Lecture 9 | Gravitational Redshift + EM

01 October 2017 01:00 PM

Gravitational Redshift

$\tilde{x}^0_{(1)}$ $\qquad$ $\tilde{x}^0_{(1)} + c$

$x^0_{(1)}$ $\qquad$ $x^0_{(1)} + c$

$\vec{x}_1$ $\qquad$ $\vec{x}_2$

Consider: a clock sitting on the point $\vec{x}_1$ in space.

$\Rightarrow \frac{dx^i}{dt} = 0$.

Assume: Time independent metric $g_{\mu\nu} = g_{\mu\nu}(\vec{x})$

~~(I think we're taking the non-relativistic limit $\Rightarrow$ weak field approx.)~~ Nope!

Recall: For a physical particle, $\frac{ds^2}{du^2} < 0$ & since u is an affine parameter $\frac{ds}{du} = \text{const}$.

This means I can always rescale to get $\frac{ds^2}{du^2} = -1$ (for time like)

$\Rightarrow du^2 = -ds^2 = d\tau^2$. So I can write

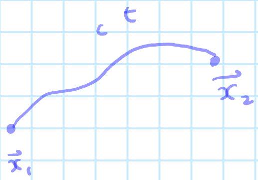
$$-\frac{ds^2}{du^2} = \frac{ds^2}{d\tau^2} \Rightarrow -g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 1 \qquad \because dx^i = 0$$

~~(In the weak field limit, one ignores $\frac{dx^i}{du}$ compared to $\frac{dx^0}{du}$. That's what I think is being done here but I need to check this)~~

$$-g_{00}\left(\frac{dx^0}{d\tau}\right)^2 = 1 \Rightarrow \Delta x^0 = \Delta\tau_1\left(-g_{00}(\vec{x}_{(1)})\right)^{-1/2}$$

(ticks)

Story: The clock emits two signals, at $x^0_{(1)}$ & $x^0_{(1)} + dx^0$. Say it took 'ct length of time' (a const, not speed of light, g) for the first signal to reach point $\vec{x}_2$. Since $g^{\mu\nu}$ is time independent, the second signal will also take as long (see the figure). The interval between the two signals at $\vec{x}_2$ will again be dx^0



I now want to find the proper time b/w two clock ticks, as seen by an observer sitting on the point $\vec{x}_{(2)}$.
Why bother? Because the clock of observer 2 is ticking according to his proper time.

Calc: $\Delta x^{(0)} = \Delta\tau_{(2)}\left(-g_{00}(\vec{x}_{(2)})\right)^{-1/2}$ $\qquad$ I wrote the eq^n for a clock sitting at $\vec{x}_{(2)}$ observing the time difference of the signals that reached $\vec{x}_{(2)}$.

$$\Rightarrow \Delta\tau_{(1)}\, g_{00}(\vec{x}_{(1)})^{-1/2} = \Delta\tau_{(2)}\, g^{00}(\vec{x}_{(2)})^{-1/2}$$

$$\Rightarrow \frac{\Delta\tau_{(1)}}{\Delta\tau_{(2)}} = \left[\frac{g_{00}(\vec{x}_{(1)})}{g_{00}(\vec{x}_{(2)})}\right]^{1/2}.$$

It seems that sir didn't actually make the weak field approximation but he does it now.

Weak Field: $g^{00} = \eta^{00} + h^{00}$ where $h^{00} = -\frac{2\phi}{c^2}$ & $\eta^{00} = -1$

$$\text{so} \quad \frac{\Delta\tau_{(1)}}{\Delta\tau_{(2)}} = \left[\frac{1 + \frac{2\phi(\vec{x}_{(1)})}{c^2}}{1 + \frac{2\phi(\vec{x}_{(2)})}{c^2}}\right]^{1/2} \simeq 1 + \frac{1}{c^2}\left[\phi(\vec{x}_1) - \phi(\vec{x}_2)\right]$$

If $\phi(\vec{x}_{(1)}) \geq \phi(\vec{x}_{(2)})$, then $\Delta\tau_{(1)} \geq \Delta\tau_{(2)}$.

The clock at $\vec{x}_{(1)}$ will be slower than the one at $\vec{x}_{(2)}$

I am, however, still a little confused about why we're using 'proper time'.
What happens if I take the clock at $\vec{x}_{(2)}$ & bring it to $\vec{x}_{(1)}$?
Shouldn't their proper times be the same?
$\hookrightarrow$ clock already at $x_{(1)}$ & the one I got from $\vec{x}_{(2)}$?

Forces + Electromagnetism in GR introduced (cont.)

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Recall: A free particle moves along a geodesic $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0$ where $g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1$

Question: If there's a "force" how do we evaluate the LHS?

Idea: We can use the principle of equivalence. While this is not "sacred" it is correct to a very good approximation.

∴ Goto a frame where $\Gamma^\mu_{\nu\rho} = 0$ so that $g^{\mu\nu} = \eta^{\mu\nu}$ by the equiv. principle.

NB: In this frame, the eq^n of motion becomes $\frac{d^2x^\mu}{d\tau^2} = f^\mu$ force in the absence of gravity

∴ in local inertial frame, the principle of equivalence says that the eq^n of motion should look exactly as though there's no gravity

More Babble: Consider having a rocket you've tested in the absence of gravity & know f, the acceleration of the rocket. The principle of equivalence tells us that the eq^n will be the same in this special local inertial frame.

NB: It isn't convenient to keep changing frames so

Consider: A general coordinate system x, where

(TODO: This I couldn't figure; need to derive the parallel transport eq^n / geodesic eq^n)

(Turns out it was an exercise there also which I remember doing)

$$\frac{d^2x'^\mu}{d\tau^2} = \partial_\nu x'^\mu\left(\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\right)$$

$$\Rightarrow \underbrace{(\partial'_\mu x^\alpha)\,\partial_\nu x'^\mu}_{\delta_\nu{}^\alpha}\left(\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}\right) = f'^\mu(\partial'_\mu x^\alpha)$$

$$\Rightarrow \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = \underbrace{\partial'_\mu x^\alpha f'^\mu}_{} =: f^\alpha$$

NB: We didn't assume that f is a vector. We just determined that f' can be computed

Def^n.

Consider. x'' is another coordinate system.

Eq^n in the x'' coordinate will be $\frac{d^2x''^\alpha}{d\tau^2} + \Gamma''^\alpha_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = f''^\alpha = \partial'_\mu x^\alpha f'^\mu$

Ex: $f''^\alpha = \partial_\beta x''^\alpha f^\beta$ (easy exercise) essentially saying that f indeed transforms like a vector.

Consider: Force due to an external field, e.g. Electromagnetic force.

Idea: Same as earlier, use principle of equivalence;

Recall: $m\frac{dx^\mu}{d\tau^2} = q\eta^{\mu\nu}F_{\nu\rho}\frac{dx^\rho}{d\tau}$; $c = 1$ unit where $F_{\nu\rho} = \partial_\nu A_\rho - \partial_\rho A_\nu$; $A_\nu = \{A_0, \underbrace{A_1, A_2, A_3}\}$

$A^\nu = \{A^0, A^1, A^2, A^3\}$ +ve

$\vee$ Electrostatic Potential; Vector potential

$E_i = -F_{0i} = \partial_i A_0 - \partial_0 A_i = -\nabla\phi - \frac{\partial A}{\partial t}$ Not sure why — consistent

$B_i = \frac{1}{2}\sum_{j,k}\epsilon_{ijk}F_{jk}$ which componentwise is $\left.\begin{array}{l} B_1 = \partial_2 A_3 - \partial_3 A_2 \\ B_2 = \partial_3 A_1 - \partial_1 A_3 \\ B_3 = \partial_1 A_2 - \partial_2 A_1\end{array}\right\}$ essentially is $B = \nabla\times A$

Consider $\mu = i$ case, $\frac{d}{d\tau}\underbrace{\left(m\frac{dx^i}{d\tau}\right)}_{= p^i} = q\eta^{ij}\left(F_{j\rho}\frac{dx^\rho}{d\tau}\right) = q\left(F_{i0}\frac{dx^0}{d\tau} + F_{ik}\frac{dx^k}{d\tau}\right)$ using $F_{i0} = -F_{0i}$ & justify this properly.

$$= q\left(E_i\frac{dx^0}{d\tau} + \epsilon_{ik\ell}B_\ell\frac{dx^k}{d\tau}\right)$$

$\frac{dp^i}{d\tau} = \frac{dp^i}{dt}\frac{dt}{d\tau}$ ($dx^0/d\tau$) $\Rightarrow \frac{dp^i}{dt} = q\left(E_i + \epsilon_{ik\ell}B_\ell\frac{dx^k}{dt}\right)$

& $\frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dx^0}{d\tau}$

NB: The p^i has τ still which is the Lorentz force law.

Electromagnetism in GR (cont.)

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So in a local inertial frame, one can write (using the principle of equivalence)

$$m \frac{d^2 x'^{\mu}}{d\tau^2} = q\, g'^{\mu\nu} F'_{\nu\rho} \frac{dx'^{\rho}}{d\tau}$$

NB: The meaning of F' would be the same as that in flat space; use test charges to measure the E & B fields.

$$m\, \partial_\nu x'^{\mu} \left(\frac{d^2 x^\nu}{d\tau^2} + \Gamma^{\nu}_{\rho\sigma} \frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} \right) = q\, \partial_\alpha x'^{\mu} \partial_\beta x'^{\nu} g^{\alpha\beta} F'_{\nu\rho} \partial_\gamma x'^{\rho} \frac{dx^\gamma}{d\tau} \Big\} \times \partial'_\mu x^\delta$$

$$\Rightarrow m \frac{d^2 x^\delta}{d\tau^2} + \Gamma^{\delta}_{\rho\sigma} \frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = q\, \cancel{\partial'_\mu x^\delta \partial_\alpha x'^{\mu}}\, \partial_\beta x'^{\nu} g^{\delta\beta} F'_{\nu\rho} \partial_\gamma x'^{\rho} \frac{dx^\gamma}{d\tau}$$

$$= q g^{\delta\beta} \left(\partial_\beta x'^{\nu} \partial_\gamma x'^{\rho} F'_{\nu\rho} \right) \frac{dx^\gamma}{d\tau}$$

Defn: $F_{\beta\gamma} := \partial_\beta x'^{\nu} \partial_\gamma x'^{\rho} F'_{\nu\rho}$

$$= q g^{\delta\beta} F_{\beta\gamma} \frac{dx^\gamma}{d\tau}$$

Recall: $F'_{\nu\rho} = \partial'_\nu A'_\rho - \partial'_\rho A'_\nu$; (Ex) Claim: Show that (using the recall) that $F_{\beta\gamma} = \partial_\beta A_\gamma - \partial_\gamma A_\beta$

where $A_\beta = \partial_\beta x'^{\mu} A'_\mu$.

Remark: Repeat the calc for another arbitrary x'' coordinate sys. & conclude

$$A''_\beta = \partial_\beta x''^{\mu} A'_\mu$$

which can be combined as

$$A''_\beta = \partial_\beta x''^{\alpha} A_\alpha$$

so this shows that A_β transforms as a rank 0,1 tensor

Proof: $F_{\beta\gamma} = \partial_\beta x'^{\nu} \partial_\gamma x'^{\rho} \left(\partial'_\nu A'_\rho - \partial'_\rho A'_\nu \right)$

$$= \partial_\gamma x'^{\rho} \partial_\beta x'^{\nu} \partial'_\nu A'_\rho - \partial_\beta x'^{\nu} \partial_\gamma x'^{\rho} \partial'_\rho A'_\nu$$

$$= \partial_\gamma x'^{\rho} \underbrace{\frac{\partial x'^{\nu}}{\partial x^\beta} \frac{\partial A'_\rho}{\partial x'^{\nu}}}_{= \partial_\beta A'_\rho} - \partial_\beta x'^{\nu} \partial_\gamma A'_\nu$$

$$= \left[\partial_\beta \left(\partial_\gamma x'^{\rho} A'_\rho \right) - \cancel{\partial_\beta \partial_\gamma x'^{\rho}}\, A'_\rho \right] - \left[\partial_\gamma \left(\partial_\beta x'^{\nu} A'_\nu \right) - \cancel{\partial_\gamma \partial_\beta x'^{\nu}}\, A'_\nu \right] \quad \square$$

Bubble: F^μ earlier was a force field. This makes sense only along the trajectory of the particle. The A^μ field however is a proper vector field defined without invoking any special trajectory.

Claim. $F_{\alpha\beta}$ is a rank $(0,2)$ tensor.

Proof: $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$

$F'_{\alpha\beta} = \partial'_\alpha A'_\beta - \partial'_\beta A'_\alpha$

$$F'_{\alpha\beta} \overset{\text{claim}}{=} \partial'_\alpha x^\mu \partial'_\beta x^\nu F_{\mu\nu}$$

$$= \partial'_\alpha x^\mu \partial'_\beta x^\nu \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right)$$

$$= \partial'_\alpha x^\mu \partial_\mu A_\nu \partial'_\beta x^\nu - \partial'_\alpha x^\mu \partial'_\beta x^\nu \partial_\nu A_\mu$$

$$= \partial'_\beta x^\nu \partial'_\alpha A_\nu - \partial'_\alpha x^\mu \partial'_\beta A_\mu$$

$$= \partial'_\alpha \left(\partial'_\beta x^\nu A_\nu \right) - \partial'_\beta \left(\partial'_\alpha x^\mu A_\mu \right)$$

$$= \partial'_\alpha A'_\beta - \partial'_\beta A'_\alpha \quad \square$$

Remark: How did they know this would work out?

Conclusion:

$$m \frac{d^2 x^\delta}{d\tau^2} + \Gamma^{\delta}_{\rho\sigma} \frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = q\, g^{\delta\beta} F_{\beta\gamma} \frac{dx^\gamma}{d\tau}$$

from the equivalence principle.

Further: To the RHS one can add

$$a\, q\, R^{\delta}{}_{\alpha\beta\gamma} \frac{dx^\alpha}{d\tau} F^{\beta\gamma}$$

but a is small (as argued earlier)

Point: Principle of equivalence is not sacred. Any term that respects general coordinate invariance might be present.

Rough (disambiguation)

07 October 2017 01:56 PM

$-d\tau^2$ is proper time.

clock

$\vec{x}_{(1)}$ $\vec{x}_{(2)}$

(1) Assume Time Independent Metric $g_{\mu\nu}(\vec{x})$.
(2) The object is at rest in this metric $\rightarrow$ $\frac{dx^i}{dt} = 0$.

$$g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du} = \frac{ds^2}{du^2} = -1$$

$-ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$

$d\tau^2$

$\frac{ds^2}{du^2} = -1$

$-d\tau^2$

$\frac{-d\tau^2}{du} = 1$

$d\tau = du$

$\frac{ds^2}{du^2} = \text{const}.$

In the weak field limit,

$$g_{00}(\vec{x})\left(\frac{dx^0}{d\tau}\right)^2 = -1$$

Let $\Delta\tau$ be the period of the clock (intrinsic $\Rightarrow$ in its rest frame).

$$\left[g_{00}(\vec{x}_{(1)})\right].(dx^0)^2 = -d\tau^2$$

$$dx^0 = d\tau\left(-g_{00}(\vec{x}_{(1)})\right)^{1/2}$$

$\eta^{\mu\nu} = \mathrm{Diag}(1, +1, +1, +1)$

$$\frac{d^2x^\mu}{d\tau^2} = \frac{d\left(\frac{dx^\mu}{d\tau}\right)}{d\tau}$$

$\frac{\partial f}{\partial x_i}dx_i = df$

$$= \frac{d}{d\tau}\left(\frac{dx'^\mu}{d\tau}\right) = \frac{d}{d\tau}\left(\partial_\nu x'^\mu \frac{dx^\nu}{d\tau}\right)$$

$$= \frac{d}{d\tau}\left[\partial_\nu x'^\mu\right]\frac{dx^\nu}{d\tau} + \partial_\nu x'^\mu \frac{d^2x^\nu}{d\tau^2}$$

$$=$$