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Lecture 4 Exercise

1) $E(aX + b) = \sum_x (ax + b)p(x) = a\sum_x xp(x) + b\sum_x p(x) = aE(X) + b \cdot 1$

$$= aE(x) + b$$

2)

cdf(x) = $\begin{cases} 0, \ x < 1 \\ 0.1, 1 \le x < 2 \\ 0.4, 2 \le x < 3 \\ 0.8, 3 \le x < 4 \\ 1, x \ge 4 \end{cases}$

3) Given:

(i) Since each battery has only 2 possible outcomes (last >= 5 hours or not), and there are 3 independent trials, we'll use Binomial distribution

$$X \sim Binomial(n = 3, p = 0.12)$$

(ii) Since there're 3 batteries/trials, the state space are {0, 1, 2, 3}, meaning there can be 0 success, 1 success, and 2 and 3.

(iii) $P(X = 1) = \binom{n}{k} p^x (1 - p)^{n-x}$

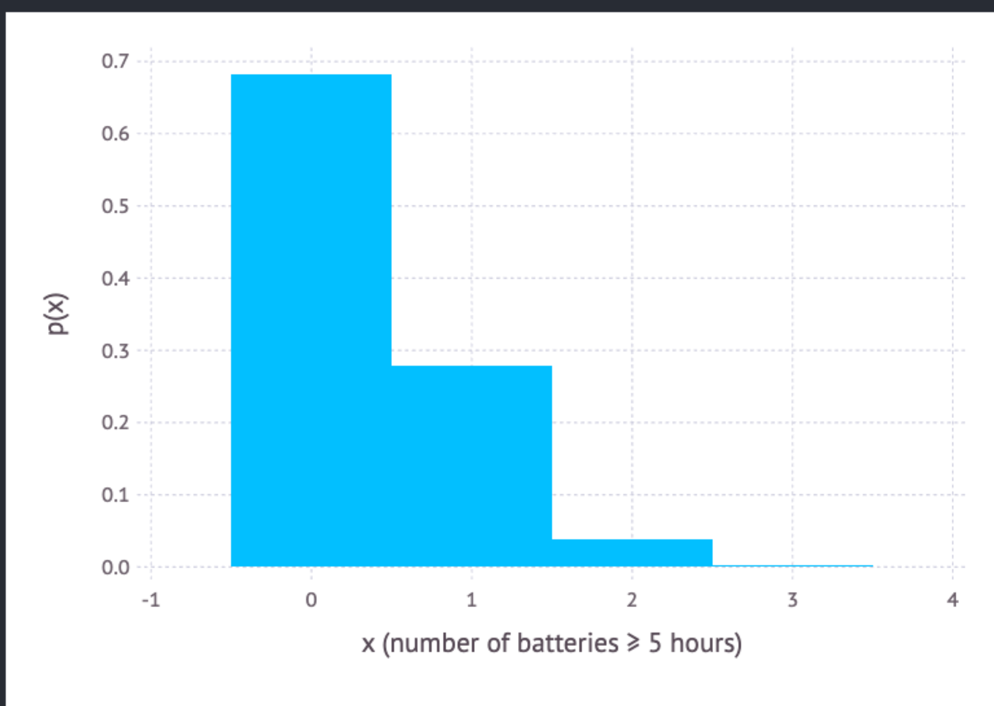
$$= \binom{3}{1} \cdot 0.12^1 \cdot (1 - 0.12)^{3-1} = 3 \cdot 0.12 \cdot 0.88^2 \approx 0.279$$

(iv)

```julia
n = 3;
p = 0.12;
d = Binomial(n, p);
y = pdf(d, 0:n)
println(y)
myplot = plot(
    x=0:n, y=y, Geom.bar,
    Guide.xlabel("x (number of batteries ≥ 5 hours)"),
    Guide.ylabel("p(x)"),
    Theme(background_color="white"),
    Coord.cartesian(ymin=0, ymax=0.7),
    Guide.yticks(ticks=0:0.1:0.7)
)
```

[74] ✓ 0.1s

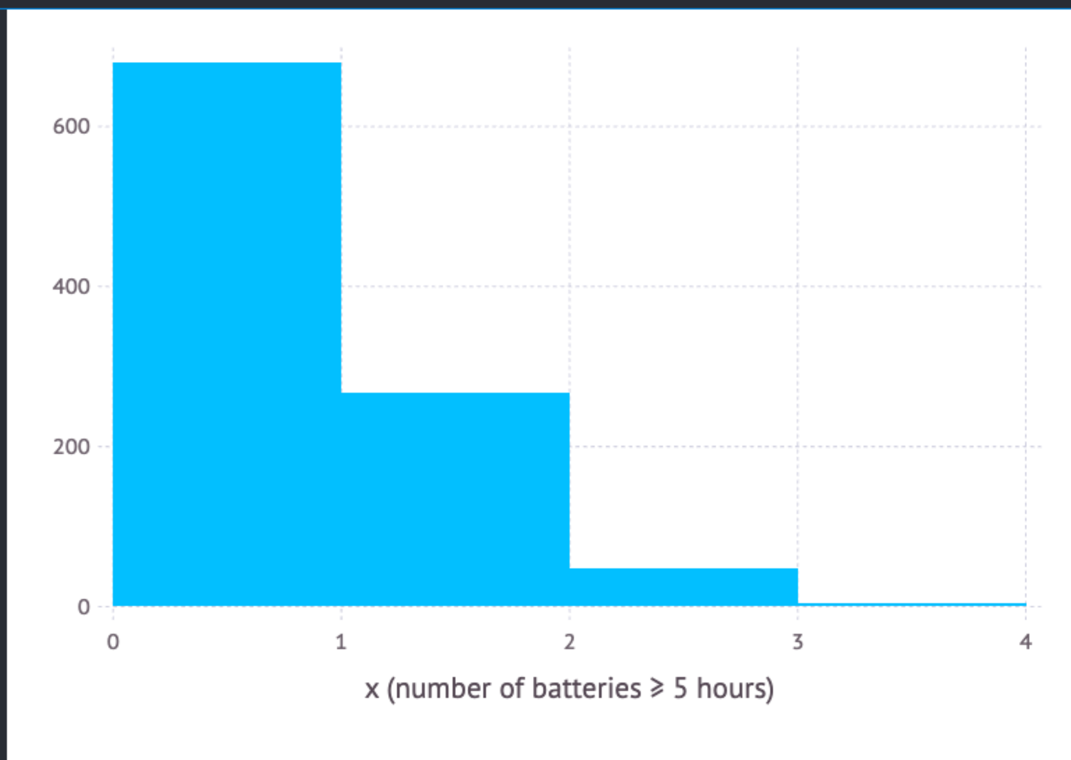
```
[0.6814720000000002, 0.27878400000000003, 0.038016000000000015, 0.0017280000000000004]
```



(v)

```
data = rand(d, 1000);
myplot = plot(
    layer(x=data, Geom.histogram(bincount=length(unique(data)))),
    Guide.xlabel("x (number of batteries ≥ 5 hours)"),
    Theme(background_color="white"),
    Coord.cartesian(xmax=4, ymin=0)
)
```

[33] ✓ 0.1s



4)

(i) Since we model the number of grasshoppers per square meter, we should use Poisson distribution

$$X \sim Poisson(\lambda = 0.5)$$

(ii) Since Poisson counts number of occurrences, the state space should be set of non-negative integers {0, 1, 2, ...}

(iii) $p(X \geq 5) = 1 - p(X < 5) = 1 - p(X \leq 4)$

$$= 1 - \big(p(X = 0) + p(X = 1) + p(X = 2) + p(X = 3) + p(X = 4)\big)$$

$$p(X = 0) = \frac{e^{-0.5}0.5^0}{0!} = 0.6065$$

$$p(X = 1) = \frac{e^{-0.5}0.5^1}{1!} = 0.3033$$

$$p(X = 2) = \frac{e^{-0.5}0.5^2}{2!} = 0.0758$$

$$p(X = 3) = \frac{e^{-0.5}0.5^3}{3!} = 0.0126$$

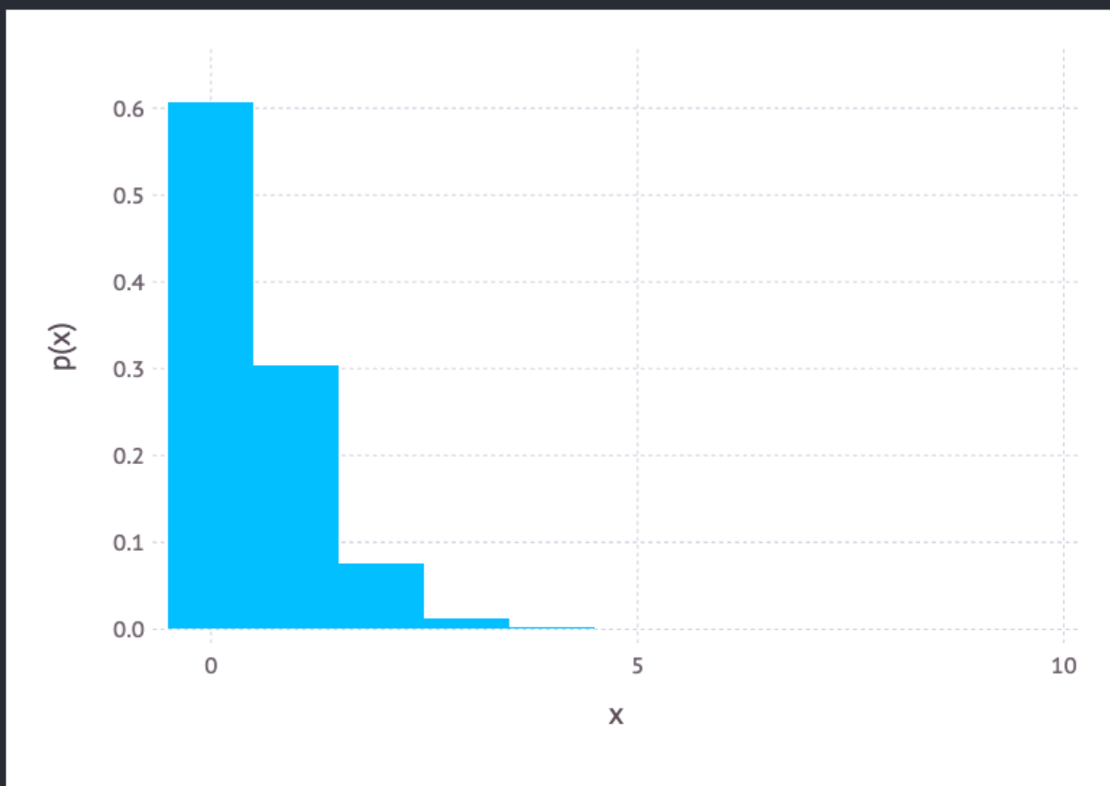
$$p(X = 4) = \frac{e^{-0.5}0.5^4}{4!} = 0.0016$$

$=> p(X \geq 5) = 1 - (0.6065 + 0.3033 + 0.0758 + 0.0126 + 0.0016) = 0.0002$

(iv)

```
n = 10;
d = Poisson(0.5);
y = pdf(d, 0:n);
myplot = plot(
    x=0:n, y=y, Geom.bar,
    Guide.xlabel("x"),
    Guide.ylabel("p(x)"),
    Theme(background_color="white"),
    Coord.cartesian(xmin=-0.5, xmax=n, ymin=0, ymax=0.65),
    Guide.yticks(ticks=0:0.1:0.6)
)
```

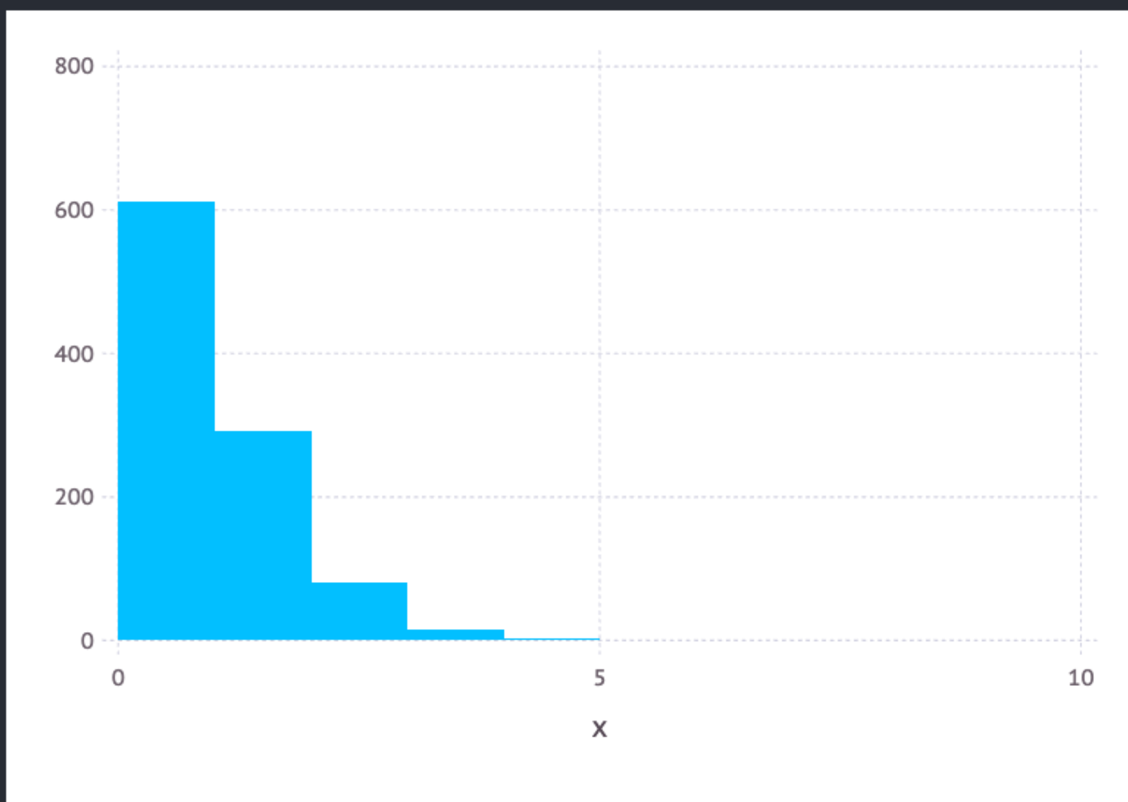
[78] ✓ 0.0s



(v)

```
data = x=rand(d, 1000);
myplot = plot(
    x=data, Geom.histogram,
    Guide.xlabel("x"),
    Theme(background_color="white"),
    Coord.cartesian(xmin=0, xmax=10)
)
```

[80] 0.1s



Bonus)

1) Let H be head and T be tail.

A wins \$1 if TT and \$2 if HH. A loses \$1 if HT or TH

$$P(X = \$2) = 0.25, P(X = \$1) = 0.25, P(X = -\$1) = 0.5$$

$E[X] = \sum_x xp(x) = 2 \cdot 0.25 + 1 \cdot 0.25 + (-1) \cdot 0.5 = 0.25$ > 0 => Not a fair game

2) $V(aX + b) = E[\left((aX + b) - E[aX + b]\right)^2]$

$$= E[(aX + b - aE[X] - b)^2] = E[(aX - aE[X])^2]$$

$$= E[a^2(X - E[X])^2] = a^2E[(X - E[X])^2]$$

$$= a^2V(X)$$

3) $p(x \geq j + k | x \geq j) = \frac{p(x \geq j+k, x \geq j)}{p(x \geq j)}$

$Since\ x \geq j + k \subset x \geq j, \qquad p(x \geq j + k, x \geq j) = p(x \geq j + k)$

Therefore,

$p(x \geq j + k | x \geq j)$

$= \frac{p(x \geq j+k)}{p(x \geq j)} = \frac{q^{j+k}}{q^j}$ (since p(x) is the probability of observing failures before a success)

$= q^k = p(x \geq k)$

4) $X \sim NegativeBinomial(6, 0.95)$

(i) p(x) = p((2 defectives + 3 good ones) and 4th good one)

$$p(x) = \binom{5}{3} 0.95^3 0.05^2 0.95 = 0.02$$

(ii)