Bell Test with Continuous Variables (Report)

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Abstract

We show the violation of the Bell inequality in a more classical setting than the usual quantum spin, viz. with continuous variables x, p. This is achieved using modular momentum and position. Assuming the state, experimental realization of the measurement scheme has also been delineated.

I. INTRODUCTION

The Bell inequality is a statement about the statistical outcomes of a certain class of experiments. The inequality was formulated by assuming the most general 'local theory'. Any violation of the inequality would imply that local realism must be false. Quantum Mechanics predicts a violation which has been confirmed experimentally. The physical situation for which this violation is usually formulated, uses entangled spin states. Here we show that a violation can be achieved without having to use spins, which is a purely quantum phenomenon; measuring a particle's position is sufficient.

II. VIOLATION IN BELL STATE

Here the violation of Bell's inequality for the typical spin case, is stated in a way that is easier to translate to the continuous variable case. We first state the Bell test in its modern form, known as the CHSH inequality.

Suppose: There are two observers[1], Alice and Bob. There are two particles, which are allowed to interact initially and are then separated. Alice takes one, Bob takes the other. Alice can measure one of two properties of her particle, call them a_1, a_2. Similarly Bob can measure b_1, b_2 of his particle.

Notatn: $\langle a_i b_j \rangle$ =Average value obtained when Alice chooses to measure a_i and Bob chooses to measure b_j

Statemnt: $|\langle a_1(b_1+b_2)+a_2(b_1-b_2)\rangle| \equiv |\langle C(a_i,b_i)\rangle| \leq 2$ if local realism is assumed. This result will be referred to as the CHSH inequality.

Quantum Mechanics predicts a violation of this in the following physical situation.

Defn: $|\psi\rangle \equiv \frac{|+-\rangle-|-+\rangle}{\sqrt{2}}$ where the $|\pm\rangle$ states are s.t. $\sigma_x|\pm\rangle = \pm|\pm\rangle$.

Consider: (a) Two particles are described by $|\psi\rangle$, (b) the first particle is with Alice, second with Bob, and (c)

	Alice		Bob	
Convention	a_1	a_2	b_1	b_2
Angle	$\theta_1=\pi/2$	$\theta_2=0$	$\phi_1=\pi/4$	$\phi_2=\pi/4+\pi/2$

Table I: Measurement Scheme

Alice measures z or x while Bob is allowed to choose between z' and x', where $z'=\frac{-z-x}{\sqrt{2}}$, $x'=\frac{z-x}{\sqrt{2}}$.

Notatn: The Pauli matrices are referred to as x, y, z.

Result: $\langle C((x,z),(x',z'))\rangle = 2\sqrt{2} \nleq 2$, which is a violation.

This value can be reproduced in a slightly modified situation.

Modifictn: (i) Due to the symmetry in the setup, we expect the same results hold for x, y and x', y' and (ii) we demand that Alice and Bob can only measure x but are allowed to perform a local unitary before the measurement.[2]

Explicitly then, consider: (a) The first qubit is with Alice and the second is with Bob. (b) Alice and Bob apply local unitaries $e^{iz\theta/2}\otimes I$ and $I\otimes e^{iz\phi/2}$ respectively and then both measure x. (c) Alice can choose $\theta\in\{0,\pi/2\}$ and Bob can choose $\phi\in\{\pi/4,\pi/4+\pi/2\}$; see I.[3]

Now it remains to show that this scheme also violates the Bell inequality (and by the same value). To that end, we state a result.

Claim(3): $\left\langle e^{-iz\theta/2}xe^{iz\theta/2}\otimes e^{-iz\phi/2}xe^{iz\phi/2}\right\rangle = -\cos(\phi-\theta)$

Given that, it is straight forward to evaluate

CHSH:

$$\begin{aligned}\langle C\rangle &= \langle a_1b_1\rangle+\langle a_1b_2\rangle+\langle a_2b_1\rangle-\langle a_2b_2\rangle\\ &= \left(-\frac{1}{\sqrt{2}}\right)+\left(-\frac{1}{\sqrt{2}}\right)+\left(-\frac{1}{\sqrt{2}}\right)-\left(+\frac{1}{\sqrt{2}}\right)\\ &= -2\sqrt{2}\end{aligned}$$

	Alice		Bob	
Convention	a_1	a_2	b_1	b_2
Angle	$\theta'_1 = 0$	$\theta'_2 = -\pi/2$	$\phi'_1 = 3\pi/8$	$\phi'_2 = 5\pi/4$

Table II: 'Corrected' Measurement Scheme

Remark: This is a violation since $|\langle C\rangle| = 2\sqrt{2} \nleq 2$.

However, to convert this to a more conventional form, we simply redefine the angles θ and ϕ.[4]

Substn: $\phi \to \phi' = \phi + \pi/2$, $\theta \to \theta' = \theta - \pi/2$; see II

Now we have $\langle C\rangle = 2\sqrt{2}$.

Concln: The violation of the Bell spin state can be equivalently be obtained by the measurement of x after applying a rotation along z by the aforesaid angles.

III. VIOLATION IN CONTINUOUS VARIABLE STATE

We start with constructing continuous variable kets that satisfy the same orthogonality and mutual relations as $|0\rangle, |1\rangle, |+\rangle, |-\rangle$.[5][6] We then construct the relavent operators and finally evaluate $\langle C\rangle$ to show the violation.

To construct the states, we start with some definitions.

Unassume: x, y, z refer to pauli matrices.

Defn: $L \equiv$some length scale, $N \equiv$number of slits

Consider: A state $|\varphi\rangle$ s.t. (a) $\langle x'|\varphi\rangle = \langle -x'|\varphi\rangle$ and (b) it's spread is $\ll L$

Defn: $S_N \equiv \{0, 1, 2, \ldots, (N-1)\}$, $T_N \equiv \{n + \frac{1}{2} \,|\, n \in S_N\}$ and $|\varphi_n\rangle \equiv \int dx' \langle x' - n2L|\varphi\rangle |x'\rangle$

Defn:

$$
\begin{aligned}
|0'\rangle &\equiv \frac{1}{\sqrt{N}} \sum_{t \in T_N} |\varphi_t\rangle \\
|1'\rangle &\equiv \frac{1}{\sqrt{N}} \sum_{s \in S_N} |\varphi_s\rangle
\end{aligned}
$$

Using these states, we construct the analogues of the $|+\rangle$ and $|-\rangle$ states.

Remark: Definition of $|0'\rangle$ and $|1'\rangle$ justify the word 'slit' in the definition of N.

Defn: $|+'\rangle \equiv \frac{|0'\rangle + |1'\rangle}{\sqrt{2}}$, $|-'\rangle \equiv \frac{|0'\rangle - |1'\rangle}{\sqrt{2}}$

NB: $\langle 0'|1'\rangle = 0, \implies \langle +'|-'\rangle = 0$

With these states, we're in a position to construct the relevant Hermitian operators.

Defn:

$$
X \equiv \frac{e^{ipL/\hbar} + e^{-ipL/\hbar}}{2}
$$

NB: For sufficiently large N,

$$
\begin{aligned}
X |+'\rangle &= +|+'\rangle \\
X |-'\rangle &= -|-'\rangle
\end{aligned}
$$

just as was with σ_x.

We have almost all the cards, except for one that allows for setting $\{\phi, \theta\}$. In the spin case, this was achieved by operating $e^{i\sigma_z\phi/2}$ on the state before measurement. Correspondingly, here we define this operator by it's action on the relevant state.

Defn: $U(\phi)$ is defined to be s.t.

$$
\begin{aligned}
U(\phi)|0'\rangle &= e^{i\phi/2}|0'\rangle \\
U(\phi)|1'\rangle &= e^{-i\phi/2}|1'\rangle
\end{aligned}
$$

This can achieved in principle using alternate glass slabs (capacitors) for photons (fermions) to adjust the relative phase as desired. Alternatively, we could also define $U(\phi)$ in its analytic form as

Alt Defn:

$$
U(\phi) = e^{iZ(x_{mod2L})\phi/2}
$$

where $Z(x') \equiv \begin{cases} 1 & 0 < x' \mod 2L \le L \\ -1 & L < x' \mod 2L \le 2L \end{cases}$ when Z acts on an eigenstate of x.[7]

NB: $Z(x_{mod2L})|0'\rangle = |0'\rangle$ and $Z(x_{mod2L})|1'\rangle = -|1'\rangle$ viz. $Z(x_{mod2L})$ plays the role of σ_z

To evaluate the CHSH inequality, we again need terms of the form $\left\langle U^\dagger(\phi_i)XU(\phi_i) \otimes U^\dagger(\theta_i)XU(\theta_i)\right\rangle$. If we assume $N \to \infty$, these can be evaluated easily to be the same as before $-\cos(\phi_i - \theta_i)$. Thus, the same analysis as before yields $\langle C\rangle = 2\sqrt{2}$ which is the same maximal violation, in a completely different setup. If N is not assumed to be arbitrarily large, then it can be shown that[8]

$$
\left\langle U^\dagger(\phi_i)XU(\phi_i) \otimes U^\dagger(\theta_i)XU(\theta_i)\right\rangle = -\left(\frac{N-1}{N}\right)^2 \cos(\phi_i - \theta_i)
$$

Thus obviously then, $\langle C\rangle = \left(\frac{N-1}{N}\right)^2 2\sqrt{2}$. If we ask for the smallest N s.t. $\langle C\rangle > 2$, we get $N \ge 7$.

A. Validating phase space behaviour

In the past, it has been shown that one can arrive at the violation of Bell inequalities in continuous variable systems. One issue concerning these is that the wigner function corresponding to the observable operators, are unbounded. An example of this is parity. Note that the Wigner function for a physical state ρ has an upper bound. Here however, we are simply using the prescription to evaluate it for unitary operators.

We care because wigner functions maybe thought of as a specific case of the Wigner-Weyl correspondence, which relates classical functions to their corresponding quantum operators. Thus, in a phase-space description of quantum mechanics, it is important to show that it is not necessary to have unboundedness to show a violation of the Bell's inequality, and thereby non-locality.

Our set of permissible states is manifestly bounded. We must show that the observable in our case, given by $U^\dagger(\phi)XU(\phi)$ is also bounded. To that end, we note that

$$\begin{aligned} W_1(q',p') &= \frac{1}{2\pi\hbar}\int dq e^{ip'q/\hbar}\left\langle q'-\frac{q}{2}\right| e^{-iZ(x_{mod2L})\phi/2} \\ & \quad e^{ip\frac{L}{\hbar}}e^{iZ(x_{mod2L})\phi/2}\left| q'+\frac{q}{2}\right\rangle \\ &= \frac{1}{2\pi\hbar}e^{-iZ_1\phi/2}e^{ip'\frac{L}{\hbar}}e^{iZ_2\phi/2} \end{aligned}$$

where $Z_{1/2} = Z[(q' \mp \frac{L}{2}) \mod 2L]$. Similarly one can evaluate W_2 and obtain $W = (W_1+W_2)/\sqrt{2}$ which corresponds to the observable. Clearly $|W| \leq (2\pi\hbar)^{-1}$.

B. Source of Violation

In the Bell test done with spins, the source of violation hinges on the anti-commutativity of the pauli matrices. It is known that $C^2 = 4\mathbb{I} + [a_1,a_2]\otimes[b_1,b_2]$. To obtain a violation, it is therefore necessary that the commutations don't vanish. This is indeed true in our case. Explicitly, I must show that $[U^\dagger XU, U'^\dagger XU'] \neq 0$.

Remark: The non-commutativity in this case is between U and X, which arises from $[x_{mod2L}, p_{modL}] \neq 0$.

TODO: Complete this

IV. CONCLUSION

We have shown how to realize a Bell test in continuous variables position and momentum using specifically chosen and physically realizable states along with modular variables.

V. APPENDIX

Remarks

Illustration(1):

This is trival if one uses the Bloch sphere picture. Instead of measuring along an arbitrary axis, we rotate the Bloch sphere appropriately, and then measure x. To illustrate this, consider

Questn: $\exists\, \mathrm{a}\, U$, *s.t.* if $|\chi\rangle \to |\chi'\rangle = U|\chi\rangle$ then $\langle\chi|x|\chi\rangle = \langle\chi'|y|\chi'\rangle$?

Explicitly, we have

$$\begin{aligned} y &= U^\dagger xU = e^{-iz\theta/2}xe^{iz\theta/2} \\ &= xe^{iz\theta} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \end{aligned}$$

for $\theta = \pi/2$ as one would guess geometrically.

Proofs

Claim(1): If$|\psi\rangle \equiv \frac{|+-\rangle-|-+\rangle}{\sqrt{2}}$, then$|\psi\rangle = \frac{|10\rangle-|01\rangle}{\sqrt{2}}$

Proof: Trivial.

Claim(2): $\langle x\otimes x\rangle = -1$, for $|\psi\rangle$ in Claim(1)

Proof: Trivial

Claim(3): $\left\langle e^{-iz\theta/2}xe^{iz\theta/2}\otimes e^{-iz\phi/2}xe^{iz\phi/2}\right\rangle = -\cos(\phi-\theta)$

Proof:

$$\begin{aligned} \text{LHS} &= \left\langle xe^{-iz\theta}\otimes xe^{iz\phi}\right\rangle \\ &= \left[\frac{\langle 10|-\langle 01|}{\sqrt{2}}\right]\left[xe^{iz\theta}\otimes xe^{iz\phi}\right]\left[\frac{|10\rangle-|01\rangle}{\sqrt{2}}\right] \\ &= \langle\psi|\, x\otimes x \\ & \left[\frac{e^{i(\phi-\theta)}\left(\frac{|+\rangle-|-\rangle}{\sqrt{2}}\right)\left(\frac{|+\rangle+|-\rangle}{\sqrt{2}}\right)}{\sqrt{2}}\right. \\ & \left. -\frac{e^{-i(\phi-\theta)}\left(\frac{|+\rangle+|-\rangle}{\sqrt{2}}\right)\left(\frac{|+\rangle-|-\rangle}{\sqrt{2}}\right)}{\sqrt{2}}\right] \end{aligned}$$

We define, $\delta \equiv \phi-\theta$ and using claim(1), it follows that only terms like $|+-\rangle$ or $|-+\rangle$; so

$$\begin{aligned} \text{LHS} &= \langle\psi|\, x\otimes x \\ & \left[\frac{e^{i\delta}\left(\frac{|+-\rangle-|-+\rangle}{2}\right)-e^{-i\delta}\left(\frac{-|+-\rangle+|-+\rangle}{2}\right)}{\sqrt{2}}\right] \\ &= \langle\psi|\, x\otimes x\left[\frac{e^{i\delta}\left(\frac{|\psi\rangle}{\sqrt{2}}\right)+e^{-i\delta}\left(\frac{|\psi\rangle}{\sqrt{2}}\right)}{\sqrt{2}}\right] \\ &= -\frac{e^{i\delta}+e^{-i\delta}}{2} \\ &= -\cos(\phi-\theta) \end{aligned}$$

where we've used claim(2).

Claim(4): With definitions from section 3, without taking the large N limit,
$\langle + | X | + \rangle = \frac{N-1}{N}$, $\langle - | X | - \rangle = -\frac{N-1}{N}$
$\langle 0 | X | 0 \rangle = 0$, $\langle 1 | X | 1 \rangle = 0$
$\langle 1 | X | 0 \rangle = \frac{\frac{N-1}{N} + \frac{N}{N}}{2} = \frac{2N-1}{2N} = \langle 0 | X | 1 \rangle$
$\langle - | X | + \rangle = \frac{-\langle 1|X|0\rangle + \langle 0|X|1\rangle}{2} = 0 = \langle + | X | - \rangle$
$\langle \psi | X \otimes X | \psi \rangle = \frac{\langle +|X|+\rangle\langle -|X|-\rangle + \langle -|X|-\rangle\langle +|X|+\rangle}{2} = -\left(\frac{N-1}{N}\right)^2$

Proof: Trivial.

Claim(5): $\left\langle U^\dagger(\phi_i) X U(\phi_i) \otimes U^\dagger(\theta_i) X U(\theta_i) \right\rangle = -\left(\frac{N-1}{N}\right)^2 \cos(\phi_i - \theta_i)$

Proof: We start with defining $\phi \equiv \phi_i$, $\theta \equiv \theta_i$, $\delta \equiv \phi - \theta$, $\delta' \equiv \delta/2$. Next, we note that LHS $= \langle \psi' | X \otimes X | \psi' \rangle$ where $|\psi'\rangle = U(\phi_i) \otimes U(\theta_i) |\psi\rangle$.

$$
\begin{aligned}
|\psi'\rangle &= \frac{e^{i\delta'}}{\sqrt{2}} \left(\frac{|+\rangle - |-\rangle}{\sqrt{2}} \right) \left(\frac{|+\rangle + |-\rangle}{\sqrt{2}} \right) \\
&\quad - \frac{e^{-i\delta'}}{\sqrt{2}} \left(\frac{|+\rangle + |-\rangle}{\sqrt{2}} \right) \left(\frac{|+\rangle - |-\rangle}{\sqrt{2}} \right) \\
&= \frac{e^{i\delta'}}{2\sqrt{2}} \left(|{+}{+}\rangle + |{+}{-}\rangle - |{-}{+}\rangle - |{-}{-}\rangle \right) \\
&\quad - \frac{e^{-i\delta'}}{2\sqrt{2}} \left(|{+}{+}\rangle - |{+}{-}\rangle + |{-}{+}\rangle - |{-}{-}\rangle \right) \\
&= \frac{e^{i\delta'} - e^{-i\delta'}}{2\sqrt{2}} |{+}{+}\rangle + \frac{e^{i\delta'} + e^{-i\delta'}}{2\sqrt{2}} |{+}{-}\rangle \\
&\quad - \left(\frac{e^{i\delta'} + e^{-i\delta'}}{2\sqrt{2}} \right) |{-}{+}\rangle - \left(\frac{e^{i\delta'} - e^{-i\delta'}}{2\sqrt{2}} \right) |{-}{-}\rangle
\end{aligned}
$$

Now using claim(4), we have

$$
\begin{aligned}
\text{LHS} &= \langle \psi' | X \otimes X | \psi' \rangle \\
&= \frac{1}{2} \left(\frac{N-1}{N} \right)^2 \left[\left| \frac{e^{i\delta'} - e^{-i\delta'}}{2} \right|^2 \right. \\
&\quad - \left| \frac{e^{i\delta'} + e^{-i\delta'}}{2} \right|^2 - \left| \frac{e^{i\delta'} + e^{-i\delta'}}{2} \right|^2 \\
&\quad \left. + \left| \frac{e^{i\delta'} - e^{-i\delta'}}{2} \right|^2 \right] \\
&= -\left(\frac{N-1}{N} \right)^2 \frac{1}{2} \left[2 \left(\cos^2 \delta/2 - \sin^2 \delta/2 \right) \right] \\
&= -\left(\frac{N-1}{N} \right)^2 \cos(\delta)
\end{aligned}
$$

Claim(8): Action of x_{mod2L} can be defined explicitly

Proof: $x_{mod2L} \equiv \int dx' x'_{mod2L} |x'\rangle \langle x'|$. To arrive at this more carefully, consider the operator $e^{ix\frac{2\pi}{2L}}$. Note that $e^{ix\frac{2\pi}{2L}} |x'\rangle = e^{ix'\frac{2\pi}{2L}} |x'\rangle = e^{ix'_{mod2L}\frac{2\pi}{2L}} |x'\rangle$. Thus, $x_{mod2L} |x'\rangle = x'_{mod2L} |x'\rangle$, consequently on the most general state $|f\rangle \equiv \int dx' f_{x'} |x'\rangle$ then, we'd have $x_{mod2L} |f\rangle = \int dx' f_{x'} x'_{mod2L} |x'\rangle$.

Remark: One needn't necessarily consider eigenstates of x to define the action. Eigenstates of $e^{ix\frac{2\pi}{2L}}$ maybe considered instead; they can be expressed as (a) $|\varphi\rangle$, $s.t.$ $\left\langle p + \frac{h}{2L} | \varphi \right\rangle = \langle p | \varphi \rangle$, $\forall\, p \in \mathbb{R}$ or (b) $|\bar{x}\rangle \propto \sum_{n \in \mathbb{Z}} |\bar{x} + n2L\rangle$. Using the second expression, we have $e^{ix\frac{2\pi}{2L}} |\bar{x}\rangle = e^{i\bar{x}\frac{2\pi}{2L}} |\bar{x}\rangle$. Thus on a more general state, $|c\rangle \equiv \int_0^{2L} d\bar{x} c_{\bar{x}} |\bar{x}\rangle$, we have $x_{mod2L} |c\rangle = \int_0^{2L} d\bar{x} c_{\bar{x}} \bar{x} |\bar{x}\rangle$.

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[1] that are often chosen to be sufficiently far away that their measurements don't influence each other
[2] Note that (ii) is equivalent to measuring along any direction. Details in the Apendix.
[3] These values are motivated by the $\pi/2$ between z and x, & $\pi/4$ between z and z' in the conventional Bell case.

[4] so that $\phi-\theta \to \phi'-\theta' = \phi-\theta+\pi = (\phi+\pi/2)-(\theta-\pi/2)$
[5] Defn: $|0\rangle$, $|1\rangle$ are s.t. $z\,|0\rangle = |0\rangle$ and $z\,|1\rangle = -\,|1\rangle$
[6] We emphasise that the kets we construct, will not be a countable superposition of eigenstates of p or x since these are highly idealized and strictly, not even a part of the Hilbert space.
[7] x_{mod2L} is well defined, see claim 8
[8] Claim(5)