II. Conservation Laws

§ 6. Energy

Integrals of Motion ≡ Functions of the $2s$ quantities (s is the no. of DOF) whose values remain constant during the motion.

Claim: The no. of independent integrals of motion for a closed mechanical system ~~is~~ (with s DOF) is $2s-1$.

(look at the text for details. based on $t+t_0$)

Remark: Some constants of motion that arise from homogenity & isotropy of space & time, are of special importance as they 'tend to be' additive.

Homogenity of Time

By virtue of this homogenity, the Lagrangian doesn't explicitly depend on time.

$$\Rightarrow \quad \frac{dL}{dt} = \sum \frac{\partial L}{\partial q_i}\dot{q}_i + \sum \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$

$$= \sum \dot{q}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \sum \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$

$$= \sum \frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) = \frac{d}{dt}\sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right) = 0$$

$$E \equiv \sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

Note: The additivity follows from linearity in L of E.

Conservative Systems ≡ Mechanical systems whose energy is conserved. ~~are~~

Remark: The result is valid (viz E is const) even for a time independent external field. ($\frac{\partial L}{\partial t}$ is still $= 0$)

The Lagrangian of a closed system is of the form
$L = T(q,\dot{q}) - U(q)$, where T is a quadratic f^n of the velocities.

Now from Euler's theorem (figure this)

$$\sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \sum \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T$$

(I can verify this for a special case though)

Substituting this ~~for~~ in the expression for energy, we get (and $L = T(q,\dot{q}) - U(q)$)

$$E = T(q,\dot{q}) + U(q)$$

In Cartesian of course then,

$$E = \sum \frac{1}{2} m_a v_a^2 + U(\vec{r}_1, \vec{r}_2, \ldots)$$

§ 7. Momentum

Homogenity of space

A parallel displacement by $\vec{\epsilon}$ (infinitesimal) should leave the Lagrangian unchanged. Lets ~~that~~ find the condition required.

$$\vec{r} \rightarrow \vec{r} + \vec{\epsilon}$$

$$\delta L = \sum \frac{\partial L}{\partial \vec{r}_a} \delta \vec{r}_a = \vec{\epsilon} \cdot \sum \frac{\partial L}{\partial \vec{r}_a}$$

$$\Rightarrow \quad \delta L = 0 \Rightarrow \sum \frac{\partial L}{\partial \vec{r}_a} = 0$$

$$\Rightarrow \sum \frac{d}{dt} \frac{\partial L}{\partial \vec{v}_a} = \frac{d}{dt} \sum \frac{\partial L}{\partial \vec{v}_a} = 0$$

$\Rightarrow \quad \vec{P} \equiv \sum \frac{\partial L}{\partial \vec{v}_a}$ is conserved for a closed system.

If $L = \sum \frac{1}{2} m_a v_a^2 - U(\vec{r}_1, \vec{r}_2, \ldots)$ we get

$$\vec{P} = \sum m_a \vec{v}_a$$

Remark: The momentum of the system is the sum of momentum of its individual parts $p_a = m_a \vec{v}_a$ (unlike energy in the sense of ~~the~~ whether there's an interaction or not)

Remark 2: All 3 components of $\vec{P}$ are conserved in the absence of external field. If $\exists$ an external field, then

{ If the P.E. in the field doesn't depend on all the Cartezian co-ordinates then

{ Those co-ordinates' corresponding momentum is conserved }

}

Remark 2(b): Physically, $\frac{\partial L}{\partial \vec{r}_a} = -\frac{\partial U}{\partial \vec{r}_a} = \vec{F}_a$ (The force on the a^{th} particle)

The conservation shows that the net force in a closed system is zero. $(\sum \vec{F}_a = 0)$

Generalized Momenta: (If the motion is described by generalized co-ordinates q_i, then) $p_i = \frac{\partial L}{\partial \dot{q}_i}$

Generalized Forces: $F_i = \frac{\partial L}{\partial q_i}$

And the Lagrange's eqn's become $\dot{p}_i = F_i$

Remark: In cartesian co-ordinates, the Generalized Momenta are components of $\vec{P}_a$.

However, in general, p_i are linear homogeneous f^ns of the generalized velocities $\dot{q}_i$, and do not reduce to products of mass & velocity

§ 8. Centre of Mass

(Remark) The momentum of a closed system has a different value in different inertial frames of reference.

Let K' & K be two inertial frames. K' moves with $\vec{V}$ wrt. K. For a particle then

$$\vec{v}_a = \vec{v}'_a + \vec{V}$$

Thus the relation b/w the moments may be given as

$$\vec{P} = \sum m_a \vec{v}_a = \sum m_a \vec{v}_a' + \vec{V} \sum m_a$$

$$\boxed{\Rightarrow \quad \vec{P} = \vec{P}' + \vec{V} \sum m_a}$$

Thus, $\exists$ a frame in which the total momentum is zero.

If we put $\vec{P}' = 0$ for this condition, we get

$$\vec{V} = \vec{P} / \sum m_a = \frac{\sum m_a \vec{v}_a}{\sum m_a}$$

Generalizing the concept of rest & velocity to a system:

At rest $\equiv$ If the total momentum of a mechanical system is zero in a given frame of reference, the system is said to be so (at rest) relative to the said frame.

Remark: The velocity $\vec{V}$ calculated above, gives the velocity of the "system as a whole"!

From $\vec{V}$, we can write

$$\vec{R} \equiv \frac{\sum m_a \vec{r}_a}{\sum m_a} \qquad \text{so that } \vec{V} = \dot{\vec{R}}$$

Centre of Mass $\equiv$ The point at which $\vec{R}$ points. ~~is called the~~

Remark: The law of conservation of momentum is equivalent to saying that the COM moves with a const. velocity (along a straight line) ← redundant

Internal Energy $\doteq E_i \equiv$ Energy of a mechanical system in a frame ~~at~~ in which its at rest.

Remark: E_i includes the Kinetic Energy of the relative motion of the particles in the system & the potential energy of their interaction.

Claim: The total Energy (when the system's moving with $\vec{V}$ as a whole) is

$$E = \frac{1}{2}\mu V^2 + E_i$$

Proof: Consider the same old frames, K & K'

$$\boxed{E} = \frac{1}{2}\sum_a m_a v_a^2 + U$$

$$= \frac{1}{2}\sum_a m_a (\vec{v}_a' + \vec{V})^2 + U$$

$$= \frac{1}{2}\mu V^2 + \vec{V}\cdot\sum_a m_a \vec{v}_a' + \frac{1}{2}\sum_a m_a v_a'^2 + U$$

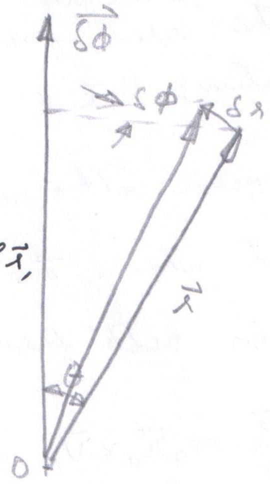
$$\boxed{= E' + \vec{V}\cdot\vec{P}' + \frac{1}{2}\mu V^2}$$

Now if K' is the rest frame, $\vec{P}' = 0$ & $E' = E_i$ & this proves it.

§ 9. Angular Momentum

Isotropy of space |

$\vec{\delta\phi} = \delta\phi\ \hat{o}$ ↳ axis of rotation, angle of rotation



Observe $\delta r = r\sin\theta\,\delta\phi$

and since δr is $\perp$ to the plane containing $\vec{\delta\phi}$ & $\vec{r}$,

$$\vec{\delta r} = \vec{\delta\phi}\times\vec{r}$$

$$\Rightarrow \vec{\delta v} = \vec{\delta\phi}\times\vec{v}$$

If these are substituted in

$$\delta L = \sum_a\left(\frac{\partial L}{\partial \vec{r}_a}\cdot\delta\vec{r}_a + \frac{\partial L}{\partial \vec{v}_a}\cdot\delta\vec{v}_a\right) = 0$$

the condition for invariance of the Lagrangian on rotation, we get

(replace also $\frac{\partial L}{\partial \vec{v}_a} = \vec{p}_a$ & $\frac{\partial L}{\partial \vec{r}_a} = \dot{\vec{p}}_a$)

~~We get~~ $$\sum_a\left(\dot{\vec{p}}_a\cdot\vec{\delta\phi}\times\vec{r}_a + \vec{p}_a\cdot\delta\phi\times\vec{v}_a\right) = 0$$

$$\Rightarrow \vec{\delta\phi}\cdot\sum_a\left(\vec{r}_a\times\dot{\vec{p}}_a + \vec{v}_a\times\vec{p}_a\right) = 0 = \vec{\delta\phi}\cdot\frac{d}{dt}\sum_a\vec{r}_a\times\vec{p}_a$$

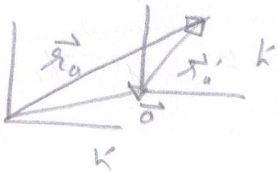
$\Rightarrow$ $M \equiv \sum_a \vec{r}_a\times\vec{p}_a$ is conserved. (The Angular Momentum)

This is also additive like momentum, regardless of interaction.

Claim: There aren't any more additive integrals of motion.
Thus every closed sys has seven such integrals; 3 comp. of momentum, 3 of angular momentum & energy.

Remark: Value of $\vec{M}$ depends on the choice of origin (because $\vec{r}$ is used)

Let $\vec{r}_a$ & $\vec{r}'_a$ be the radius vectors of a point in frames K & K', with origins related by $\vec{a}$ as shown.
Then $\quad \vec{r}_a = \vec{a} + \vec{r}'_a$



So then
$$\vec{M} = \sum_a \vec{r}_a \times \vec{p}_a$$
$$= \sum (\vec{r}'_a + \vec{a}) \times \vec{p}_a = \sum \vec{r}'_a \times \vec{p}_a + \vec{a} \times \sum \vec{p}_a$$
$$= \vec{M}' + \vec{a} \times \vec{P}$$

Notice how $\vec{M}$ becomes independent of the choice of origin if K (and therefore K', or vice versa) is the rest frame of the system ($\vec{P} = 0$)
(clarification; K & K' are at rest wrt each other)

what if $\vec{a} = \vec{a}(t)$?

Let K' move with a velocity $\vec{V}$ wrt the frame K. Also, assume that instant when $\vec{a} = 0$. ~~Thus~~ Thus the radius vectors will be the same in both frames, while $\vec{v}_a = \vec{v}_a' + \vec{V}$, thus

$$\boxed{\vec{M} = \sum_a m_a \vec{r}_a \times \vec{v}_a = \sum_a m_a \vec{r}_a \times \vec{v}_a' + \sum_a m_a \vec{r}_a \times \vec{V} = \vec{M}' + \mu \vec{R} \times \vec{V}}$$

$$\left(\because \vec{R} \equiv \frac{\sum m_a \vec{r}_a}{\sum m_a} \ \& \ \mu = \sum m_a\right)$$

Further if K' is the rest frame of the system, then
$$\mu \vec{V} = \vec{P}, \text{ thus we get}$$

$$\vec{M} = \underbrace{\vec{M}'}_{\text{The intrinsic angular momentum.}} + \underbrace{\vec{R} \times \vec{P}}_{\text{Angular momentum due to the motion as a whole.}}$$

Remark:
All three components of the Angular Momentum will be conserved, so long as the system is closed.

Claim:
If the system is not closed, but $\exists$ ~~the~~ a field that is symmetric about some axis, then the angular momentum along this axis will be conserved. (granted the origin is on the ~~axis of~~ said axis)

Proof: The Lagrangian ~~along the span~~ is unaffected by rotation about the axis of symmetry of the field, the only consideration that yielded the result. (conservation of angular momentum)

Eg. 1) Centrally Symmetric Field: Potential depends on the dist. from a fixed 'centre'.

Here angular momentum passing through the 'centre' will be conserved.

2) Homogeneous Field in z-direction: Here M_z will always be conserved, regardless of point of origin.

Further, we have

$$M_z = \sum \frac{\partial L}{\partial \dot{\phi}_a}$$

where ϕ_a is the angle of rotation about the z-axis.

(How does it follow from our previous discussion?)

Alternate proof: In cylindrical co-ordinates, we have r, ϕ, z as coordinates

$$(x_a = r_a \cos\phi_a, \ y_a = r_a \sin\phi_a)$$

$$M_z = \sum_a m_a (x_a \dot{y}_a - y_a \dot{x}_a) \qquad \left(\text{using } \sum m_a r_a \times v_a = \vec{M}\right)$$

$$= \sum_a m_a r_a^2 \dot{\phi}_a$$

The Lagrangian in these co-ordinates then is (using $dl^2 = dr^2 + r^2 d\phi^2 + dz^2$)

$$L = \frac{1}{2} \sum_a m_a (\dot{r}_a^2 + r_a^2 \dot{\phi}_a^2 + \dot{z}_a^2) - U$$

& M_z is indeed $= \dfrac{\partial L}{\partial \dot{\phi}_a}$

§ 10. Mechanical Similarity

Note:

Multiplication of the Lagrangian by a const. doesn't affect the eq^n of motion. (This can be harnessed in a powerful way)

Consider cases where the potential energy is a homogeneous f^n of the co-ordinates, viz.

$$U(\alpha \vec{r}_1, \alpha \vec{r}_2 \ldots, \alpha \vec{r}_n) = \alpha^K U(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n)$$

where α is a const., & K is the degree of homogenity.

Let's perform the following transformations:

$$\vec{r}_a \to \alpha \vec{r}_a \quad \& \quad t \to \beta t$$

Then all $\vec{v}_a = \frac{d\vec{r}_a}{dt} \to \frac{\alpha}{\beta} \vec{v}_a$ & therefore $T \to \left(\frac{\alpha}{\beta}\right)^2 T$

and as discussed (and assumed) $U \to \alpha^K U$

Now for equations of motion to be unaltered, we must have the Lagrangian multiplied by a const. viz. $\left(\frac{\alpha}{\beta}\right)^2 = \alpha^K$

$$\Rightarrow \beta = \alpha^{1-\frac{1}{2}K}$$

(This would result in multiplying the Lagrangian by a factor of α^K)

<near magic, plaussible anyway>

$$\frac{t'}{t} = \beta = \alpha^{1-\frac{1}{2}K} = \left(\frac{l'}{l}\right)^{1-\frac{1}{2}K}$$

similarly for velocities, we can write

$$\frac{v'}{v} = \frac{l'/t'}{l/t} = \frac{l'}{l} \cdot \frac{t}{t'} = \frac{l'}{l}\left(\frac{l}{l'}\right)^{1-\frac{1}{2}K} = \left(\frac{l'}{l}\right)^{\frac{1}{2}K}$$

then for energy we'll have

$$\frac{E'}{E} = \alpha^K = \left(\frac{l'}{l}\right)^K \quad \Big| \quad M = \frac{l'v'}{lv} = \left(\frac{l'}{l}\right)\left(\frac{l'}{l}\right)^{\frac{1}{2}K} = \left(\frac{l'}{l}\right)^{1+\frac{1}{2}K}$$

Let look at some applications.

1) Small Oscillations: The potential's quadratic in the pos. coordinate.
$\Rightarrow k = 2$. (using $\frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-\frac{1}{2}k}$)
This itself shows that the time period of such oscillations is independent of their amplitude.

2) Uniform Field: The potential Energy's a linear f^n of the co-ordinate, i.e. $k = 1$.
Then $\frac{t'}{t} = \sqrt{\frac{l'}{l}}$, like the time of free fall is goes as square root of the initial height.

3) Newtonian Attraction or Coulomb Attraction : $k = -1$

$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{3/2} \quad \text{(Kepler's Third law)}$$

Virial Theorem

Assumption: The potential energy is a ~~f^n of~~ homogeneous f^n of coordinates.

(From Euler's theorem on homogeneous f^ns, we have
$$\sum v_a \cdot \frac{\partial T}{\partial v_a} = 2T)$$

with $\frac{\partial T}{\partial \vec{v}_a} = \vec{p}_a$, $\quad 2T = \sum \vec{p}_a \cdot \vec{v}_a = \frac{d}{dt}\left(\sum \vec{p}_a \cdot \vec{r}_a\right) - \sum \vec{r}_a \cdot \dot{\vec{p}}_a$

$$= \underbrace{\frac{d}{dt}\left(\sum \vec{p}_a \cdot \vec{r}_a\right)}_{} + \sum \vec{r}_a \cdot \frac{\partial U}{\partial \vec{r}_a}$$

→ This drops out of the avg.

$\therefore$

$$\bar{f} = \lim_{\tau \to \infty} \frac{1}{\tau}\int_0^\tau f(t)\,dt$$

If $f(t) = \frac{d F(t)}{dt}$ & $F(t)$ is bounded, then

$$\bar{f} = \lim_{\tau \to \infty} \frac{F(\tau) - F(0)}{\tau} = 0$$

Now we take time average to obtain

$$2\bar{T} = \overline{\sum \vec{r}_a \cdot \frac{\partial U}{\partial \vec{r}_a}} = k\bar{U}$$

$2\bar{T} = k\bar{U}$ can be further written as

$$\bar{U} = \frac{2E}{k+2}, \quad \bar{T} = \frac{kE}{(k+2)}$$

using $\bar{T} + \bar{U} = \bar{E} = E$