Lecture 7 (Monodromy and R)

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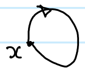
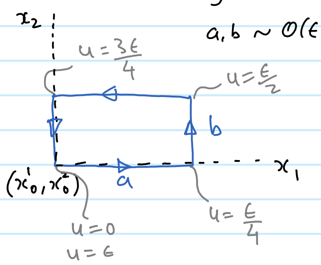
Recall: 
: Loop was of length $\mathcal{O}(\epsilon)$.
: Tangent vector at x, parallel transported along the curve.
: We can't comment beyond $\mathcal{O}(\epsilon^2)$ at the moment.

$$n^i(\epsilon) = n^i_{(0)} - \frac{1}{2} R^i{}_{jkl}\, n^j_0 \oint (\underbrace{x^l(u) - x^l_{(0)}}_{\text{has a } \int du}) \frac{d}{du}(x^k(u) - x^k_{(0)}) + \mathcal{O}(\epsilon^2)$$

Strategy: Calculate the integral for a special curve; Deduce some properties from there.

Consider: A rectangle in the $x_1 - x_2$ plane.



$a, b \sim \mathcal{O}(\epsilon)$ NB: a, b are just coordinate distances

: u is "divided" equally across the edges of the rectangle.

NB: $x^i - x^i_{(0)} = 0 \quad \forall\, i \neq 1, 2$
$\Rightarrow$ The integral will vanish unless l & k are 1 or 2.
Also, the tensor is anti-symmetric in these indexes $\therefore$ we need only evaluate one term, say $l=1$, $k=2$

We write the $x^1\, x^2$ coordinates for all segments.

Segment 1: $x^1(u) = x^1_{(0)} + a u \frac{4}{\epsilon}$
$x^2(u) = x^2_{(0)}$
Integral: $0 \quad \because$ its of the form $\int x^1 dx^2$ & $dx^2 = 0$.

Segment 2: $x^1(u) = x^1_{(0)} + a$
$x^2(u) = x^2_{(0)} + b\left(u - \frac{\epsilon}{4}\right)\left(\frac{4}{\epsilon}\right)$ $\therefore$ the integral $= \int (x^1 - x^1_{(0)}) \frac{d(x^2 - x^2_{(0)})}{du} du$
$= \int a \cdot \frac{d}{du}\left(b\left(u - \frac{\epsilon}{4}\right)\left(\frac{4}{\epsilon}\right)\right) du$
$= ab \cancel{\frac{4}{\epsilon}} \cancel{\frac{\epsilon}{4}} = ab$

Segment 3: $x^2 = x^2_{(0)} + b$
$x^1(u) = \ldots$ doesn't matter because again $dx^2 = 0$
& the integral is of the form $\int \ldots dx^2$.

Segment 4: $x^1 = x^1_{(0)}$ ($\because$ it returned)
Note that the integral is of the form $\int (x^1 - x^1_{(0)})\, d(x^2 - x^2_{(0)})$
but $x^1 - x^1_{(0)} = 0$ so the integral vanishes regardless of x^2.

$\Rightarrow$ The full integral becomes $= ab$.

TODO: check spelling!

Recall. $n^i(\epsilon) = M(\vec{x}_0, \vec{x}_0, c)^i{}_j\, n^j_{(0)}$ (The Monodromi Matrix)

So then $M^i{}_j(\vec{x}_0, \vec{x}_0, c)^i{}_j =$

$$\delta^i{}_j - \frac{1}{2} R^i{}_{jkl}(\vec{x}_{(0)}) \cdot \underbrace{\oint (x^l(u) - x^l_{(0)}) \frac{d}{du}(x^k(u) - x^k_{(0)})}$$

NB: (We just showed) that the integral $= 0$ unless $(k, l) = (1,2)$ or $(2,1)$
$= ab$ if $k=2,\ l=1$
& $-ab$ if $k=1,\ l=2$.

NB: So instead of a sum (in $R^i{}_{jkl}$) you get only one term, $R^i{}_{j12} \cdot (2ab)$.
Advantage: We don't have to worry about any cancellation of terms.
If the expression (integral) is zero, then this term must vanish.
If not, then this term can't vanish.

Story: We use it for proving, roughly, that if the monodrom is zero for all curves, then the Reimann Tensor $R = 0$.

Cont. (2)

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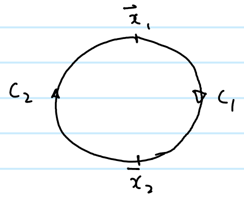
Given: For all curves, the monodrom matrix = $\mathbb{1}$.
Claim: $R^i{}_{jkl} = 0$ in that case.
Proof Idea: Choose a curve along different planes (as before) to show that each component of $R^i{}_{jkl} = 0$.
NB: Higher order terms in ϵ may give further conditions but they can't cancel the $R^i{}_{jkl}$ term so at least R must vanish if $M(x, x, c) = \mathbb{1}$.

Given: $R^i{}_{jkl} = 0$
Claim (eventual): $M = \mathbb{1}$ (under some extra assumptions)

Some additional Results

Consider:



a non-small curve. Can calculate M by starting at x_1. Can also start at x_2.
Question: Relation b/w these

: $M(x_1, x_1, c_1 + c_2)$ vs. $M(x_2, x_2, c_2 + c_1)$ [Tip: Reverse order]

Recall: $M(x_2, x_1, c_2)\ \underline{M(x_1, x_2, c_1)} \neq M(x_1, x_2, c_1)\, M(x_2, x_1, c_2)$ in general.

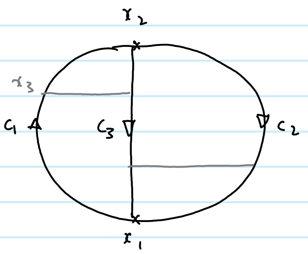
$= M(x_2, x_1, c_2)\, M(x_2, x_2, c_2 + c_1)\, M(x_2, x_1, -c_2)$ ← I think

Essentially, note that the second term (highlighted) is contained in $M(x_2, x_2, c_2 + c_1)$. We remove the extra term by multiplying by its inverse (recall $M(\dots c)^{-1} = M(\dots -c)$).

NB: The two monodrome matrices are related by conjugation.
Recall: A & B are conjugates if $A = SBS^{-1}$ for some S.
NB 2: If $A = \mathbb{1}$, then $B = \mathbb{1}$ also $\Rightarrow$ If monodrome is $\mathbb{1}$, doesn't matter which point you start from.

Comment: We use it in a different way.
NB 3: If $B = \mathbb{1} + O(\epsilon)$ then $A = SBS^{-1} = \mathbb{1} + S\,O(\epsilon)S^{-1} = \mathbb{1} + O(\epsilon)$ $\Leftrightarrow$ If B is close to $\mathbb{1}$, so is A.

Consider:



Idea: Use results from small loops & the result above.
· Break the loop.

$M(x_1, x_1, c_1 + c_2)$ ← For the full loop.

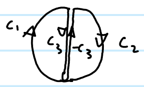
$= M(x_2, x_1, c_2)\,.\,M(x_1, x_2, c_1)$

(non-small loop) $= M(x_2, x_1, c_2)\,.\,\underbrace{M(x_1, x_2, -c_3)\,M(x_2, x_1, c_3)}_{\mathbb{1}}\,.\,M(x_1, x_2, c_1)$

$\because M(x_1, x_2, -c_3) = M(x_2, x_1, c_3)^{-1}$

start, end

Idea: go into a smaller loop. e.g.



$= M(x_1, x_1, c_2 - c_3)\;\; M(x_1, x_1, c_1 + c_3)$

& similarly for the other

Now, divide further & to get the loop to start at some other point, conjugate. start at x_3 & repeat the aforesaid.

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Assume: We've divided the original loop into n parts (roughly same size).
$\Rightarrow$ Each part has an area $\sim \frac{1}{n} \Rightarrow$ circumference $\sim \frac{1}{\sqrt{n}}$

Cont. + Parallel Transport (3)

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Monodromy (when $R^i{}_{jkl}=0$) $= \mathbb{1} + O\left(\frac{1}{n^{3/2}}\right)$ ($\because$ it was $\mathbb{1}+O(\epsilon^3)$ where ϵ was length of curve) if $R^i{}_{jkl}=0$

Final Argument: Look at structure. Several matrices (product).

. . . . each $\mathbb{1} + \mathcal{O}\left(\frac{1}{n^{3/2}}\right)$

& in between there would be conjugations, e.g. $S_2^{-1}\cdot S_{12}\cdot \; S_1^{-1}\cdot \cdots S^{-1}$.
One can bring this into the form $S\cdot S^{-1}\;\; S\cdot S^{-1}\;\; S\cdot S^{-1}\;\; S\cdot S^{-1}$ by multiplying by SS^{-1} where needed.
Recall: Conjugation doesn't change the $\mathbb{1}+O\left(\frac{1}{n^{3/2}}\right)$ part
$\Rightarrow$ In the end I can remove the S matrices & wright

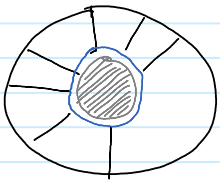
. . . . each $\mathbb{1} + O\left(\frac{1}{n^{3/2}}\right)$

& there're n matrices.
Note: $(1+x_1)(1+x_2)(1+x_3)\cdots(1+x_n)$ with $x \sim \frac{1}{n^{3/2}}$,

$$1 + n\cdot O(x) + {}^nC_2\, O(x^2)$$
$$= 1 + \cancel{n}\cdot\frac{1}{\cancel{n}\, n^{1/2}} = 1 + \frac{1}{n^{1/2}}.$$

$\Rightarrow$ The product above would be $\mathbb{1} + O\left(\frac{1}{n^2}\right)$.
NB: If we had for each $\cdot\ \mathbb{1} + O\left(\frac{1}{n}\right)$, the argument won't work.
This in turn followed $\because$ we had $\mathbb{1} + O(\epsilon^2)$ which justifies the necessity & sufficiency of the hardwork from lecture 6.
CAVEAT: We implicitly assumed that there's a surface enclosed by the curve ($\because$ we are drawing curves on it & dividing).
e.g. 2 For a conical defect, even for a small loop near the singularity (enclosing it), the monodromy $\mathbb{1}+O(\epsilon)$.
$\because$ the argument (for the infinitesimal case) assumed this
e.g. 1 (but I can't see it...)

Illustrⁿ



singularity
can't say anything about this curve, can't make it infinitesimal

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Parallel Transport Intuition: Embedding in a Larger Euclidean Space.

Idea: Surface of a sphere can be described using $x^2+y^2+z^2 = r^2$.
Generalise this, imagine this can be done.

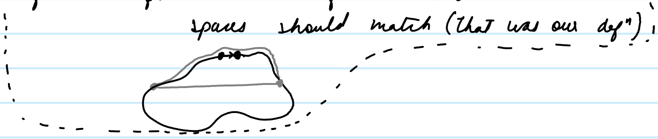
Consider: Coordinates of some manifold $\{x_1, \ldots x_N\}$ embedded in a flat space $\{y_1, \cdots y_D\}$ with $D > N$ in general.

We write $y^\alpha = f^\alpha(\vec{x})$ where $\alpha = 1, \ldots D$ & fs are a set of f's.
Remark: For a typical Riemanian surface, D is very large.

e.g.: 2-d sphere ($\vec{x} = (\theta,\phi)$), $y^1 = a\sin\theta\cos\phi$, $y^2 = a\sin\theta\sin\phi$, $y^3 = a\cos\theta$ | equiv. to saying $(y^1)^2 + (y^2)^2 + (y^3)^2 = a^2$

$$ds^2 = \sum_\alpha dy^\alpha\, dy^\alpha = \sum_\alpha \frac{\partial f^\alpha}{\partial x^i}dx^i\, \frac{\partial f}{\partial x^j}dx^j = \sum_\alpha \underbrace{\left(\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\right)}_{g_{ij}(x)} dx^i dx^j$$

Infinitesimally, the notion of distance in the two spaces should match (that was our dy^α),



Cont. (4)

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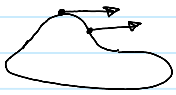

Suppose: ① You have a surface ② Pick a tangent to this surface (in general there'd be a tangent space, pick any tangent in this space) at some point ③ Using the embedded space, parallelly move the tangent vector to the neighbouring point ④ Project it along the tangent space at the new point (it is a well defined notion, we'll see)

Remark: Norm to first order remains preserved despite the projection.

motivation:  $\cos\theta = 1 + O(\theta^2)$, we're doing till $O(\theta)$.

Claim: The projected vector is a parallel transport

Question: Is this notion of parallel transport equivalent to our earlier notion that didn't refer to any embedding space?

Answer: Yes.

Proof (sketch only):

$\vec{n}(u)$ original tangent vector (N-dimensional)

$$m^\alpha = \left.\frac{\partial f^\alpha}{\partial x^i}\right|_{\vec{x}(u)} n^i(u) \longrightarrow m^\alpha \text{ at } \vec{x} + \delta\vec{x} = \vec{x}(u+\delta u)$$

(see the picture)

$$n^i(u+\delta u) = \delta_{\alpha\beta}\, m^\beta \left.\frac{\partial f^\alpha}{\partial x^j}\right|_{\vec{x}(u+\delta u)} g^{ij}(\vec{x}(u+\delta u))$$

$$= m^\beta \left(\left.\frac{\partial f^\alpha}{\partial x^j}\right|_{\vec{x}(u)} + \frac{\partial^2 f^\alpha}{\partial x^j \partial x^k}\delta x^k \right)\left(\left. g^{ij}\right|_{\vec{x}(u)} + \partial_l g^{ij}\delta x^l \right)$$

Consistency check: if you take $\delta x = 0$, you should get the same vector viz. parallel transport & come back, expect same vector

$$= m^\alpha \frac{\partial f^\alpha}{\partial x^j} g^{ij} + O(\delta x)$$

$$= n^i \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\alpha}{\partial x_j} g^{ij}$$

$$= n^i g_{ij} g^{ij}$$

$$= n^i$$

Claim: $\frac{dn^i}{du} + \Gamma^i_{jk} n^j \frac{dx^i}{du} = 0$; hint: All $\frac{\partial f}{\partial x}$ are related to the metric.

Remark: This matches the notion of parallel transport we had derived without the notion of embedding space.