GR

Newton $\vec{F} = \frac{-GM_1 m_2}{r^2}\hat{r}$

Newton → GR

Electrostatics → Maxwell's eq^n

Underlying mathematics of GR
→ Reimannian geometry
↑
Generalization of euclidian geometry.

Euclidian Geometry

Two points
(x^1, x^2, x^3) $(x^1+dx^1, x^2+dx^2, x^3+dx^3)$

In Euclidian,

$$ds = \sqrt{(dx)^2 + (dx^2)^2 + (dx^3)^2}$$

also $$ds^2 = (dx)^2 + (dx^2)^2 + (dx^3)^2$$

In N-dimension,

$$ds^2 = (dx^1)^2 + (dx^2)^2 + \ldots (dx^N)^2$$
$$= \sum_{i=1}^{N} (dx^i)^2$$

In Remanian Geometry

$$ds^2 = \sum_{i,j=1}^{N} g_{ij}(\vec{x})\, dx^i dx^j$$

$g_{ij}(\vec{x})$ is a f^n of $(x, \ldots x_n)$

for each pair (i,j)

~~We can choose~~ g_{ij} must be symmetric

$g_{ij} = g_{ji}$ ≡ referred to as a metric

- Euclidian geometry is where $g_{ij} = \delta_{ij}$

At times two different $g_{ij}(\vec{x})$ may describe the same space.

$$ds^2 = \sum_{i,j=1}^{N} g_{ij}(\vec{x})\, dx^i dx^j$$

Instead of $(x^1, \ldots x^N)$ we choose a different set of coordinates

$$ds^2 = \sum g_{ij}(\vec{x})\, dx^i dx^j$$

$(x^1 \ldots x^N)$ instead choose $(x'^1, \ldots x'^N)$

$x'^1 = f^1(\vec{x})$ $\quad x^1 = g^1(\vec{x}')$

$x'^2 = f^2(\vec{x})$ $\quad x^2 = g^2(\vec{x}')$

$\vdots$

$x'^N = f^N(\vec{x})$ $\quad x^N = g^N(\vec{x}')$

not the same as g^{ij} can call these $h^1 \ldots$ etc.

$\vec{x}'$ $\quad \vec{x}' + d\vec{x}'$

$$dx^i = \sum_k \frac{\partial x^i}{\partial x'^k} dx'^k$$

$$ds^2 = \sum_{i,j=1}^{N} g_{ij}\left(\sum_{k=1}^{n} \frac{\partial x^i}{\partial x'^k} dx'^k\right)\left(\sum_{l=1}^{n} \frac{\partial x^j}{\partial x'^l} dx'^l\right)$$

$$= \sum_{k,l} \underbrace{\sum_{i,j} g_{ij}(\vec{x}) \frac{\partial x^i}{\partial x'^k}\frac{\partial x^j}{\partial x'^l}}_{g'_{kl}(\vec{x}')} dx'^k dx'^l$$

$$= \sum_{ij} g'_{ij}(\vec{x}')\, dx'^i dx'^j$$

Examples

Euclidian metric

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

$x^1 = r \sin\theta \cos\phi$

$x^2 = r \sin\theta \sin\phi$

$x^3 = r\cos\theta$

$dx^1 = \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi$

$dx^2 = \ldots$

$dx^3 = \ldots$

$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

$\Rightarrow ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$

Ex.

$g'_{rr} = 1$, $g'_{\theta\theta} = r^2$, $g'_{\phi\phi} = r^2\sin^2\theta$

6 ~~others~~ = 0.

Example of a non-euclidian sphere.

→ surface of a 2d sphere:

$x^2+y^2+z^2=a^2$

$ds^2 = dx^2+dy^2+dz^2$

→

Method 1

$(x^1)^2+(x^2)^2+(x^3)^2 = a^2$

① $x^3 = \pm\sqrt{a^2-(x^1)^2-(x^2)^2}$

$$ds^2 = (dx^1)^2+(dx^2)^2+(dx^3)^2$$

$$= (dx^1)^2+(dx^2)^2+\left\{\pm\frac{-x^1dx^1-x^2dx^2}{\sqrt{a^2-(x^1)^2-(x^2)^2}}\right\}^2$$

$$= \left\{1+\frac{(x^1)^2}{a^2-(x^1)^2-(x^2)^2}\right\}(dx^1)^2 + \left\{1+\frac{(x^2)^2}{a^2-(x^1)^2-(x^2)^2}\right\}dx^{2\,2}$$

$$+ \frac{2x^1x^2}{a^2-(x^1)^2-(x^2)^2}\,dx^1dx^2$$

$$g_{11} = 1+\frac{(x^1)^2}{a^2-(x^1)^2-(x^2)^2} \qquad g_{22} = 1+\frac{(x^2)^2}{a^2-(x^1)^2-(x^2)^2}$$

NB: 2 goes away

$$g_{12} = \frac{x^1x^2}{a^2-(x^1)^2-(x^2)^2} \cancel{dx^1dx^2}$$

Method 2

② First go to spherical polar.

go from $x_1\ x_2\ x_3 \to r,\theta,\phi$

$r^2=a^2 \Rightarrow r=a$ (no $\pm$)

$$ds^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

$$= a^2(d\theta^2+\sin^2\theta\, d\phi^2) \longrightarrow \text{you get this after subst. } x^1\ x^2 \text{ for } ds^2$$

$x^1 = a\sin\theta\cos\phi$ Ex: $ds^2|_{(x^1x^2)} = ds^2|_{(\theta,\phi)}$

$x^2 = a\sin\theta\sin\phi$

Q. Given only the ~~coordina~~ metric, how to conclude whether they're the same space?

O. ~~As to~~ Strategy: Find appropriate linear combinations of the metric & its derivatives which are invariant under coordinate transformations.

Conventions

a. Index i of a coordinate will be a super script eg. x^i

b. $\frac{\partial}{\partial x^i} \equiv \partial_i$

c. Summation Convention: Any index, appearing twice in a formula, once as sub & once as super is summed over.

d. Index of a matrix appears as subscript

① $ds = \sum_{i,j=1}^{N} g_{ij}(\vec{x})\, dx^i\, dx^j \longrightarrow g_{ij}(\vec{x})\, dx^i\, dx^j$

② $g'_{kl}(\vec{x}') = \sum_{ij} g_{ij} \frac{\partial x^i}{\partial x'^k} \frac{\partial x^j}{\partial x'^l} = g_{ij}\, \partial'_k x^i\, \partial'_l x^j$

Tensor Fields

Consider some combination $A_{i_1 \ldots i_Q}(\vec{x})$ of the metric & its derivatives which transform as follows:

$$A'_{i_1 \ldots i_Q}(\vec{x}') = \qquad\qquad A_{j_1 \ldots j_Q}(\vec{x})$$

$$C'^{i_1 \dots i_p}{}_{j_1 \dots j_q}(\vec{x}') = \partial_{k_1} x'^{i_1} \dots \partial_{k_p} x'^{i_p} \cdot \partial'_{j_1} x^{l_1} \dots \partial'_{j_q} x^{l_q}\, C^{k_1 \dots k_p}{}_{l_1 \dots l_q}(\vec{x})$$

Rank

(0,0) Scalar $\phi'(\vec{x}') = \phi(\vec{x})$

(1,0) Contravariant vector

(0,1) Covariant vector

$A^{i_1 \dots i_p}{}_{j_1 \dots j_q} B^{k_1 \dots k_r}{}_{l_1 \dots l_s}$ — This transforms as a $(p+r,\ q+s)$ tensor

$A^{i_1 \dots i_p}{}_{j_1 \dots j_q}$

$$A'^{i_1}{}_{i_1}(\vec{x}') = \underbrace{\partial_{k_1} x'^{i_1}\, \partial'_{i_1} x^{l_1}}\ A^{k_1}{}_{l_1}(\vec{x})$$

$$= \frac{\partial x'^{i_1}}{\partial x^{k_1}} \cdot \frac{\partial x^{l_1}}{\partial x'^{i_1}} = \frac{\partial x^{l_1}}{\partial x^{k_1}} = \delta^{l_1}{}_{k_1}$$

$= A^{k_1}{}_{k_1}(\vec{x})$ which is consistent

Claim: Take some point p on the manifold. $\vec{x}_0$ is the coordinate of p in some coordinate system.

Then $\exists$ a coordinate transf $\vec{x} \to \vec{x}'$ s.t. $g'_{ij}(x'_0) = \mathrm{diag}(\pm 1, \pm 1, \dots \pm 1)$

NB: For continuous metrics, the number of + & − entries will be constant.

$= f^i(\vec{x}')$ so $x^i_0 \equiv f^i(\vec{x}'_0)$

$$x^i = x^i_{(0)} + A^i{}_k(x'^k - x'^k_0) + B^i{}_{kl}(x'^k - x'^k_0)(x'^l - x'^l_0) + \dots$$

Here we expand about x_0

$$g'_{kl}(\vec{x}'_0) = \partial'_k x^i\, \partial'_l x^j\, g_{ij} \Big|_{\substack{\vec{x}' = \vec{x}'_0 \\ \vec{x} = \vec{x}_0}}$$

$= A^i{}_k A^j{}_l\, g_{ij}(\vec{x}_0)$ (take ∂' & should be clear)

$= (A^T g A)_{kl}$ (in matrix notation)

Notation: Regard g_{ij} as matrix; i as row index, j as column index

– Recall: S is symmetric; Thus can be diagonalized with orthogonal matrices.

viz. $g = S^T g_d S$ s.t. $S^T S = \mathbb{1}$

$$g_d = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Non-singular metric only, i.e. eigenvalues are non-zero.

Def^n: $R \equiv R^T \equiv \begin{pmatrix} \sqrt{|\lambda_1|} & & 0 \\ & \ddots & \\ 0 & & \sqrt{|\lambda_n|} \end{pmatrix}$; $\Rightarrow R^T \eta R = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ where $\eta = \begin{pmatrix} \frac{\lambda_1}{|\lambda_1|} & & \\ & \ddots & \\ & & \frac{\lambda_n}{|\lambda_n|} \end{pmatrix}$

Thus, $g = S^T g_d S = S^T R^T \eta R S$

$g'_{kl} = (A^T g A)_{kl} = (A^T S^T R^T \eta R S A)_{kl}$, $\Rightarrow$ for $A = (RS)^{-1}$ we'll get $g'_{kl}(\vec{x}'_0) = \eta_{kl}$

This is not unique; eg. $\eta = 1$, then $\forall A = (URS)^{-1}$ we'll get the right form, where

Signature:
When all $+$, then eucledian
when one $-$, then Lorentzian

NB: (1) Locally then, there's no more information
(2) This will not distinguish b/w a plane & the surface of a sphere.
(3) This is more than diagonalization; A is more than just a $R^T R$ like matrix.

Def^n: $K_{ijk} = \partial_i g_{jk}$

$$K'_{i_1 i_2 i_3} = \partial'_{i_1}\left(g'_{i_2 i_3}(\vec{x}')\right)$$
$$= \partial'_{i_1}\left(\partial'_{i_2} x^{j_2} \partial'_{i_3} x^{j_3} g_{j_2 j_3}(\vec{x}')\right)$$
$$= \partial'_{i_1}\partial'_{i_2} x^{j_2}\, \partial'_{i_3} x^{j_3}\, g_{j_2 j_3} + \partial'_{i_2} x^{j_2}\, \partial'_{i_1}\partial'_{i_3} x^{j_3}\, g_{j_2 j_3} + \underbrace{\partial'_{i_2} x^{j_2}\, \partial'_{i_3} x^{j_3}\, \partial'_{i_1} g_{j_2 j_3}}_{\text{This is the term we want (why? we'll see soon)}}$$

chain rule

$$= \partial'_{i_2} x^{j_2}\, \partial'_{i_3} x^{j_3}\, \partial'_{i_1} x^{j_1}\, \partial_{j_1} g_{j_2 j_3} \qquad \left(\because \frac{\partial x^{j_1}}{\partial x'_{i_1}} \frac{\partial g_{j_2 j_3}}{\partial x^{j_1}} = \frac{\partial}{\partial x'_{i_1}} g_{j_2 j_3}\right)$$

rearrange

$$= \underbrace{\partial'_{i_1} x^{j_1}\, \partial'_{i_2} x^{j_2}\, \partial'_{i_3} x^{j_3}}\, \cancel{\partial_{j_1} g_{j_2 j_3}} \rightarrow K_{j_1 j_2 j_3}$$

This is how K_{ijk} should transform if K_{ijk} was a tensor.
However, since the other terms are non-zero, $\Rightarrow K_{ijk}$ is not a tensor under general coordinate transformation.

Idea: Take a linear combination of K_{ijk} to construct a tensor. (or even polynomials of K_{ijk})
$\because$ This is not possible.
Proof sketch: Assume we can show that given a point, $\exists$ a coordinate transformation (CT) s.t. K vanishes. Therefore, it must vanish in all coordinates ($\because$ tensors transform in that way). Therefore its impossible to find such a tensor (it'll be zero essentially), (if this argument is repeated for all points)

Recall: $x^i = x^i_0 + A^i{}_j (x'^j - x'^j_0) + B^i_{jk}(x'^j - x'^j_0)(x'^k - x'^k_0) + \dots$

We need 2 derivatives $\Rightarrow B^i_{jk}$ will be important

B^i_{jk} will be unique.

We want $K'_{ijk} = 0$, so $\frac{(n)(n+1)}{2} \cdot n$ eq^n's are required to be zero.
$\#$ constants is also $\frac{(n)(n+1)}{2} \cdot n$ ($\because B^i_{jk}$ also has similar symmetric aspects as K_{ijk})

Exercise: Find B^i_{jk} explicitly, to make $K'^{i}_{j,k} = 0$

Idea: Take 2 derivatives of g_{ij}

$$S_{ijkl} = \partial_i \partial_j g_{kl}$$

(Claim: This itself is not a tensor.)

: Use the old argument. See if it can be shown to be zero in some coordinate sys.

At $\vec{x}_0$, # of components of $S_{ijkl} = \frac{n(n+1)}{2} \cdot \frac{n(n+1)}{2}$

Recall: $x^i = x^i_0 + A^i_j(\ldots)^j + B^i_{jk}(\ldots)^{jk} + C^i_{jkl}(\ldots)^{jkl}$

Now # of parameters in $C^i_{jkl} = n \cdot \frac{n(n+1)(n+2)}{3!}$ & $<$ (for large N)

$\Rightarrow$ There're more constraints than parameters. This means one can't in general get $S_{ijkl} = 0$.

Result: So $\frac{n^2(n+1)^2}{4} - \frac{n^2(n+1)(n+2)}{6} = \frac{1}{12} n^2(n^2-1)$ — 1 for $n=2$, 6 for $n=3$, 20 for $n=4$

which means the tensor so constructed will have the aforesaid # of free parameters.

Def^n: $\Gamma^i_{jk} =: \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}) :=$ Christoffel symbol / Connection

$R^i_{jkl} =: \partial_l \Gamma^i_{jk} - \partial_k \Gamma^i_{jl} + \Gamma^m_{jk} \Gamma^i_{lm} - \Gamma^m_{jl} \Gamma^i_{km} :=$ Riemann Tensor

Exercise: $\Gamma'^l_{mn}(\vec{x}') = \partial_i x'^l \partial'_m x^j \partial'_n x^k \Gamma^i_{jk}(\vec{x}) + \partial_k x^l \partial'_m \partial'_n x^k$

NB: This is not a tensor $\because$ of the second term

: R^i_{jkl} is a tensor of rank (1,3)

: R^i_{jkl} has $\frac{1}{12} n^2(n^2-1)$ independent parameters.