Review of Statmech

Goal: Z partition f^n

Ω — region

H_Ω — Hamiltonian

$V(\Omega)$ — vol.

$S(\Omega)$ — surface Area

L length

d dimension

$V(\Omega) - L^d$

$S(\Omega) - L^{d-1}$

B.C. (1) Periodic B.C.

(2) Hard Walls

* Fluid Continuum
* Magnet Lattice

coupling const

$$-\frac{H_\Omega}{k_B T} = \sum_n K_n \theta_n$$

local operator

$$\text{eg. } H_\Omega = \sum_{i=1}^{N}\left[\frac{p_i^2}{2m} + U_1(x_i)\right] + \frac{1}{2!}\sum_{i \neq j} U_2(x_i, x_j) + \frac{1}{3!}\sum_{i \neq j \neq k} U_3(\ldots$$

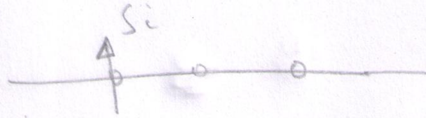
$K_1 = H$, $\theta_1 = \sum_i S_i$

$K_2 = J$, $\theta_2 = \sum_{i,j} S_i S_j$

ISING MODEL

$$-\frac{H_\Omega}{k_B T} = H\sum_i S_i + J\sum_{\langle i,j\rangle} S_i S_j$$

$$T_r \equiv \sum_{S_1}\sum_{S_2} \cdots \sum_{S_{N(\Omega)}}$$



$H\sum_i S_i$

When Ω is finite, no phase transition.

For finite systems, $F_\Omega \propto V(\Omega)$

Partition f^n

$$Z[\{K_n\}] = T_r\, e^{-\beta H_\Omega} \qquad \beta = \frac{1}{k_B T}$$

$$F_\Omega[\{K\}] = -k_B T \ln Z$$

$$\frac{\partial F_\Omega}{\partial K_n}, \quad \frac{\partial^2 F_\Omega}{\partial K_n \partial K_m}, \ldots$$

For finite system

$$F_\Omega = V(\Omega) f_b + S(\Omega) f_s + O(L^{d-2})$$

f_b: bulk free energy density

f_s: surface ~~surface~~ free energy per unit area

$$f_b[K] = \lim_{V(\Omega)\to\infty} \frac{F_\Omega}{V(\Omega)} = \lim_{N(\Omega)\to\infty} \frac{F_\Omega}{N(\Omega)}$$

lt exists & independent of Ω.

$$f_s[k] = \lim_{s(\Omega)\to\infty} \left\{ \frac{F_\Omega[k] - V(\Omega) f_b[k]}{s(\Omega)} \right\}$$

$V(\Omega) \to \infty$

$N(\Omega) \to \infty$

$\nu = \frac{N}{V}$

δ

$$\lim_{\substack{x\to\infty \\ x^2\to\infty}} \frac{x}{x^2} = 0$$

Thermodynamic lt of a charged system

$\rho \leftarrow$ uniformly charged density in 3 Dimension

$U(r) = \frac{A}{r}$ ~ worst

Coulomb's Law | at $T=0$,

Energy
$$E(R) = \int_0^R \left(\frac{4}{3}\pi r^3 \rho\right) \frac{A}{r} \left(4\pi r^2 \rho \, dr\right)$$

$$E(R) = A \frac{(4\pi)^2}{15} \rho^2 R^5$$

$$E_b = \frac{E(R)}{V(R)} = \frac{A\,4\pi}{5} \rho^2 R^2 \longrightarrow \text{diverges as } R \to \infty$$

~~E~~= $f = E - TS^{\to 0}$

THERMODYNAMICS

0th law: Transitivity of equilibrium

1st law: $đQ + đW = dE$

Force constants: (infinitesimal) ratio of displacement to force

Eg. Isothermal compressibility $\kappa_T = -\dfrac{\left.\frac{\partial V}{\partial F}\right|_T}{V}$

Susceptibility of a magnet $\chi_T = \dfrac{\left.\frac{\partial M}{\partial B}\right|_T}{V}$

2nd law:

Revise: Kelvin's & Clausius's Statement
$(Q_c = 0)$ $(W = 0)$

Carnot Engine:

Entropy: $\oint \frac{đQ}{T} \leq 0$; $\Delta S = \int_A^B \frac{đQ_{rev}}{T}$; $\Rightarrow dS \geq \frac{đQ}{T}$

Thermodynamic Potentials:

$H = E - J \cdot x$

$F = E - TS$

$G = E - TS - J \cdot x$

$\mathcal{G} = E - TS - \mu \cdot N$

	$đQ = 0$	const T
$đW = 0$	$dS \geq 0$	$\delta F \leq 0$
const J	$\delta H \leq 0$	$\delta G \leq 0$

Useful Math Results: If extensivity holds, viz $E(\lambda S, \lambda \underline{x}, \lambda \underline{N}) = \lambda E(S, \underline{x}, \underline{N})$ then

1) $E = TS + J \cdot x + \mu \cdot N$ ≡ Fundamental eqⁿ of thermodynamics

2) $S dT + x \cdot dJ + N \cdot d\mu = 0$ ≡ Gibbs-Duhem relation

Probability
(Sheldon Ross)

summary

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Chapter #2

§2.2 | Any subset E of the sample space is an *event*

Commutation $\quad E \cup F = F \cup E \qquad EF = FE$

Associative $\quad (E \cup F) \cup G = E \cup (F \cup G) \qquad (EF)G = E(FG)$

Distrib $\quad (E \cup F)G = EG \cup FG \qquad EF \cup G = (E \cup G)(F \cup G)$

DeMorgan's laws:

$$\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c$$

$$\left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c$$

Axioms of Probability:
1. $0 \le P(E) \le 1$
2. $P(S) = 1$
3. For any seq of mutually exclusive events $E_1 E_2 \ldots$ (i.e. $E_iE_j = \phi$ for $i \neq j$)

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Propositions
1. $P(E^c) = 1 - P(E)$
2. If $E \subset F$, then $P(E) \le P(F)$
3. $P(E \cup F) = P(E) + P(F) - P(EF)$
4. <page 42>

Propositition: If $\{E_n, n \ge 1\}$ is either an increasing or decreasing seq. of events, then

$$\lim_{n\to\infty} P(E_n) = P\left(\lim_{n\to\infty} E_n\right)$$

Chapter #3

Def^n:

If $P(F) > 0$ then

$$P(E|F) = \frac{P(EF)}{P(F)}$$

Multiplication Rule:

$$P(E_1E_2\ldots E_n) = P(E_1)\,P(E_2|E_1)\,P(E_3|E_1E_2)\ldots P(E_n|E_1E_2\ldots E_{n-1})$$

Baye's Formula

$$P(E) = P(EF) + P(EF^c) = P(E|F)\,P(F) + P(E|F^c)\,(1-P(F))$$

Def^n: Odds of event: $\dfrac{P(A)}{P(A^c)} = \dfrac{P(A)}{1-P(A)}$

Prop: $P(F_j|E) = \dfrac{P(EF_j)}{P(E)} = \dfrac{P(E|F_j)\,P(F_j)}{\sum_{i=1}^{n} P(E|F_i)\,P(F_i)}$ where F_i are mutually exclusive s.t. $\bigcup_{i=1}^{n} F_i = S$

Def^n:

Events E & F are independent if $P(EF) = P(E)\,P(F)$

If E & F are not independent, then they're dependent

Prop: If E & F are independent, then so are F & F^c

Def^n:

3 events E, F, G are independent if

$P(EFG) = P(E)\,P(F)\,P(G)$

$P(EF) = P(E)\,P(F)$

$P(EG) = P(E)\,P(G)$

$P(GF) = P(G)\,P(F)$

Prop. $P(\cdot|F)$ is a probability
- a) $0 \le P(E|F) \le 1$
- b) $P(S|F) = 1$
- c) If E_i are mutually exclusive then

$$P\left(\bigcup_i^{\infty} E_i|F\right) = \sum_i^{\infty} P(E_i|F)$$

Chapter #4

Random Variable: A real-valued f^n defined on the sample space.

Discrete Random Variable:

$p(a) = P(\{X = a\}) \doteq$ Probability mass f^n

$F(x) = P(\{X \le x\}) \doteq$ cumulative distribution f^n of X.

Defⁿ:
Expectation value of X

$$E[X] = \sum_{x:\, p(x)>0} x\, p(x)$$

prop. $E[g(X)] = \sum g(x_i)\, p(x_i)$

Defⁿ:
If X is a random variable with mean μ, then the variance of X is

$$\mathrm{var}(X) \equiv E[(X-\mu)^2]$$

Prop:

$$\mathrm{var}(X) = E[X^2] - (E[X])^2$$

Defⁿ:

$$\text{Standard Deviation}(X) = \sqrt{\mathrm{Var}(X)}$$

Binomial

$p(i) = \binom{n}{i} p^i (1-p)^{n-i}$ | $E[X] = np$ $\quad \mathrm{Var}(X) = np(1-p)$

Poisson

$p(i) = \dfrac{e^{-\lambda}\lambda^i}{i!}$ | $E[X] = \mathrm{Var}(X) = \lambda$

Geometric

$p(i) = p(1-p)^{i-1}$ | $E(X) = \dfrac{1}{p}$ $\quad \mathrm{var}(X) = \dfrac{1-p}{p^2}$

Chapter #5 CONTINUOUS RANDOM VARIABLES

$P\{X \in B\} = \int_B f(x)\,dx$ where $f(x) \equiv$ probability density f^n

$$F(a) \equiv \int_{-\infty}^{a} f(x)\,dx$$

$$E[X] \equiv \int_{-\infty}^{\infty} x f(x)\,dx$$

Prop.

$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$ (Tricky to prove)

Corollary

$E[aX+b] = aE[X] + b$

Old results

$$\mathrm{var}(X) = E[(X-\mu)^2] = E[X^2] - (E[X])^2$$

$$\mathrm{var}(aX+b) = a^2\,\mathrm{Var}(X)$$

Chapter #6 | JOINTLY DISTRIBUTED RANDOM VARIABLES

$F(a,b) \equiv P\{X \le a, Y \le b\} \doteq$ Joint cumulative probability distribution

$F_X(a) \equiv P\{X \le a\} = P\{X \le a, Y < \infty\} = P\left(\lim_{b\to\infty} \{X \le a, Y \le b\}\right)$

$= \lim_{b\to\infty} P\{X \le a, Y \le b\})$

$= \lim_{b\to\infty} F(a,b) = F(a,\infty)$

$F_Y(b) \equiv F(\infty, b)$ | $F_X, F_Y \doteq$ Marginal Distributions

When X & Y are discrete,

$p(x,y) = P\{X=x, Y=y\} \doteq$ joint probability mass f^n.

$p_X(x) = P\{X=x\} = \sum_{y:\,p(x,y)>0} p(x,y)$ | $p_Y(y) = P\{Y=y\} = \sum_{x:\,p(x,y)>0} p(x,y)$

§6.2 Independent Random Variables (Earlier we talked about independent events)

Def^n: X & Y are independent if $P\{X \in A, Y \in B\} = P\{X \in A\} P\{Y \in B\}$

viz (some work) X & Y are independent if $\forall A, B$, the events $E_A = \{X \in A\}$ & $E_B = \{Y \in B\}$ are independent

$\Rightarrow P\{X \le a, Y \le b\} = P\{X \le a\} P\{Y \le b\}$

$\Rightarrow F(a,b) = F_X(a) F_Y(b)$

For discrete

$p(a,b) = p_X(x) p_Y(y)$ | For continuous $f(x,y) = f_X(x) f_Y(y)$

these follow

Def^n: X & Y are jointly continuous if $\exists f(x,y)$ ($\forall x,y$) s.t. for every set C of pairs of real numbers,

$$P\{(X,Y) \in C\} = \iint_{(x,y)\in C} f(x,y)\,dx\,dy$$

$\|$

joint probability density f^n

Missed it

page (250) for more insight

$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ | $f_Y(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ } follows.

Prop: The continuous (dis) random var X & Y are independent if & only if their joint probability density (mass) f^n can be expressed as

$f_{X,Y}(x,y) = h(x) g(y)$