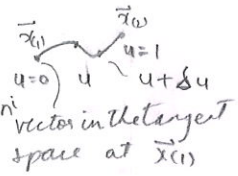


aim: see what this vector becomes upon parallel transport.

$\delta n^i = 0 \cdot \delta u + O(\delta u^2)$

$n^i(u) \to$ known (say)

$\frac{\delta n^i}{\delta u} = 0$

$n^i(u+\delta u) \to$ calculate

Goto $\vec{x}(u)$ & find a new coordinate sys $\vec{x}'$ s.t. $\partial_i' g_{jk}' = 0$ at $\vec{x}' = \vec{x}'_{(0)}$

$n'^i(u+\delta u) = n'^i(u) + O(\delta u^2)$ (by rules of parallel transport)

Since you don't want to keep changing coordinates (its inconvenient), $\therefore$ we see what parallel transport amounts in some arbitrary frame.

Translate to x'

$$n^i(u) = \left.\frac{\partial x^i}{\partial x'^k}\right|_{\vec{x}'(u)} n'^k(u)$$

$$n'^k(u) + O(\delta u^2)$$

$$n^i(u+\delta u) = \left.\frac{\partial x^i}{\partial x'^k}\right|_{\vec{x}'(u+\delta u)} n'^k(u+\delta u)$$

$$= \left.\frac{\partial x^i}{\partial x'^k}\right|_{\vec{x}'(u)} + \frac{\partial^2 x^i}{\partial x'^l \partial x'^k} \cdot \delta x'^l + \dots \qquad \& \qquad \delta x'^l = \frac{\partial x'^l}{\partial u} \cdot \delta u$$

$$\Rightarrow n^i(u+\delta u) = n^i(u) + \frac{\partial^2 x^i}{\partial x'^l \partial x'^k} \frac{\partial x'^l}{\partial u} \delta u \, n'^k(u) + O(\delta u^2)$$

$$\frac{dn^i}{du} = \frac{\partial^2 x^i}{\partial x'^l \partial x'^k} \frac{\partial x'^l}{\partial u} n'^k(u)$$

We still have dependence on the primed coordinate. We wish to get rid of it.

$$\Gamma'^l_{mn}(\vec{x}') = \frac{\partial x'^l}{\partial x^i} \frac{\partial x^j}{\partial x'^m} \frac{\partial x^k}{\partial x'^n} \Gamma^i_{jk}(\vec{x}) + \frac{\partial x'^l}{\partial x^k} \frac{\partial^2 x^k}{\partial x'^m \partial x'^n} = 0 \quad \text{(for the primed frame)}$$

$$\frac{\partial x^p}{\partial x'^l} \Gamma'^l_{mn}(\vec{x}') = \frac{\partial x^j}{\partial x'^m} \frac{\partial x^k}{\partial x'^n} \Gamma^p_{jk}(\vec{x}) + \frac{\partial^2 x^p}{\partial x'^m \partial x'^n} = 0$$

also, recall: $n'^k(u) = \frac{\partial x'^k}{\partial x^r} n^r(u)$

$$\therefore \frac{dx'^i}{du} = \frac{\partial x'^i}{\partial x^s} \frac{\partial x^s}{\partial u}$$

fix this

$$\Rightarrow \frac{dn^i}{du} = -\frac{\partial x^r}{\partial x'^l} \frac{\partial x^s}{\partial x'^n} \Gamma^i_{rs} \frac{\partial x'^l}{du} n'^k$$

$$= -\Gamma^i_{rs} \left[\frac{\partial x^r}{\partial x'^l} \frac{\partial x^s}{\partial x'^n} \frac{\partial x'^k}{\partial x^m} \frac{\partial x'^l}{\partial x^\sigma} \right] \frac{\partial x^\sigma}{\partial u} n^m$$

$$= -\Gamma^i_{rs} \delta^l_n \delta^r_m n^m$$

Using these (Ex: check) we get

$$\frac{dn^i}{du} + \Gamma^i_{pq}(\vec{x}(u)) \, n^p \frac{dx^q}{du} = 0$$

(The primes will disappear (as kronecker delta))

For a d-dimensional space, there're d- first order d.e. Thus given the initial vector, you can solve it for u.

NB: u needn't describe a geodesic.

Claim: The eq^n is parametrization independent.

Proof: $u = f(v)$

$$\left.\begin{aligned} \frac{dn^i}{du} &= \frac{dv}{du}\cdot\frac{dn^i}{dv} \\ \frac{dx^q}{du} &= \frac{dx^q}{dv}\cdot\frac{dv}{du} \end{aligned}\right\} \Rightarrow \frac{dv}{du}\left[\frac{dn^i}{dv} + \Gamma^i_{pq}(\vec{x}(v))\, n^p \frac{dx^q}{dv}\right] = 0$$

NB: unlike the geodesic eq^n which works in a specific parametrization.

Now choose an arbitrary coordinate sys $\vec{x}''$ (where Γ'' needn't be zero)

Recall: $n^i = n''^k \frac{\partial x^i}{\partial x''^k}$

$\because \Gamma$ the transformation

NB: $\frac{dx^q}{du} = \frac{\partial x^q}{\partial x''^r}\frac{dx''^r}{du}$

substitute in Ex: $\Rightarrow \frac{\partial x^i}{\partial x''^s}\left(\frac{dn''^s}{du} + \Gamma''^s_{pq}\, n''^p \frac{dx''^q}{du}\right) = 0$

Therefore its evident that the parallel transport eq^{n}s don't depend on the coordinate frame.

NB: Its not surprising $\because$ x coordinate sys to start with, was arbitrary. Therefore infact its a consistency check.

Ex: $\frac{d}{du}\left(g_{ij}(\vec{x}(u))\, n^i(u)\, n^j(u)\right) \overset{\text{claim}}{=} 0$ viz. norm is preserved under parallel transport.

Proof sketch: (direct) $\frac{dn^i}{du}$ is known now, $\frac{\partial g_{ij}}{\partial x^k}\frac{dx^k}{du} = \frac{dg_{ij}}{du}$ & thats known.

(neat) (scalar: the quantity is invariant under coordinate transformations same at a given point) Thus, goto the primed coordinates. Now first derivative of g is zero (by def^n of the primed coordinate). Also, for a primed coordinate, $\frac{dn'^i}{du} = 0$. So that does it.

$\Gamma' = 0$

$\frac{dn'}{du} + \Gamma' \ldots^{0} = 0$

$\Rightarrow \frac{dn'}{du} = 0$

Recall:

Geodesic eq^n

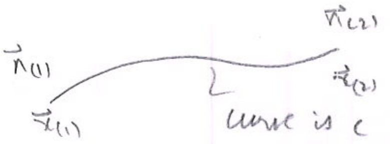
$$\frac{d^2x^i}{du^2} + \Gamma^i_{jk}\frac{dx^k}{du}\frac{dx^j}{du} = 0$$

; Z. Def^n: $n^i(u) := \frac{dx^i}{du}$ (the tangent vector)

So now, $$\frac{dn^i}{du} + \Gamma^i_{jk}\, n^j \frac{dx^k}{du} = 0$$

NB!: This is exactly the parallel transport eq^n

Alternate def^n of geodesic: curve s.t. its tangent vectors are parallel transported from any given tangent vector on the curve.



NB: The transport eq^n is linear.

$\Rightarrow \quad \alpha \vec{n}_{(1)} + \beta \vec{n}'_{(1)} \longrightarrow \alpha \vec{n}_{(2)} + \beta \vec{n}'_{(2)}$

(Not entirely certain how this follows)

$T(\alpha n_{(1)} + \beta m_{(1)}) = \alpha T(n_{(1)}) + \beta T(m_{(1)})$

$$n_{(2)}{}^i = M^i{}_j(\vec{x}_{(1)}, \vec{x}_{(2)}, c)\, n^j_{(1)}$$

independent of $n_{(1)}$ & $n_{(2)}$

Def^n: $-c :=$ Path c in the reverse direction.

Q: What can we say about $M^i{}_j(\vec{x}_{(2)}, \vec{x}_{(1)}, -c)$?

NB: The parallel transport eq^n. is reversible (independent of parametrization)

We reparametrize: $v := 1-u$; this will do the job. $u=1 \Leftrightarrow v=0$, $u=0 \Leftrightarrow v=1$

$dv = -du$

$$\frac{dn^i}{dv} + \Gamma^i_{jk}\, n^j(v) \frac{dx^k}{dv} = 0 \longrightarrow \frac{dn^i}{du} + \Gamma^i_{jk}\, n^j(u)\frac{dx^k}{du} = 0$$

So we solve the same old eq^n with boundary conditions reserved.

Thus we must have

$$\frac{d^2x}{dt^2} = F(x)$$

$$dt \to -dt$$

$$\frac{d^2x}{dt^2} = \alpha \frac{dx}{dt}$$

$$n_{(1)}{}^i = \left(M(\vec{x}_{(1)}, \vec{x}_{(2)}, c)^{-1}\right)^i{}_j\, n^j_{(2)}$$

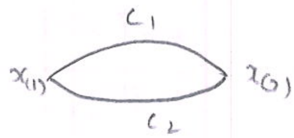
$$\overset{\text{by Def}^n}{=} M(\vec{x}_{(2)}, \vec{x}_{(1)}, -c)^i{}_j\, n^j_{(2)}$$

$\Rightarrow \because n^i_{(1)}$ is arbitrary, (?)

$$M(\vec{x}_{(2)}, \vec{x}_{(1)}, -c) = M(\vec{x}_{(1)}, \vec{x}_{(2)}, c)^{-1}$$

NB: If we had $\frac{dn^i}{du} + \Gamma^i_{jk}\, n^i \left|\frac{dx^j}{du}\right| = 0$, the eq^n won't be reversible.

Take 2 curves joining $x_{(1)}$ & $x_{(2)}$, c_1 & c_2



In general, $M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_1) \neq M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_2)$

Consider: $M(\vec{x}_{(1)}, \vec{x}_{(1)}, c_1 - c_2) = M(\vec{x}_{(2)}, \vec{x}_{(1)}, -c_2) \cdot M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_1)$

$$= M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_2)^{-1} M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_1)$$

$\neq \mathbb{1}$ in general.

Def^n: Monodromy Matrix around C

$:= M(\vec{x}, \vec{x}, C)$ for some closed curve C

In general, $\tilde{n}^i = M(\vec{x}, \vec{x}, C)^i{}_j\, n^j$

initial vector at $\vec{x}$.

final vector, after coming back to $\vec{x}$.

Now lets change the coordinate system:

Recall: $\tilde{n}'^i = \frac{\partial x'^i}{\partial x^p}\tilde{n}^p = \frac{\partial x'^i}{\partial x^p} \cdot M(x,x,c)^p{}_q\, n^q = \underbrace{\frac{\partial x'^i}{\partial x^p} M(x,x,c)^p{}_q \frac{\partial x^q}{\partial x'^j}}_{M'(x,x,c)} n'^j$

If M was $\mathbb{1}$, then you'd get

$$\tilde{n}'^i = n'^i$$

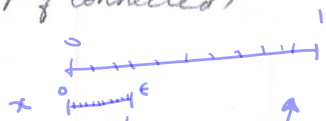
ie. M will remain identity even in the new coordinate sys.

NB: This can be proved the other way also. If $\tilde{n}'^i = n'^i$ then M must be $\mathbb{1}$. $\because n'^i$ is arbitrary $\Rightarrow \frac{\partial x'^i}{\partial x^p} M^p_q \frac{\partial x^q}{\partial x'^j} = \delta^i{}_j \Rightarrow \underbrace{\partial'_i x^m \partial_p x'^i}_{\delta^m{}_p} M^p{}_q \underbrace{\partial_n x'^j \partial'_j x^q}_{\delta_n{}^q}$

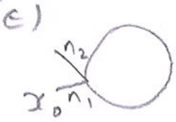
NB2: This analysis will not work for $M(\vec{x}_1, \vec{x}_2, c)$ $\because$ the $\frac{\partial x}{\partial x'} M \frac{\partial x'}{\partial x}$ factor won't become kronecker (they're evaluated at different points), even if $M = \mathbb{1}$. $= \delta^i{}_j \partial'_i x^m \partial_n x^j = \delta^m{}_n \Rightarrow M^m{}_n = \delta^m{}_n$

Claim: When $R = 0$, $M = \mathbb{1}$; the reverse is true when there're no singularities (& simply connected)

Proof: Start with a small curve; length $\sim O(\epsilon)$



Convention: $0 < u < \epsilon$ so that $\frac{dx^i}{du} \sim 1$



If u went from 0 to 1, (see the picture) $\frac{\delta x}{\delta u} \sim O(\epsilon)$

recall: $\frac{dn^i}{du} = -\Gamma^i_{jk}\, n^j(u) \frac{dx^j}{du}$

$\Gamma = g(\partial g + \partial g - \partial g)$

idea: Keep terms of order ~~ϵ~~ 1 & then integrate, you get the result to order ϵ. Then plugin the result & re-evaluate to get order ϵ^2.

$$\frac{dn^i}{du} = \overbrace{-\Gamma^i{}_{jk}(x_{(0)})}^{\text{correct to order } 0 \ (\epsilon^0 = 1)} \overbrace{n^j_{(0)}}^{} \overbrace{\frac{dx^k}{du}}^{\text{order 1 itself}} + O(\epsilon)$$

can't take it at its original value $\because$ the tangent changes quick

$$\Rightarrow n^i(u) = n^i_{(0)} - \Gamma^i{}_{jk}\Big|_{x_{(0)}} n^j_{(0)} \underbrace{(x^j(u) - x^j_{(0)})} + O(\epsilon^2) \qquad (\because \epsilon u \approx \epsilon^2)$$

$$\frac{dn^i}{du} = -\Big\{\overbrace{\Gamma^i{}_{jk}(\vec{x}_{(0)})}^{O(1)} + \partial_l \Gamma^i{}_{jk}(\vec{x}_{(0)}) \overbrace{(x^l - x^l_{(0)})}^{O(\epsilon)}\Big\} \times \Big\{\overbrace{n^j_{(0)}}^{O(1)} - \Gamma^j{}_{pq}(\vec{x}_0)\, n^p(\vec{x}_0)$$

$$\underbrace{(x^q - x^q_{(0)})}_{O(\epsilon)}\Big\} \underbrace{\frac{dx^k}{du}}_{O(1)}$$

Now keeping upto $O(\epsilon^2)$, we have

$$\frac{dn^i}{du} = -\Big\{\Gamma^i{}_{jk}\, n^j \frac{dx^k}{du} - \Gamma^i{}_{jk}\Gamma^j{}_{pq}\, n^p (x^q - x^q_{(0)}) \frac{dx^k}{du} + \partial_l \Gamma^i{}_{jk} \cdot (x^l - x^l_{(0)}) \cdot n^j_{(0)} \cdot \frac{dx^k}{du}\Big\} + O(\epsilon^2)$$

$(k) - x_0$

$\int_0^\epsilon dx(u) = x(\epsilon) - x(0) = 0$ for closed curve

Integrating from $(0,\epsilon)$, we note that term 1 is zero!

$$n^i(\epsilon) = n^i_{(0)} + \underset{\text{3rd term}}{\partial_l \Gamma^i_{jk}} \, n^j_{(0)} \oint_0^\epsilon (x^l - x^l_0) \frac{d(x^k - x^k_0)}{du} \quad \underset{\text{2nd term}}{\Gamma^i_{jk}\Gamma^j_{pq}} n^p \oint_0^\epsilon (x^q - x^q_{(0)}) \frac{d(x^k - x^k_{(0)})}{du}$$

(Lets try to match indices for aesthetics)

$$= n^i_{(0)} + \partial_q \Gamma^i_{pk} \, n^p_{(0)} \oint_0^\epsilon (x^q - x^q_0) \frac{d}{du}(x^k - x^k_{(0)}) - \Gamma^i_{jk}\Gamma^j_{pq} n^p \oint_0^\epsilon (x^q - x^q_{(0)}) \frac{d}{du}(x^k - x^k_{(0)})$$

$$= n^i_{(0)} + \left[\partial_q \Gamma^i_{pk} - \Gamma^i_{jk}\Gamma^j_{pq} \right] n^p_{(0)} \oint_0^\epsilon (x^q - x^q_{(0)}) \frac{d}{du}(x^k - x^k_{(0)})$$

NB: The boundary terms are zero. The derivative can be $\therefore$ shifted with a minus sign.
NB2: This means the indices q & k are anti-symmetric
NB3: Thus only the symmetric part of the remaining part must contribute

$$= n^i_{(0)} + \frac{1}{2}\left[\partial_q \Gamma^i_{pk} - \partial_k \Gamma^i_{pq} - \Gamma^i_{jk}\Gamma^j_{pq} + \Gamma^i_{jq}\Gamma^j_{pk} \right] n^p_{(0)} \cdot \oint_0^\epsilon (x^q - x^q_{(0)}) \frac{d}{du}(x^k - x^k_{(0)})$$

$$\int x^q dx^k + dx^q x^k = \int d(x^q x^k) = 0$$

$$= n^i_{(0)} + \frac{1}{2} R^i{}_{pkq}\Big|_{\vec{x}_0} \cdot \oint (x^q - x^q_{(0)}) \frac{d}{du}(x^k - x^k_{(0)})$$

$$\Rightarrow \int x^q dx^k = -\int dx^q x^k$$

$$n^i_{(\epsilon)} = n^i_{(0)} + \mathcal{O}(\epsilon^3) \quad \text{if} \quad R^i{}_{jkl}\Big|_{\vec{x}_0} = 0$$

$$\Rightarrow \quad M^i{}_j(\vec{x}_0, \vec{x}_0, c) = \delta^i_j + O(\epsilon^3) \quad \text{if} \quad R^i{}_{jkl}\Big|_{\vec{x}_0} = 0.$$