# Lecture 10 | field equations

Saturday, October 28, 2017 4:02 PM

Recall: $m\left(\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\right) = q F^\mu{}_\rho \frac{dx^\rho}{d\tau} + a R^\mu{}_{\alpha\beta\gamma} F_{\alpha\beta}\frac{dx^\gamma}{d\tau}$

NB: Both sides transform the same way ⇒ general coordinate invariance is respected.

Recall: $a$ from dimensional analysis is $\sim 10^{-36}$ cm; so not easy to observe

Remark: One can try to write terms with derivatives as well $\nabla^\nu R_{\nu\rho\sigma}{}^\mu \frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau}$, these would be present even in the absence of the EM fields but the term added above is due to the EM fields (zero otherwise)

## EM Field Equations

We start with flat spacetime

Recall: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$;

$\therefore \Rightarrow \partial_{[\mu} F_{\nu\rho]} = 0$ (which means you add all the cyclic permutations; totally anti-symmetric combination is zero) $\because F_{\mu\nu} = -F_{\nu\mu}$

viz. $\partial_\mu F_{\nu\rho} + \partial_\nu F_{\mu\rho} + \partial_\rho F_{\mu\nu} = 0$; Also called the Bianchi Identity.

Remark: Sometimes one doesn't like $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ & directly imposes the Bianchi Identity.

Remark 2: This (the Bianchi Identity) corresponds to $\nabla\cdot B = 0$ & $\nabla\times E + \frac{\partial B}{\partial t} = 0$

Story: Then there's another set of eq's that don't follow from $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This is a statement about the source of the fields

"Fact": $\partial^\mu F_{\mu\nu} = -J_\nu$ ; $J^0 = \rho$ ; Electric Charge Density
$J^i$ ; Current Density

Claim: These correspond to

$\nabla\cdot E = \rho$ ; $\nabla\times B - \frac{\partial E}{\partial t} = \vec{J}$ ($c=1$ unit)

Story: In a local inertial frame, one must have ① $\partial'_{[\mu} F'_{\nu\rho]} = 0$ & ② $\partial'^\mu F'_{\mu\nu} = -J'_\nu$
($x'$ coordinate) (instead of using $F'_{\mu\nu} = \partial'_\mu A'_\nu - \partial'_\nu A'_\mu$)

Further, $J'$ can also be evaluated (in $x'$) as it would be in the absence of gravity.

NB: In a general frame, $F'_{\mu\nu} = \partial'_\mu x^\rho \partial'_\nu x^\sigma F_{\rho\sigma}$

$\partial' \rightarrow$ in terms of $\partial$ as before

Def$^n$: $J_\nu = \partial_\nu x'^\sigma J'_\sigma$ NB: This makes $J_\nu$ a tensor.

Claim: $\nabla^\mu F_{\mu\nu} = -J_\nu$ NB: We had to use $\nabla$ to preserve tensorial behaviour (as discussed earlier).

NB 2: $F'_{\mu\nu}$ use $\partial'$ $\because D' \leftrightarrow \partial'$ in that case.

Claim: $F_{\mu\nu} \overset{\text{def}^n}{=} D_\mu A_\nu - D_\nu A_\mu \overset{\text{claim}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$

($\because$ $\Gamma$ terms get cancelled).

NB: One could start with defining $A_\mu$ using $A'_\mu$
$F_{\mu\nu}$ independently using $F'_{\mu\nu}$
use the rel$^n$ b/w $F'$ & $A'$ to derive $F$ in terms of $A$

This would yield $F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu$.

# ... + completed for EM field + ...

Saturday, November 18, 2017   5:39 PM

Deeper NB: This is important in ensuring the principle of equivalence holds

$\because$, say, $F^{\mu\nu}$ involved $\Gamma$.
$\nabla^\mu$ has a $\partial$. This would act on $\Gamma$
Thus the LHS of $\nabla^\mu F_{\mu\nu} = -J_\nu$ would involve $\partial\Gamma$ like terms.
Now in a local inertial frame, $\Gamma = 0$ but $\partial\Gamma$ is not zero
Thus this will not reduce to the standard eq$^n$ in the local inertial frame.

Remark (about $\nabla^\mu F_{\mu\nu} = -J_\nu$):

aim: Evaluate $\underbrace{\nabla^\nu \nabla^\mu F_{\mu\nu}} = -\nabla^\nu J_\nu$

$= \frac{1}{2}\left[\nabla^\nu, \nabla^\mu\right] F_{\mu\nu}$ $\quad\because = \frac{1}{2}\left(\nabla^\nu\nabla^\mu F_{\mu\nu} + \nabla^\mu\nabla^\nu F_{\nu\mu}\right)$

$= \frac{1}{2}\left(\nabla^\nu\nabla^\mu - \nabla^\mu\nabla^\nu\right] F_{\mu\nu} \quad \because F_{\nu\mu} = -F_{\mu\nu}$

recall: we had "derived" identities involving $[D,D]A = RA + RA + \ldots$

consider: $[\nabla_\nu, \nabla_\mu] F_{\alpha\beta} = -R_{\alpha\mu\nu}{}^{\rho} F_{\rho\beta} - R_{\beta\mu\nu}{}^{\rho} F_{\alpha\rho}$

Now we raise index ($\because$ it can be done inside covariant derivative) & replace $\alpha\beta$ by $\mu\nu$.

$$[\nabla^\nu, \nabla^\mu] F_{\mu\nu} = -R_\mu{}^{\mu\nu\rho} F_{\rho\nu} - R_\nu{}^{\mu\nu\rho} F_{\mu\rho}$$

$$= \underbrace{-R^{\nu\rho} F_{\rho\nu}}_{=0} + \underbrace{R^{\mu\rho} F_{\mu\rho}}_{=0} = 0$$

$\because R$ is symmetric & $F$ is anti-symmetric

$\Rightarrow \quad \nabla^\nu\nabla^\mu F_{\mu\nu} = 0$ for the LHS

RHS? In local inertial $\quad \partial'_\mu J'^\mu = 0$ (by construction)

Claim: $\quad \nabla_\mu J^\mu = 0 \Leftarrow \partial'_\mu J'^\mu = 0$

Remark: If $\partial'_\mu J'^\mu \neq 0$, then there'd be inconsistencies

For eg. $\partial'^\mu F'_{\mu\nu} = -J'_\nu$ is given

$\not\Rightarrow \quad \partial'^\nu\left(\partial'^\mu F'_{\mu\nu}\right) = -\partial'^\nu\left(J'_\nu\right) = 0$

$\because$ second derivative would involve neighbouring point which may not be locally inertial. (but then $\partial'^\mu F'_{\mu\nu}$ also had 2 derivatives, why couldn't we object then? $\vdash \because$ even at $F'$, the $\Gamma$ disappears

this is correct because (as calculated above) the $R$ part doesn't contribute.

Conclusion: The equivalence principle is working due to the extra condition enforced by electrodynamics.

# ... + GR Field Equations ...

Saturday, 25 November 2017 05:36 PM

GR Field Eqⁿ

NB: Starting from the equivalence principle is pointless because gravity is zero in a local inertial frame.

Intuition: We try to figure out what eqⁿ would give, in the appropriate limit,

$$\nabla^2 \phi = 4\pi G \rho_m$$

NB: The LHS we saw was related to the metric; (recall $h_{00} = \frac{-2\phi}{c^2}$)
It must be related to two derivatives of the metric.

NB: Two derivatives of the metric have the same # free parameters (ones that can't be killed by coordinate transformⁿ) as a tensor of rank 2. The only choice we have is $R_{\mu\nu}$ & $R g_{\mu\nu}$ — why not $g_{\mu\nu}$ alone? ∵ you need two derivatives of the metric.

NB: Using mass density is troublesome.

Justifⁿ: For an H atom, the mass $m_H \neq m_p + m_e$

proton mass, electron mass

One must take the binding energy into account, else the mass we get would be different from the inertial (gravitational?) mass.

Conclⁿ: Use energy density

"Corollary": Energy by itself is not covariant; one should use the energy-momentum tensor, $T_{\mu\nu}$. (More on evaluating $T_{\mu\nu}$ in the next lecture)

"⇒" $\quad b R_{\mu\nu} + c R g_{\mu\nu} = a T_{\mu\nu}$

⇔ $\quad R_{\mu\nu} + c R g_{\mu\nu} = a T_{\mu\nu} \quad$ (for $b \neq 0$ & $\frac{c}{b} \to c$, $\frac{a}{b} \to a$)

Claim. $\partial'^{\mu} T'_{\mu\nu} = 0$ (in a local inertial frame) $\Rightarrow D^{\mu} T_{\mu\nu} = 0$

NB: $\Rightarrow D^{\mu}(R_{\mu\nu} + c R g_{\mu\nu}) = 0$

This we demand to hold identically because if we try to demand this as an additional condition, apparently, the solⁿ gets very specific.

Recall: $D^{\mu}\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) = 0$ identically

Conclusion: $c = -\frac{1}{2}$.

Strategy: To find "a" we use a consistency check with Newtonian gravity (in the weak field limit)

Recall: $g_{00} \simeq -1 - 2\phi$ (in the weak field limit); $\Gamma^0_{00} \simeq 0$, $\Gamma^i_{00} \simeq \partial_i \phi$

$T_{00} = \rho_m$

$\Rightarrow R_{00} = ?$
$\Rightarrow R = ?$ Exercises

Claim: in the weak field limit the 00 component becomes

$$-2\nabla^2\phi = a\rho_m$$

$\Rightarrow a = 8\pi G$.

Remark. One can add to $T_{\mu\nu} \to T_{\mu\nu} + \Lambda g_{\mu\nu}$.

Conclusion: $\quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$ (Sign convention matches Weinberg's)