GR

Newton $\vec{F} = -\frac{G M_1 M_2}{r^2}\hat{r}$

Newton → GR

Electrostatics → Maxwell's eq^n

Underlying mathematics of GR
→ Reimannian geometry
↑
Generalization of euclidean geometry.

Euclidian Geometry

Two points
(x^1, x^2, x^3) $(x^1+dx^1, x^2+dx^2, x^3+dx^3)$

In Euclidean,
$$ds = \sqrt{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}$$

also $$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

In N-dimension,
$$ds^2 = (dx^1)^2 + (dx^2)^2 + \dots (dx^N)^2$$
$$= \sum_{i=1}^{N} (dx^i)^2$$

In Riemanian Geometry
$$ds^2 = \sum_{i,j=1}^{N} g_{ij}(\vec{x})\, dx^i dx^j$$

$g_{ij}(\vec{x})$ is a f^n of $(x_1, \dots x_n)$
for each pair (i,j)

→ ~~We can choose~~ g_{ij} must be symmetric
$g_{ij} = g_{ji} \equiv$ refered to as a metric

Euclidian geometry is where $g_{ij} = \delta_{ij}$

At times two different $g_{ij}(\vec{x})$ may describe the same space.

$$ds^2 = \sum_{i,j=1}^{N} g_{ij}(\vec{x})\, dx^i dx^j$$

Instead of $(x^1, \dots x^N)$ we choose a different set of coordinates

$$ds^2 = \sum g_{ij}(\vec{x})\, dx^i dx^j$$
$(x^1 \dots x^N)$ ~~costed~~ choose $(x'^1, \dots x'^N)$

$x'^1 = f^1(\vec{x})$ $\qquad x^1 = g^1(\vec{x}')$ not the same as g^{ij}
$x'^2 = f^2(\vec{x})$ $\qquad x^2 = g^2(\vec{x}')$ can call these $h^1 \dots$ etc.
$\vdots$
$x'^N = f^N(\vec{x})$ $\qquad x^N = g^N(\vec{x}')$

$\vec{x}'$ $\qquad \vec{x}' + d\vec{x}'$
$$dx^i = \sum_k \frac{\partial x^i}{\partial x'^k} dx'^k$$
$$ds^2 = \sum_{i,j=1}^{N} g_{ij}\left(\sum_{k=1}^{n} \frac{\partial x^i}{\partial x'^k} dx'^k\right)\left(\sum_{l=1}^{n} \frac{\partial x^j}{\partial x'^l} dx'^l\right)$$

$$= \sum_{k,l} \underbrace{\sum_{i,j} g_{ij}(\vec{x}) \frac{\partial x^i}{\partial x'^k}\frac{\partial x^j}{\partial x'^l}}_{g'_{kl}(\vec{x}')} dx'^k dx'^l$$

$$= \sum_{ij} g'_{ij}(\vec{x}')\, dx'^i dx'^j$$

Examples

Euclidean metric
$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

$x^1 = r \sin\theta \cos\phi$
$x^2 = r \sin\theta \sin\phi$
$x^3 = r\cos\theta$

$dx^1 = \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi$

$dx^2 = \dots$

$dx^3 = \dots$

$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

$\Rightarrow ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$

Ex.
$g'_{rr} = 1$, $g'_{\theta\theta} = r^2$, $g'_{\phi\phi} = r^2\sin^2\theta$

6 others $= 0$

Example of a non-euclidean sphere.

$\to$ surface of a 2d sphere:

$x^2+y^2+z^2=a^2$

$ds^2=dx^2+dy^2+dz^2$

$(x^1)^2+(x^2)^2+(x^3)^2=a^2$

$\to$

Method 1

① $x^3 = \pm\sqrt{a^2-(x^1)^2-(x^2)^2}$

$$ds^2 = (dx^1)^2+(dx^2)^2+(dx^3)^2$$
$$= (dx^1)^2+(dx^2)^2+\left\{\pm\frac{-x^1dx^1-x^2dx^2}{\sqrt{a^2-(x^1)^2-(x^2)^2}}\right\}^2$$

$$= \left\{1+\frac{(x^1)^2}{a^2-(x^1)^2-(x^2)^2}\right\}(dx^1)^2+\left\{1+\frac{(x^2)^2}{a^2-(x^1)^2-(x^2)^2}\right\}dx^{2\,2}$$
$$+\frac{2x^1x^2}{a^2-(x^1)^2-(x^2)^2}dx^1dx^2$$

$$g_{11}=1+\frac{(x^1)^2}{a^2-(x^1)^2-(x^2)^2} \qquad g_{22}=1+\frac{(x^2)^2}{a^2-(x^1)^2-(x^2)^2}$$

NB: 2 goes away

$$g_{12}=\frac{x^1x^2}{a^2-(x^1)^2-(x^2)^2}\ \cancel{dx^1dx^2}$$

Method 2

② First go to spherical polar.

go from $x_1\,x_2\,x_3 \to r,\theta,\phi$

$r^2=a^2 \Rightarrow r=a$ (no $\pm$)

$$ds^2 = dr^2+r^2d\theta^2+r^2\sin^2\theta\,d\phi^2$$
$$= a^2(d\theta^2+\sin^2\theta\,d\phi^2) \to \text{you get this after subst. } x^1\ x^2 \text{ for } ds^2$$

$x^1 = a\sin\theta\cos\phi$ Ex: $ds^2\big|_{(x^1x^2)} = ds^2\big|_{(\theta,\phi)}$

$x^2 = a\sin\theta\sin\phi$

Q. Given only the ~~coordina~~ metric, how to conclude whether they're the same space?

O. ~~As so~~ Strategy: Find appropriate linear combinations of the metric & its derivatives which are invariant under coordinate transformations.

Conventions

a. Index i of a coordinate will be a super script eg. x^i

b. $\frac{\partial}{\partial x^i} \equiv \partial_i$

c. Summation Convention: Any index, appearing twice in a formula, once as sub & once as super is summed over.

d. Index of a matrix appears as subscript

① $ds = \sum_{i,j=1}^{N} g_{ij}(\vec{x})\, dx^i dx^j \longrightarrow g_{ij}(\vec{x})\, dx^i dx^j$

② $g'_{kl}(\vec{x}') = \sum_{ij} g_{ij} \frac{\partial x^i}{\partial x'^k} \frac{\partial x^j}{\partial x'^l} = g_{ij}\, \partial'_k x^i \partial'_l x^j$

Tensor Fields

Consider some combination $A_{i_1 \dots i_Q}(\vec{x})$ of the metric & its derivatives which transform as follows:

$$A'_{i_1 \dots i_Q}(\vec{x}') = \qquad A_{j_1 \dots j_Q}(\vec{x})$$

Rough Supplement

$$A_i \, dx^i = A'_i \, dx'^i$$

$$dx^i = \cancel{\partial' x^i} \; \frac{\partial x^i}{\partial x'^j} dx'^j$$

$$= \partial'^j x^i \, dx'^j$$

$$\cancel{= (A'_i \, \partial'^j x^i) \, dx'^j}$$

$$\underbrace{A_i \, (\partial'^j x^i)}\, dx'^j = A'_j \, dx'^j$$

$$\rightarrow \; A'_j = \cancel{A_i} \boxed{\partial'_j x^i} \, A_i$$

$$dx^i = \partial'_j x^i \, dx'^j$$

$$dx'^i = \boxed{\partial_j x'^i} \, dx^j \leftrightarrow \frac{\partial x'^i}{\partial x^j} dx^j$$

$$\frac{\partial x^i}{\partial x'^j} \quad \Bigg| \quad \frac{\partial x'^i}{\partial x^j}$$

$$\cancel{\frac{\partial}{\partial x^j} = \partial_j}$$