Towards macroscopic Bell test with modular variables

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We show the violation of the Bell inequality in a more classical setting than the usual quantum spin, viz. with continuous variables x, p. This is achieved using displacement operators, or equivalently modular momentum and position. Two possible experimental realizations of the scheme have also been delineated, one uses photons, while the other is based on electrons.

I. INTRODUCTION

In 1935, Einstein, Podolsky, and Rosen argued in a thought experiment that the quantum-mechanical description of physical reality is not complete, and thus may be superseded by a more complete realistic theory which reproduces the quantum mechanical predictions, and at the same time, obeys the locality condition [1]. In his groundbreaking paper in 1964 [**?**] Bell derived an experimentally testable inequality, which bounds the correlations between bipartite measurements for any such local hidden-variable theory, but which is violated by quantum mechanics. This was a major breakthrough towards empirical tests of quantum mechanics as apposed to common sense theories. Since then, the results constraining the permissible types of hidden variable models of quantum mechanics have attracted much attention and have been reformulated as the problem of contextual measurements by Kochen and Specker [**?**] and in terms of temporal correlations by Leggett and Garg [**?**]. Today, these concepts have mainly been formulated for intrinsic quantum degree of freedom of microscopic particles such as spins ... and tested in various experiments with photons [**?**], ions [**?**], impurity spins [**? ?**] or superconducting qubits [**?**]. All experimental observations confirmed the validity of quantum nonlocality on this level.

It is, however, more tempting to formulate analogous tests for truly spatial degree of freedom and test the nonlocality in phase space as a natural extension to macroscopic local realism tests [ ]. That's why, the original EPR argument uses phase space descriptions to better address the locality problem of quantum mechanics in such manifold. Moreover, phase space is a natural concept in classical theory and it is equivalent to the state space. The outstanding challenge is to come up with local realism test which verify or disprove quantum mechanics predictions also with most "classical-like" measurements and operations in phase space.

The paper is organized as follows. In Sec. we introduce our framework for Bell nonlocality test aiming to use most classical-like variables and measurements, and thus pave the way towards macroscopic test of local realism. For our purpose we use continuous modular variables, characterized by continuum spectral eigenvalues. These variables can be clearly demonstrated in Multi-slit interfrometric setup. In Sec. We construct the CV Bell operator of CHSH form from modular variables. In Sec. we proceed with identifying the relevant entangled state leading to the violation of the Bell inequality. Finally, in Sec. we propose a possible implementation of the desired entangled states

II. FRAMEWORK FOR MACROSCOPIC BELL TEST

The central problem here is to construct a suitable Bell-operator expressed in terms of local continuous variable operators $\hat{A}$. A general form of CHSH expression of such a Bell-operator would be generally written as

$$\hat{\mathcal{B}} = \hat{A}_1 \otimes (\hat{A}_2 + \hat{A}_2') + \hat{A}_1' \otimes (\hat{A}_2 - \hat{A}_2') \tag{1}$$

In order to apply the CHSH criterion we need to use variables which take on a limited range of values to be able to determine a well-defined classical bound. We therefore aim to construct a class of bounded observables of continuous variable (CV) systems which have the following properties. Using proper rescaling for corresponding quantum eigenvalues of the observable we have

$$|a_j| \leq 1 \tag{2}$$

One obvious example is parity operator $\Pi = (-1)^{a^\dagger a}$. This is enough to construct a valid Bell operator with a well-defined classical bound. We however demand of an extra constraint required for probing Nonlocality in phase space. We impose that the observable $\hat{A}$ corresponds to a bounded c-number function in phase space using Wigner-Weyl correspondence. That is,

$$\mathcal{W}_{\hat{A}}(q,p) = \frac{1}{2\pi\hbar}\int dq' e^{ipq'} \langle q - \frac{q'}{2} |\, \hat{A}\, | q + \frac{q'}{2} \rangle \leq 1 \tag{3}$$

Accordingly, the inequality should hold for any state described by valid (positive) probability distribution.

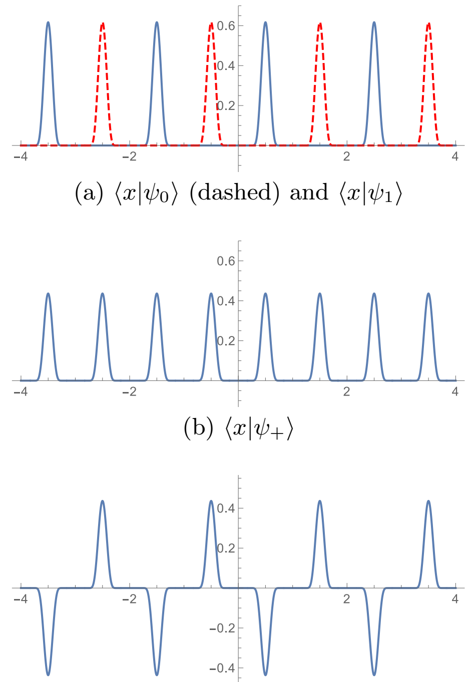
$$\begin{aligned}\mathcal{W}_{\hat{\mathcal{B}}}(q,p) =& \mathcal{W}_{A_1}(q,p)[\mathcal{W}_{A_2}(q,p) + \mathcal{W}_{A_2'}(q,p)] \\ &+ \mathcal{W}_{A_1'}(q,p)[\mathcal{W}_{A_2}(q,p) - \mathcal{W}_{A_2'}(q,p)] \leq 2\end{aligned} \tag{4}$$

Therefore,

$$\langle \hat{\mathcal{B}} \rangle = \int \mathcal{W}_{\hat{\rho}}(q,p)\mathcal{W}_{\hat{\mathcal{B}}}(q,p) dq dp \leq 2 \tag{5}$$

We that we ascertain that any state described by positive definite Wigner function, that is compatible with local hidden phase space variable, satisfy above inequality. leading to the violation of this version of Bell inequality must be characterized by negative Wigner function deviating from proper probability distribution in hidden phase space coordinates. That's why, John Bell remarked that as EPR state described by positive Wigner function is compatible with a local hidden variable model, and thus it does not demonstrate nonlocality. This claim can be justified in the sense of above.

It is well recognized and it can be shown that .. that there is a tight relation between nonlocality and noncommutativity of operators. The violation occurs for choices of settings whose corresponding observable do not commute. It would be interesting the framework for the Bell test clearly demonstrates that the source of the violation directly originates from the noncommutativity of the phase space quadratures in quantum mechanics, $[q, p] = i$. For macroscopic test the measurement scheme used for evaluating the correlations should have a classical analog for better understanding the result in classical limit. The parity operator measurements are defined in terms of resolving the consecutive eigen values with no classical analog. Therefore we do not consider it as "classical-like" measurement.



(a) $\langle x|\psi_0\rangle$ (dashed) and $\langle x|\psi_1\rangle$

(b) $\langle x|\psi_+\rangle$

(c) $\langle x|\psi_-\rangle$

Figure 1: Illustration of $|\psi_\pm\rangle$ and $|\psi_0\rangle$, $|\psi_1\rangle$ states for $N = 8$, $L = 1$ unit. The $|\psi_0\rangle$, $|\psi_1\rangle$ states are not normalized.

A. Phase space translation operators and modular variables

Particular class of bounded observable can be constructed from the quantum mechanical phase space translation operator

This observable is not Hermitian therefore it does not directly correspond to an observable.

The observables admissible in the CHSH inequality must $\in [-1, 1]$. If we consider the *real variable* $p \in (-\infty, \infty)$, then one way of obtaining a bound variable is to use $p_{\text{mod}} \equiv p \mod h/L$, where h/L has been introduced for dimensional reasons. This forces $p_{\text{mod}} \in [0, h/L)$. If we further consider, say $\cos(p_{\text{mod}} L/\hbar)$, then we get precisely the bound needed for the CHSH inequality. We chose the constant factors so that for $p_{\text{mod}} = h/L$, the cos function also completes a period. Note that $\cos(p_{\text{mod}} L/\hbar) = \cos(pL/\hbar) = \frac{e^{ipL/\hbar} + e^{-ipL/\hbar}}{2}$.

Next, recall that in quantum mechanics, the momentum that's quantized, is one that generates infinitesimal translations. Formally, this means that $(1 - i\hat{p}\delta L/\hbar)\,|x\rangle = |x + \delta L\rangle$. If we want a finite translation, we can do the following procedure: $\lim_{N\to\infty} \left(1 - i\frac{\hat{p}}{\hbar}\frac{L}{N}\right)^N |x\rangle = e^{-i\hat{p}L/\hbar}\,|x\rangle = |x + L\rangle$. This operator is aptly called the displacement operator. We must add that $e^{i\hat{p}a}e^{i\hat{x}b} = e^{i\hbar ab}e^{i\hat{x}b}e^{i\hat{p}a}$. Thus for appropriate choices of a, b, the displacement operators can be made to commute or anti-commute. In the former case, it means one can simultaneously measure modular position and momentum (which is striking) and in the latter case, one can define pauli matrix like commutation.

The idea was that the obvious generalization for obtaining an appropriate operator for the CHSH inequality is $\cos(\hat{p}L/\hbar) = \frac{e^{i\hat{p}L/\hbar} + e^{-i\hat{p}L/\hbar}}{2}$. While by construction this is bounded, the action of displacement operators will guide us in creating the appropriate states in the next sub section.

B. Violation in Continuous Variable State

We start with constructing continuous variable kets[2] that satisfy the same orthogonality and mutual relations as $|0\rangle\,, |1\rangle\,, |+\rangle\,, |-\rangle$.[3] We then construct the relevant operators and finally evaluate $\langle C\rangle$ to show the violation.

To construct the states, we start with some definitions.

Forgt: $\hat{x}$, $\hat{y}$, $\hat{z}$ represent Pauli matrices

Defn: L ≡some length scale, N ≡number of slits

Consider: A state $|\varphi\rangle$ symmetric about $x = L/2$, with spread $\ll L$. Formally, $|\varphi\rangle \equiv \int dx\, \langle x - L/2|\varphi'\rangle\,|x\rangle$ where $|\varphi'\rangle$ s.t. (a) $\langle x|\varphi'\rangle = \langle -x|\varphi'\rangle$ and (b) it's spread is $\ll L$

Defn: $|\varphi_n\rangle \equiv \int dx\, \langle x - nL|\varphi\rangle\,|x\rangle$

Defn: $I_N = \{-\frac{N}{2}, \dots, \frac{N}{2}-1, \frac{N}{2}-1\}$, defined for even N only (for simplicity)

Defn:

$$|\psi_0\rangle \equiv \frac{1}{\sqrt{N'}} \sum_{n\in I_{N'}} |\varphi_{2n+1}\rangle$$

$$|\psi_1\rangle \equiv \frac{1}{\sqrt{N'}} \sum_{n\in I_{N'}} |\varphi_{2n}\rangle$$

where $N' \equiv N/2$

Using these states, as illustrated in figure 1, we construct the analogues of the $|\psi_+\rangle$ and $|\psi_-\rangle$ states.

Remark: Definition of $|\psi_\pm\rangle$ justify the word 'slit' in the definition of N.

Defn: $|\psi_+\rangle \equiv \frac{|\psi_0\rangle+|\psi_1\rangle}{\sqrt{2}}$, $|\psi_-\rangle \equiv \frac{|\psi_0\rangle-|\psi_1\rangle}{\sqrt{2}}$

NB: $\langle\psi_0|\psi_1\rangle = 0, \implies \langle\psi_+|\psi_-\rangle = 0$

Analogous to the spin case, we construct the entangled state

Defn: $|\Psi\rangle \equiv \frac{|\psi_+\psi_-\rangle - |\psi_-\psi_+\rangle}{\sqrt{2}}$

These states were constructed with a partial translational symmetry which is appropriate to the hermitian operator, $\cos(\hat{p}L/\hbar)$ discussed earlier.

Defn:

$$\hat{X} \equiv \frac{e^{i\hat{p}L/\hbar} + e^{-i\hat{p}L/\hbar}}{2}$$

NB: Only in the limit $N \to \infty$,

$$\hat{X}|\psi_+\rangle = +|\psi_+\rangle$$
$$\hat{X}|\psi_-\rangle = -|\psi_-\rangle$$

just as was with $\hat{\sigma}_x$. In general however, the relation doesn't hold.

We have almost all the cards, except for one that allows for setting $\{\phi, \theta\}$. In the spin case, this was achieved by operating $e^{i\hat{\sigma}_z\phi/2}$ on the state before measurement. Correspondingly, here we define this operator by it's action on the relevant state.

Defn: $\hat{U}(\phi)$ is defined to be s.t.

$$\hat{U}(\phi)|\psi_0\rangle = e^{i\phi/2}|\psi_0\rangle$$
$$\hat{U}(\phi)|\psi_1\rangle = e^{-i\phi/2}|\psi_1\rangle$$

This can achieved in principle using alternate glass slabs for photons, to adjust the relative phase as desired, since $|\psi_0\rangle$ and $|\psi_1\rangle$ are spatially separated. Alternatively, we could also define $\hat{U}(\phi)$ in its analytic form. First we claim that $\exists$ an operator $\hat{Z}$ s.t. it satisfies $\hat{Z}|\psi_0\rangle = |\psi_0\rangle$ and

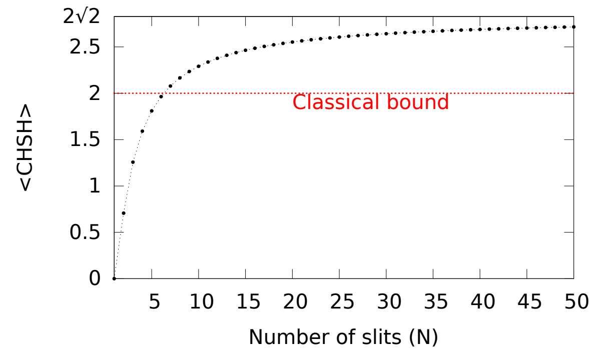


Figure 2: Practically, the number of slits, N will be finite. The plot shows $\langle C\rangle$ as a function of N. To get a violation, we need merely 7 slits, while with 50 we are almost at $2\sqrt{2}$.

$\hat{Z}|\psi_1\rangle = -|\psi_1\rangle$. Before explicitly defining $\hat{Z}$, we observe that given this operator, we can naturally define the analytic form of $\hat{U}(\phi)$.

Alt Defn:

$$\hat{U}(\phi) = e^{i\hat{Z}\phi/2}$$

in close analogy with the spin case. It is suggestive from this form that $\hat{Z}$ must play the role of $\hat{\sigma}_z$. We know that $\hat{X}$ is a function of $\hat{p}_{\text{mod}}$. It is thus natural to expect $\hat{Z}$ to be a function of $\hat{x}_{\text{mod}}$. From it's action, we also note that $\hat{Z}$ must differentiate between spatial wavefunctions $\langle x|\varphi\rangle$ and $\langle x-L|\varphi\rangle$. Thus the appropriate operator must be $\hat{x}_{\text{mod}2L}$ and not $\hat{x}_{\text{mod}L}$. This is supported by the fact that $[\hat{p}_{\text{mod}}, \hat{x}_{\text{mod}L}] = 0$. Now since the action of $\hat{Z}$ must work for arbitrary $|\varphi\rangle$ and N, we define

$$\hat{Z} \equiv Z(\hat{x}_{\text{mod}2L})$$

where Z is some appropriate function, so that $\hat{Z}|x\rangle = Z(\hat{x}_{\text{mod}2L})|x\rangle = Z(x_{\text{mod}2L})|x\rangle$. From the constraints on $\hat{Z}$, we conclude that for consistency, we must define[4]

$$Z(x) \equiv \begin{cases} 1 & 0 < x \leq L \\ -1 & L < x \leq 2L \end{cases}$$

To evaluate the CHSH inequality, we again need terms of the form $\left\langle \hat{U}^\dagger(\phi_i)\hat{X}\hat{U}(\phi_i) \otimes \hat{U}^\dagger(\theta_i)\hat{X}\hat{U}(\theta_i)\right\rangle$. If we assume $N \to \infty$, these can be evaluated easily to be the same as before $-\cos(\phi_i - \theta_i)$. Thus, the same analysis as before yields $\langle C\rangle = 2\sqrt{2}$ which is the same maximal violation, in a completely different setup. If N is not assumed to be arbitrarily large, then it can be shown that (see section V 5)

$$\left\langle \hat{U}^\dagger(\phi_i)\hat{X}\hat{U}(\phi_i) \otimes \hat{U}^\dagger(\theta_i)\hat{X}\hat{U}(\theta_i)\right\rangle = -\left(\frac{N-1}{N}\right)^2 \cos(\phi_i - \theta_i)$$

Thus obviously then, $\langle C\rangle = \left(\frac{N-1}{N}\right)^2 2\sqrt{2}$. If we ask for the smallest N s.t. $\langle C\rangle > 2$, we get $N \geq 7$; see figure 2.

C. Validating phase space behaviour

In the past, it has been shown that one can arrive at the violation of the Bell inequality in continuous variable systems. One issue concerning some of these approaches is that the wavefunction used is highly idealized and not normalizable, such as $|p\rangle$ and $|q\rangle$ or their countable superposition. Another issue in certain other approaches, is that the wigner functions corresponding to the observables, are unbounded. An example of this is parity. Note that while the Wigner function for a physical state ρ has an upper bound, here we are simply using the prescription to evaluate the same for hermitian operators.

We care because wigner functions maybe thought of as a specific case of the Wigner-Weyl correspondence, which is a one-one map between classical functions and their corresponding quantum operators. Thus, in a phase-space description of quantum mechanics, it is important to show that it is not necessary to have unboundedness in observables to arrive a violation of the Bell's inequality, and thereby non-locality.

In our case, $|\Psi\rangle$ is manifestly normalized. We must show that the observable in our case, given by $\hat{U}^\dagger(\phi)\hat{X}\hat{U}(\phi)$ is also bounded. To that end, we note that

$$\begin{aligned} W_1(q,p) &= \frac{1}{2\pi\hbar}\int dq' e^{ipq'/\hbar}\left\langle q-\frac{q'}{2}\right| e^{-iZ(\hat{x}_{mod2L})\phi/2} \\ & \quad e^{i\hat{p}\frac{L}{\hbar}} e^{iZ(\hat{x}_{mod2L})\phi/2}\left| q+\frac{q'}{2}\right\rangle \\ &= \frac{1}{2\pi\hbar} e^{-iZ_1(q)\phi/2} e^{ip\frac{L}{\hbar}} e^{iZ_2(q)\phi/2} \\ &= \frac{1}{2\pi\hbar} e^{ipL/\hbar} e^{iZ_2(q)\phi} \end{aligned}$$

where $Z_{1/2}(q) = Z[(q \mp \frac{L}{2})\mathrm{mod}2L]$ and we used the fact that $Z(x) = -Z(x+L)$ (omitting the mod2L). Similarly one can evaluate W_2 and obtain

$$W = (W_1 + W_2)/2 = \frac{1}{2\pi\hbar}\cos(pL/\hbar)e^{\pm i\phi}$$

which corresponds to the observable (we used that $Z_2 = \pm 1$). Clearly $|W| \leq (2\pi\hbar)^{-1}$.

D. Source of Violation

In the Bell test done with spins, the source of violation hinges on the non-commutativity of the Pauli matrices. It is known that $C^2 = 4\mathbb{I} + [a_1, a_2] \otimes [b_1, b_2]$, when $a_i, b_i \in \{-1, 1\}$. To obtain a violation in general (where the first term maybe less than $4\mathbb{I}$), it is therefore necessary that the commutations don't vanish. This is indeed true in our case. Explicitly, we must have $[\hat{U}^\dagger\hat{X}\hat{U}, \hat{U}'^\dagger\hat{X}\hat{U}'] \neq 0$. To arrive at the desired expression we first evaluate some simpler commutation and anti-commutation relations explicitly. Let's start with $[\hat{Z}, e^{i\hat{p}L/\hbar}] = \hat{Z}e^{i\hat{p}L/\hbar} - e^{i\hat{p}L/\hbar}\hat{Z}$. To evaluate it, we multiply the second term with $\int dx\,|x\rangle\langle x|$ and obtain $Ze^{i\hat{p}L/\hbar} + Ze^{i\hat{p}L/\hbar}$ where we've used $Z(\hat{x}_{\mathrm{mod}2L}) = -Z((\hat{x} \pm L)_{\mathrm{mod}2L})$ and $e^{i\hat{p}L/\hbar}|x\rangle = |x - L\rangle$. Thus we have

$$[\hat{Z}, \hat{X}] = 2\hat{Z}\hat{X} = -2i\hat{Y}$$

where $\hat{Y} = i\hat{Z}\hat{X}$. Here i was introduced to ensure $\hat{Y}^\dagger = \hat{Y}$, since $\hat{X}^\dagger = \hat{X}$ and $\hat{Z}^\dagger = \hat{Z}$. Similarly it follows that $\{\hat{Z}, \hat{X}\} = 0$. From the definition of Y and the anti-commutation, $\{\hat{Y}, \hat{X}\} = 0$ and $\{\hat{Y}, \hat{Z}\} = 0$ also follow trivially. While we're at it, we may state that while $\hat{Z}^2 = 1$, it is not a sum of a 2 state projector and $\hat{X}^2 \neq 1$ in general. This manifests in the other commutation relations

$$\begin{aligned} [\hat{X}, \hat{Y}] &= -2i\hat{Z}\hat{X}^2 = -2i\hat{X}^2\hat{Z} \\ [\hat{Y}, \hat{Z}] &= -2i\hat{X} \end{aligned}$$

Further, note that $e^{-i\hat{Z}\phi/2}\hat{X}e^{i\hat{Z}\phi/2} = \hat{X}e^{i\hat{Z}\phi}$ and that $e^{i\hat{Z}\phi} = \cos\phi + i\hat{Z}\sin\phi$. With all these, it is easy to show that

$$\begin{aligned} [\hat{U}^\dagger(\frac{\phi}{2})\hat{X}\hat{U}(\frac{\phi}{2}), \hat{U}^\dagger(\frac{\phi'}{2})\hat{X}\hat{U}(\frac{\phi'}{2})] &= \\ (\cos\phi\sin\phi' - \sin\phi\cos\phi')\,\hat{Z}\hat{X}^2 &\neq 0 \end{aligned}$$

as was necessary. Ofcourse this is not sufficient, however we have shown explicitly the violation. This evaluation was meant to illuminate that the source of violation essentially hinges on $[\hat{x}_{\mathrm{mod}2L}, \hat{p}_{\mathrm{mod}}] \neq 0$. It is apparent that the complete pauli algebra is unnecessary for showing a violation.

E. Advantage of asymmetry in $\hat{X}$ and $\hat{Z}$

Considering the operator (non-hermitian for simplicity) $\hat{X} = e^{i\hat{p}L/\hbar}$, defining $\hat{Z} = e^{i\hat{x}2\pi/2L}$ is more natural. They also follow the desired anti-commutation $\{\hat{X}, \hat{Z}\} = 0$ and we could define $\hat{Y} = i\hat{Z}\hat{X}$ to get a more natural generalization. The question is why did $\hat{Z} = Z(\hat{x}_{\mathrm{mod}2L})$ appear in the analysis. The cause of this asymmetry hinges on the preferential treatment of position space. We could have constructed states of the form $|\psi_0\rangle = \sum_n |x + nd\rangle$ and used the natural definition of $\hat{Z}$ to obtain the violation. The issue is that this forces us to choose a countable superposition of position eigenkets as our desired state. Such a state is strictly not even in the Hilbert space. In the scheme proposed, this issue is resolved by $Z(\hat{x}_{\mathrm{mod}2L})$ which causes a larger set of physically realizable states to become admissible.

Thus while our construction is motivated by the spin formalism, it is not just a relabelling of states and operators achieved by a severe restriction of the Hilbert space.

III. PHYSICAL IMPLEMENTATION

We show that this scheme can be implemented rather easily using photons. We harness the two degrees of freedom of a photon, it's polarization and it's position to construct the required state. With a slightly modified setup, it is possible to do the same with spin and position for matter waves (see section V 7). The overall setup (see figure 3) is such that we need only consider the quantum mechanical description along the x-axis.

A. Creation of the entangled state

The desired entangled state is $|\Psi\rangle \equiv \frac{|\psi_+\psi_-\rangle - |\psi_-\psi_+\rangle}{\sqrt{2}}$. We start with noting the triviality of constructing a

$$|\psi_+\rangle = \frac{1}{\sqrt{N}} \sum_{n\in I_N} |\varphi_n\rangle$$

state. One simply needs a source and a grating, viz. a screen with N slits of width $a \ll L$, separated by a distance L (centre to centre). The

$$|\psi_-\rangle = \frac{1}{\sqrt{N}} \sum_{n\in I_N} (-1)^{n+1} |\varphi_n\rangle$$

state can be similarly constructed by using glass slabs at alternate slits, such that the phase introduced is $\phi = \pi \implies e^{i\phi} = -1$. In figure 3, if you consider only one particle, and disregard everything after the grating, then the setup is expected to produce a $|\psi_+\rangle$ state, just after the grating. To produce the desired entangled state, we start with two entangled photons, such that their polarization state can be expressed as $|\chi\rangle = \frac{|HV\rangle - |VH\rangle}{\sqrt{2}}$. Their spatial description (along x-axis) is initially assumed to be $|\zeta\rangle = |\gamma\rangle \otimes |\gamma\rangle = |\gamma\gamma\rangle$. Thus the initial state is

$$|\chi\rangle |\zeta\rangle = \frac{|HV\rangle - |VH\rangle}{\sqrt{2}} |\gamma\gamma\rangle$$

The state $|\gamma\rangle$ maybe considered to be a Gaussian with $\sigma \gg 2NL$ so that after the grating, the spatial state can be written as

$$|\chi\rangle |\zeta_{\text{grating}}\rangle = \frac{|HV\rangle - |VH\rangle}{\sqrt{2}} |\psi_+\psi_+\rangle$$

Now comes the interesting part. To produce a $|\psi_-\rangle$ state, we already know we needed alternate glass slabs. If we had glass slabs, whose refractive index (given some orientation) was say $\eta_H = 1$ for a horizontally polarized beam and $\eta_V = \eta \neq 1$ for vertical polarization, then we could harness the entangled polarization state to create the required spatially entangled state. Birefringent crystals have such polarization dependent refractive indices. Assume that alternating birefringent crystals have been placed after both the gratings as shown in figure 3 (in the diagram, they're not alternating though as will be explained shortly). These have width such that the resultant phase it introduces is π and the state is

$$|\chi\rangle |\zeta_{\text{birefringent}}\rangle = \frac{|HV\rangle |\psi_+\psi_-\rangle - |VH\rangle |\psi_-\psi_+\rangle}{\sqrt{2}}$$

At this stage, if we were to trace out the polarization state, we'd end up with a mixed state. That is useless for our test. Ignoring the glass slabs at the moment in figure 3, we observe the action of the polarisers on the state. After the 45° polariser, the state becomes (see section V 8)

$$|\chi_{45}\rangle |\zeta_{\text{birefringent}}\rangle \propto |H'H'\rangle \left(\frac{|\psi_+\psi_-\rangle - |\psi_-\psi_+\rangle}{\sqrt{2}} \right)$$

where $|H'\rangle \equiv (|H\rangle + |V\rangle)/\sqrt{2}$. Now if we trace out the polarization state, we'd be left with the target entangled state. As a remark, it maybe be stated that although to arrive at this result we assumed that $\eta_H = 1$, which is unreasonable physically, we can compensate for $\eta_H \neq 1$ by putting appropriate glass slabs at the alternate empty slits, to produce zero relative phase when the polarization is horizontal.

B. Measurement Settings

In our scheme, Alice and Bob can choose $\phi \in \{\phi_1, \phi_2\}$ and $\theta \in \{\theta_1, \theta_2\}$ respectively. To implement this, we note that the action of $\hat{U}(\theta)$ is to introduce a phase difference between the $|\psi_0\rangle$ and $|\psi_1\rangle$ states. By construction, $|\psi_0\rangle$ and $|\psi_1\rangle$ are spatially disjoint; thus the operation of $\hat{U}(\theta)$ can be implemented by simply placing alternating glass slabs at the slits, with widths such that $|\psi_0\rangle \to e^{i\theta/2} |\psi_0\rangle$ and $|\psi_1\rangle \to e^{-i\theta/2} |\psi_1\rangle$. The same operation maybe done for $\hat{U}(\phi)$ for the second particle. In the overall scheme, as shown in figure 3, this is done by the glass slabs right after the birefringent crystals. The state after application of the unitaries is given by $|\Psi_f\rangle$, where f is used to index which of the four possible measurement settings was used.

C. Measurement

The scheme requires us to evaluate the expectation value of $\hat{X} \otimes \hat{X}$ (after the measurement settings have been applied by the appropriate unitaries). If from the experiment, we can obtain the probability $|\langle p_A, p_B | \Psi_f \rangle|^2$, then evaluating $\left\langle \hat{X} \otimes \hat{X} \right\rangle = \langle \cos(\hat{p}L/\hbar) \otimes \cos(\hat{p}L/\hbar) \rangle$ simply amounts to $\int dp_A dp_B \cos(p_A L/\hbar) \cos(p_B L/\hbar) |\langle p_A, p_B | \Psi_f \rangle|^2$.

Therefore it is sufficient to explain how to obtain the joint momentum probability distribution from the experiment, for a given measurement setting. We start with

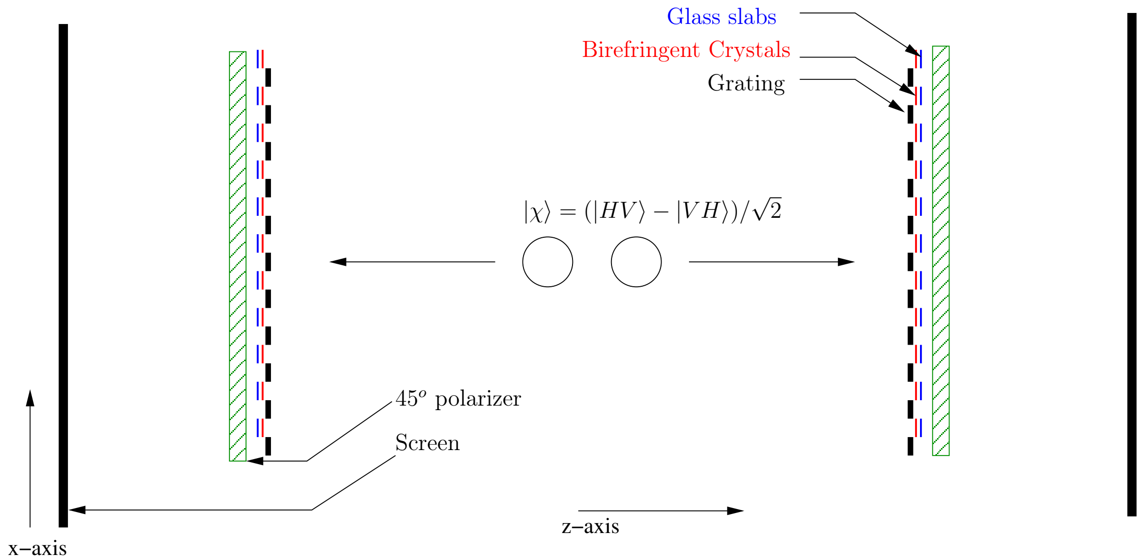


Figure 3: Scheme for creating the state

stating the result that in the far-field approximation,

$$|\langle x_A, x_B|\Psi_{f\,\mathrm{screen}}\rangle|^2 \propto \left|\left\langle p_A = \frac{p_z x_A}{D}, p_B = \frac{p_z x_B}{D}|\Psi_f\right\rangle\right|^2$$

where $|\Psi_{f\,\mathrm{screen}}\rangle$ is the state of the system at the screen, D is the distance between the gratings and the screens and p_z is the z component of momentum of the particle. For a photon, $p_z = h/\lambda$ while for an electron, $p_z = m_e D/T$, where T is the time taken to arrive at the screen from the grating. The idea is simply that the momentum distribution at the grating can be recovered by observing the spatial distribution at a screen, sufficiently far away.

In an experiment then, there are two possibilities.
(a) Given f, Alice and Bob both note the position at which they obtain their particle. After repeating the experiment sufficiently many times, they share their list to create a sequence $\{\left(x_A^{(i)}, x_B^{(i)}\right)\}$. From this sequence, they create a 2D histogram, by simply counting how many time they got (x'_A, x'_B) to lie inside a given cell. The normalized result is essentially $|\langle x_A, x_B|\Psi_{f\,\mathrm{screen}}\rangle|^2$ from which $\left\langle \hat{X}\otimes\hat{X}\right\rangle$ can be computed as discussed. f is changed to evaluate all the 4 terms to finally obtain $\langle C\rangle$ experimentally.
(b) The other simpler, equivalent and direct possibility is that Alice and Bob both obtain the position, $\{x_A^{(i)}, x_B^{(i)}\}$ for the ith repetition and evaluate $\cos(p_A^{(i)}L/\hbar)$ and $\cos(p_B^{(i)}L/\hbar)$. After sufficient iterations (for all values of f), they exchange their observations to simply evaluate the average, viz.

$$\frac{\sum_i \cos(p_A^{(i)}L/\hbar)\cos(p_B^{(i)}L/\hbar)}{\sum_i} = \langle C\rangle$$

IV. CONCLUSION

We have shown how to realize a Bell test in continuous variables position and momentum using specifically chosen and physically realizable states. In addition it has been demonstrated that the exact pauli commutation relations aren't necessary to capture the essence of non-locality; infact by relaxing the similarity, the set of states that show a violation is made considerably larger.

V. APPENDIX

Illustration (I)

This is trival if one uses the Bloch sphere picture. Instead of measuring along an arbitrary axis, we rotate the Bloch sphere appropriately, and then measure x. To illustrate this, consider

Questn: $\exists\,\mathrm{a}\,\hat{U}$, $s.t.$ if $|\chi\rangle \to |\chi'\rangle = \hat{U}\,|\chi\rangle$ then $\langle\chi\,|\hat{x}|\,\chi\rangle = \langle\chi'\,|\hat{y}|\,\chi'\rangle$?
Explicitly, we have

$$\begin{aligned}\hat{y} &= \hat{U}^\dagger\hat{x}\hat{U} = e^{-i\hat{z}\theta/2}\hat{x}e^{i\hat{z}\theta/2}\\ &= \hat{x}e^{i\hat{z}\theta} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{bmatrix}\\ &= \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\end{aligned}$$

for $\theta = \pi/2$ as one would guess geometrically.

Claims

1. If $|\psi\rangle \equiv \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$*, then* $|\psi\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$

2. $\langle x \otimes x\rangle = -1$*, for* $|\psi\rangle$ *in Claim(1)*

3. $\left\langle e^{-iz\theta/2} x e^{iz\theta/2} \otimes e^{-iz\phi/2} x e^{iz\phi/2}\right\rangle = -\cos(\phi-\theta)$

Proof:

$$\begin{aligned}
\text{LHS} &= \left\langle x e^{-iz\theta} \otimes x e^{iz\phi}\right\rangle \\
&= \left[\frac{\langle 10| - \langle 01|}{\sqrt{2}}\right]\left[x e^{iz\theta} \otimes x e^{iz\phi}\right]\left[\frac{|10\rangle - |01\rangle}{\sqrt{2}}\right] \\
&= \langle\psi|\, x \otimes x \\
&\quad \left[\frac{e^{i(\phi-\theta)}\left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right)\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right)}{\sqrt{2}}\right. \\
&\quad \left. - \frac{e^{-i(\phi-\theta)}\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right)\left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right)}{\sqrt{2}}\right]
\end{aligned}$$

We define, $\delta \equiv \phi - \theta$ and using section V 1, it follows that only terms like $|+-\rangle$ or $|-+\rangle$; so

$$\begin{aligned}
\text{LHS} &= \langle\psi|\, x \otimes x \\
&\quad \left[\frac{e^{i\delta}\left(\frac{|+-\rangle - |-+\rangle}{2}\right) - e^{-i\delta}\left(\frac{-|+-\rangle + |-+\rangle}{2}\right)}{\sqrt{2}}\right] \\
&= \langle\psi|\, x \otimes x \left[\frac{e^{i\delta}\left(\frac{|\psi\rangle}{\sqrt{2}}\right) + e^{-i\delta}\left(\frac{|\psi\rangle}{\sqrt{2}}\right)}{\sqrt{2}}\right] \\
&= -\ \frac{e^{i\delta} + e^{-i\delta}}{2} \\
&= -\ \cos(\phi-\theta)
\end{aligned}$$

where we've used section V 2.

4. Without taking the large N limit

$\left\langle \psi_+ \middle| \hat{X} \middle| \psi_+ \right\rangle = \frac{N-1}{N}$, $\left\langle \psi_- \middle| \hat{X} \middle| \psi_- \right\rangle = -\frac{N-1}{N}$
$\left\langle \psi_0 \middle| \hat{X} \middle| \psi_0 \right\rangle = 0$, $\left\langle \psi_1 \middle| \hat{X} \middle| \psi_1 \right\rangle = 0$
$\left\langle \psi_1 \middle| \hat{X} \middle| \psi_0 \right\rangle = \frac{\frac{N-1}{N} + \frac{N}{N}}{2} = \frac{2N-1}{2N} = \left\langle \psi_0 \middle| \hat{X} \middle| \psi_1 \right\rangle$
$\left\langle \psi_- \middle| \hat{X} \middle| \psi_+ \right\rangle = \frac{-\langle\psi_1|\hat{X}|\psi_0\rangle + \langle\psi_0|\hat{X}|\psi_1\rangle}{2} = 0 = \left\langle \psi_+ \middle| \hat{X} \middle| \psi_- \right\rangle$

$$\begin{aligned}
\left\langle \Psi \middle| \hat{X}\otimes\hat{X} \middle| \Psi \right\rangle &= \frac{1}{2}\Big(\left\langle \psi_- \middle| \hat{X} \middle| \psi_- \right\rangle \left\langle \psi_+ \middle| \hat{X} \middle| \psi_+ \right\rangle + \\
&\quad \left\langle \psi_+ \middle| \hat{X} \middle| \psi_+ \right\rangle \left\langle \psi_- \middle| \hat{X} \middle| \psi_- \right\rangle\Big) \\
&= -\left(\frac{N-1}{N}\right)^2
\end{aligned}$$

5. For arbitrary θ_i *and* ϕ_i
$\left\langle \hat{U}^\dagger(\phi_i)\hat{X}\hat{U}(\phi_i) \otimes \hat{U}^\dagger(\theta_i)\hat{X}\hat{U}(\theta_i)\right\rangle = -\left(\frac{N-1}{N}\right)^2 \cos(\phi_i - \theta_i)$

Proof: We start with defining $\phi \equiv \phi_i$, $\theta \equiv \theta_i$, $\delta \equiv \phi - \theta$, $\delta' \equiv \delta/2$. Next, we note that LHS $= \left\langle \Psi' \middle| \hat{X}\otimes\hat{X} \middle| \Psi' \right\rangle$ where $|\Psi'\rangle = \hat{U}(\phi_i) \otimes \hat{U}(\theta_i)\,|\Psi\rangle$.

$$\begin{aligned}
|\Psi'\rangle &= \frac{e^{i\delta'}}{\sqrt{2}}\left(\frac{|\psi_+\rangle - |\psi_-\rangle}{\sqrt{2}}\right)\left(\frac{|\psi_+\rangle + |\psi_-\rangle}{\sqrt{2}}\right) \\
&\quad - \frac{e^{-i\delta'}}{\sqrt{2}}\left(\frac{|\psi_+\rangle + |\psi_-\rangle}{\sqrt{2}}\right)\left(\frac{|\psi_+\rangle - |\psi_-\rangle}{\sqrt{2}}\right) \\
&= \frac{e^{i\delta'}}{2\sqrt{2}}\left(|\psi_+\psi_+\rangle + |\psi_+\psi_-\rangle - |\psi_-\psi_+\rangle - |\psi_-\psi_-\rangle\right) \\
&\quad - \frac{e^{-i\delta'}}{2\sqrt{2}}\left(|\psi_+\psi_+\rangle - |\psi_+\psi_-\rangle + |\psi_-\psi_+\rangle - |\psi_-\psi_-\rangle\right) \\
&= \frac{e^{i\delta'} - e^{-i\delta'}}{2\sqrt{2}}\,|\psi_+\psi_+\rangle + \frac{e^{i\delta'} + e^{-i\delta'}}{2\sqrt{2}}\,|\psi_+\psi_-\rangle \\
&\quad - \left(\frac{e^{i\delta'} + e^{-i\delta'}}{2\sqrt{2}}\right)|\psi_-\psi_+\rangle - \left(\frac{e^{i\delta'} - e^{-i\delta'}}{2\sqrt{2}}\right)|\psi_-\psi_-\rangle
\end{aligned}$$

Now using section V 4, we have

$$\begin{aligned}
\text{LHS} &= \left\langle \Psi' \middle| \hat{X}\otimes\hat{X} \middle| \Psi' \right\rangle \\
&= \frac{1}{2}\left(\frac{N-1}{N}\right)^2 \left[\left|\frac{e^{i\delta'} - e^{-i\delta'}}{2}\right|^2\right. \\
&\quad - \left|\frac{e^{i\delta'} + e^{-i\delta'}}{2}\right|^2 - \left|\frac{e^{i\delta'} + e^{-i\delta'}}{2}\right|^2 \\
&\quad \left. + \left|\frac{e^{i\delta'} - e^{-i\delta'}}{2}\right|^2\right] \\
&= -\left(\frac{N-1}{N}\right)^2 \frac{1}{2}\left[2\left(\cos^2\delta/2 - \sin^2\delta/2\right)\right] \\
&= -\left(\frac{N-1}{N}\right)^2 \cos(\delta)
\end{aligned}$$

6. Action of $\hat{x}_{mod2L}$ *can be defined explicitly*

Proof: $\hat{x}_{\mathrm{mod}2L} \equiv \int dx x_{mod2L}\,|x\rangle\langle x|$. To arrive at this more carefully, consider the operator $e^{i\hat{x}\frac{2\pi}{2L}}$. Note that $e^{i\hat{x}\frac{2\pi}{2L}}\,|x\rangle = e^{ix\frac{2\pi}{2L}}\,|x\rangle = e^{ix_{mod2L}\frac{2\pi}{2L}}\,|x\rangle$. Thus, $\hat{x}_{\mathrm{mod}2L}\,|x\rangle = x_{mod2L}\,|x\rangle$, consequently on the most general state $|f\rangle \equiv \int dx f_x\,|x\rangle$ then, we'd have $\hat{x}_{\mathrm{mod}2L}\,|f\rangle = \int dx f_x x_{mod2L}\,|x\rangle$.

Remark: One needn't necessarily consider eigenstates of $\hat{x}$ to define the action. Eigenstates of $e^{i\hat{x}\frac{2\pi}{2L}}$ maybe considered instead; they can be expressed as (a) $|\varphi\rangle$, *s.t.* $\langle p + \frac{h}{2L}|\varphi\rangle = \langle p|\varphi\rangle$, $\forall\, p \in \mathbb{R}$ or (b) $|\bar{x}\rangle \propto \sum_{n\in\mathbb{Z}} |\bar{x} + n2L\rangle$. Using the second expression, we have

$e^{i\hat{x}\frac{2\pi}{2L}} |\bar{x}\rangle = e^{i\bar{x}\frac{2\pi}{2L}} |\bar{x}\rangle$. Thus on a more general state, $|c\rangle \equiv \int_0^{2L} d\bar{x} c_{\bar{x}} |\bar{x}\rangle$, we have $\hat{x}_{\text{mod}2L} |c\rangle = \int_0^{2L} d\bar{x} c_{\bar{x}} \bar{x} |\bar{x}\rangle$.

7. Physical implementation with electrons is also possible

If we can show that the basic components used to describe the photon setup can be translated to the electron setup, then in principle we are through. (a) Glass slab: The equivalent is the electric AB effect. We need to simply put a capacitor after the slit and the two components will pick up a phase difference. (b) Polariser: The Stern Gerlach setup is the classic analogue. We simply block the orthogonal component. (c) Birefringent crystal: This is slightly tricky. It can be modelled by using a combination of gradient of magnetic field (as in Stern Gerlach) and a capacitor. We start with an equivalent superposition of spin states, $\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} |\psi_+\psi_+\rangle$. To construct the spin dependent $|\psi_-\rangle$ state, we use the magnetic field gradient to spatially separate the $|\uparrow\rangle$ and $|\downarrow\rangle$ states. We place capacitors as described at the spatial position corresponding to $|\downarrow\rangle$ say. Thereafter, we remove the magnetic field gradient and allow the beams to meet again. This will effectively act as a Birefringent crystal, since the phase difference is spin dependent.

8. Action of a polariser

If we define $|H'\rangle \equiv \frac{|H\rangle + |V\rangle}{\sqrt{2}}$, $|V'\rangle \equiv -\frac{|H\rangle - |V\rangle}{\sqrt{2}}$ and the 45° projector as $|H'\rangle \langle H'|$, then both $|H\rangle \rightarrow |H'\rangle$ and $|V\rangle \rightarrow |H'\rangle$ where ofcourse with a probability $1/2$, the photon will be lost.

[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935), URL `http://link.aps.org/doi/10.1103/PhysRev.47.777`.

[2] We emphasise that the kets we construct, will not be a countable superposition of eigenstates of p or x since these are highly idealized and strictly, not even a part of the Hilbert space.

[3] Defn: $|0\rangle$, $|1\rangle$ are s.t. $\hat{z}|0\rangle = |0\rangle$ and $\hat{z}|1\rangle = -|1\rangle$

[4] $\hat{x}_{\text{mod}2L}$ is well defined, see section V 6