The counting

$$x'^j = x^j_{(0)} + A^j{}_k (x'^k - x'^k_0) + B^j{}_{kl} (x'^k - x'^k_0)(x'^l - x'^l_0) + \dots$$

Q: Can we use this to set $\partial_i g_{jk} = 0$?
$\partial_i \partial_j g_{kl} = 0$?

Consider:
$C_{i_1 \dots i_N}$, we want to show $C'_{i_1 \dots i_n} = 0$ in some frame for some choices of As & Bs.

$$x''^i = x'^i_0 + S^i{}_j (x''^j - x''^j_{(0)})$$

$$C''_{i_1 \dots i_n} = \partial''_{i_1} x'^{j_1} \partial''_{i_N} x'^{j_n} C'_{j_1 \dots j_n}$$

$$= S_{i_1}{}^{j_1} \dots S_{i_n}{}^{j_n} C'_{j_1 \dots j_n}$$

$$= 0 \qquad (\because C'_{j_1 \dots j_n} = 0)$$

$$x''^i = x^i_0 + \underbrace{A^i{}_j S^j{}_k}_{\tilde{A}^i{}_k} (x'^k - x'^k_0) + \dots \qquad \text{(not too sure)}$$

Claim: This works for $\partial_i g_{jk}$ also. If its zero in the primed frame, remains zero in the unprimed (This is for linear transformations only)

NB: S is arbitrary, choose it to be A^{-1}.

Argument: Since $\tilde{A}^i{}_k = \delta^i_k$, therefore if x was zero, then it remains zero in a coordinate sys where X is arbitrary (even δ^i_k). Thus x being zero, doesn't depend on the 'linear term' & therefore not included in the counting.

A: To set $\partial_i g_{jk} = 0$, you need $B^j{}_{kl}$. That uses up all the freedom. Then $\partial_i \partial_j g_{kl}$ can't be set to zero in general.

NB: To compare flat space & curved space, just evaluate the Riemann Tensor. If zero, flat else curved.

: To compare b/w curved spaces, construct scalars (for Rieman Tensor, 20 diff scalars) from the Rieman Tensor & compare.

Covariant Derivative

If A_i is a Tensor, then $\partial_i A_j$ is not a tensor.

However, $D_i A_j = \partial_i A_j - \Gamma^k_{ij} A_k$ transforms like one $(0,1)$

Similarly $D_i A^j = \partial_i A^j + \Gamma^j_{ik} A_k$ transforms like a $(1,1)$ Tensor.

Generalized to an arbitrary tensor.

$\partial_k A^{i_1 \dots i_p}{}_{j_1 \dots j_q}$ is NOT a Tensor.

$$D_k A^{i_1 \dots i_p}{}_{j_1 \dots j_q} = \partial_k A^{i_1 \dots i_p}{}_{j_1 \dots j_q} + \left(\Gamma^{i_1}_{kl} A^{l i_2 \dots i_p}{}_{j_1 \dots j_q} + \Gamma^{i_2}_{kl} \dots \right)$$

$$- \left(\Gamma^l_{k i_1} A^{i_1 \dots i_p}{}_{l j_2 \dots j_q} + \Gamma^l_{k i_2} A \quad \dots \right)$$

is a Tensor, rank $(p, q+1)$

Ex: Convince yourself.

Raising & Lowering

$$A^{i_1 \dots i_p}{}_{j_1 \dots j_q} B^{m_1 \dots m_r}{}_{n_1 \dots n_s} =: C^{i_1 \dots i_p m_1 \dots m_r}{}_{j_1 \dots j_q n_1 \dots n_s}$$

is of rank $(p+r, q+s)$

If $C^{i_1 \dots i_p}{}_{j_1 \dots j_q}$ is rank (p, q) then

$C^{i_1 \dots i_p}{}_{i_1 \dots j_q}$ is rank $(p-1, q-1)$ tensor

---

If $A_{j_1 \dots j_q} g^{i_1 i_2}$ is a $(2, q)$ tensor then

$A_{i_1 j_2 \dots j_q} g^{i_1 i_2}$ is a $(2-1, q-1)$ Tensor & its written as

$A^{i_2}{}_{j_2 \dots j_q}$

Eg. (a) $A_{j_1 i_1 \dots j_q} g^{i_1 i_2} = A_{j_1}{}^{i_2}{}_{j_3 \dots j_q}$

(b) $R^i{}_{jkl}$; $R_{ijkl} = g_{im} R^m{}_{jkl}$

(c) $g^{ij} R_{ijkl} = 0$ $\because$ $R_{ij\dots} = -R_{ji\dots}$ & $g^{ij} = g^{ji}$

Kronecker

$\delta^i{}_k = 1$ if $i=k$
$\phantom{\delta^i{}_k =} 0$ if $i\neq k$

(x - coordinate
x' - coordinate

$$\delta'^i{}_k = \partial_m x'^i \, \partial'_k x^n \, \delta^n{}_m$$

$$= \partial_m x'^i \, \partial'_k x^m$$

$$= \partial'_k x'^i = \delta^i{}_k$$

Exercise : (a) g^{ij} is a tensor
(b) δ_{ik} is not a tensor.

Recall:

$$C^{i_1 \dots i_p k_1 \dots k_s}{}_{j_1 \dots j_q l_1 \dots l_s} := A^{i_1 \dots i_p}{}_{j_1 \dots j_q} B^{k_1 \dots k_s}{}_{l_1 \dots l_s}$$

Claim: $D_k C^{i_1 \dots i_p k_1 \dots k_s}{}_{j_1 \dots j_q l_1 \dots l_s} = \left(D_k A^{i_1 \dots i_p}{}_{j_1 \dots j_q}\right)\left(B^{k_1 \dots k_s}{}_{l_1 \dots l_s}\right)$
$\qquad + \left(A^{i_1 \dots i_p}{}_{j_1 \dots j_q}\right)\left(D_k B^{k_1 \dots k_s}{}_{l_1 \dots l_s}\right)$

proof: Trivial.

NB: $D_i g_{jk} = \partial_i g_{jk} - \Gamma^l_{ij} g_{lk} - \Gamma^l_{ik} g_{jl}$; This has only 1st derivatives of g_{ij}

Recall: (a) We proved that g_{ij}'s first derivatives can't form a tensor

(b) D_i of a tensor is a tensor.

Result : $D_i g_{jk} = 0$ (must hold, can be checked) Exercise

Similarly: $D_i g^{jk} = 0$

Proof: $D_i g^{jk} = \partial_i g^{jk} + \Gamma^j_{il} g^{lk} + \Gamma^k_{il} g^{jl}$

Claim: $\partial_i (g^{-1}) = - g^{-1} \partial_i g \, g^{-1}$
(proved soon)

Claim : $D_i (\delta^j{}_k) = 0$

Proof : $\underbrace{\partial_i \delta^j{}_k}_{=0} - \Gamma^l_{ik} \delta^j{}_l + \Gamma^j_{il} \delta^l{}_k$

$= - \Gamma^j_{ik} + \Gamma^j_{ik} = 0$

$A^{i_1 \dots i_p}{}_{j_1 \dots j_q}$

$$C^{i_1 \dots i_p m}{}_{j_1 \dots j_q n} := A^{i_1 \dots i_p}{}_{j_1 \dots j_q} \delta^m{}_n$$

$$D_k \left(C^{i_1 \dots i_p m}{}_{j_1 \dots j_q n}\right) = \left(D_k A^{i_1 \dots i_p}{}_{j_1 \dots j_q}\right) \delta^m{}_n + A \cancel{D_k(\delta^m{}_n)} \to 0$$

For $(m = j_1 \; ; \; n = i_1)$ we have

$$D_k A^{i_1 \dots i_p}{}_{i_1 j_2 \dots j_q} = D_k \left(A^{i_1 \dots i_p}{}_{j_1 \dots j_q}\right) \cdot \delta^{j_1}{}_{i_1}$$

Statement: Contracting before or after taking a covariant derivative are equivalent.

Eg: LHS: $D_k (A^i{}_i) = \partial_k A^i{}_i$

RHS: $D_k (A^i{}_j) \delta^j{}_i = \left(\partial_k A^i{}_j - \Gamma^l_{kj} A^i{}_l + \Gamma^i_{kl} A^l{}_j\right) \delta^j{}_i$

$$= \partial_k A^i{}_j \cancel{- \Gamma^l_{ki} A^i{}_l} \cancel{+ \Gamma^i_{kl} A^l{}_i}$$

$$= \partial_k A^i{}_j$$

Claim: $D_k \left(g^{i_1 j_1} A^{i_1 \dots i_p}{}_{j_1 \dots j_q}\right) = g^{i_1 j_1} D_k \left(A^{i_1 \dots i_p}{}_{j_1 \dots j_q}\right)$

claim: $\partial_i M^{-1} = -M^{-1} \partial_i M \, M^{-1}$

proof: $\partial_i (M M^{-1}) = \partial_i(\mathbb{1}) = 0$

$$(\partial_i M) M^{-1} + M \partial_i M^{-1} = 0$$

$$-M^{-1} (\partial_i M) M^{-1} = \partial_i M^{-1}$$