

# Sen's Lectures General Relativity

*Volume 1*

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## Foreword

The content is due to Prof. Ashoke Sen while all errors in its presentation here are mine. Strictly then this contains not a word of Ashoke Sen and effectively no thought of mine. That was the disclaimer, wait, one more thing: there's nothing useful any further in this section. So with that I am free to express my motivation for writing this. Ashok Sen needs no introduction. I did not get a chance to learn GR at IISER due to my strange selection of electives. This, however, is something I always wanted to learn. Instead of following my usual route of picking up a text book, I thought perhaps watching Ashoke Sen's lectures, delivered at IISc (I think) which some student (Anurag I think) was kind enough to upload online, would be quicker. Watching the lectures was a little challenging because the video quality sometimes distorts the text on the blackboard and one has to make educated guesses. Of course Prof Sen speaks out everything he writes, well almost everything, so one can check the logical consistency with his cues to be sure that everything is correct. That's a good exercise. I have tried making notes and then presenting them in a neat format here as a way to make the subject clear to myself but I suspect this would be of use to others as well. Prof. Sen has also delivered more lectures on many other topics, both advanced and basic, which I might try to cover in my subsequent efforts.

## Part 1

# Riemannian Geometry

## Metric and Tensors

### 1.1. Motivation

Recall that Newton's law for the force of gravity

$$\vec{F} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

has an uncanny resemblance to that of electrostatic attraction between two charged particles. From this perspective generalisation of electrostatics to Maxwell's theory of electrodynamics is similar to the generalisation of Newtonian gravity to Einstein's general theory of gravity (GR). In particular, both electrostatics and Newtonian gravity have the property of instantaneous propagation of effects as they both depend on the relative distance of particles at the instant of time when the force is to be evaluated. In electrodynamics this instantaneity is replaced with propagation of electromagnetic waves and one of the main goals in generalising Newtonian gravity is to find a similar explanation.

Just as vector calculus is the underlying mathematics of Maxwell's electromagnetism, Riemannian Geometry – a generalisation of Euclidean Geometry – is that of GR. Thus one must familiarise oneself with at least the basic concepts of Riemannian Geometry as a prerequisite to learning GR.

### 1.2. Euclidean Geometry

Consider two points  $(x^1, x^2, x^3)$  and  $(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$ . According to Euclidean geometry the distance between these points is given by  $ds = \sqrt{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}$ . This can be generalised to  $N$ -dimensions as

$$ds^2 = \sum_{i=1}^N (dx^i)^2.$$

In Riemannian geometry the notion of distance between two close by points is generalised to

$$ds^2 = \sum_{i,j=1}^N g_{ij}(\vec{x}) dx^i dx^j$$

where  $g_{ij}(\vec{x})$  is called the *metric* and is a function of  $(x^1, x^2 \dots x^N)$ . Since  $dx^i dx^j = dx^j dx^i$ , viz. it is symmetric under the exchange of  $i$  and  $j$ , it follows that  $g_{ij} = g_{ji}$  (see exercise 2). Euclidean geometry can be obtained as the special case where  $g_{ij} = \delta_{ij}$ , the Kroneker delta, which is defined as

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

It is possible, however, that two different  $g$ s represent the same space. Consider another co-ordinate system which labels the same point  $\vec{x}$  as  $\vec{x}'$  with components  $(x'^1, x'^2 \dots x'^N)$ . Again consider two points,  $\vec{x}'$  and, a point close to it,  $\vec{x}' + d\vec{x}'$  where the old  $d\vec{x}$  is related to the new  $d\vec{x}'$  by

$$dx^i = \sum_k \frac{\partial x^i}{\partial x'^k} dx'^k.$$

Substituting for  $dx^i$  in the expression for  $ds^2$  one gets

$$ds^2 = \sum_{k,l} \underbrace{\left( \sum_{i,j} g_{ij} \frac{\partial x^i}{\partial x'^k} \frac{\partial x^j}{\partial x'^l} \right)}_{g'_{kl}} dx'^k dx'^l = \sum_{k,l} g'_{kl} dx'^k dx'^l.$$

Thus a transformation change can cause  $g$  to change even though they both describe the same object.

EXAMPLE 1. In cartesian coordinates  $ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$ . Using the polar coordinates one can write

$$\begin{aligned} x^1 &= r \sin \theta \cos \phi \\ x^2 &= r \sin \theta \sin \phi \\ x^3 &= r \cos \theta \end{aligned}$$

which can be used to determine  $dx^1 = \sin \theta \cos \phi dr + r \cos \theta \cos \phi d\theta - r \sin \theta \sin \phi d\phi$ ,  $dx^2 = \dots$  and  $dx^3 = \dots$ . When plugged into the expression for  $ds^2$  one obtains  $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$  (this can also be obtained more easily by writing  $d\vec{r} = dr\hat{r} + r d\theta\hat{\theta} + r \sin \theta d\phi\hat{\phi}$  and then using  $ds^2 = d\vec{r} \cdot d\vec{r}$  as described in Landau's book). Manifestly then  $g_{rr} = 1$ ,  $g_{\theta\theta} = r^2$ ,  $g_{\phi\phi} = r^2 \sin^2 \theta$  while in the cartesian case  $g_{11} = g_{22} = g_{33} = 1$  even though they both describe the same space.

EXERCISE 2. Consider a symmetric matrix  $A$  with elements  $A_{ij} = A_{ji}$ . If one wishes to evaluate  $\sum_{i,j} A_{ij} B_{ij}$  then show that it suffices to assume  $B$  is also symmetric.

SOLUTION. Assume  $B_{ij}^{(s)} := \frac{B_{ij} + B_{ji}}{2} = B_{ji}^{(s)}$  and  $B_{ij}^{(a)} := \frac{B_{ij} - B_{ji}}{2} = -B_{ji}^{(a)}$ . Evidently  $B_{ij} = B_{ij}^{(s)} + B_{ij}^{(a)}$  so it suffices to show that  $\sum_{i,j} A_{ij} B_{ij}^{(a)} = 0$  to prove the claim. This follows from  $\sum_{i,j} \frac{1}{2} (A_{ij} B_{ij}^{(a)} + A_{ji} B_{ij}^{(a)}) = \sum_{i,j} \frac{1}{2} (A_{ij} B_{ij}^{(a)} - A_{ji} B_{ji}^{(a)}) = 0$ .

## Part 2

# General Relativity

## CHAPTER 2

# Conventions, Axioms and the Weak Field Limit