Lecture 8 (identities + GR started!)

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Useful Identities

(will not be proved here)

1(a) $D_i D_j A_k - D_j D_i A_k$ (also written as $[D_i, D_j] A_k$)
$= -R^l{}_{kji} A_l$

defined as the usual covariant derivative with the Christoffel symbol

These can be proved (claim) by looking at the definitions

(b) $D_i D_j A^k - D_j D_i A^k = R^k{}_{lji} A^l$

(c) Combining the aforesaid

$$D_i D_j A^{k_1 \dots k_p}{}_{l_1 \dots l_q} - D_j D_i A^{k_1 \dots k_p}{}_{l_1 \dots l_q} = R^{k_1}{}_{lji} A^{l k_2 \dots k_p}{}_{l_1 \dots l_q} + \dots + R^{k_p}{}_{lji} A^{k_1 k_2 \dots k_{p-1} l}{}_{l_1 \dots l_q}$$

$$- \{ R^l{}_{l_1 ji} A^{k_1 \dots k_p}{}_{l l_2 \dots l_q} + \dots + R^l{}_{l_q ji} A^{k_1 \dots k_p}{}_{l_1 \dots l_{q-1} l} \}$$

Bianchi Identities

$D_s R_{ijkl} + D_k R_{ijls} + D_l R_{ijsk} = 0$

structure: completely anti-symmetrize in $s\,k\,l$. NB: kl are already anti-symmetric; Instead of 6 terms, need to write only 3 terms.

proof idea: goto a frame in which $\Gamma = 0$.

can derive more by contracting, e.g. g^{js}

$\Rightarrow D^j R_{ijkl} + D_k R_{il} - D_l R_{ik}$

where the Ricci tensor was defined as $g^{ik} R_{ijkl} = R_{jl}$. NB: contracting the 2^{nd} & 4^{th} index is also the same $\because$ R_{ijkl} is anti-symmetric in (i,j) & (k,l)

further contract with g^{ik}.

$\Rightarrow D^j R_{jl} + D^i R_{il} - D_l R = 0$

same terms

$\Rightarrow 2D^j R_{jl} - D_l R = 0 \quad \Rightarrow 2D^j \Big(\underbrace{R_{jl} - \tfrac{1}{2} R g_{jl}}_{\text{this is called Einstein's tensor.}} \Big) = 0$

where in the last step, D doesn't act on g $\because$ $Dg = 0$. It acts on R, then lowers the index.

This completes our general discussion of manifolds. Now we start General Relativity.

Conventions

Signature: Now we focus on manifolds with signature $(n,1)$ (here $n=3$ for 3d space)

Recall. Signature $(3,1) = (- + + +)$

Indices: x^μ $\mu = 0,1,2,3$ coordinates.

functions: $f(\xi) \longrightarrow f(x)$

e.g.
$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$

$g_{\mu\nu} = \eta_{\mu\nu}$

Recall: In special relativity, proper time is given by $-ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$ where $x^0 = ct$.

Question. Does the metric $g_{\mu\nu}(x)$ as generalized, describe anything physical?
Answer: Yes, it describes spacetime in the presence of gravity.

General Relativity

$g_{\mu\nu}(x)$ describes space-time in presence of a gravitational field.

Particles in a gravitational field move along geodesics in the absence of other fields.

GR (cont.)

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Remark: How the gravitational field is produced we'll come to later, second part of general relativity.

Recall: Geodesic Eq^n

$$\frac{d^2x^\mu}{du^2} + \Gamma^\mu{}_{\nu\rho}(x)\frac{dx^\nu}{du}\frac{dx^\rho}{du} = 0. \quad \leftarrow$$ Describes the motion of the particle in a gravitational field.

Story: When the metric has all + signature, $ds^2 \geq 0$
In our case, 3 + & 1 −, ds^2 can be ≥ 0, $= 0$, ≤ 0.
Therefore we can divide the geodesic into 3 parts.

① $\frac{ds^2}{du^2} = g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du} < 0$ (Time like)

which can be made by rescaling $= -1$

"proof": ∵ u is an affine parameter, $\frac{ds^2}{du^2} = \text{const}$.
Let $\frac{ds^2}{du^2} = -c$ and rescale
$u \to u\sqrt{c}^{-1}u'$
$\Rightarrow \left(\frac{ds}{du}\right)^2 = \frac{ds^2}{c^{-1}du'^2} \Rightarrow \frac{ds^2}{du'^2} = -1.$

NB: du can be associated with $d\tau$ (proper time)
∵ $\frac{ds^2}{du^2} = g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du} = -1$

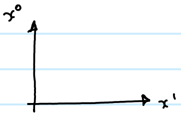
then for $g_{\mu\nu} = \eta_{\mu\nu}$
$-dx^{(0)2} + dx^{(1)2} + dx^{(2)2} + dx^{(3)2} = \underbrace{-du^2}_{-d\tau^2}$ proper time. (see the prev. page).

NB: the geodesic eq^n is invariant under scaling of u ∴ it is ok to scale.

(2) Space-like $\frac{ds^2}{du^2} = g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du} > 0$

(3) Light-like $g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du} = 0$

Story: The analogue of this in Minkoski space is trivial. Basically says every particle travels at a speed less than the speed of light.



$\left|\frac{dx^1}{dx^0}\right| < 1$ or in general $\left|\frac{d\vec{x}}{dx^0}\right| < 1$

$\Rightarrow ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -dx^{(0)2} + dx^{(1)2} < 0$

In GR we say this in a more coordinate invariant way.

For massless particles, in special relativity, $\left|\frac{d\vec{x}}{dx^0}\right| = 1$, must move at the speed of light. This entails $ds^2 = 0$
This also generalises to GR. Massless particles must move along light like geodesics.

Nothing of course travels along the space like geodesic; in SR, this corresponds to $\left|\frac{d\vec{x}}{dx^0}\right| > 1$, faster than speed of light.

Story 2 We'll see that these under the right circumstances reduce to Newton's Laws.
However, already we can see that the notion of signals not travelling faster than light is built in, as this theory a generalization of special relativity.

Newtonian Limit

Motivation: Physics without gravity, $\eta_{\mu\nu}$ describes the proper time (don't know the precise meaning of this).
Now even though gravity changes $\eta^{\mu\nu}$, that change must be small.

Newtonian Limit + ...

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We assume ① Weak Field : $g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{h_{\mu\nu}(x)}_{\text{small}}$

② Non-relativistic limit : $x^0 = ct$; write everything in terms of t & take c large.
(shouldn't be valid in the relativistic limit, allows instantaneous prop of signals)

NB: Ofcourse c has dimensions; what does taking c large mean? ∵ a dimensionful parameter being large means it must be large wrt something. Here, it means compared to velocities of objects.

Consider : $\frac{dx^\mu}{du} = \left(\frac{dx^0}{du}, \frac{dx^i}{du}\right) = \left(c\frac{dt}{du}, \frac{dx^i}{du}\right)$ for large c we ignore the $\frac{dx^i}{du}$ term compared to $c\frac{dt}{du}$, ie the velocities are small compared to c

· $\frac{\partial}{\partial x^0}$ vs $\frac{\partial}{\partial x^i}$, $\frac{1}{c}\frac{\partial}{\partial t}$ vs. $\frac{\partial}{\partial x^i}$ & again with c large,

the first term will be neglected.

Consequences: $h_{\mu\nu}(x)$ can be taken to be static ∵ $\frac{\partial}{\partial x^0}$ will be neglected compared to $\frac{\partial}{\partial x^i}$

Remark: Later we'll see the non-relativistic limit also implies the weak field limit

Recall $\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{du}\frac{dx^\rho}{du} = 0$ (geodesic eqⁿ)

$g_{\mu\nu}(x)\frac{dx^\mu}{du}\frac{dx^\nu}{du} = -1$ (defines the scaling)

Under the assumptions : $\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{00}(x)\frac{dx^0}{du}\frac{dx^0}{du} \simeq 0$

recall $\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho}\right)$

$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\sigma}\left(\underbrace{\partial_0 g_{\sigma 0} + \partial_0 g_{\sigma 0}}_{} - \partial_\sigma g_{00}\right)$

ignore ∵ they're time derivatives (see above)
(also for ∂_σ, ignore ∂_0)

$\simeq -\frac{1}{2}g^{\mu i}\partial_i g_{00} = -\frac{1}{2}g^{\mu i}\partial_i h_{00}$

NB: If we include the h dependent part, in this, then it will go 2nd order in h (overall) Thus we replace $g^{\mu i} = \eta^{\mu i}$.

NB 2: $\eta^{\mu i}$ is diagonal. $\eta^{\mu i} = 0$ for $\mu = 0$
$= \delta_{ij}$ for $\mu = j$ (signature is $-1, 1, 1, 1$)

$\Gamma^0_{00} \simeq 0$ & $\Gamma^i_{00} \simeq -\frac{1}{2}\partial_i h_{00}$.

Plugging Back : $\frac{d^2x^0}{du^2} \simeq 0$ & $\frac{d^2x^i}{du^2} - \frac{1}{2}\partial_i h_{00}\left(\frac{dx^0}{du}\right)^2 \simeq 0$

The second eqⁿ (with leading order in $g^{\mu\nu}$) gives us

$$-\left(\frac{dx^0}{du}\right)^2 + \underbrace{\sum_{i=1}^{3}\left(\frac{dx^i}{du}\right)^2}_{\text{small compared to } dx^0/du} = -1 \Rightarrow \frac{dx^0}{du} \simeq 1.$$

NB: When you integrate $\frac{d^2x^0}{du^2} \simeq 0$, you get $\frac{dx^0}{du} \simeq$ const & that is fixed to be 1.

$\Rightarrow x^0 = u$ (we don't worry about the constant).

NB 2: We took the leading part in the 2nd eqⁿ because in the first, that's already multiplied by h_{00}.

... + Newtonian Limit + Principle of Equivalence

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This yield finally $\frac{d^2x^i}{du^2} - \frac{1}{2}\partial_i h_{00} \simeq 0 \quad (\because \frac{dx^0}{du} \simeq 1)$

$\Rightarrow \frac{1}{\cancel{c^2}} \frac{d^2x^i}{dt^2} \simeq \frac{c^2}{2}\partial_i h_{00}$

Comparison to Newton's Law of Gravity: $\frac{d^2x^i}{dt^2} = -\partial_i \phi$

$\Rightarrow \phi = -\frac{c^2}{2} h_{00}$ or $h_{00} = -\frac{2}{c^2}\phi$

Remark: This shows that in the appropriate limit, GR does reduce to Newton's laws where I set $h_{00} = -\frac{2}{c^2}\phi$.

NB: For large c, automatically h_{00} is small. It means that the weak field assumption is not independent.

NB 2: One could add a "large const" in $h_{00} = -\frac{2}{c^2}\phi + \text{const}$ which would only scale η which in turn scales coordinates. So it plays no role.

Conclusions using Remainian Geometry

Consider: a point x_0; in general $\Gamma^{\mu}_{\nu\rho}(x_0) \neq 0$

Intuition: Think of Γ as a "force" in the geodesic eq^n

Idea: $\exists$ a coordinate system x' s.t $\Gamma'^{\mu}_{\nu\rho}(x'_{0}) = 0$ (see lecture 2)

Consequence: The eq^n of motion looks like that of a free particle (put $\phi = 0$ to compare) at that particular space-time point.

: This can be done for all points (even though one needs to pick a different coordinate system). These frames are called local inertial frames.

: This principle that it's possible to find a frame where the effect of gravity disappears is called the principle of equivalence.

General Remark: The equation of motion doesn't depend on the composition of the particle you're considering.
e.g. a dust particle of copper & gold both move the same under gravity.

Deeper Remark(s): The general coordinate invariance is "sacred". $\leftrightarrow$ run into inconsistencies
The principle of equivalence can be removed in further generalizations

e.g the eq^n of motion is given by

$$\frac{d^2x^\mu}{du^2} + \Gamma^{\mu}_{\nu\rho}\frac{dx^\nu}{du}\frac{dx^\rho}{du} + \alpha\, \nabla^{\mu} R_{\nu\rho}\frac{dx^\nu}{du}\frac{dx^\rho}{du} = 0$$

say this depends on the particle type.

This will violate the principle of equivalence but this is still valid (coordinate invariant)
Infact any "quantum theory" e.g. string theory, generates this term even in the classical level.

... + Dimensional Analysis

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: Dimensional Analysis (to show α must be small)

Γ has one length derivative, R has two length derivatives

$\Rightarrow \alpha$ has dimensions length square

$$\frac{d^2 x^\mu}{du^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{du} \frac{dx^\rho}{du} + \alpha \, \Delta^\mu R_{\nu\rho} \frac{dx^\nu}{du} \frac{dx^\rho}{du} = 0$$

$$[L] \qquad \cancel{[L]^{-1}} [L] \cancel{[L]} \qquad [L]^2 [L^{-1}] \cancel{[L^{-2}]} \cancel{[L]} \cancel{[L]}$$

The length scale of interest is the planck scale & square of this would essentially have no easily observable effect.

Rough Work

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$ds^2 \le 0$

$\frac{ds^2}{du^2} \le 0$ $\because$ $\frac{ds^2}{du^2}$ is const $\therefore$ can make $\frac{ds^2}{du^2} = -1$

How do I know this? ; $\because$ signature is $+---$ or $-+++$ so it's possible always.

How about this? Should I use limits? $\Delta s^2 \le 0$

$\Rightarrow \frac{\Delta s^2}{\Delta u \cdot \Delta u} \le 0$

$\lim \frac{ds^2}{du^2} \le 0$

$= g_{\mu\nu} \frac{dx^\mu}{du} \frac{dx^\nu}{du}$

From the geodesic eq^n $\Rightarrow$ $\frac{d}{du}\left(\left(\frac{ds}{du}\right)^2\right) = 0$
(or is it a consequence of the affine parametrization?)

$\frac{ds}{du} = \text{const}$ $\because$ u is an affine parameter.