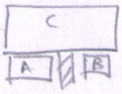
THERMODYNAMICS | KARDAR

§1.2 The zeroth law

If two systems are in equilibrium separately with a third system, then they are ~~as~~ at equilibrium with each other. (Transitivity)



$f_{AC}(\underline{A}, \underline{C}) = 0$

$f_{BC}(\underline{B}, \underline{C}) = 0$

We may restate this as

$c_1 = F_{AC}(\underline{A}, c_2 \dots)$

$c_1 = F_{BC}(\underline{B}, c_2 \dots)$

$\Rightarrow F_{AC}(\underline{A}, c_2 \dots) = F_{BC}(\underline{B}, c_2 \dots)$ — (1)

Now invoking the zeroth law

$f_{AB}(\underline{A}, \underline{B}) = 0$

(1) holds for the surface constrained by $f_{AB} = 0$.

$\Rightarrow F_{AC} = F_{AC}(\underline{A}) = F_{BC}(\underline{B})$

Renaming $\theta_{AC}(\underline{A}) = \theta_{BC}(\underline{B})$

$\uparrow$

These specify equation of state

NOTE: The zeroth law severely restricts the types of f's (of constraint eq^n)

Temperature Scale (K):

→ Emperically PV is const for isotherms of dilute gasses

$$T(K) \equiv 273.16 \times \frac{\lim_{P\to 0} (PV)_{sys}}{\lim_{P \to 0} (PV)_{ice\text{-}water}}$$

§1.3 The first law

For adiabatic: $\Delta W = E(X_f) - E(X_i)$

Heat intake: $\Delta Q = \Delta E - \Delta W$

$đQ = dE - đW$

NOTE: Signs of heat & work are s.t. energy is added to the system.

Quasi Static: Sufficiently slow, s.t. all steps are at equilibrium.

$đW = \sum_i J_i \, dx_i$ ← valid only for quasi-static (J_i - Gen. force, dx_i - Gen. displacement)

NOTE: P is calculated from the force exerted on the walls, by the system. Think. Thus we have a minus sign.

Extensive Quantities: proportional to system size (Gen. forces)

Intensive Quantities: independent of size (Gen. displacements) (eq. relation)

→ Temp ↔ (displacement!) (like Gen. Force)

Response f^n.

Heat Capacities: $C_P > C_V$ (physical arg.)

$$C_V = \left.\frac{đQ}{dT}\right|_V = \left.\frac{\partial E}{\partial T}\right|_V \; ; \; C_P = \left.\frac{đQ}{dT}\right|_P = \left.\frac{\partial E}{\partial T}\right|_P + P\left.\frac{\partial V}{\partial T}\right|_P$$

Force Constants: (infinitesimal) ratio of displacement to force.

Isothermal compressibility: $K_T = -\dfrac{\left.\frac{\partial V}{\partial P}\right|_T}{V}$

Susceptibility of a magnet: $\chi_T = \dfrac{\left.\frac{\partial M}{\partial B}\right|_T}{V}$ (comes from extensivity)

Thermal Responses: Change of thermodynamic coordinates with temperature.

$\alpha_P = \dfrac{\left.\frac{\partial V}{\partial T}\right|_P}{V}$ $\left(= \frac{1}{T} \text{ for ideal gasses}\right)$

For ideal gas:

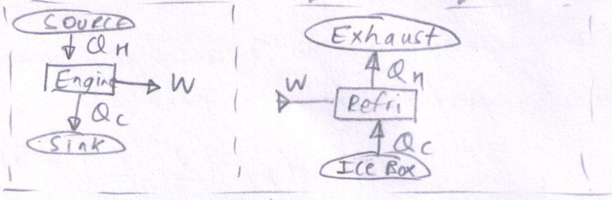
$C_P - C_V = P\left.\frac{\partial V}{\partial T}\right|_P$

$= PV\alpha_P$

$= \frac{PV}{T} = Nk_B$

§1.4 The second law

Remark: Considering efficiencies of heat engines leads to the concept of entropy.



Heat Engine Efficiency: $\eta = \dfrac{W}{Q_H} = \dfrac{Q_H - Q_C}{Q_H} \leq 1$

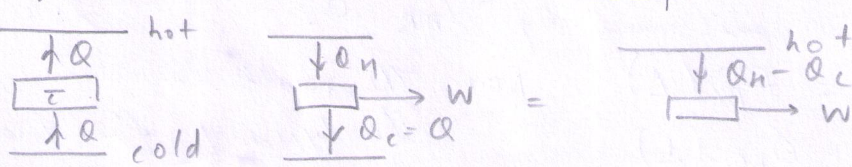
'Figure of Merit' of refrigerator: $w = \dfrac{Q_C}{W} = \dfrac{Q_C}{Q_H - Q_C}$

Kelvin's Statement: No process is possible whose sole result is the complete conversion of heat into work. ($Q_C = 0$ in Engine)

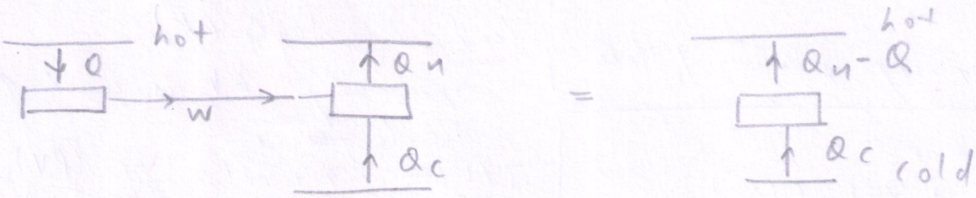
Clausius's Statement: No process is possible whose sole result is the transfer of heat from a colder to a hotter body. ($W = 0$ in refri)

∴ Equivalence of Kelvin's & Clausius's Statements

(a)

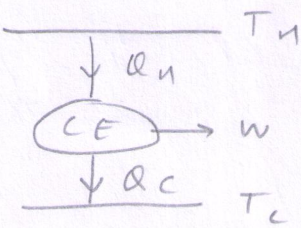


(b)



§1.5 Carnot Engines

A carnot engine: One that is ① reversible, ② runs in a cycle, ③ with heat exchanges only at a source T_H & a sink T_C.



Reversible: One that can run backwards by simply reversing inputs & outputs.

Remark: It's the thermodynamic equivalent of frictionless motion.

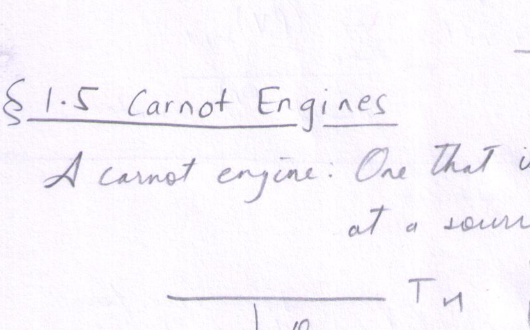
Remark: Since heat is not a f^n of state, we don't know how to construct adiabats in general.

Internal working substance: ideal gas

$$E = \frac{3}{2} N k_B T = \frac{3}{2} PV$$

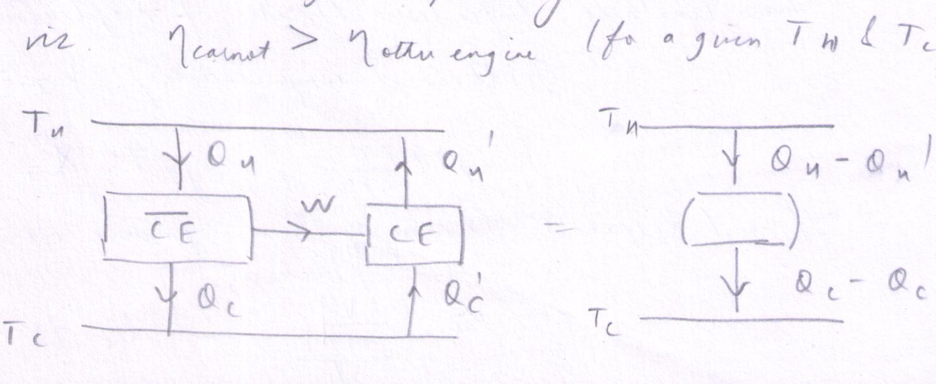
$đQ = dE - đW$ for $đQ = 0$ & E as said

$$\frac{dP}{P} + \frac{5}{3}\frac{dV}{V} = 0 \Rightarrow PV^\gamma = \text{const where } \gamma = 5/3$$



Carnot's Theorem: No engine operating b/w two reservoirs (at temperatures T_H & T_C) is more efficient than a Carnot engine operating b/w them

viz $\eta_{carnot} > \eta_{other\ engine}$ (for a given T_H & T_C)

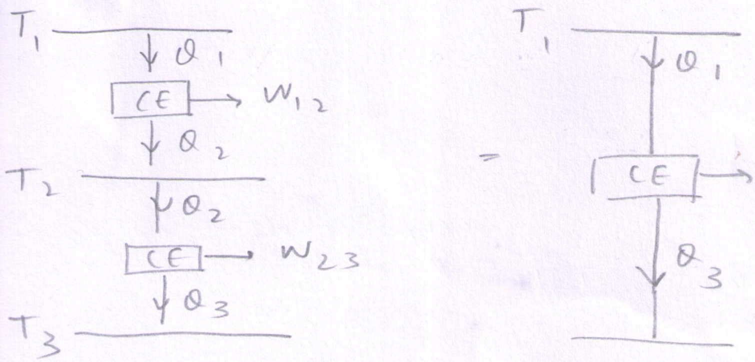


Thus using the 2nd law

$Q_H - Q_H' \geq 0$,

$\Rightarrow Q_H \geq Q_H'$

Remark: We've shown that max. efficiency is only dependent on the two temperatures; we thus use it to construct a temperature scale.



Objective: Find the form of $\eta(T_1, T_2)$

$$Q_2 = Q_1 - W_{12} = Q_1[1-\eta(T_1T_2)]$$

$$Q_3 = Q_2 - W_{23} = Q_2[1-\eta(T_2T_3)]$$

$$= Q_1[1-\eta(T_1T_2)][1-\eta(T_2T_3)]$$ using

$$Q_3 = Q_1 - W_{13} = Q_1[1-\eta(T_1T_3)]$$

$$\Rightarrow \quad [1-\eta(T_1T_3)] = [1-\eta(T_1,T_2)][1-\eta(T_2T_3)]$$ using

? (convention)

$\Rightarrow$ $1-\eta(T_1,T_2)$ is of the form $\dfrac{f(T_2)}{f(T_1)} = \dfrac{T_2}{T_1}$

$$\Rightarrow \quad 1-\eta(T_1,T_2) = \frac{Q_2}{Q_1} \equiv \frac{T_2}{T_1}$$

$$\Rightarrow \quad \eta(T_H, T_C) = \frac{T_H - T_C}{T_H}$$

(Woah! Efficiency for defining temp upto a prop. const.)

Claim: θ & T can be proved equivalent

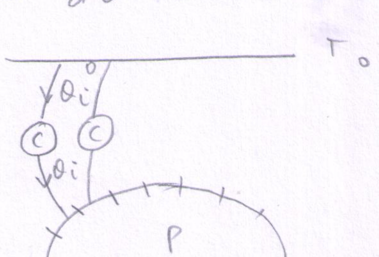
θ: ideal gas temp

T: efficiency based def^n of temp.

Note: ~~Thus~~ $T > 0$ else Kelvin's statement is violated.

§ 1.6 Entropy

Clausius's Theorem: For any cyclic transformation (reversible or not), $\oint \frac{đQ}{T} \leq 0$, where $đQ$ is the heat supplied to the system at temperature T.



For the whole system:

$$W_{tot} = W + \sum (Q_i^0 - Q_i)$$

$$= \sum Q_i^0 = \sum \frac{Q_i}{T_i} T_0$$

also $\dfrac{Q_i^0}{Q_i} = \dfrac{T_0}{T_i}$

$$= T_0 \sum \frac{Q_i}{T_i}$$

For the cyclic process

$$W = \sum Q_i$$

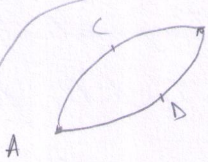
Now from 2nd law

$$W_{tot} = \sum Q_i^0 = Q_{tot} \leq 0$$

$$\Rightarrow T_0 \sum \frac{Q_i}{T_i} \leq 0 \Rightarrow \oint \frac{đQ}{T} \leq 0$$

For reversible

1.) $\oint \frac{đQ_{rev}}{T} = 0$ (you reverse the prev calc.) → and you again get



$\oint \frac{đQ'}{T} \le 0$; now since $đQ' = -dQ$ (you reverse) thus $\oint \frac{đQ}{T} \geq 0$.)

$$\Rightarrow \int_{ACB} \frac{đQ_{rev}}{T} = \int_{ADB} \frac{đQ_{rev}}{T}$$

2.) $= S(B) - S(A) = \int_A^B \frac{đQ_{rev}}{T}$

defined upto an arbitrary const.

$đQ_{rev} = T\,dS$ ← using this $dS = 0$, we find adiabats.

Thus

3. $dE = T\,dS + \sum_j J_j\,dx_j$

(true even when non-reversible!) <not proven>

Remark: after 1.24 is subtle in the text.

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4. # independent variables are $n+1$ if $\exists$ n conjugate pairs (for doing work)

Eg. $(E, \{x_i\})$ Then $\left.\frac{\partial S}{\partial E}\right|_x = \frac{1}{T}$, $\left.\frac{\partial S}{\partial x_i}\right|_{E, x_{j\neq i}} = \frac{-J_i}{T}$

5. $\int_A^B \frac{đQ}{T} + \int_B^A \frac{đQ_{rev}}{T} \le 0 \Rightarrow \int_A^B \frac{đQ}{T} \le S(B) - S(A)$

$$\Rightarrow \boxed{dS \geq \frac{đQ}{T}}$$

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§ 1.7 Approaches to equilibrium and thermodynamic Potentials

Enthalpy $đQ = 0$, eq. to const. external force (J)

$đW \le J \cdot \delta x$ ($J = J_i$ for quasi static, in general there'll be a dissipation loss)

work input to the system ↑ external force ↑ work done by 'us'

Now since $đQ = 0$, we have

$\delta E = đW \le J \cdot \delta x$

$\Rightarrow \delta H \le 0$ where $H = E - J \cdot x$

At equilibrium then, <TODO: fill>

Helmholtz free energy

Isothermal in absence of mechanical work

$T = $ const. $đW = 0$

$T\delta S \geq đQ \Rightarrow \delta E = đQ + đW \le T\delta S$

$\Rightarrow \delta F \le 0$ where $F = E - TS$

At eq. <TODO: fill>

Gibbs free energy

(const T) Isothermal with mech work at const external force (const J)

$$đW \le J\cdot \delta x \quad \& \quad đQ \le T\delta S$$

$$\delta E \le T\cdot\delta S + J\cdot \delta x$$

$$\Rightarrow \quad \delta G \le 0 \quad \text{where} \quad G = E - TS - J\cdot x$$

Grand Potential

$$\mathcal{G} = E - TS - \mu\cdot N$$

$\mu = \{\mu_1, \mu_2 \ldots\}$ chemical pot.
$N = \{N_1, N_2 \ldots\}$ # particles of each species

$$d\mathcal{G} = -SdT + J\cdot dx - N\cdot d\mu$$

$đW = \mu . dN$ specifies change in internal energy upon the addition of 'molecular' species

§1.8 Useful mathematical results

(1) Extensivity

$$dE = TdS + J\cdot dx + \mu\cdot dN$$

For fixed intensive co-ordinates, the extensive coordinates are prop to sys size (or particle #)

i.e. $$E(\lambda S, \lambda \underline{x}, \lambda \underline{N}) = \lambda E(S, \underline{x}, \underline{N})$$

Evaluating the derivatives w.r.t. λ we have & putting $\lambda = 1$

$$\left.\frac{\partial E}{\partial S}\right|_{x,N} S + \sum \left.\frac{\partial E}{\partial x_i}\right|_{S, x_{j\neq i}, N} x_i + \sum \left.\frac{\partial E}{\partial N_\alpha}\right|_{S, x, N_{\beta\neq\alpha}} N_\alpha = E$$

Side note

let $\lambda S = x(\lambda)$
$\lambda \underline{x} = y(\lambda)$
$\lambda \underline{N} = z(\lambda)$

then

$$\frac{dE}{d\lambda} = \frac{\partial E}{\partial x}\frac{\partial x}{\partial \lambda} + \frac{\partial E}{\partial y}\frac{\partial y}{\partial \lambda} + \frac{\partial E}{\partial z}\frac{\partial z}{\partial \lambda}$$

$$= \frac{\partial E}{\partial(\lambda S)}\frac{\partial \lambda S}{\partial \lambda}^{S} + \frac{\partial E}{\partial(\lambda \underline{x})}\underline{x} + \frac{\partial E}{\partial(\lambda \underline{N})}\underline{N}$$

The partials from are T, J_i & μ_α respectively

thus $$E = TS + J\cdot x + \mu\cdot N$$

differentiate & use to get

$$SdT + x\cdot dJ + N\cdot d\mu = 0$$

called the Gibbs-Duhem relation

Its sometimes called the "fundamental eq^n of thermodynamics" (holds only when extensivity holds)

but is more fundamental.

Kardar page 37. Feb 3, 2014

$$\langle F(x)\rangle = \int_{-\infty}^{\infty} p(x) F(x)\, dx$$

Claim $F(x)$ itself is a random variable with PDF

$$p_F(f)\, df = \text{prob}\left(F(x) \in [f, f+df]\right)$$

$F(x) = f$ may have multiple solns. let the solns be x_i

$$p_F(f)\, df = \sum_i p(x_i)\, dx_i \Rightarrow p_F(f) = \sum_i p(x_i) \frac{dx_i}{df} = \sum_i p(x_i) \frac{dx}{dF}\Big|_{x=x_i} \Big]?$$

$\left|\frac{dx}{dF}\right|$ are the Jacobians associated with change of variables from x to f.

Eg. $p(x) = \lambda e^{-\frac{\lambda|x|}{2}}$

$F(x) = x^2$ $\quad f(x) = f$ has 2 solns $x_\pm = \pm\sqrt{f}$

with the Jacobian $\left|\pm \frac{f^{-1/2}}{2}\right|$

So $p_F(f) = \frac{\lambda}{2} e^{-\lambda\sqrt{f}}\left(\left|\frac{1}{2\sqrt{f}}\right| + \left|-\frac{1}{2\sqrt{f}}\right|\right) = \frac{\lambda e^{-\lambda\sqrt{f}}}{2\sqrt{f}}$

for $f>0$ & $p_F(f) = 0$ for $f<0$.

Def^n ★ The n^{th} moment is

Def^n $\quad m_n = \langle x^n\rangle = \int dx\, p(x) x^n$

★ The characteristic f^n is the generator of moments of the distribution

It's the fourier transform of the PDF

$$\tilde{p}(k) = \langle e^{-ikx}\rangle = \int dx\, p(x) e^{-ikx}$$

The PDF can be recovered as

$$p(x) = \frac{1}{2\pi}\int dk\, \tilde{p}(k) e^{ikx} \quad \Big| \quad = \frac{1}{2\pi}\iint dk\, dx\, p(x) e^{-ikx} e^{ikx}$$

$$= \frac{1}{2\pi}\int dk\, e^{ikx}\int dx'\, p(x') e^{-ikx'}$$

★ Moments of the dist are obtained by expanding $\tilde{p}(k)$ in powers of k.

$$\tilde{p}(k) = \left\langle \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} x^n \right\rangle = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!}\langle x^n\rangle$$

Also,

$$e^{ikx_0}\tilde{p}(k) = \langle e^{-ik(x-x_0)}\rangle = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \langle (x-x_0)^n\rangle$$

★[24] The cumulant generating $f^n \doteq \ln(\text{characteristic } f^n)$ & its expansion generates cumulants.

$$\ln(\tilde{p}(k)) \equiv \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} \langle x^n\rangle_c \quad \Big| \text{def}^n$$

$f(x) = f(0) + f'(x)\big|_0 x + \frac{f''(x)}{2!}\big|_0$

$\ln x = \frac{1}{x}\Big|_0$ oops.

$\ln(1) = x$

$$\ln\left(1 + \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!}\langle x^n\rangle\right)$$

$$= \sum_{m=1}^{\infty} (-1)^{m+1} \left[\sum_{n=1}^{\infty} \frac{(-ik)^n}{n!}\langle x^n\rangle\right]^m$$

= (from Def^n of $\ln(\tilde{p}(k))$)

$$\sum_{n=1}^{\infty} \frac{(-ik)^n}{n!}\langle x^n\rangle_c$$

power of k

~~Not clear on what is big compared~~

$\langle x\rangle_c = \langle x\rangle$

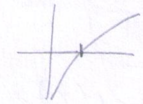
$\langle x^2\rangle_c = \langle x^2\rangle - \langle x\rangle^2 \ldots$

$$\ln(1+x) = \ln(1) + \frac{1}{1+x}\Big|_{x=0} x + \frac{-1}{(1+x)^2}\Big|_{x=0} \frac{x^2}{2!} + \ldots$$

$e^x = y$

$\ln e^x = x = \ln y$

$e^x = 1 \quad |x=0$



$$\frac{d}{dx}(1+x)^{-1} = \frac{-1}{(1+x)^2}\cdot 1$$

$$\frac{d}{dx}(1+x)^{-2} = \frac{-2}{(1+x)^3}\cdot 1$$

$$\ln(1+\epsilon) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{\epsilon^n}{n} \quad ✓$$

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§ 2.3 Some Important prob. distributions

(1) Normal

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\left[e^{-\frac{(x-\lambda)^2}{2\sigma^2}}\right]$$

$$\tilde{p}(k) = \int dx \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left[-\frac{(x-\lambda)^2}{2\sigma^2} - ikx\right]}$$

$$\overset{?}{=} e^{-ik\lambda - \frac{k^2\sigma^2}{2}}$$

$\int A e^{-ikx}\,dx$

$\int e^{-ikx}\,dx =$

$$\ln\tilde{p}(k) = -ik\lambda - \frac{k^2\sigma^2}{2}$$

$$= \sum_{n=1}^{\infty} \frac{(ik)^n}{n!}\langle x^n\rangle_c$$

$\Rightarrow \langle x\rangle_c = \lambda \quad \langle x^2\rangle_c = \sigma^2 \quad | \quad \langle x^3\rangle_c = \langle x^4\rangle_c = \ldots = 0$

Binomial 2

$$p_N(N_A) = \binom{N}{N_A} p_A^{N_A} p_B^{N-N_A}$$

$$\tilde{p}_N(k) = \langle e^{-ikN_A}\rangle = \sum_{N_A=0}^{N} \frac{N!}{N_A!(N-N_A)!} p_A^N p_B^{N-N_A} e^{-ikN_A}$$

$$= (p_A e^{-ik} + p_B)^N$$

$$\ln \tilde{p}_N(k) = N \ln(p_A e^{-ik} + p_B) = N \ln(\tilde{p}_1(k))$$

From the def^n of expectation &

$N_A = 0$ with prob p_B &

$N_A = 1$ with prob p_A, we have

$$\langle N_A^l \rangle = p_A \quad \forall\, l > 0$$

For only 1 trial

$$\tilde{p}(k) = \int e^{-ik\cancel{x}N_A} dx\, p_N(N_A)$$

$$\tilde{p}(k) = \langle e^{-ikN_A} \rangle$$

$$\ln \tilde{p}(k) = \langle \ln e^{-ikN_A} \rangle$$

$$= -ik \langle N_A \rangle$$

$$= -ik\, p_A$$

MAY/SHOULD BE WRONG

In general then

$$\langle N_A \rangle_c = N p_A$$

$$\langle N_A^2 \rangle_c = N(p_A - p_A^2)$$

$$\langle N_A^2 \rangle_c = \langle N_A^2 \rangle - \langle N_A \rangle^2$$

$$\ln \tilde{p}(k) = \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} \langle x^n \rangle_c$$

$$\ln \tilde{p}_N(k) = \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} \langle (N_A^n) \rangle_c = N \ln(\tilde{p}_1 k)$$

$$= -ik\, p_A N$$

$$\langle N_A^1 \rangle_c = N p_A$$

$$\langle N_A^2 \rangle_c = 0$$

$$p_N(N_A) = \binom{N}{N_A} p_A^{N_A} p_B^{1-N_A}$$

$$\tilde{p}_N(k) = \langle e^{-ikN_A} \rangle = \sum_{N_A=0}^{N} \frac{N!}{N_A!\,(N-N_A)!} p_A^N p_B^{N-N_A} e^{-ikN_A}$$

$$= (p_A e^{-ik} + p_B)^N$$

$$\ln \tilde{p}_N(k) = N(p_A e^{-ik} + p_B) = N \tilde{p}_1(k)$$

$$\cancel{\ast}\langle \ln \tilde{p}_N(k) \rangle = N \langle \tilde{p}_1(k) \rangle = N \langle (p_A e^{-ik} + p_B) \rangle$$ ~~Not~~ helping

$$\tilde{p}_1(k) = \langle e^{-iN_A k} \rangle = p_A e^{-ik} + p_B$$

$$p_1(N_A) = p_A$$

$\ln \tilde{P}_N(K) = N \tilde{P}_1(K)$

This shows that we need to calc. everything only for 1 trial & the cummulants will become N times that of the 1 trial case.

Now for 1 trial

$$\langle N_A \rangle = 0 \cdot P_B + 1 \cdot P_A = P_A$$

$$\langle N_A^2 \rangle = 0 \cdot P_B + 1^2 \cdot P_A = P_A$$

$$\vdots$$

$$\langle N_A^l \rangle = P_A$$

Thus, for 1 trial

$$\langle N_A \rangle_c = \langle N_A \rangle = P_A$$

$$\langle N_A^2 \rangle_c = \langle N_A^2 \rangle - (\langle N_A \rangle)^2 = P_A - P_A^2$$

Thus for N trials

$$\langle N_A \rangle_c = N P_A$$

$$\langle N_A^2 \rangle_c = N(P_A - P_A^2)$$

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(3) The Poisson Distribution

(a) Prob of one & only one event (decay) in the interval ~~[t, t+dt~~ $[t, t+dt]$ is proportional to dt as $dt \to 0$.

(b) The prob. of events at different intervals are independent of each other.

The prob. of observing exactly M decays in the interval T is given by the Poisson distribution.

It is obtained by dividing the interval into $N = T/dt \gg 1$, as a limit of the binomial distribution.

For each segment, an event occurs with the prob. p ~~A~~ = ~~T/dt~~ αdt & no event with $q = 1 - \alpha dt$.

$$\tilde{p}(K) = (p e^{-iK} + q)^N = \lim_{dt \to 0} [\alpha dt\, e^{-iK} + 1 - \alpha dt]^N$$

$$= \lim_{dt \to 0} [(e^{-iK} - 1)\alpha dt + 1]^{T/dt}$$

$$= e^{\alpha(e^{-iK}-1)T}$$

The poisson PDF is obtained as

$$p(x) = \int \frac{dk}{2\pi} e^{\alpha(e^{-ik}-1)T + ikx}$$

$$= e^{-\alpha T}\int \frac{dk}{2\pi} e^{ikx} e^{\alpha T e^{-ik}}$$

$$= e^{-\alpha T}\int \frac{dk}{2\pi} e^{ikx} \sum_{M=0}^{\infty} \frac{(\alpha T)^M}{M!} e^{-ikM}$$

$$\int \frac{dk}{2\pi} e^{ik(x-M)} = \delta(x-M)$$

$$\Rightarrow \quad p_{\alpha T}(x) = \sum_{M=0}^{\infty} e^{-\alpha T} \frac{(\alpha T)^M}{M!} \delta(x-M)$$

$\Rightarrow$ x are integers only. Thus for $x =$ some int say M

$$p_{\alpha T}(M) = e^{-\alpha T}\frac{(\alpha T)^M}{M!}$$

<u>Cumulants</u> are obtained

$$\ln \tilde{p}_{\alpha T}(k) = \alpha T(e^{-ik}-1) = \alpha T \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!}$$

$$\Rightarrow \quad \langle M^n \rangle_c = \alpha T$$

Many Random Variables | § 2.4

- The joint PDF $p(\vec{x})$ is prob. density of an outcome in the vol. element $d^N\vec{x} = \prod_{i=1}^{N} dx_i$ around $\vec{x} = \{x_1, x_2 \ldots x_N\}$
- Normalized s.t

$$p_{\vec{x}}(S) = \int d^N\vec{x}\, p(\vec{x}) = 1$$

Iff events are independent,

$$p(\vec{x}) = \prod_{i=1}^{N} p_i(x_i)$$

→ Unconditional PDF : describes subset of random variables

$$p(x_1 \dots x_m) = \int \prod_{i=m+1}^{N} dx_i \; p(x_1 \dots x_N)$$

→ Conditional PDF :

$$p(x_1 \dots x_m | x_{m+1} \dots x_N) = \frac{p(x_1 \dots x_N)}{p(x_{m+1} \dots x_N)}$$

$$p(\vec{v}|\vec{x}) = \frac{p(\vec{x},\vec{v})}{p(\vec{x})}$$

If $\vec{v}$ & $\vec{x}$ are independent then $p(\vec{v}|\vec{x})$ should be $p(\vec{v})$

$\therefore \; p(\vec{x},\vec{v}) = p(\vec{x})\,p(\vec{v})$

then $p(\vec{v}|\vec{x}) = \frac{\cancel{p(\vec{x})}\,p(\vec{v})}{\cancel{p(\vec{x})}} = p(\vec{v})$

→ expectation

$$\langle F(\vec{x})\rangle = \int d^N\vec{x}\; p(\vec{x})F(\vec{x})$$

→ Joint characteristic f^n

$$\tilde{p}(\vec{k}) = \left\langle \exp\left(-i\sum_{j=1}^{N} k_j x_j\right)\right\rangle$$

<Missing stuff>

§2.5 Sums of random variables & the central limit Theorem.

$$X = \sum_{i=1}^{N} x_i$$

$$p_X(\vec{x}) = \int d^N\vec{x}\; p(\vec{x})\,\delta\left(X - \sum x_i\right)$$

$$= \int d\prod_{i=1}^{N-1} p(x_1, x_2 \dots, X - x_1 - x_2 \dots x_{N-1})$$

$$\tilde{p}_X(k) = \left\langle e^{-ik\sum x_j}\right\rangle$$

$$= \tilde{p}(k_1 = k_2 \dots = k_N = k)$$

$$\ln \tilde{p} = -ik \sum_{i_1=1}^{N} \langle x_{i_1}\rangle_c + \frac{(-ik)^2}{2}\sum_{i_1 i_2} \langle x_{i_1} x_{i_2}\rangle_c$$

$$+ \dots$$

as

$$\langle X\rangle_c = \left\langle \sum x_i \right\rangle = \sum \langle x_i\rangle$$

$$\langle X^2\rangle_c = \left\langle \left(\sum x_i\right)^2\right\rangle = \sum_{ij} \langle x_i x_j\rangle$$

If the random variables are ~~random,~~ independent, then

$$p(\vec{x}) = \Pi\, p_i(x_i)$$
$$\tilde{p}_x(k) = \Pi\, \tilde{p}_i(k)$$

The cross terms in $\ln \tilde{p}$'s expansion vanish. thus

$$\langle x^n \rangle_c = \sum_{i=1}^{N} \langle x_i^n \rangle_c$$