Sen's Lectures
General Relativity

Volume 1

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Foreword

The content is due to Prof. Ashoke Sen while all errors in its presentation here are mine. Strictly then this contains not a word of Ashoke Sen and effectively no thought of mine. That was the disclaimer, wait, one more thing: there's nothing useful any further in this section. So with that I am free to express my motivation for writing this. Ashok Sen needs no introduction. I did not get a chance to learn GR at IISER due to my strange selection of electives. This, however, is something I always wanted to learn. Instead of following my usual route of picking up a text book, I thought perhaps watching Ashoke Sen's lectures, delivered at IISc (I think) which some student (Anurag I think) was kind enough to upload online, would be quicker. Watching the lectures was a little challenging because the video quality sometimes distorts the text on the blackboard and one has to make educated guesses. Of course Prof Sen speaks out everything he writes, well almost everything, so one can check the logical consistency with his cues to be sure that everything is correct. That's a good exercise. I have tried making notes and then presenting them in a neat format here as a way to make the subject clear to myself but I suspect this would be of use to others as well. Prof. Sen has also delivered more lectures on many other topics, both advanced and basic, which I might try to cover in my subsequent efforts.

Part 1

Reimannian Geometry

CHAPTER 1

Metric and Tensors

1.1. Motivation

Recall that Newton's law for the force of gravity

$$\vec{F} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

has an uncanny resemblance to that of electrostatic attraction between two charged particles. From this perspective generalisation of electrostatics to Maxwell's theory of electrodynamics is similar to the generalisation of Newtonian gravity to Einstein's general theory of gravity (GR). In particular, both electrostatics and Newtonian gravity have the property of instantaneous propagation of effects as they both depend on the relative distance of particles at the instant of time when the force is to be evaluated. In electrodynamics this instantaneity is replaced with propagation of electromagnetic waves and one of the main goals in generalising Newtonian gravity is to find a similar explanation.

Just as vector calculus is the underlying mathematics of Maxwell's electromagnetism, Reimannian Geometry – a generalisation of Euclidean Geometry – is that of GR. Thus one must familiarise oneself with at least the basic concepts of Reimannian Geometry as a prerequisite to learning GR.

1.2. Euclidean Geometry, Metric, Tensors

Consider two points (x^1, x^2, x^3) and $(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$. According to Euclidean geometry the distance between these points is given by $ds = \sqrt{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}$. This can be generalised to N-dimensions as

$$ds^2 = \sum_{i=1}^{N} (dx^i)^2.$$

In Remanian geometry the notion of distance between two close by points is generalised to

$$ds^2 = \sum_{i,j=1}^{N} g_{ij}(\vec{x}) dx^i dx^j$$

where $g_{ij}(\vec{x})$ is called the *metric* and is a function of $\left(x^1, x^2 \dots x^N\right)$. Since $dx^i dx^j = dx^j dx^i$, viz. it is symmetric under the exchange of i and j, it follows that $g_{ij} = g_{ji}$ (see exercise 3). Euclidean geometry can be obtained as the special case where $g_{ij} = \delta_{ij}$, the Kroneker delta, which is defined as

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

It is possible, however, that two different metrics represent the same space. Consider another co-ordinate system which labels the same point $\vec{x}$ as $\vec{x}'$ with components $\left(x'^1, x'^2 \dots x'^N\right)$. Again consider two points, $\vec{x'}$ and, a point close to it, $\vec{x'} + \vec{dx}'$ where the old $\vec{dx}$ is related to the new $\vec{dx}'$ by

$$dx^i = \sum_k \frac{\partial x^i}{\partial x'^k} dx'^k.$$

Substituting for dx^i in the expression for ds^2 one gets

$$ds^2 = \sum_{k,l} \underbrace{\left(\sum_{i,j} g_{ij} \frac{\partial x^i}{\partial x'^k} \frac{\partial x^j}{\partial x'^l}\right)}_{g'_{kl}} dx'^k dx'^l = \sum_{k,l} g'_{kl} dx'^k dx'^l.$$

Thus a transformation change can cause g to change even though they both describe the same object.

Example 1 (Flat Space). In cartesian coordinates $ds^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2$. Using the polar coordinates one can write

$$x^1 = r\sin\theta\cos\phi$$
$$x^2 = r\sin\theta\sin\phi$$
$$x^3 = r\cos\theta$$

which can be used to determine $dx^1 = \sin\theta\cos\phi dr + r\cos\theta\cos\phi d\theta - r\sin\theta\sin\phi d\phi$, $dx^2 = \ldots$ and $dx^3 = \ldots$. When plugged into the expression for ds^2 one obtains $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$ (this can also be obtained more easily by writing $d\vec{r} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$ and then using $ds^2 = d\vec{r}.d\vec{r}$ as described in Landau's book). Manifestly then $g_{rr} = 1$, $g_{\theta\theta} = r^2$, $g_{\phi\phi} = r^2\sin^2\theta$ while in the cartesian case $g_{11} = g_{22} = g_{33} = 1$ even though they both describe the same space.

Example 2 (Surface of Sphere). Consider the surface of a sphere described by $\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 = a^2$. One parametrisation is to use x^3 as a dependent variable given by $x^3 = \pm\sqrt{a^2 - \left(x^1\right)^2 - \left(x^2\right)^2}$. One can substitute dx^3 in the expression for $ds^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2$ to obtain

$$ds^2 = \underbrace{\left(1 + \frac{\left(x^1\right)^2}{a^2 - \left(x^1\right)^2 - \left(x^2\right)^2}\right)}_{g_{11}} \left(dx^1\right)^2 + \underbrace{\left(1 + \frac{\left(x^2\right)^2}{a^2 - \left(x^1\right)^2 - \left(x^2\right)^2}\right)}_{g_{22}} \left(dx^2\right)^2 + \underbrace{\left(\frac{2x^1x^2}{a^2 - \left(x^1\right)^2 - \left(x^2\right)^2}\right)}_{g_{12}} dx^1dx^2.$$

Alternatively, one could've started with the polar coordinate system and set $r = a$ to obtain $ds^2 = a^2d\theta^2 + a^2\sin^2\theta d\phi^2$ which entails $g_{\theta\theta} = a^2$, $g_{\phi\phi} = a^2\sin^2\theta$. Note that g is diagonal in the polar representation.

Given only the metric how can one conclude whether or not they represent the same space? Our strategy would be to find appropriate linear combinations of the metric and its derivatives which are invariant under coordinate transformations. To proceed we define the following convention.

(1) Index of a coordinate is given by a superscript, e.g. x^i.
(2) Shorthand for derivatives:

$$\partial_i := \frac{\partial}{\partial x^i}.$$

(3) Summation: Any index appearing twice in a formula, once as a subscript and once as a superscript is summed over.
(4) Index of a matrix appears as a subscript.

Under this convention we have

$$ds^2 = \sum_{i,j} g_{ij}dx^idx^j \to g_{ij}dx^idx^j$$

and

$$g'_{kl}(\vec{x'}) = \sum_{i,j} g_{ij}\frac{\partial x^i}{\partial x'^k}\frac{\partial x^j}{\partial x'^l} \to g_{ij}\partial'_k x^i\partial'_l x^j.$$

We can now capture this

Exercise 3. Consider a symmetric matrix A with elements $A_{ij} = A_{ji}$. If one wishes to evaluate $\sum_{i,j} A_{ij}B_{ij}$ then show that it suffices to assume B is also symmetric.

Solution. Assume $B_{ij}^{(s)} := \frac{B_{ij}+B_{ji}}{2} = B_{ji}^{(s)}$ and $B_{ij}^{(a)} := \frac{B_{ij}-B_{ji}}{2} = -B_{ji}^{(a)}$. Evidently $B_{ij} = B_{ij}^{(s)} + B_{ij}^{(a)}$ so it suffices to show that $\sum_{i,j} A_{ij}B_{ij}^{(a)} = 0$ to prove the claim. This follows from $\sum_{i,j}\frac{1}{2}\left(A_{ij}B_{ij}^{(a)} + A_{ji}B_{ij}^{(a)}\right) = \sum_{i,j}\frac{1}{2}\left(A_{ij}B_{ij}^{(a)} - A_{ji}B_{ji}^{(a)}\right) = 0$.

We capture the generalisation of objects that transform this way by the following definition.

Definition 4 (Tensors). An object that transforms as

$$C'^{i_1\ldots i_p}{}_{g_1\ldots g_q}(\vec{x}') = \partial_{k_1}x'^{i_1}\ldots\partial_{k_p}x'^{i_p}.\partial'_{g_1}x^{l_1}\ldots\partial'_{g_q}x^{l_q}C^{k_1\ldots k_p}{}_{l_1\ldots l_q}(\vec{x})$$

will be termed a rank (p,q) tensor.

Definition 5 (Scalars, Contravariant and Covariant vectors). A rank

- (0,0) tensor is called a scalar
- (1,0) tensor is called a contravariant vector
- (0,1) tensor is called a covariant vector.

Some observations: Note that $A^{i_1\ldots i_p}{}_{j_1\ldots j_q}B^{k_1\ldots k_r}{}_{l_1\ldots l_S}$ transforms as a $(p+r, q+s)$ tensor. Note also that for consistency $A^{i_1}{}_{i_1}$ should behave as a scalar. This follows directly from the transformation law and chain rule as

$$\begin{aligned} A'^{i_1}{}_{i_1} &= \underbrace{\partial_{k_1}x'^{i_1}\partial'_{i_1}x^{l_1}}A^{k_1}{}_{l_1} \\ &= \delta^{l_1}_{k_1}A^{k_1}{}_{l_1} = A^{l_1}{}_{l_1}. \end{aligned}$$

1.3. Understanding the Metric: Christophel Symbol, Reimann Tensor

We try to extract all essential information about the space as described by the metric. The main source of redundancy here is the freedom in choosing the coordinate system. Let us proceed systematically with first simplifying the metric itself. Since $g_{ij} = g_{ji}$ and because we restrict ourselves to real numbers, it follows that g_{ij} can always be diagonalised by an appropriate orthogonal matrix transformation (at a given point of course). However, one is not restricted to orthogonal matrices in this case (the transformation doesn't have to preserve distances) and so one can do better.

Proposition 6. *Take a point p and assume its coordinate is given by $\vec{x}_0$ in some coordinate system. Then, there exists a coordinate transformation $\vec{x} \to \vec{x}'$ such that $g_{ij}(\vec{x}_0) \to g'_{ij}(\vec{x}'_0) = diag\{\pm, \pm \dots, \pm\}$.*

The proof is straight forward as we shall see. It is interesting to note that for every continuous metric[1] will have a fixed number of $+$ and $-$ entries throughout because otherwise there would have to be a jump for a $+$ to turn into a $-$ (which is forbidden by the continuity assumption).

The following expansion will be used several times in this sub-section. Consider a point denoted by $\vec{x}_0$ and $\vec{x}'_0$ by two coordinate systems which are related by $x^i = f^i(\vec{x}')$. I wish to write an expression for a point near $\vec{x}_0$ which I can write as

$$x^i = f^i(\vec{x}'_0 + (\vec{x}' - \vec{x}'_0)) = \cancelto{x_0^i}{f^i(\vec{x}'_0)} + A^i{}_j(x'^j - x'^j_0) + B^i{}_{jk}(x'^j - x'^j_0)(x'^k - x'^k_0) + \dots$$

where we used the assumption that x_0 and x'_0 represent the same point. Note that $A^i{}_j = \partial'_j f^i(x')|_{\vec{x}'=\vec{x}'_0} = \partial'_j x^i|_{\vec{x}'=\vec{x}'_0}$ and similarly $B^i{}_{jk}$ would consist of two derivatives and so on.

The basic idea will be that instead of specifying f^i directly we would specify f partially by fixing the constants A, B, ... This will become clear in the following proof of the proposition.

[1]one that changes only a little when the coordinate is changed a little, roughly speaking

Part 2

General Relativity

CHAPTER 2

Conventions, Axioms and the Weak Field Limit