$C'^{i_1 \dots i_p}{}_{g_1 \dots g_q}(\vec{x}') =$

$= \partial_{k_1} x'^{i_1} \dots \partial_{k_p} x'^{i_p} \cdot \partial'_{g_1} x^{l_1} \dots \partial'_{g_q} x^{l_q} \, C^{k_1 \dots k_p}{}_{l_1 \dots l_q}(\vec{x})$

Rank

(0,0) Scalar $\phi'(\vec{x}') = \phi(\vec{x})$

(1,0) Contravariant vector

(0,1) Covariant vector

$A^{i_1 \dots i_p}{}_{j_1 \dots j_q} B^{k_1 \dots k_r}{}_{l_1 \dots l_s}$ — This transforms as a $(p+r, q+s)$ tensor

$A^{i_1 \dots i_p}{}_{j_1 \dots j_q}$

$A'^{i_1}{}_{i_1}(\vec{x}') = \underbrace{\partial_{k_1} x'^{i_1} \, \partial'_{i_1} x^{l_1}}_{\frac{\partial x'^{i_1}}{\partial x^{k_1}} \cdot \frac{\partial x^{l_1}}{\partial x'^{i_1}} = \frac{\partial x^{l_1}}{\partial x^{k_1}} = \delta^{l_1}{}_{k_1}} A^{k_1}{}_{l_1}(\vec{x})$

$= A^{k_1}{}_{k_1}(\vec{x})$ which is consistent

Claim: Take some point p on the manifold. $\vec{x}_0$ is the coordinate of p in some coordinate system.

Then $\exists$ a coordinate transf $\vec{x} \to \vec{x}'$ s.t. $g'_{ij}(x'_0) = \mathrm{diag}(\pm 1, \pm 1, \dots \pm 1)$

NB: For continuous metrics, the number of + & − entries will be constant.

$x^i = f^i(\vec{x}')$ so $x^i_0 \equiv f^i(\vec{x}'_0)$

$x^i = x^i_{(0)} + A^i{}_k (x'^k - x'^k_0) + B^i{}_{kl}(x'^k - x'^k_0)(x'^l - x'^l_0) + \dots$

Here we expand about x_0

$\vec{x} = f(\vec{x}') = f(\vec{x}'_0) + \frac{\partial f}{\partial x}(x' - x'_0) + \dots$ ($\vec{x}'_0$, $\vec{x}'$)

$g'_{kl}(\vec{x}'_0) = \partial'_k x^i \, \partial'_l x^j \, g_{ij} \Big|_{\vec{x}' = \vec{x}'_0}$

$= A^i{}_k A^j{}_l \, g_{ij}(\vec{x}_0)$ ($\vec{x} = \vec{x}_0$) (Take ∂' & should be clear)

$= (A^T g A)_{kl}$ (in matrix notation)

Notation: Regard g_{ij} as matrix; i as row index, j as column index

- Recall: S is symmetric; Thus can be diagonalized with orthogonal matrices.

viz. $g = S^T g_d S$ s.t. $S^T S = \mathbb{1}$

$g_d = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ Non-singular metric only, viz. eigenvalues are non-zero.

Def^n: $R = R^T = \begin{pmatrix} \sqrt{|\lambda_1|} & & 0 \\ & \ddots & \\ 0 & & \sqrt{|\lambda_n|} \end{pmatrix}$ $\Rightarrow R^T \eta R = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ where $\eta = \begin{pmatrix} \frac{\lambda_1}{|\lambda_1|} & & \\ & \ddots & \\ & & \frac{\lambda_n}{|\lambda_n|} \end{pmatrix}$

Thus, $g = S^T g_d S = S^T R^T \eta R S$

$g'_{kl} = (A^T g A)_{kl} = (A^T S^T R^T \eta R S A)_{kl}$, $\Rightarrow$ for $A = (RS)^{-1}$ we'll get $g'_{kl}(\vec{x}'_0) = \eta_{kl}$

This is not unique; eg. $\gamma = 1$, then if $A = (URS)^{-1}$ we'll get the right form, where

Signature:
When all +, then euclidean
when one -, then lorentzian

NB: (1) locally then, there's no more information
(2) This will not distinguish b/w a plane & the surface of a sphere.
(3) This is more than diagonalization; A is more than just a $R^T R$ like matrix.

Def^n: $K_{ijk} = \partial_i (g_{jk})$ $\quad K \to K'$

$$K'_{i_1 i_2 i_3} = \partial'_{i_1}\left(g'_{i_2 i_3}(\vec{x}')\right)$$

$$= \partial'_{i_1}\left(\partial'_{i_2} x^{j_2}\, \partial'_{i_3} x^{j_3}\, g_{j_2 j_3}(\vec{x}')\right)$$

$$= \partial'_{i_1}\partial'_{i_2} x^{j_2}\; \partial'_{i_3} x^{j_3}\, g_{j_2 j_3} + \partial'_{i_2} x^{j_2}\, \partial'_{i_1}\partial'_{i_3} x^{j_3}\, g_{j_2 j_3} + \underline{\partial'_{i_2} x^{j_2}\, \partial'_{i_3} x^{j_3}\, \partial'_{i_1} g_{j_2 j_3}}$$

This is the term we want (why? we'll see soon)

chain rule

$$= \partial'_{i_2} x^{j_2}\, \partial'_{i_3} x^{j_3}\, \partial'_{i_1} x^{j_1}\, \partial_{j_1} g_{j_2 j_3}$$

$$\left(\because \frac{\partial x^{j_1}}{\partial x'^{i_1}}\,\frac{\partial g_{j_2 j_3}}{\partial x^{j_1}} = \frac{\partial}{\partial x'^{i_1}} g_{j_2 j_3}\right)$$

rearrange

$$= \underbrace{\partial'_{i_1} x^{j_1}\, \partial'_{i_2} x^{j_2}\, \partial'_{i_3} x^{j_3}\, \partial_{j_1}}\, g_{j_2 j_3} \to K_{j_1 j_2 j_3}$$

This is how K_{ijk} should transform if K_{ijk} was a tensor.

However, since the other terms are non-zero, $\Rightarrow K_{ijk}$ is not a tensor under general coordinate transformations.

Idea: Take a linear combination of K_{ijk} to construct a tensor. (or even polynomials of K_{ijk})

: This is not possible.

Proof sketch: Assume we can show that given a point, $\exists$ a coordinate transformation (CT) s.t. K vanishes. Therefore, it must vanish in all coordinates ($\because$ tensors transform in that way). Therefore its impossible to find such a tensor (it'll be zero essentially), (if this argument is repeated for all points.)

Recall: $x^i = x^i_0 + A^i{}_j (x'^j - x'^j_0) + B^i_{jk}(x'^j - x'^j_0)(x'^k - x'^k_0) + \ldots$

We need 2 derivatives $\Rightarrow B^i_{jk}$ will be important

B^i_{jk} will be unique.

We want $K'_{ijk} = 0$, so $\frac{(n)(n+1)}{2} \cdot n$ eq^{n}s are required to be zero.

constants is also $\frac{(n)(n+1)}{2} \cdot n$ ($\because$ B^i_{jk} also has similar symmetric aspects as K_{ijk})

Exercise: Find B^i_{jk} explicitly, to make $K'^i_{j,k} = 0$

Idea: Take 2 derivatives of g_{ij}

$$S_{ijkl} = \partial_i \partial_j g_{kl}$$

(Claim: This itself is not a tensor.)

: Use the old argument. See if it can be shown to be zero in some coordinate sys.

At $\vec{x}_0$, # of components of $S_{ijkl} = \frac{n(n+1)}{2} \cdot \frac{n(n+1)}{2}$

Recall: $x'^i = x'^i_0 + A^i_j (\dots)^j + B^i_{jk} (\dots)^{jk} + C^i_{jkl} (\dots)^{jkl}$

Now # of parameters in $C^i_{jkl} = n \frac{n(n+1)(n+2)}{3!}$ & $<$ (for large N)

$\Rightarrow$ There're more constraints than parameters. This means one can't in general get $S_{ijkl} = 0$.

Result: So $\frac{n^2(n+1)^2}{4} - \frac{n^2(n+1)(n+2)}{6} = \frac{1}{12} n^2(n^2-1)$ — $\begin{matrix} 1 & \text{for } n=2 \\ 6 & \text{for } n=3 \\ >0 & \text{for } n=4 \end{matrix}$

which means the tensor so constructed will have the aforesaid # of free parameters.

Def^n: $\Gamma^i_{jk} =: \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}) :=$ Christoffel symbol / connection

$$R^i_{jkl} =: \partial_l \Gamma^i_{jk} - \partial_k \Gamma^i_{jl} + \Gamma^m_{jk} \Gamma^i_{lm} - \Gamma^m_{jl} \Gamma^i_{km} :=$$ Riemann Tensor

Exercise: $\Gamma'^l_{mn}(\vec{x}') = \partial_i x'^l \partial'_m x^i \partial'_n x^k \Gamma^i_{jk}(\vec{x}) + \partial_k x^l \partial'_m \partial'_n x^k$

NB: This is not a tensor : of the second term.

: R^i_{jkl} is a tensor of rank $(1,3)$

: R^i_{jkl} has $\frac{1}{12} n^2(n^2-1)$ independent parameters.