Contextuality in a Deterministic Theory

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Motivation

- Two theories: Quantum Mechanics (QM) & Bohmian Mechanics (BM).
- Fundamental difference: BM is deterministic, viz. positions (q) and momenta (p) are well defined.
- Initial Aim: Construct a theoretical situation that defies determinism using QM and analyse using BM.

Notation I

Definition

Notation:

(a) $\psi \in \mathcal{H}$ would typically represent the quantum mechanical state of the system (assumed pure),

(b) $\hat{\mathcal{H}}$ is defined to mean $\mathcal{H} \otimes \mathcal{H}^{\dagger}$,

(c) $[\mathcal{H}]$ is defined to mean $(\mathcal{H},\ \mathbb{R}^{\otimes})$, which represents the state of the system including hidden variables,

(d) $[\psi] \in [\mathcal{H}]$ will represent the state of the system, including hidden variables,

(e) a prediction map is $m : \hat{\mathcal{H}}, [\mathcal{H}] \to \mathbb{R}$,

(f) a sequence map is $s : \hat{\mathcal{H}}, [\mathcal{H}], \mathbb{R} \to [\mathcal{H}]$,

(g) the set of all experimental setups is denoted by $\mathcal{E}$,

(h) a setup map is $e : \hat{\mathcal{H}} \to \mathcal{E}$.

Determinism Tests I

Theorem (GHZ)

Let a map $m : \hat{\mathcal{H}} \to \mathbb{R}$, be s.t. (a) $m(\hat{\mathbb{I}}) = 1$, (b) $m(f(\hat{A}_1, \hat{A}_2, \dots)) = f(m(\hat{A}_1), m(\hat{A}_2), \dots)$, for any arbitrary function f, where $\hat{A}_i$ are arbitrary Hermitian operators. If m is assumed to describe the outcomes of measurements, then no m exists which is consistent with all predictions of Quantum Mechanics.

Proof (GHZ [GHSZ90]):

- 3 observers with one particle each.
- Two properties, X and Y, with outcomes ± 1.
- State $\sqrt{2}\,|\chi_G\rangle = |000\rangle - |111\rangle$
- $\hat{A} := \hat{\sigma}_x \otimes \hat{\sigma}_y \otimes \hat{\sigma}_y$, $\hat{A}\,|\chi_G\rangle = |\chi_G\rangle$, $\hat{B} := \hat{\sigma}_y \otimes \hat{\sigma}_x \otimes \hat{\sigma}_y$ and $\hat{C} := \hat{\sigma}_y \otimes \hat{\sigma}_y \otimes \hat{\sigma}_z$; $\implies m(\hat{A}) = m(\hat{B}) = m(\hat{C}) = +1$.

Determinism Tests II

$$\begin{aligned} m(\hat{A})m(\hat{B})m(\hat{C}) &= 1 \\ \implies 1 = m(\hat{A}\hat{B}\hat{C}) &= m(\hat{\sigma}_x \otimes \hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_y \otimes \hat{\sigma}_x) \\ &= m(\hat{\sigma}_x^{(1)})m(\hat{\sigma}_y^{(2)}\hat{\sigma}_x^{(2)}\hat{\sigma}_y^{(2)})m(\hat{\sigma}_x^{(3)}) \\ &= m(\hat{\sigma}_x^{(1)})m(\hat{\sigma}_x^{(2)})m(\hat{\sigma}_x^{(3)}) \\ &= m(\hat{D} \equiv \hat{\sigma}_x \otimes \hat{\sigma}_x \otimes \hat{\sigma}_x), \end{aligned}$$

- where $\hat{\sigma}_x^{(1)} \equiv \hat{\sigma}_x \otimes \hat{\mathbb{I}} \otimes \hat{\mathbb{I}}$ and so on.
- However, $\hat{D}\,|\chi_G\rangle = -\,|\chi_G\rangle, \implies m(\hat{D}) = -1.$
- Contradiction!

Contextuality Tests I

Theorem (KS)

Let a map $m : \hat{\mathcal{H}} \to \mathbb{R}$, *be s.t. (a)* $m(\hat{\mathbb{I}}) = 1$, *(b)* $m(f(\hat{B}_1, \hat{B}_2, \dots)) = f(m(\hat{B}_1), m(\hat{B}_2), \dots)$, *for any arbitrary function* f, *where* $\hat{B}_i$ *are mutually commuting Hermitian operators. If* m *is assumed to describe the outcomes of measurements, then no* m *exists which is consistent with all predictions of Quantum Mechanics.*

Proof (new; $|\mathcal{H}| \geq 6$): GHZ generalized; Consider

$$\hat{H}_{ij} \doteq \begin{bmatrix} \hat{\sigma}_x \otimes \hat{\mathbb{I}} \otimes \hat{\mathbb{I}}^{(a)} & \hat{\mathbb{I}} \otimes \hat{\sigma}_y \otimes \hat{\mathbb{I}}^{(2)} & \hat{\mathbb{I}} \otimes \hat{\mathbb{I}} \otimes \hat{\sigma}_y^{(3)} \\ \hat{\sigma}_y \otimes \hat{\mathbb{I}} \otimes \hat{\mathbb{I}}^{(1)} & \hat{\mathbb{I}} \otimes \hat{\sigma}_x \otimes \hat{\mathbb{I}}^{(b)} & \hat{\mathbb{I}} \otimes \hat{\mathbb{I}} \otimes \hat{\sigma}_y^{(3)} \\ \hat{\sigma}_y \otimes \hat{\mathbb{I}} \otimes \hat{\mathbb{I}}^{(1)} & \hat{\mathbb{I}} \otimes \hat{\sigma}_y \otimes \hat{\mathbb{I}}^{(2)} & \hat{\mathbb{I}} \otimes \hat{\mathbb{I}} \otimes \hat{\sigma}_x^{(c)} \\ \hat{\sigma}_x \otimes \hat{\mathbb{I}} \otimes \hat{\mathbb{I}}^{(a)} & \hat{\mathbb{I}} \otimes \hat{\sigma}_x \otimes \hat{\mathbb{I}}^{(b)} & \hat{\mathbb{I}} \otimes \hat{\mathbb{I}} \otimes \hat{\sigma}_x^{(c)} \end{bmatrix},$$

Contextuality Tests II

where $m(\hat{A}) = m(\hat{H}_{11})m(\hat{H}_{12})m(\hat{H}_{13})$ should be $+1$ (similarly for row 2 and 3). $m(\hat{D}) = -1$ imposes row 4 must be -1.

$$H_{ij} \doteq \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \\ 1 & 1 & -1 \end{bmatrix}, H_{ij} \doteq \begin{bmatrix} 1 & \pm 1 & \pm 1 \\ \pm 1 & 1 & \pm 1 \\ \pm 1 & \pm 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Proof (Mermin's [Mer90]; $|\mathcal{H}| \geq 4$): Consider

$$\hat{A}_{ij} \doteq \begin{bmatrix} \hat{\mathbb{I}} \otimes \hat{\sigma}_x & \hat{\sigma}_x \otimes \hat{\mathbb{I}} & \hat{\sigma}_x \otimes \hat{\sigma}_x \\ \hat{\sigma}_y \otimes \hat{\mathbb{I}} & \hat{\mathbb{I}} \otimes \hat{\sigma}_y & \hat{\sigma}_y \otimes \hat{\sigma}_y \\ \hat{\sigma}_y \otimes \hat{\sigma}_x & \hat{\sigma}_x \otimes \hat{\sigma}_y & \hat{\sigma}_z \otimes \hat{\sigma}_z \end{bmatrix},$$

$\hat{R}_i = \mathbb{I}$ and $\hat{C}_j = \mathbb{I}\,(j \neq 3)$, $\hat{C}_3 = -\mathbb{I}$, $(\forall\, i, j)$ where $\hat{R}_i \equiv \prod_j \hat{A}_{ij}$, $\hat{C}_j \equiv \prod_i \hat{A}_{ij}$.

(a) No possible assignment can exist.

Contextuality Tests III

(b) For any map m, we'd have

$$\langle \hat{\chi}_{\mathrm{PM}} \rangle = \left\langle \hat{R}_1 \right\rangle + \left\langle \hat{R}_2 \right\rangle + \left\langle \hat{R}_3 \right\rangle + \left\langle \hat{C}_1 \right\rangle + \left\langle \hat{C}_2 \right\rangle - \left\langle \hat{C}_3 \right\rangle \leq 4,$$

whereas QM yields $\langle \hat{\chi}_{\mathrm{PM}} \rangle = 6 \nleq 4$.
Proof (Kochen Specker's [KS67]; $|\mathcal{H}| \geq 3$);

Definition

A theory is non-contextual, if it provides a map $m : \hat{\mathcal{H}}, [\mathcal{H}] \to \mathbb{R}$.

Remark

Non-contextual maps are not consistent with quantum mechanics.

Bohmian Mechanics I

Condensed Introduction [Boh52]

A particle is associated with (1) a position, q & momentum, p, precisely defined and (2) a wavefunction $\psi = Re^{iS/\hbar}$.

Postulates (one-dimensional):

1. Evolution of the wavefunction, is governed by Schrödinger's equation: $i\hbar\partial\psi/\partial t = -(\hbar^2/2m)\nabla^2\psi + V\psi$.
2. The particle is guided by the wavefunction: $\dot{q} = p/m$ where $p = \nabla S = \hbar \mathrm{Im}(\nabla\psi/\psi)$.
3. The initial distribution of the particles is given by $\rho(x) = |\psi|^2$.

NB: $\frac{R^2\nabla S}{m} = \frac{\hbar}{2mi}(\psi^* \overleftrightarrow{\nabla}\psi) = j$ (the probability current density)
$\implies |\psi(t)|^2$ holds $\forall\, t$ if it holds initially.

Bohmian Mechanics II

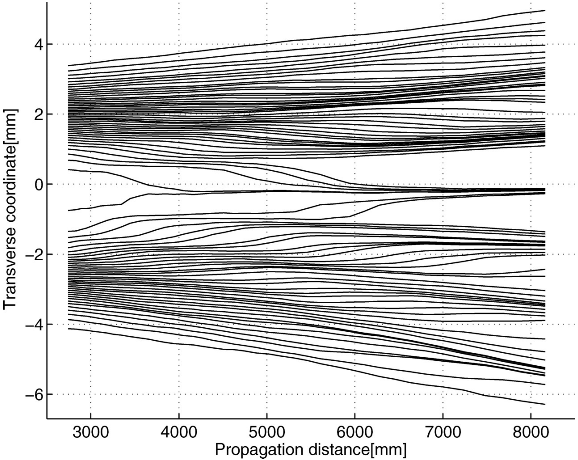


Figure: Experimentally observed average single-photon trajectories. "In the case of single-particle quantum mechanics, the trajectories measured in this fashion reproduce those predicted in the Bohm–de Broglie interpretation of quantum mechanics" [KBR+11]

Bohmian Mechanics III

- For N interacting particles, $p_i = \nabla_i S(q_1, q_2, \dots, q_N)$.
  NB: BM is an explicitly non-local, but complete description.
- Spins can also be included in BM.
  For a spinor, say $\psi \equiv (\psi_+, \psi_-)^T$, $p = \hbar \mathrm{Im}((\psi, \nabla\psi)/(\psi, \psi))$ where $(.,.)$ represents inner product in the spin space $\mathbb{C}^2$.

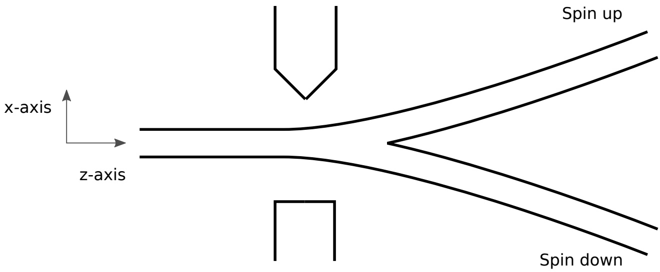


Figure: Contour plot of $|\Psi(q_x, t)|^2$ plotted for various $t = q_z/v_z$, illustrating a Stern-Gerlach measurement.

Bohmian Mechanics IV

Theorem

$\exists$ a (Bohmian) map $m_B : \hat{\mathcal{H}}, [\mathcal{H}], \mathcal{E} \to \mathbb{R}$, and a sequence map s, s.t. if m_B & s are assumed to describe the outcomes of measurements & the resultant state respectively, then they are consistent with all predictions of QM.

Proposition

For discrete $\mathcal{H}$, $\exists$ a map $e : \hat{\mathcal{H}} \to \mathcal{E}$.

Proposition

The Bohmian map, m_B can be restricted to a prediction map, we call 'Bohmian prediction map' as $m(\hat{\mathcal{H}}, [\mathcal{H}]) = m_B(\hat{\mathcal{H}}, [\mathcal{H}], e(\hat{\mathcal{H}}))$.

Multiplicativity I

Definition

A prediction map m is *multiplicative* iff

$$m(f(\hat{B}_1, \hat{B}_2, \dots \hat{B}_N), [\psi]) = f(m(\hat{B}_1, [\psi]), m(\hat{B}_2, [\psi]), \dots m(\hat{B}_N, [\psi])),$$

where $\hat{B}_i \in \hat{\mathcal{H}}$ are arbitrary mutually commuting operators, $f : \hat{\mathcal{H}}^{\otimes N} \to \hat{\mathcal{H}}$ and $[\psi] \in [\mathcal{H}]$.

Definition

A *non*-multiplicative map is one that is not multiplicative.

Proposition

The Bohmian prediction map, m must be non-multiplicative.

Multiplicativity II

Definition

A prediction map m is *sequentially multiplicative* for a given sequence map s, iff

$$m(f(\hat{B}_1, \hat{B}_2, \ldots \hat{B}_N), [\psi_1]) = f(m(\hat{B}_1, [\psi_{k_1}]), m(\hat{B}_2, [\psi_{k_2}]), \ldots, m(\hat{B}_N, [\psi_{k_N}$$

where $\mathbf{k} = (k_1, k_2, \ldots, k_N) \in \{(1, 2, 3 \ldots N), (2, 1, 3 \ldots N) +$ permutations, making $N!$ terms$\}$, $\hat{B}_i \in \hat{\mathcal{H}}$ are arbitrary mutually commuting observables, $[\psi_i] \in [\mathcal{H}]$, $f : \hat{\mathcal{H}}^{\otimes N} \to \hat{\mathcal{H}}$ and $[\psi_{k+1}] \equiv s(\hat{B}_k, [\psi_k], m(\hat{B}_k, [\psi_k]))$, $\forall\, [\psi_1]$.

Multiplicativity III

Proposition

A prediction map m *must be sequentially multiplicative for a sequence map* s*, for states* $|\psi\rangle$ *s.t. a measurement of* $f(\hat{B}_1, \hat{B}_2, \ldots \hat{B}_N)$ *yields repeatable results, to be consistent with quantum mechanics. Here* $\hat{B}_i \in \hat{\mathcal{H}}$ *are mutually commuting observables and* $f : \hat{\mathcal{H}}^{\otimes N} \to \hat{\mathcal{H}}$*.*

Toy-Model I

Peres Mermin Square:

$$\hat{A}_{ij} \doteq \begin{bmatrix} \hat{\mathbb{I}} \otimes \hat{\sigma}_x & \hat{\sigma}_x \otimes \hat{\mathbb{I}} & \hat{\sigma}_x \otimes \hat{\sigma}_x \\ \hat{\sigma}_y \otimes \hat{\mathbb{I}} & \hat{\mathbb{I}} \otimes \hat{\sigma}_y & \hat{\sigma}_y \otimes \hat{\sigma}_y \\ \hat{\sigma}_y \otimes \hat{\sigma}_x & \hat{\sigma}_x \otimes \hat{\sigma}_y & \hat{\sigma}_z \otimes \hat{\sigma}_z \end{bmatrix}.$$

The assignments are made by a three step process.

1. Initial State: Choose an appropriate initial state $|\psi\rangle$ (say $|00\rangle$).
2. Hidden Variable (HV): Toss a coin and assign $c = +1$ for heads, else assign $c = -1$.
3. Predictions/Assignments: For an operator $\hat{p}' \in \{\hat{A}_{ij}, \hat{R}_i, \hat{C}_j \, (\forall\, i, j)\}$ check if $\exists$ a λ, s.t. $\hat{p}' |\psi\rangle = \lambda |\psi\rangle$. If $\exists$ a λ, then assign λ as the value. Else, assign c.

Toy-Model II

4. Update: Say $\hat{p}$ was observed. If $\hat{p}$ is s.t. $\hat{p}\,|\psi\rangle = \lambda\,|\psi\rangle$, then leave the state unchanged. Else, find $|p_{\pm}\rangle$ (eigenkets of $\hat{p}$), s.t. $\hat{p}\,|p_{\pm}\rangle = \pm\,|p_{\pm}\rangle$ and update the state $|\psi\rangle \to |p_c\rangle$.

Remark

Observe that the Bohmian map m_B is contextual, but the Bohmian prediction map m is non-contextual. Similarly the toy-model was also non-contextual.

Conclusion

Contextuality is not a necessary feature of Quantum Mechanics.

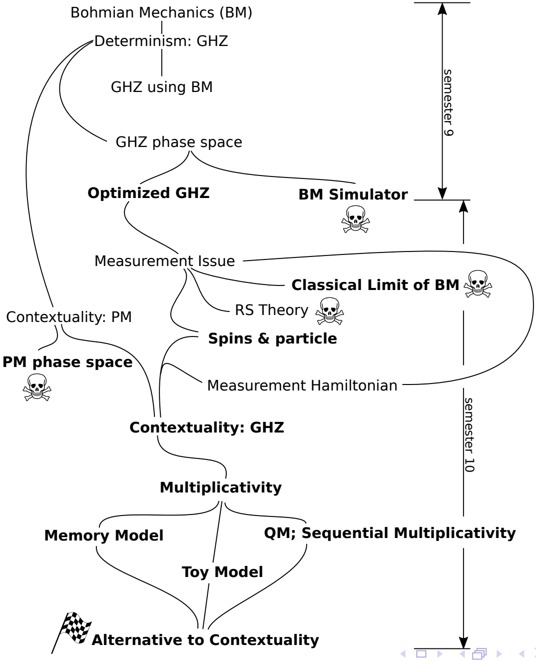
Summary of Results I

- Bohmian Mechanics
  - Generalized the Hamiltonian based measurement scheme to continuous variables
  - Analytic/graphical proof of consistency check using position measurements
  - Analytic/graphical solution to measuring entangled spins using SG & using the Hamiltonian scheme
  - Alternative proof of spins can't be associated with particles, and must only be a property of the wavefunction
  - BM simulator with many trajectories (one particle, one dimensional, arbitrary potential)
- Tests of Determinism and Contextuality
  - Optimized the phase space GHZ test
  - GHZ $\rightarrow$ a test of contextuality
  - PM extension to phase space (independently re-discovered)

Summary of Results II

- The Contextuality situation
  - BM was shown consistent with PM & restricted BM was argued to be non-contextual
  - ‘Multiplicativity’ and ‘Sequential Multiplicativity’ identified, defined and proven where they hold
  - Demonstrated ‘non-multiplicativity’ as an alternative to contextuality, by constructing a toy model
  - Proposed the ‘discretely c-ingle’ HV theory to non-contextually explain spins

Summary of Results III



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