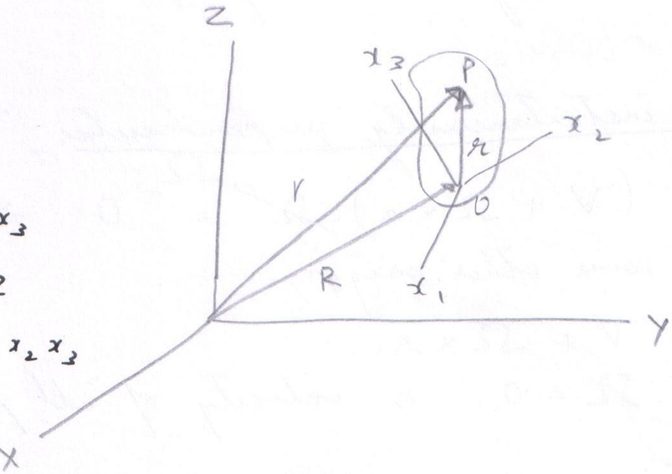
Motion of a rigid body

§31. Angular Velocity

Fixed / Inertial : XYZ
Moving System : $x_1 x_2 x_3$
(Body Fixed) $= x\ y\ z$

r : Position vector of P in $x_1 x_2 x_3$
r : Position " " " XYZ
R : Centre of Mass / Origin of $x_1 x_2 x_3$



Let us now consider an infinitesimal arbitrary displacement of the rigid body. The infinitesimal displacement of P consists of a displacement dR (same as that of the COM), and $d\phi \times r$, relative to the COM as a result of an infinitesimal rotation.

$$\Rightarrow \quad dr = dR + d\phi \times r$$

$$\Rightarrow \quad \underset{\substack{\| \\ \frac{dr}{dt}}}{dv} = \underset{\substack{\| \\ \frac{dR}{dt}}}{dV} + \underset{\substack{\| \\ \frac{d\phi}{dt}}}{\underline{d\Omega}} \times r$$

$\frac{d\phi}{dt}$ — Angular Velocity

$\frac{dR}{dt}$ — Velocity of the centre of Mass is called the translational velocity.

$\frac{dr}{dt}$ — (hehe, this has no name)

Let us now assume that the Body Fixed coordinate is not origined at the COM, O, but at O' (position a).
The corresponding labels are as shown.

Now let $\frac{dR'}{dt} = V'$ & the angular velocity of the new system be Ω' $\left(= \frac{d\phi'}{dt}\right)$

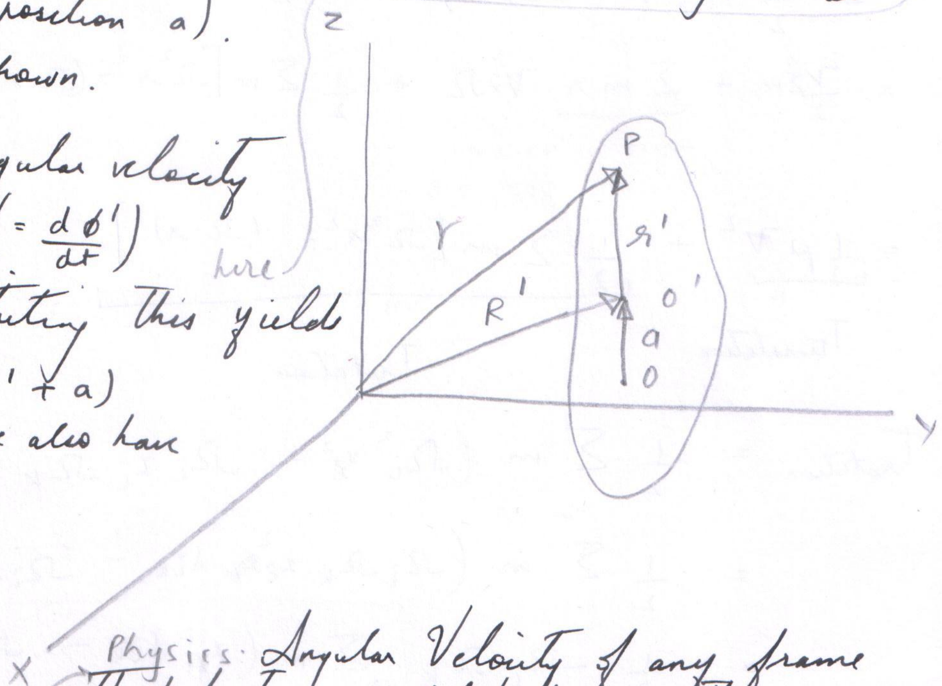
here

Observe $r = r' + a$. Substituting this yields

$$dv = dV + d\Omega \times (r' + a)$$

From the def^n of V' & Ω' we also have

$$dv = dV' + \Omega' \times r'$$



$$\Rightarrow \boxed{V' = V + \Omega \times a}$$

&

$$\Rightarrow \Omega \times r' = \Omega' \times r'$$

$$\hookrightarrow \Rightarrow \boxed{\Omega = \Omega'}$$

Physics: Angular Velocity of any frame attached to a rigid body is the same at a given instant.

Now as all such frames have the same $\underline{\Omega}$ (dir & magnitude), thus we call Ω the angular velocity of the body.

Instantaneous Axis of Rotation

<u>If V & Ω are instantaneously perpendicular</u>, then

1) $V' \cdot \Omega' = (V + \Omega \times a) \cdot \cancel{\Omega'}\,\Omega = 0 \Rightarrow V'$ & Ω' are also perpendicular for some other origin.

2) Since $v = V + \Omega \times r$

$\Rightarrow v \cdot \Omega = 0 \Rightarrow$ velocity of all points of the body ~~is~~ are $\perp \Omega$.

~~and if~~ $\Downarrow$

$\exists$ some point O' s.t. $V' = 0$. Motion through this point may be considered as a pure rotation with the axis passing through this point.

end if

§ 32. The Inertia Tensor

Henceforth, the origin is on the COM (of the moving frame) & Ω may vary during the motion.

$T = \sum \frac{1}{2} m v^2$ (we simply add the contribution from all points making the body.)

$$= \sum \frac{1}{2} m (V + \Omega \times r)^2 = \sum \frac{1}{2} m V^2 + \sum m V \cdot \Omega \times r + \sum \frac{1}{2} m (\Omega \times r)^2$$

$$= \frac{V^2}{2} \sum m + \underbrace{\sum m r}_{\text{since we're on the COM, } = 0} \cdot V \times \Omega + \frac{1}{2} \sum m \left[\Omega^2 r^2 - (\Omega \cdot r)^2\right]$$

Figure the expansion of (can do this using tensors)

$$(\Omega \times r)^2 = \varepsilon_{ijk} \Omega_i r_j \varepsilon_{lmk} \Omega_l r_m$$
$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \Omega_i r_j \Omega_l r_m$$
$$= \Omega_i^2 r_j^2 - \Omega_i r_i r_j \Omega_j$$
$$= \Omega^2 r^2 - (\Omega \cdot r)^2$$

$$= \underbrace{\frac{1}{2} \mu V^2}_{= T_{\text{translation}}} + \underbrace{\frac{1}{2} \sum m \left[\Omega^2 r^2 - (\Omega \cdot r)^2\right]}_{= T_{\text{rotation}}}$$

$$T_{\text{rotation}} = \frac{1}{2} \sum m \left(\Omega_i^2 x_l^2 - \Omega_i x_i \Omega_k x_k\right)$$

$$= \frac{1}{2} \sum m \left(\Omega_i \Omega_k x_l^2 \delta_{ik} - \Omega_i \Omega_k x_i x_k\right)$$

$$= \frac{1}{2} \Omega_i \Omega_k \boxed{\underbrace{\sum m \left(x_l^2 \delta_{ik} - x_i x_k\right)}_{= I_{ik}}}$$

So then we have: $T = \frac{1}{2}\mu V^2 + \frac{1}{2} I_{ik}\Omega_i\Omega_k$

We can get $L = T - U$.

I_{ik} is the Inertia Tensor & it is symmetric ($I_{ik} = I_{ki}$).

Explicitly

$$I_{ik} = \begin{bmatrix} \sum m(y^2+z^2) & -\sum mxy & -\sum mxz \\ -\sum myx & \sum m(x^2+z^2) & -\sum myz \\ -\sum mzx & -\sum mzy & \sum m(x^2+y^2) \end{bmatrix}$$

I_{xx}, I_{yy} & I_{zz} are called moments of Inertia about the corresponding axis.

NOTE: The inertia tensor is additive: for a body, it is sum of its parts.

For the continuous case we have

$$I_{ik} = \int \rho \, (x_l^2 \delta_{ik} - x_i x_k) dV.$$

The tensor can be diagonalized by an appropriate $x_1 x_2 x_3$. In such a case, the diagonal components are called the principal moments of Inertia.

Then $T_{rot} = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2)$

NOTE: $I_1 + I_2 = \sum m(x_1^2 + x_2^2 + 2x_3^2) \geq \sum m(x_1^2 + x_2^2) = I_3$

Asymmetrical Top: $I_1 \neq I_2 \neq I_3$

Symmetric Top: $I_1 = I_2 \neq I_3$

Spherical Top: $I_1 = I_2 = I_3$ (This will hold for any arbitrary mutually perpendicular axes. Can easily prove this (think of its a cI and transform it to a different basis)

Tip: You may calculate a 'similar Tensor' at a point O'

$$I'_{ik} = \sum m(x'^2_l \delta_{ik} - x'_i x'_k)$$

Now let OO' be $\vec{a}$, then $r = r' + a \Rightarrow x_i = x'_i + a_i$.

Use $\sum m r = 0$ (as O is the COM), we have

$$\boxed{I'_{ik} = I_{ik} + \mu(a^2\delta_{ik} - a_i a_k)}$$

§ 33. Angular Momentum of a rigid body

M: Angular momentum defined about the origin of the moving frame (COM of the body)

Unproved assertion: When the origin is at the COM, the angular momentum is due to the motion relative to the COM. use $A\times(B\times C) = B(A\cdot C) - C(A\cdot B)$

$$M = \sum m\, r \times (v) = \sum m\, r\times(\Omega \times r) = \sum m\left[r^2 \Omega - r(r\cdot\Omega)\right]$$

$$= \sum m\left(x_l^2 \Omega_i - x_i x_k \Omega_k\right) = \sum m\left(x_l^2 \Omega_k \delta_{ik} - x_i x_k \Omega_k\right)$$

$$= \Omega_k \sum m\left(x_l^2 \delta_{ik} - x_i x_k\right) = \boxed{I_{ik}\Omega_k \equiv M_i}$$

WRONG/INCOMPLETE

$e_{ijk}\, r_i\, e_{lmj}\, \Omega_l\, r_m$

$= -e_{ikj}\, e_{lmj}\, r_m r_j \Omega_l$

$= -(\delta_{il}\delta_{km} - \delta_{im}\delta_{kl})\, r_m r_j \Omega_l$

$= \delta_{im}\delta_{kl}\, r_m r_j \Omega_l$

(K) $- \delta_{il}\delta_{km}\, r_m r_j \Omega_l$

$= \Omega_k (r_i r_j) - r_k (r_j \Omega_j)$

If the axes $x_1\ x_2\ x_3$ are principal, then

$$M_1 = I_1 \Omega_1 \qquad M_2 = I_2 \Omega_2 \qquad M_3 = I_3 \Omega_3$$

Specifically further if $I_1 = I_2 = I_3$ (symmetrical top)

$$M = I\Omega$$

NOTE: In general, M is not in the same direction as Ω. That happens when the rotation is along one of the principal axes of inertia.

Free Rotation of the Body (no external forces)

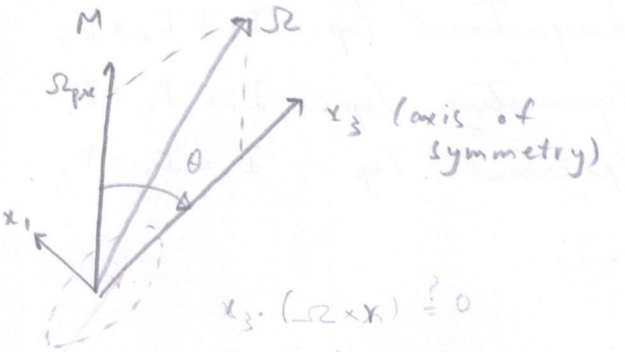
(Ignore any uniform translation motion)

Thus M = const.

→ For a spherical top, → Ω is const.

→ For a rotator,  here too $M = I\omega$ (why not some other axis)
(one of the I is zero)

→ For a symmetric top (symmetry axis x_3),
we choose x_2 to be along ~~M~~ $x_3 \times M$.
consequently $M_2 = 0$
and also (since $x_1\ x_2\ x_3$ are principal)
$\Omega_2 = 0$
(not clear)



CLARIFY LATER : (Most likely the origin is at the COM)

$x_3 \cdot (\Omega \times X) \stackrel{?}{=} 0$

$(x_3 \times \Omega)\cdot X = 0$

$$\Omega_3 = \frac{M_3}{I_3} = \frac{M\cos\theta}{I_3}$$

Now we resolve Ω along M & x_3. Ω along x_3 wouldn't affect the axis x_3. Thus Ω_{prec} must change the axis x_3 (thus the term precession).

Now $\quad \Omega_{prec} \sin\theta = \Omega_1$

Also $\quad \Omega_1 = \frac{M_1}{I_1} = \frac{M \sin\theta}{I_1} \Rightarrow \boxed{\Omega_{prec} = \frac{M}{I_1}}$

§ 34. The equations of motion of a rigid body

§ 35. Eulerian Angles

Note:

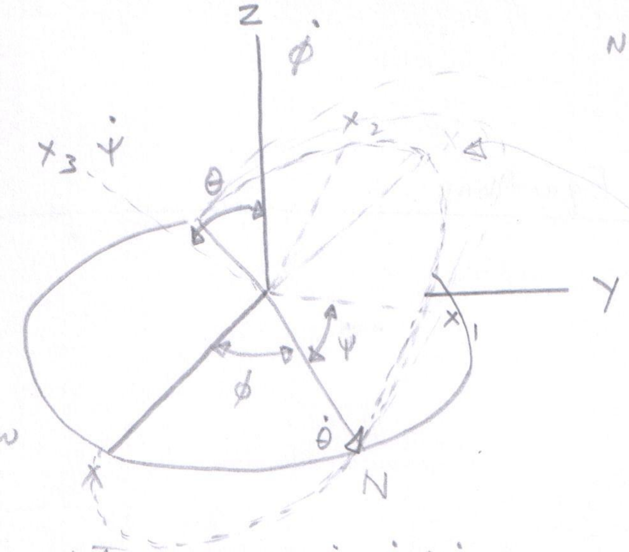
ON ⊥ z & x_3

ON ≡ z × x_3

$\theta \equiv \angle$ b/w z & x_3 – along ON
$\phi \equiv \angle$ b/w X & ON – along z
$\psi \equiv \angle$ b/w x_1 & ON – along x_3

$0 \le \theta \le \pi$, $0 \le \phi, \psi \le 2\pi$

directions are given by the corkscrew rule.



Note: X_i & x_i are identical

To find Ω along $x_1 x_2 x_3$ we just need to express $\dot\theta, \dot\phi$ & $\dot\psi$ in $x_1, x_2 x_3$.

$\dot\theta_1 = \dot\theta \cos\psi$ $\quad \dot\theta_2 = \dot\theta \sin\psi$ $\quad \dot\theta_3 = 0$ (∵ $\dot\theta$ is along ON)

$\dot\phi_3 = \dot\phi \cos\theta$ ($\dot\phi$ is along z)

in the $x_1 x_2$ plane then $\dot\phi$ will have a $\dot\phi \sin\theta$ component

⇒ $\dot\phi_1 = \dot\phi \sin\theta \sin\psi$ $\quad \dot\phi_2 = \dot\phi \sin\theta \cos\psi$

$\dot\psi$ is going to be along x_3 as is anyway clear

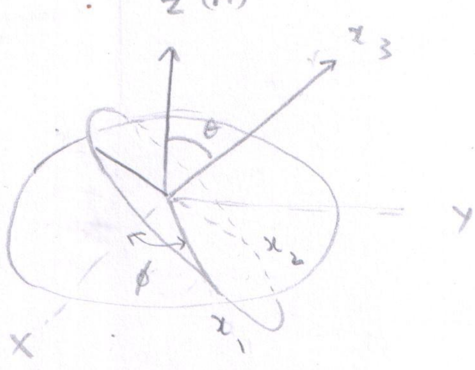
⇒ $\dot\psi_3 = \dot\psi$ $\quad \dot\psi_2 = 0$ & $\dot\psi_1 = 0$.

So finally then, collecting components along each axis.

$$\Omega_1 = \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi$$
$$\Omega_2 = \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi$$
$$\Omega_3 = \dot\phi \cos\theta + \dot\psi$$

If $x_1 x_2 x_3$ are principal, then $T_{rot} = I_i \Omega_i^2$

Eg. The symmetric top again



So then for $\psi = 0$, we have

$$\Omega_1 = \dot\theta$$
$$\Omega_2 = \dot\phi \sin\theta$$
$$\Omega_3 = \dot\phi \cos\theta + \dot\psi$$

⇒ $M_1 = I_1 \Omega_1 = I_1 \dot\theta$

$M_2 = I_1 \Omega_2 = I_1 \dot\phi \sin\theta$

$M_3 = I_3 \Omega_3 = I_3(\dot\phi \cos\theta + \dot\psi)$

Since $M \perp x_1$ thus

$M_1 = 0$, $M_2 = M \sin\theta$, $M_3 = M\cos\theta$ on comparing we get $\dot\theta = 0$, $I_1 \dot\phi = M$ & $I_3(\dot\phi\cos\theta + \dot\psi) = M\cos\theta$

$\Rightarrow \theta = \text{const}$

$\dot{\phi} = M/I_1$ (precession) [same as before!]

& $\Omega_3 = \dfrac{M \cos\theta}{I_3}$ (angular vel. with which the top rotates about its own axis)

§ 36. Euler's Equations