Lecture 18 - (Oct 26, 2015) Feb 3, 2017

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§1 Overview

Motivation: Measures of information & correlations present in a quantum system.

Topic: Von Neumann Entropy — information qubit

Remark: Def^ns are analogous — departure soon, e.g.,

Topic: Coherent Information — negative relative entropy

Topic: Other quantum information measures, e.g. mutual information

strange ∵ 2 bits of quant. mutual information in a maximally entangled state.

§2 Quantum Entropy

Motivation: Quantum Entropy depends only on ρ ∵ ρ captures both classical uncertainties (mixtures) & quantum uncertainties (uncertainty principle).

Def^n: Quantum Entropy: $H(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A)$

NB: For $\rho_A = \sum_x p_X(x) |x\rangle\langle x|$; $\underbrace{H(\rho_A)}_{\text{von Neumann}} = \underbrace{H(p_X)}_{\text{shannon}}$

Intuition: Alice sends a state $|\psi_y\rangle$ with prob $p_Y(y)$ & sends to Bob. Bobs "expected" operator will be $\sigma = \mathbb{E}_Y\{|\psi_y\rangle\langle\psi_y|\}$. When he recieves the state, his avg. information gain will be $H(\sigma)$ qubits

§2.1 Mathematical Properties of Quantum Entropy

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Property 3 (non-negativity). von Neumann entropy $H(\rho) \geq 0$

proof: $\rho > 0$ & shannon's entropy > 0

Property 4 (Min value). Min. value $= 0$, occurs for pure states (ρ is rank 1).

proof: $\log \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} = \log(1) = 0$

Remark: Quantum uncertainty ≠ sensible when you don't know the state. When you do, measure $|\psi\rangle\langle\psi|$ & $\mathbb{1} - |\psi\rangle\langle\psi|$ to confirm. You gain no information.

Property 5 (Max. Value). Max is $\log d$ where d is dimension of the system.

proof: same as classical.

Prop. 6 (concavity). $H\left(\sum_x p(x)\rho_x\right) \geq \sum_x p(x) H(\rho_x)$

proof delayed.

Prop. 7 (Isometric Invariance). $H(\rho) = H(U\rho U^\dagger)$

proof eigen unchanged.

§3 Joint Quantum Entropy

$$H(AB)_\rho := -\mathrm{tr}(\rho_{AB} \log \rho_{AB}) \quad ; \quad \rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$$

NB: When $\rho_{ABC} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$, then use $\rho_{AB} := \mathrm{tr}_C(\rho_{ABC})$

§3.1 Marginal Entropies of a Pure Bipartite State

NB: Classically $H(X,Y) \geq H(X)$ & $H(X,Y) \geq H(Y)$

$$\left(\begin{array}{lll} \text{proof:} & H(X,Y) & = H(X|Y) + H(Y) \\ \& \ H \geq 0 & & = H(Y|X) + H(X) \end{array}\right)$$

: Quantumly we have

Thm 8: Marginal entropy $H(A)_\phi$ & $H(B)_\phi$ for a pure bipartite state $|\phi\rangle_{AB}$ are equal; $H(A)_\phi = H(B)_\phi$,

whereas the joint entropy $H(AB)_\phi = 0$.

proof: $|\phi\rangle_{AB} = \sum_i \sqrt{\lambda_i}\, |i\rangle_A |i\rangle_B$ (Schmidt decomposition)

where $\{|i\rangle_A\}$ orthonormal in A &

$\{|i\rangle_B\}$ orthonormal in B.

$$\rho_A = \sum_i \lambda_i |i\rangle\langle i|_A \qquad \rho_B = \sum_i \lambda_i |i\rangle\langle i|_B$$

$\Rightarrow H(A)_\rho = H(B)_\rho$ while $H(AB)_\rho = 0$ from prop 4.

□

NB: The Thm applies regardless of how a system is cut. For e.g. $|\phi\rangle_{ABCDE}$ entails

$$H(A)_\phi = H(BCDE)_\phi$$
$$H(AB)_\phi = H(CDE)_\phi$$
$$\vdots$$
$$H(ABCD)_\phi = H(E)_\phi.$$

§ 3.2 Additivity

Prop 9 (Additivity). $H(\rho_A \otimes \sigma_B) = H(\rho_A) + H(\sigma_B)$

Proof: simultaneous diagonaliz^n of ρ_A & σ_B.

§ 3.3 Joint Quantum Entropy of a Classical-Quantum State

Assume: The state is $\rho_{XB} := \sum_x p_X(x)\, |x\rangle\langle x|_X \otimes \rho_B^x$

Thm 10: $H(XB)_\rho = H(X) + \sum_x p_X(x)\, H(\rho_B^x)$

proof: $H(XB)_\rho = -\mathrm{tr}\left(\rho_{XB} \log \rho_{XB}\right)$

$$\log \rho_{XB} = \log\left(\sum_x p(x)\, |x\rangle\langle x| \otimes \rho_B^x\right)$$
$$= \log\left(\sum_x |x\rangle\langle x| \otimes p(x)\, \rho_B^x\right)$$
$$= \sum_x |x\rangle\langle x| \otimes \log\left(p(x)\, \rho_B^x\right)$$

$$\Rightarrow -\mathrm{tr}\left(\rho_{XB} \log \rho_{XB}\right)$$
$$= -\mathrm{tr}\left\{\left[\sum_x p(x)\, |x\rangle\langle x| \otimes \rho_B^x\right]\left[\sum_{x'} |x'\rangle\langle x'| \otimes \log p(x')\, \rho_B^{x'}\right]\right\}$$
$$= -\mathrm{tr}\left\{\sum_x p(x)\, |x\rangle\langle x| \otimes \rho_B^x \log(p(x)\rho_B^x)\right\}$$
$$= -\sum_x p(x)\, \mathrm{tr}\left[\rho_B^x \log\left(p(x)\, \rho_B^x\right)\right]$$

$\dagger$NB: $\log\left(p(x)\rho_B^x\right) = \log(p(x))\,\mathbb{1} + \log(\rho_B^x)$

$$= -\sum_x p(x)\left\{\mathrm{tr}\left[\rho_B^x \log p(x)\right] + \mathrm{tr}\left[\rho_B^x \log \rho_B^x\right]\right\}$$
$$= -\sum_x p(x) \log p(x) + \sum_x p(x)\, \mathrm{tr}\left\{\rho_B^x \log \rho_B^x\right\}$$
$$= H(X) + \sum_x p(x)\, H(\rho_B^x) \qquad \square$$

§ 4 Conditional Quantum Entropy

Defⁿ (Conditional Quantum Entropy). $H(A|B)_\rho := H(AB)_\rho - H(B)_\rho$

Thm 12 (Conditioning doesn't increase entropy). $H(A)_\rho \geq H(A|B)_\rho$

§ 4.1 Conditional Quantum Entropy for Classical-Quantum States

$$\begin{aligned} H(B|X)_\rho &= H(XB)_\rho - H(X)_\rho \\ &= \cancel{H(X)_\rho} + \sum_x p(x) H(\rho_B^x) \cancel{- H(X)_\rho} \\ &= \sum_x p(x) H(\rho_B^x) \end{aligned}$$

§ 4.2 Negative Conditional Quantum Entropy.

Consider: $|\psi\rangle_{AB} = (|00\rangle_{AB} + |11\rangle_{AB}) / \sqrt{2}$

$$\begin{aligned} H(A|B)_\psi &= H(A,B)_\psi - H(B)_\psi \\ &= \underset{\text{(pure state)}}{0} - \underset{\text{(equal weighted mixed state)}}{1} \end{aligned}$$

$H(B)_\psi = \frac{1}{2}\left(-\log_2\left(\frac{1}{2}\right) - \log_2\left(\frac{1}{2}\right)\right) = 1)$

$$= -1$$

Intuitⁿ: We have a more certain descr of the whole state than its parts.

§ 5 Coherent Information

Defⁿ (Coherent Infoⁿ). $I(A\rangle B)_\rho := H(A)_\rho - H(A,B)_\rho$

NB: $I(A\rangle B) = - H(A|B)$;

Justⁿ: Truly an info quantity (see below)

: $I(A\rangle B)$ (a) quantum
(b) Alice to Bob (see below)

Claim: $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$; Consider a purificⁿ $|\psi\rangle_{ABE}$ to some environment E.

We have $I(A\rangle B) = H(B)_\psi - H(E)_\psi$

$(\because H(B)_\psi - H(A,B)_\psi = H(B)_\psi - H(E)_\psi)$

: Remark: Coherent information measures (here) the difference in uncertainty of Bob & that of the environment.

Claim 15: $-H(A|B)_\rho = I(A>B)_\rho = H(A|E)_\psi$ for $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ & for $\mathrm{tr}_E(|\psi\rangle_{ABE}) = \rho_{AB}$.

proof: $-H(A|B) = I(A>B)$ is trivial.

$$I(A>B) = H(B)_\rho - H(AB)_\rho = H(B)_\psi - H(E)_\psi \quad \text{(we used thm 8)}$$
$$= H(AE)_\psi - H(E)_\psi$$
$$= H(A|E)_\psi$$

Thm 16: $|H(A|B)_\rho| \leq \log \dim(\mathcal{H}_A)$ where saturation is achieved for

(a) $\rho_{AB} = \pi_A \otimes \sigma_B$ where π_A is maximally mixed, $\sigma_B \in D(\mathcal{H}_B)$

(b) $\rho_{AB} = \Phi_{AB}$, maximally entangled.

proof: $H(A|B)_\rho \underset{\text{thm 12}}{\leq} H(A)_\rho \leq \log \dim(\mathcal{H}_A)$

For the other case, let $\mathrm{tr}_E(|\psi\rangle_{EAB}) = \rho_{AB}$.

$$H(A|B)_\rho = -H(A|E)_\psi \quad \text{(from the claim above)}$$
$$\geq -H(A)_\rho$$
$$\geq -\log \dim(\mathcal{H}_A)$$

□

Claim 17 (Conditional Coherent Information). $I(A>BC)_\rho = I(A>B|C)_\rho$

where $I(A>B|C)_\rho := H(B|C)_\rho - H(AB|C)_\rho$

proof: $I(A>BC)_\rho = H(BC)_\rho - H(ABC)_\rho$

$$I(A>B|C)_\rho = H(B|C)_\rho - H(AB|C)_\rho$$
$$= H(BC)_\rho - \cancel{H(C)_\rho} - H(ABC)_\rho + \cancel{H(C)_\rho}$$

□

Claim 18 (Conditional Coherent Information of a Classical Quantum State). For $\sigma_{XAB} = \sum_x p_X(x)\, |x\rangle\langle x| \otimes \rho^x_{AB}$

$$I(A>BX)_\sigma = \sum_x p_X(x)\, I(A>B)_{\sigma^x_{AB}}$$

proof: <skipped for now>

§ 6 Quantum Mutual Information

Def^n (Quantum Mutual Inform^n). $I(A;B)_\rho := H(A)_\rho + H(B)_\rho - H(AB)_\rho$
$$= H(A)_\rho - H(A|B)_\rho$$
$$= H(B)_\rho - H(B|A)_\rho$$

NB: $I(A;B) = H(A) + I(A \rangle B)$
$$= H(B) + I(B \rangle A)$$

Thm 20 (Non-Negativity of Quantum Mutual Information).
$$I(A;B)_\rho \geq 0.$$
proof: !! not provided.

NB: Thm 12 gets proved by Thm 20.

Claim 22 (Bound on Quantum Mutual Info^n).
$$I(A;B)_\rho \leq 2\log\left[\min\{\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)\}\right]$$
proof: Use the bound on conditional entropy.

§ 7 Conditional Quantum Mutual Info^n (CQMI)

Def^n (Conditional Quantum Mutual Info^n):
$$I(A;B|C)_\rho := H(A|C)_\rho + H(B|C)_\rho - H(AB|C)_\rho$$

Prop 23 (Chain Rule for Quantum Mutual Information).
$$I(A;BC)_\rho = I(A;B)_\rho + I(A;C|B)_\rho$$

proof: LHS $= I(A;BC)_\rho = -H(ABC) + H(A) + H(BC)$

RHS $= I(A;B) + I(A;C|B) = H(A) + H(B) - H(AB)$
$$+ H(A|B) + H(C|B) - H(AC|B)$$
$$= H(A) \cancel{+ H(B)} - \cancel{H(AB)}$$
$$+ \cancel{H(AB)} - \cancel{H(B)} + H(CB) - \cancel{H(B)}$$
$$+ \cancel{H(B)} - H(ABC)$$
$$= H(A) + H(CB) - H(ABC)$$
$\square$

Intuⁿ: First make correlation with B, then with C given B is known.

Claim: $I(A;BC)_\rho = I(AC;B) + I(A;C) - I(B;C)$

proof: $I(A;BC) - I(B;AC) = I(A;C) - I(B;C)$

$$\begin{aligned} \text{LHS} &= I(A;CB) - I(B;CA) \\ &= I(A;C) + \cancel{I(A;B|C)} \\ &\quad - I(B;C) - \cancel{I(B;A|C)} \\ &= \text{RHS}. \end{aligned}$$

□

§ 7.1 Non-negativity of CQMI

Thm 25 (Non-negativity of CQMI). $I(A;B|C)_\rho \geq 0$

Remark: This is a foundational result (bedrock of quantum info theory).

Claim 26 (CQMI of Classical-Quantum States).

$$I(A;B|X)_{\sigma_{ABX}} = \sum_x P_x(x) I(A;B)_{\sigma^x_{AB}}$$

proof: <later>

NB: Non-negativity in this case is trivial

Claim 27 (Conditioning doesn't increase entropy) $H(B|C) \geq H(B|AC)$

NB: Stronger than thm 12

proof: $I(A;B|C) \geq 0$

$\Rightarrow H(A|C) - H(A|BC) \geq 0$ □.

Exercise: $\rho_A = \sum_x P_X(x)\ |x\rangle\langle x|_A$

ROUGH

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$$\log(\rho_A) = \sum_x \log P_X(x)\ |x\rangle\langle x|_A$$

$$\mathrm{tr}\ \rho_A \cdot \log(\rho_A) = \sum_x P_X(x) \log(x)$$

$$H(X,Y) = -\sum_{x,y} P(x,y) \log\left[P(x,y)\right]$$

$$H(X) = -\sum_x P(x) \log P(x)$$

$$H(X,Y) = -\sum_{x,y} P(x)\ P(y|X=x)\ \log\left(P(x)\, P(\cancel{x}\, y|X=x)\right)$$

$$\cancel{P(x)}$$

$$= -\sum P(x)\ P(y|X=x)\ \log(P(x))$$

$$-\sum P(x)\ P(y|X=x)\ \log(P(y|X=x))$$

$$\cancel{\geq -\sum P(x)}$$

$$\geq -\sum P(x) \log(P(x))$$

$\rho_{AB} \in D(H_A \otimes H_B)$

$$H(A|B) := H(AB) - H(B)$$

$$H(\cancel{x} X|Y) = H(XY) - H(Y)$$

$$= P(x,y) \log P(x,y)$$

$$- p(y) \log P(y)$$

$$< H(XY) - H(Y) ?$$

$x f(a) + (1-x) f(b)$

$$P(x|y) \log P(x|y)$$

$$\frac{P(x,y)}{P(y)} \log\left(\frac{P(x,y)}{P(y)}\right)$$

$$\frac{P(x,y)}{P(y)} \log P(x,y) - \frac{P(x,y)}{P(y)} \log P(y)$$

$$\frac{P(x,y)}{P(y)} \log$$

$$H(X|Y) = \langle P(X|Y) \rangle$$

$$\sum P(x|y) \log(P(x|y)) \times$$

or

$$\sum P(x,y) \log(P(x|y)) \checkmark$$

I'm trying to prove: $H(X) \geq H(X|Y)$

$H(x) = \langle x \log x \rangle$

ROUGH

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$$p(X|Y) = p(x,y) - p(y)$$

$$\langle H(X) \rangle = \langle H(X,Y) \rangle - \langle H(Y) \rangle$$

$$-\sum_{x,y} p(x,y) \log p(x,y) = \sum_{x,y} p(x|y)\, p(y) \log p(x,y)$$

$$\geq \sum_{x,y} p(x|y) \log p(x,y)$$

$$\langle \log p(x|y) \rangle \equiv \sum p(x,y) \log \underbrace{p(x|y)}$$

&

$$= -\sum p(x,y) \log \left(\frac{p(x,y)}{p(y)} \right)$$

$$H(x) = \sum_{y,x} p(x,y) \left[-\log p(x) \right]$$

$$H(X|Y) = \sum_y p(y)\, H(X|Y=y)$$

$$- \log \Big(\underbrace{\sum_y p(x,y)}_{p(x)} \Big) > -\log \frac{p(x,y)}{p(y)}$$

$$\Rightarrow \langle -\log p(x) \rangle > \langle -\log p(x|y) \rangle$$

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$\cancel{H(X|Y)}$ $H(Y|X) + H(X) \geq H(X)$

Prove: $H(X,Y) \geq H(X)$

$\cancel{H}$ $\langle \log p_{X,Y}(x,y) \rangle$

$p_{X,Y}(x,y) \qquad p_X(x)\, p_Y(y)$

$\log \cancel{p_{X,Y}}($

$\langle \log [p(x|y)\, p(y)] \rangle$

$\log$

$$H(X) - H(X|Y) \geq 0$$

$$H(X) - \big(H(XY) - H(Y)\big) \geq 0$$

$$H(X) + H(Y) \geq H(X,Y)$$