An alternative to contextuality

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We reconsider the no-go theorems on hidden variables and conclude that the notion of contextuality is not a necessary feature of quantum mechanics. We provide an alternative view, motivated by consistency requirements between Bohmian Mechanics and the said theorems. This is accomplished by identifying a new classical property, we call 'multiplicativity'. We propose a non-contextual hidden variable model, consistent with all predictions of QM (for discrete spaces). Advantages of this view are illuminated by discussion of implications of non-multiplicativity to non-locality and its relation to contextuality.

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I. INTRODUCTION

Background: Since Einstein's [1] work on completeness of quantum mechanics and the subsequent seminal work of Bell, assessing compatibility of hidden variable models has proven to be repeatedly useful. It has had foundational implications, for instance it gave a precise meaning to locality, and engendered pragmatic advantages, such as security, randomness certification etc. While Bell showed that local hidden variable models can not describe nature, the Kochen Specker theorem [2] (and related work [3–6]) ruled out an even larger class of hidden variable models. The notion of contextuality was identified which also, in recent times has been harnessed for computation and cryptography. Not all hidden variable models were, however, proved to be incompatible. Bohm's work [7, 8] is one such (in)famous example. This approach has allowed for a precise interpretation of quantum mechanics.

Previous work [the essentials I have already discussed]: In this paper, we take a closer look at the aforesaid no-go theorems and find that contextuality is in fact not a necessary feature of quantum mechanics. This has been shown [9] by taking disturbances due to measurements into consideration. Here, however, we construct a significantly simpler toy model for illustrating not only the aforesaid but to also delineate a complete alternative to contextuality. The toy model is a specific case of what we call a c-ingle variable theory. Its predictions match with those of quantum mechanics for systems described by arbitrary (discrete) Hilbert spaces.

Triangle: <TODO: ask arvind sir>

Work and results: <summarise what is done in each section>

II. CONTEXT

The notation used henceforth has been defined below.

Definition 1. Notation: (a) $\psi \in \mathcal{H}$ would typically represent the quantum mechanical state of the system (assumed pure), residing in the Hilbert space $\mathcal{H}$, (b) $\hat{\mathcal{H}}$ is defined to mean $\mathcal{H} \otimes \mathcal{H}^\dagger$, (c) $[\mathcal{H}]$ is defined to mean $(\mathcal{H}, \mathbb{R}^{\otimes})$, which represents the state of the system including hidden variables, (d) $[\psi] \in [\mathcal{H}]$ will represent the state of the system, including hidden variables, (e) a prediction map is $m : \hat{\mathcal{H}}, [\mathcal{H}] \to \mathbb{R}$, (f) a sequence map is $s : \hat{\mathcal{H}}, [\mathcal{H}], \mathbb{R} \to [\mathcal{H}]$, (g) the set of all experimental setups is denoted by $\mathcal{E}$, (h) a setup map is $e : \hat{\mathcal{H}} \to \mathcal{E}$.

In addition to the notation, we use the following definition of non-contextuality.

Definition 2. A theory is non-contextual, if it provides a map $m : \hat{\mathcal{H}}, [\mathcal{H}] \to \mathbb{R}$ to explain measurement outcomes. A theory which is not non-contextual is contextual.

Remark 3. Prediction maps are non-contextual.

In the literature broader definitions have been suggested which declare a larger set of theories as non-contextual. For our purposes however, this restricted definition will suffice. The term contextual is used to suggest that the value an operator takes might depend on which other compatible observable it is measured with. Since quantum mechanics (QM) yields deterministic results only in specific situations, it is imperative to analyse the meaning of contextuality by 'completing' QM. Hidden variable theories do precisely this and we have at our disposal the heretic theory of Bohm. Circumventing the details of the theory, we note its salient feature; with each particle, one associates precisely defined (q, p) and a wavefunction ψ. Consequently, the outcomes of all experiment are predictable, granted the initial conditions of the system and the apparatus are known (the 'hidden variables'). The essence of Bohmian Mechanics relevant here can be captured by the following theorem.

Theorem 4. *$\exists$ a (Bohmian) map $m_B : \hat{\mathcal{H}}, [\mathcal{H}], \mathcal{E} \to \mathbb{R}$, and a sequence map s, s.t. if m_B & s are assumed to describe the outcomes of measurements & the resultant state respectively, then they are consistent with all predictions of QM.*

Observe that given a Stern-Gerlach setup and the initial conditions (position & wavefunction of the particle), according to the theorem, Bohmian mechanics can predict the value of the spin (say $\hat{\sigma}_z$) of the particle. Assume that the state of the particle was $(|0\rangle + |1\rangle)/\sqrt{2}$ and that the measurement outcome was spin up. Interestingly, it can be shown [10] that if the direction of the magnetic field gradient is flipped while everything else is unchanged, the measurement outcome will now be spin down. This emphasises the fact that m_B is not a prediction map. More importantly it entails that m_B is contextual. It must be stressed, however, that the contextuality in this case is not a statement about the context in terms of compatible observables. Here no two observables are compatible (excluding identity).

III. NON-MULTIPLICATIVITY: AN ALTERNATIVE TO CONTEXTUALITY

One can show [2] that under reasonable assumptions, no non-contextual theory can explain all predictions of QM. We re-state these results and show that nowhere does contextuality appear to be a logical necessity. Instead we note that non-contextual theories with certain properties, we call *multiplicativity* are incompatible with QM. Viewed from this perspective, non-*multiplicativity* is the non-classical alternative to contextuality in QM. Precise definitions of this and a related property *sequential multiplicativity*, are given below.

Definition 5. A prediction map m is *multiplicative* iff

$$m(f(\hat{B}_1,\hat{B}_2,\dots\hat{B}_N),[\psi]) = f(m(\hat{B}_1,[\psi]),m(\hat{B}_2,[\psi]),\dots m(\hat{B}_N,[\psi])),$$

where $\hat{B}_i \in \hat{\mathcal{H}}$ are arbitrary mutually commuting observables, $f:\hat{\mathcal{H}}^{\otimes N}\to\hat{\mathcal{H}}$ and $[\psi]\in[\mathcal{H}]$. A non-*multiplicative* map is one that is not multiplicative.

Note that if m is taken to represent the measurement outcome (in QM), then for states of the system which are simultaneous eigenkets of $\hat{B}_i$s, m must clearly be multiplicative. It is, however, not obvious that this property must always hold. To motivate this, consider the state $|11\rangle := |1\rangle\otimes|1\rangle$ written in the computational basis, $\hat{B}_1 = \hat{\sigma}_x\otimes\hat{\sigma}_x$, $\hat{B}_2 = \hat{\sigma}_y\otimes\hat{\sigma}_y$ and $\hat{C} = \hat{B}_1\hat{B}_2 = -\hat{\sigma}_z\otimes\hat{\sigma}_z$, we have $m(\hat{C}) = -1$ while $m(\hat{B}_1) = \pm1$ and $m(\hat{B}_2) = \pm1$, independently, according to QM, with probability half. Contrary to this, from a quick computation, it is clear that if one first measures $\hat{B}_1$ and subsequently measures $\hat{B}_2$, then the product of the results must be -1. This is consistent with measuring $\hat{C}$. To capture this property we define *sequential multiplicativity* as follows.

Definition 6. A prediction map m is *sequentially multiplicative* for a given sequence map s, iff

$$m(f(\hat{B}_1,\hat{B}_2,\dots\hat{B}_N),[\psi_1]) = f(m(\hat{B}_1,[\psi_{k_1}]),m(\hat{B}_2,[\psi_{k_2}]),\dots,m(\hat{B}_N,[\psi_{k_N}])),$$

where $\mathbf{k} = (k_1,k_2,\dots,k_N)\in\{(1,2,3\dots N),(2,1,3\dots N)+$ permutations, making $N!$ terms$\}$, $\hat{B}_i\in\hat{\mathcal{H}}$ are arbitrary mutually commuting observables, $[\psi_i]\in[\mathcal{H}]$, $f:\hat{\mathcal{H}}^{\otimes N}\to\hat{\mathcal{H}}$ and $[\psi_{k+1}]\equiv s(\hat{B}_k,[\psi_k],m(\hat{B}_k,[\psi_k]))$, $\forall\,[\psi_1]$.

We now discuss the 'proofs of contextuality' in the purview of the aforesaid.

Theorem 7. *Let a map $m:\hat{\mathcal{H}}\to\mathbb{R}$, be s.t. (a) $m(\hat{\mathbb{I}}) = 1$, (b) $m(f(\hat{A}_1,\hat{A}_2,\dots)) = f(m(\hat{A}_1),m(\hat{A}_2),\dots)$, for any arbitrary function f, where $\hat{A}_i$ are arbitrary Hermitian operators. If m is assumed to describe the outcomes of measurements, then no m exists which is consistent with all predictions of Quantum Mechanics.*

Proof. Bell construction ($|\mathcal{H}|\geq 4$): □

Proof. GHZ construction ($|\mathcal{H}|\geq 6$): Consider three observers with one particle each. Each observer can measure two properties, call them X and Y, with outcomes ±1. The associated operators are $\hat{\sigma}_x$ and $\hat{\sigma}_y$. The state of the system is $\sqrt{2}\,|\chi_G\rangle = |000\rangle - |111\rangle$ and we also define $\hat{A} := \hat{\sigma}_x\otimes\hat{\sigma}_y\otimes\hat{\sigma}_y$, $\hat{A}\,|\chi_G\rangle = |\chi_G\rangle$, $\hat{B} := \hat{\sigma}_y\otimes\hat{\sigma}_x\otimes\hat{\sigma}_y$ and $\hat{C} := \hat{\sigma}_y\otimes\hat{\sigma}_y\otimes\hat{\sigma}_z$. A measurement of $\hat{A}$ would correspond to measurement of property X, Y and Y by the three observers respectively. For map m consistent with measurement outcomes of QM, we must have $m(\hat{A}) = m(\hat{B}) = m(\hat{C}) = +1$. Property (b) of the map $\implies 1 = m(\hat{A}\hat{B}\hat{C}) = m(\hat{\sigma}_x\otimes\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_y\otimes\hat{\sigma}_x) = m(\hat{\sigma}_x^{(1)})m(\hat{\sigma}_y^{(2)}\hat{\sigma}_x^{(2)}\hat{\sigma}_y^{(2)})m(\hat{\sigma}_x^{(3)}) = m(\hat{\sigma}_x^{(1)})m(\hat{\sigma}_x^{(2)})m(\hat{\sigma}_x^{(3)}) = m(\hat{D}\equiv\hat{\sigma}_x\otimes\hat{\sigma}_x\otimes\hat{\sigma}_x)$, where $\hat{\sigma}_x^{(1)}\equiv\hat{\sigma}_x\otimes\hat{\mathbb{I}}\otimes\hat{\mathbb{I}}$ and so on. It must be stressed that although $\hat{A}$, $\hat{B}$ and $\hat{C}$ mutually commute, $\hat{\sigma}_2^{(2)},\hat{\sigma}_y^{(2)}$ do not. We used property (b) over both these sets of operators in the previous step. Returning to the argument, note that $\hat{D}\,|\chi_G\rangle = -\,|\chi_G\rangle$, $\implies m(\hat{D}) = -1$ which yields a contradiction. We therefore conclude that no m with the said properties exists which is consistent with all predictions of QM. □

Here m maybe viewed as a specific class of prediction maps, that don't depend on the state $[\psi]$. It follows then that the maps ruled out by the theorem are non-contextual. The set of maps ruled out, however, can be enlarged by imposing fewer restrictions on it. This is achieved by the following theorem.

Theorem 8. *Let a map $m:\hat{\mathcal{H}}\to\mathbb{R}$, be s.t. (a) $m(\hat{\mathbb{I}}) = 1$, (b) $m(f(\hat{B}_1,\hat{B}_2,\dots)) = f(m(\hat{B}_1),m(\hat{B}_2),\dots)$, for any arbitrary function f, where $\hat{B}_i$ are mutually commuting Hermitian operators. If m is assumed to describe the outcomes of measurements, then no m exists which is consistent with all predictions of Quantum Mechanics.*

Proof. Bell construction ($|\mathcal{H}|\geq 4$): □

Proof. Generalized GHZ construction ($|\mathcal{H}|\geq 6$): □

Consider again, three observers with one quantum particle each. Each observer can measure the properties corresponding to $\hat{\sigma}_x$ and $\hat{\sigma}_y$ of his/her particle. The quantum state of these particles is also the same as before, $\sqrt{2}\,|\chi_G\rangle = |000\rangle - |111\rangle$. We define as earlier $\hat{A} := \hat{\sigma}_x \otimes \hat{\sigma}_y \otimes \hat{\sigma}_y \implies \hat{A}\,|\chi_G\rangle = |\chi_G\rangle$, $\hat{B} := \hat{\sigma}_y \otimes \hat{\sigma}_x \otimes \hat{\sigma}_y$ and $\hat{C} := \hat{\sigma}_y \otimes \hat{\sigma}_y \otimes \hat{\sigma}_z$. Consider in addition the following operators, represented by $\hat{H}_{ij}$.

$$\hat{H}_{ij} \doteq \begin{bmatrix} \hat{\sigma}_x \otimes \hat{\mathbb{I}} \otimes \hat{\mathbb{I}}^{(a)} & \hat{\mathbb{I}} \otimes \hat{\sigma}_y \otimes \hat{\mathbb{I}}^{(2)} & \hat{\mathbb{I}} \otimes \hat{\mathbb{I}} \otimes \hat{\sigma}_y^{(3)} \\ \hat{\sigma}_y \otimes \hat{\mathbb{I}} \otimes \hat{\mathbb{I}}^{(1)} & \hat{\mathbb{I}} \otimes \hat{\sigma}_x \otimes \hat{\mathbb{I}}^{(b)} & \hat{\mathbb{I}} \otimes \hat{\mathbb{I}} \otimes \hat{\sigma}_y^{(3)} \\ \hat{\sigma}_y \otimes \hat{\mathbb{I}} \otimes \hat{\mathbb{I}}^{(1)} & \hat{\mathbb{I}} \otimes \hat{\sigma}_y \otimes \hat{\mathbb{I}}^{(2)} & \hat{\mathbb{I}} \otimes \hat{\mathbb{I}} \otimes \hat{\sigma}_x^{(c)} \\ \hat{\sigma}_x \otimes \hat{\mathbb{I}} \otimes \hat{\mathbb{I}}^{(a)} & \hat{\mathbb{I}} \otimes \hat{\sigma}_x \otimes \hat{\mathbb{I}}^{(b)} & \hat{\mathbb{I}} \otimes \hat{\mathbb{I}} \otimes \hat{\sigma}_x^{(c)} \end{bmatrix},$$

where from property (b) it follows that $m(\hat{A}) = m(\hat{H}_{11})m(\hat{H}_{12})m(\hat{H}_{13})$. Note that $[\hat{H}_{ij}, \hat{H}_{ik}] = 0$, viz. the operators along a row commute. To be consistent with QM, $m(\hat{A}) = +1$ (similarly for row 2 and 3). $m(\hat{D}) = -1$ imposes that the product $m(\hat{H}_{41})m(\hat{H}_{42})m(\hat{H}_{43})$, row 4, must be -1. Using the notation $m(\hat{H}_{ij}) = H_{ij}$, we can start with the first case. For the last row to yield -1, we must have either one -1 assignments or three -1 assignments. Let us look at the first case. Without loss of generality, we can assume that $H_{41} = H_{42} = +1$ and $H_{43} = -1$. Further, $H_{11} = H_{41}$, $H_{22} = H_{42}$ and $H_{33} = H_{43}$ by construction (see below).

$$H_{ij} \doteq \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \\ 1 & 1 & -1 \end{bmatrix}, H_{ij} \doteq \begin{bmatrix} 1 & \pm 1 & \pm 1 \\ \pm 1 & 1 & \pm 1 \\ \pm 1 & \pm 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Next, one must ensure that the first row yields $+1$ which requires us to either set $H_{12} = H_{13}$ to -1 or $+1$. Let us write this as $H_{12} = H_{13} = \pm 1$ which would entail that $H_{23} = \pm 1$ and $H_{32} = \pm 1$. [Note that the notation implies that when we choose $+1$, all these are $+1$ and when we choose -1, all are -1. We are not allowed to set, say $H_{12} = -1$ and $H_{13} = +1$.] For the second row to be $+1$, we conclude that $H_{21} = \pm 1$. This entails, in turn, that $H_{31} = \pm 1$. We now note that the third row has become -1 whereas according to QM, it must be $+1$. It is left upto the reader to see that similar reasoning also yields results in inconsistencies with QM. This completes the proof.

The following proof is not only more general, but also independent of the initial state of the system.

Proof. Peres Mermin ($|\mathcal{H}| \geq 4$):

Consider the following set of operators

$$\hat{A}_{ij} \doteq \begin{bmatrix} \hat{\mathbb{I}} \otimes \hat{\sigma}_x & \hat{\sigma}_x \otimes \hat{\mathbb{I}} & \hat{\sigma}_x \otimes \hat{\sigma}_x \\ \hat{\sigma}_y \otimes \hat{\mathbb{I}} & \hat{\mathbb{I}} \otimes \hat{\sigma}_y & \hat{\sigma}_y \otimes \hat{\sigma}_y \\ \hat{\sigma}_y \otimes \hat{\sigma}_x & \hat{\sigma}_x \otimes \hat{\sigma}_y & \hat{\sigma}_z \otimes \hat{\sigma}_z \end{bmatrix}$$

which have the property that all operators along a row (and along a column) commute. It is trivial to see that this holds for the first two rows and the first two columns. To see that this holds also for the last row and column, note the anti-commutation relation, $\{\hat{\sigma}_x, \hat{\sigma}_y\} = 0$ and that $\sigma_z = i\sigma_y\sigma_x$, which one may check explicitly. Another property is that the product of rows (columns) yield $\hat{R}_i = \mathbb{I}$ and $\hat{C}_j = \mathbb{I}\,(j \neq 3)$, $\hat{C}_3 = -\mathbb{I}$, $(\forall\, i, j)$ where $\hat{R}_i \equiv \prod_j \hat{A}_{ij}$, $\hat{C}_j \equiv \prod_i \hat{A}_{ij}$. This can be verified easily by using the aforesaid relations and the fact that $\sigma^2 = 1$, for every Pauli matrix. Using this property of $\hat{R}_i$ and $\hat{C}_j$, it is easy to show that no map m with the stated properties can exist. Let us assume on the contrary that m exists. From property (b) of the map, to get $m(\hat{C}_3) = -1$ (used property (a)), we must have an odd number of -1 assignments in the third column. In the remaining columns, the number of -1 assignments must be even for each column. Thus, in the entire square, the number of -1 assignments must be odd. Let us use the same reasoning, but along the rows. Since each $m(\hat{R}_i) = 1$, we must have even number of -1 assignments along each row. Thus, in the entire square, the number of -1 assignments must be even. We have arrived at a contradiction and therefore we conclude that our assumption that m exists, must be wrong. ☐

Proof. Kochen Specker ($|\mathcal{H}| \geq 3$): The original proof given by [2] is rather involved. Here we will satisfy ourselves with assuming its validity and applying our constructions to the aforesaid proof(s). ☐

Remark 9. One could in principle assume m, to be s.t. (a) $m(\hat{\mathbb{I}}) = 1$, (b) $m(\alpha\hat{B}_i) = \alpha m(\hat{B}_i)$, for $\alpha \in \mathbb{R}$, (c) $m(\hat{B}_i^2) = m(\hat{B}_i)^2$, (d) $m(\hat{B}_i + \hat{B}_j) = m(\hat{B}_i) + m(\hat{B}_j)$, to deduce (d') $m(\hat{B}_i\hat{B}_j) = m(\hat{B}_i)m(\hat{B}_j)$ and that $m(\hat{B}_i) \in$ spectrum of $\hat{B}_i$. Effectively then, condition (b) listed in the theorem is satisfied as a consequence. Therefore, assuming (a)-(d) as listed here, rules out a larger class of m.

According to theorem 8, non-contextual maps which are *multiplicative* must be incompatible with QM. It is apparent that non-contextual maps which are non-*multiplicative* might be consistent with QM. To that end, in the following section we first show that this is indeed the case by removing contextuality from Bohmian Mechanics and supplement it by an explicit construction subsequently.

Before proceeding we note, however, that QM enforces *sequential multiplicativity* at least in certain cases. This result will be used to assess accuracy of any hidden variable model we might propose.

Proposition. *Let the system be in a state, s.t. measurement of $\hat{C}$ yields repeatable results (same result each time). Then according to QM,* sequential multiplicativity *holds, where $\hat{C} \equiv f(\hat{B}_1, \hat{B}_2, \dots \hat{B}_n)$, and $\hat{B}_i$ are as defined.*

Proof. Assume without loss of generality that $\hat{B}_1, \hat{B}_2, \dots \hat{B}_n$ are mutually compatible (commuting) and complete set of operators. If say for instance the set is not complete, then one can add the missing operators and label them as aforesaid. It follows that $\exists \left|\mathbf{b} = \left(b_1^{(l_1)}, b_2^{(l_2)}, \dots b_n^{(l_n)}\right)\right\rangle$ s.t. $\hat{B}_i \left|\mathbf{b}\right\rangle = b_i^{(l_i)} \left|\mathbf{b}\right\rangle$, where l_i indexes the eigenvalues corresponding to $\hat{B}_i$ and that $\sum_{\mathbf{b}} \left|\mathbf{b}\right\rangle \left\langle \mathbf{b}\right| = \hat{\mathbb{I}}$. Let the state of the system be given by $\left|\psi\right\rangle$ and it must be s.t. $\hat{C}\left|\psi\right\rangle = c\left|\psi\right\rangle$, by assumption. For the statement to follow, one need only show that $\left|\psi\right\rangle$ must be made of only those $\left|\mathbf{b}\right\rangle$s, which satisfy $c = f(b_1^{(l_1)}, b_2^{(l_2)}, \dots b_n^{(l_n)})$. This is the crucial step. Proving this is so is trivial. We start with $\hat{C}\left|\psi\right\rangle = c\left|\psi\right\rangle$ and take its inner product with $\left\langle \mathbf{b}\right|$ to get

$$\begin{aligned}
\left\langle \mathbf{b}\right| \hat{C} \left|\psi\right\rangle &= c \left\langle \mathbf{b}|\psi\right\rangle , \\
\left\langle \mathbf{b}\right| f(\hat{B}_1, \hat{B}_2, \dots \hat{B}_n) \left|\psi\right\rangle &= c \left\langle \mathbf{b}|\psi\right\rangle , \\
f(b_1^{(l_1)}, b_2^{(l_2)}, \dots b_n^{(l_n)}) \left\langle \mathbf{b}|\psi\right\rangle &= c \left\langle \mathbf{b}|\psi\right\rangle .
\end{aligned}$$

Also, we have $\left|\psi\right\rangle = \sum_{\mathbf{b}} \left\langle \mathbf{b}|\psi\right\rangle \left|\mathbf{b}\right\rangle$, from completeness. If we consider $\left|\mathbf{b}\right\rangle$s for which $\left\langle \mathbf{b}|\psi\right\rangle \neq 0$, then we can conclude that indeed $c = f(b_1^{(l_1)}, b_2^{(l_2)}, \dots b_n^{(l_n)})$. However, when $\left\langle \mathbf{b}|\psi\right\rangle = 0$, viz. $\left|\mathbf{b}\right\rangle$s that are orthogonal to $\left|\psi\right\rangle$, then nothing can be said. We can thus conclude that $\left|\psi\right\rangle$ is made only of those $\left|\mathbf{b}\right\rangle$s that satisfy the required relation. That completes the proof. □

It is worth noting that in the PM case, where $\hat{R}_i$ and $\hat{C}_j$ are just $\pm\hat{\mathbb{I}}$, it follows that all states are their eigenstates. Consequently, for these operators *sequential multiplicativity* must always hold.

A. Non-contextual Bohmain Mechanics

Recall that contextuality in Bohmian Mechanics arises from the mapping between an operator and the experimental setup. Consider the following proposition.

Lemma 10. *For discrete $\mathcal{H}$, $\exists$ a map $e : \hat{\mathcal{H}} \to \mathcal{E}$.*

Proof. To describe an experimental setup corresponding to a measurement, we use a measuring particle (mass m_L, say). It is made to interact with the system for a short duration appropriately and subsequently we measure the resultant position of the particle, to learn about the value of the observable of interest. Say for instance, we wish to measure the observable $\hat{L}$, then the required interaction Hamiltonian is given by $\hat{H}_{\text{int}} = a\hat{L} \otimes \hat{p}$. Here the first operator acts on the system and second (after the tensor product symbol) acts on the measuring particle. a quantifies the interaction strength. [TODO: check this] It has dimensions of frequency, if L has dimensions of length. Let us assume that the system in the state $\left|\psi\right\rangle$. Then it is known that one can express $\left|\psi\right\rangle = \sum_l \left\langle l|\psi\right\rangle \left|l\right\rangle$, where $\left|l\right\rangle$ are the eigenstates of $\hat{L}$ with eigenvalue l (we have assumed non-degeneracy for simplicity, but its removal doesn't cause any significant difficulty). $\left|\Psi_S(t)\right\rangle = \hat{U}(t)\left|\psi\right\rangle \otimes \left|\varphi\right\rangle$, where

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\left[-\hbar^2 \frac{\nabla_1^2}{2m} - \hbar^2 \frac{\nabla_2^2}{2m_L} + \hat{H}_{\text{int}}\right] t}$$

and $\left|\varphi\right\rangle$ is the state of the measuring particle, given by a Guassian centred at the origin, $\varphi(q) = (1/\sqrt{2\pi}\sigma) e^{-q^2/2\sigma^2}$. If t is very small, and a very large, s.t. $at = \lambda$ is a finite number, then one can neglect the free evolution, $\nabla^2 t$ terms compared to $\hat{H}_{\text{int}} t$. We then have

$$\begin{aligned}
\left|\Psi_S(t)\right\rangle &= e^{-\frac{i}{\hbar} a\hat{L}\otimes\hat{p} t} \left|\Psi_S\right\rangle \\
&= \sum_l \left\langle l|\psi\right\rangle \left|l\right\rangle \otimes e^{-\frac{i}{\hbar}\lambda l \, \hat{p}} \left|\varphi\right\rangle \\
&= \sum_l \left\langle l|\psi\right\rangle \left|l\right\rangle \otimes \left|\varphi_{\lambda l}\right\rangle ,
\end{aligned}$$

where $|\varphi_{q_0}\rangle = \int dq\varphi(q-q_0)|q\rangle$. This interaction, effectively entangles the measuring particle, with the possible 'outcomes', eigenstates of the observable $\hat{L}$ of interest. If $\sigma \ll \lambda l$, then according to QM itself, a position measurement of the measuring particle, would correspond, in a one-to-one way, to the eigenstate/eigenvalue of $\hat{L}$, to which the system will collapse. Therefore, given an observable, one can construct the aforesaid experiment to measure its value. We have, in effect, explicitly constructed the map $e : \hat{\mathcal{H}} \to \mathcal{E}$. □

The proof of this proposition can also be motivated by a simple construction. Consider that the system of interest are spin half particles. To measure spin along any direction, there are two directions along which the magnetic field gradient can be set (as was discussed in the introduction). The experimentalist agrees to always use a specific direction, thereby mapping an operator to an experimental setup uniquely. Given this, one can immediately derive a consistent non-multiplicative prediction map m.

Proposition 11. *The Bohmian map, m_B can be restricted to a prediction map m, we call 'Bohmian prediction map'.*

Proof. $m(\hat{\mathcal{H}}, [\mathcal{H}]) = m_B(\hat{\mathcal{H}}, [\mathcal{H}], e(\hat{\mathcal{H}}))$. □

Proposition 12. *The Bohmian prediction map, m must be non-multiplicative.*

Proof. Since Bohmian Mechanics is consistent with all predictions of QM, the Bohmian prediction map m must also be. According to theorem 8, m must be non-multiplicative. □

B. C-ingle model | An Explicit Construction

For readers skeptic about Bohmian Mechanics, the discussion related to it can be replaced by the following. Let the state of the system be $|\chi\rangle$, defined on a discrete Hilbert space (spin like) and we wish to assign a value to an arbitrary operator $\hat{A} = \sum_a a|a\rangle\langle a|$, which has eigenvalues $\{a_{\min} = a_1 \le a_2 \cdots \le a_n = a_{\max}\}$. This model has the following postulates:
1. Initial Hidden Variable: Pick a $c \in [0,1]$, from a uniform random distribution.
2. Assignment/Prediction: The value assigned to $\hat{A}$ is given by finding the smallest a s.t.

$$c \le \sum_{a'=a_{\min}}^{a} |\langle a'|\chi\rangle|^2 ,$$

viz. we have specified a prediction map, $m(\hat{A}) = a$.
3. Update: After measuring an operator, the state must be updated (collapsed) in accordance with the rules of QM. This completely specifies the sequence map s.

To see how this works, we restrict ourselves to a single spin case. Say $|\chi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$, and $\hat{A} = \hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$. Now, according to the postulates of this theory, $m(\hat{A}) = +1$, if $c \le \cos^2\theta$, else $\hat{A}$ will be assigned -1. It follows then, from c being uniformly random in $[0,1]$, that the statistics agree with predictions of QM. The reader can convince him(her)self that the said scheme works in general. A more interesting application of this to the PM situation follows.

We can clearly see that the assignment, described by the prediction map m, is non-contextual since given an operator and a state (+ the hidden variable), the value is uniquely assigned. The map m is non-multiplicative which will be clarified by the following.

Example 13. [C-ingle model applied to the Peres Mermin situation] We construct a simplified hidden variable model for the Peres Mermin situation (TODO: Add as a test of contextuality?) and explicitly show that the associated prediction map m is non-multiplicative. We subsequently show that this model is a special case of the C-ingle model.

The simplified model uses the following algorithm for generating the map prediction map m.
1. Initial State: Choose an appropriate initial state $|\psi\rangle$ (to start with, for illustration assume $|\psi\rangle = |00\rangle$; subsequently use step 4).
2. Hidden Variable (HV): Toss a coin and assign $c = +1$ for heads, else assign $c = -1$.
3. Predictions/Assignments: For an operator $\hat{p}' \in \{\hat{A}_{ij}, \hat{R}_i, \hat{C}_j\ (\forall\, i,j)\}$ check if $\exists$ a λ, s.t. $\hat{p}'|\psi\rangle = \lambda|\psi\rangle$. If $\exists$ a λ, then assign $m(\hat{p}') = \lambda$ as the value. Else, assign $m(\hat{p}') = c$.
So far the model has specified only the prediction map m. The sequence map s is again defined to be the collapse rule of QM. Explicitly we have the following.

4. Update: Say $\hat{p}$ was observed. If $\hat{p}$ is s.t. $\hat{p}\,|\psi\rangle = \lambda\,|\psi\rangle$, then leave the state unchanged. Else, find $|p_\pm\rangle$ (eigenkets of $\hat{p}$), s.t. $\hat{p}\,|p_\pm\rangle = \pm\,|p_\pm\rangle$ and update the state $|\psi\rangle \to |p_c\rangle$.

Let us apply the aforesaid algorithm, to the state $|\psi\rangle = |00\rangle$ for illustration as stated. Say we obtained tails, and thus $c = -1$. To arrive at the assignments, note that $|00\rangle$ is an eigenket of only $\hat{R}_i, \hat{C}_j$ and $\hat{A}_{33} = \hat{\sigma}_z \otimes \hat{\sigma}_z$. Thus, in the first iteration, all these should be assigned their respective eigenvalues. The remaining operators must be assigned c (see equation (1)). Two remarks are in order. First, this model is *non*-multiplicative, for $m_1(\hat{C}_3) = 1 \neq m_1(\hat{A}_{13})m_1(\hat{A}_{23})m_1(\hat{A}_{33}) = -1$, where the subscript has been introduced to index the iteration. Second, we must impose sequential multiplicativity as a consistency check of the model, which in particular entails that $m_1(\hat{C}_3) = m_1(\hat{A}_{33})m_2(\hat{A}_{23})m_3(\hat{A}_{13})$. To illustrate this, we must choose to measure $\hat{A}_{33}$. According to step 4, since $|00\rangle$ is an eigenstate of $\hat{A}_{33}$, the final state remains $|00\rangle$.

$$
\begin{array}{c|ccc}
\text{Iteration} & i=1 & i=2 & i=3 \\
|\psi_{\text{init}}\rangle & |00\rangle & |00\rangle & \frac{|00\rangle+|11\rangle}{\sqrt{2}} \\
\text{HV/Toss} & c=-1 & c=-1 & c=+1 \\
 & & & \\
\text{Predictions} & m_1(\hat{A}_{ij}) \doteq \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & +1 \end{bmatrix} & m_2(\hat{A}_{ij}) \doteq \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & +1 \end{bmatrix} & m_3(\hat{A}_{ij}) \doteq \begin{bmatrix} +1 & +1 & +1 \\ +1 & +1 & -1 \\ +1 & +1 & +1 \end{bmatrix} \\
\text{(Assignments)} & & & \\
 & m_1(\hat{R}_i), m_1(\hat{C}_j) = +1\,(j\neq 3) & m_2(\hat{R}_i), m_2(\hat{C}_j) = +1\,(j\neq 3) & m_3(\hat{R}_i), m_3(\hat{C}_j) = +1\,(j\neq 3) \\
 & m_1(\hat{C}_3) = -1 & m_2(\hat{C}_3) = -1 & m_3(\hat{C}_3) = -1 \\
\text{Operator} & & & \\
\text{Measured} & \hat{A}_{13} = \hat{\sigma}_z \otimes \hat{\sigma}_z; m_1(\hat{A}_{13}) = +1 & \hat{A}_{23} = \hat{\sigma}_y \otimes \hat{\sigma}_y; m_2(\hat{A}_{23}) = -1 & \hat{A}_{33} = \hat{\sigma}_x \otimes \hat{\sigma}_x; m_3(\hat{A}_{33}) = +1 \\
 & & & \\
|\psi_{\text{final}}\rangle & |00\rangle & \frac{|00\rangle+|11\rangle}{\sqrt{2}} & \frac{|00\rangle+|11\rangle}{\sqrt{2}}
\end{array} \tag{1}
$$

For the next iteration, $i = 2$, viz. after the first measurement has been performed, say we again yield $c = -1$. Since $|\psi\rangle$ is also unchanged the assignment remains invariant. For the final step we choose to measure $\hat{p} = \hat{A}_{23} (= \hat{\sigma}_y \otimes \hat{\sigma}_y)$, to proceed with sequentially measuring $\hat{C}_3$. To simplify calculations, we note

$$|00\rangle = \frac{(|\tilde{+}\tilde{-}\rangle + |\tilde{-}\tilde{+}\rangle)/\sqrt{2} + (|\tilde{+}\tilde{+}\rangle + |\tilde{-}\tilde{-}\rangle)/\sqrt{2}}{\sqrt{2}},$$

where $|\tilde{\pm}\rangle = |0\rangle \pm i\,|1\rangle$ (eigenkets of $\hat{\sigma}_y$). Since $|00\rangle$ is manifestly not an eigenket of $\hat{p}$, we must find $|p_-\rangle$, since $c = -1$. It is immediate that $|p_-\rangle = \left(|\tilde{+}\tilde{-}\rangle + |\tilde{-}\tilde{+}\rangle\right)/\sqrt{2} = (|00\rangle + |11\rangle)/\sqrt{2}$, which becomes the final state.

For the final iteration, $i = 3$, say we yield $c = 1$. So far, we have $m_1(\hat{A}_{33}) = 1$ and $m_2(\hat{A}_{23}) = -1$. [TODO: Fix this statement] We must obtain $m_3(\hat{A}_{13}) = 1$, independent of the value of c, to be consistent. Let's check that. According to step 3, since $\hat{\sigma}_x \otimes \hat{\sigma}_x (|00\rangle + |11\rangle)/\sqrt{2} = 1\,(|00\rangle + |11\rangle)/\sqrt{2}$, $m_3(\hat{A}_{13}) = 1$ indeed. As a remark, it maybe emphasised that the $m_2(\hat{A}_{33}) = m_3(\hat{A}_{33})$ and $m_2(\hat{A}_{23}) = m_3(\hat{A}_{23})$, which essentially expresses compatibility of these observables, viz. measurement of $\hat{A}_{13}$ doesn't affect the result one would obtain by measuring operators compatible to it (granted they have been measured once before).

IV. IMPLICATIONS

Definition 14. A theory is *local* iff two spatially distinct points can not be influenced instantaneously.

Proposition 15. *Consider a product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, corresponding to two particles. $\mathcal{H}_A$ and $\mathcal{H}_B$ are discrete (correspond to spins say). Let mutually commuting observables $\hat{B}_i$ be s.t. $\hat{B}_i = \hat{\mathbb{I}} \otimes \hat{A}_{Bi}$ or $\hat{B}_i = \hat{A}_{Ai} \otimes \hat{\mathbb{I}}$, where $A_{Ai} \in \hat{\mathcal{H}}_A$ and $A_{Bi} \in \hat{\mathcal{H}}_B$ are arbitrary observables. If locality is assumed, then a prediction map m and it's associated sequence map s must be s.t. over these observables, sequential multiplicativity and multiplicativity become identical.*

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