

$\vec{x}_{(0)}$ vector in target space at $\vec{x}_{(0)}$

aim: see what this vector becomes upon parallel transport. $O(\delta u^2)$

$n^i(u) \rightarrow$ known (say)

$$\delta n^i = O(\delta u^2)$$

$n^i(u+\delta u) \rightarrow$ calculate

$$\frac{\delta n^i}{\delta u} = 0$$

goto $\vec{x}(u)$ & find a new coordinate sys \vec{x}' s.t. $\partial_i g_{jk}' = 0$ at $\vec{x}' = \vec{x}'_{(0)}$

$n'^i(u+\delta u) = n^i(u) + O(\delta u^2)$ (by rules of parallel transport)

since you don't want to keep changing coordinates (its inconvenient), we see what parallel transport amounts in some arbitrary frame.

Translate to \vec{x}'

$$n^i(u) = \left. \frac{\partial x^i}{\partial x'^k} \right|_{\vec{x}'(u)} n'^k(u)$$

$$n'^k(u) + O(\delta u^2)$$

$$n^i(u+\delta u) = \left. \frac{\partial x^i}{\partial x'^k} \right|_{\vec{x}'(u+\delta u)} n'^k(u+\delta u)$$

$$\left. \frac{\partial x^i}{\partial x'^k} \right|_{\vec{x}'(u)} + \frac{\partial^2 x^i}{\partial x'^l \partial x'^k} \cdot \delta x'^l + \dots \quad \& \quad \delta x'^l = \frac{\partial x'^l}{\partial u} \cdot \delta u$$

$$\Rightarrow n^i(u+\delta u) = n^i(u) + \frac{\partial^2 x^i}{\partial x'^l \partial x'^k} \frac{\partial x'^l}{\partial u} \delta u \cdot n'^k(u) + O(\delta u^2)$$

$$\frac{dn^i}{du} = \frac{\partial^2 x^i}{\partial x'^l \partial x'^k} \frac{\partial x'^l}{\partial u} n'^k(u)$$

We still have dependence on the primed coordinate. We wish to get rid of it.

$$\Gamma'_{mn}^l(\vec{x}') = \frac{\partial x^l}{\partial x^i} \frac{\partial x^j}{\partial x'^m} \frac{\partial x^k}{\partial x'^n} \Gamma_{j,k}^i(\vec{x}) + \frac{\partial x'^l}{\partial x^k} \frac{\partial^2 x^k}{\partial x'^m \partial x'^n} = 0 \quad (\text{for the primed frame})$$

$$\frac{\partial x^P}{\partial x'^l} \Gamma'_{mn}^l(\vec{x}') = \frac{\partial x^j}{\partial x'^m} \frac{\partial x^k}{\partial x'^n} \Gamma_{j,k}^P(\vec{x}) + \frac{\partial^2 x^P}{\partial x'^m \partial x'^n} = 0$$

$$\text{also, recall: } n'^k(u) = \frac{\partial x'^k}{\partial x^i} n^i(u)$$

$$\therefore \frac{dx'^i}{du} = \frac{\partial x'^i}{\partial x^s} \frac{\partial x^s}{\partial u}$$

$$\begin{aligned} \frac{dn^i}{du} &= - \frac{\partial x^m}{\partial x'^l} \frac{\partial x^s}{\partial x'^n} \Gamma_{s,n}^i \\ &= - \Gamma_{s,n}^i \left[\frac{\partial x^l}{\partial x'^m} \frac{\partial x^s}{\partial x'^n} \frac{\partial x'^k}{\partial x^m} \frac{\partial x'^l}{\partial x'^n} \right] \frac{\partial x^m}{\partial u} \\ &= - \tilde{\Gamma}_{s,n}^l \delta_{m,n} \end{aligned}$$

Using this (ex: check) we get

$$\frac{dn^i}{du} + \Gamma_{P,q}^i(\vec{x}(u)) n^P \frac{dx^q}{du} = 0$$

(The primes will disappear (as kronecker delta)).

For a d-dimensional space,
there're d-1st order d.e.

Thus given the initial vector, you can
solve it for u.

NB: u needn't describe a geodesic.

claim: The eqⁿ is parametrization independent.

Proof: $u = f(v)$

$$\left. \begin{aligned} \frac{du}{dv} &= \frac{dv}{du} \cdot \frac{dn^i}{dv} \\ \frac{dx^q}{du} &= \frac{dx^q}{dv} \cdot \frac{dv}{du} \end{aligned} \right\} \Rightarrow \frac{dv}{du} \left[\frac{dn^i}{dv} + \Gamma_{pq}^i (\bar{x}(v)) n^p \frac{dx^q}{dv} \right] = 0$$

NB: unlike the geodesic eqⁿ which works in a specific parametrization.

Now choose an arbitrary coordinate sys \bar{x}'' (where Γ'' needn't be zero)

$$\left. \begin{aligned} \text{Recall: } n^i &= n''^k \frac{\partial x^i}{\partial x''^k} \\ \because \Gamma &\text{ the transformation} \end{aligned} \right\} \text{ substitute in}$$
$$\Rightarrow \frac{\partial x^i}{\partial x''^k} \left(\frac{dn''^k}{du} + \Gamma''^{kl} n''^l \frac{\partial x^q}{\partial u} \right) = 0$$

NB: $\frac{dx^q}{du} = \frac{\partial x^q}{\partial x''^k} \frac{dx''^k}{du}$

Therefore it's evident that the parallel transport eqⁿ's don't depend on the coordinate frame.

NB: It's not surprising: a coordinate sys to start with, was arbitrary.
Therefore in fact it's a consistency check.

Ex: $\frac{d}{du} (g_{ij}(\bar{x}(u)) n^i(u) n^j(u)) \stackrel{\text{claim}}{=} 0$ i.e. norm is preserved under parallel transport.

Proof sketch: (direct) $\frac{dn^i}{du}$ is known now, $\frac{\partial g_{ij}}{\partial x''^k} \frac{dx^k}{du} = \frac{dg_{ij}}{du}$ is that known.

(neat) the quantity is invariant under coordinate transformation.

$\Gamma^i_{jk} = 0$ $\Rightarrow \frac{\partial g_{ij}}{\partial x''^k} = 0$ Thus, goto the primed coordinates. Now first derivative of g is zero (by defⁿ of the primed coordinate). Also, $\frac{dn^i}{du} = 0$ for a primed coordinate, $\frac{dn'^i}{du} = 0$. So that does it.

Recall:

Geodesic eqⁿ

$$\frac{d^2 x^i}{du^2} + \Gamma_{jk}^i \frac{dx^k}{du} \frac{dx^j}{du} = 0 \quad ; \quad \text{2. Def}^n: n^i(u) = \frac{dx^i}{du} \quad (\text{the tangent vector})$$

So now, $\frac{dn^i}{du} + \Gamma_{jk}^i n^j \frac{dx^k}{du} = 0$

N.B!: This is exactly the parallel transport eqⁿ parallel

Alternate defⁿ of geodesic: curve s.t. its tangent vectors are transported from any given tangent vector on the curve.

$\vec{n}_{(1)}$ — after parallel transport

$\vec{n}_{(2)}$
curve is c

NB: the transport \vec{g}^* is linear.

$$\Rightarrow \alpha \vec{n}_{(1)} + \beta \vec{n}'_{(1)} \rightarrow \alpha \vec{n}_{(2)} + \beta \vec{n}'_{(2)}$$

$$\vec{n}_{(2)}^i = M_{ij}^i(\vec{x}_{(1)}, \vec{x}_{(2)}, c) \vec{n}_{(1)}^j$$

(Not entirely certain how this follows)
independent of $n_{(1)}$ and $n_{(2)}$)

$$T(\alpha n_{(1)} + \beta n_{(2)}) = \alpha T(n_{(1)}) + \beta T(n_{(2)})$$

Defⁿ: $-c$:= Path c in the reverse direction.

Q: What can we say about $M_{ij}^i(\vec{x}_{(2)}, \vec{x}_{(1)}, -c)$?

NB: The parallel transport eqⁿ. is reversible (independent of parametrization)

We reparametrize: $v := 1-u$; this will do the job. $u=1 \Leftrightarrow v=0$
 $u=0 \Leftrightarrow v=1$

$$\frac{dn^i}{dv} + \Gamma_{jk}^i n^j(v) \frac{dx^k}{dv} = 0 \rightarrow \frac{dn^i}{du} + \Gamma_{jk}^i n^j(u) \frac{dx^k}{du} = 0$$

So we solve the same old eqⁿ with boundary conditions reversed.

Thus we must have

$$\vec{n}_{(1)}^i = (M(\vec{x}_{(1)}, \vec{x}_{(2)}, c)^{-1})^i_j \vec{n}_{(2)}^j$$

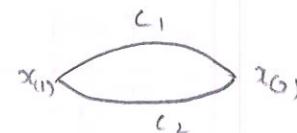
$$\stackrel{\text{by Def}^n}{=} M(\vec{x}_{(2)}, \vec{x}_{(1)}, -c)^i_j \vec{n}_{(2)}^j$$

$$\begin{aligned} \frac{d^2x}{dt^2} &= f(t) \\ dt &\rightarrow -dt \\ \frac{d^2x}{dt^2} &= -\frac{dx}{dt} \end{aligned}$$

$\therefore \vec{n}_{(1)}^i$ is arbitrary,
 $M(\vec{x}_{(2)}, \vec{x}_{(1)}, -c) = M(\vec{x}_{(1)}, \vec{x}_{(2)}, c)$

NB: If we had $\frac{dn^i}{du} + \Gamma_{jk}^i n^j \left| \frac{dx^j}{du} \right| = 0$, the eqⁿ won't be reversible.

Take 2 curves joining $\vec{x}_{(1)}$ & $\vec{x}_{(2)}$, c_1 & c_2



In general, $M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_1) + M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_2)$

$$\begin{aligned} \text{consider: } M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_1 - c_2) &= M(\vec{x}_{(2)}, \vec{x}_{(1)}, -c_2) \cdot M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_1) \\ &= M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_2)^{-1} M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_1) \end{aligned}$$

Defⁿ: Monodromy Matrix around $c \neq \emptyset$ in general.

$\hat{M} := M(\vec{x}, \vec{x}, c)$ for some closed curve c

In general,

$$\vec{n}^i = M(\vec{x}, \vec{x}, c)^i_j \vec{n}^j$$

initial vector at \vec{x} .
final vector, after coming back to \vec{x} .

$$m^i(x, x, c)$$

Now lets change the coordinate system:

$$\vec{n}^i = \frac{\partial x^i}{\partial x^P} \vec{n}^P = \frac{\partial x^i}{\partial x^P} \cdot M(x, x, c)^P_q \vec{n}^q = \underbrace{\frac{\partial x^i}{\partial x^P} M(x, x, c)^P_q}_{\frac{\partial x^i}{\partial x^j}} \vec{n}^j$$

If M was 1, then you'd get

$$\tilde{n}'^i = n'^i$$

i.e. M will remain identity even in the new coordinate sys.

NB: This can be proved the other

1. $\because n'^i$ is arbitrary \Rightarrow

way also, if $\tilde{n}'^i = n'^i$ then M must be

$$\frac{\partial \tilde{x}^i}{\partial x^p} M_{ij} \frac{\partial x^j}{\partial x'^k} = \delta^i_j \quad j \rightarrow \underbrace{\delta^i_j}_{\delta^m_p} x^m \frac{\partial x'^j}{\partial x^k}$$

NB 2: This analysis will not

$$\frac{\partial x}{\partial x'} M \frac{\partial x'}{\partial x}$$

work for $M(\tilde{x}_1, \tilde{x}_2, c)$ \therefore the $= \delta^i_j \delta^m_p \delta^m_n$

factors won't become kroneckers (they're evaluated at different points), even if $M = I$.

Claim: When $R=0$, $M=I$; the reverse is true when there're no singularities (i.e. simply connected)

Proof: Start with a small curve; length $\sim O(\epsilon)$

Convention: $0 < u < \epsilon$ so that

$$\frac{dx^i}{du} \sim 1$$



if a went from 0 to 1, $\frac{dx}{du} \sim O(\epsilon)$ (see the picture)

$$\text{recall: } \frac{dn^i}{du} = -\Gamma_{jk}^i n^j(u) \frac{dx^j}{du}$$

$$r = g(\partial_3 + \partial_3 - \partial_1)$$

Idea: Keep terms of order ϵ^1 & then integrate, you get the result to order ϵ . Then plugin the result & re-evaluate to get order ϵ^2 .

correct to order ϵ^0 ($\epsilon^0 = 1$)

$$\frac{dn^i}{du} = -\overline{\Gamma_{jk}^i(x_{(0)})} \left[\overline{n^j}_{(0)} \right] \frac{dx^k}{du} \quad \begin{array}{l} \text{order 1 itself} \\ \text{can't take it at its original value.} \end{array} \quad \therefore \quad \text{the tangent changes quick}$$

$$\Rightarrow n^i(u) = \overline{n^i}_{(0)} - \overline{\Gamma_{jk}^i(x_{(0)})} \left[\overline{n^j}_{(0)} \right] (x^j(u) - \overline{x^j}_{(0)}) + O(\epsilon^2) \quad (\because \epsilon u \approx \epsilon^2)$$

$$\frac{dn^i}{du} = -\left\{ \overline{\Gamma_{jk}^i(x_{(0)})} + \partial_l \overline{\Gamma_{jk}^i(x_{(0)})} \left[(x^l - \overline{x^l}_{(0)}) \right] \right\} \times \left\{ \overline{n^j}_{(0)} - \overline{\Gamma_{pq}^j(x_{(0)})} n^p(x_{(0)}) \right. \\ \left. \frac{(x^q - \overline{x^q}_{(0)})}{O(\epsilon)} \right\} \frac{dx^k}{du}$$

Now keeping upto $O(\epsilon^1)$, we have

$$\frac{dn^i}{du} = -\left\{ \overline{\Gamma_{jk}^i} n^j \frac{dx^k}{du} \right\} - \left\{ \overline{\Gamma_{ik}^l \Gamma_{pq}^j} n^p (x^q - \overline{x^q}_{(0)}) \frac{dx^k}{du} \right\} \\ \int_0^{\epsilon} dx^k(u) = x^k(\epsilon) - x^k(0) = 0 \quad \text{for closed curves} \quad + \left\{ \partial_l \overline{\Gamma_{jk}^i} (x^l - \overline{x^l}_{(0)}) \cdot \overline{n^j}_{(0)} \cdot \frac{dx^k}{du} \right\} + O(\epsilon^2)$$

Integrating from $(0, \epsilon)$, we note that term 1 is zero!

$$n^i(\epsilon) = n_{(0)}^i + \overset{3rd\ term}{\partial_l \Gamma_{ijk}^i} n_{(0)}^j \int_0^\epsilon (x^l - x_{(0)}^l) \frac{d(x^k - x_{(0)}^k)}{du} - \overset{2nd\ term}{\Gamma_{ijk}^i \Gamma_{pq}^{jk} n^p} \int_0^\epsilon (x^q - x_{(0)}^q) \frac{d(x^k - x_{(0)}^k)}{du}$$

(lets try to match indices for aesthetics)

$$= n_{(0)}^i + \partial_q \Gamma_{pk}^i n_{(0)}^p \int_0^\epsilon (x^q - x_{(0)}^q) \frac{d(x^k - x_{(0)}^k)}{du} - \Gamma_{jk}^i \Gamma_{pq}^{jk} n^p \int_0^\epsilon (x^q - x_{(0)}^q) \frac{d(x^k - x_{(0)}^k)}{du}$$

$$= n_{(0)}^i + [\partial_q \Gamma_{pk}^i - \Gamma_{jk}^i \Gamma_{pq}^{jk}] n_{(0)}^p \int_0^\epsilon (x^q - x_{(0)}^q) \frac{d(x^k - x_{(0)}^k)}{du}$$

NB: The boundary terms are zero. The derivative can be \therefore shifted with a minus sign.

NB2: This means the indices q, k are anti-symmetric

NB3: Thus only the symmetric part of the remaining part must contribute

$$= n_{(0)}^i + \frac{1}{2} [\partial_q \Gamma_{pk}^i - \partial_k \Gamma_{pq}^i - \Gamma_{jk}^i \Gamma_{pq}^{jk} + \Gamma_{jq}^i \Gamma_{pk}^{jq}] n_{(0)}^p$$

$$\cdot \int_0^\epsilon (x^q - x_{(0)}^q) \frac{d(x^k - x_{(0)}^k)}{du}$$

$$= n_{(0)}^i + \frac{1}{2} R_{pkqj}^i |_{\tilde{x}_0} \cdot \int_0^\epsilon (x^q - x_{(0)}^q) \frac{d(x^k - x_{(0)}^k)}{du}$$

$$n_{(0)}^i = n_{(0)}^i + O(\epsilon^3) \quad \text{if} \quad R_{ijkl}^i |_{\tilde{x}_0} = 0$$

$$\Rightarrow M_{ij}^i(\tilde{x}_0, \tilde{x}_0, c) = \delta_{ij} + O(\epsilon^3) \quad \text{if} \quad R_{ijkl}^i |_{\tilde{x}_0} = 0.$$

