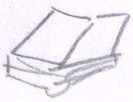
§ 7.1



$$(-1)^P \equiv \begin{cases} +1 & \text{if } P \text{ involves even \# of exchanges} \\ -1 & \text{if } P \text{ involves odd \# of exchanges} \end{cases}$$

$|\psi(1,2)\rangle = +|\psi(2,1)\rangle$ or

$|\psi(1,2)\rangle = -|\psi(2,1)\rangle$

(1) Bosons

$$P|\psi(1,\ldots N)\rangle = +|\psi(1,\ldots N)\rangle$$

(2) Fermions

$$P|\psi(1,\ldots N)\rangle = (-1)^P|\psi(1,\ldots N)\rangle$$

$\mathcal{H}$ itself is symmetric for identical particles

$P\mathcal{H} = \mathcal{H}$.

Eg. Consider N non-interacting particles, in a box of vol. V.

$$\mathcal{H} = \sum_{\alpha=1}^{N} \mathcal{H}_\alpha = \sum_{\alpha=1}^{N}\left(-\frac{\hbar^2}{2m}\nabla_\alpha^2\right)$$

Each $\mathcal{H}_\alpha$ can be diagonalized seperately with plane wave states $\{|\vec{k}\rangle\}$ & corresponding energies $\mathcal{E}(\vec{k}) = \frac{\hbar^2 k^2}{2m}$.

Using sums & products of these states we can construct

(1) the product Hilbert space:

$$|\vec{k}_1, \ldots \vec{k}_N\rangle_{\otimes} \equiv |\vec{k}_1\rangle \ldots |\vec{k}_N\rangle$$

$$= |\vec{k}_1, \ldots \vec{k}_N\rangle$$

In coordinate representation,

$$\langle \vec{x}_1 \ldots \vec{x}_N | \vec{k}_1, \ldots \vec{k}_N\rangle = \frac{1}{V^{N/2}} \exp\left(i \sum_{\alpha=1}^{N} \vec{k}_\alpha \cdot \vec{x}_\alpha\right)$$

&

$$\mathcal{H} |\vec{k}_1, \ldots \vec{k}_N\rangle = \left(\sum_{\alpha=1}^{N} \frac{\hbar^2}{2m} k_\alpha^2\right) |\vec{k}_1, \ldots \vec{k}_N\rangle$$

However the product states don't satisfy the (anti)-symmetrization properties.

(2) Fermionic subspace:

$$|\vec{k}_1 \ldots \vec{k}_N\rangle_- = \frac{1}{\sqrt{N_-}} \sum_P (-1)^P P |\vec{k}_1 \ldots \vec{k}_N\rangle$$

where the sum is over all $N!$ permutations

$\rightarrow N_- = N!$

$\rightarrow$ If two $\vec{k}$s are equal, the state vanishes

$\Rightarrow n_{\vec{k}} = 0$ or 1 (# states with momentum $\vec{k}$)

(3) Bosonic Subspace:

$$|\vec{k}_1 \ldots \vec{k}_N\rangle_+ = \frac{1}{\sqrt{N_+}} \sum_P P(|\vec{k}_1 \ldots \vec{k}_N\rangle)$$

$$\Gamma \quad {}_+\langle\{\vec{k}\}|\{\vec{k}\}\rangle_+ = \frac{1}{N_+} \sum_{P,P'} \langle P\{\vec{k}\}|P'\{\vec{k}\}\rangle$$

(IDK how) $$= \frac{N!}{N_+} \sum_P \langle\{\vec{k}\}|P\{\vec{k}\}\rangle = \frac{N! \prod_{\vec{k}} n_{\vec{k}}!}{N_+} = 1$$

$\Rightarrow N_+ = N! \prod_{\vec{k}} n_{\vec{k}}!$ | Also, $N_- = N! \cancel{\prod_{\vec{k}} n_{\vec{k}}!} \to 1$

$\Rightarrow N_- = N!$

$\longrightarrow N_+ = N! \prod_{\vec{k}} (n_{\vec{k}}!)$

Eg. A bosonic state for 2 particles in $|\alpha\rangle$, 1 particle in $|\beta\rangle$ will have

$$|\alpha\alpha\beta\rangle_+ = \frac{|\alpha\alpha\beta\rangle + |\alpha\beta\alpha\rangle + |\beta\alpha\alpha\rangle + |\alpha\alpha\beta\rangle + |\beta\alpha\alpha\rangle + |\alpha\beta\alpha\rangle}{\sqrt{12}}$$

$$= \frac{|\alpha\alpha\beta\rangle + |\alpha\beta\alpha\rangle + |\beta\alpha\alpha\rangle}{\sqrt{3}}$$

$\longrightarrow n_{\vec{k}}$ can be any int > 0.

$$|\{\vec{k}\}\rangle_\eta = \frac{1}{\sqrt{N_\eta}} \sum_P \eta^P P|\{\vec{k}\}\rangle, \text{ with } \eta = \begin{cases} +1 & \text{for Bosons} \\ -1 & \text{for Fermions} \end{cases}$$

§ 7.3 GRAND CANONICAL FORMULATION

N particles in a box, volume V.

$\sum_{\{K\}} \equiv \sum_{K_1, K_2, K_3 \ldots K_N}$ with each going from 0 to ∞ ? DOUBT what when all are zero

$$Z_N = \operatorname{tr}(e^{-\beta\mathcal{H}}) = \sum_{\{K_\alpha\}} \langle\{K_\alpha\}| e^{-\beta\mathcal{H}} |\{K_\alpha\}\rangle$$

tensor product state $\quad \varepsilon(\vec{k}) = \frac{\hbar^2 k^2}{2m}$

$$= \sum_{\{K_\alpha\}} e^{-\beta \sum_{\alpha=1}^{N} \varepsilon(\vec{k}_\alpha)}$$

$$= \sum_{\{n_{\vec{k}}\}} e^{-\beta \sum_{\vec{k}} \varepsilon(\vec{k}) n_{\vec{k}}}$$

$\because$ the particles are identical, I can only say how many have what momentum.

Remarks: Hard to find, $\because$ of restrictions of $\{n_{\vec{k}}\}$.

1) $\sum_{\vec{k}} n_{\vec{k}} = N$

2) (A) $n_{\vec{k}} = 0$ or 1 (Fermion)
(B) $n_{\vec{k}} = 0, 1, 2 \ldots$ (Bosons)

These contraints can be removed by looking at

$$\Omega_\eta(T,\mu) = \sum_{N=0}^{\infty} e^{+\beta\mu N} \sum_{\substack{\{n_{\vec{k}}\} \\ \text{s.t.} \sum_{\vec{k}} n_{\vec{k}} = N}} e^{-(\beta \sum_{\vec{k}} \varepsilon(\vec{k}) n_{\vec{k}})}$$

(claim) $$= \sum_{\{n_{\vec{k}}\}}^{\eta} \prod_{\vec{k}} e^{-\beta(\varepsilon(\vec{k})-\mu)n_{\vec{k}}}$$

Proof: $$\sum_{N=0}^{\infty} \sum_{\substack{\{n_{\vec{k}}\} \\ \sum_{\vec{k}} n_{\vec{k}} = N}}^{\eta} = \sum_{\substack{\{n_{\vec{k}}\} \\ \sum_{\vec{k}} n_{\vec{k}} = 0}}^{\eta} + \sum_{\substack{\{n_{\vec{k}}\} \\ \sum_{\vec{k}} n_{\vec{k}} = 1}}^{\eta} + \cdots$$

$$= \sum_{\{n_{\vec{k}}\}}^{\eta}$$

So $$\Omega_\eta(T,\mu) = \sum_{\{n_{\vec{k}}\}}^{\eta} e^{+\beta\mu \sum_{\vec{k}} n_{\vec{k}} - \beta \sum_{\vec{k}} \varepsilon(\vec{k}) n_{\vec{k}}}$$

$$= \sum_{\{n_{\vec{k}}\}}^{\eta} e^{\sum_{\vec{k}} \beta(\mu - \varepsilon(\vec{k}))n_{\vec{k}}}$$

$$= \sum_{\{n_{\vec{k}}\}}^{\eta} \prod_{\vec{k}} e^{-\beta(\varepsilon(\vec{k})-\mu)n_{\vec{k}}}$$

Not sure

$$= \prod_{\vec{k}} \sum_{n_{\vec{k}}}^{\eta} e^{-\beta(\varepsilon(\vec{k})-\mu)n_{\vec{k}}}$$

Remarks:

1) Fermions: $n_{\vec{k}} = 0$ or 1.

$$\Omega_- = \prod_{\vec{k}} \left[1 + e^{-\beta(\varepsilon(\vec{k})-\mu)}\right]$$

2) Bosons: $n_{\vec{k}} = 0, 1, 2 \ldots$

$$\Omega_+ = \prod_{\vec{k}} \frac{1}{\left[1 + e^{-\beta(\varepsilon(\vec{k})-\mu)}\right]}$$

$f(x) = a + ax + ax^2 \ldots$

$a + xf(x) = f(x)$

$\Rightarrow \frac{a}{1-x} = f(x)$

Combined

$$\boxed{\ln \Omega_\eta = -\eta \sum_{\vec{k}} \ln\left[1 - \eta e^{-\beta(\varepsilon(\vec{k})-\mu)}\right]}$$

In the grand canonical formulation, different one-particle states are occupied independently. Thus we have the following

$$P_\eta(\{n(\vec{k})\}) = \frac{1}{\underset{\uparrow}{\mathcal{Q}_\eta}} \prod_{\vec{k}} \left[e^{-\beta(\varepsilon(\vec{k})-\mu)n_{\vec{k}}}\right]$$

we just calculated this

joint probability

The average occupation number of a state with energy $\varepsilon(\vec{k})$ is given by

$$\boxed{\langle n_{\vec{k}}\rangle_\eta = -\frac{\partial \ln \mathcal{Q}_\eta}{\partial(\beta\varepsilon(\vec{k}))} \overset{\text{claim}}{=} \frac{1}{z^{-1}e^{\beta\varepsilon(\vec{k})}-\eta}}$$

where $z = e^{\beta\mu}$

[Proof $\displaystyle = \eta \frac{\partial \sum_{\vec{k}} \ln[1-\eta e^{-\beta(\varepsilon(\vec{k})-\mu)}]}{\partial(\beta\varepsilon(\vec{k}))}$

$$= \cancel{\eta}\, \frac{1}{1-\eta e^{-\beta(\varepsilon(\vec{k})-\mu)}} \cdot (\cancel{-}\eta)\cdot(\cancel{-1})\, e^{-\beta(\varepsilon(\vec{k})-\mu)}$$

$$= \frac{1}{e^{\beta(\varepsilon(\vec{k})-\mu)}-\eta}$$

$\square$

The average value of particle # & internal energy are then given by

$$\boxed{\begin{aligned} N_\eta &= \sum_{\vec{k}} \langle n_{\vec{k}}\rangle_\eta = \sum_{\vec{k}} \frac{1}{z^{-1}e^{\beta\varepsilon(\vec{k})}-\eta} \\ E_\eta &= \sum_{\vec{k}} \varepsilon(\vec{k})\langle n_{\vec{k}}\rangle_\eta = \sum_{\vec{k}} \frac{\varepsilon(\vec{k})}{z^{-1}e^{\beta\varepsilon(k)}-\eta} \end{aligned}}$$

§ 7.4 | Non-relativistic gas

- Quantum Particles are further characterized by a spin.
  ↳ In the absence of a magnetic field, different spin states have the same energy.
  Thus a spin degeneracy factor

$$g = 2S+1$$

is multiplied to, in the $\ln \Omega_\eta$ eq^n.

→ Further, Taking V to be large, we have $\sum_{\vec{k}} \to \frac{V}{(2\pi)^3}\int d^3\vec{k}$

also anyway, $\varepsilon(\vec{k}) = \frac{\hbar^2 k^2}{2m}$

$$\beta \overset{\text{pressure}}{P_\eta} = \frac{\ln \Omega_\eta}{V} = -\eta g \int \frac{d^3\vec{k}}{(2\pi)^3} \ln\left[1 - \eta \xi e^{-\frac{\beta \hbar^3 k^2}{2m}}\right]$$

$-\beta \mathcal{G} = \ln \Omega$
$E = TS + \mu N - PV$
$\mathcal{G} = E - TS - \mu N$
$\to \mathcal{G} = -PV$

$$n_\eta \equiv \frac{N_\eta}{V} = g \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\left(\xi^{-1} e^{\frac{\beta \hbar^2 k^2}{2m}} - \eta\right)}$$

density of states?

$$\varepsilon_\eta \equiv \frac{E_\eta}{V} = g \int \frac{d^3\vec{k}}{(2\pi)^3} \cancel{\varepsilon(\vec{k})} \frac{\hbar^2 k^2}{2m} \cdot \frac{1}{\left(\xi^{-1} e^{\frac{\beta \hbar^2 k^2}{2m}} - \eta\right)}$$

Energy density?

To simplify, we change variables to $x = \frac{\beta \hbar^2 k^2}{2m}$

so that $k = \frac{\sqrt{2m k_B T}}{\hbar} x^{1/2} = \frac{2\pi^{1/2}}{\lambda} x^{1/2} \Rightarrow dk = \frac{\pi^{1/2}}{\lambda} x^{-1/2} dx$

Then

$$\beta P_\eta \overset{\text{claim}}{=} -\eta \frac{g}{2\pi^2}\cdot\frac{4\pi^{3/2}}{\lambda^3}\int_0^\infty dx\, x^{1/2}\ln(1-\eta\xi e^{-x})$$

$$= \frac{g}{\lambda^3}\,\frac{4}{3\sqrt{\pi}}\int_0^\infty \frac{dx\ \ x^{1/2}}{\xi^{-1}e^{x}-\eta} \qquad \text{(Integration by parts)}$$

$$\beta P_\eta = -\eta g\int \frac{d^3\vec{k}}{(2\pi)^3}\ \ln\left[1-\eta\xi e^{-\frac{\beta\hbar^2k^2}{2m}}\right]$$

$$= -\eta g\, 4\pi\int\frac{k^2dk}{(2\pi)^3}\ \ln\left[1-\eta\xi e^{-\frac{\beta\hbar^2k^2}{2m}}\right]$$

$$= -\eta g\,\frac{4\pi}{(2\pi)^3}\int\left(\frac{4\pi}{\lambda^2}x\right)\cdot\left(x^{-1/2}dx\cdot\frac{\pi^{1/2}}{\lambda}\right)\ln\left[1-\eta\xi e^{-\cancel{\frac{\beta\hbar^2k^2}{2m}}}\right]$$

claim 1 □ $= -\frac{\eta g}{2\pi^2}\cdot\frac{4\pi^{3/2}}{\lambda^3}\int_0^\infty dx\ x^{1/2}\ \ln\left[1-\eta\xi e^{-x}\right]$

$$\int_0^\infty dx\ x^{1/2}\ \ln\left[1-\eta\xi e^{-x}\right]$$

(Claim, not justified, can try (Hospital))

$$\int_a^b uv\,dx = \left[u\int v\,dx\right]_a^b + \int_a^b u'\left(\int v\,dx\right)dx$$

$$= \left[\ln\left[1-\eta\xi e^{-x}\right]\cdot\frac{x^{3/2}}{3/2}\right]_0^\infty$$

$$-\int_0^\infty dx\ \frac{x^{3/2}}{3/2}\cdot\frac{1}{(1-\eta\xi e^{-x})}\cdot\eta\xi e^{-x}$$

$$= +\frac{\cancel{\eta}g}{\cancel{2}\pi^{2-3/2}}\ \frac{\cancel{4}\cancel{\pi^{3/2}}}{\lambda^3}\cdot\frac{2}{3}\,\cancel{\eta}\,\xi\int_0^\infty\frac{dx\ \ x^{3/2}}{\eta\,\xi^{-1}e^{x}-\eta}$$

□

$$n_\eta \overset{\text{claim}}{=} \frac{g}{\lambda^3}\,\frac{2}{\sqrt{\pi}}\int_0^\infty\frac{dx\ \ x^{1/2}}{\xi^{-1}e^{x}-\eta}$$

$$= \frac{gV}{(2\pi)^3}\int_0^\infty 4\pi k^2\,dk\ \frac{1}{\xi^{-1}e^{x}-\eta}$$

$$= \frac{2g}{2\cdot 4\cancel{\pi^3}\sqrt{\pi}}\int_0^\infty \cancel{4\pi}\left(\frac{\sqrt{\pi}}{\lambda}\ x^{-1/2}dx\right)\left(\frac{4\pi x}{\lambda^2}\right)\frac{1}{\xi^{-1}e^{x}-\eta}$$

$$= \frac{2gV}{\lambda^3\sqrt{\pi}}\int_0^\infty\frac{dx\ \ x^{1/2}}{\xi^{-1}e^{x}-\eta}$$

□

$$\beta \varepsilon_\eta \overset{\text{claim}}{=} \frac{g}{\lambda^3} \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{dx\, x^{3/2}}{\xi^{-1} e^x - \eta}$$

Γ obvious $\square$

We define

$$f_m^\eta(z) = \frac{1}{(m-1)!} \int_0^\infty \frac{dx\, x^{m-1}}{\xi^{-1} e^x - \eta}$$

$$\Gamma \quad m! \equiv \Gamma(m+1) = \int_0^\infty dx\, x^m e^{-x}$$

claim: $\left(\frac{1}{2}\right)! = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$

$$\Rightarrow \left(\frac{3}{2}\right)! = \frac{3}{2} \frac{\sqrt{\pi}}{2}$$

$\llcorner$

$$\Rightarrow \quad \beta P_\eta = \frac{g}{\lambda^3} f_{5/2}^\eta(\xi)$$

$$n_\eta = \frac{g}{\lambda^3} f_{3/2}^\eta(\xi)$$

$$\varepsilon_\eta = \frac{3}{2} P_\eta$$

To find the eq^n of state explicitly, we need to understand

$f_m^\eta(\xi) =$

sum of gp or else binomial expand

$$\frac{1}{(m-1)!} \int_0^\infty \frac{dx\, x^{m-1}}{\xi^{-1} e^x - \eta} = \frac{1}{(m-1)!} \int_0^\infty dx\, x^{m-1} (\xi e^{-x}) (1 - \eta \xi e^{-x})^{-1}$$

$$= \frac{1}{(m-1)!} \int_0^\infty dx\, x^{m-1} (\xi e^{-x}) \sum_{\alpha=0}^\infty (\eta)^\alpha (\xi e^{-x})^\alpha = \sum_{\alpha=1}^\infty \eta^{\alpha+1} (\xi)^\alpha \left[\frac{1}{(m-1)!} \int_0^\infty dx\, x^{m-1} e^{-\alpha x} \right]$$

$$\Gamma\ \xi e^{-x} \sum_{\alpha=0}^\infty (\eta)^\alpha (\xi e^{-x})^\alpha$$

$$= \sum_{\alpha=0}^\infty (\eta)^\alpha (\xi e^{-x})^{\alpha+1}$$

$$= \sum_{\alpha=1}^\infty (\eta)^{\alpha-1} (\xi e^{-x})^\alpha (\eta)^2$$

$$= \sum_{\alpha=1}^\infty (\eta)^{\alpha+1} (\xi e^{-x})^\alpha$$

$\llcorner$

$$\Gamma \quad \frac{1}{\alpha^m} \int_0^\infty \alpha\, dx\, (\alpha x)^{m-1} e^{-\alpha x} = \frac{(m-1)!}{\alpha^m}$$

$$\int_0^\infty dx\, x^{m-1} e^{-\alpha x}$$

$\llcorner$

$$= \sum_{\alpha=1}^{\infty} \eta^{\alpha+1} \frac{\xi^\alpha}{\alpha^m} \cancel{\frac{(m-1)!}{(m-1)!}}$$

$$= \xi + \eta \frac{\xi^2}{2^m} + \frac{\xi^3}{3^m} + \eta \frac{\xi^4}{4^m} + \cdots$$

We thus have

$$\frac{n_\eta \lambda^3}{g} = f^\eta_{3/2}(\xi) = \xi + \eta \frac{\xi^2}{2^{3/2}} + \frac{\xi^3}{3^{3/2}} + \eta \frac{\xi^4}{4^{3/2}} + \cdots$$

$$\beta \frac{P_\eta \lambda^3}{g} = f^\eta_{5/2}(\xi) = \xi + \eta \frac{\xi^2}{2^{5/2}} + \frac{\xi^3}{3^{5/2}} + \eta \frac{\xi^4}{4^{5/2}} + \cdots$$

We solve perturbatively using recursion

$$\xi = \frac{n_\eta \lambda^3}{g} - \eta \frac{\xi^2}{2^{3/2}} - \frac{\xi^3}{3^{3/2}} + \mathcal{O}(\xi^4)$$

$$= \left(\frac{n_\eta \lambda^3}{g}\right) - \frac{\eta}{2^{3/2}} \left(\frac{n_\eta \lambda^3}{g}\right)^2 - \frac{\xi^3}{3^{3/2}} + \mathcal{O}(\xi^4)$$

IDK how, but I am close

$$= \left(\frac{n_\eta \lambda^3}{g}\right) - \frac{\eta}{2^{3/2}} \left(\frac{n_\eta \lambda^3}{g}\right)^2 + \left(\frac{1}{4} - \frac{1}{3^{3/2}}\right) \left(\frac{n_\eta \lambda^3}{g}\right)^3$$

$$\xi^2 \approx \left(\frac{n_\eta \lambda^3}{g} - \eta \frac{\xi^2}{2^{3/2}}\right)^2$$

$$\approx \left(\frac{n_\eta \lambda^3}{g}\right)^2 - 2\left(\eta \frac{\xi^2}{2^{3/2}}\right)\left(\frac{n_\eta \lambda^3}{g}\right) \approx \left(\frac{n_\eta \lambda^3}{g}\right)^2 - 2\left(\frac{\eta}{2^{3/2}}\left(\frac{n_\eta \lambda^3}{g}\right)^2\right)\left(\frac{n_\eta \lambda^3}{g}\right)$$

$$\approx \left(\frac{n_\eta \lambda^3}{g}\right)^2 - \frac{2\eta}{2^{3/2}} \left(\frac{n_\eta \lambda^3}{g}\right)^3$$

$$\xi^3 \approx \left(\frac{n_\eta \lambda^3}{g} + \frac{\eta \xi^2}{2^{3/2}}\right)^3$$

$$\approx \left(\frac{n_\eta \lambda^3}{g}\right)^3 + 3\left(\frac{n_\eta \lambda^3}{g}\right)^2 \frac{\eta \xi^2}{2^{3/2}} + \mathcal{O}(\xi^3)$$

Note : $\xi^2 \approx \left(\frac{n_\eta \lambda^3}{g}\right)^2$

$$\xi^3 \approx -\left(\frac{1}{4} - \frac{1}{3^{3/2}}\right)\left(\frac{n_\eta \lambda^3}{g}\right)^3 \cdot 3^{3/2}$$

(just look at the series we obtained & look at the contribution & where its coming from)

Now in

$$\frac{\beta P_\eta \lambda^3}{g} = \xi + \eta \frac{\xi^2}{2^{5/2}} + \frac{\xi^3}{3^{5/2}} + \dots$$

we substitute for ξ, ξ^2 & ξ^3 as just calculated.

$$\frac{\beta P_\eta \lambda^3}{g} = \left[\left(\frac{n_\eta \lambda^3}{g}\right) - \frac{\eta}{2^{3/2}}\left(\frac{n_\eta \lambda^3}{g}\right)^2 + \left(\frac{1}{4} - \frac{1}{3^{3/2}}\right)\left(\frac{n_\eta \lambda^3}{g}\right)^3 \dots\right]$$

$$+ \frac{\eta}{2^{5/2}}\left[\left(\frac{n_\eta \lambda^3}{g}\right)^2 + \dots\right] + \frac{1}{3^{5/2}}\left[-\left(\frac{1}{4} - \frac{1}{3^{3/2}}\right)\left(\frac{n_\eta \lambda^3}{g}\right)^3 \cdot 3^{3/2} + \dots\right]$$

$$\left[P = k_B T n_\eta \left(n_\eta \frac{\lambda^3}{g}\right)^{-1} \left\{ \left[\left(\frac{n_\eta \lambda^3}{g}\right) - \dots \right] \right\} \right]$$

$$P = k_B T n_\eta \left\{ \left[1 - \frac{\eta}{2^{3/2}}\left(\frac{n_\eta \lambda^3}{g}\right) + \left(\frac{1}{4} - \frac{1}{3^{3/2}}\right)\left(\frac{n_\eta \lambda^3}{g}\right)^2 \dots \right] \right.$$

$$\left. + \frac{\eta}{2^{5/2}}\left[\left(\frac{n_\eta \lambda^3}{g}\right) + \dots\right] + \frac{1}{3^{5/2}}\left[\left(\frac{1}{3^{3/2}} - \frac{1}{4}\right) \cdot 3^{3/2}\left(\frac{n_\eta \lambda^3}{g}\right)^2 + \dots\right] \right\}$$

just adding coefficients we get

$$\boxed{P = n_\eta k_B T \left[1 - \frac{\eta}{2^{5/2}}\left(\frac{n_\eta \lambda^3}{g}\right) + \left(\frac{1}{8} - \frac{2}{3^{5/2}}\right)\left(\frac{n_\eta \lambda^3}{g}\right)^2 + \dots\right]}$$

$\frac{n_\eta \lambda^3}{g} \geq 1$ (The quantum degenerate lt)

This is when quantum fx becomes important.

§7.5 THE DEGENERATE FERMI GAS

At zero temperature limit, the fermi occupation #

$$\langle n_{\vec{k}}\rangle_- = \frac{1}{e^{\beta(\varepsilon(\vec{k})-\mu)}+1} = \begin{cases} 1 & \varepsilon(\vec{k})-\mu<0 \\ 0 & \text{else} \end{cases}$$

The limiting value of μ at zero temperature is called the fermi energy, ε_F

All states with energy less than ε_F are occupied, forming a fermi sea.

For the ideal gas, with $\varepsilon(\vec{k}) = \frac{\hbar^2 k^2}{2m}$, there's a corresponding Fermi Wavenumber k_F

$$N = \sum_{|\vec{k}|\le k_F} (2s+1) = gV\int^{k<k_F}\frac{d^3\vec{k}}{(2\pi)^3} = g\frac{V}{6\pi^2}k_F^3$$

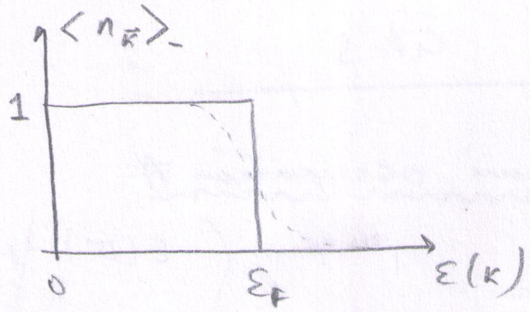
$$\frac{gV\,4\cancel{\pi}}{2(\cancel{2}\pi)^{\cancel{3}2}}\int^{k<k_F} k^2\,dk$$

$$= \frac{gV}{2\pi^2}\cdot\frac{k_F^3}{3}$$

In terms of # density $n = \frac{N}{V}$,

$$k_F = \left(\frac{6\pi^2 n}{g}\right)^{1/3} \Rightarrow \varepsilon_F(n) = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{2/3}$$

Doubt



To see how the fermi sea is modified at low temperatures, we need to know the behaviour of $f_m^-(z)$ for large z, which is

$$f_m^-(z) \overset{\text{claim}}{=} \frac{1}{m!}\int_0^\infty dx\, x^m \frac{d}{dx}\left(\frac{-1}{z^{-1}e^x+1}\right)$$

$$\Gamma\; f_m^\eta(z) = \frac{1}{(m-1)!}\int_0^\infty \frac{dx\; x^{m-1}}{z^{-1}e^x - \eta}$$

$$= \frac{1}{(m-1)!}\left[\left[\frac{1}{z^{-1}e^x-\eta}\cdot\frac{x^m}{m}\right]_0^{\infty \to 0} - \int_0^\infty \frac{d}{dx}\left(\frac{1}{z^{-1}e^x-\eta}\right)\frac{x^m}{m}\,dx\right]$$

$\lrcorner\,\square$

Since the derivative in the above equation is sharply peaked, we can expand around this peak.

Let $x = \ln z + t$ & t ranges from $-\infty$ to ∞

$$f_m^-(z) \approx \frac{1}{m!}\int_{-\infty}^\infty dt\,(\ln z + t)^m \frac{d}{dt}\left(\frac{-1}{e^t+1}\right)$$

$$\left[\; e^x = e^t \cdot z \right.$$

→ on the generalized binomial expansion

$$= \frac{1}{m!}\int_{-\infty}^\infty dt \sum_{\alpha=0}^\infty \binom{m}{\alpha} t^\alpha (\ln z)^{m-\alpha} \frac{d}{dt}\left(\frac{-1}{e^t+1}\right)$$

$$= \frac{(\ln z)^m}{m!}\sum_{\alpha=0}^\infty \frac{m!}{\alpha!\,(m-\alpha)!}(\ln z)^{-\alpha}\int_{-\infty}^\infty dt\; t^\alpha \frac{d}{dt}\left(\frac{-1}{e^t+1}\right)$$

$$\int_a^b uv\, dx = \left[u \int v\, dx\right]_a^b - \int_a^b u' \left(\int v\, dx\right) dx$$

When $\left[u \int v\, dx\right]_a^b = 0$,

$$\int_a^b uv\, dx = -\int_a^b u' \left(\int v\, dx\right) dx$$

$$\int_{-\infty}^{\infty} dt \quad \frac{d}{dt}\left(\frac{-1}{e^t+1}\right) \quad t^\alpha$$

$$u = \frac{-1}{e^t+1}$$

$$v = \alpha t^{\alpha-1}$$

$$= \int_{-\infty}^{\infty} dt \quad \frac{\alpha t^{\alpha-1}}{e^t+1}$$

Using the (anti-)symmetry of the integral under $t \to -t$, & undoing integration by parts, we have

$$\frac{1}{\alpha!} \int_{-\infty}^{\infty} dt \quad t^\alpha \quad \frac{d}{dt}\left(\frac{-1}{e^t+1}\right) = \begin{cases} 0 & \text{for odd } \alpha \\ \dfrac{2}{(\alpha-1)!} \displaystyle\int_0^{\infty} dt \ \frac{t^{\alpha-1}}{e^t+1} & \text{for even } \alpha \end{cases}$$

$$= 2 f_\alpha^-(1)$$

Plugging this in we get

$$\lim_{z\to\infty} f_m^-(z) = \frac{(\ln z)^m}{m!} \sum_{\substack{\alpha=0 \\ (\text{even})}} 2 f_\alpha^-(1) \frac{m!}{(m-\alpha)!} (\ln z)^{-\alpha}$$

looking up integrals (tabulated)

$$= \frac{(\ln z)^m}{m!}\left[1 + \frac{\pi^2}{6} \frac{m(m-1)}{(\ln z)^2} + \frac{7\pi^4}{360} \frac{m(m-1)(m-2)(m-3)}{(\ln z)^4} + \cdots\right]$$

Now we have in the "degenerate lt",

$$\frac{n\lambda^3}{g} = f_{3/2}^-(z) = \frac{(\ln z)^{3/2}}{(3/2)!}\left[1 + \frac{\pi^2}{6} \cdot \frac{3}{2} \cdot \frac{1}{2} (\ln z)^{-2} + \cdots\right] \gg 1$$

The lowest order result produces the eq^n for Fermi energy.

$$\lim_{T\to 0} \ln z = \left(\frac{3}{2}!\ \frac{n\lambda^3}{g}\right)^{2/3} = \left(\frac{3}{4\sqrt{\pi}}\lambda^3\frac{n}{g}\right)^{2/3}$$

Putting $\lambda = \dfrac{h}{\sqrt{2\pi m k_B T}}$

$$\left(\frac{3}{4\sqrt{\pi}}\frac{(2\pi\hbar)^3}{(2\pi m k_B T)^{3/2}}\frac{n}{g}\right)^{2/3}$$

$$\left(\frac{3}{4\sqrt{\pi}}\frac{(2\pi\hbar)}{(2\pi m k_B T)^{3/2}}\frac{n}{g}\right)^{2/3}$$

$$= \frac{\beta\hbar^2}{2m}\left(\frac{3}{\cancel{4\sqrt{\pi}}}\ 2^{\cancel{3}}\ \frac{(\pi)^{-1}}{\cancel{(\pi)^{3/2}}}\frac{n}{g}\right)^{2/3}$$

$$= \frac{\beta\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{2/3}$$

□

$$= \frac{\beta\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{2/3} = \beta\varepsilon_F$$

as expected.

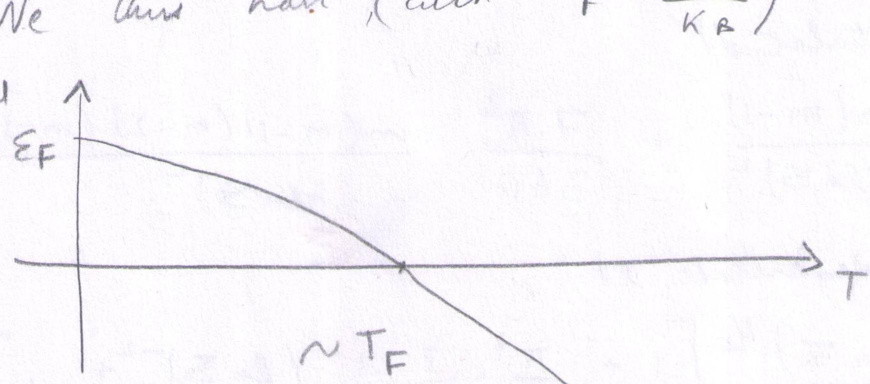
Inverting the zero-temp limit, we have the first order

$$\ln z = \beta\varepsilon_F\left[1 + \frac{\pi^2}{8}\left(\frac{k_BT}{\varepsilon_F}\right)^2 + \dots\right]^{-2/3}$$

$$\approx \beta\varepsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_BT}{\varepsilon_F}\right)^2 + \dots\right]$$

Now since $z = e^{\beta\mu}$ $\Rightarrow$ $\ln z = \beta\mu$
$\Rightarrow$ $k_BT\ln z = \mu$

We thus have (with $T_F = \dfrac{\varepsilon_F}{k_B}$)



At high temperature $\ln z$ is -ve.

DOUBT
Why/How?

Similarly, the low temp expansion for the pressure is

$$\beta P = \frac{g}{\lambda^3} f^-_{5/2}(z)$$

$$= \frac{g}{\lambda^3} \frac{(\ln z)^{5/2}}{(5/2)!}\left[1 + \frac{\pi^2}{\cancel{6}_2}\frac{5}{2}\frac{\cancel{3}}{2}(\ln z)^{-2} + \dots\right]$$

$$= \frac{g}{\lambda^3}\frac{8}{15}\frac{(\beta\varepsilon_F)^{5/2}}{\sqrt{\pi}}\left[1 - \frac{5}{2}\frac{\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \dots\right]\cdot\left[1 + \frac{5\pi^2}{8}\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \dots\right]$$

using these

$$\frac{5}{2}! = \frac{5}{2}\cdot\frac{3}{2}\cdot\left(\frac{1}{2}\right)!$$

$$= \frac{15}{4}\frac{\sqrt{\pi}}{2} = \frac{15}{8}\sqrt{\pi}$$

$$\ln z = \beta\varepsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \dots\right]$$

$$(\ln z)^{5/2} = \beta\varepsilon_F\left[1 - \frac{5}{2}\frac{\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \dots\right]$$

Next

$$= \beta\frac{g}{(2\pi\hbar)^3}\frac{(2\pi \cancel{k_B T} m)^{3/2}}{\cancel{(\beta)^{3/2}}}\frac{(\cancel{\beta}\varepsilon_F)^{5/2}}{\sqrt{\pi}}\cdot\frac{8}{15}\cdot\left[1 + \underbrace{\left(-\frac{5}{2}\frac{\pi^2}{12} + \frac{5\pi^2}{8}\right)}_{\frac{5}{12}\pi^2}\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \dots\right]$$

$$n\beta\frac{\pi^2\pi^{3/2}}{\sqrt{\pi}\cdot(2\pi)^3(6\hbar^3\pi^2 n)}\left(g\,(2m)^{3/2}\right)\varepsilon_F^{5/2}\frac{\cancel{6}\cdot\cancel{8}^2}{2^3\,\cancel{15}_5}$$

$$= n\beta\frac{\varepsilon_F^{\cancel{5/2}}}{\cancel{\varepsilon_F^{3/2}}}\cdot\frac{2}{5}$$

$$= \beta\left(\frac{2}{5}n\varepsilon_F\right)$$

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{2/3}$$

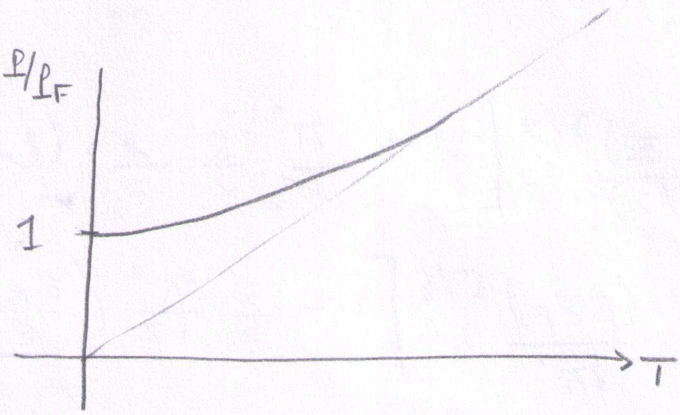
$$\varepsilon_F^{3/2} = \frac{\hbar^3}{(2m)^{3/2}}\frac{6\pi^2 n}{g}$$

$$\frac{\pi^{3/2 - 1/2 - 3}}{\cancel{\pi^{1/2}}\,\cancel{\pi^3}} = \pi^{1-3}$$

$$= \beta P_F\left[1 + \frac{5}{12}\pi^2\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \dots\right]$$

where $P_F = \frac{2}{5}n\varepsilon_F$ is the Fermi Pressure

Unlike its classical counterpart, the Fermi Gas at zero temperature has finite pressure & internal energy.



As has already been derived,

$$\varepsilon = \frac{E}{V} = \frac{3}{2} P = \frac{3}{2} \underbrace{P_F}_{2/5\, n \varepsilon_F} \left[1 + \frac{5}{12} \pi^2 \left(\frac{k_B T}{\varepsilon_F} \right)^2 + \dots \right]$$

$$= \frac{3}{5} n k_B \left(\frac{\varepsilon_F}{k_B} \right) \left[1 + \frac{5}{12} \pi^2 \left(\frac{k_B T}{\varepsilon_F} \right)^2 + \dots \right]$$

$$= \frac{3}{5} n k_B T_F \left[1 + \frac{5}{12} \pi^2 \left(\frac{k_B T}{\varepsilon_F} \right)^2 + \dots \right]$$

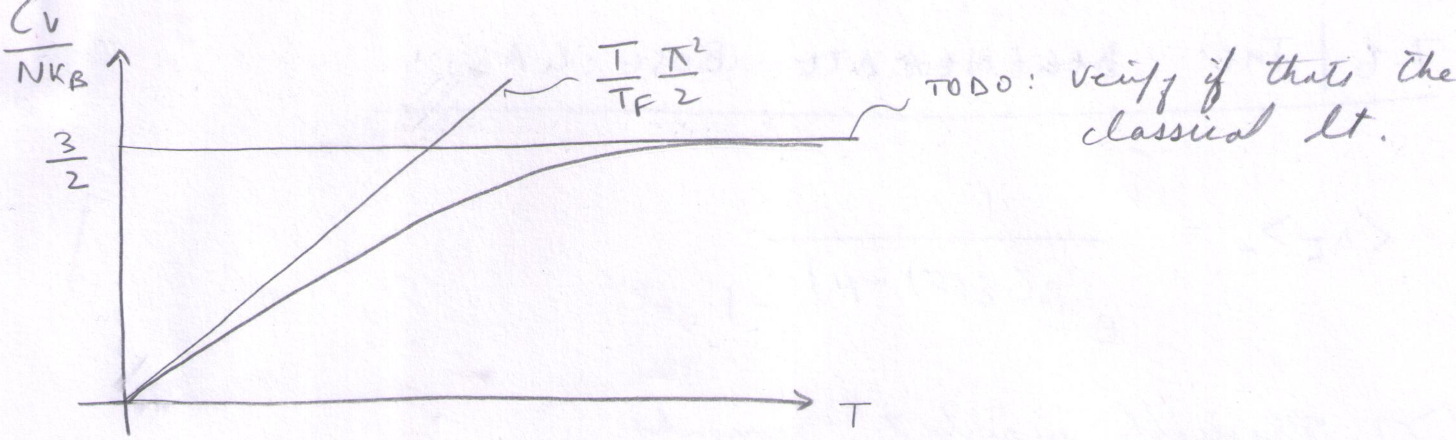
where $T_F = \frac{\varepsilon_F}{k_B}$

Thus, $C_V = \frac{dE}{dT} = \frac{\pi^2}{2} N k_B \left(\frac{T}{T_F} \right) + O\left(\frac{T}{T_F} \right)^3$

$$E = \frac{3}{5} V n k_B T_F \left[1 + \frac{5}{12} \pi^2 \left(\frac{k_B T}{\varepsilon_F} \right)^2 + \dots \right]$$

$$\frac{dE}{dT} = \frac{\cancel{3}}{\cancel{5}} V n k_B \cancel{T_F} \left[\frac{\cancel{5} \pi^2 k_B^2}{\cancel{12}_{\cancel{4}_2} \varepsilon_F^2} \cdot \cancel{2} T + \dots \right]$$

$$= \frac{\pi^2}{T_F} \cdot N k_B T$$



$$\frac{C_v}{Nk_B} = \frac{\pi^2}{2}\left(\frac{T}{T_F}\right) + \mathcal{O}\left(\frac{T}{T_F}\right)^3$$

There are important physical remarks

 TODO: complete this

§ 7.6 | THE DEGENERATE BOSE GAS

$$\langle n_{\vec{k}} \rangle_+ = \frac{1}{e^{\beta(\varepsilon(\vec{k}) - \mu)} - 1}$$

 Physically important remarks

→ $\langle n_{\vec{k}} \rangle_+$ must be positive.

Thus $\varepsilon(\vec{k}) - \mu \geq 0 \quad \forall \vec{k}$

$\Rightarrow \mu \leq \min[\varepsilon(\vec{k})]_{\vec{k}}$ $\left[\varepsilon(\vec{k}) = \frac{\hbar^2 k^2}{2m} \right.$

Claim / IDK Y

ASK → At high temperatures (classical limit),

↳ μ is large & $-$ve

↳ increases towards 0 like $k_B T \ln\left(\frac{n\lambda^3}{g}\right)$ as the temperature is dropped.

→ In the degenerate limit, μ approaches its limiting value of zero.

To see that, we look at $f_m^+(z)$

→ This increases monotonically with z in the interval $0 \leq z \leq 1$.

DOUBT
NOT SURE IF THIS IS VALID $\left[f_m^+(z) = \sum_{\alpha=1}^{\infty} \frac{z^\alpha}{\alpha^m} \right.$

DOUBT

→ The integrand attains the maximum (is it global) at $z = 1$, given by

$$\zeta_m \equiv f_m^+(1) = \frac{1}{(m-1)!}\int_0^\infty \frac{dx\ x^{m-1}}{e^x-1}$$

$$\left[\ \underset{x\to 0}{lt}\ \frac{x^{m-1}}{e^x-1} = \underset{x\to 0}{lt}\ \frac{(m-1)x^{m-2}}{e^x} \sim x^{m-2}\right.$$

→ When $x \to 0$, the integrand behaves like $\int dx\, x^{m-2}$

↳ Thus, ζ_m is finite for $m > 1$,

$$\left[\ \int dx\ x^{m-2} \sim x^{m-1}\right.$$

$$\left[\ m-1 > 0 \Rightarrow \begin{array}{l} m > 1 \to \text{finite} \\ m \leq 1 \to \text{infinite}\end{array}\right.$$

& infinite else.

→ A useful recursion property (for $m > 1$)

$$\frac{d}{dz} f_m^+(z) = \int_0^\infty dx\ \frac{x^{m-1}}{(m-1)!}\ \frac{d}{dz}\left(\frac{1}{z^{-1}e^x - 1}\right)$$

$$\left[\ f_m^+(z) = \frac{1}{(m-1)!}\int_0^\infty \frac{dx\ x^{m-1}}{z^{-1}e^x - 1}\right.$$

$$\left[\ \text{we have}\quad \frac{d}{dz} f(z^{-1}e^x) = f' \cdot \frac{-1}{z^2}\cdot e^x\ \frac{df}{dz}(z^{-1}e^x) = -\frac{1}{z}\frac{d}{dx}f(z^{-1}e^x)\right.$$

$$= \int_0^\infty dx\ \frac{x^{m-1}}{(m-1)!}\left(\frac{-1}{z}\right)\frac{d}{dx}\left(\frac{1}{z^{-1}e^x - 1}\right)$$

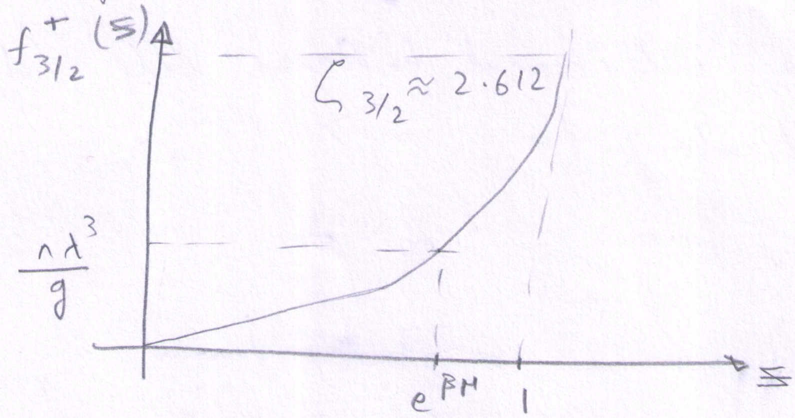
[by parts, we get $\int_a^b uv\,dx = \left[u\int v\,dx\right]_a^b - \int_a^b \left(u' \int v\,dx\right)dx$

$$= \frac{-1}{z}\left[\cancel{\frac{x^{m-1}}{(m-1)!}\ \frac{1}{z^{-1}e^x - 1}}\right]_0^{\infty \to 0} - \left(\frac{-1}{z}\right)\int_0^\infty \frac{x^{m-2}}{(m-2)!}\cdot\frac{1}{z^{-1}e^x - 1}\cdot dx$$

$$= \frac{1}{\xi} \int_0^\infty dx \; \frac{x^{m-2}}{(m-2)!} \frac{1}{\xi^{-1} e^x - 1} = \frac{1}{\xi} f^+_{m-1}(\xi)$$

↳ Hence, a sufficiently high derivative of $f^+_m(\xi)$ will be divergent at $\xi = 1$ for all m.

(Justifies the following graph!)



using $n^+ = \frac{g}{\lambda^3} f^+_{3/2}(\xi)$

where $\xi = e^{\beta\mu}$

→ The density of excited state (DOUBT) for the non-relativistic bose gas (in 3D) is thus bounded by

$$n_x = \frac{g}{\lambda^3} f^+_{3/2}(\xi) \leq n^* = \frac{g}{\lambda^3} \zeta_{3/2}$$

↳ At sufficiently high temperatures, s.t.

$$\frac{n\lambda^3}{g} = \frac{n}{g} \left(\frac{h}{\sqrt{2\pi m k_B T}} \right)^3 \leq \zeta_{3/2} \approx 2.612\ldots$$

the bound has no effect.

↳ & $n_x = n$ (DOUBT)

↳ At lowering the temperature, the limiting density of excited states is

$$\left(\frac{n}{g}\right)^{2/3} \frac{(h)^2}{2\pi m k_B T_c} = \left(\zeta_{3/2}\right)^{2/3}$$

$$\frac{h^2}{2\pi m k_B T} \left(\frac{n}{g}\right)^{2/3} = T_c(n)$$

↳ For $T \le T_c$,

(a) Ξ is stuck to ($\mu = 0$) (can't get smaller for maintaining +ivity etc.)

(b) $n^* = \dfrac{g \zeta_{3/2}}{\lambda^3} \propto T^{3/2}$ The density of excited states depends only on Temperature.

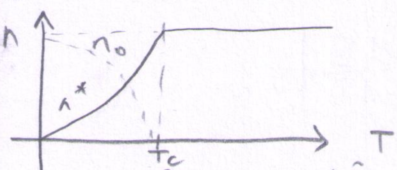
(c) $n > n^*$ when $T \le T_c$

DOUBT

why's & wouldn't the wavef^n disappear for $\vec{k} = 0$

→ $n_0 = n - n^*$ occupy the lowest energy state with $\vec{k} = 0$

THIS phenomenon of macroscopic occupation of a single one-particle state, is called Bose-Einstein condensate.



→ Properties (Unusual) of the condensate

↳ Gas pressure for $T < T_c$

DOUBT

$$\beta P = \frac{g}{\lambda^3} f^+_{5/2}(1) = \frac{g}{\lambda^3} \zeta_{5/2} \approx 1.341 \frac{g}{\lambda^3} \propto T^{3/2}$$

This is independent of density!

Its expected because only the excited n^* fraction has non-zero momentum.

↳ <missing, wasn't done in class>

→ Latent Heat Calculation <missing, not done in class>

$\longrightarrow C_V$

$$\mathcal{E} = \frac{3}{2} P \Rightarrow E = \frac{3}{2} PV$$

$$= \frac{3}{2} V \frac{g}{\lambda^3} k_B T f^+_{5/2}(\xi) \propto T^{5/2} f^+_{5/2}(\xi)$$

Now using $\frac{d}{d\xi} f^+_m(\xi) = \frac{1}{\xi} f^+_{m-1}(\xi)$ we have

$$C_{V,N} = \left.\frac{dE}{dT}\right|_{V,N} = \frac{3}{2} V \frac{g}{\lambda^3} k_B T \left[\frac{5}{2T} f^+_{5/2}(\xi) + \frac{1}{\xi} f^+_{1/2}(\xi) \left.\frac{d\xi}{dT}\right|_{V,N} \right]$$

$$= \frac{3}{2} V g k_B \left[\frac{d}{dT}\left(\frac{T}{\lambda^3}\right) f^+_{5/2}(\xi) + \frac{T}{\lambda^3} \frac{d}{dT} f^+_{3/2}(\xi) \right]$$

$$\frac{d}{dT}\left(\frac{T}{h^3} (2\pi m k_B T)^{3/2} \right)$$

$$= \frac{5}{2} \frac{(2\pi m k_B)^{3/2}}{h^3} T^{3/2-1}$$

$$= \frac{5}{2T}\left(\frac{1}{\lambda^3}\right)$$

$$\frac{d}{d\xi} f^+_{5/2}(\xi) \cdot \frac{d\xi}{dT}$$

$$= \frac{1}{\xi} f^+_{3/2}(\xi) \cdot \left.\frac{d\xi}{dT}\right|_{V,N}$$

$\square$

$\left.\frac{d\xi}{dT}\right|_{V,N}$ is found by fixing the # particles, via

$$\left.\frac{dN}{dT}\right|_V = 0 = \frac{g}{\lambda^3} V \left[\frac{3}{2T} f^+_{3/2}(\xi) + \frac{1}{\xi} f^+_{1/2}(\xi) \left.\frac{d\xi}{dT}\right|_{V,N} \right]$$

$$N = \frac{gV}{\lambda^3} f^+_{3/2}(\xi)$$

$$\left.\frac{dN}{dT}\right|_V = gV \left[\frac{d}{dT}\left(\frac{1}{\lambda^3}\right) f^+_{3/2}(\xi) + \frac{1}{\lambda^3} d f^+_{3/2}(\xi) \right]$$

$$= gV \left[\frac{3}{2T}\left(\frac{1}{\lambda^3}\right) f^+_{3/2}(\xi) + \frac{1}{\lambda^3} \frac{1}{\xi} f^+_{1/2}(\xi) \cdot \left.\frac{d\xi}{dT}\right|_{V,N} \right]$$

$\square$

Substitution of

$$-\frac{3}{2}\frac{f^+_{3/2}(\xi)}{f^+_{1/2}(\xi)} = \frac{T}{\xi}\frac{d\xi}{dT}\Big|_{V,N}$$

yields

$$\frac{C_V}{Vk_B} = \frac{3}{2}\frac{g}{\lambda^3}\left[\frac{5}{2}f^+_{5/2}(\xi) - \frac{3}{2}\frac{f^+_{3/2}(\xi)^2}{f^+_{1/2}(\xi)}\right]$$

<TODO: Expand in powers of ξ>

This is greater than the classical value at high temp.

$$\frac{C_V}{Nk_B} = \frac{3}{2}\left[1 + \frac{n\lambda^3}{2^{7/2}} + \cdots\right]$$

At low temperatures, $z = 1$ &

(IDK how) (IDK how)

$$\frac{C_V}{Nk_B} = \frac{15}{4}\frac{g}{n\lambda^3}\zeta_{5/2} = \frac{15}{4}\frac{\zeta_{5/2}}{\zeta_{3/2}}\left(\frac{T}{T_c}\right)^{3/2}$$

<discussion ASK>