GR

Newton $\vec{F} = \frac{-GM_1 m_2}{r^2}\hat{r}$

Newton → GR

Electrostatics → Maxwell's eq^n

Underlying mathematics of GR
→ Reimannian geometry
↑
generalization of euclidian geometry.

<u>Euclidian Geometry</u>

Two points
(x^1, x^2, x^3) $(x^1+dx^1, x^2+dx^2, x^3+dx^3)$

In Euclidian,

$$ds = \sqrt{(dx)^2 + (dx^2)^2 + (dx^3)^2}$$

also $ds^2 = (dx)^2 + (dx^2)^2 + (dx^3)^2$

In N-dimension,

$$ds^2 = (dx^1)^2 + (dx^2)^2 + \dots (dx^N)^2 = \sum_{i=1}^{N} (dx^i)^2$$

In <u>Riemanian Geometry</u>

$$ds^2 = \sum_{i,j=1}^{N} g_{ij}(\vec{x})\, dx^i dx^j$$

$g_{ij}(\vec{x})$ is a f^n of $(x, \dots x_n)$

for each pair (i,j)

~~We can choose~~ g_{ij} must be symmetric

$g_{ij} = g_{ji}$ ≡ referred to as a metric

- Euclidian geometry is where $g_{ij} = \delta_{ij}$

At times two different $g_{ij}(\vec{x})$ may describe the same space.

$$ds^2 = \sum_{i,j=1}^{N} g_{ij}(\vec{x})\, dx^i dx^j$$

Instead of $(x^1, \dots x^N)$ we choose a different set of coordinates

$$ds^2 = \sum g_{ij}(\vec{x})\, dx^i dx^j$$

$(x^1 \dots x^N)$ instead choose $(x'^1, \dots x'^N)$

$x'^1 = f^1(\vec{x})$

$x'^2 = f^2(\vec{x})$

$\vdots$

$x'^N = f^N(\vec{x})$

$x^1 = g^1(\vec{x}')$

$x^2 = g^2(\vec{x}')$

$x^N = g^N(\vec{x}')$

not the same as g^{ij} can call these $h^1 \dots$ etc.

$\vec{x}'$ $\quad \vec{x}' + d\vec{x}'$

$$dx^i = \sum_k \frac{\partial x^i}{\partial x'^k} dx'^k$$

$$ds^2 = \sum_{i,j=1}^{N} g_{ij}\left(\sum_{k=1}^{n} \frac{\partial x^i}{\partial x'^k} dx'^k\right)\left(\sum_{l=1}^{n} \frac{\partial x^j}{\partial x'^l} dx'^l\right)$$

$$= \sum_{k,l} \underbrace{\sum_{i,j} g_{ij}(\vec{x}) \frac{\partial x^i}{\partial x'^k}\frac{\partial x^j}{\partial x'^l}}_{g'_{kl}(\vec{x}')} dx'^k dx'^l$$

$$= \sum_{ij} g'_{ij}(\vec{x}')\, dx'^i dx'^j$$

<u>Examples</u>

Euclidian metric

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

$x^1 = r \sin\theta \cos\phi$

$x^2 = r \sin\theta \sin\phi$

$x^3 = r \cos\theta$

$$dx^1 = \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi$$

$dx^2 = \dots$

$dx^3 = \dots$

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

$$\Rightarrow ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

Ex.

$g'_{rr} = 1$, $g'_{\theta\theta} = r^2$, $g'_{\phi\phi} = r^2\sin^2\theta$

6 others $= 0$.

Example of a non-euclidean sphere.

→ surface of a 2d sphere:

$x^2+y^2+z^2=a^2$

$ds^2 = dx^2+dy^2+dz^2$

→

Method 1

$(x^1)^2+(x^2)^2+(x^3)^2=a^2$

(1) $x^3 = \pm\sqrt{a^2-(x^1)^2-(x^2)^2}$

$$ds^2 = (dx^1)^2+(dx^2)^2+(dx^3)^2$$

$$= (dx^1)^2+(dx^2)^2+\left\{\pm\frac{-x^1dx^1-x^2dx^2}{\sqrt{a^2-(x^1)^2-(x^2)^2}}\right\}^2$$

$$= \left\{1+\frac{(x^1)^2}{a^2-(x^1)^2-(x^2)^2}\right\}(dx^1)^2+\left\{1+\frac{(x^2)^2}{a^2-(x^1)^2-(x^2)^2}\right\}dx^{2\,2}$$

$$+\frac{2x^1x^2}{a^2-(x^1)^2-(x^2)^2}dx^1dx^2$$

$$g_{11} = 1+\frac{(x^1)^2}{a^2-(x^1)^2-(x^2)^2} \qquad g_{22} = 1+\frac{(x^2)^2}{a^2-(x^1)^2-(x^2)^2}$$

NB: 2 goes away

$$g_{12} = \frac{x^1x^2}{a^2-(x^1)^2-(x^2)^2} \quad \cancel{dx^1dx^2}$$

Method 2

(2) First go to spherical polar.

go from $x_1\ x_2\ x_3 \to r,\theta,\phi$

$r^2=a^2 \Rightarrow r=a$ (no $\pm$)

$$ds^2 = dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\phi^2$$

$$= a^2(d\theta^2+\sin^2\theta\, d\phi^2) \longrightarrow \text{you get this after subst. } x^1\ x^2 \text{ for } ds^2$$

$x^1 = a\sin\theta\cos\phi$ Ex. $ds^2\big|_{(x^1x^2)} = ds^2\big|_{(\theta,\phi)}$

$x^2 = a\sin\theta\sin\phi$

Q. Given only the ~~coordina~~ metric, how to conclude whether they're the same space?

O. ~~As so~~ Strategy: Find appropriate linear combinations of the metric & its derivatives which are invariant under coordinate transformations.

Conventions

a. Index i of a coordinate will be a super script eg. x^i

b. $\frac{\partial}{\partial x^i} \equiv \partial_i$

c. Summation Convention: Any index, appearing twice in a formula, once as sub & once as super is summed over.

d. Index of a matrix appears as subscript

① $ds = \sum_{i,j=1}^{N} g_{ij}(\vec{x})\, dx^i\, dx^j \longrightarrow g_{ij}(\vec{x})\, dx^i\, dx^j$

② $g'_{kl}(\vec{x}') = \sum_{ij} g_{ij} \frac{\partial x^i}{\partial x'^k} \frac{\partial x^j}{\partial x'^l} = g_{ij}\, \partial'_k x^i\, \partial'_l x^j$

Tensor Fields

Consider some combination $A_{i_1 \dots i_Q}(\vec{x})$ of the metric & its derivatives which transform as follows:

$$A'_{i_1 \dots i_Q}(\vec{x}') = \qquad\qquad A_{j_1 \dots j_Q}(\vec{x})$$