Bell Test with Continuous Variables

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Abstract

We show the violation of the Bell inequality in a more classical setting than the usual quantum spin, viz. with continuous variables x, p. This is achieved using displacement operators, or equivalently modular momentum and position. A possible experimental realization of the scheme has also been delineated.

I. INTRODUCTION

The Bell inequality is a statement about the statistical outcomes of a certain class of experiments. The inequality was formulated by assuming the most general 'local theory'. Any violation of the inequality would imply that local realism must be false. Quantum Mechanics predicts a violation which has been confirmed experimentally. The physical situation for which this violation is usually formulated, uses entangled spin states. Here we show that a violation can be achieved without having to use spins, which is a purely quantum phenomenon; measuring a particle's position is sufficient.

II. VIOLATION OF BELL INEQUALITY

A. CHSH Inequality | prior art

Here the violation of Bell's inequality for the typical spin case, is stated in a way that is easier to translate to the continuous variable case. We first state the Bell test in its modern form, known as the CHSH inequality.

Suppose: There are two observers[1], Alice and Bob. There are two particles, which are allowed to interact initially and are then separated. Alice takes one, Bob takes the other. Alice can measure one of two properties of her particle, call them a_1, a_2. Similarly Bob can measure b_1, b_2 of his particle.

Notatn: $\langle a_i b_j \rangle$ =Average value obtained when Alice chooses to measure a_i and Bob chooses to measure b_j

Statemnt: $|\langle a_1(b_1+b_2)+a_2(b_1-b_2)\rangle| \equiv |\langle C(a_i,b_i)\rangle| \leq 2$ if local realism is assumed. This result will be referred to as the CHSH inequality.

Quantum Mechanics predicts a violation of this in the following physical situation.

Defn: $|\psi\rangle \equiv \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$ where the $|\pm\rangle$ states are s.t. $\sigma_x |\pm\rangle = \pm |\pm\rangle$.

	Alice		Bob	
Convention	a_1	a_2	b_1	b_2
Angle	$\theta_1 = \pi/2$	$\theta_2 = 0$	$\phi_1 = \pi/4$	$\phi_2 = \pi/4 + \pi/2$

Table I: Measurement Scheme

Consider: (a) Two particles are described by $|\psi\rangle$, (b) the first particle is with Alice, second with Bob, and (c) Alice measures z or x while Bob is allowed to choose between z' and x', where $z' = \frac{-z-x}{\sqrt{2}}$, $x' = \frac{z-x}{\sqrt{2}}$.

Notatn: The Pauli matrices are referred to as x, y, z.

Result: $\langle C((x,z),(x',z'))\rangle = 2\sqrt{2} \nleq 2$, which is a violation.

This value can be reproduced in a slightly modified situation.

Modifictn: (i) Due to the symmetry in the setup, we expect the same results hold for x, y and x', y' and (ii) we demand that Alice and Bob can only measure x but are allowed to perform a local unitary before the measurement.[2]

Explicitly then, consider: (a) The first qubit is with Alice and the second is with Bob. (b) Alice and Bob apply local unitaries $e^{iz\theta/2} \otimes I$ and $I \otimes e^{iz\phi/2}$ respectively and then both measure x. (c) Alice can choose $\theta \in \{0, \pi/2\}$ and Bob can choose $\phi \in \{\pi/4, \pi/4+\pi/2\}$; see Table I.[3]

Now it remains to show that this scheme also violates the Bell inequality (and by the same value). To that end, we state a result.

Claim(3): $\left\langle e^{-iz\theta/2} x e^{iz\theta/2} \otimes e^{-iz\phi/2} x e^{iz\phi/2} \right\rangle = -\cos(\phi - \theta)$

Given that, it is straight forward to evaluate

CHSH:

$$\begin{aligned} \langle C\rangle &= \langle a_1 b_1\rangle + \langle a_1 b_2\rangle + \langle a_2 b_1\rangle - \langle a_2 b_2\rangle \\ &= \left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right) - \left(+\frac{1}{\sqrt{2}}\right) \\ &= -2\sqrt{2} \end{aligned}$$

	Alice		Bob	
Convention	a_1	a_2	b_1	b_2
Angle	$\theta'_1 = 0$	$\theta'_2 = -\pi/2$	$\phi'_1 = 3\pi/8$	$\phi'_2 = 5\pi/4$

Table II: 'Corrected' Measurement Scheme

Remark: This is a violation since $|\langle C\rangle| = 2\sqrt{2} \not\leq 2$.

However, to convert this to a more conventional form, we simply redefine the angles θ and ϕ.[4]

Substn: $\phi \to \phi' = \phi + \pi/2$, $\theta \to \theta' = \theta - \pi/2$; see Table II

Now we have $\langle C\rangle = 2\sqrt{2}$.

Concln: The violation of the Bell spin state can be equivalently be obtained by the measurement of x after applying a rotation along z by the aforesaid angles.

B. Modular Variables | prior art

<TODO>

C. Violation in Continuous Variable State

We start with constructing continuous variable kets that satisfy the same orthogonality and mutual relations as $|0\rangle, |1\rangle, |+\rangle, |-\rangle$.[5][6] We then construct the relavent operators and finally evaluate $\langle C\rangle$ to show the violation.

To construct the states, we start with some definitions.

Unassume: x, y, z refer to pauli matrices

Defn: $L \equiv$some length scale, $N \equiv$number of slits

Consider: A state $|\varphi\rangle$ s.t. (a) $\langle x'|\varphi\rangle = \langle -x'|\varphi\rangle$ and (b) it's spread is $\ll L$

Defn: $|\varphi_n\rangle \equiv \int dx' \langle x' - nL|\varphi\rangle |x'\rangle$

Defn:

$$\begin{aligned} |0'\rangle &\equiv \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} |\varphi_{2n+1}\rangle \\ |1'\rangle &\equiv \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} |\varphi_{2n}\rangle \end{aligned}$$

Using these states, we construct the analogues of the $|+\rangle$ and $|-\rangle$ states.

Remark: Definition of $|0'\rangle$ and $|1'\rangle$ justify the word 'slit' in the definition of N.

Defn: $|+'\rangle \equiv \frac{|0'\rangle+|1'\rangle}{\sqrt{2}}$, $|-'\rangle \equiv \frac{|0'\rangle-|1'\rangle}{\sqrt{2}}$

NB: $\langle 0'|1'\rangle = 0, \implies \langle +'|-'\rangle = 0$

Analogous to the spin case, we construct the entangled state

Defn: $|\psi'\rangle \equiv \frac{|+'-'\rangle-|-'+'\rangle}{\sqrt{2}}$

With these states, we're in a position to construct the relevant Hermitian operators.

Defn:

$$X \equiv \frac{e^{ipL/\hbar} + e^{-ipL/\hbar}}{2}$$

NB: Only in the limit $N \to \infty$,

$$\begin{aligned} X|+'\rangle &= +|+'\rangle \\ X|-'\rangle &= -|-'\rangle \end{aligned}$$

just as was with σ_x. In general however, the relation doesn't hold.

We have almost all the cards, except for one that allows for setting $\{\phi, \theta\}$. In the spin case, this was achieved by operating $e^{i\sigma_z\phi/2}$ on the state before measurement. Correspondingly, here we define this operator by it's action on the relevant state.

Defn: $U(\phi)$ is defined to be s.t.

$$\begin{aligned} U(\phi)|0'\rangle &= e^{i\phi/2}|0'\rangle \\ U(\phi)|1'\rangle &= e^{-i\phi/2}|1'\rangle \end{aligned}$$

This can achieved in principle using alternate glass slabs (capacitors) for photons (fermions) to adjust the relative phase as desired. Alternatively, we could also define $U(\phi)$ in its analytic form as

Alt Defn:

$$U(\phi) = e^{iZ(x_{mod2L})\phi/2}$$

where $Z(x') \equiv \begin{cases} 1 & 0 < x' \mod 2L \leq L \\ -1 & L < x' \mod 2L \leq 2L \end{cases}$ when Z acts on an eigenstate of x.[7]

NB: $Z(x_{mod2L})|0'\rangle = |0'\rangle$ and $Z(x_{mod2L})|1'\rangle = -|1'\rangle$ viz. $Z(x_{mod2L})$ plays the role of σ_z

To evaluate the CHSH inequality, we again need terms of the form $\left\langle U^\dagger(\phi_i)XU(\phi_i) \otimes U^\dagger(\theta_i)XU(\theta_i)\right\rangle$. If we assume $N \to \infty$, these can be evaluated easily to be the same as before $-\cos(\phi_i - \theta_i)$. Thus, the same analysis as before yields $\langle C\rangle = 2\sqrt{2}$ which is the same maximal violation, in a completely different setup. If N is not assumed to be arbitrarily large, then it can be shown that[8]

$$\left\langle U^\dagger(\phi_i)XU(\phi_i) \otimes U^\dagger(\theta_i)XU(\theta_i)\right\rangle = -\left(\frac{N-1}{N}\right)^2 \cos(\phi_i - \theta_i)$$

Thus obviously then, $\langle C\rangle = \left(\frac{N-1}{N}\right)^2 2\sqrt{2}$. If we ask for the smallest N s.t. $\langle C\rangle > 2$, we get $N \geq 7$; see Figure 1.

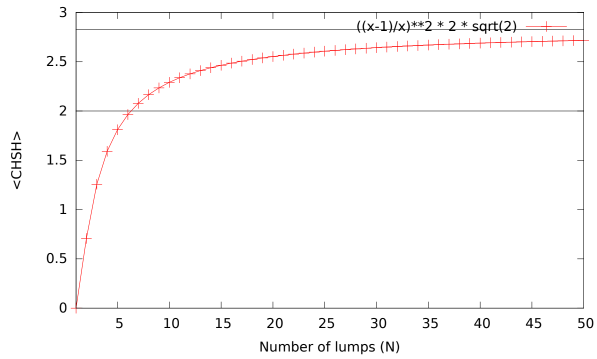

Figure 1: Practically, the number of slits, N will be finite. The plot shows $\langle C\rangle$ as a function of N. To get a violation, we need merely 7 slits, while with 50 we start approaching $2\sqrt{2}$.

D. Validating phase space behaviour

In the past, it has been shown that one can arrive at the violation of Bell inequalities in continuous variable systems. One issue concerning these is that the wigner function corresponding to the observable operators, are unbounded. An example of this is parity. Note that the Wigner function for a physical state ρ has an upper bound. Here however, we are simply using the prescription to evaluate it for unitary operators.

We care because wigner functions maybe thought of as a specific case of the Wigner-Weyl correspondence, which relates classical functions to their corresponding quantum operators. Thus, in a phase-space description of quantum mechanics, it is important to show that it is not necessary to have unboundedness to show a violation of the Bell's inequality, and thereby non-locality.

Our set of permissible states is manifestly bounded. We must show that the observable in our case, given by $U^\dagger(\phi)XU(\phi)$ is also bounded. To that end, we note that

$$\begin{aligned} W_1(q',p') &= \frac{1}{2\pi\hbar}\int dq e^{ip'q/\hbar}\left\langle q'-\frac{q}{2}\right| e^{-iZ(x_{mod2L})\phi/2} \\ & \quad e^{ip\frac{L}{\hbar}}e^{iZ(x_{mod2L})\phi/2}\left|q'+\frac{q}{2}\right\rangle \\ &= \frac{1}{2\pi\hbar}e^{-iZ_1(q')\phi/2}e^{ip'\frac{L}{\hbar}}e^{iZ_2(q')\phi/2} \end{aligned}$$

where $Z_{1/2}(q') = Z[(q'\mp\frac{L}{2}) \mod 2L]$. Similarly one can evaluate W_2 and obtain $W=(W_1+W_2)/\sqrt{2}$ which corresponds to the observable. Clearly $|W|\leq(2\pi\hbar)^{-1}$.

E. Source of Violation

In the Bell test done with spins, the source of violation hinges on the anti-commutativity of the pauli matrices. It is known that $C^2 = 4\mathbb{I}+[a_1,a_2]\otimes[b_1,b_2]$. To obtain a violation, it is therefore necessary that the commutations don't vanish. This is indeed true in our case. Explicitly, we must show that $[U^\dagger XU, U'^\dagger XU']\neq 0$.

Remark: The non-commutativity in this case is between U and X, which arises from $[x_{mod2L}, p_{modL}]\neq 0$.

TODO: Complete this

III. PHYSICAL IMPLEMENTATION

We show that this scheme can be implemented rather easily using photons. We harness the two degrees of freedom of a photon, it's polarization and it's position to construct the required state. The overall setup (see figure 2) is such that we need only consider the quantum mechanical description along the x-axis.

A. Creation of the entangled state

The desired entangled state is $|\psi'\rangle \equiv \frac{|+'-'\rangle-|-'+'\rangle}{\sqrt{2}}$. We start with noting the triviality of constructing a

$$|+'\rangle = \frac{1}{\sqrt{2N}}\sum_{n=0}^{2N-1}|\psi_n\rangle$$

state. One simply needs a source and a grating, viz. a screen with $2N$ slits of width $a \ll L$, separated by a distance L (centre to centre). The

$$|-'\rangle = \frac{1}{\sqrt{2N}}\sum_{n=0}^{2N-1}(-1)^{n+1}|\psi_n\rangle$$

state can be similarly constructed by using glass slabs at alternate slits, such that the phase introduced is $\phi=\pi \implies e^{i\phi}=-1$. In figure 2, if you consider only one particle, and disregard everything after the grating, then the setup is expected to produce a $|+'\rangle$ state, just after the grating. To produce the desired entangled state, we start with two entangled photons, such that their polarization state can be expressed as

$$|\chi\rangle = \frac{|HV\rangle-|VH\rangle}{\sqrt{2}}$$

Their spatial description (along x-axis) is initially assumed to be $|\zeta\rangle = |\gamma\rangle\otimes|\gamma\rangle = |\gamma\gamma\rangle$. The state $|\gamma\rangle$ maybe considered to be a guassian with $\sigma \gg 2NL$ so that after the grating, the spatial state can be written as $|\zeta_{\text{grating}}\rangle = |+'+'\rangle$. Now comes the interesting part. To produce a $|-'\rangle$ state, we already know we needed alternate glass slabs. If we had glass slabs, whose refractive index (given some orientation) was say $\eta_H=1$ for a horizontally polarized beam and $\eta_V=\eta\neq 0$ for vertical polarization, then we could harness the entangled polarization state to create the required spatially entangled state. Birefringent crystals have such polarization dependent refractive indices. Assume that alternating birefringent crystals have been placed after both the gratings as shown in figure 2 (in the diagram, they're not alternating though). These have width such that the resultant phase it introduces is π. The combined initial state is given by

$$|\chi\rangle |\zeta\rangle = \frac{|HV\rangle - |VH\rangle}{\sqrt{2}} |\gamma\gamma\rangle$$

After the grating, the state is

$$|\chi\rangle |\zeta_{\text{grating}}\rangle = \frac{|HV\rangle - |VH\rangle}{\sqrt{2}} |+'+'\rangle$$

And finally after the crystals, the combined state becomes

$$|\chi\rangle |\zeta_{\text{birefringent}}\rangle = \frac{|HV\rangle |+'-'\rangle - |VH\rangle |-'+'\rangle}{\sqrt{2}}$$

At this stage, if I were to trace out the polarization state, I'd end up with a mixed state. That is useless for our test. Ignoring the glass slabs at the moment in figure 2, we observe the action of the polarisers on the state. After the 45^0 polariser, the state becomes

$$\begin{aligned} |\chi_{45}\rangle |\zeta_{\text{birefringent}}\rangle &= \frac{1}{\sqrt{2}}\Bigg(\frac{|H\rangle + |V\rangle}{\sqrt{2}} \frac{|H\rangle - |V\rangle}{\sqrt{2}} |+'-'\rangle \\ &\quad - \frac{|H\rangle - |V\rangle}{\sqrt{2}} \frac{|H\rangle + |V\rangle}{\sqrt{2}} |-'+'\rangle \Bigg) \end{aligned}$$

After the final stage, the horizontal polarizer, the state becomes

$$|\chi_{\text{horzl}}\rangle |\zeta_{\text{birefringent}}\rangle \propto |HH\rangle \left(\frac{|+'-'\rangle - |-'+'\rangle}{\sqrt{2}} \right)$$

Now if I trace out the polarization state, I am left with the target entangled state. As a remark, it maybe be stated that although to arrive at this result we assumed that $\eta_H = 1$, which is unreasonable physically, we can compensate for $\eta_H \neq 0$ by putting appropriate glass slabs at the alternate empty slits, to produce zero relative phase when the polarization is horizontal.

B. Measurement Settings

In our scheme, Alice and Bob can choose $\phi \in \{\phi_1, \phi_2\}$ and $\theta \in \{\theta_1, \theta_2\}$ respectively. To implement this, we note that the action of $U(\theta)$ is to introduce a phase difference between the $|0'\rangle$ and $|1'\rangle$ states. By construction, $|0'\rangle$ and $|1'\rangle$ are spatially disjoint; thus the operation of $U(\theta)$ can be implemented by simply placing alternating glass slabs at the slits, with widths such that $|0'\rangle \to e^{i\theta} |0'\rangle$ and $|1'\rangle \to e^{-i\theta} |1'\rangle$. The same operation maybe done for $U(\phi)$ for the second particle. In the overall scheme, as shown in figure 2, this is done by the glass slabs right after the birefringent crystals.

C. Measurement

The scheme requires us to evaluate the expectation value of $X \otimes X$ (after the measurement settings have been applied by the appropriate unitaries). If from the experiment, we can obtain the probability $\left|\left\langle p'_A, p'_B | \psi'_f \right\rangle\right|^2$, then evaluating $\langle X \otimes X \rangle = \langle \cos(pL/\hbar) \otimes \cos(pL/\hbar) \rangle$ simply amounts to $\int dp'_A dp'_B \cos(p'_A L/\hbar) \cos(p'_B L/\hbar) \left|\left\langle p'_A, p'_B | \psi'_f \right\rangle\right|^2$. Here f is used to index which of the four possible measurement settings was used.

Therefore it is sufficient to explain how to obtain the joint momentum probability distribution from the experiment, for a given measurement setting. We start with stating the result that

$$|\langle x'_A, x'_B | \psi'_{f\,\text{screen}} \rangle|^2 \propto \left| \left\langle p'_A = \frac{m x'_A}{T}, p'_B = \frac{m x'_B}{T} | \psi'_f \right\rangle \right|^2$$

where $\left|\psi'_{f\,\text{screen}}\right\rangle$ is the state of the system at the screen, T is the time elapsed between leaving the gratings and arriving at the screens and m is the mass of the particle (obviously this result is for massive particles, but its analogue holds for photons also). The point is that the momentum distribution at the grating can be recovered by observing the spatial distribution at a screen, sufficiently far away.

In an experiment then, given f, Alice and Bob both note the position at which they obtain their particle. After repeating the experiment sufficiently many times, they share their list to create a sequence $\{\left(x_A^{(i)}, x_B^{(i)}\right)\}$. From this sequence, they create a 2D histogram, by simply counting how many time they got (x'_A, x'_B) to lie inside a given cell. The normalized result is essentially $\left|\left\langle x'_A, x'_B | \psi'_{f\,\text{screen}} \right\rangle\right|^2$ from which $\langle X \otimes X \rangle$ can be computed as discussed. f is changed to evaluate all the 4 terms to finally obtain $\langle C \rangle$ experimentally.

IV. CONCLUSION

We have shown how to realize a Bell test in continuous variables position and momentum using specifically chosen and physically realizable states along with modular variables.

V. APPENDIX

Remarks

Illustration(1):

This is trival if one uses the Bloch sphere picture. Instead of measuring along an arbitrary axis, we rotate the

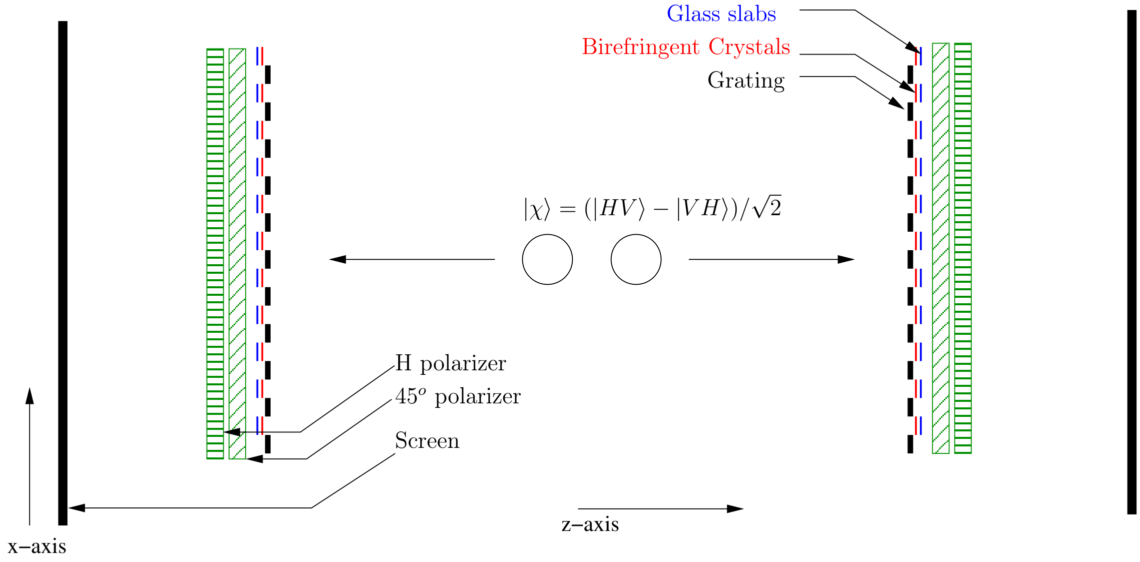

Figure 2: Scheme for creating the state

Bloch sphere appropriately, and then measure x. To illustrate this, consider

Questn: $\exists\,\mathrm{a}\,U,\ s.t.$ if $|\chi\rangle \to |\chi'\rangle = U\,|\chi\rangle$ then $\langle\chi\,|x|\,\chi\rangle = \langle\chi'\,|y|\,\chi'\rangle$?
Explicitly, we have

$$\begin{aligned} y &= U^\dagger x U = e^{-iz\theta/2} x e^{iz\theta/2} \\ &= x e^{iz\theta} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \end{aligned}$$

for $\theta = \pi/2$ as one would guess geometrically.

Proofs

Claim(1): If$|\psi\rangle \equiv \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$, then$|\psi\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$

Proof: Trivial.

Claim(2): $\langle x \otimes x\rangle = -1$, for $|\psi\rangle$ in Claim(1)

Proof: Trivial

Claim(3): $\langle e^{-iz\theta/2} x e^{iz\theta/2} \otimes e^{-iz\phi/2} x e^{iz\phi/2}\rangle = -\cos(\phi - \theta)$

Proof:

$$\begin{aligned} \mathrm{LHS} &= \langle x e^{-iz\theta} \otimes x e^{iz\phi}\rangle \\ &= \left[\frac{\langle 10| - \langle 01|}{\sqrt{2}}\right] \left[x e^{iz\theta} \otimes x e^{iz\phi}\right] \left[\frac{|10\rangle - |01\rangle}{\sqrt{2}}\right] \\ &= \langle\psi|\, x \otimes x \\ & \left[\frac{e^{i(\phi-\theta)}\left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right)\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right)}{\sqrt{2}} - \frac{e^{-i(\phi-\theta)}\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right)\left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right)}{\sqrt{2}}\right] \end{aligned}$$

We define, $\delta \equiv \phi - \theta$ and using claim(1), it follows that only terms like $|+-\rangle$ or $|-+\rangle$; so

$$\begin{aligned} \mathrm{LHS} &= \langle\psi|\, x \otimes x \\ & \left[\frac{e^{i\delta}\left(\frac{|+-\rangle - |-+\rangle}{2}\right) - e^{-i\delta}\left(\frac{-|+-\rangle + |-+\rangle}{2}\right)}{\sqrt{2}}\right] \\ &= \langle\psi|\, x \otimes x \left[\frac{e^{i\delta}\left(\frac{|\psi\rangle}{\sqrt{2}}\right) + e^{-i\delta}\left(\frac{|\psi\rangle}{\sqrt{2}}\right)}{\sqrt{2}}\right] \\ &= -\ \frac{e^{i\delta} + e^{-i\delta}}{2} \\ &= -\ \cos(\phi - \theta) \end{aligned}$$

where we've used claim(2).

Claim(4): With definitions from section 3, without taking the large N limit,
$\langle +\,|X|\,+\rangle = \frac{N-1}{N}$, $\langle -\,|X|\,-\rangle = -\frac{N-1}{N}$
$\langle 0\,|X|\,0\rangle = 0$, $\langle 1\,|X|\,1\rangle = 0$
$\langle 1\,|X|\,0\rangle = \frac{\frac{N-1}{N} + \frac{N}{N}}{2} = \frac{2N-1}{2N} = \langle 0\,|X|\,1\rangle$

$\langle - | X | + \rangle = \frac{-\langle 1|X|0\rangle + \langle 0|X|1\rangle}{2} = 0 = \langle + | X | - \rangle$
$\langle \psi | X \otimes X | \psi \rangle = \frac{\langle +|X|+\rangle\langle -|X|-\rangle + \langle -|X|-\rangle\langle +|X|+\rangle}{2} = -\left(\frac{N-1}{N}\right)^2$

Proof: Trivial.

Claim(5): $\left\langle U^\dagger(\phi_i) X U(\phi_i) \otimes U^\dagger(\theta_i) X U(\theta_i) \right\rangle = -\left(\frac{N-1}{N}\right)^2 \cos(\phi_i - \theta_i)$

Proof: We start with defining $\phi \equiv \phi_i$, $\theta \equiv \theta_i$, $\delta \equiv \phi - \theta$, $\delta' \equiv \delta/2$. Next, we note that LHS $= \left\langle \psi'_f \right| X \otimes X \left| \psi' \right\rangle$ where $\left| \psi'_f \right\rangle = U(\phi_i) \otimes U(\theta_i) \left| \psi' \right\rangle$.

$$
\begin{aligned}
\left|\psi'_f\right\rangle &= \frac{e^{i\delta'}}{\sqrt{2}} \left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right) \left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right) \\
&\quad - \frac{e^{-i\delta'}}{\sqrt{2}} \left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right) \left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right) \\
&= \frac{e^{i\delta'}}{2\sqrt{2}} \left(|++\rangle + |+-\rangle - |-+\rangle - |--\rangle\right) \\
&\quad - \frac{e^{-i\delta'}}{2\sqrt{2}} \left(|++\rangle - |+-\rangle + |-+\rangle - |--\rangle\right) \\
&= \frac{e^{i\delta'} - e^{-i\delta'}}{2\sqrt{2}} |++\rangle + \frac{e^{i\delta'} + e^{-i\delta'}}{2\sqrt{2}} |+-\rangle \\
&\quad - \left(\frac{e^{i\delta'} + e^{-i\delta'}}{2\sqrt{2}}\right) |-+\rangle - \left(\frac{e^{i\delta'} - e^{-i\delta'}}{2\sqrt{2}}\right) |--\rangle
\end{aligned}
$$

Now using claim(4), we have

$$
\begin{aligned}
\text{LHS} &= \left\langle \psi'_f \right| X \otimes X \left| \psi'_f \right\rangle \\
&= \frac{1}{2} \left(\frac{N-1}{N}\right)^2 \left[\left| \frac{e^{i\delta'} - e^{-i\delta'}}{2} \right|^2 \right. \\
&\quad - \left| \frac{e^{i\delta'} + e^{-i\delta'}}{2} \right|^2 - \left| \frac{e^{i\delta'} + e^{-i\delta'}}{2} \right|^2 \\
&\quad \left. + \left| \frac{e^{i\delta'} - e^{-i\delta'}}{2} \right|^2 \right] \\
&= -\left(\frac{N-1}{N}\right)^2 \frac{1}{2} \left[2\left(\cos^2 \delta/2 - \sin^2 \delta/2\right)\right] \\
&= -\left(\frac{N-1}{N}\right)^2 \cos(\delta)
\end{aligned}
$$

Claim(8): Action of x_{mod2L} can be defined explicitly

Proof: $x_{mod2L} \equiv \int dx' x'_{mod2L} \left|x'\right\rangle \left\langle x'\right|$. To arrive at this more carefully, consider the operator $e^{ix\frac{2\pi}{2L}}$. Note that $e^{ix\frac{2\pi}{2L}} \left|x'\right\rangle = e^{ix'\frac{2\pi}{2L}} \left|x'\right\rangle = e^{ix'_{mod2L}\frac{2\pi}{2L}} \left|x'\right\rangle$. Thus, $x_{mod2L} \left|x'\right\rangle = x'_{mod2L} \left|x'\right\rangle$, consequently on the most general state $\left|f\right\rangle \equiv \int dx' f_{x'} \left|x'\right\rangle$ then, we'd have $x_{mod2L} \left|f\right\rangle = \int dx' f_{x'} x'_{mod2L} \left|x'\right\rangle$.

Remark: One needn't necessarily consider eigenstates of x to define the action. Eigenstates of $e^{ix\frac{2\pi}{2L}}$ maybe considered instead; they can be expressed as (a) $|\varphi\rangle$, *s.t.* $\langle p + \frac{h}{2L} | \varphi \rangle = \langle p | \varphi \rangle$, $\forall\, p \in \mathbb{R}$ or (b) $|\bar{x}\rangle \propto \sum_{n\in\mathbb{Z}} |\bar{x} + n2L\rangle$. Using the second expression, we have $e^{ix\frac{2\pi}{2L}} |\bar{x}\rangle = e^{i\bar{x}\frac{2\pi}{2L}} |\bar{x}\rangle$. Thus on a more general state, $|c\rangle \equiv \int_0^{2L} d\bar{x} c_{\bar{x}} |\bar{x}\rangle$, we have $x_{mod2L} |c\rangle = \int_0^{2L} d\bar{x} c_{\bar{x}} \bar{x} |\bar{x}\rangle$.

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[1] that are often chosen to be sufficiently far away that their measurements don't influence each other
[2] Note that (ii) is equivalent to measuring along any direction. Details in the Apendix.
[3] These values are motivated by the $\pi/2$ between z and x, & $\pi/4$ between z and z' in the conventional Bell case.
[4] so that $\phi - \theta \to \phi' - \theta' = \phi - \theta + \pi = (\phi + \pi/2) - (\theta - \pi/2)$
[5] Defn: $|0\rangle$, $|1\rangle$ are s.t. $z\,|0\rangle = |0\rangle$ and $z\,|1\rangle = -\,|1\rangle$
[6] We emphasise that the kets we construct, will not be a countable superposition of eigenstates of p or x since these are highly idealized and strictly, not even a part of the Hilbert space.
[7] x_{mod2L} is well defined, see claim 8
[8] Claim(5)