Rough / Experiment Sheet



#1. $x' = \theta x_1 + (1-\theta)x_2$ → given

Let $\exists A$ s.t. $Ax_1 = b$ & $Ax_2 = b$. This is possible? → assumed

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{matrix}x_1\\x_2\end{matrix} = \begin{matrix}c_1\\c_2\end{matrix}$$

only one soln., unless.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32}\end{pmatrix}\begin{matrix}x_1\\x_2\end{matrix} = \begin{matrix}c_1\\c_2\\c_3\end{matrix}$$

$\Rightarrow$ now, $Ax' = A[\theta x_1 + (1-\theta)x_2]$
$= b$

$x \in \mathbb{R}^n$

$A^{m\times n}\ \mathbb{R}^n = \mathbb{B}^m$

Yes

$(a_{11}\ a_{12})\begin{pmatrix}x_1\\x_2\end{pmatrix} = c_1$

#2 $(v_1 - v_0)\theta_1 + (v_2 - v_0)\theta_2 + \dots$
$(v_k - v_0)\theta_n$
$[1 - (\theta_1 + \theta_2 + \theta_3 + \dots)]v_0$
$+$

#3 $X = X^T$

$n$

$n \cdot n - \frac{1}{2}(n-1)(n-1)$
$n^2 - \frac{1}{2}(n^2 - 1 + 2n)$

$-\frac{1}{2}n^2 + n$

$$n^2 - \frac{1}{2}(n-1)^2 = n^2 - \frac{1}{2}(n^2 + 1 - 2n)$$
$$= \frac{n^2}{2} - \cancel{\frac{n^2}{2}} + \frac{1}{2} - n$$

$n + \frac{(n-1)^2}{2}$

$$= \frac{n^2}{2} + \frac{1}{2} - 2\left(\frac{n}{2}\right)$$
$$= \frac{n^2}{2} + \frac{n}{2}$$

$\frac{(n-1)^2}{2} + n$

$$\boxed{\left(\frac{n^2 - n}{2}\right) + n = \frac{n^2+n}{2} = \frac{n(n+1)}{2}}$$

# of entries in the triangle is
$$\frac{n^2 - n}{2}$$
total # entries → # entries in diagonal.

So then

#4 $X = \begin{bmatrix} x & y \\ y & z\end{bmatrix} \in S^2_+$

$x + z < 2y$

$$\det\begin{pmatrix} (x-\lambda) & y \\ y & (z-\lambda)\end{pmatrix} = 0$$

$$(a\ b)\begin{pmatrix} x & y \\ y & z\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}$$

$$(a\ b)\begin{pmatrix} xa + by \\ ya + bz\end{pmatrix} = \begin{pmatrix} xa^2 + aby \\ + yab + b^2 z\end{pmatrix}$$

$\Rightarrow (x-\lambda)(z-\lambda) - y^2 = xz - \lambda x - \lambda z + \lambda^2 - y^2$
$= \lambda^2 - (x+z)\lambda - y^2 + xz = 0$

$= x a^2 + 2aby + b^2 z$

$(x+z)^2 + 4(y^2 - xz) \geq 0$

$\lambda = x + z \pm \sqrt{\phantom{x}}$

#5

$x \preceq y$ $\quad u \preceq v$

$x - y \in K$ $\quad u - v \in K$

To prove: $(x+u) - (y+v) \in K$

$x - y + u - v$

$A, B \in K \mid \theta_1 A + \theta_2 B \in K$

$\cancel{x \preceq_K}$ $x - y \in K$

$y - x \in K$

$A \in K$ & $-A \in K$ then $A = 0$ — proper cone, convex → pointedness.

$A - A \in K$

$\theta_1 A + (1-\theta)(-A)$

$\theta A \in$

If true & $A \neq 0$, then I can have a line in this cone.

(7) ✓

(8)

$x - y \in$ int$(K)$

$v - u \in K$

$v + y - x - u$

$v - u - (x - y) \in$ int$(K)$

vaguely yes

(9) $x \prec_K y \in$ int$(K)$

$(x - y) \in$ int$(K)$

$x \prec_K y$

$x - y$

$(x+u) - (y+v)$

Assume $y \neq x$ & $x$ & $y$ are both the minima.

Objective: show some contradiction

$\{x \quad 2, 2^2$

$\cancel{x \preceq y}$

$x \leq a$

$y \leq a$

$1 \leq 5$

$0 \leq 5$

$\cancel{x \preceq y \preceq 2x}$

$\cancel{x - y \leq 0}$

$\cancel{y - x \leq 0}$

$x + y \leq 2a$

$\frac{x+y}{2}$

$x < \frac{x+y}{2} < y$

$K$

#6

$(d-c)^T\left(x - \left[\left(\frac{1}{2}\right)(d+c)\right]\right)$

$\underbrace{(d-c)^T}_{a} x - \underbrace{\frac{1}{2}(d-c)^T(d+c)}_{b}$

$\frac{d}{dt}\sum_i (d_i + t(u_i - d_i) - c_i)^2$

$= \sum_i 2\,(d_i + t(u_i - d_i) - c_i) \cdot u_i - d_i$

$= 2\,(d - c)^T(u - d)$

#7

$a^T x \leq b \quad \forall x \in C$

$a^T x \geq b \quad \forall x \in D$

$\Rightarrow a^T F = 0$ & $b \leq a^T g$

I need $a$ & $b$ to specify the plane.

$a$ is evaluated from $a^T F = 0$.

& for the space $a^T x \leq b$ one can use a larger space

$a^T x \leq a^T g$

so one can use $b = a^T g$

#8 Minimize $a^T(x_0 + u)$ wrt $u$.

$a^T(x_0 + u) = a^T x_0 + a^T u$

$\frac{\partial}{\partial u_j}\, a_i x_{0i} + a_i u_i = 0$

$a_1^T x_{01} + a_2^T x_{02}$

$a^T\left(x_0 - \frac{\epsilon a}{\|a\|_2}\right)$

Is $a^T(x_0 + u)$ linear? Yes $\Rightarrow$ Extreme value must occur for extreme value of $u = \epsilon \cdot \hat{n}$

which $\hat{n}$ would I choose so that $a \cdot \hat{n}$ is highest?

$\hat{n}$ as $\hat{a}$ so $\vec{a} \cdot \hat{a}$ of course is $a$

$a^T x_0 \geq b \quad x \in B$

$a^T x \leq b \quad x \in C$

$a^T x - b \leq 0 \quad x \in C$

$\Rightarrow a^T(x_0 + u) \geq b$

$\forall$ smallest

$a^T x_0 - \epsilon\|a\|_2 \geq b$

$a^T x_0 \geq b + \epsilon\|a\|_2$

$\Rightarrow a^T x_0 > b\left(+\frac{\epsilon\|a\|_2}{2}\right)$

$a^T x_0 > b\left(+\frac{\epsilon\|a\|_2}{2}\right)$

$\Rightarrow$ for the original set $x_0$ also then $a^T x < b$

$a^T x < b + \frac{\epsilon\|a\|_2}{2}$

$\Rightarrow a^T x < b + \frac{\epsilon\|a\|_2}{2}$

so then my $b' = b + \frac{\epsilon\|a\|_2}{2}$ & I've created a strictly separating hyperplane

#9 $\lambda^T y \le \mu$ on $C$ — $b - Ax$

$\lambda^T y \ge \mu$ on $D$ — $y$ $(>0)$

$\Rightarrow A^T\lambda = 0$ & $\lambda^T b \le \mu$

$\Rightarrow ?$

show (1) $\mu \le 0$, (2) $\lambda \ge 0$ (3) $\lambda \ne 0$

(1) $\lambda^T y \ge \mu \ge \lambda^T b$ ; $(>0)$ ; can $\mu \le 0$

(2) $\lambda^T y \ge 0$ $\forall y > 0$

$y = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}$

$\Rightarrow \lambda \ge 0$

(3) $\lambda \ne 0$ from the hyperplane thm.

Ex 2.3 (a) $K^*$ is indeed a cone

If $y \in K^*$ $\Rightarrow x^T y \ge 0$ $\forall x \in K$

$\Rightarrow \theta x^T y \ge 0$ for $\theta \ge 0$

$\Rightarrow \theta y \in K^*$

$\Rightarrow K^*$ is a cone

(b) $K_1 \subseteq K_2 \Rightarrow K_2^* \subseteq K_1^*$ ✓

$\because$ more constraints are on $K_2$ compared to $K_1$;

$K_1$ $K_2$ $x^T$

(c) . . .

#10

$X = qq^T$

$\mathrm{tr}(XY) < 0$

$Y \notin S_+^n$

$y \mid x^T y \ge 0 \ \forall x \in K$

$\Rightarrow Y = qq^T \notin (S_+^n)^*$.

$X, Y \in S_+^n$

$X = \sum_{i=1}^n \lambda_i q_i q_i^T$

$\lambda_i \ge 0$, $i = \dots$

$\mathrm{tr}(YX) = \sum \mathrm{tr}(Y \lambda q_i q_i^T)$

$= \sum \lambda q^T Y q \ge 0$

$\Rightarrow Y \in (S_+^n)$

e.g. 2.25 ~~Dual of n~~

$\|\cdot\|$ ; $K = \{(x,t) \mid \|x\| \le t\}$

$K^* = \{(u,v) \mid \|u\|_* \le v\}$

$\|u\|_* = \sup\{u^T x \mid \|x\| \le 1\}$

$\|\cdot\|_a$ , $\|\cdot\|_b$
on $\mathbb{R}^m$ on $\mathbb{R}^n$

op norm on $X \in \mathbb{R}^{m\times n}$ $\|X\|_{a,b} = \sup\{\|Xu\|_a \mid \|u\| \le 1\}$

special case: a & b are Euclidean → op of $X$ $\|X\|_{ab}$ = max sing. value.

$\|X\|_2 = \sigma_{max}(X) = (\lambda_{max}(X^T X))^{1/2}$

$\|\cdot\|$ is a norm on $\mathbb{R}^n$
then the dual norm is

$\|z\|_* = \sup\{z^T x \mid \|x\| \le 1\}$

$z^T x \le \|x\| \|z\|_*$

$z^T x$ $x \|z\|_*$

$\max\{|z^T x| \mid \|x\| \le 1\}$

$x > 0$

$x = c x_0$ s.t. $\|x_0\| \le 1$

$\|c x_0\|$ $\mid \|x_0\| \le 1$

$\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$U^T X U u$

$|\lambda_1|$

~~$\lambda_{max}$~~

$|z^T x_0'| \ge |z^T x_0|$

$z^T x_0$

$z^T \frac{z}{|z|} < \frac{\|z\| \cdot \|z\|}{|z|}$

$\|z\| \le |z|$

$|z^T x|$

$= |c| |z^T x_0| = |z^T x|$

$\ge |c| |z^T x_0| \mid \|x\| \le 1\}$

$= \sup\{|z^T x| \mid \|x\| \le 1\}$

$\ge |c| |z^T x_0|$

$|z^T x|$

$|c| \|z\|_* = \max |z^T x_0|$

$z^T$ $\|z\|_*$

$|c| |z^T x_0'|$

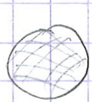
THIS WAS TRIVIAL!

$\sup\{z^T x \mid \|x\| \le 1\} = z^T x^*$ say

Now $z^T x^* \ge z^T x'$ — normalized "$\frac{x'}{\|x'\|}$" where $x'$ is arbit

$\Rightarrow \|x'\| \|z\|_* \ge z^T x'$

$\cancel{\|x\|_*}$

$D = \{ x \mid \|x\| \leq 1 \}$ —  $\langle x, y \rangle$

$\|x\| = \inf \{ \lambda > 0 \mid x \in \lambda D \}$

$D^* = \{ x \mid 1 + \langle x, y \rangle \geq 0 \ \forall y \in D \}$ — $-1 \geq \langle x, y \rangle$

$\|x\|_* = \inf \{ \lambda > 0 \mid x \in \lambda D^* \}$