IV. COLLISIONS B/W PARTICLES

§ 16. Distintegration of particles

Spontaneous disintegration ≡ Disintegration of a particle into two constituent parts (that is, one not due to external forces) that move independently thereafter.

Lets start with a frame of reference in which the particle is at rest. (before disintegration) (C-FRAME), (COM FRAME)

Conservation of momentum yields that the total momentum after must be zero. [Thus the momentum's magnitude is p_0 for both constituent.]

Conservation of energy yields

$$E_i = E_{1i} + \frac{p_0^2}{2m_1} + E_{2i} + \frac{p_0^2}{2m_2}$$

where m_1 & m_2 are the masses of the particles after the process. E_{1i} & E_{2i} are their internal energies.

Disintegration Energy $\doteq \epsilon \equiv E_i - E_{1i} - E_{2i}$ (the difference in internal energies)

From energy conservation, $\epsilon > 0$

$$\epsilon = \frac{1}{2} p_0^2 \left(\frac{1}{m_1} + \frac{1}{m_2} \right) = \frac{p_0^2}{2m}$$

where m is the reduced mass.

Consequently $v_{10} = \frac{p_0}{m_1}$ $\quad v_{20} = \frac{p_0}{m_2}$ (|Velocities| in the rest frame)

Lets now choose a frame in which the primary particle moves with V (L-FRAME), (Laboratory frame)

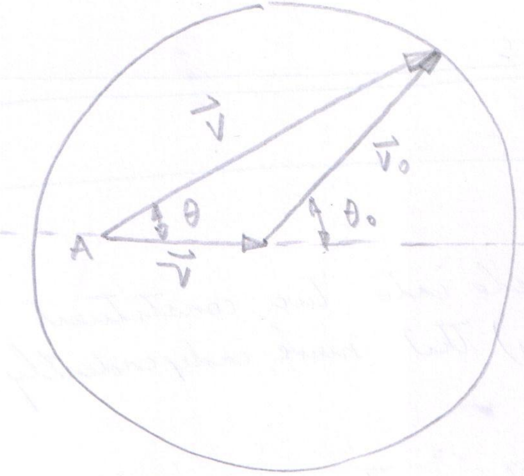
Consider now one of the resulting particles. Let its velocity be $\vec{v}$ in L & $\vec{v}_0$ in C.

$$\Rightarrow \vec{v} = \vec{V} + \vec{v}_0 \quad \Rightarrow \quad \vec{v} - \vec{V} = \vec{v}_0$$

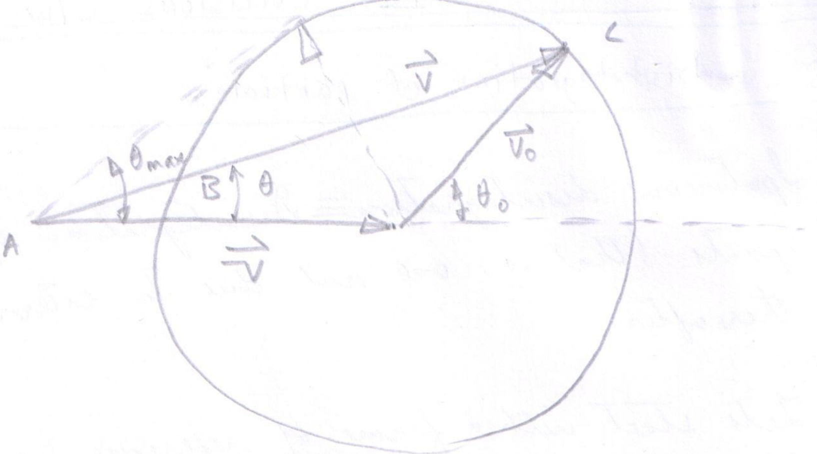
$$\Rightarrow v^2 - V^2 - 2vV\cos\theta = v_0^2$$

where θ is the angle b/w the resulting particle's velocity & the primary particle's velocity.

[Not in Landau: Assuming ϵ is fixed, p_0 is fixed, ~~too~~ $\Rightarrow v_1$ is fixed but θ can vary. The conservation laws don't tell us θ.]



(a) $V < v_0$

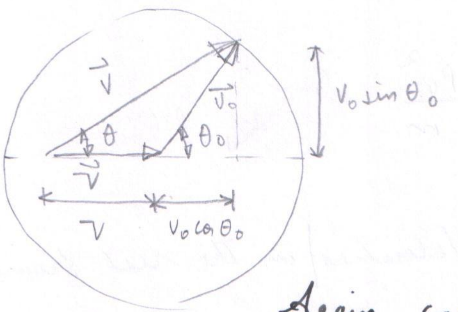


(b) $V > v_0$

→ "The velocity $\vec{v}$ is represented by a vector drawn to any point on a circle† of radius v_0 from a point A, V distance from the centre."
(Not a very good way of saying this) († on a sphere, precisely speaking)
Anyway it should be clear from the vector eq^n and the previous comment.

→ In case (a), θ can have any value. However, in case (b), θ can't exceed θ_{max}. Evidently ~~θ_{max} =~~ $\sin\theta_{max} = \frac{v_0}{V}$. (Keyword: Tangent)

→ Relation b/w θ & θ_0 may be obtained as follows:



$$\tan\theta = \frac{v_0 \sin\theta_0}{(v_0\cos\theta_0 + V)}$$

When solved for $\cos\theta_0$ we obtain

$$\cos\theta_0 = -\frac{V}{v_0}\sin^2\theta \pm \cos\theta\sqrt{1 - \frac{V^2}{v_0^2}\sin^2\theta}$$

Again case (a), θ & θ_0 are one-one. We take the + sign so that when $\theta = 0$, $\theta_0 = 0$ too.
(TODO: Find the link limiting case. Understand this better)
(just plug 'em in to verify)

For case (b), For each θ, $\exists$ 2 ~~θ_0~~ values of θ_0 and these are given by the two signs. (the points B & C should be enough said)

In physical problems, we're concerned with disintegration of not one, but many similar particles. Thus, we're interested in the distribution of the resulting particles in direction, energy etc. Motive

→ We assume that the primary particles are randomly oriented in space, i.e. isotropic on an average.
(does this mean velocity, or else what do you mean by orientation of particles?)

~~(then, it follows that)~~ Claim:

↳ In the C system, every resulting particle (of a given kind) has the same energy and their directions of motion are isotropically distributed.

The latter depends on the random orientation assumption of the primary particle.

We can mathematically express this by stating that the fraction of particles entering a solid angle element do_0 is proportional to do_0, i.e. $= \frac{do_0}{4\pi}$. The distribution then wrt θ_0 (note θ_0 was defined wrt V, which is not quite there in the COM picture (naturally that is)) is obtained by putting $do_0 = 2\pi \sin\theta_0 \, d\theta_0$

$(do_0 = \sin\theta \, d\theta \, d\phi)$

so then the corresponding fraction becomes $\frac{1}{2}\sin\theta_0 \, d\theta_0$.

↳ The L system; here the corresponding distributions can be obtained by appropriate transformations.

Eg. Kinetic Energy distribution

We know $\vec{v} = \vec{v}_0 + \vec{V} \Rightarrow v^2 = v_0^2 + V^2 + 2v_0 V \cos\theta_0$

$$\Rightarrow d(v^2) = \underbrace{d(v_0^2)}_{\text{fixed for a given kind of particle (from } E \text{ being fixed, thus } p_0^2 \text{ thus } v_0^2)} + \underbrace{d(V^2)}_{V \text{ is a const.}} + d(2v_0 V \cos\theta_0)$$

$$= 2v_0 V \, d(\cos\theta_0)$$

$$\Rightarrow \frac{d(v^2)}{2v_0 V} = d(\cos\theta_0) = \frac{d\left(\frac{1}{2}mv^2\right)}{\frac{1}{2}m\,2v_0 V} = \frac{d(T)}{mv_0 V}$$

$$\frac{dT}{mv_0 V} = d(\cos\theta_0)$$

(KE)

Therefore the required distribution is $= \frac{dT}{2mv_0 V}$

More than 2 particles after disintegration

→ In this case, there's a considerably greater freedom in the velocities of the resultant particles.

→ ∃ an upper limit to the KE of any one resulting particle.

To find the limit, consider a system formed by all particle except the one concerned (with mass m_1). The internal energy of this system is denoted by E_i' (This is obviously after the collision we're talking)

(assuming mass is additive & conserved)
And considering the 'system' as one particle

$$\frac{p_0^2}{2\left(\frac{(m_1)(M-m_1)}{M}\right)} = E = E_i - E_i' - E_{1i}$$

(using the discussion at the start)

This is NOT the disintegration energy of the whole system i.e. system of all particles

$$T_{10} = \frac{p_0^2}{2m_1} = \left(E_i - E_i' - E_{1i}\right)\frac{(M-m_1)}{M}$$

Evidently T_{10} is greatest when E_i' is least. Recalling the def^n of internal energy, we note that E_i' is least when all particles (except m_1) move with the same velocity, so then E_i' is the sum of their internal energies. ($=\sum_{n\neq 1} E_{ni}'$)

In this case,

$$T_{10,max} = \frac{(M-m_1)}{M}E$$

where E is the disintegration energy of the whole system

$$E = E_i - \sum_n E_{ni}'$$

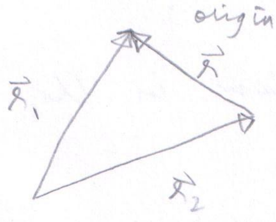
§ 17. Elastic Collisions

Elastic Collision = If the collision involves no change of their internal energy.

⇒ When the law of conservation of energy is applied, the internal energies can be ignored.

(understand this properly)

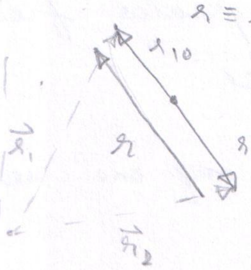
If not in the COM is the origin



If the COM is the origin

i.e. $m_1 r_{10} + m_2 r_{20} = 0$

$r \equiv r_{10} - r_{20} = r_1 - r_2$ (cool right?)



Then $r_{10} = \frac{m_2 r}{m_1+m_2}$

$r_{20} = -\frac{m_1 r}{m_1+m_2}$

Quantities in the C frame are distinguished using the subscript o.

⇒ $v_{10} = \frac{m_2 v}{m_1+m_2}$

$v_{20} = \frac{-m_1 v}{m_1+m_2}$

where $v = v_1 - v_2$

Claim: From the law of conservation of momentum, the momenta of the two particles remain equal & opposite after the collision.

In the C-frame, (before collision)

$$m_1\vec{r}_{10} + m_2\vec{r}_{20} = 0$$

$$\Rightarrow m_1\vec{v}_{10} + m_2\vec{v}_{20} = 0$$

$$\vec{p}_{10} + \vec{p}_{20} = 0 \Rightarrow \vec{p}_{10} = -\vec{p}_{20}$$

Thus by conservation of momentum, we have

$$\vec{p}_{10}' + \vec{p}_{20}' = 0 \qquad \text{Q.E.D.}$$

$$\vec{p}_{10}' = -\vec{p}_{20}'$$

Claim: Further, from conservation of energy, we know that their magnitude is unchanged.

$$2p_{10}^2 = 2p_{10}'^2 \qquad \text{(Elastic collision)}$$

$$\Rightarrow p_{10} = p_{10}'$$

∴ In the C-sys, the collision simply rotates the velocities, which remain opposite in direction & unchanged in magnitude.

We let $\vec{n}_0$ (unit vector) represent the direction of the velocity of the m_1 particle after collision. Then

$$\vec{v}_{10}' = \frac{m_2 v}{(m_1+m_2)}\vec{n}_0 \qquad \vec{v}_{20}' = -\frac{m_1 v}{(m_1+m_2)}\vec{n}_0$$

To transform to the L-frame then, we simply need to add velocity of the COM. (we're writing the vel. in terms of vel. before collision but that's okay as the vel. of COM doesn't change) TODO: show this more explicitly

$$\vec{v}_1' = \frac{m_2 v}{m_1+m_2}\vec{n}_0 + \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}$$

$$\vec{v}_2' = -\frac{m_1 v}{m_1+m_2}\vec{n}_0 + \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}$$

Remark: No further information about the collision can be obtained from the laws of conservation of momentum & energy.
∵ The direction of $\vec{n}_0$ depends on the law of interaction of the particles & on their relative pos. during collision.

Geometric Interpretation

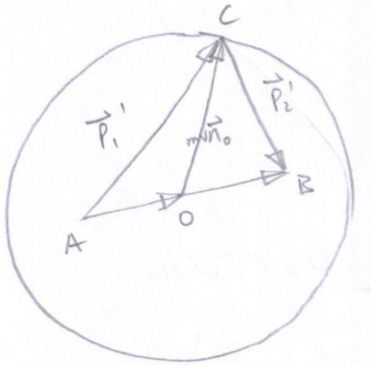
→ From the velocity eq^s, we can obtain the momenta eq^s.

$$\vec{p}_1{}' = m v \vec{n}_0 + \frac{m_1 (\vec{p}_1 + \vec{p}_2)}{(m_1 + m_2)}$$

$$\vec{p}_2{}' = - m v \vec{n}_0 + \frac{m_2 (\vec{p}_1 + \vec{p}_2)}{(m_1 + m_2)}$$

where $m = \frac{m_1 m_2}{m_1 + m_2}$

→



$$\vec{AO} = \frac{m_1}{m_1+m_2} (\vec{p}_1 + \vec{p}_2)$$

$$\vec{OB} = \frac{m_2}{m_1+m_2} (\vec{p}_1 + \vec{p}_2)$$

radius = mv

When $\vec{p}_1$ & $\vec{p}_2$ are given, then A & B are fixed. So is the radius (recall $\vec{v} = \frac{\vec{p}_1}{m_1} - \frac{\vec{p}_2}{m_2}$)

The point C may be anywhere on the circle though.

special case: m_2 is at rest before collision

(can't we always choose such a frame?)

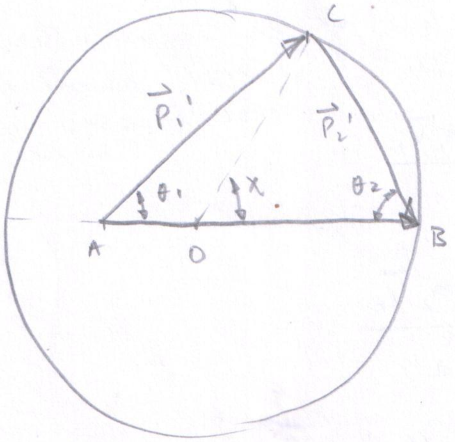
→ First, $OB = \frac{m_2 p_1}{m_1 + m_2} = m v$ = radius. ⇒ B lies on the circle.

→ $AB = AO + OB = \frac{m_1}{m_1+m_2}\vec{p}_1 + \frac{m_2}{m_1+m_2}\vec{p}_1 = \vec{p}_1$

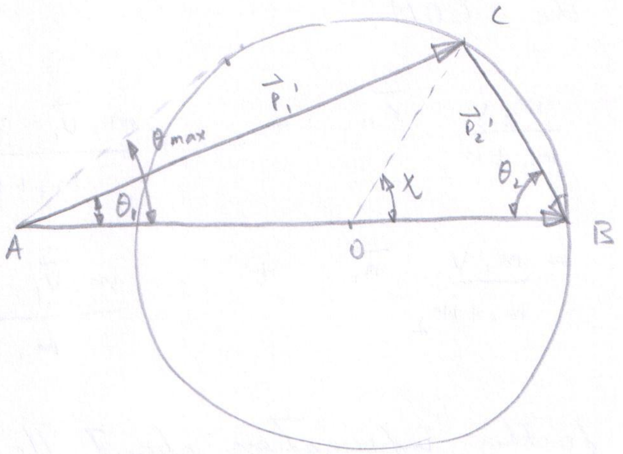
→ $\vec{OA} = \left(\frac{m_2}{m_2}\right) \frac{m_1 (m_1 \vec{v}_1)}{m_1 + m_2} = \frac{m_1}{m_2} m \vec{v}$ ⇒ $OA = \frac{m_1}{m_2} \times$ radius

⇒ A lies inside if $m_2 > m_1$ & outside if $m_1 > m_2$

→ We thus have

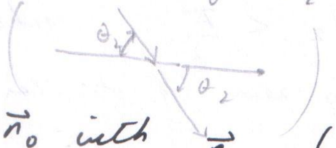


(a) $m_1 < m_2$



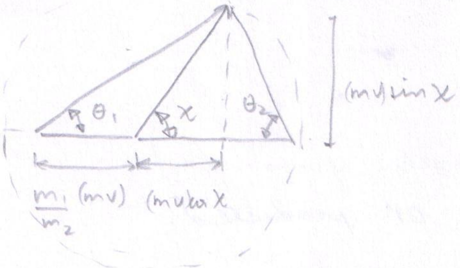
(b) $m_1 > m_2$

↳ θ_1 & θ_2 are angles b/w the directions of motion after collision, wrt the direction of impact ($\vec{p}_1$). (Note: even for θ_2 the statement is valid



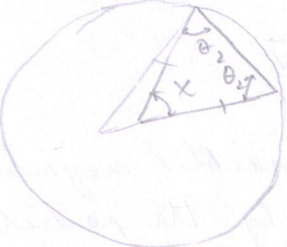
↳ Claim: χ gives the angle of $\vec{n}_0$ with $\vec{p}$. (By construction)

↳



$$\tan\theta_1 = \frac{\sin\chi}{\frac{m_1}{m_2} + \cos\chi} = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi}$$

(From fig (a) and)



$$\chi + 2\theta_2 = \pi \Rightarrow \theta_2 = \frac{1}{2}(\pi - \chi)$$

↳

$$v_1'^2 = \frac{p_1'^2}{m_1^2} = \frac{\left(\frac{m_1}{m_2} m v\right)^2 + (mv)^2 + 2\left(\left(\frac{m_1}{m_1}\right) m\right)(m) v^2 \cos\chi}{m_2}$$

$$= \frac{m^2 v^2}{m_1^2}\left(\frac{m_1^2}{m_2^2} + 1 + 2\frac{m_1}{m_2}\cos\chi\right)$$

$$= \left(\frac{m_1 m_2}{m_1 + m_2}\right)^2 \frac{v^2}{m_1^2 m_2^2}\left(m_1^2 + m_2^2 + 2 m_1 m_2 \cos\chi\right)$$

$$\Rightarrow v_1'^2 = \frac{v^2}{(m_1+m_2)^2}\left(m_1^2 + m_2^2 + 2 m_1 m_2 \cos\chi\right)$$

$$\Rightarrow v_1' = \frac{v}{m_1+m_2}\left(m_1^2 + m_2^2 + 2 m_1 m_2 \cos\chi\right)^{1/2}$$

$$v_2'^2 = \frac{(mv)^2 + (mv)^2 + 2(mv)(mv)\cos\chi}{m_2^2}$$

$$= \frac{m^2}{m_2^2} 2 v^2 (1 + \cos\chi) \overset{?}{=} \frac{2 m_1^2 v^2 (2) \sin^2\left(\frac{\chi}{2}\right)}{(m_1+m_2)^2}$$

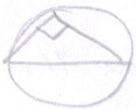
$$\Rightarrow v_2' = \frac{2 m_1 v}{m_1 + m_2} \sin\frac{\chi}{2}$$

↳ The sum $\theta_1 + \theta_2$ is the angle b/w the directions of motion ~~&~~ of the particles after collision. (just stare at it.)

Claim:

↳ Also, $\theta_1 + \theta_2 > \frac{\pi}{2}$ when $m_1 < m_2$

$\theta_1 + \theta_2 < \frac{\pi}{2}$ when $m_1 > m_2$

This can be easily seen by recalling that  and staring at the two figures.

↳ Claim: When the two particles (after collision) move in the same or opposite direction, $\chi = \pi$.

($\chi = 0$ makes $p_2' = 0$
rest, stare at the diagrams to convince yourself)

– In this case, the point C lies on OA produced. Thus after the collision, we have

$$\vec{v}_1' = \frac{m_1 - m_2}{m_1 + m_2}\vec{V}, \qquad \vec{v}_2' = \frac{2m_1}{m_1 + m_2}\vec{V}$$

– Also note, that $\vec{v}_1'$ has the greatest possible magnitude & the max energy which can be acquired by the particle initially at rest is therefore

$$E'_{2\,max} = \frac{1}{2} m_2 v'^{2}_{2\,max} = \frac{4 m_1 m_2}{(m_1 + m_2)^2} E_1$$

(E_1 is the energy of the incident particle $= \frac{1}{2} m v_1^2$)

↳ Claim: If $m_1 < m_2$, the velocity of m_1 after the collision can have any direction.
If $m_1 > m_2$, the particle m_1's velocity can't exceed θ_{max} (θ_1 is usual)

(stare at the diagram)

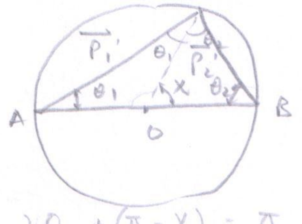
Evidently $\sin\theta_{max} = \frac{m_2}{m_1}$ $\left(= \frac{OC}{OA} = \frac{\cancel{mV}}{\frac{m_1}{m_2}\cancel{mV}} \right)$

'super special case': $m_1 = m_2$

→ both B & A lie on the circle ($\because \frac{m_2}{m_1} = 1$)

→ $\theta_1 = \frac{1}{2}\chi$

$\theta_2 = \frac{1}{2}(\pi - \chi)$

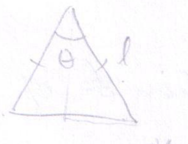


$2\theta_1 + (\pi - \chi) = \pi$

$2\theta_2 + \chi = \pi$

Remark: After the collision, the particles move at ⊥ to each other!

→ $v_1' = V\cos\left(\frac{1}{2}\chi\right)$ $\quad v_2' = V\sin\left(\frac{1}{2}\chi\right)$



$= 2l\sin\theta/2$

§18. Scattering

Motivation: A complete calculation of χ, requires soln. of eqⁿ of motion for the particular law of interaction involved.

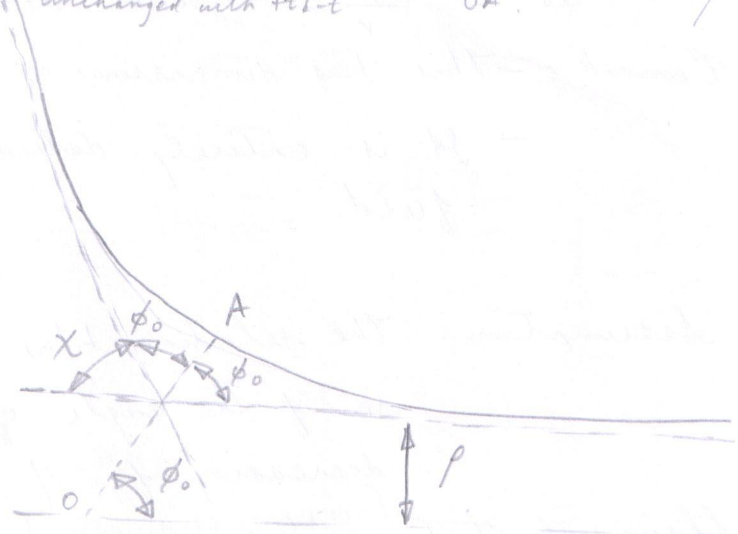
We consider the equivalent problem of the deflection of a single particle of mass m, moving in a field $U(r)$ whose centre is at rest.
(and is at the centre of mass of the two particles in the original problem)

Claim: The path of a particle in a central field is symmetric about a line from the centre to the nearest ~~ap~~ point in the orbit (OA in the figure).

($r_{min} \Rightarrow \dot{r}|_{r_{min}} = 0 \Rightarrow \dot{\phi} = \sqrt{\cdots}$ changes sign
plus path of r remains unchanged with $t \to -t$ $\Rightarrow$ symmetric about OA.)

$\Rightarrow$ The angle of the asymptotes with OA is the same.
(look at the figure)

Claim: $\chi = |\pi - 2\phi_0|$
(look at the figure)



Prior Result:

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{(M/r^2)\,dr}{\sqrt{2m(E - U(r)) - M^2/r^2}}$$

Now instead of E & M, the constants used would be ρ & v_∞ as they are more natural to the problem.

ρ = Impact Parameter ≡ The perpendicular distance of the centre from the velocity at ∞. ~~its~~

So then $E = \frac{1}{2} m v_\infty^2$
&
$M = m v_\infty \rho$

Thus

$$\phi_0 = \int_{r_{min}}^{\infty} \frac{(\rho/r^2)\,dr}{\sqrt{1 - \left(\frac{\rho^2}{r^2}\right) - \frac{2U}{m v_\infty^2}}}$$

So that does it then. These eqⁿs give χ as a fⁿ of ρ.

Motivation: Physically, we're concerned with problems involving not one, but a beam of particles (identical) incident with a uniform $\vec{v}_\infty$. ~~but~~ Different particles have different scattering parameters, and are thus scattered through different scattering angles χ.

~~Let~~ $dN \equiv$ # particles scattered per unit time, through angles b/w χ & $\chi + d\chi$

Remark: This number itself depends on the beam density, thus unsuitable for describing the scattering process.

$n \equiv$ # particles passing in unit time, through unit area of the beam cross section (The beam's assumed uniform over its cross section)

$d\sigma \equiv \frac{dN}{n}$: effective ~~cross section~~ ~~area~~ scattering cross section

Remark: → This has dimensions of area.

→ It is entirely determined from the form of the scattering field.

Assumption: The relation b/w χ & ρ is one-to-one; this is so if the angle of scattering is a monotonically decreasing f^n of the impact parameter.

(TODO: Understand this statement)

Claim: In that case
Then only those particles whose impact parameters lie b/w $\rho(\chi)$ & $\rho(\chi) + d\rho(\chi)$ are scattered at angles b/w χ & $\chi + d\chi$.

# of such ~~such~~ particles equals $dN = \cancel{2\pi(\rho + d\rho)}$
(per unit time)

$[2\pi(\rho) d\rho]\, n$

area × particles / (time × area)

$= 2\pi d\rho\, n$

(Area b/w circle of rad ρ & $\rho + d\rho$)

$\Rightarrow \quad d\sigma = 2\pi\rho\, d\rho$

Note ρ is a f^n of χ

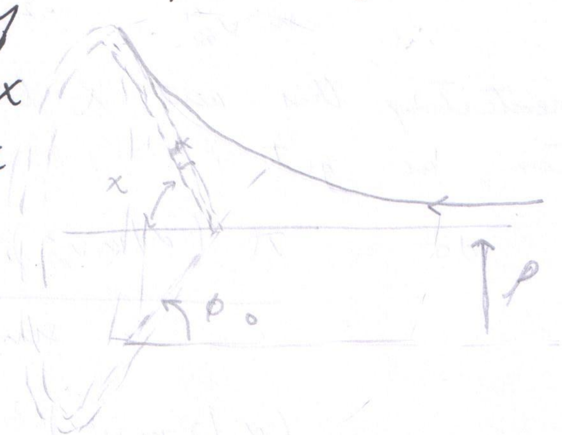
so you know the effective scattering cross section for a given $(\chi + d\chi)$ range!

$\Rightarrow \quad d\sigma = 2\pi\rho(\chi) \left|\frac{d\rho}{d\chi}\right| d\chi$

this is: $\frac{d\rho}{d\chi}$ is usually −ve subtle points - book

Convention: Often $d\sigma$ is ~~seff~~ referred to the solid angle element d_o instead of the plane angle element $d\chi$.

The solid angle b/w cones of vertical angles χ & $\chi + d\chi$ will be $d_o = 2\pi \sin\chi \, d\chi$



$$\Rightarrow d\sigma = \frac{\rho(\chi)}{\sin\chi} \left| \frac{d\rho}{d\chi} \right| d_o$$

— — —

Now back to the problem of scattering of a beam of particles, not by a central field but by other particles initially at rest.

~~We can say~~

Claim: The effective cross-section as a f^n of χ is given by

$$d\sigma = 2\pi\rho \left|\frac{d\rho}{d\chi}\right| d\chi$$

in the centre of mass system.

(TODO: Think why this is valid)

Tip:

To find the ~~scattering~~ effective cross section in terms of θ (L frame) relations derived between χ & θ ~~should~~ can be used.

This gives the scattering cross section for the incident beam (in terms of θ_1) & the particles at rest (same expression in terms of θ_2)!

§19. Rutherford's formula

Intro: For an application, consider the scattering of charged particle in a Coulomb field.

We put $U = \frac{\alpha}{R}$ in the prior result of the prev. section and integrate to get (verify this)

$$\phi_0 = \cos^{-1} \frac{\alpha / m v_\infty^2 \rho}{\sqrt{1 + (\alpha / m v_\infty^2 \rho)^2}} \Rightarrow \rho^2 = \frac{\alpha^2}{m^2 v_\infty^4} \tan^2 \phi_0$$

and using $\phi_0 = \frac{1}{2}(\pi - \chi)$

we get $\rho(\chi) =$ as

$$\rho^2 = \frac{\alpha^2}{m^2 v_\infty^4} \cos^2 \frac{\chi}{2}$$

differentiating this wrt χ & substituting in the scattering cross-section, we get (verify)

$$d\sigma = \frac{\pi \, (\alpha / m v_\infty^2)^2 \cos \frac{\chi}{2} \, d\chi}{\sin^3 \frac{\chi}{2}}$$

$$= (\alpha / 2 m v_\infty^2)^2 \frac{do}{\sin^4 \frac{\chi}{2}}$$

This then is Rutherford's Formula

Note: The effective cross-section is independent of the sign of α, so the result is equally valid for repulsive & attractive Coulomb fields.

Again as before, $d\sigma$ is given for in the COM frame of the colliding particles. To obtain the corresponding $d\sigma$ for the particle at rest, we use $\chi = \pi - 2\theta_2$ (proven earlier) to get

$$d\sigma_2 = 2\pi \left(\alpha / m v_\infty^2\right)^2 \sin\theta_2 \, d\theta_2 / \cos^3\theta_2 \qquad \text{(verify)}$$

$$= (\alpha / m v_\infty^2)^2 \, do_2 / \cos^3\theta_2$$

$d\sigma_1$ can also be found, but the result becomes complicated. So we discuss 2 particular cases.

case: $m_2 >> m_1$

$\Rightarrow \chi \sim \theta_1$ & $m \sim m_1$ $\left(\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}\right)$

$\left(\tan\theta_1 = \frac{\sin\chi}{\frac{m_1}{m_2} + \cos\chi}\right)$

then $$d\sigma_1 = (\alpha / 4 E_1)^2 \frac{do_1}{\sin^4 \frac{\theta_1}{2}}$$

where $E_1 = \frac{1}{2} m_1 v_\infty^2$ (energy of the incident particle)

case: $m_1 = m_2$ ($\Rightarrow m = \frac{m_1}{2}$) then (prior result) $\theta_1 = \frac{\chi}{2}$

then $$d\sigma_1 = 2\pi \left(\frac{\alpha}{E_1}\right)^2 \frac{\cos\theta_1 \, d\theta_1}{\sin^3\theta_1} = \left(\frac{\alpha}{E_1}\right)^2 \frac{\cos\theta_1 \, do_1}{\sin^4\theta_1}$$

Claim:
If the particles are identical (the incident & the ones at rest that is) then the total effective cross section is the sum of $d\sigma_1$ & $d\sigma_2$ & replacing θ_1 & θ_2 by their common value θ.

This is clear if we look at the result after transformation to the L frame. However I'm unclear about the result's validity if I start from $d\sigma(\chi)$.

$$d\sigma = \left(\frac{\alpha}{E_1}\right)^2\left(\frac{1}{\sin^4\theta} + \frac{1}{\cos^4\theta}\right)\cos\theta \, do$$

end case.

Claim: The (In the C frame) energy acquired by m_2 (and lost by m_1) is

$$\epsilon = \frac{1}{2} m_2 v_2'^2 = \frac{1}{2} m_2 \left[\frac{2 m_1}{m_1 + m_2}\right]^2 v_\infty^2 \sin^2\frac{\chi}{2}$$

(v' is known in terms of χ from prior discussions)

So representing $\sin\frac{\chi}{2}$ in terms of ϵ & substituting in $d\sigma$, we get

$$d\sigma = 2\pi \left(\frac{\alpha^2}{m_2 v_\infty^2}\right)\frac{d\epsilon}{\epsilon^2}$$

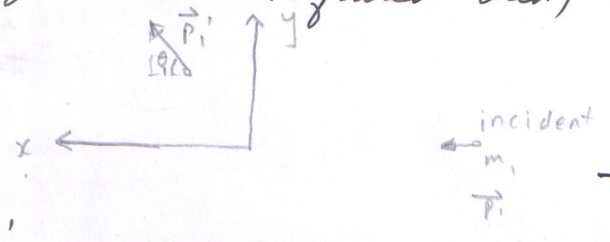
Remark ~~Claim~~: This gives the effective cross-section as a f^n of the energy loss ϵ.

Claim: $$\epsilon_{max} = \frac{2 m^2 v_\infty^2}{m_2}$$

§ 20. Small-angle scattering

Motivation: The calculation of the effective cross-section is much simplified if only those collisions are considered for which the impact parameter is so large the the field U is weak and the angles of deflection are small.

Remark: The calculation then may be done in the lab frame (the C frame isn't required then)



about the fig

→ x-axis is along the direction of the initial momentum of m_1, along $\vec{p}_1$.
→ xy plane is in the scattering ~~#~~ plane.

→ $\sin\theta_1 = \frac{p_{1y}'}{p_1'}$; For small θ_1 ; $\sin\theta_1 \sim \theta_1$ &
$p_1' \sim p_1 = m_1 v_\infty \Rightarrow \theta_1 \sim \frac{p_{1y}'}{m_1 v_\infty}$

Now,

→ $\dot{p}_{1y} = F_y$, Thus $p_{1y}' = \int_{-\infty}^{\infty} F_y \, dt$

Also, $F_y = -\frac{\partial U}{\partial y} = -\left(\frac{dU}{dr}\right)\frac{\partial r}{\partial y} = -\left(\frac{dU}{dr}\right)\frac{y}{r}$

Claim: Since the integral already has the small quantity U, it can be calculated, in the same approximation, by assuming that the particle is not deflected at all from its initial path.

viz. it moves along $y = \rho$, with v_∞.

(don't quite know why this holds, intuitively though it makes sense)

Then $p_{1y}' = \int_{-\infty}^{\infty} -\left(\frac{dU}{dr}\right)\frac{y}{r}\, dt = -\frac{\rho}{v_\infty}\int_{-\infty}^{\infty} \frac{dU}{dr}\frac{dx}{r}$

Trick: Change the integration from x to r.

In general, (in a plane) $r^2 = x^2 + y^2$

for $y = \rho$,

$r^2 = x^2 + \rho^2 \Rightarrow dx = \frac{r\,dr}{\sqrt{r^2-\rho^2}}$

Then $\theta_1 = \frac{p_{1y}'}{m_1 v_\infty} = -\frac{2\rho}{m_1 v_\infty^2}\int_{\rho}^{\infty}\frac{dU}{dr}\frac{dr}{\sqrt{r^2-\rho^2}}$

So we now have $\theta_1(\rho)$. We can find $\rho(\theta_1)$.

Then the effective cross-section simply becomes (with $\chi \to \theta_1$ & $\sin\chi \to \theta_1$ in $d\sigma = \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d_o$)

$$d\sigma = \frac{\rho(\theta_1)}{\theta_1}\left|\frac{d\rho}{d\theta_1}\right| d_o$$

DOUBT: This was derived for the C frame. How is it valid in the L frame? Shouldn't we have to transform χ to θ_1?

NOTE: For scattering we used the frame in which centre of the field was fixed & there were particles.

DOUBT related: How is this valid in the lab frame?