

Lecture 10 | field equations

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Recall: $m \left(\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} \right) = q F^\mu_{\nu} \frac{dx^\nu}{ds} + a R^\mu_{\alpha\beta\gamma} F^{\alpha\beta} \frac{dx^\gamma}{ds}$

NB: Both sides transform the same way \Rightarrow general coordinate invariance is respected.

Recall: a from dimensional analysis is $\sim 10^{-36}$ cm, so not easy to observe

Remark: One can try to write terms with derivatives as well $\nabla^\nu R_{\nu\rho}{}^\mu \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds}$, there would be present even in the absence of the EM fields but the term added above is due to the EM fields (zero otherwise)

EM Field Equations

We start with flat spacetime

Recall: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$;

$\Rightarrow \partial_{[\mu} F_{\nu\rho]} = 0$ (which means you add all its cyclic permutations; totally anti-symmetric combination is zero) $\because F_{\mu\nu} = -F_{\nu\mu}$
 i.e. $\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$; also called the Bianchi Identity.

Remark: Sometimes one doesn't like $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ & directly imposes the Bianchi Identity.

Remark 2: This (the Bianchi Identity) corresponds to $\nabla \cdot B = 0$ & $\nabla \times E + \frac{\partial B}{\partial t} = 0$

Story: Then there's another set of eq's that don't follow from $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
 This is a statement about the source of the fields

"Fact": $\partial^\mu F_{\mu\nu} = -J_\nu$; $J^0 = \rho$; Electric Charge Density
 J^i ; Current Density.

Claim: These correspond to

$\nabla \cdot E = \rho$; $\nabla \times B - \frac{\partial E}{\partial t} = \vec{J}$ ($c=1$ unit)

Story: In a local inertial frame, one must have ① $\partial_{[\mu} F_{\nu\rho]} = 0$ & ② $\partial^\mu F_{\mu\nu} = -J_\nu$
 (x' coordinate) (instead of using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$)

Further, J' can also be evaluated as it would be in the absence of gravity.
 (in x')

NB: In a general frame, ① $F'_{\mu\nu} = \partial'_\mu x^\rho \partial'_\nu x^\sigma F_{\rho\sigma}$

$\partial' \rightarrow$ in terms of ∂ as before — ∂ instead

② Def: $J_\nu = \partial_\mu x'^\sigma J'_\sigma$ NB: This makes J_ν a tensor.

Claim: $\partial^\mu F_{\mu\nu} = -J_\nu$

NB: We had to use ∂ to preserve tensorial behaviour (as discussed earlier).

NB 2: $F'_{\mu\nu}$ use ∂' $\because D' \leftrightarrow \partial'$ in that case.

Claim: $F_{\mu\nu} \stackrel{\text{def}}{=} D_\mu A_\nu - D_\nu A_\mu \stackrel{\text{claim}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$

($\because \Gamma$ terms get cancelled).

NB: One could start with defining A_μ using A'_μ
 $F_{\mu\nu}$ independently using $F'_{\mu\nu}$
 use the rel'n b/w F' & A' to
 derive F in terms of A
 This would yield $F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu$.

... + completed for EM field + ...

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Depos NB: This is important in ensuring the principle of equivalence holds

... say, $F^{\mu\nu}$ involved Γ .
 Δ^μ has a ∂ . This would act on Γ
 Thus the LHS of $\Delta^\mu F_{\mu\nu} = -J_\nu$
 would involve $\partial\Gamma$ like term.
 Now in a local inertial frame, $\Gamma=0$ but
 $\partial\Gamma$ is not zero
 Thus this will not reduce to the standard
 ∇^μ in the local inertial frame.

Remark (about $\Delta^\mu F_{\mu\nu} = -J_\nu$):

$$\begin{aligned} \text{aim: Evaluate } \Delta^\nu \Delta^\mu F_{\mu\nu} &= -\Delta^\nu J_\nu \\ &= \frac{1}{2} [\Delta^\nu, \Delta^\mu] F_{\mu\nu} \quad \because \Delta^\mu \Delta^\nu F_{\mu\nu} = \Delta^\nu \Delta^\mu F_{\mu\nu} \\ &= \frac{1}{2} (\Delta^\nu \Delta^\mu F_{\mu\nu} + \Delta^\mu \Delta^\nu F_{\nu\mu}) \\ &= \frac{1}{2} (\Delta^\nu \Delta^\mu - \Delta^\mu \Delta^\nu) F_{\mu\nu} \quad \because F_{\nu\mu} = -F_{\mu\nu} \end{aligned}$$

Recall: we had "derived" identities involving $[D_\mu, D_\nu] A = R A + R A + \dots$

$$\text{consider: } [D_\nu, D_\mu] F_{\alpha\beta} = -R_{\alpha\mu\nu}{}^\rho F_{\rho\beta} - R_{\beta\mu\nu}{}^\rho F_{\alpha\rho}$$

Now we raise index (\because it can be done inside covariant derivatives) & replace α, β by μ, ν .

$$\begin{aligned} [\Delta^\nu, \Delta^\mu] F_{\mu\nu} &= -R_{\mu}{}^{\mu\nu\rho} F_{\rho\nu} - R_{\nu}{}^{\mu\nu\rho} F_{\mu\rho} \\ &= -\underbrace{R^{\nu\rho} F_{\rho\nu}}_0 + \underbrace{R^{\mu\rho} F_{\mu\rho}}_0 = 0 \end{aligned}$$

L

$\because R$ is symmetric & F is anti-symmetric

$$\Rightarrow \Delta^\nu \Delta^\mu F_{\mu\nu} = 0 \quad \text{for the LHS}$$

$$\text{RHS? In local inertial } \partial'_\mu J'^\mu = 0 \quad (\text{by construction})$$

$$\text{Claim: } \Delta_\mu J^\mu = 0 \Leftrightarrow \partial'_\mu J'^\mu = 0$$

Remark: If $\partial'_\mu J'^\mu \neq 0$, then there'd be inconsistencies

$$\text{For eg. } \partial'^\mu F'_{\mu\nu} = -J'_\nu \quad \text{is given}$$

$$\nabla^\nu (\partial'^\mu F'_{\mu\nu}) = -\partial'^\nu J'_\nu = 0$$

second derivative would involve neighbouring points which may not be locally inertial. (but then $\partial'^\mu F'_{\mu\nu}$ also has 2 derivatives, why couldn't we object then? \because even at F' , the Γ disappears)

this is correct because (as calculated above) the R part doesn't contribute.

Conclusion: The equivalence principle is working due to the extra condition enforced by electrodynamics.

... + GR Field Equations ...

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GR Field Eqⁿ

NB: Starting from the equivalence principle is pointless because gravity is zero in a local inertial frame.

Intuition: We try to figure out what eqⁿ would give, in the appropriate limit,

$$\nabla^2 \phi = 4\pi G \rho_m$$

NB: The LHS we saw was related to the metric; (Recall $h_{00} = -2\frac{\phi}{c^2}$)
It must be related to two derivatives of the metric.

g_{μν} & g_{μν}

NB: Two derivatives of the metric have the same # free parameters (ones that can't be killed by a coordinate transform) as a tensor of rank 2. The only choice we have is $R_{\mu\nu}$ & $R g_{\mu\nu}$ — why not $g_{\mu\nu}$ alone? ∴ you need two derivatives of the metric.

NB: Using mass density is troublesome.

Justifⁿ: For an H atom, the mass $m_H \neq m_p + m_e$
proton mass → electron mass

One must take the binding energy into account, else the mass we get would be different from the inertial (gravitational?) mass.

Conclⁿ: Use energy density

"Corollary": Energy by itself is not covariant; one should use the energy-momentum tensor, $T_{\mu\nu}$. (More on evaluating $T_{\mu\nu}$ in the next lecture)

$$\Rightarrow b R_{\mu\nu} + c R g_{\mu\nu} = a T_{\mu\nu}$$

$$\Leftrightarrow R_{\mu\nu} + c R g_{\mu\nu} = a T_{\mu\nu} \quad (\text{for } b \neq 0 \text{ \& } \frac{c}{b} \rightarrow c, \frac{a}{b} \rightarrow a)$$

Claim. $\partial^\mu T_{\mu\nu} = 0$ \Rightarrow $\partial^\mu T_{\mu\nu} = 0$
(in a local inertial frame)

NB: $\Rightarrow \partial^\mu (R_{\mu\nu} + c R g_{\mu\nu}) = 0$

This we demand to hold identically because if we try to demand this as an additional condition, apparently, the solⁿ gets very specific.

Recall: $\partial^\mu (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) = 0$ identically

Conclusion: $c = -\frac{1}{2}$

Strategy: To find "a" we use a consistency check with Newtonian gravity (in the weak field limit)

Recall: $g_{00} \approx -2 - 2\phi$ (in the weak field limit); $\Gamma_{00}^0 \approx 0$, $\Gamma_{00}^i \approx \partial_i \phi$

$T_{00} = \rho_m$

$\Rightarrow R_{00} = ?$
 $\Rightarrow R = ?$

Claim: in the weak field limit the 00 component becomes
 $-2\nabla^2 \phi = a \rho_m$

$\Rightarrow a = 8\pi G$

Remark: One can add to $T_{\mu\nu} \rightarrow T_{\mu\nu} + \Lambda g_{\mu\nu}$.

Conclusion: $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$ (sign convention matches Weinberg's)