$\vec{x}_{(1)}$ $\vec{x}_{(0)}$ $u=1$
$u=0$ u $u+\delta u$
n^i vector in the tangent
space at $\vec{x}_{(1)}$

aim: see what this vectors becomes upon parallel transport.

$n^i(u) \to$ known (say)

$n^i(u+\delta u) \to$ calculate

goto $\vec{x}(u)$ & find a new coordinate sys $\vec{x}'$ s.t. $\partial'_i g'_{jk}=0$ at $\vec{x}' = \vec{x}'_{(0)}$

$n'^i(u+\delta u) = n'^i(u) + O(\delta u^2)$ (by rules of parallel transport)

Since you don't want to keep changing coordinates (its inconvenient), ∴ we see what parallel transport amounts in some arbitrary frame.

Translate to x'

$n^i(u) = \dfrac{\partial x^i}{\partial x'^k}\Big|_{\vec{x}'(u)} n'^k(u)$

$n'^k(u) + O(\delta u^2)$

$n^i(u+\delta u) = \dfrac{\partial x^i}{\partial x'^k}\Big|_{\vec{x}'(u+\delta u)} n'^k(u+\delta u)$

$\dfrac{\partial x^i}{\partial x'^k}\Big|_{\vec{x}'(u)} + \dfrac{\partial^2 x^i}{\partial x'^l \partial x'^k}\cdot \delta x'^l + \dots$ & $\delta x'^l = \dfrac{\partial x'^l}{\partial u}\cdot \delta u$

$\Rightarrow n^i(u+\delta u) = n^i(u) + \dfrac{\partial^2 x^i}{\partial x'^l \partial x'^k}\dfrac{\partial x'^l}{\partial u}\delta u\, n'^k(u) + O(\delta u^2)$

$\dfrac{dn^i}{du} = \dfrac{\partial^2 x^i}{\partial x'^l \partial x'^k}\dfrac{\partial x'^l}{\partial u} n'^k(u)$

We still have dependence on the primed coordinate. We wish to get rid of it.

$\Gamma'^l_{mn}(\vec{x}') = \dfrac{\partial x'^l}{\partial x^i}\cdot\dfrac{\partial x^j}{\partial x'^m}\dfrac{\partial x^k}{\partial x'^n}\Gamma^i_{jk}(\vec{x}) + \dfrac{\partial x'^l}{\partial x^k}\cdot\dfrac{\partial^2 x^k}{\partial x'^m \partial x'^n} = 0$ (for the primed frame)

$\dfrac{\partial x^p}{\partial x'^l}\Gamma'^l_{mn}(\vec{x}') = \dfrac{\partial x^j}{\partial x'^m}\dfrac{\partial x^k}{\partial x'^n}\Gamma^p_{jk}(\vec{x}) + \dfrac{\partial^2 x^p}{\partial x'^m \partial x'^n} = 0$

also, recall: $n'^k(u) = \dfrac{\partial x'^k}{\partial x^s} n^s(u)$

$\therefore \dfrac{dx'^i}{du} = \dfrac{\partial x'^i}{\partial x^s}\dfrac{\partial x^s}{\partial u}$

first this

$\Rightarrow \dfrac{dn^i}{du} = -\dfrac{\partial x^r}{\partial x'^l}\dfrac{\partial x^s}{\partial x'^n}\Gamma^i_{rs}\dfrac{\partial x'^l}{du}n'^k$

$= -\Gamma^i_{rs}\left[\dfrac{\partial x^r}{\partial x'^l}\dfrac{\partial x^s}{\partial x'^n}\dfrac{\partial x'^k}{\partial x^m}\dfrac{\partial x'^l}{\partial x^s}\right]\dfrac{\partial x^s}{\partial u}n^m$

$= -\Gamma^i_{rs}\delta^r_n (\dots)$

Using these (Ex: check) we get

$\dfrac{dn^i}{du} + \Gamma^i_{pq}(\vec{x}(u)) n^p \dfrac{dx^q}{du} = 0$

(The primes will disappear (as kronecker delta)).

For a d-dimensional space, there're d first order d.e. Thus given the initial vector, you can solve it for u.

NB: u needn't describe a geodesic.

claim: The eq^n is parametrization independent.

Proof: $u = f(v)$

$$\left.\begin{array}{c} \dfrac{dn^i}{du} = \dfrac{dv}{du} \cdot \dfrac{dn^i}{dv} \\[2mm] \dfrac{dx^g}{du} = \dfrac{dx^g}{dv} \cdot \dfrac{dv}{du} \end{array}\right\} \Rightarrow \dfrac{dv}{du}\left[\dfrac{dn^i}{dv} + \Gamma^i_{pg}(\vec{x}(v))\, n^p \dfrac{dx^g}{dv}\right] = 0$$

NB: unlike the geodesic eq^n which works in a specific parametrization.

Now choose an arbitrary coordinate sys $\vec{x}''$ (where Γ'' needn't be zero)

Recall: $n^i = n''^k \dfrac{\partial x^i}{\partial x'^k}$

$\therefore \Gamma$ the transformation

NB: $\dfrac{dx^g}{du} = \dfrac{\partial x^g}{\partial x''^{\lambda}} \dfrac{dx''^{\lambda}}{du}$

$\left.\begin{array}{c}\\[6mm]\end{array}\right\}$ substitute in

Ex:

$\Rightarrow \dfrac{\partial x^i}{\partial x''^{\lambda}}\left(\dfrac{dn''^{\lambda}}{du} + \Gamma''^{\lambda}_{pg}\, n''^p \dfrac{dx''^g}{du}\right) = 0$

therefore it's evident that the parallel transport eq^n's don't depend on the coordinate frame.

NB: It's not surprising : x coordinate sys to start with, was arbitrary. Therefore infact it's a consistency check.

Ex: $\dfrac{d}{du}\left(g_{ij}(\vec{x}(u))\, n^i(u)\, n^j(u)\right) \overset{\text{claim}}{=} 0$ viz. norm is preserved under parallel transport.

Proof sketch: (direct) $\dfrac{dn^i}{du}$ is known now, $\dfrac{\partial}{\partial x^k} g_{ij}\, \dfrac{dx^k}{du} = \dfrac{dg_{ij}}{du}$ & thats known.

(neat) the quantity is invariant under coordinate transformations. Thus, goto the primed coordinates. Now first derivative of g is zero (by def^n of the primed coordinate). Also, for a primed coordinate, $\dfrac{dn'^i}{du} = 0$. So that does it.

Recall:

Geodesic eq^n

$$\dfrac{d^2 x^i}{du^2} + \Gamma^i_{jk}\, \dfrac{dx^k}{du}\, \dfrac{dx^j}{du} = 0 \quad ; \quad \text{L.} \text{def}^n: \quad n^i(u) := \dfrac{dx^i}{du} \quad (\text{the tangent vector})$$

So now, $\quad \dfrac{dn^i}{du} + \Gamma^i_{jk}\, n^j \dfrac{dx^k}{du} = 0$

NB: This is exactly the parallel transport eq^n

Alternate def^n of geodesic: curve s.t. its tangent vectors are transported from any given tangent vector on the curve.

$\vec{n}_{(1)}$ ——— $\vec{n}_{(2)}$ — after parallel transport

curve is c

NB: The transport eq^n is linear.

$\Rightarrow \alpha \vec{n}_{(1)} + \beta \vec{n}'_{(1)} \longrightarrow \alpha \vec{n}_{(2)} + \beta \vec{n}'_{(2)}$

$$n_{(2)}^i = M^i{}_j(\vec{x}_{(1)}, \vec{x}_{(2)}, c)\, n_{(1)}^j$$

independent of $n_{(1)} \& n_{(2)}$

(Not entirely certain how this follows)

Def^n: $-c :=$ Path c in the reverse direction.

Q: What can we say about $M^i{}_j(\vec{x}_{(2)}, \vec{x}_{(1)}, -c)$?

NB: The parallel transport eq^n is reversible (independent of parametrization)

We reparametrize: $v := 1 - u$; this will do the job. $\quad u = 1 \Leftrightarrow v = 0$
$\quad u = 0 \Leftrightarrow v = 1$

$$\frac{dn^i}{dv} + \Gamma^i{}_{jk}\, n^j(v)\, \frac{dx^k}{dv} = 0 \quad \longrightarrow \quad \frac{dn^i}{du} + \Gamma^i{}_{jk}\, n^j(u)\, \frac{dx^k}{du} = 0$$

So we solve the same old eq^n with boundary conditions reversed.

Thus we must have

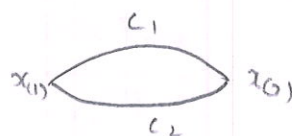$$n_{(1)}^i = \left(M(\vec{x}_{(1)}, \vec{x}_{(2)}, c)^{-1}\right)^i{}_j\, n_{(2)}^j$$

$\underset{\text{by Def}^n}{=} M(\vec{x}_{(2)}, \vec{x}_{(1)}, -c)^i{}_j\, n_{(2)}^j \quad \Rightarrow \quad \therefore\ n_{(1)}^i$ is arbitrary,

$$M(\vec{x}_{(2)}, \vec{x}_{(1)}, -c) = M(\vec{x}_{(1)}, \vec{x}_{(2)}, c)$$

NB: If we had $\quad \dfrac{dn^i}{du} + \Gamma^i{}_{jk}\, n^j \left|\dfrac{dx^j}{du}\right| = 0$, the eq^n won't be reversible.

———

Take 2 curves joining $x_{(1)} \& x_{(2)}$, $c_1 \& c_2$

In general, $\quad M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_1) \neq M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_2)$

Consider: $\quad M(\vec{x}_{(1)}, \vec{x}_{(1)}, c_1 - c_2) = M(\vec{x}_{(2)}, \vec{x}_{(1)}, -c_2) \cdot M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_1)$

$$= M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_2)^{-1}\, M(\vec{x}_{(1)}, \vec{x}_{(2)}, c_1)$$

Def^n: Monodromy Matrix around $c \neq \mathbb{1}$ in general.

$$:= M(\vec{x}, \vec{x}, c) \quad \text{for some closed curve } c$$

In general, $\quad \tilde{n}^i = M(\vec{x}, \vec{x}, c)^i{}_j\, n^j$

final vector, after coming back to $\vec{x}$.

initial vector at $\vec{x}$.

Now let's change the coordinate system:

Recall: $\quad \tilde{n}'^i = \dfrac{\partial x'^i}{\partial x^p}\, \tilde{n}^p = \dfrac{\partial x'^i}{\partial x^p} \cdot M(x, x, c)^p{}_q\, n^q = \dfrac{\partial x'^i}{\partial x^p}\, M(x,x,c)^p{}_q\, \dfrac{\partial x^q}{\partial x'^j}\, n'^j$

If M was $\mathbb{1}$, then you'd get

$$\tilde{n}'^i = n'^i$$

viz. M will remain identity even in the new coordinate sys.

NB: This can be proved the other way also. If $\tilde{n}'^i = n'^i$ then M must be 1.

NB2: This analysis will not work for $M(\vec{x}_1, \vec{x}_1, c)$ ∵ The $\frac{\partial x}{\partial x'}\, M\, \frac{\partial x'}{\partial x}$ factors won't become kronecker (they're evaluated at different points), even if $M = \mathbb{1}$.
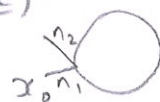
Claim: when $R = 0$, $M = \mathbb{1}$; the reverse is true when there're no singularities. (& simply connected)

Proof: Start with a small curve; length $\sim O(\epsilon)$

Convention: $0 < u < \epsilon$ so that $\frac{dx^i}{du} \sim 1$

$$\frac{dn^i}{du} = -\Gamma^i_{jk}\, n^j(u)\, \frac{dx^j}{du}$$

idea: Keep terms of order u & then integrate, you get the result to order ϵ. Then plugin the result & re-evaluate to get order ϵ^2.

$$\frac{dn^i}{du} = \overbrace{-\Gamma^i_{jk}(x_{(0)})}^{\text{correct to order } \epsilon}\ \overbrace{n^j_{(0)}}^{}\ \overbrace{\frac{dx^k}{du}}^{\substack{\text{order 1, can't take it at its}\\ \text{original value.}}} + O(\epsilon)$$

$$\Rightarrow n^i(u) = n^i_{(0)} - \Gamma^i_{jk}\Big|_{x_{(0)}} n^j_{(0)} \left(x^j(u) - x^j_{(0)}\right) + O(\epsilon^2) \qquad \left(\because \epsilon u \approx \epsilon^2\right)$$

$$\frac{dn^i}{du} = -\Big\{ \overbrace{\Gamma^i_{jk}(\vec{x}_{(0)})}^{O(1)} + \partial_\ell\, \Gamma^i_{jk}(\vec{x}_{(0)})\ \overbrace{\left(x^\ell - x^\ell_{(0)}\right)}^{O(\epsilon)} \Big\} \times \Big\{ \overbrace{n^j_{(0)}}^{O(1)} - \Gamma^j_{p q}(\vec{x}_0)\, n^p(\vec{x}_0) $$

$$\underbrace{\left(x^q - x^q_{(0)}\right)}_{O(\epsilon)} \Big\}\, \frac{dx^k}{du}$$

Now keeping upto $O(\epsilon^2)$, we have

$$\frac{dn^i}{du} = -\Big\{ \Gamma^i_{jk}\, n^j\, \frac{dx^k}{du} - \Gamma^i_{jk}\, \Gamma^j_{pq}\, n^p\, (x^q - x^q_{(0)})\, \frac{dx^k}{du}$$

$$+ \partial_\ell \Gamma^j_{jk}\, (x^\ell - x^\ell_{(0)}) \cdot n^j_{(0)} \cdot \frac{dx^k}{du} \Big\} + O(\epsilon^2)$$

Integrating from $(0, t)$, we note that term 1 is zero.

$$n^i(u) = n^i_{(0)} + \partial_l \Gamma^i_{jk} \, n^j_{(0)} \oint_0^\epsilon (x^l - x^l_0) \frac{d(x^k - x^k_{(0)})}{du} \quad\underset{\text{2nd term}}{-}\quad \Gamma^i_{jk} \Gamma^j_{pq} n^p \oint_0^\epsilon (x^q - x^q_{(0)}) \frac{d(x^k - x^k_{(0)})}{du}$$

(3rd term)

(Lets try to match indices for aesthetics)

$$= n^i_{(0)} + \partial_q \Gamma^i_{pk} \, n^p_{(0)} \oint_0^t (x^q - x^q_0) \frac{d(x^k - x^k_{(0)})}{du} - \Gamma^i_{jk} \Gamma^j_{pq} n^p \oint_0^t (x^q - x^q_{(0)}) \frac{d(x^k - x^k_{(0)})}{du}$$

$$= n^i_{(0)} + \left[\partial_q \Gamma^i_{pk} - \Gamma^i_{jk} \Gamma^j_{pq} \right] n^p_{(0)} \oint_0^t (x^q - x^q_{(0)}) \frac{d(x^k - x^k_{(0)})}{du}$$

NB: The boundary terms are zero. The derivative can be ∴ shifted with a minus sign.

NB2: This means the indices q & k are anti-symmetric

NB3: Thus only the symmetric part of the remaining part must contribute

$$= n^i_{(0)} + \frac{1}{2} \left[\partial_q \Gamma^i_{pk} - \partial_k \Gamma^i_{pq} - \Gamma^i_{jk} \Gamma^j_{pq} + \Gamma^i_{jq} \Gamma^j_{pk} \right] n^p_{(0)}$$

$$\cdot \oint_0^\epsilon (x^q - x^q_{(0)}) \frac{d}{du}(x^k - x^k_{(0)})$$

$$= n^i_{(0)} + \frac{1}{2} R^i_{pkq} \Big|_{x_0} \cdot \oint (x^q - x^q_{(0)}) \frac{d}{du}(x^k - x^k_{(0)})$$

$$n^i_{(\epsilon)} = n^i_{(0)} + \mathcal{O}(\epsilon^3) \quad \text{if} \quad R^i_{jkl}\Big|_{x_0} = 0$$

$$\Rightarrow \quad M^i_j(\vec{x}_0, \vec{x}_0, c) = \delta^i_j + \mathcal{O}(\epsilon^3) \quad \text{if} \quad R^i_{jkl}\Big|_{x_0} = 0.$$