Bell Test with Continuous Variables

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Abstract

We show the violation of the Bell inequality in a more classical setting than the usual quantum spin, viz. with continuous variables x, p. We use modular momentum and position to achieve this.

1 Introduction

The Bell inequality is a statement about the statistical outcomes of a certain class of experiments. The inequality was formulated by assuming the most general 'local theory'. Any violation of the inequality would imply that local realism must be false. Quantum Mechanics predicts a violation which has been confirmed experimentally. The physical situation for which this violation is usually formulated, uses entangled spin states. Here we show that a violation can be achieved without having to use spins, which is a purely quantum phenomenon. Using a quantum particle's position and momentum, we construct an analogous situation.

2 Violation in EPR/Bell state

Here the violation of Bell's inequality for the typical spin case, is stated in a way that is easier to translate to the continuous variable case. We first state the Bell test in its modern form, known as the CHSH inequality.

Suppose: There are two observers*, Alice and Bob. There are two particles, which are allowed to interact initially and are then separated. Alice takes one, Bob takes the other. Alice can measure one of two properties of her particle, call them a_1, a_2. Similarly Bob can measure b_1, b_2 of his particle.

Notatn: $\langle a_i b_j \rangle$ =Average value obtained when Alice chooses to measure a_i and Bob chooses to measure b_j

*that are often chosen to be sufficiently far away that their measurements don't influence each other

Statemnt: $|\langle a_1(b_1+b_2)+a_2(b_1-b_2)\rangle| \equiv |\langle C(a_i,b_i)\rangle| \leq 2$ if local realism is assumed. This result will be referred to as the CHSH inequality.

Quantum Mechanics predicts a violation of this in the following physical situation. Consider

Defn:

$$|\psi\rangle \equiv \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$$

where the $|\pm\rangle$ states are s.t. $\sigma_x |\pm\rangle = \pm |\pm\rangle$.

NB: Alternatively, this follows from the definition of σ_x in the standard basis (where σ_z is diagonal) and the following two definitions.

Defn:

$$\begin{aligned} |+\rangle &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\ |-\rangle &= \frac{|0\rangle - |1\rangle}{\sqrt{2}} \end{aligned}$$

Notatn: The Pauli matrices are referred to as x, y, z henceforth.

Defn: $|0\rangle, |1\rangle$ are s.t. $z|0\rangle = |0\rangle$ and $z|1\rangle = -|1\rangle$

NB:

$$\begin{aligned} |0\rangle &= \frac{|+\rangle + |-\rangle}{\sqrt{2}} \\ |1\rangle &= \frac{|+\rangle - |-\rangle}{\sqrt{2}} \end{aligned}$$

NB:

$$|\psi\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$$

This state is crucial for violating the CHSH inequality. For the typical spin case,

Consider: (a) The first qubit is with Alice and the second is with Bob.
(b) Alice measures z or x while Bob is allowed to

choose between z' and x', where

$$\begin{aligned} z' &= \frac{-z-x}{\sqrt{2}} \\ x' &= \frac{z-x}{\sqrt{2}} \end{aligned}$$

Result $\quad \langle C((x,z),(x',z'))\rangle = 2\sqrt{2} \nleq 2$, which is a violation.

This value can be reproduced in a slightly modified situation. We make 2 modifications.

Modifctn: (i) due to the symmetry in the setup, we expect the same results hold for x, y and x', y'
(ii) we demand that Alice and Bob can only measure x but are allowed to perform a local unitary before the measurement.

Questn: Can we obtain the same violation?

First we show that (ii) is equivalent to measuring along any direction. This is trival if one uses the Bloch sphere picture. Instead of measuring along an arbitrary axis, we rotate the Bloch sphere appropriately, and then measure x. To illustrate this, consider

Questn: $\exists\, \mathrm{a}\, U$, $s.t.$ if $|\chi\rangle \rightarrow |\chi'\rangle = U|\chi\rangle$ then $\langle \chi|x|\chi\rangle = \langle \chi'|y|\chi'\rangle$?
Explicitly, we have

$$\begin{aligned} y &= U^\dagger x U = e^{-iz\theta/2} x e^{iz\theta/2} \\ &= x e^{iz\theta} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \end{aligned}$$

for $\theta = \pi/2$ as one would guess geometrically.

Therefore we define the following situation.

Consider: (a) The first qubit is with Alice and the second is with Bob.
(b) Alice and Bob apply local unitaries $e^{iz\theta/2} \otimes I$ and $I \otimes e^{iz\phi/2}$ respectively and then both measure x.
(c) Alice can choose $\theta \in \{0, \pi/2\}$ and Bob can choose $\phi \in \{\pi/4, \pi/4 + \pi/2\}$

Remark: These values are motivated by the $\pi/2$ between z and x and $\pi/4$ between z and z' in the conventional Bell case.

	Alice		Bob	
Convention	a_1	a_2	b_1	b_2
Angle	$\theta_1 = \pi/2$	$\theta_2 = 0$	$\phi_1 = \pi/4$	$\phi_2 = \pi/4 + \pi/2$

Now it remains to show that this scheme also violates the Bell inequality (and by the same value). To that end, we state a result.

Claim(3): $\left\langle e^{-iz\theta/2} x e^{iz\theta/2} \otimes e^{-iz\phi/2} x e^{iz\phi/2} \right\rangle = -\cos(\phi - \theta)$

Given that, it is straight forward to evaluate

CHSH:

$$\begin{aligned} \langle C\rangle &= \langle a_1 b_1\rangle + \langle a_1 b_2\rangle + \langle a_2 b_1\rangle - \langle a_2 b_2\rangle \\ &= \left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right) - \left(+\frac{1}{\sqrt{2}}\right) \\ &= -2\sqrt{2} \end{aligned}$$

Remark $\quad$ This is a violation since $|\langle C\rangle| = 2\sqrt{2} \nleq 2$.

However, to convert this to a more conventional form, we simply redefine the θ and ϕ so that $\phi - \theta \rightarrow \phi' - \theta' = \phi - \theta + \pi = (\phi + \pi/2) - (\theta - \pi/2)$.

Substn:

$$\begin{aligned} \phi &\rightarrow \phi' = \phi + \pi/2 \\ \theta &\rightarrow \theta' = \theta - \pi/2 \end{aligned}$$

This will cause

$$\cos(\phi - \theta) \rightarrow \cos(\phi' - \theta') = -\cos(\phi - \theta)$$

Thus we finally define

	Alice		Bob	
Convention	a_1	a_2	b_1	b_2
Angle	$\theta_1' = 0$	$\theta_2' = -\pi/2$	$\phi_1' = 3\pi/8$	$\phi_2' = 5\pi/4$

so that we have $\langle C\rangle = 2\sqrt{2}$.

Concln: The violation of the Bell spin state can be equivalently be obtained by the measurement of x after applying a rotation along z by the aforesaid angles.

3 Violation in Continuous Variable State

We now construct continuous variable kets that satisfy the same orthogonality and mutual relations as $|0\rangle, |1\rangle, |+\rangle, |-\rangle$. We emphasise that the kets we construct, will not be eigenstates of

p or x since these are highly idealized and strictly, not even a part of the Hilbert space.

3.1 Ideal Case, infinite slits

Imagine: <TODO: Add an image of the setup>

Unassume: x, y, z refer to pauli matrices
Definitions of $|0\rangle, |1\rangle$

Defn: $L \equiv$some length scale
$N \equiv$number of slits

Consider: A normalized wavefunction $\varphi(x')$ s.t.
(a) $\varphi(x' - L/2) = \varphi(-(x' - L/2))$
(b) it's spread is $\ll L$
(c) $\varphi(x') = Re[\varphi(x')]$

Defn: $|\varphi(y')\rangle = \int dx' \varphi(x' - y' - \frac{N\cancel{2}L}{\cancel{2}}) |x'\rangle$, basically it is just φ positioned at y' if the origin of the following is assumed at the appropriate centre, to maintain symmetry.

Defn:

$$
\begin{aligned}
|0\rangle &\equiv \frac{1}{\sqrt{N}} \sum_{n=1}^{N} |\varphi[n(2L)]\rangle \\
|1\rangle &\equiv \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \left|\varphi\left[\left(n + \frac{1}{2}\right)(2L)\right]\right\rangle
\end{aligned}
$$

Remark: Definition of $|0\rangle$ and $|1\rangle$ justify the word 'slit' in the definition of N.

Recall:

$$
\begin{aligned}
|+\rangle &\equiv \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\
|-\rangle &\equiv \frac{|0\rangle - |1\rangle}{\sqrt{2}}
\end{aligned}
$$

NB: $\langle 0|1\rangle = 0, \implies \langle +|-\rangle = 0$

Defn:

$$
X \equiv \frac{e^{ipL/\hbar} + e^{-ipL/\hbar}}{2}
$$

NB: For sufficiently large N,

$$
\begin{aligned}
X|+\rangle &= +|+\rangle \\
X|-\rangle &= -|-\rangle
\end{aligned}
$$

just as was with σ_x.

Remark: We have almost all the cards, except for one that allows for setting ϕ and θ. This is done by essentially using alternate glass slabs (for photons) to adjust the relative phase as desired. We define this operator by it's action on the relevant state.

Defn: $U(\phi)$ is defined to be s.t.

$$
\begin{aligned}
U(\phi)|0\rangle &= e^{i\phi/2}|0\rangle \\
U(\phi)|1\rangle &= e^{-i\phi/2}|1\rangle
\end{aligned}
$$

Alternatively, we could also define $U(\phi)$ in its analytic form as

Alt Defn:

$$
U(\phi) = e^{iZ(x_{mod 2L})\phi/2}
$$

where $x_{mod2L} \equiv x \mod 2L$, whose operation can be defined explicitly*, and

$$
Z(x) \equiv \begin{cases} 1 & x \leq L \\ -1 & L < x \leq 2L \end{cases}
$$

when Z acts on an eigenstate of x_{mod2L}.

NB: $Z(x_{mod2L})|0\rangle = |0\rangle$ and $Z(x_{mod2L})|1\rangle = -|1\rangle$

Remark: Here, $Z(x_{mod2L})$ plays the role of σ_z in the discrete case.

To evaluate the CHSH inequality, we again need terms of the form $\langle U^\dagger(\phi_i) X U(\phi_i) \otimes U^\dagger(\theta_i) X U(\theta_i)\rangle$ which can be evaluated easily to be the same as before $-\cos(\phi_i - \theta_i)$. Thus, the same analysis as before yields

$$
\langle C\rangle = 2\sqrt{2}
$$

which is the same maximal violation, in a completely different setup.

Remark: The non-commutativity in this case is between U and X, which arises from $[x_{mod2L}, p_{modL}] \neq 0$.

3.2 Finite slits

If N is not assumed to be arbitrarily large, then it can be shown that

Claim(5): $\langle U^\dagger(\phi_i) X U(\phi_i) \otimes U^\dagger(\theta_i) X U(\theta_i)\rangle = -\left(\frac{N-1}{N}\right)^2 \cos(\phi_i - \theta_i)$

Thus obviously then, $\langle C\rangle = \left(\frac{N-1}{N}\right)^2 2\sqrt{2}$. If we ask for the smallest N s.t. $\langle C\rangle > 2$, we get $N \geq 7$.

*claim (8)

4 Conclusion

We have shown how to realize a Bell test in continuous variables position and momentum using specifically chosen and physically realizable states along with modular variables.

5 Appendix

Claim(1): If$|\psi\rangle \equiv \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$, then$|\psi\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$

Proof: Trivial.

Claim(2): $\langle x \otimes x\rangle = -1$, for $|\psi\rangle$ in Claim(1)

Proof: Trivial

Claim(3): $\left\langle e^{-iz\theta/2} x e^{iz\theta/2} \otimes e^{-iz\phi/2} x e^{iz\phi/2}\right\rangle = -\cos(\phi-\theta)$

Proof:

$$\begin{aligned}
\text{LHS} &= \left\langle x e^{-iz\theta} \otimes x e^{iz\phi}\right\rangle \\
&= \left[\frac{\langle 10| - \langle 01|}{\sqrt{2}}\right]\left[x e^{iz\theta} \otimes x e^{iz\phi}\right]\left[\frac{|10\rangle - |01\rangle}{\sqrt{2}}\right] \\
&= \langle\psi|\, x \otimes x \\
&\quad \frac{e^{i(\phi-\theta)}\left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right)\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right)}{\sqrt{2}} \\
&\quad -\frac{e^{-i(\phi-\theta)}\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right)\left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right)}{\sqrt{2}}
\end{aligned}$$

We define, $\delta \equiv \phi - \theta$ and using claim(1), it follows that only terms like $|+-\rangle$ or $|-+\rangle$; so

$$\begin{aligned}
\text{LHS} &= \langle\psi|\, x \otimes x \\
&\quad \left[\frac{e^{i\delta}\left(\frac{|+-\rangle - |-+\rangle}{2}\right) - e^{-i\delta}\left(\frac{-|+-\rangle + |-+\rangle}{2}\right)}{\sqrt{2}}\right] \\
&= \langle\psi|\, x \otimes x \left[\frac{e^{i\delta}\left(\frac{|\psi\rangle}{\sqrt{2}}\right) + e^{-i\delta}\left(\frac{|\psi\rangle}{\sqrt{2}}\right)}{\sqrt{2}}\right] \\
&= -\frac{e^{i\delta} + e^{-i\delta}}{2} \\
&= -\cos(\phi-\theta)
\end{aligned}$$

where we've used claim(2).

Claim(4): With definitions from section 3, without taking the large N limit,
$\langle + |X| + \rangle = \frac{N-1}{N}$, $\langle - |X| - \rangle = -\frac{N-1}{N}$
$\langle 0 |X| 0 \rangle = 0$, $\langle 1 |X| 1 \rangle = 0$
$\langle 1 |X| 0 \rangle = \frac{\frac{N-1}{N} + \frac{N}{N}}{2} = \frac{2N-1}{2N} = \langle 0 |X| 1 \rangle$
$\langle - |X| + \rangle = \frac{-\langle 1|X|0\rangle + \langle 0|X|1\rangle}{2} = 0 = \langle + |X| - \rangle$
$\langle \psi |X \otimes X| \psi \rangle = \frac{\langle +|X|+\rangle\langle -|X|-\rangle + \langle -|X|-\rangle\langle +|X|+\rangle}{2} = -\left(\frac{N-1}{N}\right)^2$

Proof: Trivial.

Claim(5): $\left\langle U^\dagger(\phi_i) X U(\phi_i) \otimes U^\dagger(\theta_i) X U(\theta_i)\right\rangle = -\left(\frac{N-1}{N}\right)^2 \cos(\phi_i - \theta_i)$

Proof: We start with defining $\phi \equiv \phi_i$, $\theta \equiv \theta_i$, $\delta \equiv \phi - \theta$, $\delta' \equiv \delta/2$. Next, we note that LHS $= \langle\psi'|X \otimes X|\psi'\rangle$ where $|\psi'\rangle = U(\phi_i) \otimes U(\theta_i)|\psi\rangle$.

$$\begin{aligned}
|\psi'\rangle &= \frac{e^{i\delta'}}{\sqrt{2}}\left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right)\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right) \\
&\quad -\frac{e^{-i\delta'}}{\sqrt{2}}\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right)\left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right) \\
&= \frac{e^{i\delta'}}{2\sqrt{2}}\left(|++\rangle + |+-\rangle - |-+\rangle - |--\rangle\right) \\
&\quad -\frac{e^{-i\delta'}}{2\sqrt{2}}\left(|++\rangle - |+-\rangle + |-+\rangle - |--\rangle\right) \\
&= \frac{e^{i\delta'} - e^{-i\delta'}}{2\sqrt{2}}|++\rangle + \frac{e^{i\delta'} + e^{-i\delta'}}{2\sqrt{2}}|+-\rangle \\
&\quad -\left(\frac{e^{i\delta'} + e^{-i\delta'}}{2\sqrt{2}}\right)|-+\rangle - \left(\frac{e^{i\delta'} - e^{-i\delta'}}{2\sqrt{2}}\right)|--\rangle
\end{aligned}$$

Now using claim(4), we have

$$\begin{aligned}
\text{LHS} &= \langle\psi'|X \otimes X|\psi'\rangle \\
&= \frac{1}{2}\left(\frac{N-1}{N}\right)^2\left|\frac{e^{i\delta'} - e^{-i\delta'}}{2}\right|^2 \\
&\quad -\frac{1}{2}\left(\frac{N-1}{N}\right)^2\left|\frac{e^{i\delta'} + e^{-i\delta'}}{2}\right|^2 \\
&\quad -\frac{1}{2}\left(\frac{N-1}{N}\right)^2\left|\frac{e^{i\delta'} + e^{-i\delta'}}{2}\right|^2 \\
&\quad +\frac{1}{2}\left(\frac{N-1}{N}\right)^2\left|\frac{e^{i\delta'} - e^{-i\delta'}}{2}\right|^2 \\
&= -\left(\frac{N-1}{N}\right)^2\frac{1}{2}\left[2\left(\cos^2\delta/2 - \sin^2\delta/2\right)\right] \\
&= -\left(\frac{N-1}{N}\right)^2\cos(\delta)
\end{aligned}$$

Claim(8): Action of x_{mod2L} can be defined explicitly

Proof: Consider the operator $e^{ix\frac{2\pi}{2L}}$. Note that $e^{ix\frac{2\pi}{2L}}|x'\rangle = e^{ix'\frac{2\pi}{2L}}|x'\rangle = e^{ix'_{mod2L}\frac{2\pi}{2L}}|x'\rangle$. Thus, $x_{mod2L}|x'\rangle = x'_{mod2L}|x'\rangle$, consequently on the most general state $|f\rangle \equiv \int dx' f_{x'}|x'\rangle$ then, we'd have $x_{mod2L}|f\rangle = \int dx' f_{x'} x'_{mod2L}|x'\rangle$.

Remark: One needn't necessarily consider eigenstates of x

to define the action. Eigenstates of $e^{ix\frac{2\pi}{2L}}$ maybe considered instead; they can be expressed as (a) $|\varphi\rangle$, $s.t.$ $\langle p + \frac{h}{2L} |\varphi\rangle = \langle p|\varphi\rangle$, $\forall\, p \in \mathbb{R}$ or (b) $|\bar{x}\rangle \propto \sum_{n\in\mathbb{Z}} |\bar{x} + n2L\rangle$. Using the second expression, we have $e^{ix\frac{2\pi}{2L}} |\bar{x}\rangle = e^{i\bar{x}\frac{2\pi}{2L}} |\bar{x}\rangle$. Thus on a more general state, $|c\rangle \equiv \int_0^{2L} d\bar{x} c_{\bar{x}} |\bar{x}\rangle$, we have $x_{mod2L} |c\rangle = \int_0^{2L} d\bar{x} c_{\bar{x}} \bar{x} |\bar{x}\rangle$.