Summary (till lecture 6, scanned)

09 September 2017 01:01 PM

— Lec #1 March 4 2017

$ds^2 = \sum g_{ij}\, dx^i dx^j$

$g'_{kl}(\vec{x}') = \sum_{i,j} g_{ij}(\vec{x}) \frac{\partial x^i}{\partial x'^k} \frac{\partial x^j}{\partial x'^l}$

Convention

a. Index i of a coordinate is superscript.

b. $\frac{\partial}{\partial x^i} := \partial_i$

c. Summation Convention: Any index twice; once as sub, once as super is to be summed.

d. Index of a matrix appears as subscript.

Tensor Field: $C^{i_1 \dots i_p}{}_{j_1 \dots j_q}(\vec{x})$ is a combination of the metric & its derivatives which transforms as
$\partial_{k_1} x'^{i_1} \dots \partial_{k_p} x'^{i_p}\, \partial'_{j_1} x^{l_1} \dots \partial'_{j_q} x^{l_q}\, C^{k_1 \dots k_p}{}_{l_1 \dots l_q}(\vec{x})$

— Lec #2

claim: $g'_{ij}(x') = \mathrm{diag}(\pm1, \pm1, \pm1 \dots)$

useful expansion (*): $x^i = x_0^i + A^i{}_k (x'^k - x_0'^k) + B^i{}_{jk}(x'^k - x_0'^k)(x'^j - x_0'^j) + \dots$

proof idea: write g' from g. Use (*) to express as A. Use symmetricity of g to diagonalise and then pull out (eigenvalues)$^{1/2}$ in matrices & you're done.

story: So g_{ij} can't help distinguish b/w a sphere & plane.
What about derivatives of g_{ij}?

claim: $K_{ijk} = \partial_i g_{jk} = 0$ in all frames; proof idea: Use (*) to show that # constraints = # par. in B.

claim: $S_{ijkl} = \partial_i \partial_j g_{kl} \neq 0$ in general; proof idea similar

story: Can we make a Tensor out of S_{ijkl}, K_{ijk}?

Defn: $\Gamma^i_{jk} =: \frac{1}{2} g^{il}(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}) :=$ Christoffel Symbol / Connection

Defn: $R^i{}_{jkl} =: \partial_l \Gamma^i_{jk} - \partial_k \Gamma^i_{jl} + \Gamma^r_{jk}\Gamma^i_{lm} - \Gamma^m_{jl}\Gamma^i_{km} :=$ Reimann Tensor (claim)

claim: $\Gamma'^l_{mn} = \partial_i x'^l \partial'_m x^j \partial'_n x^k \Gamma^i_{jk} + \partial_k x'^l \partial'_m \partial'_n x^k$

— Lec #3

story: $R = 0$ for flat space, $R \neq 0$ for spherical space
: Scaling can change the manifold!

NB: Symm. & anti symm. comb. of a tensor are also tensors.

claim: $R_{ijkl} = -R_{jikl}$
$R_{ijkl} = -R_{ijlk}$
$R_{ijkl} = R_{klij}$
$R_{i[jkl]} = 0$

Defn: $D_i A_j = \partial_i A_j - \Gamma^k_{ij} A_k$

claim: $D_i A_j$ transforms as rank (2,0) tensor

claim: $D_i A^j = \partial_i A^j + \Gamma^j{}_{ik} A^k$ (using g^{ij}) (∵ other is easy)

— Lec #4

claim: $D_k A^{i_1 \dots i_p}{}_{j_1 \dots j_q} = \partial_k A + (\Gamma^{i_1}_{kl} A^{l i_2 \dots i_p}{}_{j_1 \dots j_q} + \Gamma^{i_2}_{kl} A^{i_1 l i_3 \dots i_p}{}_{j_1 \dots j_q})$
$- (\Gamma^l_{kj_1} A^{i_1 \dots i_p}{}_{l j_2 \dots j_q} + \Gamma^l_{kj_2} A^{i_1 \dots i_p}{}_{j_1 l j_3 \dots j_q})$

claim: (raise and lower) $A^{i_1 \dots i_p}{}_{j_1 \dots j_q} B^{m_1 \dots m_r}{}_{n_1 \dots n_s}$ is a rank $(p+r, q+s)$ tensor
$C^{i_1 \dots i_p}{}_{i_1 j_2 \dots j_q}$ is rank $(p-1, q-1)$

defn: $A_{i_1 j_2 \dots j_q} g^{i_1 i_2} = A^{i_2}{}_{j_2 \dots j_q}$

claim: $D_k A^{i_1 i_2 \dots i_p}{}_{j_1 j_2 \dots j_q} B^{k_1 k_2 \dots k_r}{}_{l_1 \dots l_s} = (D_k A^{i_1 \dots i_p}{}_{j_1 \dots j_q})(B^{k_1 \dots k_r}{}_{l_1 \dots l_s}) +$
$(A^{i_1 \dots i_p}{}_{j_1 \dots j_q})(D_k B^{k_1 \dots k_r}{}_{l_1 \dots l_s})$

mnem: $D(AB) = D(A) \cdot B + A \cdot D(B)$

NB: $D_i g_{jk} = 0$; Defn: $g^{ij} g_{jk} = \delta^i{}_k$

claim: $D_i g^{jk} = 0$; $D_i \delta^j{}_k = 0$

— Lec #5

story: Length: $L = \int_P^Q ds$: parametrise by u & put $\delta L = 0$ to $\mathcal{O}(\epsilon)$

Defn: Geodesics: Extremal length path

Defn: Affine parametrization: Equal u, equal length $\Leftrightarrow \frac{d}{du}\left(\frac{ds}{du}\right) = 0$

Claim: Geodesic (affine parametrisation) eqn: $\frac{d^2 x^l}{du^2} + \Gamma^l_{ij} \frac{dx^i}{du} \frac{dx^j}{du} = 0$

: (*) $\Rightarrow \frac{d}{du}\left(\left[\frac{ds}{du}\right]^2 = g_{mn} \frac{dx^m}{du} \frac{dx^n}{du}\right) = 0$

: Form is preserved under coordinate transform $\Leftrightarrow \frac{d^2 x'^l}{du^2} + \Gamma'^l_{ij} \frac{dx'^i}{du} \frac{dx'^j}{du} = 0$

Story: (*) is n 2nd order D.E.s. $\Rightarrow$ 2n conditions; $\vec{x}_0$; set $u = u_0$, $x|_{u=u_0} = x_0$ (n const) + ?

Impl Defn: Tangent Vector: $n^l = \frac{dx^l}{du}\Big|_{x^l = x_0^l}$

Property: $n'^l = \partial_m x'^l n^m \Rightarrow$ Transforms as a tensor (vector).

Defn: Tangent Space: Space containing tangent vectors (at a point).

story: Notion of parallel vectors is not easily defined. Soln: Consider 2 close points & co-ordinates st. $\Gamma' = 0$ at $x' = x'_0$. Then we have
: Parallel: n^l at $\vec{x}_0$ & m^l at $\vec{x}_0 + \delta\vec{x}$ are parallel $\Leftrightarrow n'^l = \lambda m'^l$ "in co-ordinates"

Defn: Parallel transport := $\vec{m}$ is a parallel transport of $\vec{n}$ if $\vec{m}' = \vec{n}' + \mathcal{O}(\delta x_0^2)$

story: δx_0^2 ∵ to make $\Gamma' = 0$, we used freedom till B. Curvatures are undefined. We must account for these differences.

— Lec #6

story: Obtain a D.E. for parallel transport: Start with $\Gamma' = 0$ frame, translate to arbitrary frame.

The Summary

05 August 2017 11:38 AM

Lecture #6

D.E. equations for parallel transport

Translate to an arbitrary frame $\longrightarrow \dfrac{dn^i}{du} + \Gamma^i_{pq}\, n^p \dfrac{dx^q}{du} = 0$

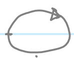
NB: Parametrization independent ($u = f(v)$, same form)

NB 2: coordinate frame independent ($\Rightarrow$ reversible $u \to 1-u$)

NB 3: Set $\frac{dx^i}{du} = n^i$ in the geodesic eq & it becomes parallel transport.

Finite Parallel Transport x_1  x_2 — the path is called c. $-c$ is opp. direction.

$M(x_1, x_2, c)^{-1} = M(x_1, x_2, -c)$ (from linearity & reversibility)

Monodrom Matrices: $M(x, x, c)$  c

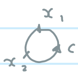
(a) If $M^i_j = \delta^i_j$ one frame, it is in all frames (use $\tilde{n} = Mn$, $\tilde{n}' = Mn'$ & use kron. delta)

(b) $R = 0 \Rightarrow M^i_j = \delta^i_j$ (proof step 1: for curve $\sim \mathcal{O}(\epsilon)$ $M^i_j = \delta^i_j + O(\epsilon^3)$)

Lecture #7

(c) $M^i_j = \delta^i_j \Rightarrow R = 0$ (proof idea: Show only R^i_{j12} contributes if curve is in the $x_1 - x_2$ plane; Ensures no cancelling happens)

(b) resumed

step 2: $M(x_1, x_1, c) = S\, M(x_2, x_2, c)\, S^{-1}$ 

step 3:  M by diving a finite loop into n parts

NB: Shouldn't have holes etc.

Area of each part $\sim \frac{1}{n} \Rightarrow$ circumf $\sim \frac{1}{\sqrt{n}}$

Recall: M for ϵ sized $= \mathbb{1} + \epsilon^3 \simeq \mathbb{1} + \frac{1}{n^{3/2}}$

$\Rightarrow M$ for finite $\simeq \mathbb{1} + \underbrace{n}_{\#\text{ terms}} \underbrace{\frac{1}{n^{3/2}}}_{\text{size of error}} = \mathbb{1} + \frac{1}{\sqrt{n}} \to \mathbb{1}$.

Embedding in Larger Spaces:

Use a Eucledian space in which the manifold can be embedded. ① Pick a point on the manifold ② Parallel (in Eucledian space) transport to nearby point on manifold ③ Project along the tangent space.

Claim: This new vector is a parallel transport (consistent with the earlier notion).

Lecture #8

Useful Identities

1) $[D_i, D_j] A^{k_1 k_2 \ldots k_p}{}_{l_1 \ldots l_q} = R^{k_1}{}_{l j i} A^{l k_2 \ldots k_p}{}_{l_1 \ldots l_q} + \ldots + R^{k_p}{}_{l j i} A^{k_1 \ldots k_{p-1} l}{}_{l_1 \ldots l_q}$
$- \{ R^l{}_{l_1 j i} A^{k_1 \ldots k_p}{}_{l\, l_2 \ldots l_q} + \ldots + R^l{}_{l_q j i} A^{k_1 \ldots k_p}{}_{l_1 \ldots l_{q-1} l} \}$ — proof hint: use def^n

2) Bianchi Identities

(a) $D_s R_{ijkl} + D_k R_{ijls} + D_l R_{ijsk} = 0$ (proof hint: use frame with $\Gamma = 0$) NB: skl anti-symmet.

(b) $D^i R_{ijkl} + D_k R_{jl} - D_l R_{jk} = 0$ (used contr^n with $\boxed{g^{js}}$)

where $R_{ik} = g^{jl} R_{ijkl}$; with g^{jk}, j^{jl} also same eq^n pops.

(c) $D^j \underbrace{(R_{jl} - \frac{1}{2} R g_{jl})}_{\text{Einstein's Tensor}} = 0$ (use contr with g^{ik})

General Relativity

Conventions

signature $(3,1) = (-+++)$; x^μ, $\mu = 0,1,2,3$

$f(\vec{x}) \longrightarrow f(x)$;

Proper Time in STR: $-ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$; $x_0 = ct$

(with $ds^2 = \eta^{\mu\nu} dx^\mu dx^\nu$)

Claims

(1) $g^{\mu\nu}$ describes spacetime in the presence of gravity

(2) Motion of a "free" particle in the gravitational field is given by the geodesic eq^n: $\dfrac{d^2 x^\mu}{du^2} + \Gamma^\mu{}_{\nu\rho} \dfrac{dx^\nu}{du} \dfrac{dx^\rho}{du} = 0$

Summary Continued

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(conventions (resumed))

$$\frac{ds^2}{du^2} = g_{\mu\nu} \frac{dx^\mu}{du} \frac{dx^\nu}{du} \begin{cases} < 0 & \text{time like} \\ = 0 & \text{light like} \\ > 0 & \text{space like} \end{cases}$$

remark: with rescaling $\frac{ds^2}{du^2} = \pm 1$ or 0.

story: Newtonian gravity under what limit?

(1) Weak Field: Without gravity $\eta^{\mu\nu}$; with, should be a small change

consequence $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

(2) Non-relativistic: $x^0 = ct$ & take c large

consequence $\frac{\partial}{\partial x^0}$ neglect compared to $\frac{\partial}{\partial x^i}$

$\frac{dx^i}{du}$ neglect compared to $\frac{dx^0}{du}$

Sketch of calcul^n: (A) Geodesic eq^n & time like condition

(B) Only Γ^μ_{00} relevant ($\because$ of (2)). simplify Γ^μ_{00} using (2)

(C) Keep only first order in h in both eq^{n}s

$$\frac{d^2x^i}{dt^2} \simeq \frac{c^2}{2} \partial_i h_{00} \text{ ; compare to Newton's law \& } h_{00} = \frac{-2}{c^2}\phi$$

NB: For large c, h_{00} is automatically small; (2) $\Rightarrow$ (1) ; NB2: Adding a const to h_{00} only scales coordinates

discussed: Local Inertial Frames, Principle of Equivalence, Modifications of GR

Lecture # 9

Lecture 7 (Monodromy and R)

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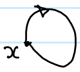
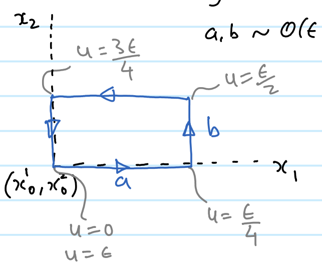
Recall: 

: Loop was of length $\mathcal{O}(\epsilon)$.
: Tangent vector at x, parallel transported along the curve.
: We can't comment beyond $\mathcal{O}(\epsilon^2)$ at the moment.

$$n^i(\epsilon) = n^i_{(0)} - \frac{1}{2} R^i{}_{jkl}\, n^j_0 \oint \underbrace{(x^l(u) - x^l_{(0)})}_{\text{has a } \int du} \frac{d}{du}(x^k(u) - x^k_{(0)}) + O(\epsilon^3)$$

Strategy: Calculate the integral for a special curve; Deduce some properties from there.

Consider: A rectangle in the $x_1 - x_2$ plane.



$a, b \sim O(\epsilon)$ NB: a, b are just coordinate distances

: u is "divided" equally across the edges of the rectangle.

NB: $x^i - x^i_{(0)} = 0 \;\forall\; i \neq 1, 2$
$\Rightarrow$ The integral will vanish unless l & k are 1 or 2.
Also, the tensor is anti-symmetric in these indexes $\therefore$ we need only evaluate one term, say $l = 1$, $k = 2$

We write the $x^1\, x^2$ coordinates for all segments.

Segment 1: $x^1(u) = x^1_{(0)} + a u \frac{4}{\epsilon}$
$x^2(u) = x^2_{(0)}$
Integral: 0 $\because$ its of the form $\int x^1 dx^2$ & $dx^2 = 0$.

Segment 2: $x^1(u) = x^1_{(0)} + a$
$x^2(u) = x^2_{(0)} + b\left(u - \frac{\epsilon}{4}\right)\left(\frac{4}{\epsilon}\right)$ $\therefore$ the integral $= \int (x^1 - x^1_{(0)}) \frac{d(x^2 - x^2_{(0)})}{du} du$
$= \int a \cdot \frac{d}{du}\left(b\left(u - \frac{\epsilon}{4}\right)\left(\frac{4}{\epsilon}\right)\right) du$
$= ab \cancel{\frac{4}{\epsilon}} \cancel{\frac{\epsilon}{4}} = ab$

Segment 3: $x^2 = x^2_{(0)} + b$
$x^1(u) = \ldots$ doesn't matter because again $dx^2 = 0$ & the integral is of the form $\int \ldots dx^2$.

Segment 4: $x^1 = x^1_{(0)}$ ($\because$ it returned)
Note that the integral is of the form $\int (x^1 - x^1_{(0)})\, d(x^2 - x^2_{(0)})$ but $x^1 - x^1_{(0)} = 0$ so the integral vanishes regardless of x^2.

$\Rightarrow$ The full integral becomes $= ab$.

TODO: check spelling!

Recall. $n^i(\epsilon) = M(\vec{x}_0, \vec{x}_0, c)^i{}_j\, n^j_{(0)}$ (The Monodromi Matrix)

So then $M^i{}_j(\vec{x}_0, \vec{x}_0, c)^i{}_j =$

$$\delta^i_j - \frac{1}{2} R^i{}_{jkl}(\vec{x}_{(0)}) \cdot \underbrace{\oint (x^l(u) - x^l_{(0)}) \frac{d}{du}(x^k(u) - x^k_{(0)})}$$

NB: (We just showed) that the integral $= 0$ unless $(k, l) = (1,2)$ or $(2,1)$
$= ab$ if $k = 2$, $l = 1$
& $-ab$ if $k = 1$, $l = 2$.

NB: So instead of a sum (in $R^i{}_{jkl}$) you get only one term, $R^i{}_{j12} \cdot (2ab)$.
Advantage: We don't have to worry about any cancellation of terms.
If the expression (integral) is zero, then this term must vanish.
If not, then this term can't vanish.

Story: We use it for proving, roughly, that if the monodrom is $\mathbb{I}$ for all curves, then the Reimann Tensor $R = 0$.

Cont. (2)

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Given: For all curves, the monodrom matrix = $\mathbb{1}$.

Claim: $R^i{}_{jkl} = 0$ in that case.

Proof Idea: Choose a curve along different planes (as before) to show that each component of $R^i{}_{jkl} = 0$.

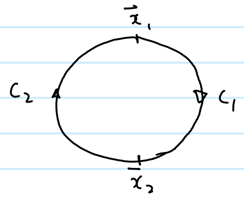
NB: Higher order terms in ϵ may give further conditions but they can't cancel the $R^i{}_{jkl}$ term so at least R must vanish if $M(x,x,c) = \mathbb{1}$.

Given: $R^i{}_{jkl} = 0$

Claim (eventual): $M = \mathbb{1}$ (under some extra assumptions)

Some additional Results

Consider:



a non-small curve.

Can calculate M by starting at x_1. Can also start at x_2.
Question: Relation b/w these

: $M(x_1, x_1, c_1 + c_2)$ vs. $M(x_2, x_2, c_2 + c_1)$ Tip: Reverse order

Recall: $\underbrace{M(x_2, x_1, c_2)\,\underline{M(x_1, x_2, c_1)}}_{=M(x_1,x_1,c_1+c_2)} \neq \underbrace{M(x_1, x_2, c_1)\,M(x_2, x_1, c_2)}_{=M(x_2,x_2,c_2+c_1)}$ in general.

$= M(x_2, x_1, c_2)\,M(x_2, x_2, c_2 + c_1)\,M(x_2, x_1, -c_2)$

Essentially, note that the second term (highlighted) is contained in $M(x_2, x_2, c_2 + c_1)$. We remove the extra term by multiplying by its inverse (recall $M(\dots c)^{-1} = M(\dots -c)$).

NB : The two monodrome matrices are related by conjugation.

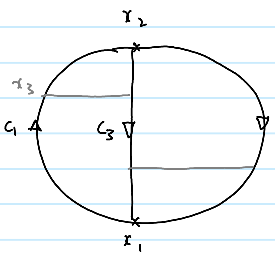
Recall: A & B are conjugates if $A = SBS^{-1}$ for some S.

NB 2: If $A = \mathbb{1}$, then $B = \mathbb{1}$ also $\Rightarrow$ If monodrome is $\mathbb{1}$, doesn't matter which point you start from.

Comment: We use it in a different way.

NB 3: If $B = \mathbb{1} + O(\epsilon)$ then
$A = SBS^{-1} = \mathbb{1} + S\,O(\epsilon)S^{-1} = \mathbb{1} + O(\epsilon)$ $\Rightarrow$ If B is close to $\mathbb{1}$, so is A.

Consider:



Idea: Use results from small loops & the result above.
· Break the loop.

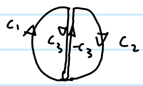
$M(x_1, x_1, c_1 + c_2)$ ← for the full loop.

$= M(x_2\, x_1\, c_2) \cdot M(x_1\, x_2\, c_1)$

(non-small loop) $= M(x_2\, x_1\, c_2) \cdot \underbrace{M(x_1\, x_2\, -c_3)\,M(x_2, x_1\, c_3)}_{\mathbb{1}} \cdot M(x_1\, x_2\, c_1)$

$\because M(x_1\, x_2\, -c_3) = M(x_2\, x_1\, c_3)^{-1}$

Idea: Go into a smaller loop. e.g.



$= M(x_1, x_1\; c_2 - c_3)\; M(x_1\; x_1\; c_1 + c_3)$

start
end

& similarly for the other

Now, divide further & To get the loop to start at some other point, conjugate. Start at x_3 & repeat the aforesaid.

- -

Assume: We've divided the original loop into n parts (roughly same size).
$\Rightarrow$ Each part has an area $\sim \frac{1}{n} \Rightarrow$ circumference $\sim \frac{1}{\sqrt{n}}$

Cont. + Parallel Transport (3)

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Monodromy (when $R^i{}_{jkl}=0$) $= \mathbb{1} + O\left(\frac{1}{n^{3/2}}\right)$ $\left(\because \text{ it was } \mathbb{1}+O(\epsilon^3) \text{ where } \epsilon \text{ was length of curve when } R=0\right)$

Final Argument: Look at structure. Several matrices (product).

$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$ $\sim$ each $\mathbb{1} + \mathcal{O}\left(\frac{1}{n^{3/2}}\right)$

& in between there would be conjugations, e.g. $S_2^{-1} \cdot S_{12} \cdot \cdot \; S_1^{-1} \cdot \cdot \; \cdot \; S^{-1}$.
One can bring this into the form $S \cdot S^{-1} \quad S \cdot S^{-1} \quad S \cdot S^{-1} \quad S \cdot S^{-1}$ by multiplying by SS^{-1} where needed.
Recall: Conjugation doesn't change the $\mathbb{1} + O\left(\frac{1}{n^{3/2}}\right)$ part
$\Rightarrow$ In the end I can remove the S matrices & write

$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$ $\sim$ each $\mathbb{1} + O\left(\frac{1}{n^{3/2}}\right)$

& there're n matrices.
Note: $(1+x_1)(1+x_2)(1+x_3) \cdots (1+x_n)$ with $x \sim \frac{1}{n^{3/2}}$,

$$1 + n \cdot \mathcal{O}(x) + {}^nC_2 \, O(x^2)$$
$$= 1 + \cancel{n} \cdot \frac{1}{\cancel{n} \, n^{1/2}} = 1 + \frac{1}{n^{1/2}} \, .$$

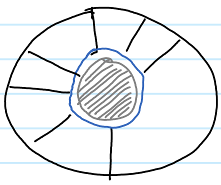
$\Rightarrow$ The product above would be $\mathbb{1} + O\left(\frac{1}{n^2}\right)$.

NB: If we had for each $\cdot$ $\mathbb{1} + \mathcal{O}\left(\frac{1}{n}\right)$, the argument won't work. This in turn followed $\because$ we had $\mathbb{1} + O(\epsilon^2)$ which justifies the necessity & sufficiency of the hardwork from lecture 6.

CAVEAT: We implicitly assumed that there's a surface enclosed by the curve ($\because$ we are drawing curves on it & dividing).
e.g. 2 For a conical defect, even for a small loop near the singularity (enclosing it), the monodromy $\mathbb{1} + O(\epsilon)$.
$\because$ the argument (for the infinitesimal case) assumed this
(but I can't see it...)

e.g. 1

Illustr^n



can't say anything about this curve, can't make it infinitesimal

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Parallel Transport Intuition: Embedding in a Larger Euclidean Space.

Idea: Surface of a sphere can be described using $x^2+y^2+z^2=r^2$. Generalise this, imagine this can be done.

Consider: Coordinates of some manifold $\{x_1 \ldots x_N\}$ embedded in a flat space $\{y_1 \cdots y_D\}$ with $D > N$ in general.

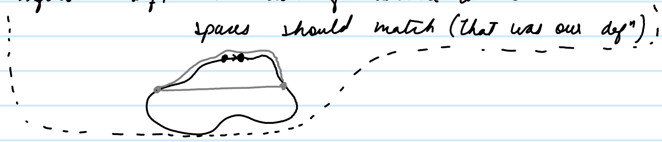
We write $y^\alpha = f^\alpha(\vec{x})$ where $\alpha = 1, \ldots D$ & fs are a set of f^ns.
Remark: For a typical Riemannian surface, D is very large.

e.g.: 2-d sphere $(\vec{x} = (\theta, \phi))$, $y^1 = a \sin\theta \cos\phi$
$y^2 = a \sin\theta \sin\phi$
$y^3 = a \cos\theta$

equiv. to saying $(y^1)^2 + (y^2)^2 + (y^3)^2 = a^2$

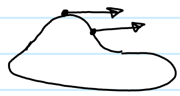
$$ds^2 = \sum_\alpha dy^\alpha dy^\alpha = \sum_\alpha \frac{\partial f^\alpha}{\partial x^i} dx^i \frac{\partial f}{\partial x^j} dx^j = \sum_\alpha \underbrace{\left(\frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j}\right)}_{g_{ij}(x)} dx^i dx^j$$

Infinitesimally, the notion of distance in the two spaces should match (that was our dy^α).



Cont. (4)

19 August 2017 11:40 PM



Suppose: ① You have a surface ② Pick a tangent to this surface (in general there'd be a tangent space, pick any tangent in this space) at some point ③ Using the embedded space, parallely move the tangent vector to the neighbouring point ④ Project it along the tangent space at the new point (it is a well defined notion, we'll see)

Remark: Norm to first order remains preserved despite the projection.

motivation: $\cos\theta = 1 + O(\theta^2)$, we're doing till $O(\theta)$.

Claim: The projected vector is a parallel transport

Question: Is this notion of parallel transport equivalent to our earlier notion that didn't refer to any embedding space?

Answer: Yes.

Proof (sketch only):

$\vec{n}(u)$ original tangent vector (N-dimensional)

$$m^\alpha = \left.\frac{\partial f^\alpha}{\partial x^i}\right|_{\vec{x}(u)} n^i(u) \longrightarrow m^\alpha \text{ at } \vec{x} + \delta\vec{x} = \vec{x}(u+\delta u)$$

(see the picture)

$$n^i(u+\delta u) = \delta_{\alpha\beta}\, m^\beta \left.\frac{\partial f^\alpha}{\partial x^j}\right|_{\vec{x}(u+\delta u)} g^{ij}(\vec{x}(u+\delta u))$$

$$= m^\beta \left(\left.\frac{\partial f^\alpha}{\partial x^j}\right|_{\vec{x}(u)} + \frac{\partial^2 f^\alpha}{\partial x^j \partial x^k}\delta x^k \right) \left(\left. g^{ij}\right|_{\vec{x}(u)} + \partial_\ell g^{ij}\delta x^\ell \right)$$

Consistency check: if you take $\delta x = 0$, you should get the same vector

viz. parallel transport & come back, expect same vector

$$= m^\alpha \frac{\partial f^\alpha}{\partial x^j} g^{ij} + O(\delta x)$$

$$= n^i \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\alpha}{\partial x^j} g^{ij}$$

$$= n^i g_{ij} g^{ij}$$

$$= n^i$$

Claim: $\frac{dn^i}{du} + \Gamma^i{}_{jk} n^j \frac{dx^i}{du} = 0$; hint: All $\frac{\partial f}{\partial x}$ are related to the metric.

Remark: This matches the notion of parallel transport we had derived without the notion of embedding space.

Lecture 8 (identities + GR started!)

02 September 2017 04:28 PM

Useful Identities

(will not be proved here)

1(a) $D_i D_j A_k - D_j D_i A_k$ (also written as $[D_i, D_j] A_k$)
$= -R^l{}_{kji} A_l$

defined as the usual covariant derivative with the Christoffel symbol

These can be proved (claim) by looking at the definitions

(b) $D_i D_j A^k - D_j D_i A^k = R^k{}_{lji} A^l$

(c) Combining the aforesaid

$$D_i D_j A^{k_1 \dots k_p}{}_{l_1 \dots l_q} - D_j D_i A^{k_1 \dots k_p}{}_{l_1 \dots l_q} = R^{k_1}{}_{lji} A^{l k_2 \dots k_p}{}_{l_1 \dots l_q} + \dots + R^{k_p}{}_{lji} A^{k_1 k_2 \dots k_{p-1} l}{}_{l_1 \dots l_q}$$
$$- \{ R^l{}_{l_1 ji} A^{k_1 \dots k_p}{}_{l l_2 \dots l_q} + \dots + R^l{}_{l_q ji} A^{k_1 \dots k_p}{}_{l_1 \dots l_{q-1} l} \}$$

Bianchi Identities

$$D_s R_{ijkl} + D_k R_{ijls} + D_l R_{ijsk} = 0$$

structure: completely anti-symmetrize in $s k l$ NB: kl are already anti-symmetric; instead of 6 terms, need to write only 3 terms.

proof idea: go to a frame in which $\Gamma = 0$.

can derive more by contracting, e.g. g^{js}

$\Rightarrow D^j R_{ijkl} + D_k R_{il} - D_l R_{ik}$

where the Ricci tensor was defined as $g^{ik} R_{ijkl} = R_{jl}$. NB: contracting the 2nd & 4th index is also the same $\because R_{ijkl}$ is anti-symmetric in (i,j) & (k,l)

further contract with g^{ik}.

$\Rightarrow D^j R_{jl} + D^i R_{il} - D_l R = 0$

same terms

$\Rightarrow 2 D^j R_{jl} - D_l R = 0 \quad \Rightarrow 2 D^j \big(\underbrace{R_{jl} - \tfrac{1}{2} R\, g_{jl}}_{\text{this is called Einstein's tensor.}} \big) = 0$

where in the last step, D doesn't act on g $\because Dg = 0$. It acts on R, then lowers the index.

This completes our general discussion of manifolds. Now we start General Relativity.

Conventions

Signature Now we focus on manifolds with signature $(n,1)$ (here $n=3$ for 3d space)

Recall. Signature $(3,1) = (- + + +)$

Indices: x^μ $\quad \mu = 0,1,2,3$ coordinates.

function: $f(\xi) \longrightarrow f(x)$

e.g.
$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$

$g_{\mu\nu} = \eta_{\mu\nu}$

Recall: In special relativity, proper time is given by $-ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$ where $x^0 = ct$.

Question. Does the metric $g_{\mu\nu}(x)$ as generalized, describe anything physical?
Answer: Yes, it describes spacetime in the presence of gravity.

General Relativity

$g_{\mu\nu}(x)$ describes space-time in presence of a gravitational field.

Particles in a gravitational field move along geodesics in the absence of other fields.

GR (cont.)

02 September 2017 05:46 PM

Remark: How the gravitational field is produced we'll come to later, second part of general relativity.

Recall: Geodesic Eq^n

$$\frac{d^2x^\mu}{du^2} + \Gamma^\mu{}_{\nu\rho}(x)\frac{dx^\nu}{du}\frac{dx^\rho}{du} = 0.$$

← Describes the motion of the particle in a gravitational field.

Story: When the metric has all + signature, $ds^2 \geq 0$

In our case, 3 + & 1 −, ds^2 can be ≥ 0, $= 0$, ≤ 0.

Therefore we can divide the geodesic into 3 parts.

① $\frac{ds^2}{du^2} = g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du} < 0$ (Time like)

which can be made by rescaling $= -1$

"proof": ∵ u is an affine parameter, $\frac{ds^2}{du^2} = \text{const}$.

Let $\frac{ds^2}{du^2} = -c$ and $\not$ rescale

$u \to u\sqrt{c}^{-1} u'$

$\Rightarrow \left(\frac{ds}{du}\right)^2 = \frac{ds^2}{c^{-1}du'^2} \Rightarrow \frac{ds^2}{du'^2} = -1.$

NB: du can be associated with $d\tau$ (proper time)

∵ $\frac{ds^2}{du^2} = g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du} = -1$

Then for $g_{\mu\nu} = \eta_{\mu\nu}$

$$-dx^{(0)2} + dx^{(1)2} + dx^{(2)2} + dx^{(3)2} = \underbrace{-du^2}_{\substack{-d\tau^2 \\ \text{proper time}}}$$

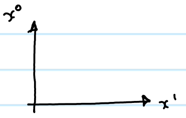
(see the prev. page).

NB: the geodesic eq^n is invariant under scaling of u: it is ok to scale.

(2) Space-like $\frac{ds^2}{du^2} = g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du} > 0$

(3) Light-like $g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du} = 0$

Story: The analogue of this in Minkoski space is trivial. Basically says every particle travels at a speed less than the speed of light.



$\left|\frac{dx^1}{dx^0}\right| < 1$ or in general $\left|\frac{d\vec{x}}{dx^0}\right| < 1$

$\Rightarrow ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -dx^{(0)2} + dx^{(1)2} < 0$

In GR we say this in a more coordinate invariant way.

For massless particles, in special relativity, $\left|\frac{d\vec{x}}{dx^0}\right| = 1$, must move at the speed of light. This entails $ds^2 = 0$

This also generalises to GR. Massless particles must move along light like geodesics.

Nothing of course travels along the space like geodesic; in SR, this corresponds to $\left|\frac{d\vec{x}}{dx^0}\right| > 1$, faster than speed of light.

Story 2 We'll see that these under the right circumstances reduce to Newton's Laws. However, already we can see that the notion of signals not travelling faster than light is built in, as this theory a generalization of special relativity.

Newtonian Limit

Motivation: Physics without gravity, $\eta_{\mu\nu}$ describes the proper time (don't know the precise meaning of this).

Now even though gravity changes $\eta^{\mu\nu}$, that change must be small.

Newtonian Limit + ...

09 September 2017 07:37 PM

We assume ① Weak Field : $g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{h_{\mu\nu}(x)}_{\text{small}}$

② Non-relativistic limit : $x^0 = ct$; write everything in terms of t & take c large.
(shouldn't be valid in the relativistic limit, allows instantaneous prop of signals)

NB: Ofcourse c has dimensions; what does taking c large mean? $\because$ a dimensionful parameter being large means it must be large wrt something. Here, it means compared to velocities of objects.

Consider : $\frac{dx^\mu}{du} = \left(\frac{dx^0}{du}, \frac{dx^i}{du}\right) = \left(c\frac{dt}{du}, \frac{dx^i}{du}\right)$ for large c we ignore the $\frac{dx^i}{du}$ term compared to $c\frac{dt}{du}$, ie the velocities are small compared to c

$\frac{\partial}{\partial x^0}$ vs $\frac{\partial}{\partial x^i}$, $\frac{1}{c}\frac{\partial}{\partial t}$ vs. $\frac{\partial}{\partial x^i}$ & again with c large, the first term will be neglected.

Consequence: $h_{\mu\nu}(x)$ can be taken to be static $\because \frac{\partial}{\partial x^0}$ will be neglected compared to $\frac{\partial}{\partial x^i}$

Remark: Later we'll see the non-relativistic limit also implies the weak field limit

Recall $\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{du}\frac{dx^\rho}{du} = 0$ (geodesic eq^n)

$g_{\mu\nu}(x)\frac{dx^\mu}{du}\frac{dx^\nu}{du} = -1$ (defines the scaling)

Under the assumptions: $\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{00}(x)\frac{dx^0}{du}\frac{dx^0}{du} \simeq 0$

recall $\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho}\right)$

$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\sigma}\left(\underbrace{\partial_0 g_{\sigma 0} + \partial_0 g_{\sigma 0}}_{\text{ignore} \because \text{they're time derivatives (see above)}} - \partial_\sigma g_{00}\right)$

(also for ∂_σ, ignore ∂_0)

$\simeq -\frac{1}{2}g^{\mu i}\partial_i g_{00} = -\frac{1}{2}g^{\mu i}\partial_i h_{00}$

NB: If we include the h dependent part, in this, then it will go 2nd order in h (overall) Thus we replace $g^{\mu i} = \eta^{\mu i}$.

NB 2: $\eta^{\mu i}$ is diagonal. $\eta^{\mu i} = 0$ for $\mu = 0$
$= \delta_{ij}$ for $\mu = j$ (signature is $-1, 1, 1, 1$)

$\Gamma^0_{00} \simeq 0$ & $\Gamma^i_{00} \simeq -\frac{1}{2}\partial_i h_{00}$.

Plugging Back: $\frac{d^2x^0}{du^2} \simeq 0$ & $\frac{d^2x^i}{du^2} - \frac{1}{2}\partial_i h_{00}\left(\frac{dx^0}{du}\right)^2 \simeq 0$

The second eq^n (with leading order in $g^{\mu\nu}$) gives us

$$-\left(\frac{dx^0}{du}\right)^2 + \underbrace{\sum_{i=1}^{3}\left(\frac{dx^i}{du}\right)^2}_{\text{small compared to } dx^0/du} = -1 \quad \Rightarrow \quad \frac{dx^0}{du} \simeq 1.$$

NB: When you integrate $\frac{d^2x^0}{du^2} \simeq 0$, you get $\frac{dx^0}{du} \simeq$ const & that is fixed to be 1.

$\Rightarrow x^0 = u$ (we don't worry about the constant).

NB 2: We took the leading part in the 2nd eq^n because in the first, that's already multiplied by h_{00}.

… + Newtonian Limit + Principle of Equivalence

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This yield finally $\frac{d^2x^i}{du^2} - \frac{1}{2}\partial_i h_{00} \simeq 0$ $\left(\because \frac{dx^0}{du} \simeq 1\right)$

$\Rightarrow \frac{1}{c^2}\frac{d^2x^i}{dt^2} \simeq \frac{c^2}{2}\partial_i h_{00}$

Comparison to Newton's Law of Gravity: $\frac{d^2x^i}{dt^2} = -\partial_i \phi$

$\Rightarrow \phi = -\frac{c^2}{2}h_{00}$ or $h_{00} = -\frac{2}{c^2}\phi$

Remark: This shows that in the appropriate limit, GR does reduce to Newton's laws where I set $h_{00} = -\frac{2}{c^2}\phi$.

NB: For large c, automatically h_{00} is small. It means that the weak field assumption is not independent.

NB 2: One could add a "large const" in $h_{00} = -\frac{2}{c^2}\phi + \text{const}$ which would only scale η which in turn scales coordinates. So it plays no role.

Conclusions using Remainian Geometry

Consider: a point x_0; in general $\Gamma^\mu_{\nu\rho}(x_0) \neq 0$

Intuition: Think of Γ as a "force" in the geodesic eq^n

Idea: $\exists$ a coordinate system x' s.t $\Gamma'^\mu_{\nu\rho}(x'_0) = 0$ (see lecture 2)

Consequence: The eq^n of motion looks like that of a free particle (put $\phi = 0$ to compare) at that particular space-time point.

: This can be done for all points (even though one needs to pick a different coordinate system). These frames are called **local inertial frames**.

: This principle that it's possible to find a frame where the effect of gravity disappears is called the **principle of equivalence**.

General Remark: The equation of motion doesn't depend on the composition of the particle you're considering.
e.g a dust particle of copper & gold both move the same under gravity.

Deeper Remark(s): The general coordinate invariance is "sacred". $\leftrightarrow$ run into inconsistencies
The principle of equivalence can be removed in further generalizations

e.g the eq^n of motion is given by

say this depends on the particle type.

$$\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{du}\frac{dx^\rho}{du} + \alpha D^\mu R_{\nu\rho}\frac{dx^\nu}{du}\frac{dx^\rho}{du} = 0$$

This will violate the principle of equivalence but this is still valid (coordinate invariant)
Infact any "quantum theory" e.g. string theory, generates this term even in the classical level.

... + Dimensional Analysis

10 September 2017 02:22 PM

: Dimensional Analysis (to show α must be small)
Γ has one length derivative, R has two length derivatives
$\Rightarrow$ α has dimensions length square

$$\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{du} \frac{dx^\rho}{du} + \alpha\, \Delta^\mu R_{\nu\rho} \frac{dx^\nu}{du} \frac{dx^\rho}{du} = 0$$

$$[L] \qquad \cancel{[L]^{-1}}\,[L]\,\cancel{[L]} \qquad [L]^2\,[L^{-1}]\,\cancel{[L^{-2}]}\,\cancel{[L]}\,\cancel{[L]}$$

The length scale of interest is the planck scale & square of this would essentially have no easily observable effect.

Rough Work

09 September 2017 06:39 PM

$ds^2 \leq 0$

$\frac{ds^2}{du^2} \leq 0$ $\because$ $\frac{ds^2}{du^2}$ is const $\therefore$ can make $\frac{ds^2}{du^2} = -1$

How do I know this? ; $\because$ signature is $+ - - -$ or $- + + +$ so it's possible always.

How about this? Should I use limits? $\Delta s^2 \leq 0$

$\Rightarrow \frac{\Delta s^2}{\Delta u \cdot \Delta u} \leq 0$

$lt \; \frac{ds^2}{du^2} \leq 0$

$= g_{\mu\nu} \frac{dx^\mu}{du} \frac{dx^\nu}{du}$

From the geodesic eq^n $\Rightarrow \frac{d}{du}\left(\left(\frac{ds}{du}\right)^2\right) = 0$

(or is it a consequence of the affine parametrization?)

$\frac{ds}{du} = \text{const}$ $\because$ u is an affine parameter.

Lecture 9 | Gravitational Redshift + EM

01 October 2017 01:00 PM

Gravitational Redshift

$\tilde{x}^0_{(1)}$ $\tilde{x}^0_{(1)}+c$

$x^0_{(1)}$ $x^0_{(1)}+c$

$\vec{x}_1$ $\vec{x}_2$

Consider: a clock sitting on the point $\vec{x}_1$ in space.
$\Rightarrow \frac{dx^i}{dt} = 0$.

Assume: Time independent metric $g_{\mu\nu} = g_{\mu\nu}(\vec{x})$
(I think we're taking the non-relativistic limit $\Rightarrow$ weak field approx.) Nope!

Recall: For a physical particle, $\frac{ds^2}{du^2} < 0$ & since u is an affine parameter $\frac{ds}{du} = \text{const}$.

This means I can always rescale to get $\frac{ds^2}{du^2} = -1$ (for time like)

$\Rightarrow du^2 = -ds^2 = d\tau^2$. So I can write

$$-\frac{ds^2}{du^2} = \frac{ds^2}{d\tau^2} = -g^{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 1$$

(In the weak field limit, one ignores $\frac{dx^i}{du}$ compared to $\frac{dx^0}{du}$. That's what I think is being done here but I need to check this)

$$-g^{00}\left(\frac{dx^0}{d\tau}\right)^2 = 1 \quad \Rightarrow \quad \Delta x^0 = \Delta\tau_{(1)}\left(-g^{00}(\vec{x}_{(1)})\right)^{-1/2}$$

(ticks) a const not speed of light, ϵ

Story: The clock emits two signals, at $x^0_{(1)}$ & $\tilde{x}^0_{(1)} + dx^0$. Say it took 'ϵt length of time' for the first signal to reach point $\vec{x}_2$. Since $g^{\mu\nu}$ is time independent, the second signal will also take as long (see the figure)
The interval between the two signals at $\vec{x}_2$ will again be dx^0

I now want to find the proper time b/w two clock ticks, as seen by an observer sitting on the point $\vec{x}_{(2)}$
Why bother? Because the clock of observer 2 is ticking according to his proper time.

Calc: $\Delta x^{(0)} = \Delta\tau_{(2)}\left(-g^{00}(\vec{x}_{(2)})\right)^{-1/2}$ I wrote the eq^n for a clock sitting at $\vec{x}_{(2)}$ observing the time difference of the signals that reached $\vec{x}_{(2)}$.

$$\Rightarrow \Delta\tau_{(1)}\, g^{00}(\vec{x}_{(1)})^{-1/2} = \Delta\tau_{(2)}\, g^{00}(\vec{x}_{(2)})^{-1/2}$$

$$\Rightarrow \frac{\Delta\tau_{(1)}}{\Delta\tau_{(2)}} = \left[\frac{g^{00}(\vec{x}_{(1)})}{g^{00}(\vec{x}_{(2)})}\right]^{1/2}.$$

It seems that sir didn't actually make the weak field approximation but he does it now.

Weak Field: $g^{00} = \eta^{00} + h^{00}$ where $h^{00} = -\frac{2\phi}{c^2}$ & $\eta^{00} = -1$

$$\text{so} \quad \frac{\Delta\tau_{(1)}}{\Delta\tau_{(2)}} = \left[\frac{1+\frac{2\phi(\vec{x}_{(1)})}{c^2}}{1+\frac{2\phi(\vec{x}_{(2)})}{c^2}}\right]^{1/2} \simeq 1 + \frac{1}{c^2}\left[\phi(\vec{x}_1) - \phi(\vec{x}_{2})\right]$$

If $\phi(\vec{x}_{(1)}) \geq \phi(\vec{x}_{(2)})$, then $\Delta\tau_{(1)} \geq \Delta\tau_{(2)}$.

The clock at $\vec{x}_{(1)}$ will be slower than the one at $\vec{x}_{(2)}$

I am, however, still a little confused about why we're using 'proper time'.
What happens if I take the clock at $\vec{x}_{(2)}$ & bring it to $\vec{x}_{(1)}$?
Shouldn't their proper times be the same?
$\hookrightarrow$ clock already at $x_{(1)}$ & the one I got from $\vec{x}_{(2)}$?

Forces + Electromagnetism in GR introduced (cont.)

22 October 2017 06:19 PM

Recall: A free particle moves along a geodesic $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0$ where $g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1$

Question: If there's a "force" how do we evaluate the LHS?

Idea: We can use the principle of equivalence. While this is not "sacred" it is correct to a very good approximation.
: Goto a frame where $\Gamma^\mu_{\nu\rho} = 0$ so that $g^{\mu\nu} = \eta^{\mu\nu}$ by the equiv. principle.

NB: In this frame, the eq^n of motion becomes $\frac{d^2x^\mu}{d\tau^2} = f^\mu$ force in the absence of gravity

∴ in local inertial frame, the principle of equivalence says that the eq^n of motion should look exactly as though there's no gravity

More Babble: Consider having a rocket you've tested in the absence of gravity & know f, the acceleration of the rocket. The principle of equivalence tells us that the eq^n will be the same in this special local inertial frame.

NB: It isn't convenient to keep changing frames so

Consider: A general coordinate system x, where

(TODO: This I couldn't figure; need to derive the parallel transport eq^n / geodesic eq^n)
(Turns out it was an exercise there also which I remember doing)

$$\frac{d^2x'^\mu}{d\tau^2} = \partial_\nu x'^\mu\left(\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\right)$$

$$\Rightarrow \underbrace{(\partial'_\mu x^\alpha)\,\partial_\nu x'^\mu}_{\delta_\nu{}^\alpha}\left(\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\right) = f'^\mu(\partial'_\mu x^\alpha)$$

$$\Rightarrow \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = \partial'_\mu x^\alpha f'^\mu =: f^\alpha$$

NB: We didn't assume that f is a vector. We just determined that f^α can be computed

Def^n.

Consider. x'' is another coordinate system.

Eq^n in the x'' coordinate will be $\frac{d^2x''^\alpha}{d\tau^2} + \Gamma''^\alpha_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = f''^\alpha = \partial'_\mu x^\alpha f'^\mu$

Ex: $f''^\alpha = \partial_\beta x''^\alpha f^\beta$ (easy exercise) essentially saying that f indeed transforms like a vector.

Consider: Force due to an external field, e.g. Electromagnetic force.

Idea: Same as earlier, use principle of equivalence;

Recall: $m\frac{d^2x^\mu}{d\tau^2} = q\,\eta^{\mu\nu}F_{\nu\rho}\frac{dx^\rho}{d\tau}$; $c = 1$ unit where $F_{\nu\rho} = \partial_\nu A_\rho - \partial_\rho A_\nu$; $A_\nu = \{\underbrace{A_0}_{\text{Electrostatic Potential}}, \underbrace{A_1, A_2, A_3}_{\text{Vector potential}}\}$

$$E_i = -F_{0i} = \partial_i A_0 - \partial_0 A_i = -\nabla\phi - \frac{\partial A}{\partial t}$$ — Not sure why —

$$B_i = \frac{1}{2}\sum_{j,k}\epsilon_{ijk}F_{jk}$$ which componentwise is $\left.\begin{aligned} B_1 &= \partial_2 A_3 - \partial_3 A_2 \\ B_2 &= \partial_3 A_1 - \partial_1 A_3 \\ B_3 &= \partial_1 A_2 - \partial_2 A_1 \end{aligned}\right\}$ essentially is $B = \nabla\times A$

Consider $\mu = i$ case, $\frac{d}{d\tau}\Big(\underbrace{m\frac{dx^i}{d\tau}}_{=\,p^i}\Big) = q\,\eta^{ij}\left(F_{j\rho}\frac{dx^\rho}{d\tau}\right) = q\left(F_{i0}\frac{dx^0}{d\tau} + F_{ik}\frac{dx^k}{d\tau}\right)$ using $F_{i0} = -F_{0i}$ & justify this properly.

$$= q\left(E_i\frac{dx^0}{d\tau} + \epsilon_{ik\ell}B_\ell\frac{dx^k}{d\tau}\right)$$

$\frac{dp^i}{d\tau} = \frac{dp^i}{dt}\overbrace{\frac{dt}{d\tau}}^{dx^0/d\tau}$ & $\frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dx^0}{d\tau}$ $\Rightarrow$ $\frac{dp^i}{dt} = q\left(E_i + \epsilon_{ik\ell}B_\ell\frac{dx^k}{dt}\right)$ NB: The p^i has τ still which is the Lorentz force law.

Electromagnetism in GR (cont.)

28 October 2017 03:01 PM

So in a local inertial frame, one can write (using the principle of equivalence)

$$m\frac{d^2x'^\mu}{d\tau^2} = q\, g'^{\mu\nu} F'_{\nu\rho} \frac{dx'^\rho}{d\tau}$$

NB: The meaning of F' would be the same as that in flat space; use test charges to measure the E & B fields.

$$m\,\partial_\nu x'^\mu \left(\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\right) = q\,\partial_\alpha x'^\mu \partial_\beta x'^\nu g^{\alpha\beta} F'_{\nu\rho}\partial_\gamma x'^\rho \frac{dx^\gamma}{d\tau} \Big\}\times \partial'_\mu x^\delta$$

$$\Rightarrow m\frac{d^2x^\delta}{d\tau^2} + \Gamma^\delta_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = q\,\cancel{\partial'_\mu x^\delta \partial_\alpha x'^\mu}\, \partial_\beta x'^\nu g^{\overset{\delta}{\cancel{\alpha}}\beta} F'_{\nu\rho}\partial_\gamma x'^\rho \frac{dx^\gamma}{d\tau}$$

$$= q g^{\delta\beta}\left(\partial_\beta x'^\nu \partial_\gamma x'^\rho F'_{\nu\rho}\right)\frac{dx^\gamma}{d\tau}$$

Defn. $F_{\beta\gamma} := \partial_\beta x'^\nu \partial_\gamma x'^\rho F'_{\nu\rho}$

$$= q g^{\delta\beta} F_{\beta\gamma}\frac{dx^\gamma}{d\tau}$$

Recall: $F'_{\nu\rho} = \partial'_\nu A'_\rho - \partial'_\rho A'_\nu$; (Ex) Claim: Show that (using the recall) that $F_{\beta\gamma} = \partial_\beta A_\gamma - \partial_\gamma A_\beta$

where $A_\beta = \partial_\beta x'^\mu A'_\mu$.

Remark: Repeat the calc for another arbitrary x'' coordinate sys. & conclude

$$A''_\beta = \partial_\beta x''^\mu A'_\mu$$

which can be combined as

$$A''_\beta = \partial_\beta x''^\alpha A_\alpha$$

so this shows that A_β transforms as a rank 0,1 tensor

Proof: $F_{\beta\gamma} = \partial_\beta x'^\nu \partial_\gamma x'^\rho (\partial'_\nu A'_\rho - \partial'_\rho A'_\nu)$

$$= \partial_\gamma x'^\rho \partial_\beta x'^\nu \partial'_\nu A'_\rho - \partial_\beta x'^\nu \partial_\gamma x'^\rho \partial'_\rho A'_\nu$$

$$= \partial_\gamma x'^\rho \underbrace{\frac{\partial x'^\nu}{\partial x^\beta}\frac{\partial A'_\rho}{\partial x'^\nu}}_{=\,\partial_\beta A'_\rho} - \partial_\beta x'^\nu \partial_\gamma A'_\nu$$

$$= \begin{bmatrix}\partial_\beta(\partial_\gamma x'^\rho A'_\rho) \\ \cancel{-\partial_\beta\partial_\gamma x'^\rho A'_\rho}\end{bmatrix} - \begin{bmatrix}\partial_\gamma(\partial_\beta x'^\nu A'_\nu) \\ \cancel{-\partial_\gamma\partial_\beta x'^\nu A'_\nu}\end{bmatrix} \quad \square$$

Bubble: F^μ earlier was a force field. This makes sense only along the trajectory of the particle. The A^μ field however is a proper vector field defined without invoking any special trajectory.

Claim. $F_{\alpha\beta}$ is a rank (0,2) tensor.

Proof: $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$

$F'_{\alpha\beta} = \partial'_\alpha A'_\beta - \partial'_\beta A'_\alpha$

$$F'_{\alpha\beta} \overset{\text{claim}}{=} \partial'_\alpha x^\mu \partial'_\beta x^\nu F_{\mu\nu}$$

$$= \partial'_\alpha x^\mu \partial'_\beta x^\nu (\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$= \partial'_\alpha x^\mu \partial_\mu A_\nu \partial'_\beta x^\nu - \partial'_\alpha x^\mu \partial'_\beta x^\nu \partial_\nu A_\mu$$

$$= \partial'_\beta x^\nu \partial'_\alpha A_\nu - \partial'_\alpha x^\mu \partial'_\beta A_\mu$$

$$= \partial'_\alpha(\partial'_\beta x^\nu A_\nu) - \partial'_\beta(\partial'_\alpha x^\mu A_\mu)$$

$$= \partial'_\alpha A'_\beta - \partial'_\beta A'_\alpha \quad \square$$

Remark: How did they know this would work out?

Conclusion:

$$m\frac{d^2x^\delta}{d\tau^2} + \Gamma^\delta_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = q g^{\delta\beta} F_{\beta\gamma}\frac{dx^\gamma}{d\tau}$$

from the equivalence principle.

Further: To the RHS one can add

$$a\, q\, R^\delta{}_{\alpha\beta\gamma}\frac{dx^\alpha}{d\tau}F^{\beta\gamma}$$

but a is small (as argued earlier)

Point: Principle of equivalence is not sacred. Any term that respects general coordinate invariance might be present.

Rough (disambiguation)

07 October 2017 01:56 PM

$-d\tau^2$ is proper time.

clock

$x_{(1)}$ $x_{(2)}$

(1) Assume Time Independent Metric $g_{\mu\nu}(\vec{x})$.
(2) The object is at rest in this metric $\Rightarrow \frac{dx^i}{dt} = 0$.

$$g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du} = \frac{ds^2}{du^2} = -1$$

$$\underbrace{-ds^2}_{d\tau^2} = c^2dt^2 - dx^2 - dy^2 - dz^2$$

$$\frac{ds^2}{du^2} = \text{const.}$$

$$\frac{ds^2}{du^2} = -1 \qquad -d\tau^2 \qquad \frac{-d\tau^2}{du} = 1 \qquad d\tau = du$$

In the weak field limit,

$$g_{00}(\vec{x})\left(\frac{dx^0}{d\tau}\right)^2 = -1$$

Let $\Delta\tau$ be the period of the clock (intrinsic $\Rightarrow$ in its rest frame).

$$[g_{00}(\vec{x}_{(1)})].(dx^0)^2 = -d\tau^2 \qquad \eta^{\mu\nu} = \mathrm{Diag}(1, +1, +1, +1)$$

$$dx^0 = d\tau\left(-g_{00}(\vec{x}_{(1)})\right)^{1/2}$$

$$\frac{\partial f}{\partial x_i}dx_i = df$$

$$\frac{d^2x'^\mu}{d\tau^2} = \frac{d\left(\frac{dx'^\mu}{d\tau}\right)}{d\tau} \qquad \frac{da}{d\tau} = \frac{dx^\sigma}{d\tau}\frac{\partial a}{\partial x^\sigma}$$

$$= \frac{d}{d\tau}\left(\frac{dx'^\mu}{d\tau}\right) = \frac{d}{d\tau}\left(\partial_\nu x'^\mu\frac{dx^\nu}{d\tau}\right)$$

$$= \frac{d}{d\tau}\left[\partial_\nu x'^\mu\right]\frac{dx^\nu}{d\tau} + \partial_\nu x'^\mu\frac{d^2x^\nu}{d\tau^2}$$

$$= \frac{dx^\sigma}{d\tau}\partial_\sigma\partial_\nu x'^\mu\frac{dx^\nu}{d\tau} + \partial_\nu x'^\mu\frac{d^2x^\nu}{d\tau^2}$$

Recall: $\Gamma'^l_{mn} = \partial_i x'^l\,\partial'_m x^j\,\partial'_n x^k\,\Gamma^i_{jk} + \partial_k x'^l\,\partial'_m\partial'_n x^k$

Ah, so I'm in a frame with $\Gamma = 0$.

Rewrite with 'on: $\Gamma^l_{mn} = \partial'_i x^l\,\partial'_m x^j\,\partial'_n x^k\,\Gamma'^i_{jk} + \partial'_k x^l\,\partial_m\partial_n x'^k$

now with $\Gamma' = 0$ (primed was the one with $\Gamma = 0$)

$$\Gamma^l_{mn} = \partial'_k x^l\,\partial_m\partial_n x'^k$$

$$\partial_l x'^j\,\Gamma^l_{mn} = \underbrace{\partial_l x'^j\,\partial'_k x^l}_{\delta^j_k}\,\partial_m\partial_n x'^k$$

$$\delta^j_k\,\partial_m\partial_n x'^k$$

$$\Rightarrow \boxed{\partial_l x'^j\,\Gamma^l_{mn} = \partial_m\partial_n x'^j}$$

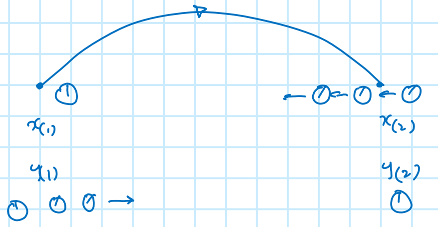
This can now be used to get $\partial_\rho x'^\mu\,\Gamma^\rho_{\sigma\nu}\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} + \partial_\rho x'^\mu\frac{d^2x^\rho}{d\tau^2}$

$$= \partial_\rho x'^\mu\left[\Gamma^\rho_{\sigma\nu}\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} + \frac{d^2x^\rho}{d\tau^2}\right]$$

Rough (second)

Saturday, 03 March 2018 11:25 AM

2 coordinate systems. x & y.



Signal 1 $x_{(1)}$ $x_{(2)}$ clock 1's coordinate sys

$y_{(1)}$ $y_{(2)}$ clock 2's coordinate sys

signal 2 $x_{(1)} + dx^0_{(1)}$ $x_{(2)} + dx^0_{(1)}$ $\because$ g is time independent

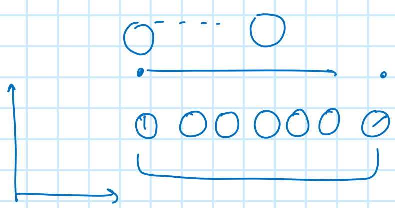
$y_{(1)} + dy^0_{(1)}$ $y_{(2)} + dy^0_{(1)}$ can't say $dy^0_{(1)} = dy^0_{(2)}$ $\because$ g may depend on time in the y coordinate system.

$d\tau^2_{(1)} = -g_{00}(dx^0_{(1)})^2$; $d\tau^2_{(2)} = -g_{00}(dy^0_{(2)})^2$

$\because$ clock 1 is at rest in x coordinate sys.

Question: relation b/w $dx^0_{(1)}$ & $dy_{(1)}$

Guess : $dy^0_{(1)} = dx^0_{(1)}$



③ 1-dim unit change scale

x^0 τ

$\frac{d}{dx^i} =$

$\frac{d}{dx^\mu} =$

$\frac{dx^\mu}{dx^\nu} = \delta^\mu$

$\partial_\nu x^\mu = \delta^\mu{}_\nu$

$\partial_\nu x^\mu = \eta^\mu{}_\nu$ ~~X~~

$x^\mu = (x_0, \vec{x})$

$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(+\frac{\partial}{\partial t}, -\frac{\partial}{\partial \vec{x}}\right)$

$\partial_\nu x^\mu = -\eta^\mu{}_\nu$ ~~X~~

$= \delta^\mu{}_\nu$

② 2 A units

② 2 k units

$2 \to 2'$

$x \to x'$

Atul $-g_{00}(dx^0)^2 = ds^2$

Kishor $-g'_{00}(dx'^0)^2 = ds'^2$

$-g_{00}(2)^2 = ds^2$ Transform x'

$-g'_{00}(2)^2 = ds'^2$ x^0

g_{00}

g'_{00}

Convention : Inertial frame : measure time.

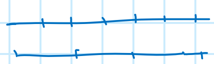
$g = \lambda g$

$-(2)^2 = ds^2 = \underline{d\tau^2}$

$du^2 = -ds^2 = d\tau^2$ STR

$\frac{ds^2}{d\tau^2} = g$ η

$g = \eta$



at rest

$ds^2 = dx^2$

$x \to \lambda x = x'$

$ds'^2 = \lambda dx^2 = \lambda ds^2$

Lecture 10 | field equations

Saturday, October 28, 2017 4:02 PM

Recall: $m\left(\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\right) = q F^\mu{}_\rho \frac{dx^\rho}{d\tau} + a R^\mu{}_{\alpha\beta\gamma} F_{\alpha\beta}\frac{dx^\gamma}{d\tau}$

NB: Both sides transform the same way ⇒ general coordinate invariance is respected.

Recall: a from dimensional analysis is $\sim 10^{-36}$ cm; so not easy to observe

Remark: One can try to write terms with derivatives as well $D^\nu R_{\nu\rho\sigma}{}^\mu \frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}$, these would be present even in the absence of the EM fields but the term added above is due to the EM fields (zero otherwise)

EM Field Equations

We start with flat spacetime

Recall: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$;

$\therefore \Rightarrow \partial_{[\mu} F_{\nu\rho]} = 0$ (which means you add all the cyclic permutations; totally anti-symmetric combination is zero) $\because F_{\mu\nu} = -F_{\nu\mu}$

viz. $\partial_\mu F_{\nu\rho} + \partial_\nu F_{\mu\rho} + \partial_\rho F_{\mu\nu} = 0$; Also called the Bianchi Identity.

Remark: Sometimes one doesn't like $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ & directly imposes the Bianchi Identity.

Remark 2: This (the Bianchi Identity) corresponds to $\nabla\cdot B = 0$ & $\nabla\times E + \frac{\partial B}{\partial t} = 0$

Story: Then there's another set of eq's that don't follow from $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This is a statement about the source of the fields

"Fact": $\partial^\mu F_{\mu\nu} = -J_\nu$; $J^0 = \rho$; Electric charge Density; J^i ; Current Density.

Claim: These correspond to

$\nabla\cdot E = \rho$; $\nabla\times B - \frac{\partial E}{\partial t} = \vec{J}$ ($c=1$ unit)

Story: In a local inertial frame (x' coordinate), one must have ① $\partial'_{[\mu} F'_{\nu\rho]} = 0$ & ② $\partial'^\mu F'_{\mu\nu} = -J'_\nu$ (instead of using $F'_{\mu\nu} = \partial'_\mu A'_\nu - \partial'_\nu A'_\mu$)

Further, J' can also be evaluated (in x') as it would be in the absence of gravity.

NB: In a general frame, $F'_{\mu\nu} = \partial'_\mu x^\rho \partial'_\nu x^\sigma F_{\rho\sigma}$

$\partial' \to$ in terms of ∂ as before

Def^n: $J_\nu = \partial_\nu x'^\sigma J'_\sigma$ NB: This makes J_ν a tensor.

Claim: $D^\mu F_{\mu\nu} = -J_\nu$ NB: We had to use D to preserve tensorial behaviour (as discussed earlier).

NB 2: $F'_{\mu\nu}$ use ∂' $\because D' \leftrightarrow \partial'$ in that case.

Claim: $F_{\mu\nu} \overset{def^n}{=} D_\mu A_\nu - D_\nu A_\mu \overset{claim}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$

($\because \Gamma$ terms get cancelled).

NB: One could start with defining A_μ using A'_μ, $F_{\mu\nu}$ independently using $F'_{\mu\nu}$, use the rel^n b/w F' & A' to derive F in terms of A

This would yield $F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu$.

... + completed for EM field + ...

Saturday, November 18, 2017 5:39 PM

Deeper NB: This is important in ensuring the principle of equivalence holds

$\because$, say, $F^{\mu\nu}$ involved Γ.
∇^μ has a ∂. This would act on Γ
Thus the LHS of $\nabla^\mu F_{\mu\nu} = -J_\nu$
would involve $\partial\Gamma$ like terms.
Now in a local inertial frame, $\Gamma = 0$ but
$\partial\Gamma$ is not zero
Thus this will not reduce to the standard eq^n in the local inertial frame.

Remark (about $\nabla^\mu F_{\mu\nu} = -J_\nu$):

aim: Evaluate $\underbrace{\nabla^\nu \nabla^\mu F_{\mu\nu}}_{= \frac{1}{2}[\nabla^\nu, \nabla^\mu] F_{\mu\nu}} = -\nabla^\nu J_\nu$

$\because = \frac{1}{2}(\nabla^\nu\nabla^\mu F_{\mu\nu} + \nabla^\mu\nabla^\nu F_{\nu\mu})$

$= \frac{1}{2}(\nabla^\nu\nabla^\mu - \nabla^\mu\nabla^\nu] F_{\mu\nu}$ $\quad \because F_{\nu\mu} = -F_{\mu\nu}$

recall: we had "derived" identities involving $[\nabla, \nabla] A = RA + RA + \ldots$

consider: $[\nabla_\nu, \nabla_\mu] F_{\alpha\beta} = -R_{\alpha\mu\nu}{}^{\rho} F_{\rho\beta} - R_{\beta\mu\nu}{}^{\rho} F_{\alpha\rho}$

Now we raise index ($\because$ it can be done inside covariant derivatives) & replace $\alpha\beta$ by $\mu\nu$.

$[\nabla^\nu, \nabla^\mu] F_{\mu\nu} = -R_\mu{}^{\mu\nu\rho} F_{\rho\nu} - R_\nu{}^{\mu\nu\rho} F_{\mu\rho}$

$= \underbrace{-R^{\nu\rho} F_{\rho\nu}}_{=0} + \underbrace{R^{\mu\rho} F_{\mu\rho}}_{=0} = 0$

$\because R$ is symmetric & F is anti-symmetric

$\Rightarrow \nabla^\nu\nabla^\mu F_{\mu\nu} = 0$ for the LHS

RHS? In local inertial $\partial'_\mu J'^\mu = 0$ (by construction)

Claim: $\nabla_\mu J^\mu = 0 \Leftarrow \partial'_\mu J'^\mu = 0$

Remark: If $\partial'_\mu J'^\mu \neq 0$, then there'd be inconsistencies

For eg. $\partial'^\mu F'_{\mu\nu} = -J'_\nu$ is given

$\Rightarrow \partial'^\nu(\partial'^\mu F'_{\mu\nu}) = -\partial'^\nu(J'_\nu) = 0$

$\because$ second derivative would involve neighbouring point which may not be locally inertial. (but then $\partial'^\mu F'_{\mu\nu}$ also had 2 derivatives, why couldn't we object then? — $\because$ even at F', the Γ disappears

this is correct because (as calculated above) the R part doesn't contribute.

Conclusion: The equivalence principle is working due to the extra condition enforced by electrodynamics.

… + GR Field Equations …

Saturday, 25 November 2017 05:36 PM

GR Field Eqⁿ

NB: Starting from the equivalence principle is pointless because gravity is zero in a local inertial frame.

Intuition: We try to figure out what eqⁿ would give, in the appropriate limit,

$$\nabla^2\phi = 4\pi G\rho_m$$

NB: The LHS we saw was related to the metric; $\left(\text{recall } h_{00} = -\frac{2\phi}{c^2}\right)$
∴ must be related to two derivatives of the metric.

NB: Two derivatives of the metric have the same # free parameters (ones that can't be killed by coordinate transformⁿ) as a tensor of rank 2. The only choice we have is $R_{\mu\nu}$ & $R g_{\mu\nu}$ — why not $g_{\mu\nu}$ alone? ∵ you need two derivatives of the metric.

NB: Using mass density is troublesome.

Justⁿ: For an H atom, the mass $m_H \neq m_p + m_e$ (proton mass; electron mass)

One must take the binding energy into account, else the mass we get would be different from the inertial (gravitational?) mass.

Conclⁿ: Use energy density

"Corollary": Energy by itself is not covariant; one should use the energy-momentum tensor, $T_{\mu\nu}$. (More on evaluating $T_{\mu\nu}$ in the next lecture)

"⇒" $$b R_{\mu\nu} + c R g_{\mu\nu} = a T_{\mu\nu}$$

$$\Leftrightarrow \quad R_{\mu\nu} + c R g_{\mu\nu} = a T_{\mu\nu} \quad \left(\text{for } b\neq 0 \text{ \& } \frac{c}{b}\to c,\ \frac{a}{b}\to a\right)$$

Claim. $\partial'^{\mu} T'_{\mu\nu} = 0$ (in a local inertial frame) $\Rightarrow D^{\mu} T_{\mu\nu} = 0$

NB: $\Rightarrow D^{\mu}\left(R_{\mu\nu} + c R g_{\mu\nu}\right) = 0$

This we demand to hold identically because if we try to demand this as an additional condition, apparantly, the solⁿ gets very specific.

Recall: $D^{\mu}\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) = 0$ identically

Conclusion: $c = -\frac{1}{2}$.

Strategy: To find "a" we use a consistency check with Newtonian gravity (in the weak field limit)

Recall: $g_{00} \simeq -1 - 2\phi$ (in the weak field limit); $\Gamma^0_{00} \simeq 0$, $\Gamma^i_{00} \simeq \partial_i\phi$

$T_{00} = \rho_m$ $\qquad \Rightarrow R_{00} = ?$, $\Rightarrow R = ?$ Exercises

Claim: in the weak field limit the 00 component becomes

$$-2\nabla^2\phi = a\rho_m$$

$\Rightarrow a = 8\pi G$.

Remark. One can add to $T_{\mu\nu} \to T_{\mu\nu} + \Lambda g_{\mu\nu}$.

Conclusion: $$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$ (Sign convention matches Weinberg's)

Exercise from Lecture 3 (or 4)

Sunday, 11 February 2018 12:18 PM

$$g'^{il} \underbrace{g'_{lj}} = \delta^i{}_j$$

$$\partial_\rho x'^j \quad \left\{ g'^{il} \, \partial'_l x^m \, \underbrace{\partial'_j x^n} \, g_{mn} = \delta^i{}_j \right.$$

$$g'^{il} \, \partial'_l x^m \, \delta^n_\rho \, g_{mn} = \partial_\rho x'^i$$

$$g^{\rho\sigma} \quad \left\{ g'^{il} \, \partial'_l x^m \, g_{m\rho} = \partial_\rho x'^i \right.$$

$$g'^{il} \, \partial'_l x^m \, \underbrace{g_{m\rho} \, g^{\rho\sigma}} = \partial_\rho x'^i \, g^{\rho\sigma}$$

$$g'^{il} \, \partial'_l x^m \, \delta_m{}^\sigma = \partial_\rho x'^i \, g^{\rho\sigma}$$

$$\partial_\sigma x'^j \quad \left\{ g'^{il} \, \partial'_l x^\sigma = \partial_\rho x'^i \, g^{\rho\sigma} \right.$$

$$g'^{il} \, \delta_l^j = \partial_\sigma x'^j \, \partial_\rho x'^i \, g^{\rho\sigma}$$

$$g'^{ij} = \partial_\sigma x'^j \, \partial_\rho x'^i \, g^{\rho\sigma}$$