Catalan's Constant

Yidan Chen

Abstract

We present a list of formulas involving different integral representations of Catalan's constant and give detailed methods to prove them. Catalan's constant is defined as:

$$G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.916$$

Contents

1 Introduction

The constant known as Catalan's constant is attributed to the French-Belgian mathematician Eugène Charles Catalan, renowned for his discovery of fast-converging series for computation. Despite not being the first discoverer of the constant, Catalan's significant contributions to mathematical fields such as number theory and combinatorics have earned him eponymous recognition..[4]

The value of Catalan's constant is approximately 0.9159656. Intriguingly, whether Catalan's constant is irrational or transcendental remains an unsolved mathematical problem.

Catalan's constant plays a crucial role for applications in conjectural formulas concerning the asymptotic approximations of prime numbers [5] and in determining the mass distribution of spiral galaxies. David M. Bradley [1] extensively lists hundreds of various expressions of this constant in his papers, ranging from single integrals to double integrals and infinite sums, among others.

In this project, we aim to provide detailed proofs of the expressions listed below for Catalan's constant.

1.1 Equation List

$$G = \int_0^1 \frac{\arctan x}{x}\, dx \tag{1}$$

$$G = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{x}{\sin x}\, dx \tag{2}$$

$$G = -\int_0^1 \frac{\ln x}{1+x^2}\, dx \tag{3}$$

$$G = -2\int_0^{\frac{\pi}{4}} \log(2\sin x)\, dx \tag{4}$$

$$G = 2\int_0^{\frac{\pi}{4}} \log(2\cos x)\,dx \tag{5}$$

$$G = \int_0^{\frac{\pi}{4}} \log(\cot x)\,dx = -\int_0^{\frac{\pi}{4}} \log(\tan x)\,dx \tag{6}$$

$$G = -\frac{3}{2}\int_0^{2-\sqrt{3}} \frac{\log x}{1+x^2}\,dx \tag{7}$$

$$G = -\frac{1}{8}\pi\log(2+\sqrt{3}) + \frac{2}{3}\int_0^{2-\sqrt{3}} \frac{\arctan x}{x}\,dx \tag{8}$$

$$\frac{2}{3}G = \sum_{n=0}^{\infty} \frac{(\sin(2n+1)\pi/6)}{(2n+1)^2} \tag{9}$$

2 Proofs

2.1 Proof of Equation (1):

$$G = \int_0^1 \frac{\arctan x}{x}\, dx$$

As shown below, the area under the curve is equal to Catalan's Constant.

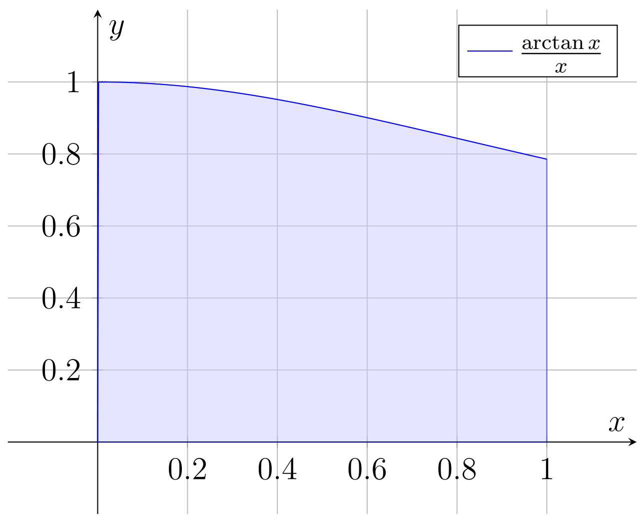


Figure 1: Graph of $y = \frac{\arctan x}{x}$

To prove that the integral is equal to Catalan's Constant, we need to use the idea of geometric sum. First, we can write $\arctan x$ in the form of the integral of $\frac{1}{1+x^2}$, and we find that integral can be written in the form of a geometric sum:

$$\begin{aligned}
\arctan x &= \int_0^x \frac{1}{1+t^2}\, dt \\
&= \int_0^x \sum_{n=0}^{\infty} (-t^2)^n dt \\
&= \sum_{n=0}^{\infty} \int_0^x (-1)^n t^{2n} dt
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

Following that, we can bring the new form to the integral and we get:

$$\begin{aligned}
\int_0^1 \frac{\arctan x}{x} dx &= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n} dx \\
&= \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n}{2n+1} x^{2n} dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \\
&= G
\end{aligned}$$

Note that $\int_0^1 \frac{\arctan x}{x}\, dx$ appears to be an improper integral, considering $x = 0$. However, by applying L'Hôpital's Rule, we find:

$$\lim_{x \to 0^+} \frac{\arctan x}{x} = \lim_{x \to 0^+} \frac{\frac{1}{x^2+1}}{1} = 1$$

Indicating that as x approaches 0, the limit exists and the discontinuity at 0 is removable. This implies that the function can be suitably defined or modified at $x = 0$ to make it continuous. Thus, the integral is not improper.

2.2 Proof of Equation (2):

$$G = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{x}{\sin x}\, dx$$

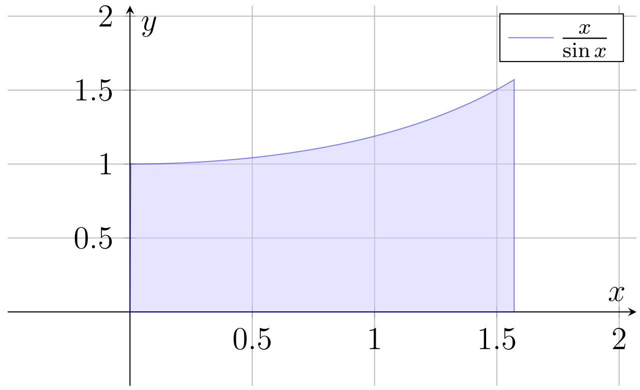


Figure 2: Graph of $y = \frac{x}{\sin x}$
The area under the curve is equal to Catalan's Constant

Starting with (1), if we let:

$$y = \tan\frac{x}{2}, x = 2\arctan y$$

and substitute, we get:

$$\begin{aligned} G &= \int_0^1 \frac{\arctan y}{y}\, dy = \int_0^{\frac{\pi}{2}} \frac{\frac{x}{2}}{\tan\frac{x}{2}} \cdot \frac{1}{2}\sec^2\frac{x}{2}\, dx \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{x}{2\sin\frac{x}{2}\cos\frac{x}{2}}\, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{x}{\sin x}\, dx \end{aligned}$$

2.3 Proof of Equation (3):

$$G = -\int_0^1 \frac{\ln x}{1+x^2}\, dx$$

Using integration by parts, if we let $u = \ln x, dv = \frac{dx}{1+x^2}$, then we have $du = \frac{1}{x}dx, v = \arctan x$ and thus:

$$\int_0^1 \frac{\ln x}{1+x^2}\, dx = (\ln x)(\arctan x)\Big|_0^1 - \int_0^1 \frac{\arctan x}{x}\, dx$$

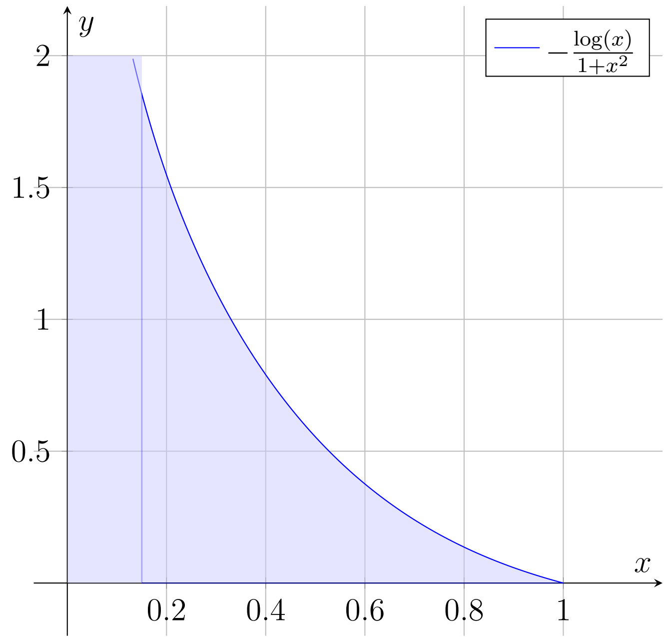


Figure 3: Graph of $\frac{\ln x}{1+x^2}$

Considering $(\ln x)(\arctan x)$, note that $\ln(0)$ is undefined. Therefore, we express the function as $\frac{\ln x}{\frac{1}{\arctan x}}$. Applying L'Hôpital's Rule, we find:

$$\begin{aligned}\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{\arctan x}} &= \lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{1}{-(\arctan x)^2(x^2+1)}}\\ &= \lim_{x\to 0^+}\frac{-(\arctan x)^2(x^2+1)}{x}\\ &= \lim_{x\to 0^+}\frac{-2\arctan x - 2x(\arctan x)^2}{1} = 0\end{aligned}$$

Thus, as x approaches 0^+, $(\ln x)(\arctan x)$ tends to 0. Additionally, since $\int_0^1 \frac{\arctan x}{x}\,dx = G$ has already been proven, then we have:

$$-\int_0^1 \frac{\ln x}{1+x^2}\,dx = -\left(0 - \int_0^1 \frac{\arctan x}{x}\,dx\right) = G$$

2.4 Proof of Equation (4):

$$G = -2\int_0^{\frac{\pi}{4}} \log(2\sin x)\, dx$$

Using the Maclaurin series

$$-\log(1-x) = \sum_{k=1}^{\infty} \frac{1}{k} x^k,$$

we can use Euler;s formula to express $\log(\sin x)$ in two different ways, the first of which we factor out $\frac{e^{ix}}{2i}$:

$$\begin{aligned}
\log(\sin x) &= \log\left(\frac{e^{ix} - e^{-ix}}{2i}\right) \\
&= \log\left(\frac{e^{ix}}{2i}\left(1 - e^{-2ix}\right)\right) \\
&= \log\frac{e^{ix}}{2i} + \log(1 - e^{-2ix}) \\
&= ix - \log(2i) - \sum_{k=1}^{\infty} \frac{1}{k} e^{-2ikx}
\end{aligned}$$

For the second we factor out $\frac{e^{-ix}}{2i}$:

$$\begin{aligned}
\log(\sin x) &= \log\left(\frac{e^{ix} - e^{-ix}}{2i}\right) \\
&= \log\left(-\frac{e^{-ix}}{2i}\left(1 - e^{2ix}\right)\right) \\
&= \log(-\frac{e^{-ix}}{2i}) + \log(1 - e^{2ix}) \\
&= -ix - \log(-2i) - \sum_{k=1}^{\infty} \frac{1}{k} e^{2ikx}
\end{aligned}$$

It is important to note that we are dealing with logarithms of complex numbers, unlike real numbers. When expressing $e^{i\pi} = \cos x + i \sin x$, it is crucial to recognize that adding multiples of $2i\pi$ yields the same result. To avoid ambiguity and simplify the computation, we use single-valued computation, i.e., we choose the principal value in $(-\pi, \pi]$ on the unit circle for log.

Next, by adding the two different expressions together, we get $2\log(\sin x)$, and by shifting the terms, we get $\log 2(\sin x)$.

$$\begin{aligned}
-(\log(\sin x) + \log(\sin x)) &= -2\log(\sin x) \\
&= \log(2i) + \log(-2i) + \sum_{k=1}^{\infty} \frac{1}{k}(e^{-2ikx} + e^{2ikx}) \\
&= \log(4) + \sum_{k=1}^{\infty} \frac{2\cos(2kx)}{k}
\end{aligned}$$

Using this equation, we are able to write $-\log(2\sin x)$ in the new form as follows.

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\cos(2kx)}{k} &= -\log(\sin x) - \log(2) \\
&= -\log(2\sin x)
\end{aligned}$$

Replace the original integral by the new form $\sum\limits_{k=1}^{\infty} \dfrac{\cos(2kx)}{k}$. We find for each even term, we have: $\frac{\sin n\pi}{n^2} = 0$, and for the odd terms, we have $\frac{\sin(\frac{n\pi+\frac{\pi}{2}}{2})}{n^2}$ oscillating between 1 and -1. Thus, we can just sum the odd terms:

$$-2\int_0^{\frac{\pi}{4}} \log(2\sin x)\, dx = \int_0^{\frac{\pi}{4}} \sum_{n=1}^{\infty} \frac{2\cos(2nx)}{n}\, dx$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{\sin(2nx)}{n^2}\Bigg|_0^{\frac{\pi}{4}} \\
&= \sum_{n=0}^{\infty} \frac{\sin(n\pi + \frac{\pi}{2})}{(2n+1)^2} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = G
\end{aligned}
$$

2.5 Proof of Equation (5):

$$G = 2\int_0^{\frac{\pi}{4}} \log(2\cos x)\, dx$$

Note that $2\log(\sin x) = 2\log(\cos(\frac{\pi}{2} - x))$. And recall (4), if we use substitution, let $u = \frac{\pi}{2} - x$, then we have:

$$
\begin{aligned}
-2\int_0^{\frac{\pi}{4}} \log(2\sin x)\, dx &= -2\int_0^{\frac{\pi}{4}} \log\left(2\cos\left(\frac{\pi}{2} - x\right)\right)\, du \\
&= 2\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \log(2\cos u)\, du \\
&= 2\int_0^{\frac{\pi}{4}} \log(2\cos u)\, du - 2\int_0^{\frac{\pi}{2}} \log(2\cos u)\, du \\
&= 2\int_0^{\frac{\pi}{4}} \log(2\cos u)\, du - 0 \\
&= G
\end{aligned}
$$

We claim that $2\int_0^{\frac{\pi}{2}} \log(2\cos u)\, du = 0$.

First, since $\cos(\frac{\pi}{2} - x) = \sin x$, we have:

$$\int_0^{\frac{\pi}{2}} \log(\cos u)\, du = \int_0^{\frac{\pi}{2}} \log(\sin u)\, du$$

Then we double $\int_0^{\frac{\pi}{2}} \log(\sin u)\,du$:

$$\begin{aligned}
2\int_0^{\frac{\pi}{2}} \log(\sin u)\,du &= \int_0^{\frac{\pi}{2}} \log(\sin u\cos u)\,du \\
&= \int_0^{\frac{\pi}{2}} \log(2\sin u\cos u)\,du - \int_0^{\frac{\pi}{2}} \log 2\,du \\
&= \int_0^{\frac{\pi}{2}} \log(\sin 2u)\,du - \frac{\pi}{2}\log 2 \\
&= \frac{1}{2}\int_0^{\pi} \log(\sin x)\,dx - \frac{\pi}{2}\log 2
\end{aligned}$$

Note that $\log(\sin(\pi - x)) = \log(\sin x)$, then:

$$\int_0^{\pi} \log(\sin x)\,dx = 2\int_0^{\frac{\pi}{2}} \log(\sin x)\,dx,$$

then we can conclude these two integral:$\int_0^{\frac{\pi}{2}} \log(\cos u)\,du$, $\int_0^{\frac{\pi}{2}} \log(\sin u)\,du$ equal to $-\frac{\pi}{2}\log(2)$:

$$\begin{aligned}
2\int_0^{\frac{\pi}{2}} \log(\sin u)\,du &= \frac{1}{2}\cdot 2\int_0^{\frac{\pi}{2}} \log(\sin x)\,dx - \frac{\pi}{2}\log 2 \\
\int_0^{\frac{\pi}{2}} \log(\cos u)\,du &= \int_0^{\frac{\pi}{2}} \log(\sin u)\,du = -\frac{\pi}{2}\log 2
\end{aligned}$$

Substituting the values back in, we get

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \log(2\cos u)\,du &= \int_0^{\frac{\pi}{2}} \log 2\,du + \int_0^{\frac{\pi}{2}} \log(\cos u)\,du \\
&= \frac{\pi}{2}\log 2 - \frac{\pi}{2}\log 2 \\
&= 0
\end{aligned}$$

2.6 Proof of Equation (6):

$$G = \int_0^{\frac{\pi}{4}} \log(\cot x)\,dx = -\int_0^{\frac{\pi}{4}} \log(\tan x)\,dx$$

Since we have already proven that $G = 2\int_0^{\frac{\pi}{4}} \log(2\cos x)dx$ and $G = -2\int_0^{\frac{\pi}{4}} \log(2\sin x)dx$, we can combine these two equations. Adding them together yields:

$$
\begin{aligned}
-2\int_0^{\frac{\pi}{4}} \log(2\sin x)\,dx + 2\int_0^{\frac{\pi}{4}} \log(2\cos x)\,dx &= 2G \\
2\int_0^{\frac{\pi}{4}} \log\left(\frac{2\cos x}{2\sin x}\right)\,dx &= 2G \\
\int_0^{\frac{\pi}{4}} \log(\cot x)\,dx &= G
\end{aligned}
$$

Since $-\log(\tan x) = \log(\cot x)$, then it following that:

$$
\int_0^{\frac{\pi}{4}} \log(\cot x)dx = -\int_0^{\frac{\pi}{4}} \log(\tan x)dx = G
$$

2.7 Proof of Equation (7):

$$
G = -\frac{3}{2}\int_0^{2-\sqrt{3}} \frac{\ln x}{1+x^2}\,dx
$$

To prove this, we need to utilize a useful trigonometric identity 5.4: $\tan(3x) = \tan x \tan\left(x + \frac{\pi}{3}\right)\tan\left(\frac{\pi}{3} - x\right)$. Additionally, recall (6): $G = -\int_0^{\frac{\pi}{4}} \ln(\tan x)dx$.

First, observe that $2 - \sqrt{3} = \tan(\frac{\pi}{12})$. If we let $u = \frac{x}{3}$, then as $x = \frac{\pi}{4}$, we have $u = \frac{x}{3}, du = \frac{dx}{3}$. Substitute in $-\int_0^{\frac{\pi}{4}} \ln(\tan x)dx$, we have:

$$
-\int_0^{\frac{\pi}{4}} \ln(\tan x)\,dx = -3\int_0^{\frac{\pi}{12}} \ln(\tan 3u)du = G
$$

We are able to divide this integral into three parts according to $\tan(3x) = \tan x \tan\left(x + \frac{\pi}{3}\right)\tan\left(\frac{\pi}{3} - x\right)$:

$$
-\frac{1}{3}G = \int_0^{\frac{\pi}{12}} \ln(\tan 3u)du
$$

$$= \int_0^{\frac{\pi}{12}} \ln\left((\tan x)\tan\left(\frac{\pi}{3} - x\right)\tan\left(\frac{\pi}{3} + x\right)\right) dx$$

$$= \int_0^{\frac{\pi}{12}} \ln(\tan x)dx + \int_0^{\frac{\pi}{12}} \tan\left(\frac{\pi}{3} - x\right) dx + \int_0^{\frac{\pi}{12}} \tan\left(\frac{\pi}{3} + x\right) dx$$

Next, we express each term in the form of $\tan x$ and update the integral boundaries, resulting in:

$$= \int_0^{\frac{\pi}{12}} \ln(\tan x)dx - \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \ln(\tan x)dx + \int_{\frac{\pi}{3}}^{\frac{5\pi}{12}} \ln(\tan x)dx$$

$$= \int_0^{\frac{\pi}{12}} \ln(\tan x)\, dx + \int_{\frac{\pi}{4}}^{\frac{5}{12}\pi} \ln(\tan x)\, dx$$

Some transformations can be applied here. If we let $w = \frac{\pi}{2} - x$ then $w = \frac{\pi}{12}$ when $x = \frac{5}{12}\pi$, which then connects to the previous integral bounds

$$= \int_0^{\frac{\pi}{12}} \ln(\tan x)\, dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{12}} \ln\left(\tan\left(\frac{\pi}{2} - x\right)\right) dx$$

$$= \int_0^{\frac{\pi}{12}} \ln(\tan x)\, dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{12}} \ln(\cot x)\, dx$$

$$= \int_0^{\frac{\pi}{12}} \ln(\tan x)\, dx - \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \ln(\tan x)\, dx$$

$$= \int_0^{\frac{\pi}{12}} \ln(\tan x)\, dx - \int_0^{\frac{\pi}{4}} \ln(\tan x)\, dx + \int_0^{\frac{\pi}{12}} \ln(\tan x)\, dx$$

$$= 2\int_0^{\frac{\pi}{12}} \ln(\tan x)\, dx + G$$

$$= -\frac{1}{3}G = \int_0^{\frac{\pi}{12}} \ln(\tan 3u)du$$

We then conclude that:

$$G = -\frac{3}{2}\int_0^{\frac{\pi}{12}} \ln(\tan x)\, dx$$

At this point, we can establish equation (7) through substitution. Let $u = \tan x$, then $x = \arctan u, dx = \frac{1}{1+u^2}du$. Thus, we obtain:

$$G = -\frac{3}{2}\int_0^{\frac{\pi}{12}} \ln(\tan x)\, dx = -\frac{3}{2}\int_0^{2-\sqrt{3}} \frac{\ln(x)}{1+x^2}dx$$

2.8 Proof of Equation (8):

$$G = \frac{1}{8}\pi\ln(2+\sqrt{3}) + \frac{3}{2}\int_0^{2-\sqrt{3}} \frac{\arctan x}{x}\, dx$$

We recall from (7) that $G = -\frac{3}{2}\int_0^{2-\sqrt{3}} \frac{\ln x}{1+x^2}\, dx$. Using integration by parts, let $u = \ln x, dv = \frac{dx}{1+x^2}$ then $du = \frac{dx}{x}, v = \arctan x$, (also note that $2-\sqrt{3} = \tan\frac{\pi}{12}$) we get:

$$\begin{aligned} G = -\frac{3}{2}\int_0^{2-\sqrt{3}} \frac{\log(x)}{1+x^2}\, dx &= -\frac{3}{2}\left((\ln x)(\arctan x)\Big|_0^{2-\sqrt{3}} - \int_0^{2-\sqrt{3}} \frac{\arctan x}{x}\, dx \right) \\ &= -\frac{1}{8}\pi\ln(2-\sqrt{3}) + \frac{3}{2}\int_0^{2-\sqrt{3}} \frac{\arctan x}{x}\, dx \\ &= \frac{1}{8}\pi\ln(2+\sqrt{3}) + \frac{3}{2}\int_0^{2-\sqrt{3}} \frac{\arctan x}{x}\, dx \end{aligned}$$

2.9 Proof of Equation (9):

$$\frac{2}{3}G = \sum_{n=0}^{\infty} \frac{\sin((2n+1)\frac{\pi}{6})}{(2n+1)^2}$$

To prove the formula, we initially observe that we can break n into three parts: $3n, 3n+1$, and $3n+2$. We begin by expressing G in its definition, representing it as the sum of these three components:

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+1)^2} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+3)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+5)^2}$$

Express (9) in a similar manner, breaking it down into three sums:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\sin((2n+1)\pi/6)}{(2n+1)^2} \\
&= \sum_{n=0}^{\infty} \frac{\sin((2(3n)+1)\frac{\pi}{6})}{(2(3n)+1)^2} + \sum_{n=0}^{\infty} \frac{\sin((2(3n+1)+1)\frac{\pi}{6})}{(2(3n+1)+1)^2} \\
&\quad + \sum_{n=0}^{\infty} \frac{\sin((2(3n+2)+1)\frac{\pi}{6})}{(2(3n+2)+1)^2} \\
&= \sum_{n=0}^{\infty} \frac{\sin(n\pi + \frac{\pi}{6})}{(6n+1)^2} + \sum_{n=0}^{\infty} \frac{\sin(n\pi + \frac{\pi}{2})}{(6n+3)^2} + \sum_{n=0}^{\infty} \frac{\sin(n\pi + \frac{5}{6}\pi)}{(6n+5)^2} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \frac{1}{2}}{(6n+1)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+3)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n \frac{1}{2}}{(6n+5)^2}
\end{aligned}$$

We notice that a portion of our equation is identical to G. To obtain G, we manipulate the equation by first factoring out $\frac{1}{2}$ and incorporating the additional subtracted part to align with G. This yields:

$$\begin{aligned}
&= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+1)^2} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+3)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+5)^2} + 3\sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+3)^2}\right) \\
&= \frac{1}{2}\left(G + \sum_{n=0}^{\infty} \frac{3(-1)^n}{3^2(2n+1)^2}\right) \\
&= \frac{1}{2}\left(G + \frac{1}{3}G\right) \\
&= \frac{2}{3}G
\end{aligned}$$

3 Alternative Proof of Equation (7):

$$G = -\frac{3}{2}\int_0^{2-\sqrt{3}} \frac{\log(x)}{1+x^2}\,dx$$

We have already shown previously that the integral is equal to Catalan's constant by using the trigonometric identity 5.4: $\tan(3x) = \tan x \tan\left(x+\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}-x\right)$. Here we provide another method to carry out the proof, by verifying the following equations:

$$\int_0^x \frac{t}{2\sin t}dt = \frac{x}{2}\log\left|\tan\left(\frac{x}{2}\right)\right| + \sum_{k=1}^{\infty}\frac{\sin((2k-1)x)}{(2k-1)^2} \tag{10}$$

$$\int_0^{\tan x} \frac{\arctan t}{t}dt = \int_0^{2x}\frac{u}{2\sin u}du \tag{11}$$

$$\int_0^{2-\sqrt{3}} \frac{\arctan x}{x}dx = \frac{\pi}{12}\ln(2-\sqrt{3}) + \frac{2}{3}G \tag{12}$$

$$\int_0^{2-\sqrt{3}} \frac{-\ln x}{1+x^2}dx = \frac{2}{3}G \tag{13}$$

$$\int_0^{\frac{\pi}{12}} \ln(\tan x)dx = -\frac{2}{3}G \tag{14}$$

To begin, we establish the validity of the first equation (10) by proving the following:

If f is continuous on $[a,b]$ and $\theta \in \mathbb{R}$, then:

$$\lim_{n\to\infty}\int_a^b f(x)\cos(nx+\theta)dx = 0.$$

Suppose $f(x) = x^k$ for some positive integer k. We use integration by parts with:

$$u = x^k \qquad\qquad dv = \cos(nx+\theta)dx$$
$$du = x^{k-1}dx \qquad\qquad v = \frac{\sin(nx+\theta)}{n}$$

Then we have:

$$\int_a^b x^k \cos(nx+\theta)\,dx$$
$$= \left.\frac{x^k \sin(nx+\theta)}{n}\right|_a^b - \int_a^b \frac{\sin(nx+\theta)kx^{k-1}}{n}\,dx$$
$$= \left(\frac{b^k \sin(nb+\theta) - a^k \sin(na+\theta)}{n}\right) - \frac{1}{n}\left[\int_a^b \sin(nx+\theta)kx^{k-1}\,dx\right]$$

Since $|\sin(nx+\theta)| \leq 1$ for all x, and for all n, there exists $M \in \mathbb{R}$ such that $\left|\int_a^b \sin(nx+\theta)kx^{k-1}\,dx\right| \leq M$ for all n. Thus, as n approach to infinity, the integral with $\frac{1}{n}$ goes to 0

$$\left|\int_a^b x^k \cos(nx+\theta)\,dx\right| \leq \frac{|b|^k + |a|^k + M}{n}$$
$$\lim_{n\to\infty} \frac{|b|^k + |a|^k + M}{n} = 0$$

Thus, as n approach to infinity, $\int_a^b a_k x^k \cos(nx+\theta)\,dx$ goes to 0.

Furthermore, since we have shown that for any term $a_k x^k$, the integral $\int_a^b x^k \cos(nx+\theta)\,dx$ tends to 0 as n approach to infinity. Hence, for any polynomial $p(x) = \sum_{k=0}^{r} a_k x^k$ we can conclude that as n goes to infinity, each term of the sum will also tend to 0.

Now, we extend our result to the case where $f(x)$ is not a polynomial. Since f is continuous, and it is a known fact that there exists a polynomial $p(x)$ such that the difference between $f(x)$ and $p(x)$ can be made arbitrarily small.

Let $\epsilon > 0$ and let $\frac{\epsilon}{b-a+1} = \epsilon_1$. There exists a polynomial $p(x)$ such that:

$|f(x) - p(x)| < \epsilon_1$ for all $x \in [a, b]$, and there exists N such that: $|\int_a^b p(x)\cos(nx+\theta)dx| < \epsilon_1$ for all $n \geq N$. For all $n \geq N$, we have:

$$\left|\int_a^b f(x)\cos(nx+\theta)dx\right| = \left|\int_a^b (f(x) - p(x) + p(x))\cos(nx+\theta)dx\right|$$
$$\leq \int_a^b |f(x) - p(x)|\,|\cos(nx+\theta)|\,dx + \left|\int_a^b p(x)\cos(nx+\theta)dx\right|$$
$$\leq \int_a^b \epsilon_1 dx + \epsilon_1$$
$$= \epsilon_1(b-a+1)$$
$$= \epsilon$$

Thus, we have proved if f is continuous on $[a, b]$ and $\theta \in \mathbb{R}$, then $\lim\limits_{n\to\infty} \int_a^b f(x)\cos(nx+\theta)dx = 0$.

3.1 First Equation (10):

$$\int_0^x \frac{t}{2\sin t}dt = \frac{x}{2}\log\left|\tan\left(\frac{x}{2}\right)\right| + \sum_{k=1}^{\infty}\frac{\sin((2k-1)x)}{(2k-1)^2}$$

Recall that $\lim\limits_{n\to\infty} \int_a^b f(x)\cos(nx+\theta)dx = 0$ and trig-identity, then we can rewrite our integral:

$$\int_0^x \frac{t}{2\sin t}dt = \int_0^x \frac{t}{2\sin t}dt - 0$$

$$
\begin{aligned}
&= \int_0^x \frac{t}{2\sin t}dt - \lim_{n\to\infty}\int_0^x \frac{t}{2\sin t}\cos(2nt)dt \\
&= \lim_{n\to\infty}\int_0^x \frac{t}{2\sin t}(1-\cos(2nt))dt \\
&= \lim_{n\to\infty}\int_0^x \frac{t}{2\sin t}\sum_{k=1}^{n} 2\sin t\sin((2k-1)t)dt \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\int_0^x t\sin((2k-1)t)dt
\end{aligned}
$$

To evaluated the integral, we let $u = t, dv = \sin((2k-1)t)dt$, then $du = dt, v = -\frac{\cos((2k-1)t)}{(2k-1)}$, we have:

$$
\begin{aligned}
&= \lim_{n\to\infty}\sum_{k=1}^{n}\left(-t\frac{\cos((2k-1)t)}{(2k-1)}\bigg|_0^x + \int_0^x \frac{\cos((2k-1)t)}{(2k-1)}dt\right) \\
&= -\frac{x}{2}\log\left|\tan\left(\frac{x}{2}\right)\right| + \sum_{k=1}^{\infty}\left(\frac{\sin((2k-1)x)}{(2k-1)^2}\right)
\end{aligned}
$$

Two steps are worth noting here. First, the identity

$$
\sum_{k=1}^{N} 2\sin t\sin((2k-1)t) = 1-\cos(2Nt).
$$

is verified in the appendix 5.5

The second point is showing that:

$$
\sum_{k=1}^{\infty}\frac{\cos((2k-1)x)}{(2k-1)} = -\frac{1}{2}\log\left|\tan\left(\frac{x}{2}\right)\right|
$$

.

Recall that $-\log|2\sin x| = \sum_{k=1}^{\infty}\frac{\cos(2kx)}{k}$, as shown in (4), then we have:

$$
-\frac{1}{2}\log|2\sin x| = \sum_{k=1}^{\infty}\frac{\cos((2kx)}{2k}
$$

$$-\log\left|2\sin\left(\frac{x}{2}\right)\right| = \sum_{k=1}^{\infty}\frac{\cos(kx)}{k}$$

We observe that all we need is the odd part of the sum of $\frac{\cos(kx)}{k}$, so we just subtract the even part from all:

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{\cos(2k-1)x)}{(2k-1)} &= \sum_{k=1}^{\infty}\frac{\cos(kx)}{k} - \sum_{k=1}^{\infty}\frac{\cos((2kx)}{2k} \\
&= -\log\left|2\sin\left(\frac{x}{2}\right)\right| + \frac{1}{2}\log|2\sin x| \\
&= \frac{1}{2}\log\left|\frac{2\sin x}{(2\sin\left(\frac{x}{2}\right))^2}\right| \\
&= \frac{1}{2}\log\left|\frac{4\sin(\frac{x}{2})\cos(\frac{x}{2})}{4\sin^2(\frac{x}{2})}\right| \\
&= \frac{1}{2}\log\left|\cot\left(\frac{x}{2}\right)\right| \\
&= -\frac{1}{2}\log\left|\tan\left(\frac{x}{2}\right)\right|
\end{aligned}$$

3.2 Second Equation (11):

$$\int_0^{\tan x}\frac{\arctan t}{t}dt = \int_0^{2x}\frac{u}{2\sin u}du$$

To prove that the two integrals are the same, we need to use substitution. If we let $u = 2\arctan t$, then $t = \tan\frac{u}{2}, dt = \sec^2(\frac{u}{2})\cdot\frac{1}{2}du$. And when $t = \tan x, u = 2x, t = 0, u = 0$, bringing these in gives us:

$$\begin{aligned}
\int_0^{\tan x}\frac{\arctan t}{t}dt &= \int_0^{2x}\frac{\frac{u}{2}}{\tan\frac{u}{2}}\cdot\sec^2\left(\frac{u}{2}\right)\cdot\frac{1}{2}du \\
&= \int_0^{2x}\frac{u}{4}\cdot\frac{1}{\cos^2\frac{u}{2}}\cdot\frac{1}{\frac{\sin\frac{u}{2}}{\cos\frac{u}{2}}}du
\end{aligned}$$

$$= \int_0^{2x} \frac{u}{2} \cdot \frac{1}{2\cos\frac{u}{2}\sin\frac{u}{2}} du$$
$$= \int_0^{2x} \frac{u}{2\sin u} du$$

3.3 Third Equation (12):

$$\int_0^{2-\sqrt{3}} \frac{\arctan x}{x} dx = \frac{\pi}{12}\ln(2-\sqrt{3}) + \frac{2}{3}G$$

To prove this equation, we need to recall (9): $\frac{2}{3}G = \sum_{n=0}^{\infty} \frac{\sin((2n+1)\frac{\pi}{6})}{(2n+1)^2}$.
Since we have shown:
$\int_0^{\tan x} \frac{\arctan t}{t} dt = \int_0^{2x} \frac{u}{2\sin u} du$,
$\int_0^{x} \frac{t}{2\sin t} dt = \frac{x}{2}\log\left|\tan\left(\frac{x}{2}\right)\right| + \sum_{k=1}^{\infty} \frac{\sin((2k-1)x)}{(2k-1)^2}$

Then if we do substitution here, let $y = 2x$, then $x = \frac{y}{2}$, then we have:

$$\int_0^{\tan x} \frac{\arctan t}{t} dt = \int_0^{2x} \frac{t}{2\sin t} dt = x\log|\tan(x)| + \sum_{k=1}^{\infty} \frac{\sin((2k-1)y)}{(2k-1)^2}$$

Also, as we shown earlier when 5.2: for $x = \frac{\pi}{12}$, $\tan x = 2-\sqrt{3}$, we have:

$$= \frac{\pi}{12}\log\left|\tan\left(\frac{\pi}{12}\right)\right| + \sum_{k=1}^{\infty} \frac{\sin((2k-1)\frac{\pi}{6})}{(2k-1)^2}$$
$$= \frac{\pi}{12}\ln(2-\sqrt{3}) + \frac{2}{3}G$$

3.4 Fourth Equation (13):

$$\int_0^{2-\sqrt{3}} \frac{-\ln x}{1+x^2} dx = \frac{2}{3}G$$

If we do substitution here, where we let $u = \arctan x, dv = \frac{dx}{x}$, then $du = \frac{dx}{1+x^2}, v = \ln x$, then:

$$\int_0^{2-\sqrt{3}} \frac{\arctan x}{x} dx = \frac{\pi}{12} \ln(2-\sqrt{3}) - \int_0^{2-\sqrt{3}} \frac{\ln x}{1+x^2} dx$$

Recall 12, we already shown that:

$$\int_0^{2-\sqrt{3}} \frac{\arctan x}{x} dx = \frac{\pi}{12} \ln(2-\sqrt{3}) + \frac{2}{3} G$$

Since these two expressions represent the same integral, where $\frac{\pi}{12} \ln(2-\sqrt{3})$ represents the common part, $-\int_0^{2-\sqrt{3}} \frac{\ln x}{1+x^2} dx$ and $\frac{2}{3}G$ represent the remaining parts, we can conclude that:

$$\int_0^{2-\sqrt{3}} \frac{-\ln x}{1+x^2} dx = \frac{2}{3} G$$

3.5 Fifth Equation (14):

$$\int_0^{\frac{\pi}{12}} \ln(\tan x) dx = -\frac{2}{3} G$$

Recall that $\int_0^{2-\sqrt{3}} \frac{-\ln x}{1+x^2} dx = \frac{2}{3}G$. If we perform the substitution $u = \tan x$, then $x = \arctan u$, $dx = \frac{1}{1+u^2} du$. When $x = \frac{\pi}{12}$, $u = 2-\sqrt{3}$, substituting these values in, we have:

$$\int_0^{\frac{\pi}{12}} \ln(\tan x) dx = \int_0^{2-\sqrt{3}} \frac{\ln(u)}{1+u^2} du = -\frac{2}{3} G$$

4 Acknowledgement

This project would not have been possible without the support and encouragement of these people.

The first and most important person I would like to thank is my supervisor, Prof. Gordon. I sincerely thank him for his patient guidance and unwavering support throughout this project. I am extremely fortunate that I have had the opportunity to work with and learn from him. His dedication and expertise have been crucial to this project. I am grateful to him for showing me more of the magic of mathematics.

I would also like to extend my heartfelt thanks to Prof. Keef, who gave me many suggestions and guidance to enrich the content and depth of this project.

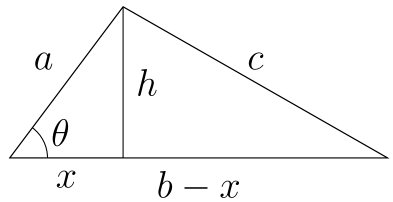
Thanks also to my peer editor and reader Holly, whose comments improved the readability and coherence of this paper.

Finally, I would like to extend special thanks to my family and friends: Sia, Tina, Yingshan, and Zihan for their trust and companionship throughout.

5 Appendix

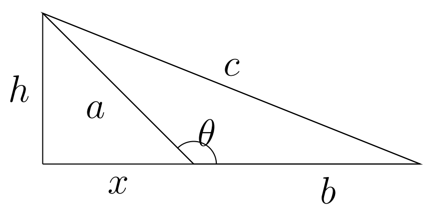
5.1 Law of Cosines

Theorem 5.1 (Law of Cosines)**.** $c^2 = a^2 + b^2 - 2ab\cos(\theta)$



Proof. Proof for acute angle. Consider $x = a \cdot \cos\theta$.

$$\begin{aligned} c^2 &= h^2 + (b-x)^2 \\ &= a^2 - x^2 + (b-x)^2 \\ &= a^2 - x^2 + b^2 - 2bx + x^2 \\ &= a^2 + b^2 - 2bx \\ &= a^2 + b^2 - 2ab\cos\theta \end{aligned}$$

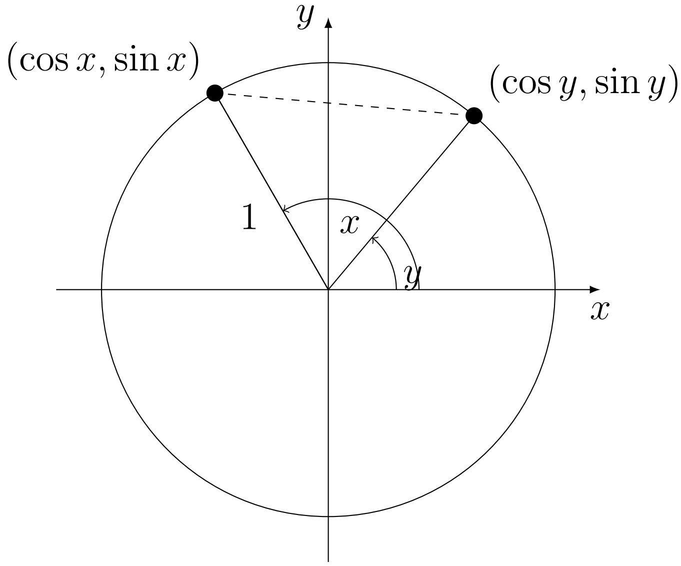


Proof for obtuse angle. Note that $x = \cos(\pi - \theta) \cdot a = -\cos\theta \cdot a$. Then we have:

$$\begin{aligned} c^2 &= h^2 + (b+x)^2 \\ &= a^2 - x^2 + (b+x)^2 \\ &= a^2 - x^2 + b^2 + 2bx - x^2 \\ &= a^2 + b^2 + 2bx \\ &= a^2 + b^2 - 2ab\cos\theta \end{aligned}$$

Thus, the law of cosines is valid for both acute and obtuse angles. $\square$

5.2 Identity of $\cos(x-y)$



Proof. Using the law of cosine, we have:

$$\begin{aligned}(\cos x - \cos y)^2 + (\sin x - \sin y)^2 &= 1 + 1 - 2\cos(x-y)\\ 2 - 2\sin y \sin x - 2\cos y \cos x &= 2 - 2\cos(x-y)\\ \cos(x-y) &= \sin y \sin x + \cos y \cos x\end{aligned}$$

□

5.3 $\cos(x+y)$, $\sin(x+y)$, and $\sin(x-y)$

Since we already proved $\cos(x-y) = \sin y \sin x + \cos y \cos x$, and we know $\sin x = \cos(x - \frac{\pi}{2})$, then we can get $\sin(x-y)$ by following:

$$\begin{aligned}\sin(x-y) &= \cos\left((x-y) - \frac{\pi}{2}\right) = \cos\left(x - \left(y + \frac{\pi}{2}\right)\right)\\ &= \sin\left(y + \frac{\pi}{2}\right)\sin x + \cos\left(y + \frac{\pi}{2}\right)\cos x\\ &= \cos y \sin x - \sin y \cos x\end{aligned}$$

In this way, we only need to flip the sign to get the formula for $\cos(x+y)$ and $\sin(x+y)$:

$$\cos(x+y) = \cos(x-(-y)) = \sin(-y)\sin x + \cos(-y)\cos x$$

$$
\begin{aligned}
&= \cos y \cos x - \sin y \sin x \\
\sin(x+y) = \sin(x-(-y)) &= \cos(-y)\sin x - \sin(-y)\cos x \\
&= \sin x \cos y + \sin y \cos x
\end{aligned}
$$

5.4 Proof $\tan 3x = \tan x \tan\left(x + \frac{\pi}{3}\right)\tan(\frac{\pi}{3} - x)$

To prove the equation we first need to get the formula for $\tan(x+y)$, recall that we already have the formula for $\cos(x+y)$ and $\sin(x+y)$, so simply by dividing $\sin(x+y)$ by $\cos(x+y)$we get:

$$
\begin{aligned}
\tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)} \\
&= \frac{\sin x \cos y + \sin y \cos x}{\cos y \cos x - \sin y \sin x} \cdot \frac{\frac{1}{\cos x \cos y}}{\frac{1}{\cos x \cos y}} \\
&= \frac{\frac{\sin x \cos y}{\cos y \cos x} + \frac{\sin y \cos x}{\cos y \cos x}}{1 - \tan y \tan x} \\
&= \frac{\tan x + \tan y}{1 - \tan y \tan x}
\end{aligned}
$$

In this way, we can apply the formula by writing $\tan(3x)$ as $\tan(x+2x)$, and $\tan(2x)$. We can derive by applying the formula by writing $\tan(x+x)$.

$$
\begin{aligned}
\tan(3x) = \tan(2x+x) &= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} \\
&= \frac{\frac{2\tan x}{1-\tan^2 x} + \tan x}{1 - \frac{2\tan x}{1-\tan^2 x}\tan x} \cdot \frac{1-\tan^2 x}{1-\tan^2 x} \\
&= \frac{2\tan x + \tan x - \tan^3 x}{1 - \tan^2 x - 2\tan^2 x} \\
&= \tan x \frac{3 - \tan^2 x}{1 - 3\tan^2 x} \\
&= \tan x \frac{\left(\sqrt{3} + \tan x\right)\left(\sqrt{3} - \tan x\right)}{\left(1 - \sqrt{3}\tan x\right)\left(1 + \sqrt{3}\tan x\right)} \\
&= \tan x \tan\left(x + \frac{\pi}{3}\right)\tan\left(\frac{\pi}{3} - x\right)
\end{aligned}
$$

5.5 $\sum_{k=1}^{N} 2\sin t \sin((2k-1)t) = 1 - \cos(2Nt)$

To show $\sum_{k=1}^{N} 2\sin t \sin((2k-1)t) = 1 - \cos(2Nt)$ we need to recall that $\cos(x-y) = \sin y \sin x + \cos y \cos x$, and telescoping sums.

$$\begin{aligned}\sum_{k=1}^{N} 2\sin t \sin((2k-1)t) &= \sum_{k=1}^{N} (\cos((2k-1)t - t) - \cos((2k-1)t + t)) \\ &= \sum_{k=1}^{N} (\cos((2k-2)t) - \cos(2kt)) \\ &= 1 - \cos(2Nt)\end{aligned}$$

References

[1] Bradley, David. (2001). *Representations of Catalan's Constant.*

[2] Penn, Michael. (2021). *A nice integral and an important ignored constant.* `https://www.youtube.com/watch?v=XPBrFqU2Fxc&ab_channel=MichaelPenn`.

[3] Wyse, A. B.; Mayall, N. U. (1942). *Distribution of Mass in the Spiral Nebulae Messier 31 and Messier 33*

[4] Goldstein, Catherine. (2015). *The mathematical achievements of Eugène Catalan*

[5] Shanks, Daniel. (1959). *A sieve method for factoring numbers of the form* $n^2 + 1$