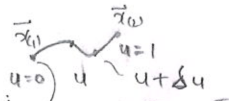


n^i vector in the tangent space at $\vec{x}_{(1)}$

aim: see what this vector becomes upon parallel transport.

$n^i(u) \to$ known (say)

$n^i(u+\delta u) \to$ calculate

Goto $\vec{x}(u)$ & find a new coordinate sys $\vec{x}'$ s.t. $\partial_i g'_{jk} = 0$ at $\vec{x}' = \vec{x}'_{(0)}$

$n'^i(u+\delta u) = n'^i(u) + O(\delta u^2)$ (by rules of parallel transport)

Since you don't want to keep changing coordinates (its inconvenient), ∴ we see what parallel transport amounts in some arbitrary frame.

Translate to x'

$$n^i(u) = \left.\frac{\partial x^i}{\partial x'^k}\right|_{\vec{x}'(u)} n'^k(u) \qquad n'^k(u) + O(\delta u^2)$$

$$n^i(u+\delta u) = \left.\frac{\partial x^i}{\partial x'^k}\right|_{\vec{x}'(u+\delta u)} n'^k(u+\delta u)$$

$$\left.\frac{\partial x^i}{\partial x'^k}\right|_{\vec{x}'(u)} + \frac{\partial^2 x^i}{\partial x'^l \partial x'^k}\cdot \delta x'^l + \dots \qquad \& \quad \delta x^l = \frac{\partial x'^l}{\partial u}\cdot \delta u$$

$$\Rightarrow n^i(u+\delta u) = n^i(u) + \frac{\partial^2 x^i}{\partial x'^l \partial x'^k}\frac{\partial x'^l}{\partial u}\delta u\, n'^k(u) + O(\delta u^2)$$

$$\frac{dn^i}{du} = \frac{\partial^2 x^i}{\partial x'^l \partial x'^k}\frac{\partial x'^l}{\partial u} n'^k(u)$$

We still have dependence on the primed coordinate. We wish to get rid of it.

$$\Gamma'^l_{mn}(\vec{x}') = \frac{\partial x'^l}{\partial x^i}\frac{\partial x^j}{\partial x'^m}\frac{\partial x^k}{\partial x'^n}\Gamma^i_{jk}(\vec{x}) + \frac{\partial x'^l}{\partial x^k}\frac{\partial^2 x^k}{\partial x'^m \partial x'^n} = 0 \quad \text{(for the primed frame)}$$

$$\frac{\partial x^p}{\partial x'^l}\Gamma'^l_{mn}(\vec{x}') = \frac{\partial x^j}{\partial x'^m}\frac{\partial x^k}{\partial x'^n}\Gamma^p_{jk}(\vec{x}) + \frac{\partial^2 x^p}{\partial x'^m \partial x'^n} = 0$$

also, recall: $n'^k(u) = \dfrac{\partial x'^k}{\partial x^r} n^r(u)$

$$\therefore \quad \frac{dx'^i}{du} = \frac{\partial x'^i}{\partial x^s}\frac{\partial x^s}{\partial u}$$

Using these (Ex: check) we get

$$\frac{dn^i}{du} + \Gamma^i_{pq}(\vec{x}(u))\, n^p \frac{dx^q}{du} = 0$$

(The primes will disappear (as kronecker delta)).

For a d-dimensional space, there're d- first order d.e. Thus given the initial vector, you can solve it for u.

NB: u needn't describe a geodesic.

Claim: The eq^n is parametrization independent.

Proof: $u = f(v)$

$$\left.\begin{aligned} \frac{dn^i}{du} &= \frac{dv}{du}\cdot\frac{dn^i}{dv} \\ \frac{dx^q}{du} &= \frac{dx^q}{dv}\cdot\frac{dv}{du} \end{aligned}\right\} \Rightarrow \frac{dv}{du}\left[\frac{dn^i}{dv} + \Gamma^i_{pq}(\vec{x}(v))\, n^p \frac{dx^q}{dv}\right] = 0$$

NB: unlike the geodesic eq^n which works in a specific parametrization.

Now choose an arbitrary coordinate sys $\vec{x}''$ (where Γ'' needn't be zero)

Recall: $n^i = n''^k \dfrac{\partial x^i}{\partial x''^k}$

$\because \Gamma$ the transformation

NB: $\dfrac{dx^q}{du} = \dfrac{\partial x^q}{\partial x''^l}\dfrac{dx''^l}{du}$

substitute in

Ex: $\Rightarrow \dfrac{\partial x^i}{\partial x''^l}\left(\dfrac{dn''^l}{du} + \Gamma''^l_{pq}\, n''^p \dfrac{dx''^q}{du}\right) = 0$

Therefore its evident that the parallel transport eq^s don't depend on the coordinate frame.

NB: Its not surprising $\because$ x coordinate sys to start with, was arbitrary. Therefore infact its a consistency check.

Ex: $\dfrac{d}{du}\left(g_{ij}(\vec{x}(u))\, n^i(u)\, n^j(u)\right) \overset{\text{claim}}{=} 0$ i.e. norm is preserved under parallel transport.

Proof sketch: (direct) $\dfrac{dn^i}{du}$ is known now, $\dfrac{\partial g_{ij}}{\partial x^k}\dfrac{dx^k}{du} = \dfrac{dg_{ij}}{du}$ & thats known.

(neat) the quantity is invariant under coordinate transformations. Thus, goto the primed coordinates. Now first derivative of g is zero (by def^n of the primed coordinate). Also, for a primed coordinate, $\dfrac{dn'^i}{du} = 0$. So that does it.

Recall:
Geodesic eq^n

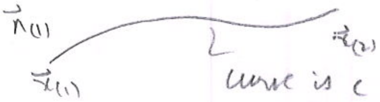
$$\frac{d^2x^i}{du^2} + \Gamma^i_{jk}\frac{dx^k}{du}\frac{dx^j}{du} = 0$$; L. Def^n: $n^i(u) := \dfrac{dx^i}{du}$ (the tangent vector)

So now, $\dfrac{dn^i}{du} + \Gamma^i_{jk}\, n^j \dfrac{dx^k}{du} = 0$

NB!: This is exactly the parallel transport eq^n

Alternate def^n of geodesic: curve s.t. its tangent vectors are transported from any given tangent vector on the curve.

$\vec{n}_{(2)}$ — after parallel transport



NB: The transport eq^n is linear.

$\Rightarrow \quad \alpha \vec{n}_{(1)} + \beta \vec{n}'_{(1)} \longrightarrow \alpha \vec{n}_{(2)} + \beta \vec{n}'_{(2)}$

$$n_{(2)}{}^i = \underbrace{M^i{}_j(\vec{x}_{(1)}, \vec{x}_{(2)}, c)}_{\text{independent of } n_{(1)} \,\&\, n_{(2)}}\, n^j_{(1)}$$

(Not entirely certain how this follows)

Def^n: $-c :=$ Path c in the reverse direction.

Q: What can we say about $M^i{}_j(\vec{x}_{(2)}, \vec{x}_{(1)}, -c)$?

NB: The parallel transport eq^n. is reversible (independent of parametrization)

We reparametrize: $v := 1-u$; this will do the job. $u=1 \Leftrightarrow v=0$, $u=0 \Leftrightarrow v=1$

$$\frac{dn^i}{dv} + \Gamma^i_{jk}\, n^j(v)\, \frac{dx^k}{dv} = 0 \longrightarrow \frac{dn^i}{du} + \Gamma^i_{jk}\, n^j(u)\, \frac{dx^k}{du} = 0$$

So we solve the same old eq^n with boundary conditions reversed.

Thus we must have

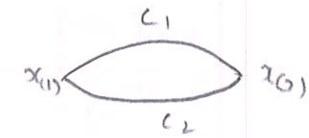
$$n_{(1)}{}^i = \left(M(\vec{x}_{(1)}, \vec{x}_{(2)}, c)^{-1}\right)^i{}_j\, n^j_{(2)}$$

$$\overset{\text{by Def}^n}{=} M(\vec{x}_{(2)}, \vec{x}_{(1)}, -c)^i{}_j\, n^j_{(2)}$$

$\Rightarrow \because n^i_{(1)}$ is arbitrary, $M(\vec{x}_{(2)}, \vec{x}_{(1)}, -c) = M(\vec{x}_{(1)}, \vec{x}_{(2)}, c)^{-1}$

NB: If we had $\displaystyle \frac{dn^i}{du} + \Gamma^i_{jk}\, n^j \left|\frac{dx^j}{du}\right| = 0$, the eq^n won't be reversible.

Take 2 curves joining $x_{(1)}$ & $x_{(2)}$, c_1 & c_2



In general, $M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_1) \neq M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_2)$

Consider: $M(\vec{x}_{(1)}, \vec{x}_{(1)}, c_1 - c_2) = M(\vec{x}_{(2)}, \vec{x}_{(1)}, -c_2) \cdot M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_1)$

$$= M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_2)^{-1}\, M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_1) \neq \mathbb{1} \text{ in general.}$$

Def^n: Monodroms Matrix around C

$:= M(\vec{x}, \vec{x}, C)$ for some closed curve C

In general, $\underbrace{\tilde{n}^i}_{\text{final vector, after coming back to } \vec{x}} = M(\vec{x}, \vec{x}, c)^i{}_j\, \underbrace{n^j}_{\text{initial vector at } \vec{x}}$.

Now lets change the coordinate system:

Recall: $\displaystyle \tilde{n}'^i = \frac{\partial x'^i}{\partial x^p}\tilde{n}^p = \frac{\partial x'^i}{\partial x^p} \cdot M(x, x, c)^p{}_q\, n^q = \frac{\partial x'^i}{\partial x^p} M(x,x,c)^p{}_q \frac{\partial x^q}{\partial x'^j}\, n'^j$

If M was 1, then you'd get

$$\tilde{n}'^{i} = n'^{i}$$

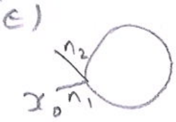
i.e. M will remain identity even in the new coordinate sys.

NB: This can be proved the other way also. If $\tilde{n}'^{i} = n'^{i}$ then M must be 1.

NB2: This analysis will not work for $M(\vec{x}_1, \vec{x}_2, c)$ $\because$ the $\frac{\partial x}{\partial x'} M \frac{\partial x'}{\partial x}$ factor won't become kronecker (they're evaluated at different points), even if $M = \mathbb{I}$.

Claim: When $R = 0$, $M = \mathbb{I}$; the reverse is true when there're no singularities (& simply connected)

Proof: Start with a small curve; length $\sim O(\epsilon)$



Convention: $0 < u < \epsilon$ so that $\frac{dx^i}{du} \sim 1$

$$\frac{dn^i}{du} = -\Gamma^i_{jk}\, n^j(u)\, \frac{dx^j}{du}$$

idea: Keep terms of order u & then integrate, you get the result to order ϵ. Then plugin the result & re-evaluate to get order ϵ^2.

$$\frac{dn^i}{du} = \overbrace{-\Gamma^i_{jk}(x_{(0)})\, n^j_{(0)}}^{\text{correct to order } \epsilon}\ \overbrace{\frac{dx^k}{du}}^{\text{order 1, can't take it at its original value}} + O(\epsilon)$$

$$\Rightarrow n^i(u) = n^i_{(0)} - \Gamma^i_{jk}\Big|_{x_{(0)}} n^j_{(0)} \left(x^j(u) - x^j_{(0)}\right) + O(\epsilon^2) \qquad (\because \epsilon u \approx \epsilon^2)$$

$$\frac{dn^i}{du} = -\Big\{\overbrace{\Gamma^i_{jk}(\vec{x}_{(0)})}^{O(1)} + \partial_l \Gamma^i_{jk}(\vec{x}_{(0)}) \overbrace{\left(x^l - x^l_{(0)}\right)}^{O(\epsilon)}\Big\} \times \Big\{ \overbrace{n^j_{(0)}}^{O(1)} - \Gamma^j_{pq}(\vec{x}_0)\, n^p(\vec{x}_0) \underbrace{\left(x^q - x^q_{(0)}\right)}_{O(\epsilon)} \Big\} \frac{dx^k}{du}$$

Now keeping upto $O(\epsilon^2)$, we have

$$\frac{dn^i}{du} = -\Big\{ \Gamma^i_{jk}\, n^j \frac{dx^k}{du} - \Gamma^i_{jk}\Gamma^j_{pq}\, n^p \left(x^q - x^q_{(0)}\right) \frac{dx^k}{du} + \partial_l \Gamma^i_{jk} \cdot \left(x^l - x^l_{(0)}\right) \cdot n^j_{(0)} \cdot \frac{dx^k}{du} \Big\} + O(\epsilon^2)$$

Integrating from $(0, \epsilon)$, we note that term 1 is zero.

$$n^i(u) = n^i_{(0)} + \underbrace{\partial_l \Gamma^i_{jk} \, n^j_{(0)} \oint_0^\epsilon (x^l - x^l_0) \frac{d(x^k - x^k_0)}{du}}_{3^{rd}\text{ term}} - \underbrace{\Gamma^i_{jk} \Gamma^j_{pq} n^p \oint_0^\epsilon (x^q - x^q_{(0)}) \frac{d(x^k - x^k_{(0)})}{du}}_{2^{nd}\text{ term}}$$

(Lets try to match indices for aesthetics)

$$= n^i_{(0)} + \partial_q \Gamma^i_{pk} \, n^p_{(0)} \oint_0^\epsilon (x^q - x^q_0) \frac{d}{du}(x^k - x^k_{(0)}) - \Gamma^i_{jk} \Gamma^j_{pq} n^p \oint_0^\epsilon (x^q - x^q_{(0)}) \frac{d}{du}(x^k - x^k_{(0)})$$

$$= n^i_{(0)} + \left[\partial_q \Gamma^i_{pk} - \Gamma^i_{jk} \Gamma^j_{pq} \right] n^p_{(0)} \oint_0^\epsilon (x^q - x^q_{(0)}) \frac{d}{du}(x^k - x^k_{(0)})$$

NB: The boundary terms are zero. The derivative can be $\therefore$ shifted with a minus sign.
NB2: This means the indices q & k are anti-symmetric
NB3: Thus only the symmetric part of the remaining part must contribute

$$= n^i_{(0)} + \frac{1}{2} \left[\partial_q \Gamma^i_{pk} - \partial_k \Gamma^i_{pq} - \Gamma^i_{jk} \Gamma^j_{pq} + \Gamma^i_{jq} \Gamma^j_{pk} \right] n^p_{(0)} \cdot \oint_0^\epsilon (x^q - x^q_{(0)}) \frac{d}{du}(x^k - x^k_{(0)})$$

$$= n^i_{(0)} + \frac{1}{2} R^i{}_{pkq} \Big|_{\vec{x}_0} \cdot \oint (x^q - x^q_{(0)}) \frac{d}{du}(x^k - x^k_{(0)})$$

$$n^i_{(\epsilon)} = n^i_{(0)} + \mathcal{O}(\epsilon^3) \quad \text{if} \quad R^i{}_{jkl} \Big|_{\vec{x}_0} = 0$$

$$\Rightarrow \quad M^i{}_j(\vec{x}_0, \vec{x}_0, c) = \delta^i_j + O(\epsilon^3) \quad \text{if} \quad R^i{}_{jkl} \Big|_{\vec{x}_0} = 0.$$