\psi_{\text{init}}\rangle$	$\lvert 00\rangle$	$\lvert 00\rangle$	$\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}}$
HV/Toss	$c = -1$	$c = -1$	$c = +1$
Predictions (Assignments)	$m_1(\hat{A}_{ij}) \doteq \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & +1 \end{bmatrix}$	$m_2(\hat{A}_{ij}) \doteq \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & +1 \end{bmatrix}$	$m_3(\hat{A}_{ij}) \doteq \begin{bmatrix} +1 & +1 & +1 \\ +1 & +1 & -1 \\ +1 & +1 & +1 \end{bmatrix}$
	$m_1(\hat{R}_i), m_1(\hat{C}_j) = +1\,(j \neq 3)$ $m_1(\hat{C}_3) = -1$	$m_2(\hat{R}_i), m_2(\hat{C}_j) = +1\,(j \neq 3)$ $m_2(\hat{C}_3) = -1$	$m_3(\hat{R}_i), m_3(\hat{C}_j) = +1\,(j \neq 3)$ $m_3(\hat{C}_3) = -1$
Operator Measured	$\hat{A}_{13} = \hat{\sigma}_z \otimes \hat{\sigma}_z; m_1(\hat{A}_{13}) = +1$	$\hat{A}_{23} = \hat{\sigma}_y \otimes \hat{\sigma}_y; m_2(\hat{A}_{23}) = -1$	$\hat{A}_{33} = \hat{\sigma}_x \otimes \hat{\sigma}_x; m_3(\hat{A}_{33}) = +1$
$\lvert\psi_{\text{final}}\rangle$	$\lvert 00\rangle$	$\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}}$	$\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}}$

(8)

V. Contextuality - Memory Model

An example of a contextual and non-multiplicative model; Sequential multiplicativity has been assumed.

- Initial: The assignment is as given in the first Mat in Eq. 9.
- Remark: The system is assumed to be capable of remembering the last three observables that were measured.
- Algorithm: Upon measurement of an observable, (i) yield the value as saved in the matrix, (ii) append the observable in the 3 element memory and (iii) update the matrix, once the context (set of commuting observables) is known, to satisfy the PM requirements.

$$m_1(\hat{A}_{ij}) = m_2(\hat{A}_{ij}) \doteq \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, m_3(\hat{A}_{ij}) \doteq \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{9}$$

- For example:

Operator	Updated Array	Assignment	Value
$\hat{A}_{33}$	$\{*, *, \hat{A}_{33}\}$	$m_1(\hat{A}_{ij})$	1
$\hat{A}_{23}$	$\{*, \hat{A}_{33}, \hat{A}_{23}\}$	$m_2(\hat{A}_{ij})$	1
$\hat{A}_{13}$	$\{\hat{A}_{33}, \hat{A}_{23}, \hat{A}_{13}\}$	$m_3(\hat{A}_{ij})$	-1

Result: $m_1(\hat{C}_3) = -1$ as required.

VI. The Toy Model

An example of a non-contextual non-multiplicative model; Sequential multiplicativity is demonstrated.

- Initial: $|\psi\rangle$.
- 'hidden variable': Choose $c = +1$ for heads, $c = -1$ for tails, after a coin toss.
- Predictions/Assignments: For an operator $\hat{\rho}' \in \{\hat{A}_{ij}, \hat{R}_i, +$ their products such as $\hat{C}_j\,(\forall\, i, j)\}$ check if $\exists$ a λ, s.t. $\hat{\rho}' |\psi\rangle = \lambda |\psi\rangle$. If $\exists$ a λ, then assign λ as the value. Else, assign c.
- Update: Say $\hat{p}$ was observed. If $\hat{p}$ is s.t. $\hat{p}|\psi\rangle = \lambda|\psi\rangle$, then leave the state unchanged. Else, find $|p_\pm\rangle$ (eigenkets of $\hat{p}$), s.t. $\hat{p}|p_\pm\rangle = \pm|p_\pm\rangle$ and update the state $|\psi\rangle \to |p_c\rangle$.

NB: This would statistically agree with QM, for a few $|\psi\rangle$s.

VII. Results and Conclusion

- Contextuality is not necessary.
  - The properties 'multiplicativity' and 'sequential multiplicativity' were identified, defined and proven where they hold.
  - Demonstrated that 'non-multiplicativity' is an alternative to 'contextuality', by constructing a 'non-contextual' theory, consistent with QM predictions.
  - Proposed a Minimalistic HV theory; simplifies predictions.
- Tests of Determinism and Contextuality
  - Optimized phase-space GHZ
  - GHZ extension to a test of contextuality
  - PM extension to phase space (independently re-discovered)
- Measurements in Bohmain Mechanics
  - Generalized the Hamiltonian based measurement scheme to continuous variables
  - Analytic/graphical solution to measuring entangled spins using SG
  - Analytic/graphical proof of consistency of position measurements
  - Alternative proof of spins can't be associated with particles, only with wavefunctions

Bohmian Mechanics, being a deterministic and precise theory, has been successfully used to probe fundamental concepts in Quantum Mechanics and has radically clarified them (to the author atleast).

References

[1] John S. Bell.
On the einstein-podolsky-rosen paradox.
Physics, 1:195–200, 1964.

[2] David Bohm.
A suggested interpretation of the quantum theory in terms of "hidden" variables. i.
Phys. Rev., 85:166–179, Jan 1952.

[3] A. Einstein, B. Podolsky, and N. Rosen.
Can quantum-mechanical description of physical reality be considered complete?
Phys. Rev., 47:777–780, May 1935.

[4] N. David Mermin.
Simple unified form for the major no-hidden-variables theorems.
Phys. Rev. Lett., 65:3373–3376, Dec 1990.

[5] A. Peres.
Quantum Theory: Concepts and Methods (Fundamental Theories of Physics).
Springer, 1993.

[6] E. Specker Simon Kochen.
The problem of hidden variables in quantum mechanics.
Indiana Univ. Math. J., 17:59–87, 1968.