Lecture 8 (identities + GR started!)

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Useful Identities

(will not be proved here)

1(a) $D_i D_j A_k - D_j D_i A_k$ (also written as $[D_i, D_j] A_k$)
$= -R^l{}_{kji} A_l$

defined as the usual covariant derivative with the Christoffel symbol

These can be proved (claim) by looking at the definitions

(b) $D_i D_j A^k - D_j D_i A^k = R^k{}_{lji} A^l$

(c) Combining the aforesaid

$$D_i D_j A^{k_1 \dots k_p}{}_{l_1 \dots l_q} - D_j D_i A^{k_1 \dots k_p}{}_{l_1 \dots l_q} = R^{k_1}{}_{lji} A^{l k_2 \dots k_p}{}_{l_1 \dots l_q} + \dots + R^{k_p}{}_{lji} A^{k_1 k_2 \dots k_{p-1} l}{}_{l_1 \dots l_q}$$

$$- \{ R^l{}_{l_1 ji} A^{k_1 \dots k_p}{}_{l l_2 \dots l_q} + \dots + R^l{}_{l_q ji} A^{k_1 \dots k_p}{}_{l_1 \dots l_{q-1} l} \}$$

Bianchi Identities

$$D_s R_{ijkl} + D_k R_{ijls} + D_l R_{ijsk} = 0$$

structure: completely anti-symmetrize in $s\,k\,l$. NB: kl are already anti-symmetric; instead of 6 terms, need to write only 3 terms.

proof idea: go to a frame in which $\Gamma = 0$.

can derive more by contracting, e.g. g^{js}

$\Rightarrow D^j R_{ijkl} + D_k R_{il} - D_l R_{ik}$

where the Ricci tensor was defined as $g^{ik} R_{ijkl} = R_{jl}$. NB: contracting the 2^{nd} & 4^{th} index is also the same $\because$ R_{ijkl} is anti-symmetric in (i,j) & (k,l)

further contract with g^{ik}.

$\Rightarrow D^j R_{jl} + D^i R_{il} - D_l R = 0$

same terms

$\Rightarrow 2 D^j R_{jl} - D_l R = 0 \quad \Rightarrow 2 D^j \big(\underbrace{R_{jl} - \tfrac{1}{2} R\, g_{jl}}_{\text{this is called Einstein's tensor.}} \big) = 0$

where in the last step, D doesn't act on g $\because$ $Dg = 0$. It acts on R, then lowers the index.

This completes our general discussion of manifolds. Now we start General Relativity.

Conventions

Signature: Now we focus on manifolds with signature $(n,1)$ (here $n=3$ for 3d space)

Recall. Signature $(3,1) = (- + + +)$

Indices: x^μ $\quad \mu = 0,1,2,3$ coordinates.

function: $f(\xi) \longrightarrow f(x)$

e.g.
$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$
$g_{\mu\nu} = \eta_{\mu\nu}$
Recall: In special relativity, proper time is given by $-ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$
where $x^0 = ct$.

Question. Does the metric $g_{\mu\nu}(x)$ as generalized, describe anything physical?
Answer: Yes, it describes spacetime in the presence of gravity.

General Relativity

$g_{\mu\nu}(x)$ describes space-time in presence of a gravitational field.

Particles in a gravitational field move along geodesics in the absence of other fields.