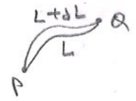


If $\delta L = 0$ to first order, then the length of the curve is extremal.
($\because$ around a maxima, first order changes must vanish).

Ways of defining a curve:

Recall: $(n-1)$ coordinates defined in terms of the n^{th}. This is usually how we describe a line.

B) NB: Define $(n-1)$ eq^{n}s in n variables. This is more symmetric (but implicit)

C) : Parametric Description

u changes monotonically as you move along the curve.

$x^n = f^n(u)$ → define the curve

good: explicit, doesn't break symmetry

bad: different parametrizations may describe the same curve.

Now a nearby curve will be given as

$$x^i = f^i(u) + \delta f^i(u)$$

$$\delta f^i(0) = 0 = \delta f^i(1)$$

Geodesic $\Rightarrow \delta X = 0$ to order δf^i

$$\text{Length} := L = \int_P^Q ds = \int du \cdot \frac{ds}{du} = \int du \cdot \sqrt{g_{ij}(x)\, \frac{dx^i}{du} \cdot \frac{dx^j}{du}}\ \Bigg|_{x^i = f^i(u)}$$

not fully certain but intuitively makes sense.

Short hand: Def^n: $\delta x^i := \delta f^i(x)$

$$\delta L = \int_0^1 \left(g_{mn} \frac{dx^m}{du} \cdot \frac{dx^n}{du} \right)^{-1/2} \cdot \frac{1}{2} \left\{ \partial_k g_{ij}\, \delta x^k\, \frac{dx^i}{du}\, \frac{dx^j}{du} \right.$$

$$\left. + 2 \cdot g_{ij} \frac{d}{du}(\delta x^i) \cdot \frac{dx^j}{du} + \cancel{g_{ij} \frac{dx^i}{du} \cdot \frac{d(\delta x^j)}{du}} \right\}$$

change $i \to k$

$$\stackrel{\text{goal}}{=} \int_0^1 du\ (\ldots)\ \delta x^k(u) + (\ldots)\, O(\delta x^2) + \ldots$$

Then we can demand this to be zero.

Def^n: Affine Parameter (Implicit):

Now choose u s.t. equal u spans equal distance. Then we have

$$\frac{d}{du} \left\{ g_{ij} \frac{dx^i}{du} \cdot \frac{dx^j}{du} \right\}^{1/2} = 0$$

$$\delta L = \int_0^1 \left(g_{mn} \frac{dx^m}{du}\frac{dx^n}{du}\right)^{-1/2} \cdot \frac{1}{2}\,\delta x^k \cdot \underbrace{\left\{ \partial_k g_{ij}\, \frac{dx^i}{du} \cdot \frac{dx^j}{du} \;\overset{\text{int. by parts}}{-\, 2\frac{d}{du}}\, g_{kj}\, \frac{dx^j}{du} - 2 g_{kj}\, \frac{d^2 x^j}{du^2} \right\}}_{\text{This must vanish}}$$

(The surface term is anyway zero $\because$ δx^k on the surface $= 0$)

$$-\frac{1}{2} g^{lk} \underbrace{\qquad\quad} = \overset{\text{(last term)}}{\frac{d^2 x^l}{du^2}} + g^{lk} \frac{dx^j}{du} \partial_m g_{kj} \frac{dx^m}{du} - \frac{1}{2} g^{lk} g_{ij} \frac{dx^i}{du}\frac{dx^j}{du} = 0$$

Ex: fill calculations & get (a) $\dfrac{d^2 x^l}{du^2} + \Gamma^l_{ij} \dfrac{dx^i}{du}\dfrac{dx^j}{du} = 0$

Ex: Show that (b) holds $\left(\text{(b)} \quad \dfrac{d}{du}\left(g_{mn} \dfrac{dx^m}{du} \dfrac{dx^n}{du} \right) = 0 \right)$ if (a) is true.

viz. the 'affineness' is automatic.

Ex: Show that in a different coordinate system, the eq^n becomes

$$\frac{d^2 x'^l}{du^2} + \Gamma'^l_{ij} \frac{dx'^i}{du} \frac{dx'^j}{du} = 0$$

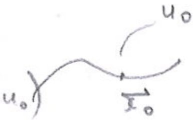
Eucledian metric $\quad g_{ij} = \delta_{ij} \quad \Gamma^i_{jk} = 0$

$$\frac{d^2 x^l}{du^2} = 0 \quad \Rightarrow \quad x^l = a^l u + b^l$$ where a & b are just constants.

Choose a point $\vec{x}_0$ & a coordinate sys x' s.t. $\Gamma'^i_{jk} = 0$ at $\vec{x}'_0$

Then $\quad \dfrac{d^2 x'^l}{du^2} = O((u-u_0))$

$(\because \Gamma'^i_{jk} = 0$ at $u = u_0)$



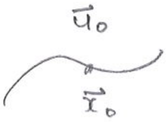
& after integration $\quad x'^l = a'^l u + b'^l + O((u-u_0)^3)$

i.e. won't have $O((u-u_0)^2)$ term.

Tangent Vectors & Tangent Space

NB: Tangent vectors are not the usual vectors built from the metric & its f^ns.

Consider $\vec{x}_0$ & a geodesic passing through $\vec{x}_0$.



Q: How many such geodesics are there that pass through it?
(# of parameters)

Recall: Usual 3-d, we use a vector to specify a specific tangent at a point.

$$\frac{d^2x^l}{du^2} + \Gamma^l_{mn}\frac{dx^m}{du}\frac{dx^n}{du} = 0 \qquad (l = 1 \dots n)$$

These are n 2nd order diff eq^{n}s, so it needs $2n$ constants.

$x^l|_{u=u_0} = x^l_0 \rightarrow n$ boundary conditions

Def^n: $\quad n^l := \left.\frac{dx^l}{du}\right|_{\vec{x}=\vec{x}_0}$; Given n^l also, we get n more vectors.

$\therefore \quad n^l :=$ Tangent vectors to a geodesic at $\vec{x}_0$.

NB: We only need the direction of n^l. $\because$ u can be scaled.

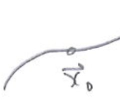
NB: If all of n^l is used (dir & magnitude) then the parametrization (scale) is also fully specified.

N. Def^n:
"Tangent Space at $\vec{x}_0$:= Space containing all such vectors $\vec{n}$.

NB: Any vector in the tangent space (of $\vec{x}_0$), specifies a geodesic through $\vec{x}_0$.

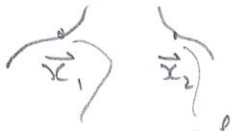
NB: It maybe so that globally, 2 $\vec{n}$ describe the same geodesic.

eg.  – great circle, one geodesic; but locally, 2 geodesics corresponding to the two tangent.



$n^l = \left.\frac{dx^l}{du}\right|_{\vec{x}_0}$ Same geodesic in a diff. coordinate.

$$n'^l = \left.\frac{dx'^l}{du}\right|_{\vec{x}'_0} = \partial_m x'^l \left.\frac{dx^m}{du}\right|_{\vec{x}_0} = \left.\partial_m x'^l\right|_{\vec{x}_0} n^m$$



are n^l & m^l parallel?

try. $\vec{n} \parallel \vec{m}$ if $n^l = \lambda m^l$ for some λ.

now $n'^l = \partial_k x'^l \big|_{\vec{x}_1} n^k$

$m'^l = \partial_k x'^l \big|_{\vec{x}_2} n^k$

NB: In the new coordinate they won't be parallel (in general)!

Looks like there's no unambiguous way to find if two tangents are parallel.

Rescue: Find a coordinate system which is "flat" to first order.
(i.e. first derivative of the metric vanishes)

Imagine two close-by points. Now if the coordinate sys has first derivative zero at point one, it will be small even at point two. Using this we can say if the tangent vectors are parallel (in the aforesaid sense).

n^l & m^l are parallel, if $n^l = \lambda m^l$ in the aforesaid coordinate sys.

(a) Choose x' coordinate sys. s.t. $\Gamma'^i_{jk} = 0$ at $\vec{x}' = \vec{x}'_0$

(b) If n^l is the tangent vector at $\vec{x}_0$ & m^l the one at $\vec{x}_0 + \delta\vec{x}_0$, then n^l & m^l are parallel iff $n'^l = \lambda m'^l$ + corrections

Def^n

Parallel Transport := $\vec{m}$ is called the parallel transport of $\vec{n}$ if $\vec{m}' = \vec{n}' + O(\delta\vec{x}_0^2)$

Recall: To fix $\Gamma'^i_{jk} = 0$, we only used the Bs. The As were anyway not affecting Γ we saw.

NB: We still have the freedom to specify the Cs. These will define different co-ordinate systems.

$\therefore$ We must $\therefore$ in our def^n of parallel vectors, ensure its invariance under such coordinate transformations.

$\because$ Else, its possible that in one $\Gamma' = 0$, $n'^l = \lambda m'^l$, whereas in $\Gamma'' = 0$, $n''^l \neq \lambda'' m''^l$ for any λ''