$$C'^{i_1 \dots i_p}{}_{j_1 \dots j_q}(\vec{x}') = \partial_{k_1} x'^{i_1} \dots \partial_{k_p} x'^{i_p} \cdot \partial'_{j_1} x^{l_1} \dots \partial'_{j_q} x^{l_q} \, C^{k_1 \dots k_p}{}_{l_1 \dots l_q}(\vec{x})$$

Rank

(0,0) Scalar $\phi'(\vec{x}') = \phi(\vec{x})$

(1,0) Contravariant vector

(0,1) Covariant vector

$A^{i_1 \dots i_p}{}_{j_1 \dots j_q} B^{k_1 \dots k_r}{}_{l_1 \dots l_s}$ — This transforms as a $(p+r,\ q+s)$ tensor

$A^{i_1 \dots i_p}{}_{j_1 \dots j_q}$

$$A'^{i_1}{}_{i_1}(\vec{x}') = \underbrace{\partial_{k_1} x'^{i_1} \, \partial'_{i_1} x^{l_1}}_{= \frac{\partial x'^{i_1}}{\partial x^{k_1}} \cdot \frac{\partial x^{l_1}}{\partial x'^{i_1}} = \frac{\partial x^{l_1}}{\partial x^{k_1}} = \delta^{l_1}{}_{k_1}} A^{k_1}{}_{l_1}(\vec{x})$$

$= A^{k_1}{}_{k_1}(\vec{x})$ which is consistent

Claim: Take some point p on the manifold. $\vec{x}_0$ is the coordinate of p in some coordinate system.

Then $\exists$ a coordinate transf $\vec{x} \longrightarrow \vec{x}'$ s.t. $g'_{ij}(x'_0) = \mathrm{diag}(\pm 1, \pm 1, \dots \pm 1)$

NB: For continuous metrics, the number of + & − entries will be constant.

$$x^i = f^i(\vec{x}') \to x^i_0 \equiv f^i(\vec{x}'_0)$$

$$x^i = x^i_{(0)} + A^i{}_k(x'^k - x'^k_0) + B^i{}_{kl}(x'^k - x'^k_0)(x'^l - x'^l_0) + \dots$$

Here we expand about x_0

$$g'_{kl}(\vec{x}'_0) = \partial'_k x^i \, \partial'_l x^j \, g_{ij} \Big|_{\substack{\vec{x}' = \vec{x}'_0 \\ \vec{x} = \vec{x}_0}}$$

$$= A^i{}_k A^j{}_l \, g_{ij}(\vec{x}_0) \quad \text{(take } \partial' \text{ \& should be clear)}$$

$$= (A^T g A)_{kl} \quad \text{(in matrix notation)}$$

Notation: Regard g_{ij} as matrix; i as row index, j as column index

Recall: g is symmetric; Thus can be diagonalized with orthogonal matrices.

viz. $g = S^T g_d S$ s.t. $S^T S = \mathbb{1}$

$$g_d = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Non-singular metric only, i.e. eigenvalues are non-zero.

Def^n: $R \equiv R^T \equiv \begin{pmatrix} \sqrt{|\lambda_1|} & & 0 \\ & \ddots & \\ 0 & & \sqrt{|\lambda_n|} \end{pmatrix}$; $\Rightarrow R^T \eta R = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ where $\eta = \begin{pmatrix} \frac{\lambda_1}{|\lambda_1|} & & \\ & \ddots & \\ & & \frac{\lambda_n}{|\lambda_n|} \end{pmatrix}$

Thus, $g = S^T g_d S^T = S^T R^T \eta R S$

$g'_{kl} = (A^T g A)_{kl} = (A^T S^T R^T \eta R S A)_{kl}$, $\Rightarrow$ for $A = (RS)^{-1}$ we'll get $g'_{kl}(\vec{x}'_0) = \eta_{kl}$

This is not unique ; eg. $\eta = 1$, then $\forall A = (URS)^{-1}$ we'll get the right form, where

Signature :
When all +, then eucledian
when one -, then lorentzian

NB : (1) Locally then, there's no more information
(2) This will not distinguish b/w a plane & the surface of a sphere.
(3) This is more than diagonalization ; A is more than just a $R^T R$ like matrix.

Def^n : $K_{ijk} = \partial_i g_{jk}$

$$K'_{i_1 i_2 i_3} = \partial'_{i_1} \left(g'_{i_2 i_3}(\vec{x}') \right)$$

$$= \partial'_{i_1} \left(\partial'_{i_2} x^{j_2} \partial'_{i_3} x^{j_3} g_{j_2 j_3}(\vec{x}') \right)$$

$$= \partial'_{i_1} \partial'_{i_2} x^{j_2} \partial'_{i_3} x^{j_3} g_{j_2 j_3} + \partial'_{i_2} x^{j_2} \partial'_{i_1} \partial'_{i_3} x^{j_3} g_{j_2 j_3} +$$

$$\underbrace{\partial'_{i_2} x^{j_2} \partial'_{i_3} x^{j_3} \partial'_{i_1} g_{j_2 j_3}}_{\text{This is the term we want (why? we'll see soon)}}$$

chain rule

$$= \partial'_{i_2} x^{j_2} \partial'_{i_3} x^{j_3} \partial'_{i_1} x^{j_1} \partial_{j_1} g_{j_2 j_3} \qquad \left(\because \frac{\partial x^{j_1}}{\partial x'_{i_1}} \frac{\partial g_{j_2 j_3}}{\partial x^{j_1}} = \frac{\partial}{\partial x'_{i_1}} g_{j_2 j_3} \right)$$

rearrange

$$= \underbrace{\partial'_{i_1} x^{j_1} \partial'_{i_2} x^{j_2} \partial'_{i_3} x^{j_3}}\ \underbrace{\partial_{j_1} g_{j_2 j_3}}_{\to K_{j_1 j_2 j_3}}$$

This is how K_{ijk} should transform if K_{ijk} was a tensor.

However, since the other terms are non-zero, $\Rightarrow K_{ijk}$ is not a tensor under general coordinate transformation.

Idea : Take a linear combination of K_{ijk} to construct a tensor.
(or even polynomials of K_{ijk})

$\therefore$ This is not possible.

Proof sketch : Assume we can show that given a point, $\exists$ a coordinate transformation (CT) s.t. K vanishes. Therefore, it must vanish in all coordinates ($\because$ tensors transform in that way). Therefore its impossible to find such a tensor (it'll be zero essentially), (if this argument is repeated for all points)

Recall : $x^i = x'^i_0 + A^i{}_j (x'^j - x'^j_0) + B^i_{jk} (x'^j - x'^j_0)(x'^k - x'^k_0) + \dots$

We need 2 derivatives $\Rightarrow B^i_{jk}$ will be important

B^i_{jk} will be unique.

We want $K'_{ijk} = 0$, so $\frac{(n)(n+1)}{2} \cdot n$ eq^n's are required to be zero.

constants is also $\frac{(n)(n+1)}{2} \cdot n$ ($\because B^i_{jk}$ also has similar symmetric aspects as K_{ijk})

Exercise: Find B^i_{jk} explicitly, to make $K'^i_{j,k} = 0$

Idea: Take 2 derivatives of g_{ij}

$$S_{ijkl} = \partial_i \partial_j g_{kl}$$

(Claim: This itself is not a tensor.)

$\therefore$ Use the old argument. See if it can be shown to be zero in some coordinate sys.

At $\vec{x}_0$, # of components of $S_{ijkl} = \frac{n(n+1)}{2} \cdot \frac{n(n+1)}{2}$

Recall: $x^i = x^i_0 + A^i_j(\ldots)^j + B^i_{jk}(\ldots)^{jk} + C^i_{jkl}(\ldots)^{jkl}$

Now # of parameters in $C^i_{jkl} = n \cdot \frac{n(n+1)(n+2)}{3!}$ & $<$ (for large N)

$\Rightarrow$ There're more constraints than parameters. This means one can't in general get $S_{ijkl} = 0$.

Result: So $\frac{n^2(n+1)^2}{4} - \frac{n^2(n+1)(n+2)}{6} = \frac{1}{12} n^2(n^2-1)$ — 1 for $n=2$, 6 for $n=3$, 20 for $n=4$

which means the tensor so constructed will have the aforesaid # of free parameters.

Def^n: $\Gamma^i_{jk} =: \frac{1}{2} g^{il}(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk})$:= Christoffel symbol / Connection

$R^i_{jkl} =: \partial_l \Gamma^i_{jk} - \partial_k \Gamma^i_{jl} + \Gamma^m_{jk}\Gamma^i_{lm} - \Gamma^m_{jl}\Gamma^i_{km}$:= Riemann Tensor

Exercise: $\Gamma'^l_{mn}(\vec{x}') = \partial_i x'^l \partial'_m x^j \partial'_n x^k \Gamma^i_{jk}(\vec{x}) + \partial_k x^l \partial'_m \partial'_n x^k$

NB: This is not a tensor $\because$ of the second term

$\therefore$ R^i_{jkl} is a tensor of rank (1,3)

$\therefore$ R^i_{jkl} has $\frac{1}{12} n^2(n^2-1)$ independent parameters.