Final-math 4931

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1 Question 1

a) Find the quantile function of X. [3 points]

Answer:

$$F(x) = x = \begin{cases} 0, & \text{for } x \leq 1 \\ 1/3, & \text{for } 1 \leq x < 2 \\ 1, & \text{for } x \geq 2 \end{cases}$$

The quantile function of X is:

$$x = F(x)^{-1} = \begin{cases} 1, & \text{for } 0 < z \leq 1/3 \\ 2, & \text{for } 1/3 < z \leq 1 \end{cases}$$

(b) Write out steps of the procedure to simulate values from the PMF p(x) using the inverse transform method. [2 points]

Answer:

Step 1: simulate z $\sim$ Uniform(0,1)
Step 2: Set

$$x = \begin{cases} 1, & \text{for } 0 < z \leq 1/3 \\ 2, & \text{for } 1/3 < z \leq 1 \end{cases}$$

(c) Suppose you simulated a random variable Y f(x) for some density function f and you then calculated a new random variable as:
Answer:

Z has Uniform distribution (0,1)
because F(Y) $\sim$ Uniform(0,1) so,
$\Longrightarrow$ 1-F(Y) also have Uniform(0,1)

2 Question 2

(a) Calculate the exact value of this integral. [1 points]

Answer:

$I = \int_1^2 x^2 dx = \frac{x^3}{3}|_1^2 = \frac{(8-1)}{3} = 7/3$

(b) Explicitly write out the steps to estimate this integral using Monte Carlo integration. [2 points]

Answer:

To estimate this integral using Monte Carlo integration:

Step 1: Simulate (iid) $X_1, X_2, , .., X_n \sim Uniform(1, 2)$

$h(x) = x^2$

$f(x) = \begin{cases} 1, & \text{if } x \in [1,2] \\ 0, & \text{Otherwise} \end{cases}$

Step 2: Calculate estimate: $\hat{I} = \frac{\sum_{i=1}^{n} h(x_i)}{n}$

(c) Explicitly write out the steps to estimate this integral using importance sampling with proposal density:

Answer:

To estimate this integral using importance sampling with proposal density:

$h(x) = x^2$

$f(x) = \begin{cases} 1, & \text{if } x \in [1,2] \\ 0, & \text{Otherwise} \end{cases}$

$g(x) = \begin{cases} \frac{4x^3}{15}, & \text{if } x \in [1,2] \\ 0, & \text{Otherwise} \end{cases}$

We also have :

$\frac{h(x)f(x)}{g(x)} = \frac{x^2.1}{4x^3/15} = \frac{15}{4x}$, for $x \in [1,2]$

Step 1: Simulate (iid) $X_1, X_2, , .., X_n \sim g(x)$

Step 2:Calculate estimate:

$\hat{I}^{imp} = \frac{1}{n}\sum_{i=1}^{n} \frac{h(x_i)f(x_i)}{g(x_i)} = \frac{1}{n}\frac{15}{4x_i}$

3 Question 3

(a) Explicitly write out the steps to simulate from density f using the inverse transform method. [3 points]

Answer:

We have : $F(x) = \int_1^x \frac{3x^2}{7} dx = \frac{x^3}{7}|_1^x = \frac{x^3-1}{7}$

$\iff 7.F(x) + 1 = x^3$

$\Longleftrightarrow x = \sqrt[3]{7.F(x)+1}$
Let $F(x) = z \sim U(0,1)$
$\longrightarrow x = F^{-1}(z) = \sqrt[3]{7.z+1}$

To simulate from density f using the inverse:

Step 1: Simulate $U \sim Uniform(0,1)$

Step 2: Take the transform $z \sim F^{-1}(u)$

b) Explicitly write out the steps to simulate from density f using sampling importance resampling, using proposal density g(x) as defined in Question 2. Assume you already know how to simulate from g. [4 points]

Answer:

We have :

$$f(x) = \begin{cases} \frac{3x^2}{7}, & \text{if } x \in [1,2] \\ 0, & \text{Otherwise} \end{cases}$$

$$g(x) = \begin{cases} \frac{4x^3}{15}, & \text{if } x \in [1,2] \\ 0, & \text{Otherwise} \end{cases}$$

The importance weights can be calculated as:
$\frac{f(x)}{g(x)} = \frac{3x^2}{7} / \frac{4x^3}{15} = \frac{45}{28.x}$

The sampling importance resampling procedure:
Step 1: Sample $Y_1, ..., Y_n \sim Uniform(0,1)$
Step 2: For i = 1, . . . , n calculate the normalized weights:
$w_i = \frac{\frac{f(x_i)}{g(x_i)}}{\sum_{i=1}^{n} \frac{f(x_i)}{g(x_i)}}$
Step 3: Draw samples with replacement from $Y_1, ..., Y_n$ with respective probabilities $w_1, ..., w_n$.

(d) Suppose you didn't know the scaling factor for the density f(x) is $\frac{3}{7}$, and you weren't sure how to calculate it. That is, suppose you only knew that:

Would you still be able to use sampling importance resampling as you described in part (b)? Explain. [2 points]

Answer:

Yes, because f(x) is a distribution, so it has to sastify the condition:

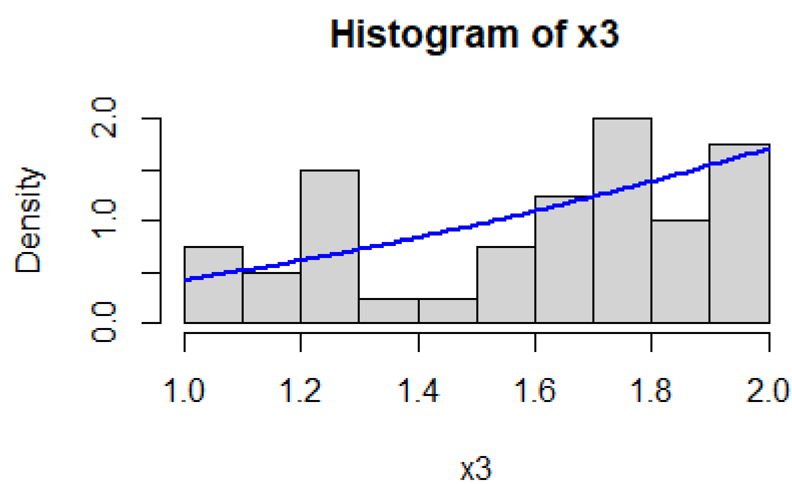
$\int_{-\infty}^{\infty} f(x) = 1$

Let $f(X) = a.x^2$, a is the scaling factor.We don't know yet.
To find a :
F(2)=1, because $x \in [1,2]$

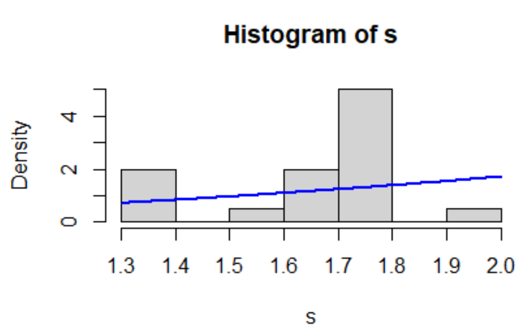
$\iff \int_1^2 a.x^2dx = 1 \iff \frac{a.x^3}{3}|_1^2 = 1$
$\iff a.\frac{8-1}{3} = 1 \iff a = 3/7$
We can find the scaling factor,when we only knew $f(x) \propto x^2$.After that we can follow the procedure in part (b) to sample the importance resampling.

(e) For each of the following simulation methods, state whether they provide a sample that exactly follows the target distribution or approximately follows the target distribution and explain:
Answer:

- Inverse transform method



The f(X) does not fit well with small sample when n=40, so, The inverse transform method **approximately** follow the target distribution.
- Sampling importance resampling



when n is set to 25, the f(x) function does not fit well .Therefore, The sampling importance resampling method **approximately** follow the target distribution.

- Accept-reject sampling

For this method, we will have poorly fit when we set small, so The Accept-reject sampling method **approximately** follow the target distribution.

4 Question 5

Suppose you want to optimize the following function:

(a) Write out the steps to find the maximum of this function using Newton-Raphson. Make sure to explicitly write out the formula to get from step k to step k + 1. [4 points]

Answer:

we have :

$h'(x) = cos(x) - x^3$
$h''(x) = -sin(x) - 3.x^2$
Step 1: we choose a starting value $x_0 \in [0, 1]$

Step 2:
$x_{k+1} = x_k - \frac{h'(X)}{h''(x)}$

Step 3: Continue repeating Step (2) for the next value of k+1 until one of our stopping rules has been triggered.

Stopping rule :
$l'(\theta_k|x) < \epsilon$