Lagrangian Field Theory

$$S = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4x = \int L\, dt$$

$\delta S = 0$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right\}$$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right)}_{\text{surface term } \& \text{ on the surface } \delta\phi = 0.} \right\}$$

Thus we have

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

Euler-Lagrange for the field.

HAMILTONIAN FIELD THEORY

Why not] ∵ in the Lagrangian formulation, terms are explicitly Lorentz invariant.

Why] Makes transition from Quantum Mechanics easier

Note carefully (for a discrete sys) [and recall]

$$p \equiv \frac{\partial L}{\partial \dot{q}} \quad (\text{not } \mathcal{L})$$

$\left(\text{where } \dot{q} = \frac{\partial q}{\partial t}\right)$ for each dynamic var. q

Then $H \equiv \sum p\dot{q} - L$

For a continuum generalization, we define

$$\boxed{p(x) = \frac{\partial L}{\partial \dot{\phi}(x)}} = \frac{\partial}{\partial \dot{\phi}(x)} \int \mathcal{L}\left(\phi(y), \dot{\phi}(y)\right) d^3y$$

$$\left(\because \int L\,dt = \int \mathcal{L}\, d^4x\right)$$

$$\sim \frac{\partial}{\partial \dot{\phi}(x)} \sum_y \mathcal{L}\left(\phi(y), \dot{\phi}(y)\right) d^3y$$

$$= \frac{\partial \mathcal{L}\left(\phi(x), \dot{\phi}(x)\right)}{\partial \dot{\phi}(x)} d^3x$$

$\Rightarrow p(x) = \pi(x)\, d^3x$ where $\boxed{\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}}$

Thus, the Hamiltonian can be written as

$$H = \sum_x p(x)\dot{\phi}(x) - L$$

in the continuum lt.

$$\sim \boxed{\int d^3x \left(\pi(x)\dot{\phi}(x) - \mathcal{L}\right) \equiv \int d^3x\, \mathcal{H}}$$

example: (Euler Lagrange)

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2$$

$$= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$$

assume ϕ is real.

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi$$

had missed this

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

So then from Euler Lagrange we have

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi = 0$$

$$= (\partial^\mu\partial_\mu + m^2)\phi = 0$$

example: (Hamiltonian)

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$$

So

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L}$$

$$= \dot{\phi}^2 - \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2\right)$$

$$= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

$$\Rightarrow H = \int d^3x\,\mathcal{H} = \int d^3x\left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right]$$

FIELD QUANTIZATION | Revision Sudipta Notes based.

Noethers theorem : To every continuous symmetry, $\exists$ a conserved charge.

Proof : Since the transformation associated with the symmetry must be continuous, in the infinitesimal form I can write

$$\phi(x) \longrightarrow \phi'(x) = \phi(x) + \alpha \Delta\phi(x)$$

and the Lagrangian Density (d^4x will remain the same) can then at most change as

$$\mathcal{L}(x) \longrightarrow \mathcal{L}(x) + \alpha \partial_\mu J^\mu(x)$$

so that the "Equation of motion" remain the same.

Lets first evaluate the change of $\mathcal{L}$ caused by ϕ

$$\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)$$

$$= \frac{\partial\mathcal{L}}{\partial\phi}\alpha\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\cdot\alpha\partial_\mu(\Delta\phi)$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\cdot\alpha\Delta\phi\right) + \frac{\partial\mathcal{L}}{\partial\phi}\alpha\Delta\phi - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\alpha\Delta\phi$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\cdot\alpha\Delta\phi\right) + \underbrace{\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]}_{=0}\alpha\Delta\phi$$

From Eq of Motion

Also, from previous arguments.

$$\Delta\mathcal{L} = \alpha\partial_\mu J^\mu(x)$$

Thus, $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\cdot\Delta\phi\right) = \partial_\mu J^\mu \Rightarrow \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - J^\mu\right) = 0$

Thus $j^\mu = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - J^\mu$ is the conserved

noether current, viz. $\partial_\mu j^\mu = 0$

Example for Noether

Consider $\quad x^\mu \longrightarrow x^\mu - a^\mu$

This can be alternatively expressed as

$$\phi(x) \longrightarrow \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

(taylored)

The Klein-Gordan Field as Harmonic Oscillator

Idea: 1) Start with a classical theory & then interpret the dynamic variables as operators satisfying canonical commutation relation.

2) Solve for eigenstates & eigenvalues

1) Objective: Quantize

ϕ & π become the canonical variables

$$[q_i, p_j] = i\delta_{ij}$$
$$[q_i, q_j] = [p_i, p_j] = 0$$

For continuous, since $\pi(\vec{x},t)$ is the momentum density, we have

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})$$
$$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0$$

(We're doing Schrodinger picture)

$H = H(\phi, \pi)$ also becomes an operator.

2) Objective: Find the spectrum of H

Klein Gordan (Motivation)

Let

$$\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \tilde{\phi}(\vec{p},t)$$

$\vec{p}$ is not the same as the old $p = \frac{\partial L}{\partial \dot{\phi}}$ yet, nor is it & the conserved 'charge'

For Klein Gordan we have

$$\left[\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right]\phi = 0$$

$$\Rightarrow \left[\frac{\partial^2}{\partial t^2} + \left(|\vec{p}|^2 + m^2\right)\right]\tilde{\phi}(\vec{p},t) = 0$$

This is the same as the eqn of motion for a SHO with

$$\omega_p^2 = |\vec{p}|^2 + m^2$$

This we already know how to solve. So,

$$H_{SHO} = \frac{1}{2} p^2 + \frac{1}{2} w^2 \phi^2$$

$$\phi = \frac{1}{\sqrt{2w}} (a + a^+) \; ; \; p = -i \sqrt{\frac{w}{2}} (a - a^+)$$

$$[\phi, p] = i \iff [a, a^+] = 1$$

Thus

$$H = w (a^+ a + \tfrac{1}{2})$$

Now $|0\rangle$ is s.t $a|0\rangle = 0$

& $H|0\rangle = \frac{1}{2} w |0\rangle$

Also using

$$[H_{SHO}, a^+] = w a^+$$

$$[H_{SHO}, a] = -wa$$

we have

$$|n\rangle \equiv (a^+)^n |0\rangle$$

which are the eigenstates of H_{SHO} with eigenvalues $(n + \frac{1}{2}) w$.

(Claim) These states exhaust the spectrum

IDK why (cf Peskin) but we can write

$$\phi(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2 w_p}} \left(a_p e^{i \vec{p} \cdot \vec{x}} + a_p^+ e^{-i \vec{p} \cdot \vec{x}} \right)$$

(for me, this is obvious) → [actually I missed the a^+_{-p}]

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2 w_p}} (a_p + a^+_{-p}) e^{i \vec{p} \cdot \vec{x}}$$

$$\pi(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{w_p}{2}} \left(a_p e^{i \vec{p} \cdot \vec{x}} - a_p^+ e^{-i \vec{p} \cdot \vec{x}} \right)$$

$$= \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{w_p}{2}} (a_p - a^+_{-p}) e^{i \vec{p} \cdot \vec{x}}$$

(again, now I'm confused)

$$[a_p, a^+_{p'}] = (2\pi)^3 \delta^{(3)} (\vec{p} - \vec{p}')$$

(ok ∵) $\Rightarrow [\phi(\vec{x}), \pi(\vec{y})] = i \delta^{(3)} (\vec{x} - \vec{y})$ (This is a long calc. see it in CALC section)

We now find H using ϕ & π in terms of a_p, $a_p^\dagger$.

$$H = \int d^3x\, \mathcal{H} = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right]$$

$$\nabla\phi = \int \frac{d^3p}{(2\pi)^3} \frac{i}{\sqrt{2\omega_p}} \vec{p}\,(a_p + a^\dagger_{-p}) e^{ip\cdot x}$$

$$= \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2} \Bigg[(-i)^2 \frac{\sqrt{\omega_p\, \omega_{p'}}}{2} e^{i(\vec{p}+\vec{p}')\cdot\vec{x}} (a_p - a^\dagger_{-p})(a_{p'} - a^\dagger_{-p'})$$

$$+ \frac{(i)^2}{2\sqrt{\omega_p \omega_{p'}}} \vec{p}\cdot\vec{p}\,' e^{i(\vec{p}+\vec{p}')\cdot\vec{x}} (a_p + a^\dagger_{-p})(a_{p'} + a^\dagger_{-p'})$$

$$+ \frac{m^2}{2\sqrt{\omega_p \omega_{p'}}} (a_p + a^\dagger_{-p})(a_{p'} + a^\dagger_{-p'}) e^{i(\vec{p}+\vec{p}')\cdot x} \Bigg]$$

$$= \int \frac{d^3p}{(2\pi)^3} d^3p'\, \delta^{(3)}(\vec{p}+\vec{p}\,').$$

$$\Bigg\{ -\frac{\sqrt{\omega_p \omega_{p'}}}{4} (a_p - a^\dagger_{-p})(a_{p'} - a^\dagger_{-p'})$$

$$+ \frac{-\vec{p}\cdot\vec{p}\,' + m^2}{4\sqrt{\omega_p \omega_{p'}}} (a_p + a^\dagger_{-p})(a_{p'} + a^\dagger_{-p'}) \Bigg\}$$

(now $p' = -p$ after integrating)

$$= \int \frac{d^3p}{(2\pi)^3} \cdot \Bigg\{ -\frac{\omega_p}{4} (a_p - a^\dagger_{-p})(a_{-p} - a^\dagger_p)$$

$$+ \frac{\overset{\omega_p}{\cancel{m^2 + p^2}}}{4\,\cancel{\omega_p}} (a_p + a^\dagger_{-p})(a_{-p} + a^\dagger_p) \Bigg\}$$

look at CALC for details

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{4} \Big\{ 2a^\dagger_{-p} a_{-p} + 2[a_p, a_p^\dagger] + 2a^\dagger_p a_p \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \omega_p \left(a^\dagger_p a_p + \frac{1}{2}[a_p, a_p^\dagger] \right)$$

So finally

$$H = \int \frac{d^3p}{(2\pi)^3}\, \omega_p \left(a^\dagger_p a_p + \frac{1}{2} \overbrace{[a_p, a^\dagger_p]}^{\sim \delta(0)} \right)$$

There's some talking (physics) here. Look at your class notes for detail.

Then we also have (ignoring the 2nd term)

$$[H, a^\dagger_p] = \int \frac{d^3p'}{(2\pi)^3} \omega_{p'} [a^\dagger_{p'} a_{p'}, a^\dagger_p]$$

$$= \int \frac{d^3p'}{(2\pi)^3} \omega_{p'} \left(a^\dagger_{p'} [a_{p'}, a^\dagger_p] + \cancelto{0}{[a^\dagger_{p'}, a^\dagger_p]} a_{p'} \right)$$

$$[a^\dagger a, a^\dagger] = a^\dagger \cancelto{1}{[a, a^\dagger]} + \cancelto{0}{[a^\dagger, a^\dagger]} a = a^\dagger$$

$$= \int \frac{d^3p' \omega_{p'}}{\cancel{(2\pi)^3}} a^\dagger_{p'} \cancel{(2\pi)^3} \delta^{(3)}(p' - p)$$

$$= \omega_p a^\dagger_p$$

So effectively (and similarly)

$$[H, a^\dagger_p] = \omega_p a^\dagger_p \quad \& \quad [H, a_p] = -\omega_p a_p$$

Assume $|0\rangle$ s.t. $a_p |0\rangle = 0 \ \forall p$ & $H|0\rangle = 0$ (ignore the infinity)

now identical to the proof for the number operator being a number we have

$$H(a^\dagger_p a^\dagger_q \ldots |0\rangle) = (\omega_p + \omega_q + \ldots)(a^\dagger_p a^\dagger_q \ldots |0\rangle)$$

proof: $H a^\dagger_p a^\dagger_q \ldots |0\rangle$ (assume $H(a^\dagger_q \ldots |0\rangle) = (\omega_q + \ldots)|0\rangle$)

$$= ([H, a^\dagger_p] + a^\dagger_p H) a^\dagger_q \ldots |0\rangle$$

$$= \omega_p a^\dagger_p + a^\dagger_p (\omega_q + \ldots) a^\dagger_q \ldots |0\rangle$$

$$= (\omega_p + \omega_q + \ldots) a^\dagger_p a^\dagger_q \ldots |0\rangle$$

& $H a^\dagger_p |0\rangle = ([H, a^\dagger_p] + a^\dagger_p H)|0\rangle$

$$= \omega_p a^\dagger_p |0\rangle \quad (\text{assuming } H|0\rangle = 0)$$

so by induction we have it.

claim: These states exhaust the spectrum.

Claim: Using ϕ, π & their commutation $(a, a^\dagger)$, we also have

$$\underline{P}^i = -\int d^3x\, \pi(\vec{x})\, \partial_i \phi(\vec{x})$$

$$\Rightarrow \vec{\underline{P}} = -\int d^3x\, \pi(\vec{x})\, \nabla\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\, \vec{p}\, a^\dagger_p a_p$$

Now $\quad [\vec{\underline{P}}, a^\dagger_p] = \vec{p}\, a^\dagger_p \quad$ & $\quad [\vec{\underline{P}}, a_p] = -\vec{p}\, a^\dagger_p$

so then by the same logic we have

$a^\dagger_p$ creates a state of momentum $\vec{p}$ & energy ω_p

$= \sqrt{|p|^2 + m^2}$

(Note $\vec{p}$ can now be interpreted as the momentum)

Similarly the state

$a^\dagger_p\, a^\dagger_q \ldots |0\rangle$ has momentum $\vec{p} + \vec{q} + \ldots$

Interpretation: These excitations are called 'particles' since they are discrete entities that have the proper relativistic energy-momentum relation
(Although it doesn't mean something that's localized in space)

Further, $E_p \equiv \omega_p$.

Note $E_p \geq 0 \Rightarrow E_p = +\sqrt{|\vec{p}|^2 + m^2}$

Remark: The state is unchanged under exchange of particles, (momenta p & p' swapped) thus they obey bose-einstein statistics.

The vacuum state $|0\rangle$ is normalized as

$$\langle 0|0\rangle = 1$$

The one-particle state $|\vec{p}\rangle \propto a^+_p|0\rangle$

is NOT normalized as $\langle \vec{p}|\vec{q}\rangle = (2\pi)^3 \delta^3(\vec{p}-\vec{q})$

as its not Lorentz invariant.

proof: Consider a boost in z-direction (3-direction)

$$p_3' = \gamma(p_3 + \beta E)$$
$$E' = \gamma(E + \beta p_3)$$

Noting $\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|}\delta(x-x_0)$ we have

$$\delta^3(\vec{p}-\vec{q}) = \delta^3(\vec{p}'-\vec{q}') \cdot \frac{1}{\left|\frac{dp_3}{dp_3'}\right|}$$

$$= \delta^{(3)}(\vec{p}'-\vec{q}') \cdot \frac{dp_3'}{dp_3} = \delta^{(3)}(\vec{p}'-\vec{q}')\,\gamma\left(1+\beta\frac{dE}{dp_3}\right)$$

$$= \delta^{(3)}(\vec{p}'-\vec{q}')\,\frac{\gamma}{E}\left(E + \beta\left(E\frac{dE}{dp_3}\right)\right)$$

Now use $E^2 = |\vec{p}|^2 + m^2$
$\Rightarrow \cancel{2}E\frac{dE}{dp_3} = \cancel{2}p_3$

$$= \delta^{(3)}(\vec{p}'-\vec{q}')\,\frac{\gamma}{E}(E + \beta p_3)$$
$$= \delta^{(3)}(\vec{p}'-\vec{q}')\,\frac{E'}{E}$$

This shows that $E\,\delta^{(3)}(\vec{p}-\vec{q})$ IS Lorentz invariant.

Thus $|p\rangle \equiv \sqrt{2E_p}\, a^+_p|0\rangle$

so that $\langle q|p\rangle = 2E_p(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})$

[some part missing]

The Lorentz invariant integral is

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} = \int \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2-m^2)\Big|_{p^0>0}$$

(Sai did a better job in class)

Now the particle in space interpretation:

$$\phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} e^{-ip\cdot x} a_p^+ |0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot x} |p\rangle$$

$\phi(x)$ acting on $|0\rangle$ creates a particle at position x.

Now "the position-space representation" of state $|p\rangle$ is

$$\langle 0|\phi(x)|\vec{p}\rangle = \langle 0|\int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p'}}} \left(a_{p'} e^{i\vec{p}'\cdot\vec{x}} + a_{p'}^+ e^{-i\vec{p}'\cdot\vec{x}}\right)$$

$$\sqrt{2E_p}\, a_p^+|0\rangle$$

$$= \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p'}}} \sqrt{2E_p} \langle 0|a_{p'} a_p^+|0\rangle e^{i\vec{p}'\cdot\vec{x}} + \underbrace{\cancel{\langle 0|a_{p'}^+} a_p^+|0\rangle}_{\sqrt{2E_p}} {}^0$$

$$= \int \frac{d^3p'}{(2\pi)^3} \frac{\sqrt{2E_p}}{\sqrt{2E_{p'}}} \langle 0|[a_{p'}, a_p^+] + \cancel{a_p^+ a_{p'}}^{\,0} |0\rangle e^{i\vec{p}'\cdot\vec{x}}$$

$$= \int \frac{d^3p'}{\cancel{(2\pi)^3}} \frac{\sqrt{2E_p}}{\sqrt{2E_{p'}}} \cancel{(2\pi)^3} \delta^{(3)}(\vec{p}' - \vec{p})\, e^{i\vec{p}'\cdot\vec{x}}$$

$$= e^{i\vec{p}\cdot\vec{x}} \cdot \cancel{\frac{2E_p}{2E_{p'}}}$$

just as $\langle x|p\rangle \propto e^{i\vec{p}\cdot\vec{x}}$ in non-relativistic QM.

§2.4

The Klein-Gordon Field in Space-Time

$$\phi(x) = \phi(\vec{x},t) = e^{iHt}\phi(\vec{x})e^{-iHt}$$

$$\pi(x) = \pi(\vec{x},t) = \cdots$$

$$i\frac{\partial}{\partial t}\mathcal{O} = [\mathcal{O}, H]$$

just differentiate this with t.

So

$$i\frac{\partial}{\partial t}\phi(\vec{x},t) = \left[\phi(x,t), \int d^3x'\left\{\frac{1}{2}\pi^2(\vec{x}',t) + \frac{1}{2}(\nabla\phi(\vec{x}',t))^2 + \frac{1}{2}m^2\phi^2(\vec{x}',t)\right\}\right]$$

$$= \int d^3x' \,\cancel{\frac{2}{2}}\,\pi(\vec{x}',t)\underbrace{[\phi(x,t),\pi(\vec{x}',t)]}_{i\delta^{(3)}(\vec{x}-\vec{x}')}$$

$$= i\pi(x,t)$$

Similarly, (IDK why this is valid) had something to do with total divergence.

$$i\frac{\partial}{\partial t}\pi(\vec{x},t) \stackrel{?}{=} \left[\pi(\vec{x},t), \int d^3x'\left\{\frac{1}{2}\pi^2(\vec{x}',t) + \frac{1}{2}\phi(\vec{x}',t)(-\nabla^2+m^2)\phi(\vec{x}',t)\right\}\right]$$

$$\stackrel{?}{=} \int d^3x'\left(-i\delta^{(3)}(\vec{x}-\vec{x}')(-\nabla^2+m^2)\phi(\vec{x}',t)\right)$$

$$= -i(-\nabla^2+m^2)\phi(\vec{x},t)$$

Combining these we have

$$\frac{\partial^2}{\partial t^2}\phi = (\nabla^2-m^2)\phi$$

which is the Klein-Gordon eq^n.

The time dependence is better understood in terms of

a, a^+

$$[H, a_p] = -E_p a_p \qquad [H, a^+_p] = +E_p a^+_p$$

$$\Rightarrow \quad H a_p = a_p (H - E_p) \qquad \Rightarrow \quad H a^+_p = a^+_p (H + E_p)$$

$$\Rightarrow \quad H^n a_p = a_p (H - E_p)^n \qquad \Rightarrow \quad H^n a^+_p = a_p (H - E_p)^n$$

sign?

Now

$$e^{iHt} a_p e^{-iHt} = \left(1 + \frac{(iHt)}{1!} + \frac{(iHt)^2}{2!} + \frac{(iHt)^3}{3!} + \cdots\right) a_p e^{-iHt}$$

$$= a_p \left(1 + i(H - E_p)t + \frac{(i(H-E_p)t)^2}{2!} + \cdots\right) e^{-iHt}$$

$$= a_p e^{-iE_p t} \qquad e^{i(A+B)} = e^{iA} e^{iB} \text{ if } [A,B] = 0$$

So $\boxed{e^{iHt} a_p e^{-iHt} = a_p e^{-iE_p t}}$; $\boxed{e^{iHt} a^+_p e^{-iHt} = a^+_p e^{iE_p t}}$

Thus

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} e^{iHt} \left(a_p e^{i\vec{p}\cdot x} + a^+_p e^{-i\vec{p}\cdot\vec{x}}\right) e^{-iHt}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p e^{i(\vec{p}\cdot\vec{x} - E_p t)} + a^+_p e^{-i(\vec{p}\cdot\vec{x} - E_p t)}$$

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p e^{-ipx} + a^+_p e^{+ipx} \Big|_{p^0 = E_p}$$

$$\pi(\vec{x}, t) = \frac{\partial}{\partial t} \phi(\vec{x}, t)$$

Relating $\phi(x)$ to $\phi(0)$

using $[\vec{P}, a_p] = -\vec{p}\, a_p$ & $[\vec{P}, a^+_p] = \vec{p}\, a^+_p$ we have

$$e^{-i\vec{P}\cdot\vec{x}} a_p e^{i\vec{P}\cdot\vec{x}} = a_p e^{i\vec{p}\cdot\vec{x}} \; ; \; e^{-i\vec{P}\cdot\vec{x}} a^+_p e^{i\vec{P}\cdot\vec{x}} = a^+_p e^{-i\vec{p}\cdot\vec{x}}$$

and therefore

$$\phi(x) \overset{\text{IDK why}}{\overset{?}{=}} e^{i(Ht - \vec{P}\cdot\vec{x})} \phi(0) e^{-i(Ht - \vec{P}\cdot\vec{x})}$$

$$= e^{i P\cdot x} \phi(0) e^{-i P\cdot x}$$

where $P^\mu = (H, \vec{P})$

Remarks: <missing>

Causality

In the given formalism (so far i.e.)
"amplitude for a particle to propagate from y to x is
$\langle 0|\phi(x)\,\phi(y)|0\rangle$"
We call this $D(x-y) = \langle 0|\phi(x)\,\phi(y)|0\rangle$

$$= \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\,e^{-ip\cdot(x-y)}$$

Remark:
It has been shown that the measure $\int \frac{d^3p}{(2\pi)^3\, 2E_p}$ is
lorentz invariant & thus if $f(p) = e^{-ip\cdot(x-y)}$
is lorentz invariant which it is, then

$$\int \frac{d^3p}{(2\pi)^3\,2E_p}\,f(p)$$ is also. (page 23, eqn 2.40)

Consider particular values of $x-y$ & find the integral

1) First, let $x-y$ be s.t.
$x^0 - y^0 = t$ & $\vec{x} - \vec{y} = 0$

(Note: If the interval from y to x is timelike, then there always a frame in which this holds)

$$D(x-y) = \frac{4\pi}{(2\pi)^3}\int_0^\infty dp\,\frac{p^2}{2\sqrt{p^2+m^2}}\,e^{-i\sqrt{p^2+m^2}\,t}$$

$$= \frac{1}{4\pi^2}\int_m^\infty dE\,\sqrt{E^2-m^2}\,e^{-iEt}$$

$$\underset{t\to\infty}{\sim} e^{-imt}$$

Next we consider $x-y$ s.t. $x^0 - y^0 = 0$, $\vec{x} - \vec{y} = \vec{r}$.

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}\cdot\vec{r}}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2E_p} \frac{e^{ipr} - e^{-ipr}}{ipr}$$

$$= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^{\infty} dp \frac{p}{\sqrt{p^2+m^2}} e^{ipr}$$

Then what peskin does, I'm not so sure: $\rho = -ip$

$$\frac{1}{8\pi^2 r} \int_{-\infty}^{\infty} d\rho \frac{\rho e^{-\rho r}}{\sqrt{m^2-\rho^2}} \underset{r\to\infty}{\sim} e^{-mr}$$

(look at page 27, 26)

its non zero.

So is causality gone?

Well, disc truncated

$[\phi(x), \phi(y)]$ should be 0 for $(x-y)^2 < 0$

then causality is preserved 'quite generally'

We already have for $x^0 = y^0$, $[\phi(x), \phi(y)] = 0$

so then

in general we have

$[\phi(x), \phi(y)]$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \times$$

$$\underbrace{\left[(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}), (a_q e^{-iq\cdot y} + a_q^\dagger e^{iq\cdot y}) \right]}$$

$$\Big[\overset{0}{\cancel{[a_p e^{-ip\cdot x}, a_q e^{-iq\cdot y}]}} + \overset{=}{[a_p e^{-ip\cdot x}, a_q^\dagger e^{iq\cdot y}]}$$

$$+ [a_p^\dagger e^{ip\cdot x}, a_q e^{-iq\cdot y}] + \overset{0}{\cancel{[a_p^\dagger e^{ip\cdot x}, a_q^\dagger e^{iq\cdot y}]}}$$

$$= e^{i(q\cdot y - p\cdot x)} (2\pi)^3 \delta(\vec{p} - \vec{q})$$

$$- e^{i(p\cdot x - q\cdot y)} (2\pi)^3 \delta(\vec{q} - \vec{p}) \Big]$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[e^{ip(y-x)} - e^{ip(x-y)} \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[e^{-ip(x-y)} - e^{-ip(y-x)} \right]$$

$$= D(x-y) - D(y-x)$$

This case

When $(x-y)^2 < 0$
(Note both terms are separately Lorentz invariant)
I can take $(x-y) \to -(x-y)$
by a continuous Lorentz Transform & then $D(x-y) - D(x-y) = 0$
However when $(x-y)^2 > 0$ then $\sim e^{-imt} - e^{imt}$
$\because \exists$ no continuous Lorentz transform to do $x \to -x$

(more remarks skipped)

TODO: CALCULATIONS for §2.3

1) Derive $a^+ a$ is infact the number operator

$$\underbrace{(a^+ a)\, a^+|0\rangle}_{\substack{\text{|||} \\ |1\rangle}} = a^+ [a, a^+] |0\rangle = (1)(a^+ |0\rangle)$$

$\therefore$ its eigenvalue is 1

Assuming $|m\rangle$ is s.t. $a^+ a|m\rangle = m|m\rangle$, then

$$(a^+ a)\underbrace{a^+|m\rangle}_{\substack{\text{|||} \\ |m+1\rangle}} = a^+[a, a^+] + a^+(a^+ a|m\rangle)$$

$$= a^+ + a^+ m|m\rangle$$

$$= (m+1)\, a^+ |m\rangle$$

Thus $a^+ a$ is the number operator

2) Verify $[a_p, a^+_{p'}] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}')$

$$\Rightarrow [\phi(\vec{x}), \pi(\vec{y})] = i\,\delta^{(3)}(\vec{x} - \vec{y})$$

$$[\phi(\vec{x}), \pi(\vec{y})] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \frac{d^3p'}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{p'}}{2}}$$

$$\left[a_p e^{i\vec{p}\cdot\vec{x}} + a^+_p e^{-i\vec{p}\cdot\vec{x}},\; a_{p'} e^{i\vec{p}'\cdot\vec{y}} - a^+_{p'} e^{-i\vec{p}'\cdot\vec{y}}\right]$$

what you remembered was for $[A, BC] = [A, B]C + B[A, C]$; $[AB, C] = A[B, C] + [A, C]B$

$$[A + B, C + D] = \cancel{A}\,[B, C+D] + [A, C+D]\cancel{B}$$

$$= \cancel{A}\left([B, C]\cancel{D} + \cancel{C}[B, D]\right) + \left([A, C]\cancel{D} + \cancel{C}[A, D]\right)\cancel{B}$$

(this is plain stupid; use the other expression for ϕ & π)

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \frac{d^3p'}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{p'}}{2}}$$

$$\underbrace{\left[a_p + a^+_{-p},\; a_{p'} - a^+_{-p'}\right]}\qquad e^{i(\vec{p}\cdot\vec{x} + \vec{p}'\cdot\vec{y})}$$

$$\| $$

$$\cancel{a_p}\,[a^+_{-p}, a_{p'}](\cancel{-a^+_{-p'}}) + (\cancel{a_{p'}})[a_p, -a^+_{-p'}](\cancel{a^+_{-p}})$$

$$= \int \frac{d^3p}{\cancel{(2\pi)^3}} \frac{1}{\sqrt{2\omega_p}} \frac{d^3p'}{(2\pi)^3} (\cancel{-}i) \sqrt{\frac{\omega_{p'}}{2}} \left[\cancel{(-1)}\cancel{(2\pi)^3}\,2\,\delta^{(3)}(\vec{p}' + \vec{p}) + \cancel{(-1)(2\pi)^3 \delta^3(\vec{p} + \vec{p}')}\right]$$

$$e^{i(\vec{p}\cdot\vec{x} + \vec{p}'\cdot\vec{y})}$$

$$= \int \frac{d^3p}{\sqrt{\cancel{2}\cancel{\omega_p}}} \frac{i}{(2\pi)^3} \sqrt{\frac{\cancel{\omega_p}}{\cancel{2}}}\; \cancel{2}\; e^{i(\vec{p})\cdot(\vec{x} - \vec{y})} \quad (\because \omega_{-p} = \omega_p) = i\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}(\vec{x} - \vec{y})}$$

$$= i\,\delta^3(\vec{x} - \vec{y})$$

3) show $\frac{1}{4}\left[-(a_p - a^+_{-p})(a_{-p} - a^+_p) + (a_p + a^+_{-p})(a_{-p} + a^+_p)\right]$

$$= a^+_p a_p + \frac{1}{2}[a_p, a_p^+]$$

under integral $\frac{d^3p}{(2\pi)^3} w_p$ s.t. $w_{-p} = w_p$

$$- a_p a_{-p} + a_p a^+_p + a^+_{-p} a_{-p} - a^+_{-p} a^+_p$$

$$+ a_p a_{-p} + a_p a^+_p + a^+_{-p} a_{-p} + a^+_{-p} a^+_p$$

$$= -\overbrace{\cancel{[a_p, a_{-p}]}}^{0} + 2 a_p a^+_p + 2 a^+_{-p} a_{-p} - \overbrace{\cancel{[a^+_{-p}, a^+_p]}}^{0}$$

$$= 2 a^+_{-p} a_{-p} + 2[a_p, a^+_p] + 2 a^+_p a_p$$

---

4) Show that: $[a^+ a, a^+] = a^+$

$$[a^+ a, a^+] = a^+ a a^+ - a^+ a^+ a$$

$$= a^+[a, a^+] + \cancel{a^+ a^+ a} - \cancel{a^+ a^+ a}$$

$$= a^+$$

or you could use

$$[a^+ a, a^+] = a^+[a, a^+] + [a^+, a^+]a$$

---

5) Show eq^n (2.33) of Peskin holds

$$\vec{P} = -\int d^3\vec{x}\ \pi(\vec{x}) \nabla \phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \vec{p}\ a_p^+ a_p$$

6) Show that $\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|} \delta(x - x_0)$

The Klein-Gordan Propagator

(stripped down version)

We know

$$[\phi(x), \phi(y)] = c\text{-no.} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right)$$

(as was shown earlier)

also $\quad \langle 0| [\phi(x), \phi(y)] |0\rangle = [\phi(x), \phi(y)]$

& we also know

$$D(x-y) = \langle 0| \phi(x) \phi(y) |0\rangle = \int \frac{d^3p}{(2\pi)^3} e^{-ip\cdot(x-y)}$$

so that

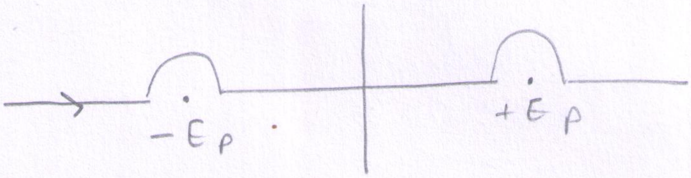
$$[\phi(x), \phi(y)] = D(x-y) - D(y-x)$$

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{2E_p} e^{-ip\cdot(x-y)} \Big|_{p^0=E_p} + \frac{1}{2(+E_p)} e^{-ip\cdot(x-y)} \Big|_{p^0=-E_p} \right\}$$

(look at (ALC) Q#7 (eqn 2.54))

$$\underset{x^0 > y^0}{=} \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi i} \frac{-1}{p^2 - m^2} e^{i\, p\cdot(x-y)}$$

given the contour



for $x^0 > y^0$ closed below.

Not needed.

For $x^0 < y^0$ (then if the events are time-like? How do we show that the future can't affect the past?)

then we choose the closed above contour & get the commutator = 0. (for space like,

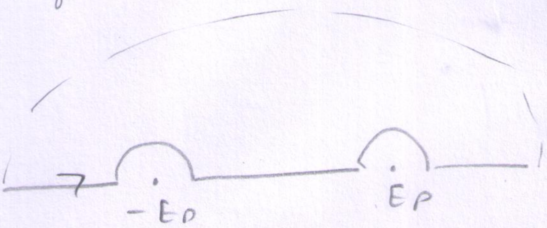
ignore this

Now consider

$$\int \frac{d^3 p}{(2\pi)^3} \int \frac{dp^0}{2\pi i} \quad \frac{-1}{p^2 - m^2} \; e^{-ip\cdot(x-y)}$$

together with

for $x^0 < y^0$ for $x^0 > y^0$



&

$$= \quad \theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle \equiv D_R(x-y)$$

its called the retared Green's function.

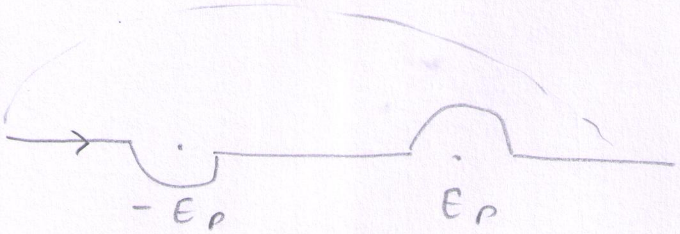
some discussion's missing

2

In fact, now we consider the same integral

$$\int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi i} \frac{-1}{p^2-m^2} e^{-ip\cdot(x-y)}$$

with for $x^0 < y^0$ the contour for $x^0 > y^0$



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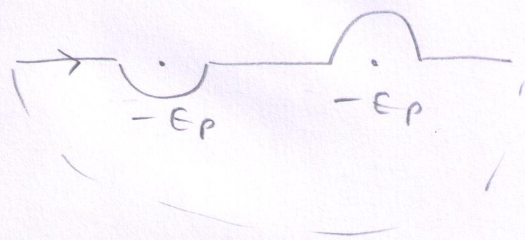


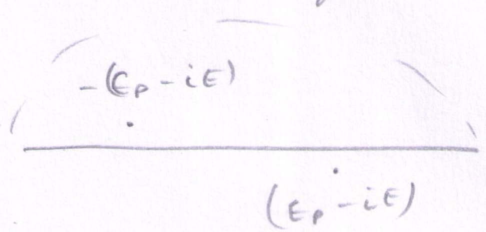
(Ah. the Feynman prescription)

a more convenient way to write this is

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)} \equiv \Delta_F(x-y)$$

so that the poles are at $p^0 = \pm(E_p - i\epsilon)$

displaced properly & use the contour



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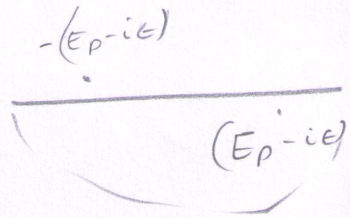


In that case, as follows from the prev. discussions,

$$D_F(x-y) = \begin{cases} \Delta(x-y) & \text{for } x^0 > y^0 \\ \Delta(y-x) & \text{for } x^0 < y^0 \end{cases}$$

— ask why there's no minus sign for this

$$= \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle$$

$$\equiv \langle 0| T\,\phi(x)\,\phi(y)|0\rangle$$

↳ Time ordering (latest on the left and so on)

√<comments missing>

CALCULATIONS

§ 2.4

Q1 Show that $[A(\vec{x}), B(\vec{x}')] = [A(\vec{x},t), B(\vec{x}',t)]$

given: $[A(\vec{x}), B(\vec{x}')] = c$ → complex no.

Q2. Show that if $H a_p = a_p(H-E_p)$ then

$$H^n a_p = a_p(H-E_p)^n$$

let $H^m a_p = a_p(H-E_p)^m$

then $H^{m+1} a_p = \underbrace{H a_p}(H-E_p)^m$

$$= a_p(H-E_p)(H-E_p)^m$$

$$= a_p(H-E_p)^{m+1}$$

Q3. <The exam question>

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{-ip\cdot(x-y)}$$

$$\langle 0|\phi(x)\phi(y)|0\rangle = \langle 0| \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_p}\sqrt{2E_{p'}}} \left(a_p e^{-ipx} + \cancelto{0}{a^\dagger_p e^{ipx}}\right) \left(\cancelto{0}{a_{p'} e^{-ip'x'}} + a^\dagger_{p'} e^{ip'x'}\right) |0\rangle$$

(assume $x'=y$)

$$= \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{2\sqrt{E_pE_{p'}}} \langle 0|a_p a^\dagger_{p'}|0\rangle e^{i(p'x'-px)}$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^{3\cdot 2}\, 2\sqrt{E_pE_{p'}}} \langle 0| [a_p, a^\dagger_{p'}] + \cancelto{0}{a^\dagger_{p'} a_p}|0\rangle e^{i(p'x'-px)}$$

$$= \int \frac{d^3p\,d^3p'}{(2\pi)^3 \cancel{(2\pi)^3}\, 2\sqrt{E_pE_{p'}}} \cancelto{1}{\langle 0|0\rangle} \cancel{(2\pi)^3}\,\delta^3(\vec p - \vec p')\, e^{i(p'x'-px)}$$

(Note: $\vec p = \vec p'$ → $E_p = E_{p'}$)

$$= \int \frac{d^3p}{(2\pi)^3}\frac{e^{ip(x-x')}}{2E_p}$$

Q4. Show $\int_0^\infty dp \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}\,t} = \int dE \sqrt{E^2-m^2}\, e^{-iEt}$

$E^2 = p^2+m^2 \to \cancel{2}E\,dE = \cancel{2}p\,dp$

$dE = \frac{p}{E}dp$

then its obvious.

Q5. Show that

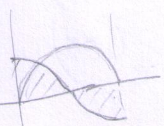
$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{i\vec{p}\cdot\vec{r}} = \frac{2\pi}{(2\pi)^3}\int_0^\infty dp \frac{p^2}{2E_p}\frac{e^{ipr}-e^{-ipr}}{ipr}$$

$$= \frac{2\pi}{(2\pi)^3}\int_0^\infty dp\frac{p^2}{2E_p}\frac{2\cancel{i}\sin(pr)}{\cancel{i}pr}$$

$$= \frac{4\pi}{(2\pi)^3}\int_0^\infty dp\frac{p^2}{2E_p}\frac{\sin(pr)}{pr}$$

$\int d^3r = \iiint d\phi \sin\theta\, r^2\, dr\, d\theta$

$$= \frac{2\pi}{(2\pi)^3}\int_0^\pi\int_0^\infty p^2\frac{dp\,d\theta\sin\theta}{2E_p} e^{ipr\cos\theta}$$



↑

Thinking & attempts (removed for clarity)

↓

$$\int_{-1}^{1} d(\cos\theta)\, e^{ipr(\cos\theta)} = \frac{e^{ipr}-e^{-ipr}}{pr}$$

$$= \frac{2\pi}{(2\pi)^3}\int_0^\infty \frac{p^2\,dp}{2E_p}\frac{e^{ipr}-e^{-ipr}}{pr}$$

Done.

Q 6.

$$\int_0^\infty dp \, \frac{p^2}{2E_p} \frac{e^{ipx}}{ipx} - \int_0^\infty dp \, \frac{p^2}{2E_p} \frac{e^{-ipx}}{ipx}$$

$$= \int_0^\infty dp \, \frac{p^2}{2E_p} \frac{e^{ipx}}{ipx} - \int_0^{-\infty} \cancel{-}dp \, \frac{p^2}{2E_p} \frac{e^{ipx}}{\cancel{-}ipx}$$

$$= \int_0^\infty dp \, \frac{p^2}{2E_p} \frac{e^{ipx}}{ipx} + \int_{-\infty}^{0} dp \, \frac{p^2}{2E_p} \frac{e^{ipx}}{ipx}$$

$$= \int_{-\infty}^{\infty} dp \, \frac{p^2}{2E_p} \frac{e^{ipx}}{ipx}$$

(Thats what had to be shown, forgot to write it as a question)

Q.7. Show eq (2.54) holds.

$$\int \frac{dp^0}{2\pi i} \frac{-1}{p^2 - m^2} e^{-i p\cdot(x-y)}$$

Assum: p^2 is the 4 vector p.

$\cancel{p^2 - m^2}$ Then $p^2 - m^2 = p_0^2 + \vec{p}^2 - m^2$

$$\int \frac{dp^0}{2\pi i} \frac{-1}{(p+m)(p-m)} e^{-ip\cdot(x-y)} \qquad p^2 \cancel{-pm+mp} - m^2$$

$$= \int \frac{dp^0}{2\pi i} \frac{1}{2m} \frac{(p+m - p+m)}{(p+m)(p-m)}$$

$$= \int \frac{dp^0}{2\pi i} \frac{1}{2m} \left[\frac{\cancel{p+m}\ 1}{\cancel{(p+m)}(p-m)} - \frac{1}{(p+m)} \right] e^{-ip\cdot(x-y)}$$

$e^{-ip\cdot(x-y)}$ $\therefore$ p is now a 4 vector

Q Where's the singularity?

$p^\mu p_\mu = m^2$ $\qquad |E|^2 - p^2 = m^2$

$|E|^2 = m^2 + \vec{p}^2$

$$(p^0)^2 - |E|^2 = (p^0)^2 - m^2 - |\vec{p}|^2$$

$$p^2 - m^2$$

$$\frac{1}{(P_0+E)(P_0-E)}$$

$$\frac{1}{2E}\left(\frac{P_0+E - P_0+E}{(P_0+E)(P_0-E)}\right)$$

$$\frac{1}{P_0-E} - \frac{1}{P_0+E}$$