

vectors in the tangent space at $\vec{x}(u)$

aim: see what this vector becomes upon parallel transport.

$n^i(u) \rightarrow$ known (say)

$n^i(u+\delta u) \rightarrow$ calculate

Go to $\vec{x}(u)$ & find a new coordinate sys \vec{x}' s.t. $\partial_i g'_{jk} = 0$ at $\vec{x}' = \vec{x}'_{(0)}$

$n^i(u+\delta u) = n^i(u) + \mathcal{O}(\delta u^2)$ (by rules of parallel transport)

Since you don't want to keep changing coordinates (it's inconvenient), \therefore we see what parallel transport amounts to in some arbitrary frame.

Translate to x'

$$n^i(u) = \left. \frac{\partial x^i}{\partial x'^k} \right|_{\vec{x}'(u)} n'^k(u)$$

$$n'^k(u) + \mathcal{O}(\delta u^2)$$

$$n^i(u+\delta u) = \left. \frac{\partial x^i}{\partial x'^k} \right|_{\vec{x}'(u+\delta u)} n'^k(u+\delta u)$$

$$\left. \frac{\partial x^i}{\partial x'^k} \right|_{\vec{x}'(u)} + \frac{\partial^2 x^i}{\partial x'^l \partial x'^k} \delta x'^l + \dots \quad \& \quad \delta x'^l = \frac{\partial x'^l}{\partial u} \delta u$$

$$\Rightarrow n^i(u+\delta u) = n^i(u) + \frac{\partial^2 x^i}{\partial x'^l \partial x'^k} \frac{\partial x'^l}{\partial u} \delta u n'^k(u) + \mathcal{O}(\delta u^2)$$

$$\frac{dn^i}{du} = \frac{\partial^2 x^i}{\partial x'^l \partial x'^k} \frac{\partial x'^l}{\partial u} n'^k(u)$$

We still have dependence on the primed coordinate. We wish to get rid of it.

$$\Gamma'^l_{mn}(\vec{x}') = \frac{\partial x'^l}{\partial x^i} \frac{\partial x^j}{\partial x'^m} \frac{\partial x^k}{\partial x'^n} \Gamma^i_{jk}(\vec{x}) + \frac{\partial x'^l}{\partial x^k} \frac{\partial^2 x^k}{\partial x'^m \partial x'^n} = 0 \quad (\text{for the primed frame})$$

$$\frac{\partial x^p}{\partial x'^l} \Gamma'^l_{mn}(\vec{x}') = \frac{\partial x^j}{\partial x'^m} \frac{\partial x^k}{\partial x'^n} \Gamma^p_{jk}(\vec{x}) + \frac{\partial^2 x^p}{\partial x'^m \partial x'^n} = 0$$

$$\text{also, recall: } n'^k(u) = \frac{\partial x'^k}{\partial x^s} n^s(u)$$

$$\therefore \frac{dx'^i}{du} = \frac{\partial x'^i}{\partial x^s} \frac{dx^s}{du}$$

Using these (ex: check) we get

$$\frac{dn^i}{du} + \Gamma^i_{pq}(\vec{x}(u)) n^p \frac{dx^q}{du} = 0$$

(The primes will disappear (as Kronecker delta)).

For a d -dimensional space,

there're d -first order d.e.

Thus given the initial vector, you can

solve it for u .

NB: u needn't describe a geodesic.

claim: The eq^n is parametrization independent.

Proof: $u = f(v)$

$$\left. \begin{aligned} \frac{dn^i}{du} &= \frac{dv}{du} \cdot \frac{dn^i}{dv} \\ \frac{dx^q}{du} &= \frac{dx^q}{dv} \cdot \frac{dv}{du} \end{aligned} \right\} \Rightarrow \frac{dv}{du} \left[\frac{dn^i}{dv} + \Gamma_{pq}^i(\vec{x}(v)) n^p \frac{dx^q}{dv} \right] = 0$$

NB: unlike the geodesic eq^n which works in a specific parametrization.

Now choose an arbitrary coordinate sys \vec{x}'' (where Γ'' needn't be zero)

Recall: $n^i = n''^k \frac{\partial x^i}{\partial x''^k}$

Γ the transformation

substitute in

Ex: $\Rightarrow \frac{\partial x^i}{\partial x''^k} \left(\frac{dn''^k}{du} + \Gamma''_{pq}^k n''^p \frac{dx'^q}{du} \right) = 0$

NB: $\frac{dx^q}{du} = \frac{\partial x^q}{\partial x''^k} \frac{dx''^k}{du}$

Therefore it's evident that the parallel transport eq^n 's don't depend on the coordinate frame.

NB: It's not surprising \because x coordinate sys to start with, was arbitrary.

Therefore infact it's a consistency check.

Ex: $\frac{d}{du} (g_{ij}(\vec{x}(u)) n^i(u) n^j(u)) \stackrel{\text{claim}}{=} 0$ viz. norm is preserved under parallel transport.

Proof sketch: (direct) $\frac{dn^i}{du}$ is known now, $\frac{\partial}{\partial x^k} g_{ij} \frac{dx^k}{du} = \frac{dg_{ij}}{du}$ & that's known.

(neat) the quantity is invariant under coordinate transformations. Thus, go to the primed coordinates. Now first derivative of g is zero (by defⁿ of the primed coordinate). Also, for a primed coordinate, $\frac{dn'^i}{du} = 0$. So that does it.

Recall:
Geodesic eq^n

$$\frac{d^2 x^i}{du^2} + \Gamma_{jk}^i \frac{dx^k}{du} \frac{dx^j}{du} = 0 \quad ; \quad \text{L. Def}^n: \quad n^i(u) := \frac{dx^i}{du} \quad (\text{the tangent vector})$$

So now, $\frac{dn^i}{du} + \Gamma_{jk}^i n^j \frac{dx^k}{du} = 0$

NB!: This is exactly the parallel transport eq^n

Alternate defⁿ of geodesic: curve s.t. its tangent vectors are transported from any given tangent vector on the curve.

$\vec{n}(u)$ — after parallel transport

$\vec{x}(u)$ — curve is c

NB: The transport eqⁿ is linear.

$$\Rightarrow \alpha \vec{n}(u) + \beta \vec{n}'(u) \rightarrow \alpha \vec{n}(u) + \beta \vec{n}'(u)$$

$$n_a^i = M^i_j(\vec{x}(u), \vec{x}(u), c) n_{(1)}^j$$

independent of $n_{(1)}$ & $n_{(2)}$

(Not entirely certain how this follows)

Defⁿ: $-c :=$ Path c in the reverse direction.

Q: What can we say about $M^i_j(\vec{x}(u), \vec{x}(u), -c)$?

NB: The parallel transport eqⁿ is reversible (independent of parametrization)

We reparametrize: $v := 1-u$; this will do the job.

$$u=1 \Leftrightarrow v=0 \\ u=0 \Leftrightarrow v=1$$

$$\frac{dn^i}{dv} + \Gamma^i_{jk} n^j(v) \frac{dx^k}{dv} = 0 \rightarrow \frac{dn^i}{du} + \Gamma^i_{jk} n^j(u) \frac{dx^k}{du} = 0$$

So we solve the same old eqⁿ with boundary conditions reversed.

Thus we must have

$$n_{(1)}^i = (M(\vec{x}(u), \vec{x}(u), c)^{-1})^i_j n_{(2)}^j$$

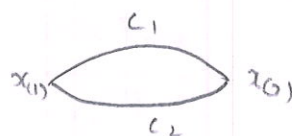
$$\stackrel{\text{by Def}^n}{=} M(\vec{x}(u), \vec{x}(u), -c)^i_j n_{(2)}^j$$

$\Rightarrow \because n_{(2)}^i$ is arbitrary,

$$M(\vec{x}(u), \vec{x}(u), -c) = M(\vec{x}(u), \vec{x}(u), c)$$

NB: If we had $\frac{dn^i}{du} + \Gamma^i_{jk} n^j \left| \frac{dx^k}{du} \right| = 0$, the eqⁿ won't be reversible.

Take 2 curves joining $x_{(1)}$ & $x_{(2)}$, c_1 & c_2



In general, $M(\vec{x}(u), \vec{x}(u), c_1) \neq M(\vec{x}(u), \vec{x}(u), c_2)$

$$\begin{aligned} \text{Consider: } M(\vec{x}(u), \vec{x}(u), c_1 - c_2) &= M(\vec{x}(u), \vec{x}(u), -c_2) \cdot M(\vec{x}(u), \vec{x}(u), c_1) \\ &= M(\vec{x}(u), \vec{x}(u), c_2)^{-1} M(\vec{x}(u), \vec{x}(u), c_1) \end{aligned}$$

Defⁿ: Monodromy Matrix around c $\neq \mathbb{I}$ in general.

$$:= M(\vec{x}, \vec{x}, c) \text{ for some closed curve } c$$

In general, $\tilde{n}^i = M(\vec{x}, \vec{x}, c)^i_j n^j$

initial vector at \vec{x} .
final vector, after coming back to \vec{x} .

Now let's change the coordinate system:

$$\text{Recall: } \tilde{n}^i = \frac{\partial x'^i}{\partial x^p} \tilde{n}^p = \frac{\partial x'^i}{\partial x^p} M(x, x, c)^p_q n^q = \frac{\partial x'^i}{\partial x^p} M(x, x, c)^p_q \frac{\partial x^q}{\partial x'^j} n'^j$$

If M was 1, then you'd get

$$\tilde{n}'^i = n'^i$$

ie. M will remain identity even in the new coordinate sys.

NB: This can be proved the other way also. If $\tilde{n}'^i = n'^i$ then M must be 1.

NB2: This analysis will not work for $M(\vec{x}', \vec{x}', c) \because$ The $\frac{\partial x}{\partial x'} M \frac{\partial x'}{\partial x}$ factors won't become Kroneckers (they're evaluated at different points), even if $M = \mathbb{I}$.

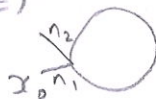
Claim: When $R=0$, $M=\mathbb{I}$; the reverse is true when there're no singularities (& simply connected)

Proof: Start with a small curve; length $\sim O(\epsilon)$

Convention:

$0 < u < \epsilon$ so that

$$\frac{dx^i}{du} \sim 1$$



$$\frac{dn^i}{du} = -\Gamma_{jk}^i n^j(u) \frac{dx^j}{du}$$

idea: Keep terms of order u & then integrate, you get the result to order ϵ . Then plug in the result & re-evaluate to get order ϵ^2 .

$$\frac{dn^i}{du} = \overbrace{-\Gamma_{jk}^i(x_0)}^{\text{correct to order } \epsilon} \overbrace{n_{(0)}^j}^{\text{order 1, can't take it at its original value.}} \overbrace{\frac{dx^k}{du}}^{\text{original value.}} + O(\epsilon)$$

$$\Rightarrow n^i(u) = \overbrace{n_{(0)}^i}^{O(1)} - \overbrace{\Gamma_{jk}^i(x_0)}^{O(1)} \overbrace{n_{(0)}^j}^{O(1)} (x^j(u) - x_{(0)}^j) + O(\epsilon^2) \quad (\because \epsilon u \approx \epsilon^2)$$

$$\frac{dn^i}{du} = -\left\{ \overbrace{\Gamma_{jk}^i(\vec{x}_0)}^{O(1)} + \partial_l \overbrace{\Gamma_{jk}^i(\vec{x}_0)}^{O(\epsilon)} \overbrace{(x^l - x_{(0)}^l)}^{O(\epsilon)} \right\} \times \left\{ \overbrace{n_{(0)}^j}^{O(1)} - \overbrace{\Gamma_{pq}^j(\vec{x}_0)}^{O(1)} \overbrace{n_{(0)}^p}^{O(1)} \overbrace{(x^q - x_{(0)}^q)}^{O(\epsilon)} \right\} \frac{dx^k}{du}$$

Now keeping upto $O(\epsilon^2)$, we have

$$\frac{dn^i}{du} = -\left\{ \Gamma_{jk}^i n^j \frac{dx^k}{du} - \Gamma_{ik}^l \Gamma_{pq}^j n^p (x^q - x_{(0)}^q) \frac{dx^k}{du} + \partial_l \Gamma_{jk}^i (x^l - x_{(0)}^l) \cdot n_{(0)}^j \cdot \frac{dx^k}{du} \right\} + O(\epsilon^2)$$

Integrating from $(0, \epsilon)$, we note that term 1 is zero.

$$n^i(u) = n_{(0)}^i + \underbrace{\partial_l \Gamma_{jk}^i}_{\text{3rd term}} n_{(0)}^j \int_0^\epsilon (x^l - x_{(0)}^l) \frac{d(x^k - x_{(0)}^k)}{du} - \underbrace{\Gamma_{jk}^i \Gamma_{pq}^j}_{\text{2nd term}} n^p \int_0^\epsilon (x^q - x_{(0)}^q) \frac{d(x^k - x_{(0)}^k)}{du}$$

(lets try to match indices for aesthetics)

$$= n_{(0)}^i + \partial_q \Gamma_{pk}^i n_{(0)}^p \int_0^\epsilon (x^q - x_{(0)}^q) \frac{d(x^k - x_{(0)}^k)}{du} - \Gamma_{jk}^i \Gamma_{pq}^j n^p \int_0^\epsilon (x^q - x_{(0)}^q) \frac{d(x^k - x_{(0)}^k)}{du}$$

$$= n_{(0)}^i + \left[\partial_q \Gamma_{pk}^i - \Gamma_{jk}^i \Gamma_{pq}^j \right] n_{(0)}^p \int_0^\epsilon (x^q - x_{(0)}^q) \frac{d(x^k - x_{(0)}^k)}{du}$$

NB: The boundary terms are zero. The derivative can be \therefore shifted with a minus sign.

NB2: This means the indices q & k are anti-symmetric

NB3: Thus only the symmetric part of the remaining part must contribute

$$= n_{(0)}^i + \frac{1}{2} \left[\partial_q \Gamma_{pk}^i - \partial_k \Gamma_{pq}^i - \Gamma_{jk}^i \Gamma_{pq}^j + \Gamma_{jq}^i \Gamma_{pk}^j \right] n_{(0)}^p \int_0^\epsilon (x^q - x_{(0)}^q) \frac{d(x^k - x_{(0)}^k)}{du}$$

$$= n_{(0)}^i + \frac{1}{2} R_{pkq}^i \Big|_{\vec{x}_0} \cdot \int_0^\epsilon (x^q - x_{(0)}^q) \frac{d(x^k - x_{(0)}^k)}{du}$$

$$n_{(\epsilon)}^i = n_{(0)}^i + O(\epsilon^3) \quad \text{if} \quad R_{ijk}^i \Big|_{\vec{x}_0} = 0$$

$$\Rightarrow M_{ij}^i(\vec{x}_0, \vec{x}_0, \epsilon) = \delta_{ij} + O(\epsilon^3) \quad \text{if} \quad R_{ijk}^i \Big|_{\vec{x}_0} = 0.$$

