

$\vec{x}(u)$   
 $u=0$   $u=1$   
 $u$   $u+\delta u$   
 vectors in the tangent space at  $\vec{x}(u)$

aim: see what this vector becomes upon parallel transport.

$n^i(u) \rightarrow$  known (say)

$n^i(u+\delta u) \rightarrow$  calculate

go to  $\vec{x}(u)$  & find a new coordinate sys  $\vec{x}'$  s.t.  $\partial_i g'_{jk} = 0$  at  $\vec{x}' = \vec{x}'(u)$

$n'^i(u+\delta u) = n'^i(u) + \mathcal{O}(\delta u^2)$  (by rules of parallel transport)

since you don't want to keep changing coordinates (it's inconvenient),  $\therefore$  we see what parallel transport amounts to in some arbitrary frame.

Translate to  $x'$

$$n^i(u) = \left. \frac{\partial x^i}{\partial x'^k} \right|_{\vec{x}'(u)} n'^k(u)$$

$$n'^k(u) + \mathcal{O}(\delta u^2)$$

$$n^i(u+\delta u) = \left. \frac{\partial x^i}{\partial x'^k} \right|_{\vec{x}'(u+\delta u)} n'^k(u+\delta u)$$

$$\left. \frac{\partial x^i}{\partial x'^k} \right|_{\vec{x}'(u)} + \frac{\partial^2 x^i}{\partial x'^l \partial x'^k} \delta x'^l + \dots \quad \& \quad \delta x'^l = \frac{\partial x'^l}{\partial u} \delta u$$

$$\Rightarrow n^i(u+\delta u) = n^i(u) + \frac{\partial^2 x^i}{\partial x'^l \partial x'^k} \frac{\partial x'^l}{\partial u} \delta u n'^k(u) + \mathcal{O}(\delta u^2)$$

$$\frac{dn^i}{du} = \frac{\partial^2 x^i}{\partial x'^l \partial x'^k} \frac{\partial x'^l}{\partial u} n'^k(u)$$

We still have dependence on the primed coordinate. We wish to get rid of it.

$$\Gamma'^l_{mn}(\vec{x}') = \frac{\partial x'^l}{\partial x^i} \frac{\partial x^j}{\partial x'^m} \frac{\partial x^k}{\partial x'^n} \Gamma^i_{jk}(\vec{x}) + \frac{\partial x'^l}{\partial x^k} \frac{\partial^2 x^k}{\partial x'^m \partial x'^n} = 0 \quad (\text{for the primed frame})$$

$$\frac{\partial x^p}{\partial x'^l} \Gamma'^l_{mn}(\vec{x}') = \frac{\partial x^j}{\partial x'^m} \frac{\partial x^k}{\partial x'^n} \Gamma^p_{jk}(\vec{x}) + \frac{\partial^2 x^p}{\partial x'^m \partial x'^n} = 0$$

also, recall:  $n'^k(u) = \frac{\partial x'^k}{\partial x^i} n^i(u)$

$$\frac{dx'^i}{du} = \frac{\partial x'^i}{\partial x^s} \frac{dx^s}{du}$$

$$\Rightarrow \frac{dn^i}{du} = - \frac{\partial x'^s}{\partial x'^l} \frac{\partial x^s}{\partial x'^n} \Gamma^i_{rs} \frac{\partial x'^l}{du} n'^k$$

$$= - \Gamma^i_{rs} \left[ \frac{\partial x'^s}{\partial x'^l} \frac{\partial x^s}{\partial x'^n} \frac{\partial x'^l}{\partial x^m} \frac{\partial x'^k}{\partial x^m} \right] \frac{dx^m}{du} n^m$$

$$= - \Gamma^i_{rs} \delta^s_n \frac{dx^r}{du} n^m$$

Using these (ex: check) we get

$$\frac{dn^i}{du} + \Gamma^i_{pq}(\vec{x}(u)) n^p \frac{dx^q}{du} = 0$$

(The primes will disappear (as Kronecker delta)).

for a d-dimensional space,

there're d first order d.e.

Thus given the initial vector, you can solve it for u.

NB:  $u$  needn't describe a geodesic.

claim: The  $eq^n$  is parametrization independent.

Proof:  $u = f(v)$

$$\left. \begin{aligned} \frac{dn^i}{du} &= \frac{dv}{du} \cdot \frac{dn^i}{dv} \\ \frac{dx^q}{du} &= \frac{dx^q}{dv} \cdot \frac{dv}{du} \end{aligned} \right\} \Rightarrow \frac{dv}{du} \left[ \frac{dn^i}{dv} + \Gamma_{pq}^i(\vec{x}(v)) n^p \frac{dx^q}{dv} \right] = 0$$

NB: unlike the geodesic  $eq^n$  which works in a specific parametrization.

Now choose an arbitrary coordinate sys  $\vec{x}''$  (where  $\Gamma''$  needn't be zero)

Recall:  $n^i = n''^k \frac{\partial x^i}{\partial x''^k}$

$\Gamma$  the transformation

substitute in

Ex:  $\Rightarrow \frac{\partial x^i}{\partial x''^k} \left( \frac{dn''^k}{du} + \Gamma''_{pq}^k n''^p \frac{dx'^q}{du} \right) = 0$

NB:  $\frac{dx^q}{du} = \frac{\partial x^q}{\partial x''^k} \frac{dx''^k}{du}$

Therefore it's evident that the parallel transport  $eq^n$ 's don't depend on the coordinate frame.

NB: It's not surprising  $\because$   $x$  coordinate sys to start with, was arbitrary.

Therefore infact it's a consistency check.

Ex:  $\frac{d}{du} (g_{ij}(\vec{x}(u)) n^i(u) n^j(u)) \stackrel{\text{claim}}{=} 0$  viz. norm is preserved under parallel transport.

Proof sketch: (direct)  $\frac{dn^i}{du}$  is known now,  $\frac{\partial}{\partial x^k} g_{ij} \frac{dx^k}{du} = \frac{dg_{ij}}{du}$  & that's known.

(neat) the quantity is invariant under coordinate transformations. Thus, go to the primed coordinates. Now first derivative of  $g$  is zero (by def<sup>n</sup> of the primed coordinate). Also, for a primed coordinate,  $\frac{dn'^i}{du} = 0$ . So that does it.

Recall:  
Geodesic  $eq^n$

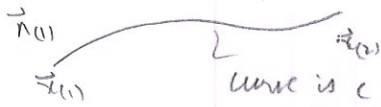
$$\frac{d^2 x^i}{du^2} + \Gamma_{jk}^i \frac{dx^j}{du} \frac{dx^k}{du} = 0 \quad ; \quad \text{L. Def}^n: \quad n^i(u) := \frac{dx^i}{du} \quad (\text{the tangent vector})$$

So now,  $\frac{dn^i}{du} + \Gamma_{jk}^i n^j \frac{dx^k}{du} = 0$

NB!: This is exactly the parallel transport  $eq^n$

Alternate def<sup>n</sup> of geodesic: curve s.t. its tangent vectors are transported from any given tangent vector on the curve.

$\vec{n}(u)$  — after parallel transport



NB: The transport eq<sup>n</sup> is linear.

$$\Rightarrow \alpha \vec{n}(u) + \beta \vec{n}'(u) \rightarrow \alpha \vec{n}(v) + \beta \vec{n}'(v)$$

(Not entirely certain how this follows)

$$n_{(v)}^i = M^i_j(\vec{x}(u), \vec{x}(v), c) n_{(u)}^j$$

independent of  $n_{(u)}$  &  $n_{(v)}$

Def<sup>n</sup>:  $-c :=$  Path  $c$  in the reverse direction.

Q: What can we say about  $M^i_j(\vec{x}(v), \vec{x}(u), -c)$ ?

NB: The parallel transport eq<sup>n</sup> is reversible (independent of parametrization)

We reparametrize:  $v := 1-u$ ; this will do the job.

$$u=1 \Leftrightarrow v=0$$

$$u=0 \Leftrightarrow v=1$$

$$\frac{dn^i}{dv} + \Gamma^i_{jk} n^j(v) \frac{dx^k}{dv} = 0 \rightarrow \frac{dn^i}{du} + \Gamma^i_{jk} n^j(u) \frac{dx^k}{du} = 0$$

So we solve the same old eq<sup>n</sup> with boundary conditions reversed.

Thus we must have

$$n_{(u)}^i = (M(\vec{x}(u), \vec{x}(v), c)^{-1})^i_j n_{(v)}^j$$

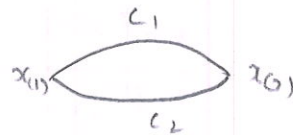
$$\stackrel{\text{by Def}^n}{=} M(\vec{x}(v), \vec{x}(u), -c)^i_j n_{(v)}^j$$

$\Rightarrow \because n_{(v)}^i$  is arbitrary,

$$M(\vec{x}(v), \vec{x}(u), -c) = M(\vec{x}(u), \vec{x}(v), c)$$

NB: If we had  $\frac{dn^i}{du} + \Gamma^i_{jk} n^j \left| \frac{dx^k}{du} \right| = 0$ , the eq<sup>n</sup> won't be reversible.

Take 2 curves joining  $x_{(u)}$  &  $x_{(v)}$ ,  $c_1$  &  $c_2$



In general,  $M(\vec{x}(u), \vec{x}(v), c_1) \neq M(\vec{x}(u), \vec{x}(v), c_2)$

$$\text{Consider: } M(\vec{x}(u), \vec{x}(v), c_1 - c_2) = M(\vec{x}(v), \vec{x}(u), -c_2) \cdot M(\vec{x}(u), \vec{x}(v), c_1)$$

$$= M(\vec{x}(u), \vec{x}(v), c_2)^{-1} M(\vec{x}(u), \vec{x}(v), c_1)$$

Def<sup>n</sup>: Monodromy Matrix around  $C$   $\neq \mathbb{I}$  in general.

$$:= M(\vec{x}, \vec{x}, C) \text{ for some closed curve } C$$

In general,  $\tilde{n}^i = M(\vec{x}, \vec{x}, C)^i_j n^j$

initial vector at  $\vec{x}$ .  
final vector, after coming back to  $\vec{x}$ .

Now let's change the coordinate system:

$$\text{Recall: } \tilde{n}^i = \frac{\partial x'^i}{\partial x^p} \tilde{n}^p = \frac{\partial x'^i}{\partial x^p} M(x, x, C)^p_q n^q = \frac{\partial x'^i}{\partial x^p} M(x, x, C)^p_q \frac{\partial x^q}{\partial x'^j} n'^j$$



If  $M$  was 1, then you'd get

$$\tilde{n}'^i = n'^i$$

ie.  $M$  will remain identity even in the new coordinate sys.

NB: This can be proved the other way also. If  $\tilde{n}'^i = n'^i$  then  $M$  must be 1.

NB2: This analysis will not work for  $M(\vec{x}', \vec{x}', c) \because$  The  $\frac{\partial x}{\partial x'} M \frac{\partial x'}{\partial x}$  factors won't become Kroneckers (they're evaluated at different points), even if  $M = \mathbb{I}$ .

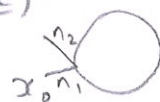
Claim: When  $R=0$ ,  $M=\mathbb{I}$ ; the reverse is true when there're no singularities (& simply connected)

Proof: Start with a small curve; length  $\sim O(\epsilon)$

Convention:

$0 < u < \epsilon$  so that

$$\frac{dx^i}{du} \sim 1$$



$$\frac{dn^i}{du} = -\Gamma_{jk}^i n^j(u) \frac{dx^j}{du}$$

idea: Keep terms of order  $u$  & then integrate, you get the result to order  $\epsilon$ . Then plug in the result & re-evaluate to get order  $\epsilon^2$ .

$$\frac{dn^i}{du} = \overbrace{-\Gamma_{jk}^i(x_0)}^{\text{correct to order } \epsilon} \overbrace{n_{(0)}^j}^{\text{order 1, can't take it at its original value.}} \overbrace{\frac{dx^k}{du}}^{\text{original value.}} + O(\epsilon)$$

$$\Rightarrow n^i(u) = \overbrace{n_{(0)}^i}^{O(1)} - \overbrace{\Gamma_{jk}^i(x_0)}^{O(1)} \overbrace{n_{(0)}^j}^{O(1)} (x^j(u) - x_{(0)}^j) + O(\epsilon^2) \quad (\because \epsilon u \approx \epsilon^2)$$

$$\frac{dn^i}{du} = -\left\{ \overbrace{\Gamma_{jk}^i(\vec{x}_0)}^{O(1)} + \partial_l \overbrace{\Gamma_{jk}^i(\vec{x}_0)}^{O(\epsilon)} \overbrace{(x^l - x_{(0)}^l)}^{O(\epsilon)} \right\} \times \left\{ \overbrace{n_{(0)}^j}^{O(1)} - \overbrace{\Gamma_{pq}^j(\vec{x}_0)}^{O(1)} \overbrace{n_{(0)}^p}^{O(1)} \overbrace{(x^q - x_{(0)}^q)}^{O(\epsilon)} \right\} \frac{dx^k}{du}$$

Now keeping upto  $O(\epsilon^2)$ , we have

$$\frac{dn^i}{du} = -\left\{ \Gamma_{jk}^i n^j \frac{dx^k}{du} - \Gamma_{ik}^l \Gamma_{pq}^j n^p (x^q - x_{(0)}^q) \frac{dx^k}{du} + \partial_l \Gamma_{jk}^i (x^l - x_{(0)}^l) \cdot n_{(0)}^j \cdot \frac{dx^k}{du} \right\} + O(\epsilon^2)$$

Integrating from  $(0, \epsilon)$ , we note that term 1 is zero.

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$$n^i(u) = n_{(0)}^i + \underbrace{\partial_l \Gamma_{jk}^i}_{\text{3rd term}} n_{(0)}^j \int_0^\epsilon (x^l - x_{(0)}^l) \frac{d(x^k - x_{(0)}^k)}{du} - \underbrace{\Gamma_{jk}^i \Gamma_{pq}^j}_{\text{2nd term}} n^p \int_0^\epsilon (x^q - x_{(0)}^q) \frac{d(x^k - x_{(0)}^k)}{du}$$

(lets try to match indices for aesthetics)

$$= n_{(0)}^i + \partial_q \Gamma_{pk}^i n_{(0)}^p \int_0^\epsilon (x^q - x_{(0)}^q) \frac{d(x^k - x_{(0)}^k)}{du} - \Gamma_{jk}^i \Gamma_{pq}^j n^p \int_0^\epsilon (x^q - x_{(0)}^q) \frac{d(x^k - x_{(0)}^k)}{du}$$

$$= n_{(0)}^i + \left[ \partial_q \Gamma_{pk}^i - \Gamma_{jk}^i \Gamma_{pq}^j \right] n_{(0)}^p \int_0^\epsilon (x^q - x_{(0)}^q) \frac{d(x^k - x_{(0)}^k)}{du}$$

NB: The boundary terms are zero. The derivative can be  $\therefore$  shifted with a minus sign.

NB2: This means the indices  $q$  &  $k$  are anti-symmetric

NB3: Thus only the symmetric part of the remaining part must contribute

$$= n_{(0)}^i + \frac{1}{2} \left[ \partial_q \Gamma_{pk}^i - \partial_k \Gamma_{pq}^i - \Gamma_{jk}^i \Gamma_{pq}^j + \Gamma_{jq}^i \Gamma_{pk}^j \right] n_{(0)}^p \int_0^\epsilon (x^q - x_{(0)}^q) \frac{d(x^k - x_{(0)}^k)}{du}$$

$$= n_{(0)}^i + \frac{1}{2} R_{pkq}^i \Big|_{\vec{x}_0} \cdot \int_0^\epsilon (x^q - x_{(0)}^q) \frac{d(x^k - x_{(0)}^k)}{du}$$

$$n_{(\epsilon)}^i = n_{(0)}^i + O(\epsilon^3) \quad \text{if} \quad R_{ijk}^i \Big|_{\vec{x}_0} = 0$$

$$\Rightarrow M_{ij}^i(\vec{x}_0, \vec{x}_0, \epsilon) = \delta_{ij} + O(\epsilon^3) \quad \text{if} \quad R_{ijk}^i \Big|_{\vec{x}_0} = 0.$$

