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Lecture 7 Exercise

1)

a) Let $D(x_1, x_2, \dots, x_5)$ denotes the 5 samples

$$L(\lambda|D) = p(x_1, x_2, \dots, x_n|\lambda) = \prod_{i=1}^{5} f(x_i|\lambda) = \prod_{i=1}^{5} \lambda e^{-\lambda x_i} = \lambda^5 e^{-\lambda \sum_{i=1}^{5} x_i}$$

b) $\ell(\lambda) = logL(\lambda) = 5\log(\lambda) - \lambda \sum_{i=1}^{5} x_i$ (we denote log as natural log)

c) $\frac{d}{d\lambda}\ell(\lambda) = \frac{5}{\lambda} - \sum_{i=1}^{5} x_i$

d) $\frac{d}{d\lambda}\ell(\lambda) = 0 \Rightarrow \frac{5}{\lambda} - \sum_{i=1}^{5} x_i = 0$

$$\Rightarrow \frac{5}{\lambda} = \sum_{i=1}^{5} x_i = x_1 + x_2 + \dots + x_5 = 2 + 3 + 1 + 3 + 4 = 13$$

$$\Rightarrow \widehat{\lambda_{MLE}} = \frac{5}{13} = 0.385$$

2)

a) $L(\theta|D) = p(20AA, 10Aa, 70aa|\theta)$

$$= p(20AA|\theta)p(10Aa|\theta)p(70aa|\theta)$$

$$= \theta^{2\cdot 20} \cdot \left(2\theta(1-\theta)\right)^{10} \cdot (1-\theta)^{2\cdot 70}$$

$$= \theta^{40} \cdot \left(2\theta(1-\theta)\right)^{10} \cdot (1-\theta)^{140}$$

b) $\ell(\theta) = logL(\theta) = 40log\theta + 10log2 + 10log\theta + 10\log(1-\theta) + 140\log(1-\theta)$

$$= 10log2 + 50log\theta + 150\log(1-\theta)$$

c) $\frac{d\ell(\theta)}{\theta} = \frac{50}{\theta} - \frac{150}{1-\theta} = 0$

d) $\frac{d\ell(\theta)}{\theta} = \frac{50}{\theta} - \frac{150}{1-\theta} = 0$

$$\Rightarrow \frac{50}{\theta} = \frac{150}{1-\theta} \Rightarrow 50(1-\theta) = 150\theta \Rightarrow \widehat{\theta_{MLE}} = \frac{50}{200} = 0.25$$

Bonus)

1) $f(x|\alpha,\beta) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}}, where\ \Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}e^{-x}dx$

$$L(\alpha,\beta|D) = p(x_1, x_2, \dots, x_n|\alpha,\beta) = \prod_{i=1}^{n} \frac{x_i^{\alpha-1}e^{-\frac{x_i}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}}$$

$$\ell(\alpha,\beta) = \sum_{i=1}^{n}\left((\alpha-1)\log(x_i) - \frac{x_i}{\beta} - \log\big(\Gamma(\alpha)\big) - \alpha\log(\beta)\right)$$

$$\frac{\partial\ell(\alpha,\beta)}{\partial\alpha} = \sum_{i=1}^{n}\left(\log(x_i) - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \log(\beta)\right)$$

$$= \sum_{i=1}^{n}\log(x_i) - \frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} - nlog(\beta) = 0 \qquad (1)$$

$$\frac{\partial\ell(\alpha,\beta)}{\partial\beta} = \sum_{i=1}^{n}\left(\frac{x_i}{\beta^2} - \frac{\alpha}{\beta}\right) = \frac{\sum_{i=1}^{n} x_i}{\beta^2} - \frac{n\alpha}{\beta} = 0$$

$$=> \hat{\beta} = \frac{\sum_{i=1}^{n} x_i}{n\alpha} = \frac{\overline{x}}{\hat{\alpha}}$$

Plug back into (1),

$$\sum_{i=1}^{n}\log(x_i) - \frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} - nlog\left(\frac{\overline{x}}{\hat{\alpha}}\right) = 0$$