§ 11. Motion in one dimension

Motion of a system with one degree of freedom is said to take place in one dimension. The most general Lagrangian for such a system would be

$$L = \frac{1}{2} a(q)\, \dot{q}^2 - U(q)$$

where $a(q)$ is some f^n of q.

If q is the cartezian co-ordinate, then

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

We needn't even write the eq^{n}s of motion. Lets start with Energy eq^n (1^{st} integral)

$$\frac{1}{2} m \dot{x}^2 + U(x) = E \Rightarrow \frac{dx}{dt} = \left(\frac{2(E - U(x))}{m} \right)^{1/2}, \text{ thus}$$

$$t = \sqrt{\tfrac{1}{2} m} \int \frac{dx}{\sqrt{[E - U(x)]}} + \text{const.}$$

The two const are E & the "const".

Now since $\frac{1}{2} m \dot{x}^2 \geq 0$, thus $E \geq U(x)$

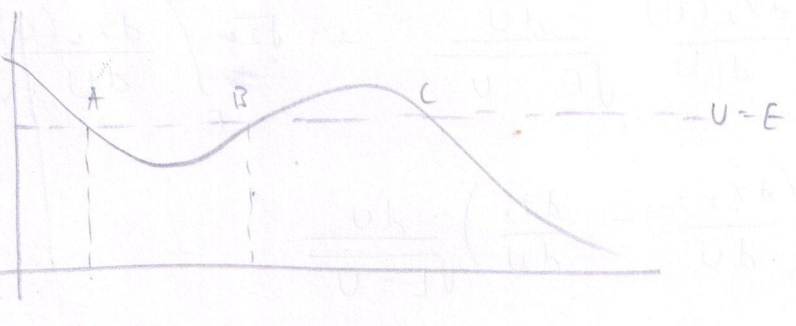
When $U(x) = E$, we get the limits of the motion. These points are called turning points

Finite $\equiv$ When the region of motion is bounded by two such points, this is what the motion is said to be.

Infinite $\equiv$ If the region of motion is limited on only one side, or neither, then the motion is said to be this.

In 1-Dimension, a finite motion is oscillatory. The time period then is twice the time to go from x_1 & x_2 (clear from both 1) the integral & 2) the time reversibility)

$$T(E) = \sqrt{2m} \int_{x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{E - U(x)}}$$ where x_1 & x_2 are roots of $U(x) = E$.



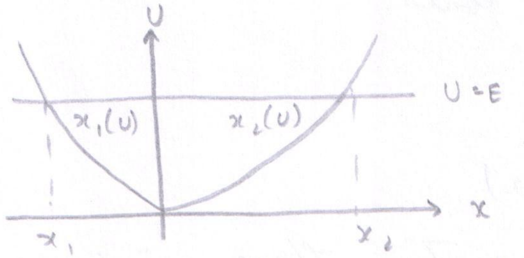
§12. Determination of the potential energy from the period of oscillation

Given period of oscillation T as a f^n of E, we wish to find $U(x)$. Mathematically then, we want to solve

$$T(E) = \sqrt{2m}\int_{x_1(E)}^{x_2(E)} \frac{dx}{\sqrt{E-U(x)}}$$, with $T(E)$ as known & $U(x)$ as unknown.

→ We start with assuming that $U(x)$ has only one ~~root~~. minima in the region (x_1, x_2). (We aren't considering solns. that may exist which don't satisfy the said condition)

→ Further, for convenience, we choose the origin at the position of the minima. ~~(Why can we do this? THINK)~~
Intuitively its clear; Mathematically, so long as the field is not dependent on time, you can write the energy conservation in the translated co-ordinates also)



Further we take this minima to be zero.
∵ The eq^ns of motion don't depend on arbitrary constants added to U.
(but $T(E)$ seems to ... THINK)

→ In the integral, we regard x as a f^n of U (unlike earlier, where we regarded U as a f^n of x).
Also, $x(U)$ is two-valued as is clear (from the graph). Thus, we write ~~x(U)~~

$$x(U) = \begin{cases} x_1(U) & x_1 < x < 0 \\ x_2(U) & 0 < x < x_2 \end{cases}$$

Further, we replace dx by $\left(\frac{dx}{dU}\right) dU$.

→ The limits of integration in terms of U, evidently become E to 0 & 0 to E. (seems more like E to E to me, but thats cause we've broken it to $E, 0$ then $0, E$)

$$\Rightarrow T(E) = \sqrt{2m}\int_0^E \frac{dx_2(U)}{dU}\frac{dU}{\sqrt{E-U}} + \sqrt{2m}\int_E^0 \frac{dx_1(U)}{dU}\frac{dU}{\sqrt{E-U}}$$

$$= \sqrt{2m}\int_0^E \left(\frac{dx_2}{dU} - \frac{dx_1}{dU}\right)\frac{dU}{\sqrt{E-U}}$$

→ Both sides are now divided by $\sqrt{\alpha - E}$ (α is a parameter) & integrated wrt E from 0 to α.

$$\Rightarrow \int_0^\alpha \frac{T(E)}{\sqrt{\alpha - E}}\, dE = \sqrt{2m} \int_0^\alpha \int_0^E \left[\frac{dx_2}{dU} - \frac{dx_1}{dU}\right] \frac{dU\, dE}{\sqrt{(\alpha - E)(E - U)}}$$

or changing the order of integration

(how still?)

$$= \sqrt{2m} \int_0^\alpha \left[\frac{dx_2}{dU} - \frac{dx_1}{dU}\right] dU \underbrace{\int_U^\alpha \frac{dE}{\sqrt{(\alpha - E)(E - U)}}}_{= \pi}$$

(why though?)

Thus we get

$$\int_0^\alpha \frac{T(E)\, dE}{\sqrt{\alpha - E}} = \pi\sqrt{2m}\left[\left(x_2(\alpha) - x_1(\alpha)\right) - \cancelto{0}{\left(x_2(0) - x_1(0)\right)}\right]$$

Replacing α with U (x_1 & x_2 are f^s of U)

$$x_2(U) - x_1(U) = \frac{1}{\pi\sqrt{2m}} \int_0^U \frac{T(E)\, dE}{\sqrt{U - E}}$$

Note / Remark: $x_2(U)$ & $x_1(U)$ are still indeterminate. ~~If we~~ Thus there are infinitely many $U = U(x)$ which satisfy the time period dependence on E.

: If we assume that the curve $U = U(x)$ is symmetric about $x = 0$ (U-axis), $\Rightarrow x_1(U) = -x_2(U) \equiv x(U)$

Then we get

$$x(U) = \frac{1}{2\pi\sqrt{2m}} \int_0^U \frac{T(E)\, dE}{\sqrt{U - E}}$$

(space for remarks on questions I'm stuck at)

§13. The reduced mass

The potential energy of the interaction of 2 particles depends only on the distance b/w them; ~~i.e~~ the Lagrangian is therefore

$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(|\vec{r}_1 - \vec{r}_2|)$$

→ We let $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$.

→ Let the origin be at the COM. Thus $m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$

use these to get $\vec{r}_1 - \vec{r} = \vec{r}_2$ $\Rightarrow m_1 \vec{r}_1 + m_2 \vec{r}_1 = m_2 \vec{r}$, $\vec{r}_1 = \frac{m_2 \vec{r}}{m_1 + m_2}$

$$\Rightarrow \quad \vec{r}_1 = \frac{m_2 \vec{r}}{m_1 + m_2} \qquad \vec{r}_2 = \frac{m_1 \vec{r}}{m_1 + m_2}$$

(minus?)

substitute in the lagrangian

$$\Rightarrow \quad L = \frac{1}{2} m \dot{\vec{r}} - U(r)$$

$$\text{where} \quad m \equiv \frac{m_1 m_2}{(m_1 + m_2)} \doteq \text{reduced mass}$$

Remark: This reduces the 2 body problem's ~~into~~ Lagrangian to that of a single particle in an ~~a~~ external field, symmetric about a fixed origin.

Remark 2: The soln. $\vec{r}(t)$ entails solns. $\vec{r}_1(t) = \frac{m_2 \vec{r}(t)}{m_1 + m_2}$ & $\vec{r}_2(t)$.

§14. Motion in a central field

→ The central field is one in which the potential energy depends only on the distance of r of ~~to~~ the particle from some fixed origin.

The force is thus

$$\vec{F} = -\frac{\partial U(r)}{\partial \vec{r}} = -\frac{\partial U(r)}{\partial r}\left(\frac{\vec{r}}{r}\right)$$

$$\because \frac{\partial U\left((x^2+y^2+z^2)^{1/2}\right)}{\partial x} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial U}{\partial r}\frac{x}{r} \text{ and so on}$$

→ As we'd already seen, angular momentum defined wrt the centre of the field, will ~~have~~ be conserved.

The angular momentum of a single particle is $\vec{M} = \vec{r} \times \vec{p}$. Since M is const, ⇒ the radius vector is always in a plane ⊥ M. Thus the motion is always constrained to a plane.

→ Using polar co-ordinates for a plane,

$$L = \frac{1}{2} m(\dot{r}^2 + r^2 \dot{\phi}^2) - U(r)$$

Note: L is independent of ϕ (explicitly)

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} = 0$$

Cyclic ≡ Any generalized co-ordinate that doesn't appear explicitly in the Lagrangian is called this.

⇒ The corresponding generalized momentum is an integral of motion.

Here, $p_\phi = M_z$ [THINK: This I know that $M_z = \frac{\partial L}{\partial \dot\phi}$ but it also the generalized momentum p_ϕ. How else can I show $M_z = p_\phi$? What is the general relation b/w them?]

From the law of conservation of ang. momentum, we have $M = mr\dot\phi = \text{const}.$

Geometrically, the area of a sector bound by two neighbouring radius vectors is

$\left(\frac{1}{2} r\right)(r\,d\phi) = df$

½ (base)(height)

$\Rightarrow \quad M = 2m\dot f$ where $\dot f$ ≡ the sectorial velocity.

Constancy of M implies $\dot f$ is const. (Keppler's second law)

The soln. to the central field problem

We use conservation of 1) Energy & 2) Angular Momentum

So,
$$E = \frac{1}{2} m(\dot r^2 + r^2\dot\phi^2) + U(r) = \frac{1}{2} m\dot r^2 + \frac{1}{2}\frac{M^2}{mr^2} + U(r)$$

$$\Rightarrow \quad \dot r \equiv \frac{dr}{dt} = \sqrt{\frac{2}{m}[E-U(r)] - \frac{M^2}{m^2r^2}}$$

$$\Rightarrow \quad t = \int \frac{dr}{\sqrt{\frac{2}{m}[E-U(r)] - \frac{M^2}{m^2r^2}}} + \text{const}.$$

Also, from $M = mr\dot\phi$, $d\phi = \frac{M\,dt}{mr^2}$

$$\Rightarrow \quad \phi = \int \frac{\frac{M}{r^2}\,dr}{\sqrt{2m[E-U(r)] - \frac{M^2}{r^2}}}$$

Remark: ϕ varies monotonically with time, since $\dot\phi$ = const ⇒ $\dot\phi$ never changes sign.

Further, the radial part of the motion can be taken as the motion in one dimension with

$$U_{eff} = U(r) + \frac{M^2}{2mr^2}$$

$\frac{M^2}{2mr^2}$ 'is called' $\equiv$ centrifugal energy.

Roots of $E = U(r) + \frac{M^2}{2mr^2}$ yield the limits of the radial motion; the distance from the ~~cr~~ centre.

Note: This $\not\Rightarrow$ that the particle is at rest when $\dot{r} = 0$ $\because$ $\dot{\phi} \neq 0$.

Case 1: r has only 1 bound, $r \geq$ min.
The particle comes from & goes to infinity.

Case 2: r is bounded: $r_{min} < r < r_{max}$
The path is bounded by corresponding circles but $\not\Rightarrow$ the path is closed. That happens in specific when $\Delta\phi = 2\pi \frac{m}{n}$ $(m, n \in I)$

where $\Delta\phi = 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E-U) - \frac{M^2}{r^2}}}$ why the factor of 2? for a complete revolution, $r_{max} - r_{min} - r_{max}$.

After n revolution, the radial part would've completed m revolution, thus the curve would close.
(look at the image given in the text)

Claim: $\exists$ only two types of central fields in which all finite motions take place in closed paths.

Remark on path construction from repeating segments: Read from the text. (It's fun)

Remark: When $M \neq 0$, (rather obvious)

From energy conservation, we have

$$\frac{1}{2} m \dot{r}^2 = E - U(r) - \frac{M^2}{2mr^2} > 0$$

$$\Rightarrow \quad r^2 U(r) + \frac{M^2}{2m} < E r^2$$

Now r can take values tending to zero if

$$\left[r^2 U(r) \right]_{r \to 0} < - \frac{M^2}{2m}$$

viz. $U(r)$ must tend to $-\infty$ either as $-\frac{\alpha}{r^2}$ with $\alpha > \frac{M^2}{2m}$

or proportionally to $-\frac{1}{r^n}$, $n > 2$.

§ 15. Kepler's Problem