V. SMALL OSCILLATIONS

§21. Free Oscillations in one dimension

→ We consider small oscillation about a position of a stable equilibrium, in a system with only one degree of freedom.

→ Stable equilibrium corresponds to a pos. where $U(q)$ is a minimum.

↳ A movement away from this point sets up a force $-\frac{dU}{dq}$ which tends to return the system to the equilibrium pos.

↳ Let $q = q_0$ be the equilibrium value of the generalized co-ordinate

↳ Then for small deviations from eq. (retains only 1st ~~order~~ non-vanishing term) we expand $U(q_0) - U(q_0)$ in powers of $q - q_0$.

$$U(q_0 + (q - q_0)) - U(q_0)$$

$$= \left(U(q_0) + \cancel{\frac{\partial U}{\partial q}\Big|_{q_0}}^{\,0} (q - q_0) + \frac{1}{2}\frac{\partial^2 U}{\partial q^2}\Big|_{q_0}(q - q_0)^2 + \ldots \right) - U(q_0)$$

$$\sim \frac{1}{2} k^2 (q - q_0)^2 \qquad \text{where} \quad k^2 = \frac{\partial^2 U}{\partial q^2}\Big|_{q_0}$$

(+ve cause it's a minima)

↳ We measure the potential from this minima, i.e. $U(q_0) = 0$

& $x \equiv q - q_0 \doteq$ deviation from equilibrium

so $U(x) = \frac{1}{2} k x^2$

→ The KE of ~~a~~ such a sys (1°) is of the form $\frac{1}{2} a(q) \dot{q}^2 = \frac{1}{2} a(q) \dot{x}^2$

↳ In the same approx. as above, $\frac{1}{2} a(q) \sim \frac{1}{2} a(q_0)$

We write $a(q_0) = m$ (it's NOT mass in general, only when cartezian co-ordinates are used)

→ The Lagrangian of this sys is then

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

→ The eqn of motion is $m\ddot{x} + kx = 0$ or $\ddot{x} + \omega^2 x = 0$ where $\omega^2 = \frac{k}{m}$

↳ The soln. are (2 independent) $\cos\omega t$ & $\sin\omega t$. Thus the general soln. is

$$x = c_1 \cos\omega t + c_2 \sin\omega t$$

which is the same as $x = a\cos(\omega t + \alpha) = a\cos\omega t \cos\alpha - a\sin\omega t \sin\alpha$

where $a^2 = c_1^2 + c_2^2$ & $\tan\alpha = -c_2/c_1$ (compare co-efficients)

↳ Thus, the system executes harmonic oscillations.

↳ Amplitude of oscillation $\doteq a$ (coefficient of the periodic factor)

Phase $\doteq \alpha$ (argument of the cosine)

Angular Freq $\doteq \omega$

often referred to as simply frequency.

Remark: The fact that ω is independent of initial conditions depends on the potential energy being a quadratic f^n of the co-ordinate.

→ The energy $E = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 = \frac{1}{2} m \omega^2 a^2$

↳ substitute the soln.

→ Complex representation

$x = re\left[A e^{i\omega t}\right]$ where $A = a e^{i\alpha}$

↑ This yields the same soln.

$A \doteq$ The Complex Amplitude

$|A| \doteq$ The ordinary amplitude & $\arg(A) \doteq$ phase.

Remark: So long as the operations are linear, re may be removed & taken in the end.

§22. Forced Oscillations

Forced oscillations $\equiv$ Oscillations of a system on which variable external force acts.

Remark: Since the oscillations are small here, the external field is implied to be weak; else the displacement can take large values.

→ Finding the new Lagrangian.

↳ Besides the existing $\frac{1}{2} kx^2$ energy, we now have $U_e(x,t)$ resulting from the field.

↳ Expanding this in powers of x, we have

$$U_e(x,t) \approx U_e(0,t) + x \left.\frac{\partial U_e}{\partial x}\right|_{x=0}$$

This can be dropped from the Lagrangian as its a f^n of time only.

↳ $\left.-\frac{\partial U_e}{\partial x}\right|_{x=0} = F(t)$ is the force acting on the sys in the eq. position.

⇒ The energy associated $= -x F(t)$

↳ $L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} K x^2 + x F(t)$

→ Eq^n of Motion is thus $\ddot{x} + \omega^2 x = \frac{F(t)}{m}$

↳ The general soln. of this inhomogeneous linear differential eq^n is

$x = x_0 + x_1$ where x_0 is the general soln. of the corresponding homogeneous eq^n (the free oscillator soln)

& x_1 is a particular soln. of the inhomogeneous eq^n.

special case: $F(t) = f\cos(\gamma t + \beta)$

↳ We try $x_1 = b\cos(\gamma t + \beta)$ (clear from the form that it'll solve)

and find $b = \frac{f}{m(\omega^2 - \gamma^2)}$ so then

↳ $x = a\cos(\omega t + \alpha) + \left[\frac{f}{m(\omega^2 - \gamma^2)}\right]\cos(\gamma t + \beta)$

where a & α are determined from init conditions

non-resonance soln.

For resonance $(\gamma = \omega)$ we rewrite

$$x = a'\cos(\omega t + \alpha) + \frac{f}{m(\omega^2 - \gamma^2)}\left[\cos(\gamma t + \beta) - \cos(\omega t + \beta)\right]$$

Apply L'Hospital on the second term (that goes 0/0 for $\gamma \to \omega$), we get

$$x = a'\cos(\omega t + \alpha) + \left(\frac{f}{2m\omega}\right) t \sin(\omega t + \beta)$$

Resonance Soln.

Remark: Thus in resonance the amplitude of oscillation increases linearly with time (until the theory becomes invalid)

We can write the general soln. as (in the complex form) (with $\gamma = \omega + \epsilon$)

$$x = Ae^{i\omega t} + Be^{i(\omega+\epsilon)t} = \underbrace{\left[A + Be^{i\epsilon t}\right]}_{\text{varies slowly compared to}} e^{i\omega t}$$

near resonance

$$C = |A + Be^{i\epsilon t}| \quad \text{where } A = ae^{i\alpha} \text{ \& } B = be^{i\beta}$$

$$\Rightarrow C^2 = a^2 + b^2 + 2ab\cos(\epsilon t + \beta - \alpha)$$

Thus C varies periodically with freq ϵ as $|a-b| \leq C \leq a+b$. This phenomenon is called beats.

Special case: end

→ $\ddot{x} + \omega^2 x = \frac{F(t)}{m}$ can be integrated directly in the general form.

We rewrite the eq^n as

$$\frac{d}{dt}(\dot{x} + i\omega x) - i\omega(\dot{x} + i\omega x) = \frac{F(t)}{m}$$

verification

$$\ddot{x} + \cancel{i\omega\dot{x}} - \cancel{i\omega\dot{x}} + i\omega x = \frac{F(t)}{m} \quad Q.E.D$$

or

$$\frac{d\xi}{dt} - i\omega\xi = \frac{F(t)}{m} \quad \text{where } \xi = \dot{x} + i\omega x$$

↳ Soln. of the corresponding homogeneous eq^n is $\xi = Ae^{i\omega t}$ with A const.

↳ Particular soln. of the inhomogeneous eq^n:

we attempt $\xi = A(t)e^{i\omega t}$ and get

$$\dot{A}(t) = \frac{F(t)}{m}e^{-i\omega t}$$

verification

$$\dot{A}e^{i\omega t} + \cancel{i\omega Ae^{i\omega t}} - \cancel{i\omega Ae^{i\omega t}} = \frac{F(t)}{m}$$

$$\dot{A} = \frac{F(t)}{m}e^{-i\omega t} \quad Q.E.D$$

↳ Thus the general soln. is

$$\xi = e^{i\omega t}\left[\int_0^t \frac{1}{m}F(t)e^{-i\omega t} + \xi_0\right]$$

TODO: Understand what ξ_0 is exactly. I think it is A. The book says its ξ at $t=0$. They seem consistent if $\xi = Ae^{i\omega t}$.

The soln. is ~~the~~ $\frac{\operatorname{im}(\xi)}{\omega}$ (from def^n of ξ)

→ The energy of the system is obviously not conserved.

$\because$ Physically: The sys. gains energy from the external field
Mathematically: $\frac{\partial L}{\partial t} \neq 0$

ASK: How?

for $t \to \infty$

$$|\xi(\infty)|^2 = \frac{1}{m^2}\left|\int_{-\infty}^{\infty} F(t)e^{-i\omega t}\,dt\right|^2$$

So then the energy is $E = \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2) = \frac{1}{2}m|\xi|^2$ (def^n of ξ)

$$\Rightarrow E = \frac{1}{2m}\left|\int_{-\infty}^{\infty} F(t)e^{-i\omega t}\,dt\right|^2$$

Remark: The transferred energy depends on the squared modulus of the Fourier component of the force $F(t)$ whose freq. is the intrinsic freq of the sys.

special case: If the force acts for a duration much shorter than $1/\omega$, then $e^{-i\omega t} \sim 1$. Then we have

$t \ll 1/\omega$

$\omega t \ll 1$

$$E = \frac{1}{2m}\left[\int_{-\infty}^{\infty} F(t)\,dt\right]^2$$

Remark: TODO: look up this in the text and understand.

§ 23. Oscillations of systems with more than one degree of freedom

Let the minima of $U(q_i)$ be at $q_i = q_{i0}$. We put $x_i = q_i - q_{i0}$ for small displacements about q_{i0} & expand U as a f^n of x_i till quadratic terms and take $U(q_{i0}) = 0$, to obtain

$$U = \frac{1}{2}\sum_{i,k} k_{ik} x_i x_k$$

Remark: This is positive definite.

$$F(x+\Delta x, y+\Delta y, z+\Delta z) = F(x,y,z) + \frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial y}\Delta y + \frac{\partial F}{\partial z}\Delta z + \frac{\partial^2 F}{\partial x \partial y}\Delta x\Delta y + \frac{\partial^2 F}{\partial y \partial z}\Delta y\Delta z + \frac{\partial^2 F}{\partial z \partial x}\Delta x\Delta z + \cdots$$

thus $k_{ik} = k_{ki}$ as they multiply the same quantity $x_i x_k$ from the Taylor expansion

→ Kinetic Energy

In general, it is $\frac{1}{2}\sum a_{ik}(q)\,\dot{q}_i\,\dot{q}_k$

As a first approximation, we put $a_{ik}(q) \approx a_{ik}(q_0) = m_{ik}$

We thus have $\frac{1}{2}\sum m_{ik}\,\dot{x}_i\,\dot{x}_k$

Claim: $m_{ik} = m_{ki}$ & that the K.E is of a positive definite quadratic form.

→ Lagrangian

$$L = \frac{1}{2}\sum_{ik}\left(m_{ik}\dot{x}_i\dot{x}_k - k_{ik}x_i x_k\right)$$

~~↳ Eq^n of~~

→ Equations of Motion

We start with $dL = \frac{1}{2}\sum_{ik}\left(m_{ik}\dot{x}_i\,d\dot{x}_k + m_{ik}\,d\dot{x}_i\,\dot{x}_k - k_{ik}x_i\,dx_k - k_{ik}x_k\,dx_i\right)$

Now since in the summation, indexes are just labels for individual groups, they may be swapped. Also $m_{ik} = m_{ki}$ & $k_{ik} = k_{ki}$. Thus we have

$$dL = \sum_{ik}\left(m_{ik}\dot{x}_k\,d\dot{x}_i - k_{ik}x_k\,dx_i\right)$$

Hence, $\frac{\partial L}{\partial \dot{x}_i} = \sum_k m_{ik}\dot{x}_k$; $\frac{\partial L}{\partial x_i} = -\sum_k k_{ik}x_k$

↳ Lagrange's Eq^ns are thus $\sum_k m_{ik}\ddot{x}_k + \sum_k k_{ik}x_k = 0 \quad (i = \{1, 2 \dots s\})$

Remark: They form a set of s homogeneous differential eq^ns with const co-efficients.

We plug in $x_k = A_k e^{i\omega t}$, cancel $e^{i\omega t}$ & get a set of linear homogeneous algebraic eq^ns, to be satisfied by A_k

$$\sum_k\left(-\omega^2 m_{aik} + k_{ik}\right)A_k = 0$$

For a non zero soln., the det of the coefficients must vanish.

$$-\omega^2 m_{ik} + k_{ik} = \text{Matrix}$$

$$A_k = \text{Column}$$

$$\sum_k\left(-\omega^2 m_{ik} + k_{ik}\right)A_k = \text{Matrix} \times \text{Column} = 0$$

$$|\text{Matrix}| = 0$$

ie. $|k_{ik} - \omega^2 m_{ik}| = 0$

Remark: This is the characteristic eq^n & is of deg s in ω^2.

Claim: In general, it has s different real positive roots ω_α^2 $(\alpha = \{1, 2 \dots s\})$ however in particular cases, some may co-incide

why exactly will be shown soon

The quantities ω_α so ~~det~~ found are ≡ Characteristic frequencies
or
Eigenfrequencies (of the system)

Why are ω_α real and positive?

(only for the real bit) Physical: If ω_α was complex, then in $x_k = A_k e^{i\omega_\alpha t}$, there would be an exponentially increasing or decreasing factor in x_k. [This is not admissable for the $E = U + T$ would vary with time instead of being conserved.] I'm not fully convinced

Mathematically: We start with

$$\sum_k (-\omega^2 m_{ik} + k_{ik}) A_k = 0$$

then we multiply to it A_i^* & sum over i

$$\sum (-\omega^2 m_{ik} + k_{ik}) A_i^* A_k = 0$$

$$\Rightarrow \omega^2 = \frac{\sum k_{ik} A_i^* A_k}{\sum m_{ik} A_i^* A_k}$$

Claim: Both the numerator & denominator are real.

$\because$ $\left(\sum k_{ik} A_i^* A_k\right)^* = \sum k_{ik} A_i A_k^* = \sum k_{ik} A_i^* A_k$ ↳ interchange variables & use $k_{ki} = k_{ik}$

similarly for the denominator.

Claim: Both are also positive

~~dts~~. a quadratic form with k_{ik} (m_{ik}) coefficients is positive definite. (look at the starting part)

If A_k are written as $a_k + ib_k$, we have

$$\sum k_{ik} A_i^* A_k = \sum k_{ik}(a_i - ib_i)(a_k + ib_k)$$

$$= \sum k_{ik} a_i a_k + \sum k_{ik} b_i b_k$$

both of which are positive. Similarly for the denomenator.

With the eigenfrequencies (ω_α) known, we simply put it back in

$$\sum_k (-\omega_\alpha^2 m_{ik} + k_{ik}) A_k = 0 \quad \text{with } i = \{1, 2 .. s\}$$

to obtain the corresponding A_k.

Claim: If ω_α are distinct, then A_k are proportional to the minors of 'the det' with $\omega = \omega_\alpha$. (Prove this later)

Let these minors be $\Delta_{K\alpha}$. Then a particular soln. would be

$$x_K = \Delta_{K\alpha} C_\alpha e^{i\omega_\alpha t} \qquad \text{where } C_\alpha \text{ is a complex const.}$$

Claim: The general soln. is the sum of s particular solns.
(prove this later)

TIP: You may want to stop here and look at the big picture.

1) We had s linear homogenous eq^ns in which for x_K we'd put $A_K e^{i\omega t}$ to get

$$\begin{pmatrix} K_{iK} - \omega^2 m_{iK} \end{pmatrix}_{s\times s} \begin{pmatrix} A_1 \\ \vdots \\ A_s \end{pmatrix}_{s\times 1} = 0 \qquad (s \text{ eigenfreq.})$$

2) From this, you get s different $\{A_1 \dots A_s\}$ as a f^n of ω_α which we represent as $\{A_{1\alpha}, A_{2\alpha} \dots A_{s\alpha}\} \equiv A_{K\alpha}$

3) Now the general soln. $x_K = \sum_{\alpha=1}^{s} A_{K\alpha} e^{i\omega_\alpha t}$

$$x_K = \mathrm{re} \sum_{\alpha=1}^{s} \Delta_{K\alpha} C_\alpha e^{i\omega_\alpha t} \equiv \sum_\alpha \Delta_{K\alpha} \Theta_\alpha$$

$$\text{where} \quad \Theta_\alpha = \mathrm{re}\left[C_\alpha e^{i\omega_\alpha t}\right]$$

DOUBT: If C_α is arbitrary, isn't $\Delta_{K\alpha}$ redundant?

$$x_1 = \Delta_{11} C_1 e^{i\omega_1 t} + \Delta_{12} C_2 e^{i\omega_2 t} + \Delta_{13} C_3 e^{i\omega_3 t}$$

$$x_2 = \Delta_{21} C_1 e^{i\omega_1 t} + \Delta_{22} C_2 e^{i\omega_2 t} + \Delta_{23} C_3 e^{i\omega_3 t}$$

$$x_3 = \Delta_{31} C_1 e^{i\omega_1 t} + \Delta_{32} C_2 e^{i\omega_2 t} + \Delta_{33} C_3 e^{i\omega_3 t}$$

Now do you see why $\Delta_{K\alpha}$ is not redundant?
C_α can't change the proportionality b/w (for a fixed α) x_K.

Remark: Thus the time variation of each co-ordinate of the system is a superposition of s simple harmonic oscillations $(\Theta_1, \Theta_2 \dots \Theta_s)$ with arbitrary phase & amplitude but definite frequencies.

$$\begin{pmatrix} \\ x_K \\ \\ \end{pmatrix}_{s\times 1} = \begin{pmatrix} \\ \Delta_{K\alpha} \\ \\ \end{pmatrix}_{s\times s} \begin{pmatrix} \\ \Theta_\alpha \\ \\ \end{pmatrix}_{s\times 1}$$

Motivating Question: Can generalized co-ordinates be chosen in such a way that each co-ordinate executes only one simple ~~harmonic~~ oscillation?

Well, we simply invert the relation to get Θ in terms of x.

$$\left(\Delta_{K\alpha}\right)^{-1}_{5\times 5}\left(x_K\right)_{5\times 1} = \left(\Theta_\alpha\right)_{5\times 1}$$

Normal Co-ordinates $\equiv$ Generalized co-ordinates Θ_α

Normal Oscillation $\equiv$ The simple periodic oscillation of Θ_α

The normal co-ordinates, from its def^n, satisfy

$$\ddot{\Theta}_\alpha + w_\alpha^2\,\Theta_\alpha = 0$$

$\Rightarrow$ In normal co-ordinates, the eq^n of motion become 5 independent eq^{n}s.

viz The normal oscillations of the system are completely independent.

This means that the Lagrangian would be given by

$$L = \sum_\alpha \frac{1}{2}\, m_\alpha\left(\dot{\Theta}_\alpha^2 - w_\alpha^2\,\Theta_\alpha^2\right)$$

where m_α are the constants.

Remark: Mathematically it means that the transformation $\left(\Theta_\alpha\right)_{5\times 1} = \left(\Delta_{K\alpha}\right)^{-1}_{5\times 5}\left(x_K\right)_{5\times 1}$ simultaneously puts both the quadratic forms (PE & KE) in the diagonal form.

(TODO: Think of how to obtain this without invoking the Lagrangian arguments)

Convension: The normal co-ordinates are usually so chosen, that the co-efficients of the velocity square is $1/2$.

viz. $Q_\alpha = \sqrt{m_\alpha}\;\Theta_\alpha$

so that $$L = \frac{1}{2}\sum_\alpha\left(\dot{Q}_\alpha^2 - w_\alpha^2\,Q_\alpha^2\right)$$

READ: what happens when there's a degeneracy from the Text

→ 3D oscillations of a single particle in a const external field.

Taking the origin at the minima of $U(x,y,z)$ we have

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$ (doesn't depend on the orientation of the axis)

Next we reduce the potential energy to its diagonal form by an appropriate choice of axes.

Thus

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}(k_1 x^2 + k_2 y^2 + k_3 z^2)$$

and the normal oscillations take place in the x, y & z ~~dir~~ directions with frequencies $\omega_1 = \sqrt{\frac{k_1}{m}}$, $\omega_2 = \sqrt{\frac{k_2}{m}}$

$\omega_3 = \sqrt{\frac{k_3}{m}}$

special case: If U is a central field, $k_1 = k_2 = k_3 \equiv k$, $U = \frac{1}{2} k r^2$

→ Forced Oscillations in more than 1° of freedom can be reduced to a series of problems of forced oscillations in one dimension, if normal co-ordinates are used.

In that case, the Lagrangian

$$L = L_0 + \sum_k F_k(t)\, x_k$$

where L_0 is the Lagrangian for free oscillations.

Replacing x_k by normal coordinates we have

$$L = \frac{1}{2} \sum_\alpha (\dot{Q}_\alpha^2 - \omega_\alpha^2 Q_\alpha^2) + \sum_\alpha f_\alpha(t)\, Q_\alpha$$

where $f_\alpha(t) = \sum_k \frac{F_k(t)\, \Delta_{k\alpha}}{\sqrt{m_\alpha}}$

this is coming ∵ of Q_α (instead of Θ_α)

Then the eq^s of motion

$$\ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = f_\alpha(t)$$

each with only one $Q_\alpha(t)$ as unknown.

§ 24. Vibrations of molecules

Claim: If we have a system of interacting particles, not in an external field, then not all of its degrees of freedom relate to oscillations.
(proof by example?)

Eg: Molecules: Besides motions in which the atoms oscillate about their positions of eq., the molecule as a whole can execute translational & rotational motions.

→ DOF of rotation = DOF of translation = 3 (in general).
So of the $3n$ DOF of a molecule, $3n-6$ ~~are~~ correspond to vibration

special case: collinear molecules, DOF of rotation = 2
so $3n-5$ correspond to vibration

→ It is convenient to remove the translational & rotational DOF.

Translational
↳ ⇒ COM is at rest.
Putting $\vec{r}_a = \vec{r}_{a0} + \vec{u}_a$

where $\vec{r}_{a0}$ is the radius vector of the eq. pos. of the a^{th} atom
$\vec{u}_a$ its deviation from this position.

We want $\sum m_a \vec{r}_a = \text{const} = \sum m_a \vec{r}_{a0}$

(Why is this valid?
Intuitively: At eq. $\vec{u}_a = 0$,
doubt: How do we know the whole sys is in eq. together?
rebuttal: Well, else the $\vec{r}_{a0}$ will be different.)

$\Rightarrow \sum m_a u_a = 0$

Rotational

↳ To eliminate rotation, we must equate its total angular momentum to zero.

Remark: Since the angular momentum is not the total time derivative of a f^n of the co-ordinates, the condition that $M=0$, can't be expressed by stating that some such f^n is const.

However, for small oscillations this can be done.

$$M = \sum m_a \vec{r}_0 \times \vec{v}_a$$

$$= \sum m \left(\vec{r}_{a0} + \vec{u}_a\right) \times \left(\dot{\vec{r}}_{a0} + \dot{\vec{u}}_a\right)$$

$$= \sum m \quad \vec{r}_{a0} \times \dot{\vec{r}}_{a0} + \vec{r}_{a0} \times \dot{\vec{u}}_0 + \vec{u}_0 \times \dot{\vec{r}}_{a0} + \vec{u}_a \times \dot{\vec{u}}_a$$

VERIFY THE REASONING (eg. pos remains const? then $\dot{\vec{r}}_{a0} = 0$ & $\frac{1}{2}\frac{d}{dt}(\vec{u}_0 \times \vec{u}_0)$ becomes 2nd order in $\vec{u}_0$)

They have taken $\vec{r}_a$ as const.

Doubt: How do we know $\vec{r}_a$ is const?

$$\simeq \sum m \vec{r}_0 \times \dot{\vec{u}}_0 = \frac{d}{dt} \sum m_a \vec{r}_{a0} \times \vec{u}_a$$

Thus the condition is

$$\sum m_a \vec{r}_{a0} \times \vec{u}_0 = \text{const}$$ (The book says zero, although why, I'm not so sure)

where the origin may be chosen arbitrarily.

→ Symmetry / gp theory approach is skipped (The book says!)

→ Example considerations

↳ n atoms of a molecule in a plane.

- For motion in a plane there are $2n$ DOF.
- Of these, there are 2 translational & 1 rotational DOF.
- ⇒ # normal vibrations that leave the atoms in the plane are $2n-3$
- ⇒ rest, i.e. $(3n-6)-(2n-3) = n-3$ are vibrational DOF that correspond to atoms vibrating out of plane.

↳ n atoms, linear molecule.

- For longitudinal vibrations (along the line) ~~, for n particles~~

  For n particles to stay in line $\exists\ n$ DOF, of which one is translational.

  ⇒ # DOF $= n-1$
- Since the total DOF in a linear mol. is $3n-5$, we have $(3n-5)-(n-1) = 2n-4$ DOF corresponding to vibrations that get the atoms out of line.

Remark: $2n-4$ however correspond to $n-2$ different freq $\because$ each vib can occur in 2 mutually $\perp$ planes through the axis of the molecule.

§ 25. Damped Oscillations

→ Introduction : (read it off the book, I'm just writing key terms here)
<implied motion in vacuum> <in medium, retardation, heat>
<motion no longer purely mechanical> <can't write accn as f^n of co-ordinate & velocity>
<special class, medium approximated, include small freq. compared to those of dissipative process> <can be modelled as friction f^n(velocity)>
<If vel. is small, force expanded> <zero order term force of is zero (no friction on a body at rest)> <first order is proportional to vel>

$\Rightarrow \quad f_{fr} = -\alpha \dot{x} \qquad (\alpha > 0)$ (- shows it opposes the velocity)

→ Eq^n of Motion : Adding this to the eq^n of motion of free oscillations, we have

$$m\ddot{x} = -Kx - \alpha\dot{x}$$

with $K/m = \omega_0^2$ & $\alpha/m = 2\lambda$, we have

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0$$

we put $x = e^{rt}$ and find $r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2}$ (didn't verify)
The general soln. then, is

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

case: $\lambda < \omega_0$

r has 2 complex conjugate values
The general soln. ~~of the eqn~~ is

$$x = \mathrm{re}\left[A e^{-\lambda t + i\sqrt{\omega_0^2 - \lambda^2}\, t}\right]$$

The $-i$ part is not going to give a different answer. $Re(e^{i\theta}) = Re(e^{-i\theta})$

where A is an arbitrary const (complex)

$$x = a e^{-\lambda t} \cos(\omega t + \alpha)$$

with $\omega = \sqrt{\omega_0^2 - \lambda^2}$ and a & α as real const.

Damped Oscillations ≡ The motion described by

Remarks: 1) The rate of decrease is given by λ
2) The 'frequency' ω, is $< \omega_0$ (free oscillation freq)

special case: $\lambda << \omega_0$

$$\omega = \omega_0 - \frac{1}{2}\frac{\lambda}{\sqrt{\omega_0^2 - \lambda^2}}\Bigg|_{\lambda=0} \lambda + \left(\frac{-1}{2}\frac{1}{\sqrt{\omega_0^2-\lambda^2}} + \left(-\frac{1}{2}\right)\left(\frac{-1}{2}\right)\frac{2\lambda^2}{(\omega_0^2-\lambda^2)^{3/2}}\right)\Bigg|_{\lambda=0}\lambda^2 \dots$$

$$\sim \omega_0 + \left(\frac{-1}{2\omega_0}\right)\lambda^2$$

Remark: 1) Thus the correction in ω is of second order.

2) The amplitude of the damped oscillation is almost unchanged during the period $\frac{2\pi}{\omega}$

Note: This suggests we can neglect $e^{-\lambda t}$ while taking mean.

3) Mean squares of co-ordinates & velocities will be proportional to $e^{-2\lambda t}$

(perhaps the term with λ that pops out while differentiating the coordinate to get velocity can be neglected)

Thus the energy decreases as

$$E = E_0 e^{-2\lambda t}$$

case : $\lambda \geq \omega_0$

$$r_1 = -\lambda + \underbrace{\sqrt{\lambda^2 - \omega_0^2}}_{< \lambda}$$

$$\left(\because \sqrt{\lambda^2} > \sqrt{\lambda^2 - \omega_0^2}\right)$$

$$\Rightarrow r_1 < 0$$

$$r_2 = -\lambda - \sqrt{\lambda^2 - \omega_0^2}$$

$$\Rightarrow r_2 < 0$$

The general soln. in this case is

$$x = c_1 e^{-(\lambda - \sqrt{\lambda^2 - \omega_0^2})t} + c_2 e^{-(\lambda + \sqrt{\lambda^2 - \omega_0^2})t}$$

Remark: The motion consists of a decrease in $|x|$, asymptotically as $t \to \infty$, to its equilibrium position.

Aperiodic Damping $\equiv$ This kind of Damping.

(i think: You don't see oscillations)

special case : $\lambda = \omega_0$

$$r_{1,2} = -\lambda$$

$$x = c_1 e^{-\lambda t} + c_2 e^{-\lambda t} = (c_1 + c_2) e^{-\lambda t}$$

ASK: The book has a t term there! ☹

end case

System with more than one degree of freedom

Claim: The generalized friction force is of the form (corresponding to x_i)

$$f_{fr, i} = -\sum_k \alpha_{ik} \dot{x}_k$$

From statistical physics, we have $\alpha_{ik} = \alpha_{ki}$

Thus we can write

$$f_{fr,i} = -\frac{\partial F}{\partial \dot{x}_i}$$

where F is the quadratic form

$$F = \frac{1}{2}\sum_{i,k} \alpha_{ik}\dot{x}_i \dot{x}_k \doteq \text{Dissipative } f^n.$$

~~which is called the~~

The force is added to the RHS of Lagrange's Eq^{n}s

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} - \frac{\partial F}{\partial \dot{x}_i}$$

→ Physical significance of the Dissipative f^n.

$$\frac{dE}{dt} = \frac{d}{dt}\left(\sum_i \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L\right)$$

$$= \sum_i \ddot{x}_i \frac{\partial L}{\partial \dot{x}_i} + \underbrace{\dot{x}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{dL}{dt}}$$

also $\frac{dL}{dt} = \sum \frac{\partial L}{\partial x_i}\dot{x}_i + \sum \frac{\partial L}{\partial \dot{x}_i}\ddot{x}_i$

$$= \sum_i \dot{x}_i \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i}\right)$$

$$= -\sum_i \dot{x}_i \frac{\partial F}{\partial \dot{x}_i}$$

(Since F is a quadratic f^n of velocities, Euler's Theorem on homogeneous f^n shows)

$$\frac{dE}{dt} = -2F$$

Why $F>0$: Physically, since dissipative processes lead to loss of energy.

NOTE/FIGURE: Even though $\frac{\partial L}{\partial t} = 0$, the energy is not conserved. Trace precisely why this happens. Intuitively, cause we started with adding dissipative forces to the eq^n of motion, not the Lagrangian.

— Eq^{n}s of small oscillations under friction

To $\sum_k m_{ik}\ddot{x}_k + \sum_k k_{ik}x_k = 0$, we add to the RHS the force on the x_i coordinate to get

$$\sum_k m_{ik} \ddot{x}_k + \sum_k k_{ik} x_k = -\sum_k \alpha_{ik} \dot{x}_k$$

Again putting $x_k = A_k e^{rt}$ & cancelling we get

$$\sum_k (m_{ik} r^2 + \alpha_{ik} r + k_{ik}) A_k = 0$$

Equating the det to zero, we have

$$|m_{ik} r^2 + \alpha_{ik} r + k_{ik}| = 0$$

This is an eq^n in r of deg 2s.

Claim: Since all co-efficients are real, its roots are either real or complex conjugate pairs.

Remark: The real roots < 0 & $\mathrm{Re}(\text{complex roots}) < 0$ else the energy of the sys. would increase exponentially whereas dissipative forces must lead to a decrease in energy.

§ 26. Forced Oscillations under friction

Intro: Forced oscillation under friction is analogous to forced oscillations without friction.

Special case: The external force is periodic, $f \cos \gamma t$

We have

$$\ddot{x} + 2\lambda \dot{x} + \omega_0^2 x = \left(\frac{f}{m}\right) \cos \gamma t$$

We convert / resort to e to get

$$\ddot{x} + 2\lambda \dot{x} + \omega_0^2 x = \left(\frac{f}{m}\right) e^{i\gamma t}$$

For a particular soln. we try $x = Be^{i\gamma t}$ & find

$$B = \frac{f}{m} (\omega_0^2 - \gamma^2 - 2i\lambda\gamma) \quad \text{(not verified)}$$

With $B = be^{i\delta}$, we have

$$b = \frac{f}{m} \left[(\omega_0^2 - \gamma^2)^2 + 4\lambda^2\gamma^2\right] \quad (\;\text{"}\;)$$

$$\delta = 2\lambda\gamma / (\gamma^2 - \omega_0^2)$$

Taking only the real part of the soln. & adding it to the ('homogeneous' soln.) soln. for RHS = 0, we get (and assuming $\omega_0 > \lambda$)

$$x = a e^{-\lambda t} \cos(\omega t + \alpha) + b \cos(\gamma t + \delta)$$

The first term decreases exponentially, so after ~~some~~ sufficient time, only the second term remains.

$$x = b \cos(\gamma t + \delta)$$