THE CANONICAL EQUATIONS

§40 Hamilton's Equations

Motivatⁿ: We assumed that the mechanical state is described by $\{x_i, v_i\}$.

Sometimes it is better described as

$\{x_i, p_i\}$ canonical momentum.

$\therefore$ equations in x, p?

Lagandre Transfⁿ: Means by which one passes from one set of independent variables to another.

Conversion (in this case):

NB: $dL = \sum_i \frac{\partial L}{\partial q} dq_i + \sum \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i$

Recall: $p_i = \frac{\partial L}{\partial \dot{q}_i}$ (by defⁿ)

$\frac{\partial L}{\partial q_i} = \dot{p}_i$ (by Lagrange eqⁿs)

$$\Rightarrow \quad dL = \sum \dot{p}_i dq_i + \sum p_i d\dot{q}_i \qquad (*)$$

NB: $\sum p_i d\dot{q}_i = d\left(\sum p_i \dot{q}_i\right) - \sum \dot{q}_i dp_i$

$$\Rightarrow \quad -\sum \dot{p}_i dq_i + \sum \dot{q}_i dp_i = d\left(p_i \dot{q}_i - L\right) \qquad (1)$$

Recall: $H = p_i \dot{q}_i - L$ is the energy of the syst.

Def^n: Hamiltonian or Hamilton's F^n := Energy expressed in terms of (q, p).

NB from (1): $dH = -\sum \dot{p}_i \, dq_i + \sum \dot{q}_i \, dp_i$

$$\left.\begin{aligned} -\frac{\partial H}{\partial q_i} &= \dot{p}_i \\ \frac{\partial H}{\partial \dot{q}_i} &= \dot{q}_i \end{aligned}\right] \quad \text{Hamilton's Eq}^n\text{s}$$

Remark 1: Form $2s$ first-order eq^n for $2s$ unknowns (p, q) compared to s second-order eq^{n}s earlier.

Remark 2: Also called "Canonical eq^{n}s" due to symmetry & simplicity.

NB: We didn't include an explicit time dependence in L (will do something more general shortly).

NB 2:
$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \underbrace{\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i}_{\text{This using Hamilton's eq}^n \; = 0}$$

$$\Rightarrow \frac{dH}{dt} = \frac{\partial H}{\partial t}$$ manifestly, if no explicit time dependence, Energy is conserved.

Consider: Some additional parameter (encoding e.g. an external λ field etc.)

$$dL = \sum p_i dq_i + \sum p_i d\dot{q}_i + \frac{\partial L}{\partial \lambda} d\lambda$$

using this instead of (*) I would get

$$dH = -\sum \dot{p}_i dq_i + \sum \dot{q}_i dp_i - \frac{\partial L}{\partial \lambda} d\lambda$$

$$\Rightarrow \quad \left.\frac{\partial H}{\partial \lambda}\right|_{p,q} = -\left.\frac{\partial L}{\partial \lambda}\right|_{q,\dot{q}}$$

NB: λ could even be explicit time dependence

$$\Rightarrow \quad \left.\frac{\partial H}{\partial t}\right|_{p,q} = -\left.\frac{\partial L}{\partial t}\right|_{q,\dot{q}} .$$