GR #3

N-dimensional manifold: N coordinates are needed to describe a point on the manifold.

Eg: Surface of a sphere.

(θ, ϕ) or (x^1, x^2) ; It's possible that a given coordinate system becomes "bad"

such as north pole: $\theta = 0$, ϕ is arbitrary

$x^1 x^2$ is well defined.

At the equator: θ, ϕ is good,

$x^1 x^2$ is not (metric diverges at the equator)

Therefore use θ, ϕ is used near the equator,

use $x^1 x^2$ near the poles.

So to describe the manifold, we need 2 systems & their relation.

Thus in general, in a "good" coordinate system, you need n-points to describe an n-dimensional manifold.

The manifold itself maybe singular, i.e. it may have a point, s.t. a small neighbourhood around the point, can't be described regularly by a "good" coordinate system.

One way to check is to construct a scalar of the metric. Now if the scalar $= \infty$, then regardless of the coordinate, scalar $= \infty$. Thus the manifold is singular.

In Newtonian gravity, we have $\vec{F} = -m \vec{\nabla} \phi$

Recall: $ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$ (distance in minkoski space) | straight line $\downarrow$ minimal length path

Now: $ds^2 = \underbrace{g_{\mu\nu}(x)}_{\text{effect of gravity}}\, dx^\mu dx^\nu$ (Riemanian)

$(\vec{x}, t)$

In GR: The notion of a gravitational potential, is replaced by the notion of a metric in the underlying space.

: There's no concept of force; for no force, the particle travels along the geodesic ($\equiv$ minimum length path)

Remark: This is why Riemanian geometry is important.

Newtonian $\nabla^2 \phi = \rho$ gives the field as a f^n of mass distr (ρ) density | $\to$ in GR we'll have an eq^n to describe g as a f^n ρ.

Γ^i_{jk} $\quad\quad$ $R^i_{\ jkl}$

$\hookrightarrow$ Not a Tensor $\quad\quad$ $\hookrightarrow$ transforms as a $(1,3)$ Tensor

$\because \ \Gamma'^{\,i}_{jk}(\vec{x}') = \partial_m x'^i \, \partial'_j x^n \, \partial'_k x^p \, \Gamma^m_{np}(\vec{x}) + \partial_l x'^i \, \partial'_j \partial'_k x^l$

Def^n: $R_{ijkl} = g_{im} R^m_{\ jkl}$ $\quad$ | $g_{im} R^n_{\ jkl}$ - rank $(1,5)$ tensor

for $n=m$ $\quad$ rank $(0,4)$ Tensor

(A) $ds^2 = (dx^1)^2 + (dx^2)^2$

$= dr^2 + r^2 d\theta^2$

$R_{ijkl} = 0$ $\quad$ ($\because$ well, $g_{ij} = $ const)

(& $\because$ R_{ijkl} is a tensor its zero for the polar coordinates even though Γ in polar $\neq 0$)

This holds true of 3d systems also.

(B) $ds^2 = (d\theta^2 + \sin^2\theta \, d\phi^2) a^2$

$g_{\theta\theta} = a^2 \quad\quad g_{\phi\phi} = \sin^2\theta . a^2$

You'll see then that $R_{ijkl} \neq 0$.

Now you see why (B) & (A) are not the same manifolds.

Does scaling leave the manifolds invariant?

(A) its obvious.

(B) its not as obvious; we'll try to construct a scalar to check that.

$g^{ik} g^{jl} R_{ijkl}$ $\quad$ | NB: $g^{ij} g^{kl} R_{ijkl} = 0$

where $g^{ij} \equiv (g_{ij})^{-1}$

so $g^{ik} g^{jl} R_{ijkl} \overset{\text{claim}}{=} a^{-2}.\text{const}$

Now since its a scalar, $S(\vec{x}) = S'(\vec{x}')$ under coordinate transformations. If $S(\vec{x}) = $ const, then it mustn't change under CT.

Therefore "scaling" changes the manifold.

In general its hard to compare metrics to check if they describe the same manifold.

Symmetries of R

1. $R_{ijkl} = -R_{jikl}$
2. $R_{ijkl} = -R_{ijlk}$
3. $R_{ijkl} = R_{klij}$
4. $R_{i[jkl]} = 0$

$\rightarrow$ These can be proved from their def^{n}s.

$\frac{1}{6}(R_{ijkl} - R_{ijlk} + R_{iljk} - R_{ilkj} + R_{iklj} - R_{ikjl})$

$A_{ijkl} = R_{ijkl} + R_{jikl}$

$R'_{ijkl} = \partial'_i x^m \, \partial'_j x^n \, \partial'_k x^p \, \partial'_l x^q \, R_{mnpq}$

$A'_{ijkl} = \partial'_i x^m \, \partial'_j x^n \, \partial'_k x^p \, \partial'_l x^q \, R_{mnpq} + \underbrace{\partial'_j x^m \, \partial'_i x^n \, \partial'_k x^p \, \partial'_l x^q \, R_{mnpq}}_{= \ \partial'_j x^n \, \partial'_i x^m \, \partial'_k x^p \, \partial'_l x^q \, R_{nmpq}}$

$= \partial'_i x^m \, \partial'_j x^n \, \partial'_k x^p \, \partial'_l x^q \, A_{mnpq}$

Therefore A_{ijkl} is indeed a tensor.

Recall: That at a given point, a coordinate frame can be found s.t. the first derivative of the metric is zero.

<Exercise Alert>

Using this information, one can evaluate A_{ijkl} & show that $A_{ijkl} = 0$ in that frame & by extension, a zero Tensor.

NB: This hinges on the fact that R is a tensor (& that took a lot of work)

Recall: R_{ijkl} := Riemann Tensor

Def^n: $g^{ik} R_{ijkl} = R_{jl}$:= Ricci Tensor

NB: $R_{jl} = R_{lj}$, rank $(0,2)$

: $g^{lj} R_{lj} = R$:= Ricci Scalar (or curvature scalar)

NB: $R'(x') = R(x)$

Ex: $A_i = \partial_i R$ show this is a covariant vector. (rank $(0,1)$)

Covariant Derivatives

Recall: A_i ; a tensor of rank $(0,1)$

Def^n: $B_{ij} \equiv \partial_i A_j$; $B'_{ij}(x') = \partial'_i A'_j(x') = \partial'_i \cdot (\partial'_j x^l A_l)$

$$= \partial'_i \partial'_j x^l A_l + \partial'_j x^l \partial'_i A_l$$

$$= \partial'_i \partial'_j x^l A_l + \underbrace{\partial'_j x^l \partial'_i x^k \partial_k A_l}_{\text{how a tensor transforms.}}$$

Def^n: $C_{ij} \equiv D_i A_j := \partial_i A_j - \Gamma^k_{ij} A_k$

substitute

$$C'_{ij} = \partial'_i A'_j - \Gamma'^k_{ij} A'_k$$

$$= \underbrace{\partial'_i \partial'_j x^l A_l}_{\text{I}} + \partial'_j x^l \partial'_i A_l - \partial_m x'^k \partial'_i x^n \partial'_j x^p \Gamma^m_{np} \cdot \underbrace{\partial_m x'^k \partial'_k x^l}_{= \delta^l_m} A_l$$

$$- \underbrace{\partial_m x'^k \partial'_i \partial'_j x^m \cdot \partial'_k x^l A_l}_{\text{II}}, \quad \partial_m x'^k \partial'_k x^l = \delta^l_m$$

I & II cancel & we get

$$= \partial'_j x^l \partial'_i x^k \partial_k A_l - \partial'_i x^n \partial'_j x^p \Gamma^l_{np} A_l$$

$$= \partial'_i x^m \partial'_j x^p \left(\partial_m A_p - \Gamma^k_{mp} A_k \right)$$

$$= \partial'_i x^m \partial'_j x^p \; C_{mp}$$; $\Rightarrow C_{ij}$ transforms as a tensor.

Claim: $D_i A^j \equiv \partial_i A^j + \Gamma^j_{ik} A^k$ transforms as rank $(1,1)$ tensor

(and exercise)

$$\Gamma'^{i}_{jk} = \partial_l x'^{i} \, \partial'_p x^{p}_{j} \, \partial'_q x^{q}_{k} \, \Gamma^{l}$$

$$A'^{i} = \frac{\partial A'^{i}}{\partial x^{j}} dx^{j}$$

$$\boxed{dx'}^{i} = \frac{\partial x'^{i}}{\partial x^{j}} \boxed{dx}^{j}$$

$$A'^{i} = \frac{\partial x'^{i}}{\partial x^{j}} A^{j}$$

$$A'^{i} = \partial_j x'^{i} A^{j}$$

Prov: $g^{kl} R_{ijkl} = 0$

Doubt: $g^{ij} := (g_{ij})^{-1}$ inverse of matrix or element?

$R_{ijkl} = + R_{jikl}$ (confirm)