Contextuality in a Deterministic Quantum Theory

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Abstract

Einstein started with showing that Quantum Mechanics (QM) must be incomplete if one assumes (a) realism and (b) locality. He wanted us to believe that there must be some theory which satisifed both (a) & (b) and can produce the intended results. Bell later showed that certain correlations between measurable properties must be bounded, given (a) and (b) hold. QM he showed violates this and therefore it doesn't satisfy atleast one among (a) and (b). This was confirmed experimentally.
Our story begins with a theory which satisfies (a) but not (b), while being completely identical with QM in terms of experimental predictions. Such a theory should exist is by itself rather discomforting. Infact, the said theory, known as Bohmian Mechanics (BM), is deterministic. Which entails that while it maybe expected that the correlations can arise from the explicit non locality in the theory, what happens to questions related to contextuality? It is this that will be explored in the remainder of the text.

For the reader who's more familiar, it would already be known that usual formulations of contextuality (and other determinism tests) are for discrete degrees of freedom, viz. spins (for massive particles). Since Bohmian Mechanics doesn't claim that spins have pre-defined values, it is not surprising that both are compatible, yet the analysis is illuminating. It sheds light on how non-locality and contextuality are related. Further, tests of contextuality maybe extended to phase space (q, p). In this setting then, the tests show that (q, p) can't be deterministic, roughly speaking, while antithetically, BM is infact deterministic in (q, p)! To explore this apparent contradiction, we construct an appropriate test of determinism and analyse it in the framework of BM. The analysis is expected to provide further insights into the relation between non-locality and contextuality, especially in continuous variables, where the current understanding is limited. Additionally, we might glimpses into learning the fundamental difference between spins (discrete) and phase space variables (continuous), if at all there is any.

Part I

Prior Art

Chapter 1

Classical Mechanics

It is worthwhile spending some time trying reformulate classical mechanics (CM) into a form that is closest to quantum mechanics. Although this is not necessary for understanding BM (and may very well be skipped), doing so brings with it it's own perspective and excitement.

1.1 The Hamilton Jacobi Theory

Let's start with a Lagrangian, $L = L(q, \dot{q}, t)$, where $p_i := \partial L/\partial \dot{q}_i$, $H := p_i \dot{q}_i - L$, q_i represents the position of the ith particle and $q, \dot{q}$ refer to the set $\{q_1, q_2, \dots\}$, $\{\dot{q}_1, \dot{q}_2, \dots\}$ respectivley. If we imagine a path that the particle takes from a point q_0, t_0 to q, t, then we can define a quantity $I(q, t; q_0, t_0) := \int_{q_0,t_0}^{q,t} L(q, \dot{q}, t)dt$ where the integral is taken along the aforesaid path. We know from experiments that the path the particle actually takes, given L, is one which extremizes I. This entails that, just like at the extremum value of the function, it's derivative vanishes, the first order variation of I for a path that extremizes I must vanish; viz. $\delta I = 0$ for any extremal path. As the reader would recall, after some calculation, this yields

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0.$$

This can be expressed in terms of H instead of L by substituting for L in the definition of I and changing the independent variables from $(q, \dot{q})$ to (q, p) & varying them, where p refers to $\{p_1, p_2, \dots\}$. The result is the familiar Hamilton equations (this again requires some calculation and may not always be possible)

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Defn: Canoncial Transformation: $q, p \to Q = Q(q, p, t)$, $P = P(q, p, t)$ is a a canonical transformation if Q, P preserve the form of the Hamilton Equations, viz. if $K = K(Q, P, t)$ is the new H, then we must have $\dot{Q}_i = \frac{\partial K}{\partial P_i}$, $\dot{P}_i = -\frac{\partial K}{\partial Q_i}$ which could've come from $I = \int P_i dQ_i - K dt$.

It is known that if two Ls differ by $\frac{df}{dt}$, where f [check: where $f = f(q, \dot{q}, t)$] is some function, then they yield identical equations of motion. [) This follows at once by recalling that $I_1 = \int_{q_0,t}^{q,t} L_1 dt$ and noting that $I_2 = \int_{q_0,t}^{q,t} L_1 dt + F - F_0$ entails that the variation $\delta I_2 = \delta I_1 + \cancelto{0}{\delta F} - \cancelto{0}{\delta F_0} = \delta I_1$(]. Thus if we write the Lagrangian in terms of the Hamiltonian and impose

$$P_i dQ_i - K(Q, P)dt + \frac{dF}{dt} = p_i dq_i - H(q, p)dt,$$

then we're guaranteed that the equation of motion will be identical and hence describe the same physical system. Assuming a specific functional form, viz. $F = F(q, Q, t)$, one can evaluate

$$
\begin{aligned}
p_i =& \frac{\partial F}{\partial q_i},\\
P_i =& \frac{\partial F}{\partial Q_i},\\
K =& H + \frac{\partial F}{\partial t}.
\end{aligned}
$$

[) This follows from the restatement $\left(P_i + \frac{\partial F}{\partial q_i}\right)\dot{Q}_i - \left(p_i - \frac{\partial F}{\partial q_i}\right)\dot{q}_i + \left(K - H - \frac{\partial F}{\partial t}\right) = 0$ and by choosing q, Q to be the independent variables. (] Note that the first equation will yield Q in terms of p and q. The following equation yields P in terms of Q and q. The last equation yields K, since Q and P are known (it's not obvious though that writing $K = K(Q, P)$ will be straight forward).

Thus, given $F(q, Q, t)$ we can find the canonical transformation $Q = Q(q, p, t)$, $P = P(q, p, t)$ and even $K = K(q, p, t)$. Defn: (by example)F is a generating function.

Next let us consider a case where the transformed variables are almost the same as the original set, viz.

$$
\begin{aligned}
Q_i =& q_i + \delta q_i,\\
P_i =& p_i + \delta p_i.
\end{aligned}
$$

1.2 Densities

1. To understand the continuity equation $\rho + \nabla.(\rho v) = 0$, examples of v are taken

   a) $v = v(x)$, a solution is obtained as

   $$\rho(x,t) = \frac{1}{v(x)} v\left[x\left(t - \int \frac{dx}{v}\right)\right]\rho_0\left[x\left(t - \int \frac{dx}{v}\right)\right]$$

   in which further assuming that $\rho(x)$ results in $\rho = A/\left|v(x)\right|$

      i. $v = v(t)$ then we get

      $$\rho(x,t) = \rho_0\left(x - \int v dt\right)$$

      which means that ρ is constant along particle trajectories

   b) connection with Liuoville's equation

      i. $f(x, p, t)$ is defined instead of $\rho(x, t)$. Pure and mixed states are defined accordingly as $f(x, p, t) = \rho(x, t)\delta(p - \nabla S(x, t))$ being pure and the remaining as mixed.

      ii. $\frac{df}{dt} = \partial_t f + \frac{1}{m}\sum p_i \partial_{x_i} f - \sum \partial_{x_i} V \partial_{p_i} f = 0$ is the Liuoville's equation (which holds since we can show that the volume doesn't change under Hamiltonian evolution and particles inside the volume stay inside; $f(p', q', t + \delta t) = f(p, q, t)$ is essentially the statement $\frac{df}{dt} = 0$) which is linear in f.

      iii. One may project out the moment space. They define equivalent of ρ as $P(x) = \int f d^3p$, mean momentum as $\overline{p_i(x)} = \frac{\int p_i f d^3p}{P(x)}$ and $\overline{p_i p_j(x)} = \frac{\int p_i p_j f d^3p}{P(x)}$. The louvielle equation can then be expressed in terms of these spatial variables. Integrating it we get

      $$\partial_t P + \frac{1}{m}\sum \partial_{x_i}(P\overline{p_i}) = 0.$$

      To get the momentum transport equation, after multiplying the louvielle equation with p_i and integrating, we get

      $$\partial_t(P\overline{p_i}) + \frac{1}{m}\sum \partial_{x_j}(P\overline{p_i p_j}) + P\partial_{x_i} V = 0$$

(apparently integrated by parts and assumed $f \to 0$ as $p_i \to \infty$)

While f is constant along a phase space trajectory, the spatial density P (equivalent of ρ) is not. It's apparent from the derivation of the continuity equation; either we start with a fixed volume or a fixed number of particles, not both.

If you substitute $f = \rho\delta(p - \nabla S)$ as stated earlier, you'd get $P = \rho$, $\overline{p_i} = \partial_{x_i} S$, $\overline{p_i p_j} = \partial_{x_i} S \partial_{x_i} S$ as expected. The substitution also yields what's called a field theoretic version of Newton's Laws given by

$$\partial_t \rho + \frac{1}{m}\nabla.(\rho \nabla S) = 0$$

and

$$\left[\partial_t + \frac{1}{m}\sum \partial_{x_i} S \partial_{x_j}\right] \partial_{x_i} S = 0$$

iv. Remarks:

A. It's not obvious that if we start with a state that has well defined momentum (delta distribution) but the positions are given by $\rho(x)$, then they will continue to be well defined in momentum. This happens only exceptionally. In general, a pure state maybe sent to a mixed state. We'll see examples of these. [todo: ensure examples make sense]

B. Can we decompose any mixed ensamble into a linear combination of pure ones? The answer's no. [proof?] Say there are many solutions of the Hamilton-Jacobi equation, given by S_i. Thus, we can construct a linear combination as $f(x,p,t) = \sum P_i \rho_i(x,t)\delta\left(p - \nabla S_i(x,t)\right)$ where P_i (degenerate notation) refers to the distribution of momenta at a given point. $\sum P_i = 1$ is assumed for normalization. Claim is that this is not in general possible to decompose a state into this form. An explicit example is that of reflecting through a potential barrier (in CM) [todo: ensure the example works]

C. While this is not particularly useful in CM (the pure and mixed states), the formalism helps in comparison with QM.

c) Pure and Mixed States

i. Illustration: We see that $f_0(x,p) = \delta(x - x_0)\delta(p - p_0)$ remains sharp (it can be checked by inserting it in the louviel equation) to yield $f(x,p) = \delta(x - x(t,x_0,p_0))\delta(p - p(t,x_0,p_0))$ [this is expected, since you're in essence saying there's only one particle]

ii. Illustration 2: We want to see what happens to a Gaussian like state, does it spread? We start with $\rho_0 = \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$ and $S_0 = px$ with σ and p constant. This form of S_0 has already been solved for and tells us $\rho = \frac{e^{-(x-vt)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$. There's no spreading classically! We'll see for the same initial conditions, what happens quantum mechanically.

iii. Illustration 3: What initial conditions yield a spreading Guassian? We start with the same ρ_0 but use $S_0 = \frac{m(x-x_0)^2}{2t}$, in which case the solution we saw the result is

Chapter 2

Deterministic Quantum Theory

2.1 Aside

So we start with $\oint dS = \oint \nabla S.dx = \oint p.dx = \int \nabla \times p.da$. If $\oint dS = nh$, then we must have $\nabla \times p = \sum_a \Gamma_a \int_{\gamma_a} \delta(x - x_a)dx_a$ where γ_a is the nodal line. If we assume $\oint dS = nh$ holds, then can we construct some example of the same? Let's first see how $\oint dS = nh$ can be derived. If the only condition is that ψ is single valued, then we know that at any point, S' and S both yield the same ψ, where $S' = S + nh$. If one considers a loop, then say we start from a point S_a. Then after completing some distance, the change in S is given by ΔS. So the value of S starting from $S_a$will be $S_a + \Delta S$. Now if we come back to the point a, then from uniqueness of ψ, we only demand $S_a + \Delta S = S_a + nh$. If S itself was unique, then we'd say $S_a + \Delta S = S_a$. Now at this point itself I seem to have trouble. I have tacitly assumed that S is single valued when I'm evaluating the 'change is S' along the curve.

Talked to manu for a while and made some progress, then figured it was non-sense and made some more progress. Finally, Manu found a document that helped clarify a few things. The issue was still that they had used a vector field and not a potential. And it wasn't clear to me what potential must I use in that case.

The potential is $V = k\theta$. Note how this is itself, as a function of position is multi valued and yet we never have any issues integrating this (as we'll see shortly). While V is multivalued, $\nabla V = \frac{k}{r}\hat{\theta}$ is happily single valued :) And not just that, check this; $\oint_\gamma \nabla V.dx = 2\pi$ (simply because γ is chosen to be a circle and then $dx = rd\theta\hat{\theta}$). Since in the domain of interest, everything is well defined, I can write $\oint_\gamma \nabla V.dx = \oint_\gamma dV = 2\pi$. And one can show independently (I know only a simple minded proof with discritizing the function) that $\oint dV = 0$ whenever V is single valued (or a function). So what does this example show? Various rather peculiar things. (I) that $\oint dV$ maybe non zero for a reasonable physical situation by virtue of multivaluedness of V. Yes, V is multivalued and yet we can integrate the said expression without ambiguity. (II) that there happens to be a singularity within the loop, over which the integral is non-zero. (III) The curl, $\nabla \times \nabla V \neq 0$ at the center and $= 0$ else.
Now we've made plausible various things which would've seemed arbitrary otherwise.

Chapter 3

Bohm's original proposal - A suggested interpretation of Quantum Theory in terms of "Hidden Variables" I

The usual interpretation of quantum mechanics (QM) is self consistent. It assumes however that the most complete possible specification of an individual system is in terms of ψ that yields only probabilistic results. To investigate if this assumption is accurate, it is reasonable to expect an experimental test to exist. However, as will be shown, no experiment can test this interpretation. The other alternative then, is to attempt constructing a theory, which has extra variables which in principle completely specify the system but in practice get averaged over. If such a theory existed then it is obvious that the assumption false. In his paper, Bohm describes such a theory. This theory yields the same results as QM in all physical situations studied and is shown to be broader conceptually. The mathematics used is shown to be more general. Bohm's idea was that while QM fails at high energies, his theory might work. In retrospection though, one realizes how both BM and QM suffer from the same problem which requires quantum field theory (QFT) to remedy.

3.1 Introduction

The usual interpretation assumes that the wavefunction is the most complete description. This description was criticized by Einstein for he believed that there must exist a better, precise theory that describes nature as opposed to QM that claims nature is fundamentally probabilistic. Einstein's views have themselves been criticised to be irrelevant because the usual QM interpretation is in excellent agreement with experiments.

Chapter 4

Theory and it's location

From the point of view of QM, the latter is just an appropriate limit of the former, while from the perspective of BM, the latter is treated significantly differently from the former. Perhaps then one can imagine a more 'secular' formulation.