2 Probability

Def^n:
Random variable x, has a set of possible outcomes S.
Event ≡ any subset of the outcomes
$E \subset S$

Axioms: (i) $p(E) \geq 0$
(ii) $p(A \text{ or } B) = p(A) + p(B)$ if A & B are disjoint
(iii) $p(S) = 1$

Assignment of Prob (i) objective (ii) subjective

One random Variable

$S_x = \{\infty < x < \infty\}$

- cumulative prob. f^n: $P(x) = \text{prob}(E \subset [-\infty, x])$
  - monotonic inc.
  - $P(-\infty) = 0$, $P(\infty) = 1$
- prob. density f^n (PDF) $p(x) = \frac{dP(x)}{dx}$
  - $p(x)\,dx = \text{prob}(E \in [x, x+dx])$
  - $\text{prob}(S) = \int_{-\infty}^{\infty} dx\, p(x) = 1$
  - $0 < p(x) < \infty$ (no upper bound)

shortened

Def^n $\langle F(x) \rangle \equiv \int_{-\infty}^{\infty} p(x) F(x)\, dx$ | $F(x) = f$; $p_F(f)\,df = \sum_i p(x_i)\,dx_i$ | ie $p_F(f) = \sum_i p(x_i) \left.\frac{dx}{dF}\right|_{x=x_i}$

Def^n n^{th} moment $\equiv m_n \equiv \langle x^n \rangle$

Def^n Characteristic f^n $\equiv \tilde{p}(k) = \langle e^{-ikx} \rangle$

NB: $p(x) = \frac{1}{2\pi} \int dk\, \tilde{p}(k) e^{ikx}$

NB: $\tilde{p}(k) = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \langle x^n \rangle$

Def^n: Cumulant generating f^n $\equiv \ln(\tilde{p}(k))$

Def^n: (implicit) $\ln(\tilde{p}(k)) \equiv \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} \underbrace{\langle x^n \rangle_c}_{\text{cumulant}}$

NB: Using $\ln(1+\epsilon)$ we can show

$\langle x \rangle_c = \langle x \rangle$

$\langle x^2 \rangle_c = \langle x^2 \rangle - \langle x \rangle^2$

$\langle x^3 \rangle_c = \langle x^3 \rangle - 3\langle x^2 \rangle \langle x \rangle + 2\langle x \rangle^3$

$\langle x^4 \rangle_c = \langle x^4 \rangle - 4\langle x^3 \rangle \langle x \rangle - 3\langle x^2 \rangle^2 + 12\langle x^2 \rangle \langle x \rangle^2 - 6\langle x \rangle^4$

Def^n: PDF of (a) Normal Dist^n $\equiv p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\lambda)^2}{2\sigma^2}}$

(b) Binomial $\equiv p_N(N_A) = \binom{N}{N_A} p_A^{N_A} p_B^{N-N_A}$

(c) Poisson $\equiv p_{\alpha T}(M) = e^{-\alpha T} \frac{(\alpha T)^M}{M!}$

Many Variables

Defⁿ: $d^N\vec{x} \equiv \prod_{i=1}^{N} dx_i$

NB: Iff events are independent

$$p(\vec{x}) = \prod_{i=1}^{N} p_i(x_i)$$

Defⁿ: Unconditional PDF $\equiv p(x_1 \dots x_m) = \int \prod_{i=m+1}^{N} dx_i \; p(x_1 \dots x_N)$

: Conditional PDF $\equiv p(x_1 \dots x_m | x_{m+1} \dots x_N)$

$$= \frac{p(x_1 \dots x_N)}{p(x_{m+1} \dots x_N)}$$

: Joint Characteristic fⁿ $\tilde{p}(\vec{k}) \equiv \langle \exp(-i \sum_{j=1}^{N} k_j x_j) \rangle$