I. The Equations of Motion

§1. Generalized co-ordinates

Particle $\equiv$ A body whose dimensions may be neglected in describing its motion.

$\vec{r} \doteq$ position vector

$\vec{v} \equiv \frac{d\vec{r}}{dt} \doteq$ velocity of the particle

$\frac{d^2\vec{r}}{dt} \doteq$ acceleration

Degree of Freedom $\equiv$ The number of independent quantities which must be specified in order to define uniquely the position of any ~~system given~~ given system.

Generalized Co-ordinates $\equiv$ Any s (where s is the DOF) co-ordinates (not necessarily position) which completely define the position of a system (with ~~s~~ s DOF).
say $\{q_1, q_2 \cdots q_s\}$ (represented by q)

Generalized Velocities $\doteq \{\dot{q}_1, \dot{q}_2 \cdots \dot{q}_s\}$ (represented by $\dot{q}$)

Experimental Claim: If all the co-ordinates & velocities are simultaneously specified, its ~~s~~ state is completely determined viz., its subsequent motion, can in principle be calculated.

[Mathematically]: If q & $\dot{q}$ are given, then the accelerations $\ddot{q}$ are uniquely defined, at that instant.

Equations of Motion $\equiv$ The relations between the acceleration, velocities & co-ordinates. ~~are called this.~~

Remark: They are second order differential equation for the functions $q(t)$. In principle, their ~~only~~ integration ~~yields~~ determines these function & thus their path of integration.

§2. The principle of least action

Hamilton's principle or principle of least Action:

(Claim: Every mechanical system is characterized by a definite f^n $L(q, \dot{q}, t)$

$$\underset{(\text{Action})}{S} \equiv \int_{t_1}^{t_2} L(q, \dot{q}, t)\,dt \; ;$$) If the system occupies a position $q^{(1)}$ & $q^{(2)}$ at time t_1 & t_2 then the motion is s.t. S is extremized (Ask about the complete path problem)

Derivation of equations of motion.

Let $q = q(t)$ be the f^n for which S is minimum. This implies if $q(t)$ is replaced by $q(t) + \delta q(t)$, S will increase (where $\delta q(t)$ is small for $t \in (t_1, t_2)$) $\delta q(t)$ is called a variation of $q(t)$.

Also, we don't wish for ~~q~~ $q(t_1)$ & $q(t_2)$ to change. Thus we have $\delta q(t_1) = \delta q(t_2) = 0$.

Now how does S change,
$$\int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta\dot{q}, t)\,dt - \int_{t_1} L(q, \dot{q}, t)\,dt$$

Upon expansion in powers of δq & $\delta \dot{q}$, ~~these~~ the leading terms are of first order. Thus, the necessary condition for minima (or extrema) is that these terms are zero.

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\,\delta q + \underbrace{\frac{\partial L}{\partial \dot{q}}\,\delta \dot{q}} \right) dt = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\,dt = \delta S$$

Now the condition mathematically is $\delta S = 0$

Integrating by parts the second term, we have (note $\delta \dot{q} = \frac{d}{dt}(\delta q)$)

$$\delta S = \cancel{\left[\frac{\partial L}{\partial \dot{q}}\,\delta q \right]_{t_1}^{t_2}}^{\,0} + \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q\,dt = 0$$

Thus we have
$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) = 0 \quad \forall\, \delta q$$

For $s > 1$,
$$\left. \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \quad \right] \leftarrow \text{Lagrange Equations}$$

These are a set of s 2nd order differential equations in $q_i(t)$. Thus 2s constants will appear which can be found from initial condition such as all q_i & $\dot{q}_i$ at $t=t_0$.

ASK about lim $L = L_A + L_B$

Units remark

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(q, t)$$

(why not $\dot{q}$?)
∵ only $\delta q(t_1) = \delta q(t_2) = 0$

Let this be a new lagrangian.

$$S' = \int_{t_1}^{t_2} L'(q, \dot{q}, t)\,dt = \int_{t_1}^{t_2} L(q, \dot{q}, t)\,dt + f(q^{(2)}, t_2) - f(q^{(1)}, t_1)$$

$$= S + f(q^{(2)}, t_2) - f(q^{(1)}, t_1)$$

$$\delta S' = \delta S + \delta\left(f(q^{(2)}, t_2) - f(q^{(1)}, t_1)\right)$$

$$= \delta S + \Big(\big(f(q^{(2)} + \delta q(t_2), t_2) - f(q^{(2)}, t_2)\big) - \big(f(q^{(1)} + \delta q(t_1), t_1) - f(q^{(1)}, t_1)\big)\Big)$$

$$= \delta S$$

Thus both Lagrangians yield the same equations and are therefore defined upto an arbitrary aditive total time derivative of any f^n of coordinates & time.

§3. Galileo's Relativity Principle

TODO: Fill in the details

Claim: $\exists$ always, a frame of reference in which space is homogeneous & isotropic & time is homogeneous. This is called an inertial frame. In particular, a free body in this frame, which is at rest, will always remain at rest.

Lagrangian of a free particle

From the homogenity of space & time (in the inertial frame) we have
$\rightarrow$ The Lagrangian can't depend on the radius vector $\vec{r}$, nor on t.
Thus the Lagrangian must be a f^n of $\vec{v}$ only.
From isotropy of space, the Lag must be a f^n of its magnitude only, viz $\vec{v}^2 = v^2$

$$L = L(v^2)$$

Now since L is independent of $\vec{r}$, we have $\dfrac{\partial L}{\partial \vec{r}} = 0$
(derivative wrt a vector means a vector whose components are derivatives of the ~~Quantity~~ scalar with the components of the vector)
Thus Lagrange's Eq^{n}s become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) = 0 \Rightarrow \frac{\partial L}{\partial \vec{v}} = \text{const.}$$

Now since L is a f^n of $\vec{v}^2$ only, it follows that

$$\vec{v} = \text{constant}$$

Claim: Not only are the laws of motion of a free particle same, ~~two~~ the 2 frames are equivalent in all mechanical respects, where the first is an inertial frame & the second that moves with a const. velocity wrt the first.

This is Galileo's relativity principle.

$\vec{r}' = \vec{r}' + \vec{V}t$ & $t' = t$ are the Galilean transformations.

§4. The Lagrangian ~~of~~ for a free particle

Aim: To find the form of dependence of L on $\vec{v}^2$ for a free particle.

Consider a frame K' & K (both inertial). K is moving with velocity $\vec{\epsilon}$ wrt K'. Then

$$\vec{v}' = \vec{v} + \vec{\epsilon} \quad \Rightarrow \quad \vec{v}'^2 = v^2 + \epsilon^2 + 2\vec{v}\cdot\vec{\epsilon}$$

Since the equations of motion must have the same form in every inertial ~~where~~ frame, ~~and~~ ~~have~~, $L(v^2)$ must be transformed into $L'(v'^2)$, ~~if at all~~ and should be different from $L(v^2)$, if at all, only by a ~~tota~~ total time derivative of a f^n of position & time.

Thus, $\quad L' = L(v'^2) = L(v^2 + 2\vec{v}\cdot\vec{\epsilon} + \epsilon^2)$

Neglecting higher powers of ϵ & expanding, we have

$$L(v'^2) = L(v^2) + \underbrace{\frac{\partial L}{\partial v^2}\, 2\vec{v}\cdot\vec{\epsilon}} + \mathcal{O}(\epsilon^2)$$

This term is a total derivative of time, only if $\frac{\partial L}{\partial v^2}$ is independent of v.

i.e. $L = \frac{1}{2} m v^2$

$\Rightarrow \quad L'(v'^2) = \frac{1}{2} m v'^2$

ASK: How from this infinitesimal case it follows that even for ~~infinitesimal~~ finite, rel velocities, the Lagrangian is invariant?

$$L' = \frac{1}{2} m v'^2 = \frac{1}{2} m(\vec{v} + \vec{V})^2 = \frac{1}{2} m v^2 + m\vec{v}\cdot\vec{V} + \frac{1}{2} m V^2$$

or

$$L' = L + \frac{d}{dt}\left(m\vec{x}\cdot\vec{V} + \frac{1}{2} m V^2 t\right)$$

The second term may be ommitted.

The const. m depends on units of mass measurement.

From the additive property of ~~th~~ Lagrangians we get

$$L = \sum \frac{1}{2} m_a v_a^2$$

Note: m is meaningful only under addition. ~~Else the~~ Its ratios of masses matter physically, thus different units exist.

Also, $m > 0$ else for $m < 0$, (we any know

$$S = \int_{t_1}^{t_2} \frac{1}{2} m v^2 \, dt \quad)$$

S could take arbitrarily small values.

It is useful to note that $v^2 = \left(\frac{dl}{dt}\right)^2 = \frac{dl^2}{dt^2}$ (Figure: why this holds)

§ 5. The Lagrangian for a System of Particles

Closed System ≡ A system of particles which interact with one another but with no other bodies.

Claim: Interaction b/w particles can be described by adding to the Lagrangian of the non-interacting particles, a certain f^n of co-ordinates which depends on the nature of the interaction. ~~Do~~

$$L = \underbrace{\sum \tfrac{1}{2} m_a v_a^2}_{\substack{\text{III} \\ T}} - \underbrace{U(\vec{r}_1, \vec{r}_2 \ldots)}_{\text{III}}$$

The Kinetic Energy — The Potential Energy

(Note the instaneity of prop. of interactions.)

Remarks: $t \to -t$

U is defined to a const | particular case of $\frac{d}{dt} f(q,t)$ | defined s.t. $U \to 0$ as dist b/w particles $\to \infty$.

If arbitrary generalized coordinates are used to describe instead of cartesian, the motion, ~~then the new~~

$$x_a = f_a(q_1, q_2 \cdots q_s) \quad , \quad \dot{x}_a = \sum_k \frac{\partial f_a}{\partial q_k} \dot{q}_k$$

then the new Lagrangian upon substitution becomes from

$$L = \frac{1}{2} \sum m_a (\dot{x}_a^2 + \dot{y}_a^2 + \dot{z}_a^2) - U \quad \text{to}$$

$$L = \frac{1}{2} \sum_{ik} a_{ik}(q) \dot{q}_i \dot{q}_k - U(q) \quad \text{where } a_{ik} \text{ is a } f^n \text{ of co-ordinates only.}$$

Now we consider a system A which is not closed. It interacts with another system B that's executing a given motion.
We can find the equation of motion for the system A by using L for A & B as a whole, and varying q_A & replacing the known soln. of ~~q_A~~ q_B as f^ns of time.
Assuming A+B to be closed, we have

$$L = T_A(q_A, \dot{q}_A) + T_B(q_B, \dot{q}_B) - U(q_A, q_B)$$

T_B is a f^n of time only (after substituting for $q_B(t)$) & can be therefore ommitted.

$$L = T_A(q_A, \dot{q}_A) - U(q_A, q_B(t))$$

The form is the same, except the potential energy becomes a f^n of time, in general.

A Uniform field ≡ The same force $\vec{F}$ acts on a particle, independent of its location.

U for such a field is $-\vec{F}\cdot\vec{r}$