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Lecture 6 Exercise

1)

a) $P(X \leq 2) = cdf(2)$

```
alpha = 4
beta = 1.5

d = Gamma(alpha, beta)
prob = cdf(d, 2)
```

✓ 0.8s

```
0.046494302865334014
```

b)

- $E(l) = E(30x + 2) = 30E(x) + 2 = 30 \cdot \alpha\beta + 2 = 30 \cdot 4 \cdot 1.5 + 2 = 182$
- $Var(l) = E(l^2) - E(l)^2 = E((30x + 2)^2) - 182^2$

$$= E(900x^2 + 120x + 4) - 182^2$$

$$= 900E(x^2) + 120E(x) + 4 - 182^2$$

$$= 900(Var(x) + E(x)^2) + 120E(x) + 4 - 182^2$$

$$= 900(\alpha\beta^2 + (\alpha\beta)^2) + 120\alpha\beta + 4 - 182^2$$

$$= 900(4 \cdot 1.5^2 + (4 \cdot 1.5)^2) + 120 \cdot 4 \cdot 1.5 + 4 - 182^2$$

$$= 8100$$

2)

a) $P(X > 700) = 1 - P(X \leq 700) = 1 - cdf(700)$

```
mu = 600
sigma = 40

d = Normal(mu, sigma)
prob = 1 - cdf(d, 700)
```

✓ 0.0s

```
0.006209665325776159
```

b) This below is trying to find the smallest x value such that $cdf(x) \geq 0.9$

```
x = 300:0.01:1000
y = pdf.(d, x)

for i in length(x):-1:1
    if cdf(d, x[i]) < 0.9
        budget = x[i+1]
        print(budget)
        break
    end
end
```

✓ 0.0s

```
651.27
```

3)

a) $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1$

Since $f(x) = 20x^3(1-x),\ \alpha = 4, \beta = 2$

$P(X > 0.3) = 1 - P(X < 0.3) = 1 - cdf(0.3)$

```
alpha = 4
beta = 2

d = Beta(alpha, beta)
prob = 1 - cdf(d, 0.3)
```

✓ 0.0s

```
0.96922
```

b)

- $E(v) = E(10 - 0.75x) = 10 - 0.75E(x)$

$$= 10 - 0.75\frac{\alpha}{\alpha+\beta} = 10 - 0.75 \cdot \frac{4}{4+2} = 9.5$$

- $Var(v) = E(v^2) - E(v)^2$

$$= E(0.75^2x^2 - 15x + 100) - 9.5^2$$

$$= 0.75^2E(x^2) - 15E(x) + 100 - 9.5^2$$

$$= 0.75^2(Var(x) + E(x)^2) - 15E(x) + 9.75$$

$$= 0.75^2\left(\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + \left(\frac{2}{3}\right)^2\right) - 15 \cdot \frac{2}{3} + 9.75$$

$$= 0.75^2\left(\frac{4 \cdot 2}{(4+2)^2(4+2+1)} + \frac{4}{9}\right) - 0.25 \approx 0.0179$$

4)

a) $P(X < 2) = cdf(2)$

```
gamma = 2
theta = 4

d = Weibull(gamma, theta)
prob = cdf(d, 2)
```

✓ 0.0s

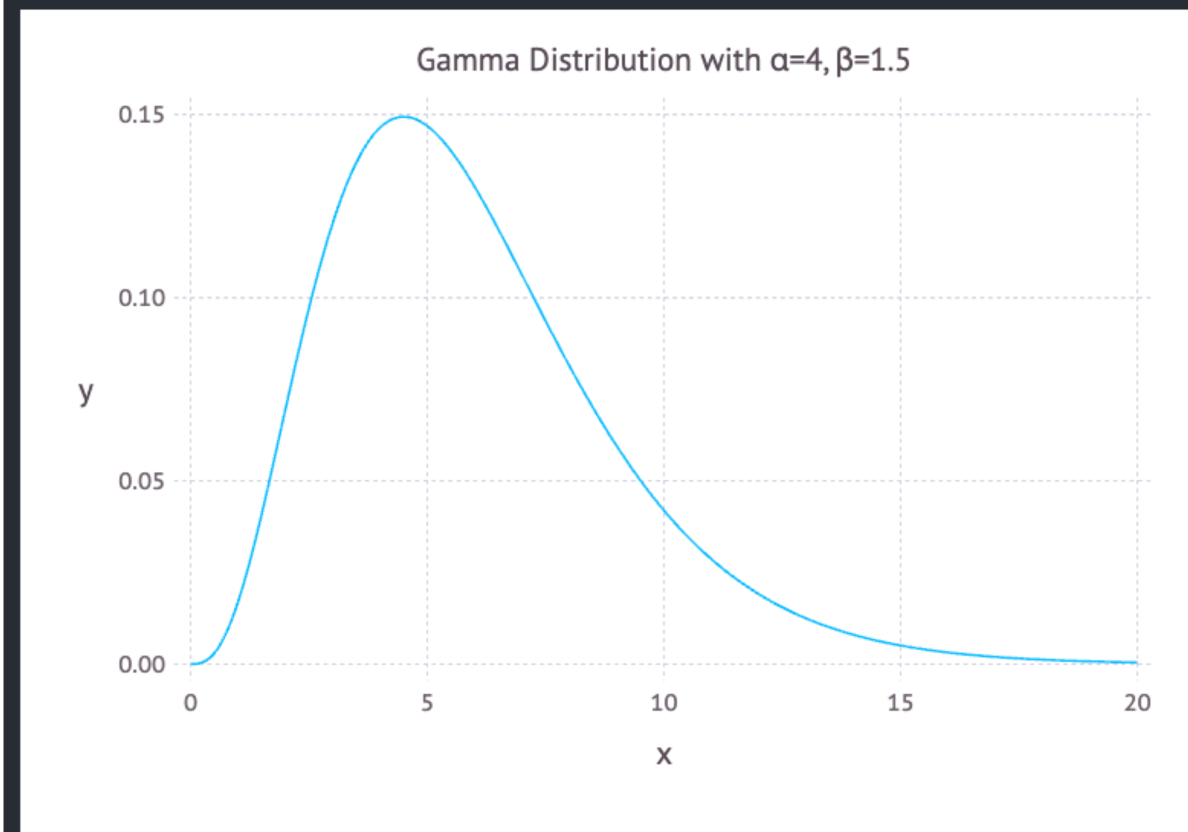
```
0.22119921692859512
```

b) $E(x) = \theta^{\frac{1}{\gamma}}\Gamma\left(1 + \frac{1}{\gamma}\right) = 4^{\frac{1}{2}}\Gamma\left(1 + \frac{1}{2}\right) = 2\Gamma(1.5)$

5)

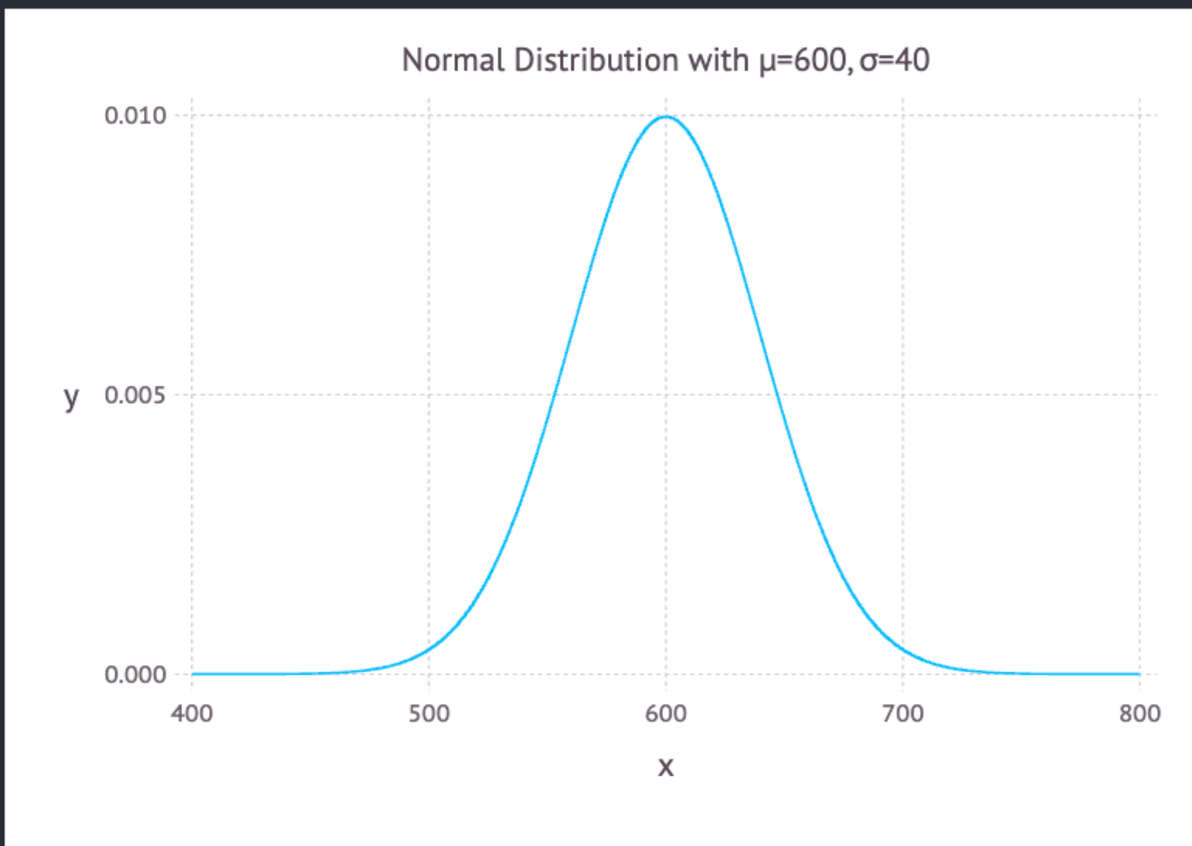
```julia
x1 = 0:0.01:20
y_gamma = pdf.(d_gamma, x1)
plot_gamma = plot(
    x=x1, y=y_gamma,
    Geom.line,
    Theme(background_color="white"),
    Guide.title("Gamma Distribution with α=4, β=1.5"),
)
```

✓ 0.1s



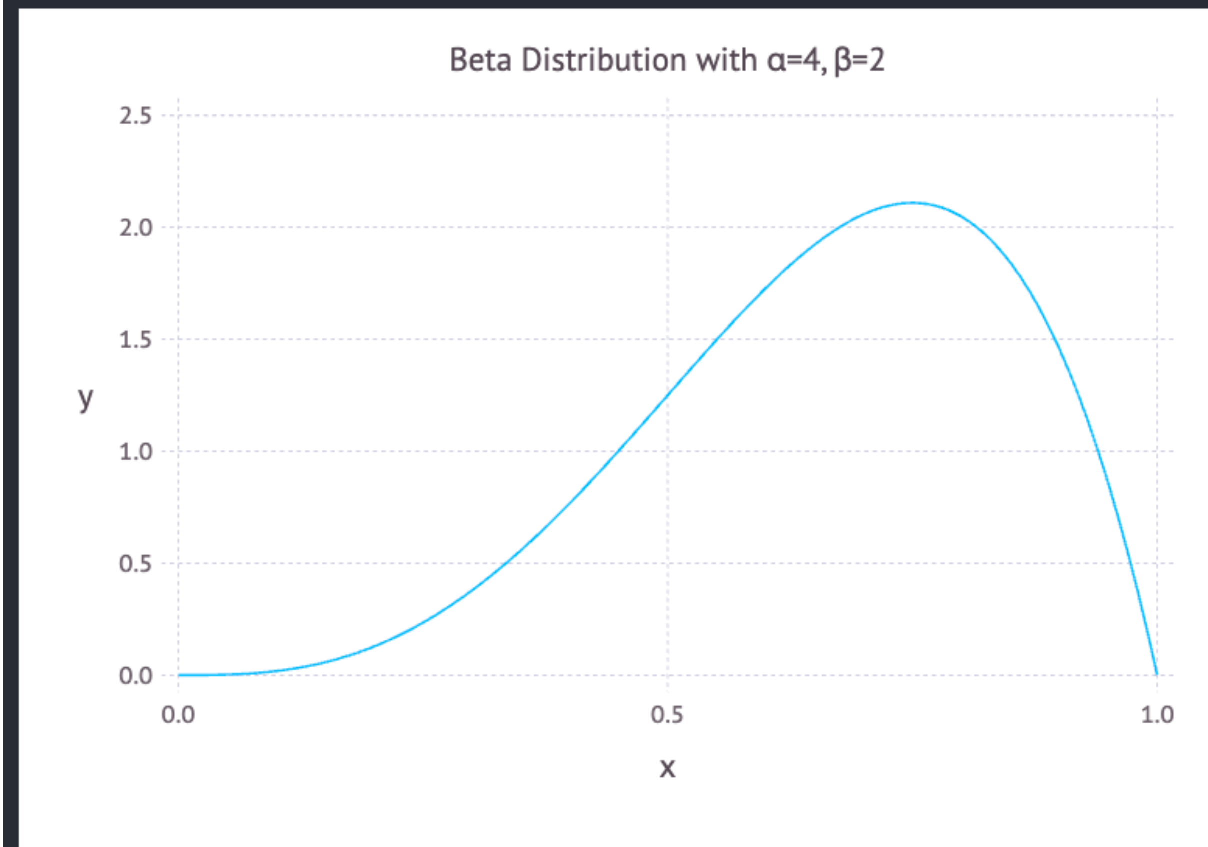
```
x2 = 400:0.01:800
y_normal = pdf.(d_normal, x2)
plot_normal = plot(
    x=x2, y=y_normal,
    Geom.line,
    Theme(background_color="white"),
    Guide.title("Normal Distribution with μ=600, σ=40"),
    Coord.cartesian(xmin=400, xmax=800)
)
```

✓ 0.3s



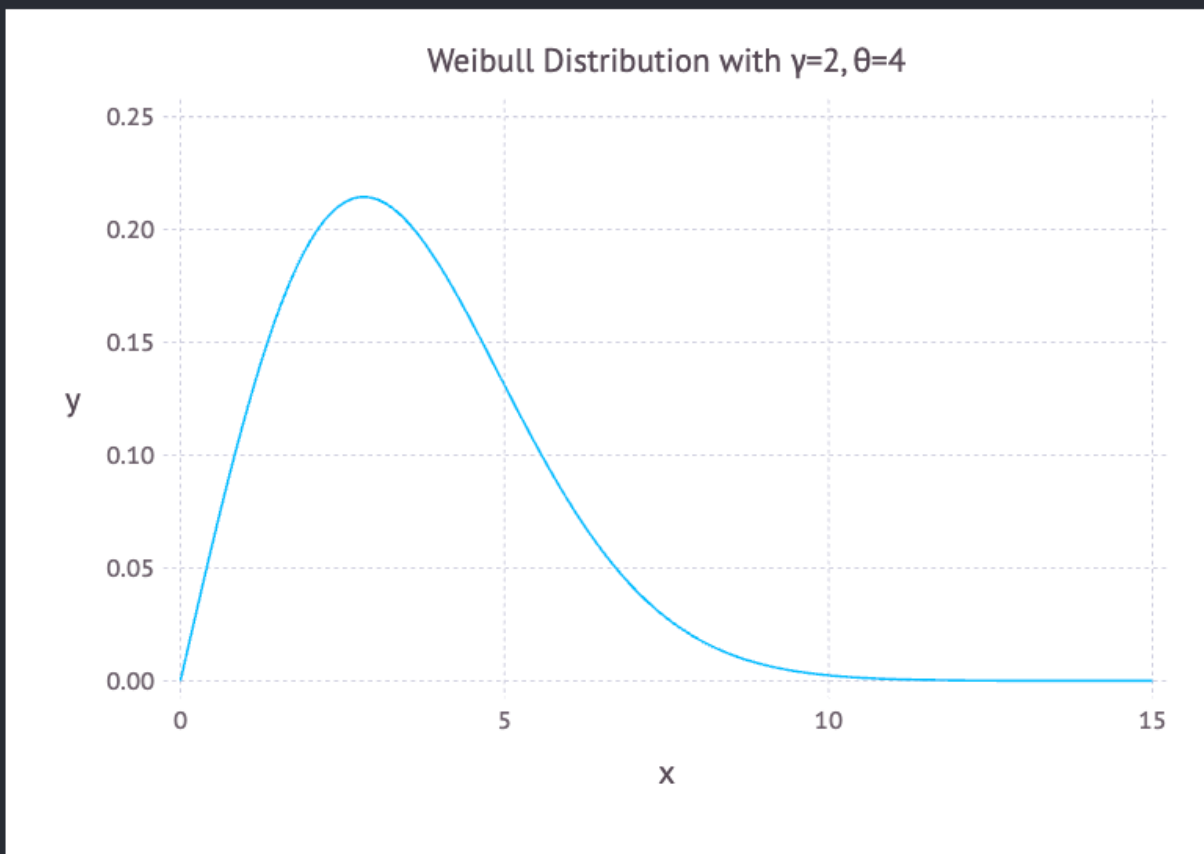
```
x3 = 0:0.01:1
y_beta = pdf.(d_beta, x3)
plot_beta = plot(
    x=x3, y=y_beta,
    Geom.line,
    Theme(background_color="white"),
    Guide.title("Beta Distribution with α=4, β=2"),
)
```

✓ 0.3s



```
x4 = 0:0.01:15
y_weibull = pdf.(d_weibull, x4)
plot_weibull = plot(
    x=x4, y=y_weibull,
    Geom.line,
    Theme(background_color="white"),
    Guide.title("Weibull Distribution with γ=2, θ=4"),
)
```

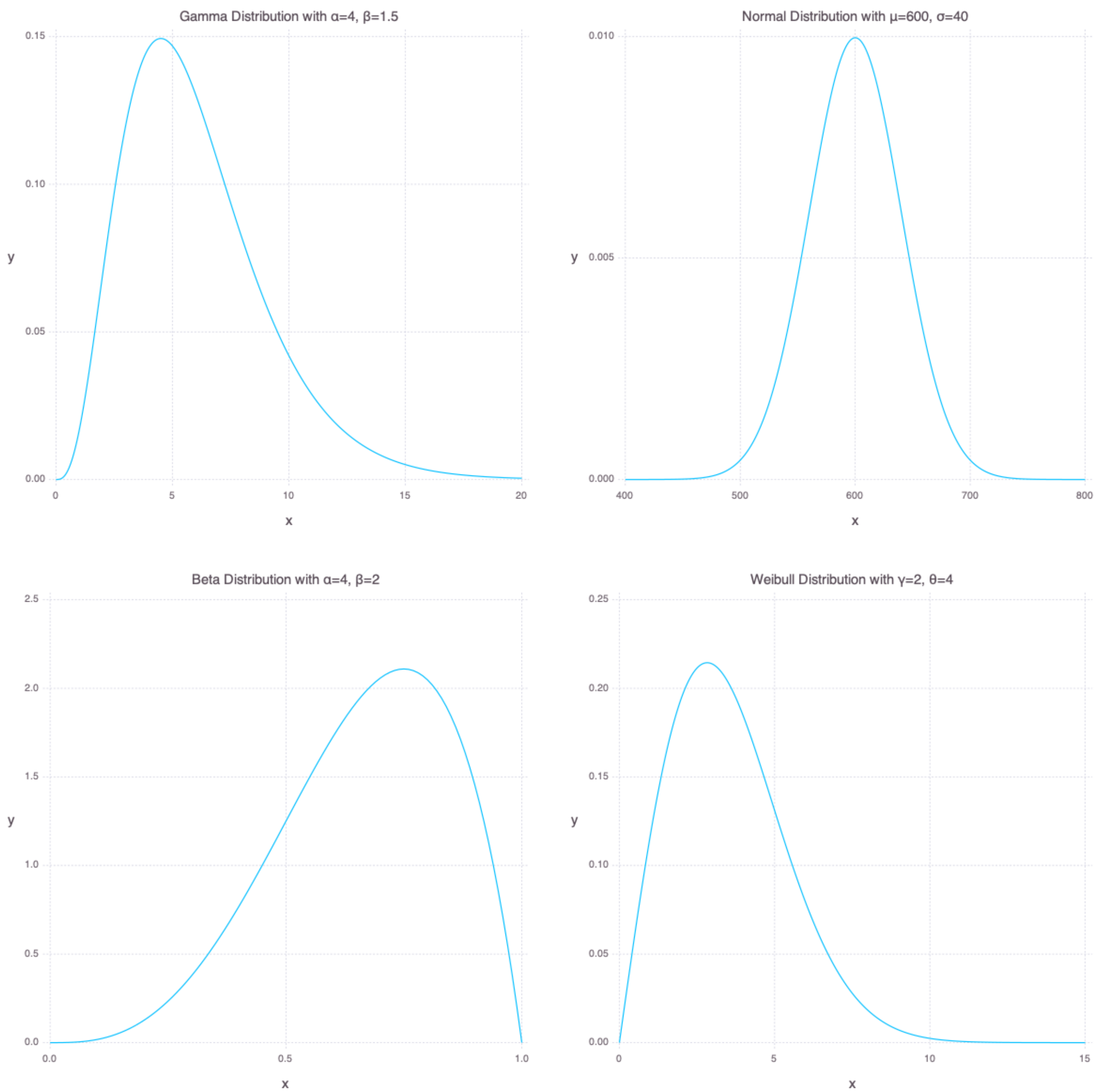
✓ 0.3s



```
using Cairo
using Compose

plot_all = vstack(
    hstack(plot_gamma, plot_normal),
    hstack(plot_beta, plot_weibull)
)
draw(PNG("all_plots.png", 12inch, 12inch), plot_all)
```

✓ 0.7s



Bonus)

1)

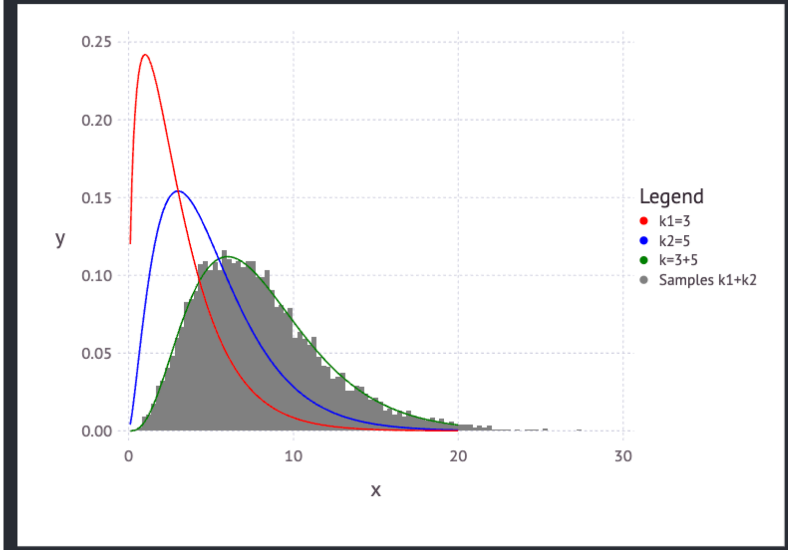
```
k1 = 3
k2 = 5
x = 0.1:0.1:20
n_samples = 10000

pdf1 = pdf.(Chisq(k1), x)
pdf2 = pdf.(Chisq(k2), x)
samples1 = rand(Chisq(k1), n_samples)
samples2 = rand(Chisq(k2), n_samples)
samples_sum = samples1 .+ samples2

pdf_sum = pdf.(Chisq(k1 + k2), x)

myplot = plot(
    layer(x=x, y=pdf1, Geom.line, Theme(default_color="red")),
    layer(x=x, y=pdf2, Geom.line, Theme(default_color="blue")),
    layer(x=x, y=pdf_sum, Geom.line, Theme(default_color="green")),
    layer(x=samples_sum, Geom.histogram(bincount=100, density=true), Theme(default_color="gray")),
    Guide.manual_color_key("Legend", ["k1=3", "k2=5", "k=3+5", "Samples k1+k2"], ["red", "blue", "green", "gray"]),
    Theme(background_color="white")
)
draw(PNG("chi_squared.png", 6inch, 4inch), myplot)
myplot
```

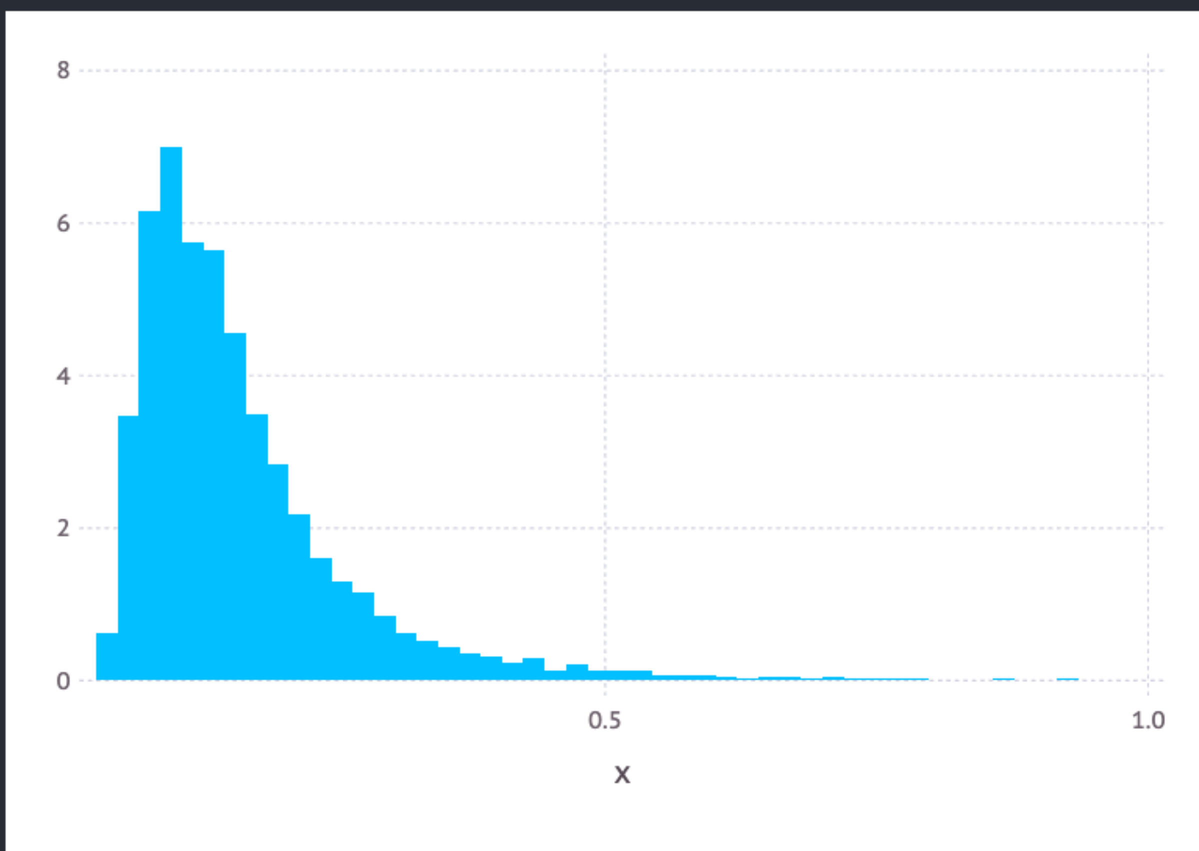
✓ 0.2s



2)

```
n_samples = 10000
k = 8
d = InverseGamma(k/2, 1/2) # beta^-1
samples = rand(d, n_samples)
myplot = plot(
    x=samples,
    Geom.histogram(bincount=100, density=true),
    Theme(background_color="white"),
    Coord.cartesian(xmax=1)
)
```

✓ 0.1s



4)

First, from $f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$,

$$\int_0^1 f(x)dx = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}dx = 1$$

$$=> \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad (1)$$

$$\sigma^2 = E(X^2) - E(X)^2 \qquad (2)$$

$$E(X^2) = \int_0^1 x^2 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x^{\alpha+1}(1-x)^{\beta-1}dx$$

The part inside the integral is similar to (1), therefore

$$E(X^2) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)}$$

Since $\Gamma(x+2) = (x+1)x\Gamma(x)$,

$$E(X^2) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{(\alpha+1)\alpha\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta)} = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} \qquad (3)$$

Plug (3) into (2),

$$\sigma^2 = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2} = \frac{(\alpha^2+\alpha)(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \frac{\alpha^3+\alpha^2+\alpha^2\beta+\alpha\beta-(\alpha^3+\alpha^2\beta+\alpha^2)}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

5) $f(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$

$$E(X) = \int_{-\infty}^{\infty} x \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} dx, \qquad Let\ t = \frac{x-\mu}{\sigma}\ , x = \sigma t + \mu, dx = \sigma dt$$

$$= \int_{-\infty}^{\infty} (\sigma t + \mu) \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt = \int_{-\infty}^{\infty} \sigma t \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt + \int_{-\infty}^{\infty} \mu \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt$$

Since integral of odd equation is 0 (the areas cancels out), the left integral is 0.

Therefore,

$$E(X) = \int_{-\infty}^{\infty} \mu \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt$$

Since this is even function,

$$E(X) = \int_{0}^{\infty} 2\mu \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt = \frac{2\mu}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{t^2}{2}} dt =$$