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Lecture 5 Exercise

1) f(x) follows an Exponential distribution

a) $P(X < 2\ or\ X > 4) = P(X < 2) + P(X > 4)$

$$= \int_{-\infty}^{2} f(x)dx + \int_{4}^{\infty} f(x)dx$$

$$= \int_{0}^{2} \frac{1}{2} e^{-\frac{x}{2}} dx + \int_{4}^{\infty} \frac{1}{2} e^{-\frac{x}{2}} dx$$

$$= -e^{-\frac{x}{2}} |_0^2 + \left(-e^{-\frac{x}{2}}\right) |_4^{\infty} = (-e^{-1} + 1) + (0 + e^{-2})$$

$$= e^{-2} - e^{-1} + 1 = 0.7675$$

b) $P(X > 3 \mid X > 2) = P(X > 2 + 1 \mid X > 2)$

$$= P(X > 1)\ (Memoryless\ property\ of\ exponential\ distribution)$$

$$= \int_1^{\infty} \frac{1}{2} e^{-\frac{x}{2}} dx = -e^{-\frac{x}{2}} |_1^{\infty} = e^{-\frac{1}{2}} = 0.6065$$

2)

$$E(x) = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{0} xf(x)dx + \int_{0}^{2} xf(x)dx + \int_{2}^{3} xf(x)dx + \int_{3}^{\infty} xf(x)dx$$

$$= 0 + \int_{0}^{2} x\left(\frac{x}{4}\right) dx + \int_{2}^{3} x\left(\frac{1}{2}\right) dx + 0$$

$$= \frac{x^3}{12} |_0^2 + \frac{x^2}{4} |_2^3 = \frac{2}{3} + \frac{9}{4} - 1 = 1.9167$$

3) Let $X \sim Uniform(0, 2)$ be launch time in hours in 2-hour window

Within 10 minutes (or 1/6 hours) of the center of the launch window means the interval of interest is [5/6, 7/6]

$$=> P\left(\frac{5}{6} < X < \frac{7}{6}\right) = cdf\left(\frac{7}{6}\right) - cdf\left(\frac{5}{6}\right) = \frac{\frac{7}{6} - 0}{2 - 0} - \frac{\frac{5}{6} - 0}{2 - 0} = \frac{7}{12} - \frac{5}{12} = \frac{1}{6} = 0.167$$

4) $Mean = \theta = 2.4,$ $\quad f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$

$$cdf(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\frac{x}{\theta}}, & x \geq 0 \end{cases}$$

a) $P(X \leq 2.5) = cdf(2.5) = 1 - e^{-\frac{2.5}{2.4}} = 0.6471$

b) $P(2 \leq X \leq 3) = cdf(3) - cdf(2) = \left(1 - e^{-\frac{3}{2.4}}\right) - \left(1 - e^{-\frac{2}{2.4}}\right) = 0.1481$

Bonus)

1)

2) $\quad f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$

$$E(x^k) = \int_{-\infty}^{\infty} x^k f(x) dx = \int_0^{\infty} \frac{x^k}{\theta} e^{-\frac{x}{\theta}} dx = \frac{1}{\theta} \int_0^{\infty} x^k e^{-\frac{x}{\theta}} dx$$

Let $t = \frac{x}{\theta} => x = \theta t, \quad dx = \theta dt$

$$E(x^k) = \frac{1}{\theta} \int_0^{\infty} (\theta t)^k e^{-t} \theta dt = \theta^k \int_0^{\infty} t^k e^{-t} dt$$

Since $\Gamma(\alpha + 1) = \int_0^{\infty} x^{\alpha} e^{-x} dx$, which is similar to the above equation:

$$E(x^k) = \theta^k \Gamma(k+1) = k!\,\theta^k \qquad (property\ of\ Gamma\ function)$$

3) $Mean = \theta = 10,$ $\quad f(R) = \frac{1}{\theta} e^{-\frac{R}{\theta}} = 0.1e^{-0.1R}$

Area = πR^2

$$E(Area) = E(\pi R^2) = \pi E(R^2) = \pi \int_0^{\infty} \frac{R^2}{\theta} e^{-\frac{R}{\theta}} dR = \frac{\pi}{\theta} \int_0^{\infty} R^2 e^{-\frac{R}{\theta}} dR$$

$$= \frac{\pi}{\theta} \theta^3 \Gamma(3) \qquad (Gamma\ function\ with\ \alpha = 3, \beta = \theta)$$

$$= \pi \theta^2 2! = 200\pi$$

$$\sigma^2 = Var(Area) = E(Area^2) - E(Area)^2 = E(\pi^2 R^4) - (200\pi)^2 \qquad (1)$$

$$E(\pi^2 R^4) = \pi^2 E(R^4) = \pi^2 \int_0^\infty \frac{R^4}{\theta} e^{-\frac{R}{\theta}} dR = \frac{\pi^2}{\theta} \int_0^\infty R^4 e^{-\frac{R}{\theta}} dR$$

$$= \frac{\pi^2}{\theta} \theta^5 \Gamma(5) \qquad (Gamma\ function\ with\ \alpha = 5, \beta = \theta)$$

$$= \pi^2 \theta^4 4! = 240{,}000\pi^2 \qquad (2)$$

Plug (2) into (1):

$$\sigma^2 = 240{,}000\pi^2 - 40{,}000\pi^2 = 200{,}000\pi^2$$