Lecture 19 - (Oct 28, 2015) Feb 7, 2017

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§1 Overview

Motivation : Entropy inequalities – Monotonicity of relative entropy, strong subadditivity, quantum data processing inequality & continuation of quantum entropy.

§2 Quantum Relative Entropy

Motivation : Quantum Relative Entropy can be used to express many quantities of interest. We as before establish properties of it first

Def^n (Kernel & Support). For $A \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$, the Kernel is

$$\ker(A) := \{ |\psi\rangle \in \mathcal{H} : A|\psi\rangle = 0 \}.$$

The support of A is the orthogonal subspace to $\ker(A)$,

$$\operatorname{supp}(A) := \{ |\psi\rangle \in \mathcal{H} : A|\psi\rangle \neq 0 \}.$$

For A Hermitian with $A = \sum_i a_i |i\rangle\langle i|$

$$\operatorname{supp}(A) := \operatorname{span}\{ |i\rangle : a_i \neq 0 \}.$$

Projection into the support of A is,

$$\Pi_A := \sum_{i : a_i \neq 0} |i\rangle\langle i| .$$

Def^n (Quantum Relative Entropy). $D(\rho\|\sigma) := \operatorname{tr}(\rho(\log\rho - \log\sigma))$

for $\rho \in D(\mathcal{H})$ & a positive semi-definite operator $\sigma \in \mathcal{L}(\mathcal{H})$

with $\operatorname{supp}(\rho) \subseteq \operatorname{supp}(\sigma)$.

Else $D(\rho\|\sigma) = +\infty$.

Remark : This def^n is consistent with the classical case.

Remark 2: Well, $D'(\rho\|\sigma) := \operatorname{tr}(\rho \log(\rho^{1/2}\sigma^{-1}\rho^{1/2}))$ is also consistent in the classical case.

"We" single out "our" def^n $\because$ it is meaningful as (a) it answers a sensible quantum-information processing task (what?) & (b) it reduces to answers of previous questions we've already asked.

Intui^n: This is intuitively like a distance measure b/w quantum states.

NB/: Not a mathematical distance (doesn't satisfy the
Claim triangle inequality.)

Proposition 3. $\rho \in D(\mathcal{H})$, $\sigma \geq 0 \in \mathcal{L}(\mathcal{H})$

$$D(\rho\|\sigma) = \lim_{\epsilon \downarrow 0} D(\rho\|\sigma + \epsilon \mathbb{I}).$$

proof: Perhaps given later.

Remark: Justifies further the choice of Def^n of D.

Thm 4 (Monotonicity of Quantum Relative Entropy).

For $\rho \in D(\mathcal{H})$, $\sigma \geq 0 \in \mathcal{L}(\mathcal{H})$ & $\mathcal{N}: \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}')$, a quantum channel,

$$D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

proof: comes a little later.

Claim: Thm 4 $\Rightarrow$ non-negativity of relative entropy in certain cases.

Thm 5 (Non-negativity).

For $\rho \in D(\mathcal{H})$, $\sigma \geq 0 \in \mathcal{L}(\mathcal{H})$ with $\mathrm{tr}\{\sigma\} \leq 1$,

$$D(\rho\|\sigma) \geq 0$$

& $D(\rho\|\sigma) = 0 \iff \rho = \sigma$.

proof. Apply Thm 4 using the trace map.

$$D(\rho\|\sigma) \geq D(\mathrm{tr}(\rho)\|\mathrm{tr}(\sigma)) = \cancel{\mathrm{tr}(\rho)}^{1} \log\left(\frac{\mathrm{tr}(\rho)}{\mathrm{tr}(\sigma)}\right)$$

$$\geq 0$$

NB : $\rho = \sigma \Rightarrow \log \rho - \log \sigma = 0 \Rightarrow D(\rho\|\sigma) = 0$

NB : $D(\rho\|\sigma) = 0 \Rightarrow$ the inequality is saturated. $\Rightarrow \text{tr}\{\sigma\} = \text{tr}(\rho) = 1$

$\Rightarrow \sigma$ is a valid density operator.

Applying Thm 4, we have $D(\mathcal{M}(\rho)\|\mathcal{M}(\sigma)) \leq D(\rho\|\sigma) = 0$

also we just proved $D(\ \|\ ) \geq 0$

$\Rightarrow D(\mathcal{M}(\rho)\|\mathcal{M}(\sigma)) = 0 \Rightarrow \mathcal{M}(\rho) = \mathcal{M}(\sigma)$

$\Rightarrow \rho = \sigma$ [$\because$ we can choose $\mathcal{M}$ to be the optimal measurement $\Rightarrow \max_{\mathcal{M}} \|\mathcal{M}(\rho) - \mathcal{M}(\sigma)\|_1 = \|\rho - \sigma\|_1 = 0 \Rightarrow \rho = \sigma$] $\square$

§ 2.1 Deriving Other Entropies from Quantum Relative Entropy

Motiv^n: We show how Relative Entropy is a "parent quantity" for other entropies/information.

Claim 6: For $P_A \geq 0 \in \mathcal{L}(\mathcal{H}_A)$, $Q_B \geq 0 \in \mathcal{L}(\mathcal{H}_B)$

$$\log(P_A \otimes Q_B) = \log(P_A) \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \log(Q_B)$$

proof: Simultaneously diagonalize & use common sense $\square$.

Claim 7: (Mutual Information & Relative Entropy).

Let $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ then

$$\begin{aligned} I(A;B)_\rho &= D(\rho_{AB}\|\rho_A \otimes \rho_B) \\ &= \min_{\sigma_B} D(\rho_{AB}\|\rho_A \otimes \sigma_B) \\ &= \min_{\omega_A} D(\rho_{AB}\|\omega_A \otimes \rho_B) \\ &= \min_{\omega_A, \sigma_B} D(\rho_{AB}\|\omega_A \otimes \sigma_B) \end{aligned}$$

where the optimisation is wrt $\omega_A \in \mathcal{D}(\mathcal{H}_A)$, $\sigma_B \in \mathcal{D}(\mathcal{H}_B)$.

proof: see rough, page 1.

Claim 8 (Conditional Entropy & Relative Entropy).

For $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$

$$I(A \rangle B)_\rho = D(\rho_{AB} \| \mathbb{I}_A \otimes \rho_B)$$
$$= \min_{\sigma_B \in D(\mathcal{H}_B)} D(\rho_{AB} \| \mathbb{I}_A \otimes \sigma_B)$$

proof: see rough, page 1

Claim 9 (Relative Entropy of Classical-Quantum States).

For $\rho_{XB} := \sum p_X(x) |x\rangle\langle x|_X \otimes \rho_B^x$ & $\sigma_{XB} := \sum_x p_X(x) |x\rangle\langle x|_X \otimes \sigma_B^x$

$$D(\rho_{XB} \| \sigma_{XB}) = \sum_x p_X(x) D(\rho_B^x \| \sigma_B^x)$$

proof: <skipped for now>.

§3 Quantum Entropy Inequalities

Remark: Monotonicity of quantum relative entropy entails many corollaries

Corollary 10 (Strong Subadditivity). $I(A;B|C)_\rho \geq 0$

$$\Leftrightarrow \quad H(AC)_\rho + H(BC)_\rho \geq H(ABC)_\rho + H(C)_\rho$$

proof: For $I(A;B|C)_\rho = H(AC)_\rho + H(BC)_\rho - H(ABC)_\rho - H(C)_\rho$

$$= H(B|C) - H(B|AC)_\rho$$

From claim 8, we have $-H(B|AC)_\rho = D(\rho_{ABC} \| \mathbb{1}_B \otimes \rho_{AC})$

& $H(B|C)_\rho = -D(\rho_{BC} \| \mathbb{1} \otimes \rho_C)$

Also we have (from Thm 1) $D(\rho_{ABC} \| \mathbb{1}_B \otimes \rho_{AC}) \geq D(\mathrm{tr}_A(\rho_{ABC}) \| \mathrm{tr}_A(\mathbb{1}_B \otimes \rho_{AC}))$

$$= D(\rho_{BC} \| \mathbb{1}_B \otimes \rho_C)$$

$$\Rightarrow \quad H(B|C) \geq H(B|AC) \Rightarrow I(A;B|C)_\rho \geq 0 \qquad \square$$

Corollary 11 (Joint Convexity of Quantum Relative Entropy).

For $\rho^x \in D(\mathcal{H}), \sigma^x \geq 0 \in \mathcal{L}(\mathcal{H})$, let $\rho := \sum_x p_X(x) \rho^x$

$\sigma := \sum_x p_X(x) \sigma^x$ then

$$D(\rho \| \sigma) \leq \sum_x p_X(x) D(\rho^x \| \sigma^x).$$

proof. Use $\rho_{XB} := \sum_x P_X(x) |x\rangle\langle x|_X \otimes \rho^x_B$

$\sigma_{XB} := \sum_x P_X(x) |x\rangle\langle x|_X \otimes \sigma^x_B$

so that $\displaystyle\sum_x P_X(x) D(\rho^x_B \| \sigma^x_B) \underset{\text{claim 9}}{=} D(\rho_{XB} \| \sigma_{XB})$

$\overset{\text{thm 4}}{\geqslant} D(\rho_B \| \sigma_B)$. □

Corollary 12 (Unital Channels Increase Entropy).

For $\rho \in D(\mathcal{H})$ & $\mathcal{N} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, a unital channel,

$$H(\mathcal{N}(\rho)) > H(\rho)$$

proof.

$$H(\rho) = -D(\rho \| \mathbb{1})$$
$$H(\mathcal{N}(\rho)) = -D(\mathcal{N}(\rho) \| \mathbb{1})$$
$$= -D(\mathcal{N}(\rho) \| \mathcal{N}(\mathbb{1})) \quad \because \mathcal{N} \text{ is unital}$$
$$-D(\mathcal{N}(\rho) \| \mathcal{N}(\mathbb{1})) \geqslant -D(\rho \| \mathbb{1}) \quad \because \text{thm 4}$$
$$\Rightarrow H(\mathcal{N}(\rho)) \geqslant H(\rho)$$ □

§ 3.1 Quantum Data Processing

Intuiⁿ: Similar to the classical case, performing quantum data processing reduces quantum correlations (we'll show this)

Situation: Alice & Bob share a state $\rho_{AB'}$ $I(A \rangle B)$ is reduced if Bob applies some map to his part & the state is $\sigma_{AB'}$, say, thereafter. More precisely

$$I(A \rangle B)_\rho \geqslant I(A \rangle B')_\sigma$$

Thm 13 (Quantum Data Processing for Coherent Information).

For $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$, $\mathcal{N} : \mathcal{L}(\mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_{B'})$ (some quantum channel), define $\sigma_{AB'} := \mathcal{N}_{B \to B'}(\rho_{AB})$.

Then $$I(A \rangle B)_\rho \geqslant I(A \rangle B')_\sigma$$

proof. $I(A\rangle B)_\rho \overset{\text{claim 8}}{=} D(\rho_{AB} \| \mathbb{1}_A \otimes \rho_B)$

$$I(A\rangle B')_\sigma = D(\sigma_{AB} \| \mathbb{1}_A \otimes \sigma_{B'})$$

$$= D(\mathcal{N}_{B\to B'}\, \sigma_{AB} \| \mathcal{N}_{B\to B'}(\mathbb{1}_A \otimes \rho_B))$$

$$\overset{\text{thm 4}}{\Rightarrow} I(A\rangle B)_\rho \geq I(A\rangle B')_\sigma \qquad \square$$

Thm 14 (*Quantum Data Processing for Mutual Information*).

For $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$, $\mathcal{N}: \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_{A'})$ (quantum channel)

& $\mathcal{M}: \mathcal{L}(\mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_{B'})$ (another quantum channel)

define $\sigma_{A'B'} := (\mathcal{N}_{A\to A'} \otimes \mathcal{N}_{B\to B'})\, \rho_{AB}$.

Then $$I(A;B)_\rho \geq I(A';B')_\sigma$$

proof. use claim 7 & thm 4 as above. $\square$

§ 4 Continuity of Entropy

<skipped for now>

Rough: $H(A)_\rho = -\mathrm{tr}(\rho \log \rho)$

$$I(A;B) = H(A) - H(A|B)$$
$$= H(A) + H(B) - H(A,B)$$

$$D(\rho_{AB} \| \rho_A \otimes \rho_B) = \mathrm{tr}\left(\rho_{AB}\left[\log(\rho_{AB}) - \log(\rho_A \otimes \rho_B)\right]\right)$$
$$= \mathrm{tr}\left[\rho_{AB} \log(\rho_{AB})\right] -$$
$$\mathrm{tr}\left[\rho_{AB} \log(\rho_A) \otimes \mathbb{1}_B\right] -$$
$$\mathrm{tr}\left[\rho_{AB}\, \mathbb{1}_A \otimes \underline{\log(\rho_B)}\right]$$
$$= -\left[H(A,B) - H(A) - H(B)\right]$$
$$= -H(A,B) + H(A) + H(B).$$

$$(\rho_{AB} \| \rho_A \otimes \sigma_B) = \mathrm{tr}\left[\rho_{AB}\left(\log(\rho_{AB}) - \log(\rho_A \otimes \sigma_B)\right)\right]$$
$$= \mathrm{tr}\left[\rho_{AB} \log(\rho_{AB})\right] - \mathrm{tr}\left[\rho_{AB} \log(\rho_A \otimes \sigma_B)\right] \quad \sigma_B$$
$$= -H(A,B) - \mathrm{tr}\left[\rho_{AB} \log(\rho_A)\otimes\mathbb{1}\right] - \mathrm{tr}\left[\rho_{AB}\, \mathbb{1}\otimes\log(\sigma_B)\right]$$
$$= -H(A,B)$$

$$\mathrm{tr}\left[\rho_{AB}\, \mathbb{1} \otimes \log(\sigma_B)\right]$$
$$-\mathrm{tr}\left[\rho_B \log(\sigma_B)\right]$$
$$\parallel$$
$$H(B)$$

$$-\mathrm{tr}\left[\rho_B \log \rho_B\right] \leq -\mathrm{tr}(\rho_B \log \sigma_B)$$

$$I(A\rangle B) = D(\rho_{AB} \| \mathbb{1}_A \otimes \rho_B)$$
$$= \min_{\sigma_B \in D(\mathcal{H}_B)} D(\rho_{AB} \| \mathbb{1}_A \otimes \sigma_B)$$
$$\mathrm{tr}\left(\rho_{AB}\left(\log \rho_{AB} - \log(\mathbb{1} \otimes \rho_B)\right)\right)$$
$$\cancel{\mathrm{tr}(\rho_A}$$
$$-H(AB) + H(B)$$
$$= H(B|A)$$

$$\mathrm{tr}\left[\rho_B \log \rho_B\right] \geq \mathrm{tr}(\rho_B \log \sigma_B)$$
$$\mathrm{tr}\left[\rho_B\left(\log \rho_B - \log \sigma_B\right)\right] \geq 0$$
$$\because\ D(\rho_B \| \sigma_B) \geq 0$$

$$\mathrm{tr}\left[\rho_{XB}\left(\log\rho_{XB} - \log\sigma_{XB}\right)\right]$$

$$= \mathrm{tr}\Big[\sum_x P_X(x)\,|x\rangle\langle x|_X \otimes \rho_B^x \ \infty\ \log\sum_x P_X($$ <skipped for now>