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Lecture 3 Exercise

1) Assume each friend orders a pizza, the total number of pizza to be ordered is 7. Since there are 4 types of pizza, and the chef only cares about the number of each type (such as 2 pizza #1, 2 pizza #2, 2 pizza #3 and 1 pizza #4), the order here doesn't matter, for example, AABBCCD is the same as ABABCCD (assuming ABCD is pizza type).

Let's denote pizza as an asterisk (*), we can have this diagram to visualize the problem easier: Pizza #1 Pizza #2 Pizza #3 Pizza #4

* * | * * | * * | *

Using star and bar approach, there are 7 pizza (7 asterisks *) and 3 bars (since there are 4 types), so there are 10 total available spots. The bars can be in anywhere in the 10 spots, therefore, the total number of possible combined orders are:

$$\binom{10}{3} = \frac{10!}{3!\,(10-3)!} = \frac{10!}{3!\,7!} = 120$$

2)

a) (i) The random variable is a

(ii) The domain is dom(a) = {1, 2, 3}

(iii) Since $\sum_{a \in dom(a)} p(a = a) = 0.6 + 0.04 + 0.34 = 0.98 < 1$, it is not a valid probability distribution

b) (i) The random variable is x

(ii) The domain is dom(x) = {a, b, c}

(iii) Since $\sum_{x \in dom(x)} p(x = x) = 0.46 + 0.16 + 0.38 = 1$, it is a valid probability distribution

c) (i) Random variable is z

(ii) dom(z) = {aa, bc, de}

(iii) $\sum_{z \in dom(z)} p(z = z) = 0.4 + 0.1 + 0.5 = 1$, it is a valid probability distribution

3) Given:

Box 1: 3 red, 5 white (8 total)   Box 2: 2 red, 5 white (7 total)

A box is chosen randomly p(box 1) = p(box 2) = 0.5

A ball chosen randomly from this box is red

p(box 1 | red) = ?

$p(box\ 1 \mid red) = \frac{p(red \mid box\ 1)\, p(box\ 1)}{p(red)}$ (1) (Bayes' rule)

$p(box\ 1) = 0.5$

$p(red \mid box\ 1) = \frac{3}{8}, \qquad p(red \mid box\ 2) = \frac{2}{7}$

$p(red) = \sum_{box} p(red, box)$ (Marginalize over box)

$$= p(red \mid box\ 1)p(box\ 1) + p(red \mid box\ 2)p(box\ 2)$$

$$= \frac{3}{8} * 0.5 + \frac{2}{7} * 0.5 = \frac{37}{112}$$

Plug all the above back into (1):

$$p(box\ 1 \mid red) = \frac{\frac{3}{8} * 0.5}{\frac{37}{112}} = \frac{21}{37}$$

4)

(i) $p(x|z) = \frac{p(x,z)}{p(z)}$ (Bayes' rule)

$= \frac{\sum_y p(x,y,z)}{p(z)}$ (Marginalize over y)

$= \frac{\sum_y p(x|y,z)\, p(y,z)}{p(z)}$ (Bayes' rule)

$= \frac{\sum_y p(x|y,z) p(y|z) p(z)}{p(z)} = \frac{p(z) \sum_y p(x|y,z) p(y|z)}{p(z)}$ (Bayes' rule)

$= \sum_y p(x|y,z) p(y|z)$ (1)

Similarly, p(x | y, z) can be expanded using the same rule as in (1):

$p(x|y,z) = \sum_w p(x|w,y,z) p(w|y,z)$ (2)

Plug (2) into (1):

$$(1) = \sum_y \left(\sum_w p(x|w,y,z) p(w|y,z)\right) p(y|z)$$

$$= \sum_{y,w} p(x|w,y,z) p(w|y,z) p(y|z)$$

(ii) Since y is unconditionally independent of z, p(y | z) = p(y)

Plug it into equation (1) from (i):

$$p(x|z) = \sum_y p(x|y,z) p(y|z) = \sum_y p(x|y,z) p(y)$$

(iii) Since x is conditionally independent of y, given z, p(x | y, z) = p(x | z)

Plug it into equation (1) from (i):

$$p(x|z) = \sum_y p(x|y,z) p(y|z) = \sum_y p(x|z) p(y|z)$$

5)

	$b1$	$b2$	$b3$
$a1$	0.42	0.05	0.02
$a2$	0.02	0.02	0.01
$a3$	0.02	0.02	0.42

$$p(A) = \sum_{B} p(A,B) = \begin{pmatrix} 0.42 + 0.05 + 0.02 \\ 0.02 + 0.02 + 0.01 \\ 0.02 + 0.02 + 0.42 \end{pmatrix} = \begin{pmatrix} 0.49 \\ 0.05 \\ 0.46 \end{pmatrix}$$

0.49 + 0.05 + 0.46 = 1 => Valid probability distribution

	$b1$	$b2$	$b3$
$a1$	0.42	0.05	0.02
$a2$	0.02	0.02	0.01
$a3$	0.02	0.02	0.42

$$p(B) = \sum_{A} p(A,B) = \begin{pmatrix} 0.42 + 0.02 + 0.02 \\ 0.05 + 0.02 + 0.02 \\ 0.02 + 0.01 + 0.42 \end{pmatrix} = \begin{pmatrix} 0.46 \\ 0.09 \\ 0.45 \end{pmatrix}$$

0.46 + 0.09 + 0.45 = 1 => Valid

6)

$p(A = a1)p(B = b1) = 0.49 * 0.46 = 0.2254$

$p(A = a1, B = b1) = 0.42 \neq 0.2254$

Therefore, A and B are not independent

Bonus)

Let x1, x2, x3, x4 be the number of pizza #1, #2, #3, and #4, respectively

Also, x1, x2, x3, x4 >= 0 and x1 + x2 + x3 + x4 = 7 (7 people => 7 pizza)

First, since each person has 4 options, the sample space is 4^7

Let x = $\{x1, x2, x3, x4\}$ be the variable storing the correct proportion of each pizza

Assume an ordered pizza sequence looks like _ _ _ _ _ _ _ (7 slots for 7 people choices)

=> If given x1, x2, x3, x3, the number of possible pizza sequences = $\frac{7!}{x1!x2!x3!x4!}$

=> Probability that the 7 friends choose this exact composition x1, x2, x3, x4:

$$\mathrm{P(X\ =\ x)} = \frac{7!}{4^7 x1!\, x2!\, x3!\, x4!}$$

Similarly, the probability that the chef chooses the correct pizza counts:

$$\mathrm{P(Chef = x)} = \frac{7!}{4^7 x1!\, x2!\, x3!\, x4!}$$

The probability that the chef matches the correct order:

$$\mathrm{P} = \sum\nolimits_{\mathrm{x}} P(X = x)P(Chef\ = x) = \sum\nolimits_{x} \left(\frac{7!}{4^7 x1! x2! x3! x4!}\right)^2 \quad (1)$$

From question 1, we know that there are 120 combinations of {x1, x2, x3, x4}

Therefore, to calculate the equation (1) above, we'll loop through all 120 combinations, which will take too long, so I will use Julia to calculate the result as below:

$$P = \sum_{x} \left(\frac{7!}{4^7 x1!\, x2!\, x3!\, x4!} \right)^2 = \frac{\left(\frac{7!}{x1!\, x2!\, x3!\, x4!} \right)^2}{4^{14}}$$

module_1 > bonus_question.jl

```julia
n = 7;
result = 0.0;
for x1 in 0:n
    for x2 in 0:(n-x1)
        for x3 in 0:(n-x1-x2)
            x4 = n - x1 - x2 - x3;
            global result += (factorial(n) / (factorial(x1) * factorial(x2) * factorial(x3) * factorial(x4)))^2;
        end
    end
end
result /= 4.0^14;
println("Probability that the chef matches the correct order: $result")
```

PROBLEMS OUTPUT DEBUG CONSOLE TERMINAL PORTS Julia REPL (v1.6.5)

```
Probability that the chef matches the correct order: 0.018445968627929688
julia>
```

Therefore, the probability that the chef delivers the correct order is 0.018