The counting (homework) Show that the first term doesn't contribute to the counting.

$$x'^j = x^j_{(0)} + A^j{}_k (x'^k - x'^k_0) + B^j{}_{kl}(x'^k - x'^k_0)(x'^l - x'^l_0) + \dots$$

Q: Can we use this to set $\partial_i g_{jk} = 0$?
$\partial_i \partial_j g_{kl} = 0$?

Objective: show A can't help in setting a tensor to zero.

Consider:
$C_{i_1 \dots i_N}$, ~~we want to show~~ and suppose $C'_{i_1 \dots i_n} = 0$ in some frame for some choices of As & Bs.

Consider a further change. We take this to be a linear "coordinate system" change.

$$x''^i = x'^i_0 + S^i{}_j (x''^j - x''^j_{(0)}) \qquad S \leftrightarrow \frac{\partial x'}{\partial x''}$$

$x \xrightarrow{A} x'$

$C \longrightarrow C' = A\,C$
$= 0$

$C'' = BC$

$$C''_{i_1 \dots i_n} = \partial''_{i_1} x'^{j_1} \dots \partial''_{i_N} x'^{j_n} C'_{j_1 \dots j_n}$$

$$= S_{i_1}{}^{j_1} \dots S_{i_n}{}^{j_n} C'_{j_1 \dots j_n}$$

$$= 0 \qquad (\because C'_{j_1 \dots j_n} = 0)$$

$\frac{\partial x}{\partial x'}, \frac{\partial x'}{\partial x''}$

$$x''^i = x^i_0 + \underbrace{A^i{}_j S^j{}_k}_{\tilde{A}^i{}_k} (x'^k - x'^k_0) + \dots \qquad \text{(not too sure)}$$

$$\frac{\partial x^i}{\partial x'^j} = \delta^i{}_j$$

Claim: This works for $\partial_i g_{jk}$ also. If its zero in the primed frame, remains zero in the unprimed (This is for linear transformations only)

NB: S is arbitrary, choose it to be A^{-1}.

REDO from video.

Argument: Since $\tilde{A}^i{}_k = \delta^i{}_k$, therefore if ~~x~~ C was zero, then it remains zero in a coordinate sys where $\tilde{A}$ is arbitrary (even $\delta^i{}_k$). Thus ~~x~~ C being zero, doesn't depend on the 'linear term' & therefore not included in the counting.

A: To set $\partial_i g_{jk} = 0$, you need $B^j{}_{kl}$. That uses up all the freedom. Then $\partial_i \partial_j g_{kl}$ can't be set to zero in general.

NB: To compare flat space & curved space, just evaluate the Riemann Tensor. If zero, flat else curved. If flat then 0. rest I don't know.

: To compare b/w curved spaces, construct scalars (for Rieman Tensor, 20 diff scalars) from the Rieman Tensor & compare.

Covariant Derivative

If A_i is a Tensor, then $\partial_i A_j$ is not a tensor.

However, $D_i A_j = \partial_i A_j - \Gamma^k_{ij} A_k$ transforms like one $(0,1)$

Similarly $D_i A^j = \partial_i A^j + \Gamma^j_{ik} A_k$ transforms like a $(1,1)$ Tensor.

Generalized to an arbitrary tensor.

$\partial_k A^{i_1 \ldots i_p}{}_{j_1 \ldots j_q}$ is NOT a tensor.

$$D_k A^{i_1 \ldots i_p}{}_{j_1 \ldots j_q} = \partial_k A^{i_1 \ldots i_p}{}_{j_1 \ldots j_q} + \left(\Gamma^{i_1}_{k\ell} A^{\ell i_2 \ldots i_p}{}_{j_1 \ldots j_q} + \Gamma^{i_2}_{k\ell} \ldots \right)$$

$$- \left(\Gamma^{\ell}_{k i_1} A^{i_1 \ldots i_p}{}_{\ell j_2 \ldots j_q} + \Gamma^{\ell}_{k i_2} A \quad \ldots \right)$$

is a Tensor, rank $(p, q+1)$

Ex: Convince yourself.

Raising & Lowering

$$A^{i_1 \ldots i_p}{}_{j_1 \ldots j_q} B^{m_1 \ldots m_r}{}_{n_1 \ldots n_s} =: C^{i_1 \ldots i_p m_1 \ldots m_r}{}_{j_1 \ldots j_q n_1 \ldots n_s}$$

is of rank $(p+r, q+s)$

If $C^{i_1 \ldots i_p}{}_{j_1 \ldots j_q}$ is rank (p, q) then

$C^{i_1 \ldots i_p}{}_{i_1 \ldots j_q}$ is rank $(p-1, q-1)$ tensor

---

If $A_{j_1 \ldots j_q} g^{i_1 i_2}$ is a $(2, q)$ tensor then

$A_{i_1 j_2 \ldots j_q} g^{i_1 i_2}$ is a $(2-1, q-1)$ Tensor & its written as

$A^{i_2}{}_{j_2 \ldots j_q}$

Eg. (a) $A_{j_1 i_1 \ldots j_q} g^{i_1 i_2} = A_{j_1}{}^{i_2}{}_{j_3 \ldots j_q}$

(b) $R^i{}_{jkl}$; $R_{ijkl} = g_{im} R^m{}_{jkl}$

(c) $g^{ij} R_{ijkl} = 0 \quad \because R_{ij\ldots} = -R_{ji\ldots}$ & $g^{ij} = g^{ji}$

$g^{ij} g_{jk} = \delta^i{}_k$

Kronecker

$\delta^i{}_k = \begin{cases} 1 & \text{if } i=k \\ 0 & \text{if } i\neq k \end{cases}$

x - coordinate

x' - coordinate

$$\delta'^i{}_k = \partial_m x'^i \, \partial'_k x^n \, \delta^n{}_m$$

$$= \partial_m x'^i \, \partial'_k x^m$$

$$= \partial'_k x'^i = \delta^i{}_k$$

$\delta^i{}_k = \partial_k x^i$

$\delta'^i{}_k = \partial'_k x'^i = \delta^i{}_k$

$\delta' = \delta$

$\delta^i{}_k = \partial_k x^i \qquad \frac{\partial x^i}{\partial x^k}$

$\delta'^i{}_k = \partial'_k x'^i$

$= \partial_k x^i = \delta^i{}_k$

$\delta^i{}_k = \partial'_k x'^i = \partial_k x^n \, \underline{\delta^m{}_n}$

$= \underline{\partial_m x'^i \, \partial'_k x^n}$

Exercise: (a) g^{ij} is a tensor — see study notes.

(b) δ_{ik} is not a tensor.

Recall:

$$C^{i_1\ldots i_p k_1\ldots k_s}{}_{j_1\ldots j_q l_1\ldots l_s} := A^{i_1\ldots i_p}{}_{j_1\ldots j_q} B^{k_1\ldots k_s}{}_{l_1\ldots l_s}$$

Claim: $$D_k C^{i_1\ldots i_p k_1\ldots k_s}{}_{j_1\ldots j_q l_1\ldots l_s} = \left(D_k A^{i_1\ldots i_p}{}_{j_1\ldots j_q}\right)\left(B^{k_1\ldots k_s}{}_{l_1\ldots l_s}\right) + \left(A^{i_1\ldots i_p}{}_{j_1\ldots j_q}\right)\left(D_k B^{k_1\ldots k_s}{}_{l_1\ldots l_s}\right)$$

proof: Trivial.

NB: $D_i g_{jk} = \partial_i g_{jk} - \Gamma^l{}_{ij} g_{lk} - \Gamma^l{}_{ik} g_{jl}$; This has only 1st derivatives of g_{ij}

Recall: (a) We proved that g_{ij}'s first derivatives can't form a tensor

(b) D_i of a tensor is a tensor.

Result: $D_i g_{jk} = 0$ (must hold, can be checked) Exercise

Similarly: $D_i g^{jk} = 0$

Proof: $D_i g^{jk} = \partial_i g^{jk} + \Gamma^j{}_{il} g^{lk} + \Gamma^k{}_{il} g^{jl}$ (complete?)

Claim: (proved soon) $\partial_i (g^{-1}) = -g^{-1} \partial_i g \, g^{-1}$

Claim: $D_i(\delta^j{}_k) = 0 \longrightarrow 0 = D(\underbrace{g^{ij} g_{jk}}_{\delta^i{}_k})$

$= D(g^{ij}) g_{jk} + g^{ij} \cancel{D(g_{jk})}^{0}$

suggestive $D(g^{ij}) = 0$

Proof: $= \cancel{\partial_i \delta^j{}_k}^{0} - \Gamma^l{}_{ik}\delta^j{}_l + \Gamma^j{}_{il}\delta^l{}_k$

$$= -\Gamma^j{}_{ik} + \Gamma^j{}_{ik} = 0$$

$A^{i_1 \dots i_p}{}_{j_1 \dots j_q}$

$$C^{i_1 \dots i_p m}{}_{j_1 \dots j_q n} := A^{i_1 \dots i_p}{}_{j_1 \dots j_q} \delta^m{}_n$$

$$D_k \left(C^{i_1 \dots i_p m}{}_{j_1 \dots j_q n} \right) = \left(D_k A^{i_1 \dots i_p}{}_{j_1 \dots j_q} \right) \delta^m{}_n + A \cancel{D_k (\delta^m{}_n)} \to 0$$

For $(m = j_1 \; ; \; n = i_1)$ we have

$$D_k A^{i_1 \dots i_p}{}_{i_1 j_2 \dots j_q} = D_k \left(A^{i_1 \dots i_p}{}_{j_1 \dots j_q} \right) \cdot \delta^{j_1}{}_{i_1}$$

Statement: Contracting before or after taking a covariant derivative are equivalent.

Eg: LHS: $D_k (A^i{}_i) = \partial_k A^i{}_i$ ($A^i{}_j$)

RHS: $D_k (A^i{}_j) \delta^j{}_i = \left(\partial_k A^i{}_j - \Gamma^l_{kj} A^i{}_l + \Gamma^i_{kl} A^l{}_j \right) \delta^j{}_i$

$$= \partial_k A^i{}_i - \cancel{\Gamma^l_{ki} A^i{}_l} \; \cancel{+ \Gamma^i_{kl} A^l{}_i}$$

$$= \partial_k A^i{}_i$$

Claim: $D_k \left(g^{i_1 j_1} A^{i_1 \dots i_p}{}_{j_1 \dots j_q} \right) = g^{i_1 j_1} D_k \left(A^{i_1 \dots i_p}{}_{j_1 \dots j_q} \right)$

$\because D_k (g^{i_1 j_1}) = 0$

& use chain rule.

claim: $\partial_i M^{-1} = - M^{-1} \partial_i M \, M^{-1}$

proof: $\partial_i (M M^{-1}) = \partial_i (\mathbb{1}) = 0$

$$(\partial_i M) M^{-1} + M \partial_i M^{-1} = 0$$

$$- M^{-1} (\partial_i M) M^{-1} = \partial_i M^{-1} \checkmark$$