Light-Atom Quantum Evolution

We are looking for the dynamics of a quantum mechanical atom when it is exposed to classical light field.

⟹ **Time Evolution**

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \left(\hat{H}_0 + \hat{H}_I(t) \right) |\psi(t)\rangle$$

General Ansatz

$$|\psi(t)\rangle = \sum_n c_n(t)\, e^{-iE_n t/\hbar} |n\rangle$$

Atomic Eigenstates

$$\hat{H}_0 |n\rangle = E_n |n\rangle$$

Atomic Hamiltonian

Inserting into Schrodinger equation (both sides)

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = i\hbar \sum_n \dot{c}_n(t)\, e^{-iE_n t/\hbar} |n\rangle +$$

$$\cancel{i\hbar} \sum_n c_n(t) \left(\frac{\cancel{-i} E_n}{\cancel{\hbar}} \right) e^{-iE_n t/\hbar} \quad — (1)$$

$$\left(\hat{H}_0 + \hat{H}_I(t) \right) |\psi(t)\rangle =$$

$$\hat{H}_0 \left(\sum_n c_n(t)\, e^{-iE_n t/\hbar} |n\rangle \right) +$$

$$\hat{H}_I(t) \left(\sum_n c_n(t)\, e^{-iE_n t/\hbar} |n\rangle \right)$$

$$= \sum_n c_n(t)\, E_n\, e^{-iE_n t/\hbar} |n\rangle + \sum_n c_n(t)\, e^{-iE_n t/\hbar} \ldots$$

Comparing 1 and 2

$$\hat{H}_I(t)\,|\Psi(t)\rangle = i\hbar \sum_n \dot{c}_n(t)\, e^{-iE_n t/\hbar}\,|n\rangle$$

$$\sum_n c_n(t)\, e^{-iE_n t/\hbar}\, \hat{H}_I(t)\,|n\rangle =$$

$$i\hbar \sum_n \dot{c}_n(t)\, e^{-iE_n t/\hbar}\,|n\rangle$$

Using $\langle k|n\rangle = \delta_{kn}$

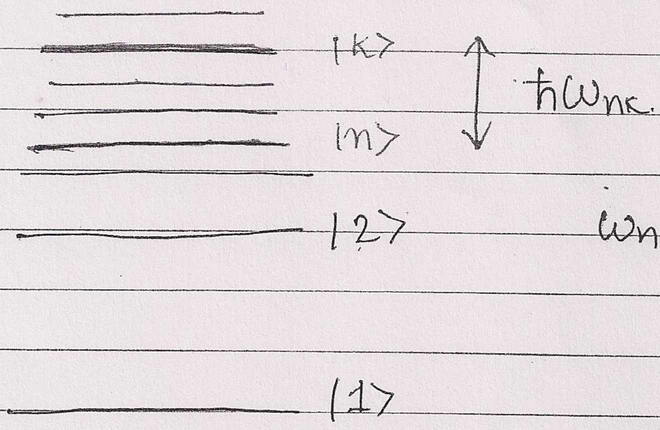
$$\boxed{i\hbar\, \dot{c}_k\, e^{-iE_k t/\hbar} = \sum_n c_n(t)\, e^{-iE_n t/\hbar}\, \langle k|\hat{H}_I(t)|n\rangle}$$

tells us how strong the nth state is linked to the kth state.

$$\boxed{i\hbar\, \dot{c}_k = \sum_n c_n(t)\, e^{-i\underbrace{(E_n - E_k)}_{\text{Call it } E_{nk}}\, t/\hbar}\, \langle k|\hat{H}_I(t)|n\rangle}$$

$$\boxed{i\hbar\, \dot{c}_k = \sum_n c_n(t)\, e^{-i\omega_{nk}}\, \underbrace{\langle k|\hat{H}_I(t)|n\rangle}_{\text{Matrix Element}}}$$

Time Dependent Perturbation Theory



$$\omega_{nk} = \frac{(E_n - E_k)}{\hbar}$$

Time evolution

$$\boxed{i\hbar \dot{c}_k = \sum_n c_n(t)\, e^{-i\omega_{nk} t} \langle k | \hat{H}_I(t) | n \rangle}$$

Simplification (Perturbation theory)

(1). System only in state $|1\rangle$ at $t = 0$.

i.e $C_1(0) = 1$

(2) Perturbative treatment of interaction term

i.e weak perturbation.

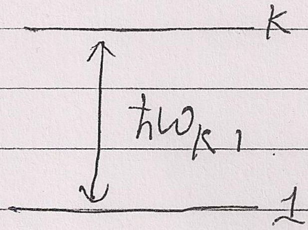
$\forall t \quad |C_k(t)|^2 << 1$

Under these approximatns, the eqns Simplifies

Thus the sum simplifies as

$\cancel{i\hbar \dot{c}_k = e^{-i\omega_{k}}}$

$$\boxed{i\hbar \dot{c}_k = e^{-i\omega_{1k} t} \, \langle k | \hat{H}_I(t) | 1 \rangle}$$



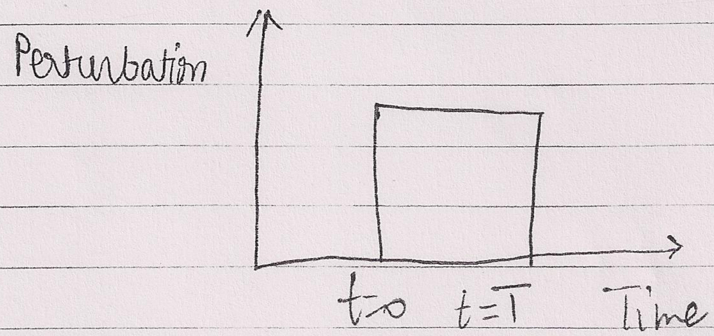
Using $c_k(0) = 0$, we get

$$\boxed{c_k(t) = \frac{1}{i\hbar} \int_0^t e^{-i\omega_{1k} t} \, \langle k | \hat{H}_I(t') | 1 \rangle \, dt'}$$

Let us be a little specific

Example:- (Sinusoidal perturbation)

$$\hat{H}_I(t) = \hat{H}_I \, e^{-i\omega t}$$



$$c_k(t) = \frac{1}{i\hbar} \int_0^t e^{-i\omega_{1k} t} \, \langle k | \hat{H}_I(t') | 1 \rangle \, dt'$$

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$$= \frac{1}{i\hbar}\int_0^T e^{-i\omega_{1k}t'} \langle k|\hat{H}_I|1\rangle e^{i\omega t'} dt'$$

$$= \frac{1}{i\hbar}\int_0^T e^{-i\Delta\omega t'} \langle k|\hat{H}_I|1\rangle dt'$$

$$\boxed{\Delta\omega = \omega - \omega_{k1}}$$

Detuning.

And the transition probability turns out to be

$$P_{k1}(T) = |C_k(T)|^2 = \frac{1}{\hbar^2}|\langle k|\hat{H}_I|1\rangle|^2 \, Y(\Delta\omega, T)$$

Where

$$Y(\Delta\omega, T) = \frac{\sin^2(\Delta\omega T/2)}{(\Delta\omega/2)^2}$$

$$\sim \operatorname{sinc}^2 x$$