

Time and Space Complexity Analysis

Objective:

At the end of this lecture, students will understand :

- Time Complexity
- Space Complexity
- Importance of them in Competitive Programming

Time and Space Complexity:

Sometimes, there are more than one way to solve a problem. We need to learn how to compare the performance of different algorithms and choose the best one to solve a particular problem. While analysing an algorithm, we mostly consider time complexity and space complexity.

Time and Space Complexity: (Contd.)

- Time complexity of an algorithm quantifies the amount of time taken by an algorithm to run as a function of the length of the input.
- Similarly, Space complexity of an algorithm quantifies the amount of space or memory taken by an algorithm to run as a function of the length of the input.
- Time and space complexity depends on lots of things like hardware, operating system, processors, etc.

Order of growth:

- Order of growth of an algorithm is a way of saying /predicting how execution time of a program and the space/memory occupied by it changes with the input size.
- The most famous way is the Big-Oh notation. It gives the worst case possibility for an algorithm.

Order of growth: (Contd.)

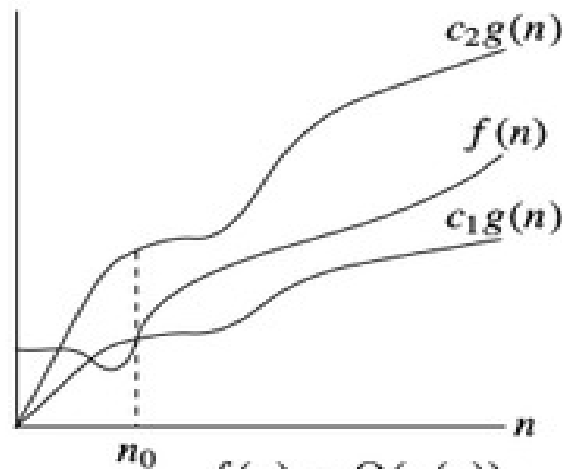
- E.g. a program takes $O(n)$ time means it takes at most n operations to compute the answer. Now if your input for n is 10^6 , it takes 10^6 operations, while a $O(n^2)$ algorithm takes 10^{12} operations for the same input.

Order of growth: (Contd.)

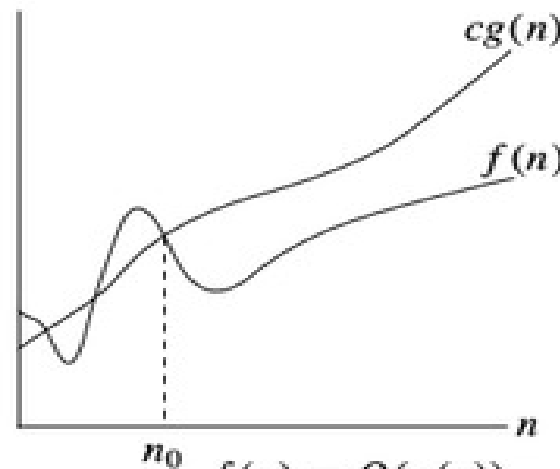
O-notation:

To denote asymptotic upper bound, we use *O*-notation. For a given function $g(n)$, we denote by $O(g(n))$ (pronounced "big-oh of g of n") the set of functions:

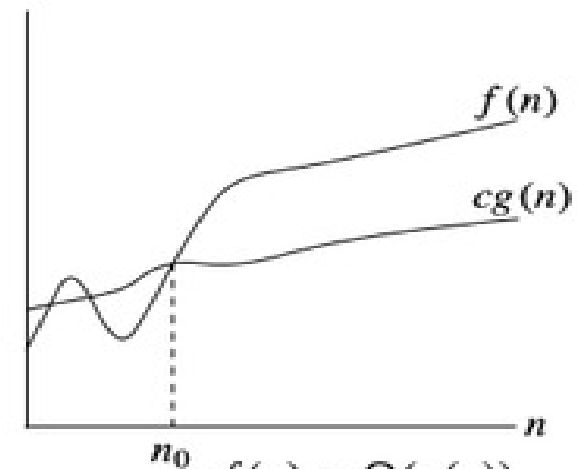
$O(g(n)) = \{ f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \leq f(n) \leq c * g(n) \text{ for all } n \geq n_0 \}$



(a)



(b)



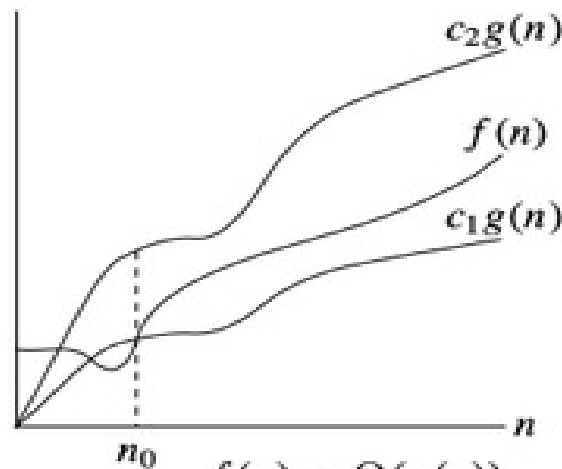
(c)

Order of growth: (Contd.)

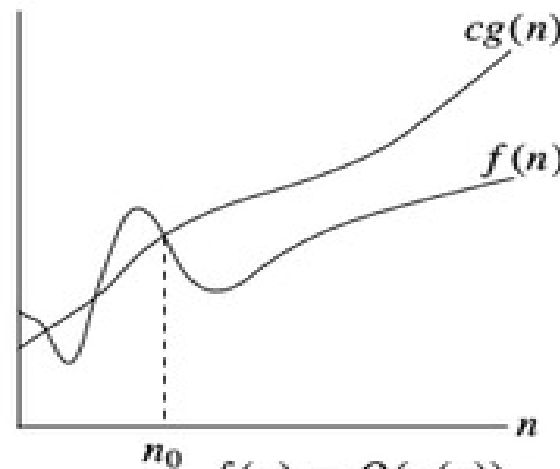
Ω -notation:

To denote asymptotic lower bound, we use Ω -notation. For a given function $g(n)$, we denote by $\Omega(g(n))$ (pronounced “big-omega of g of n”) the set of functions:

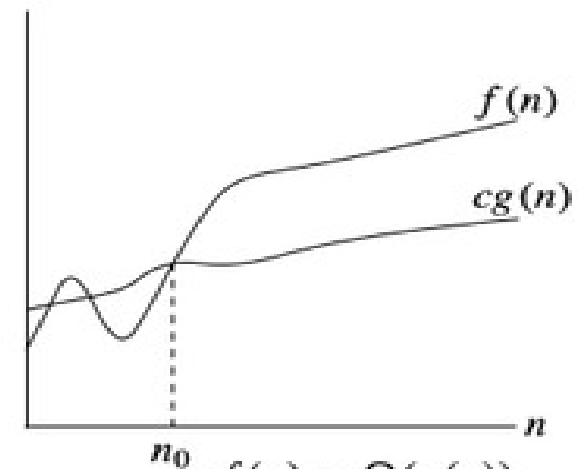
$\Omega(g(n)) = \{ f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \leq c * g(n) \leq f(n) \text{ for all } n \geq n_0 \}$



(a)



(b)



(c)

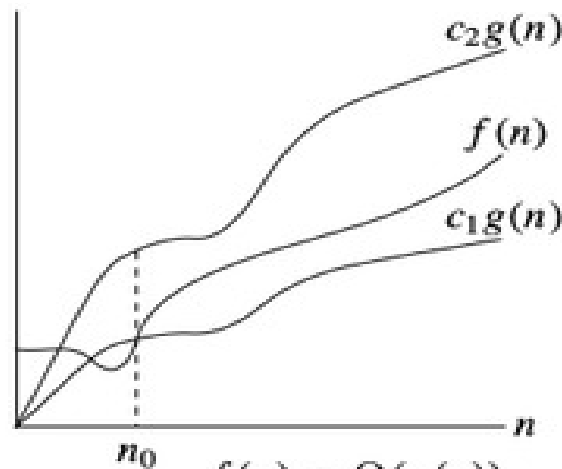
Order of growth: (Contd.)

Θ -notation:

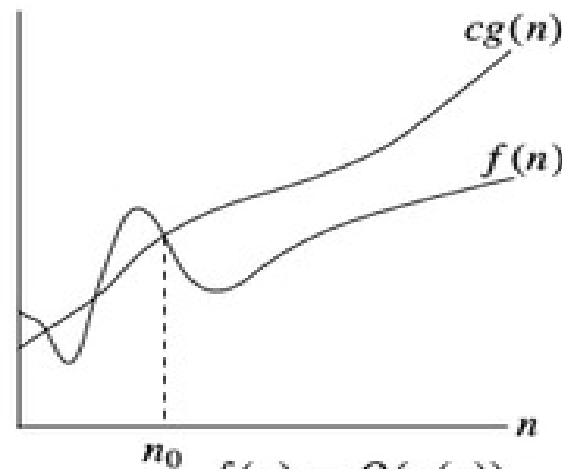
To denote asymptotic tight bound, we use Θ -notation. For a given function $g(n)$, we denote by $\Theta(g(n))$

(pronounced "big-theta of g of n") the set of functions:

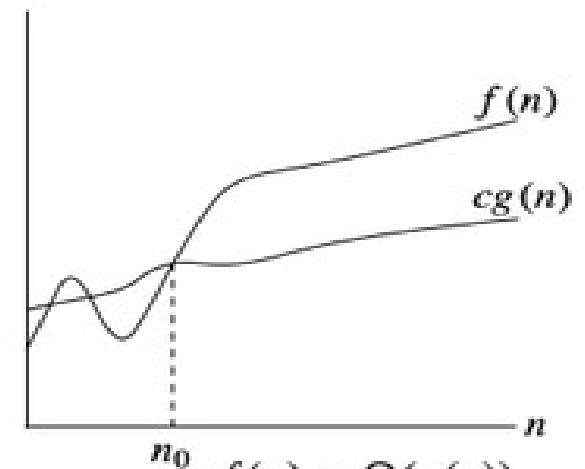
$\Theta(g(n)) = \{ f(n) : \text{there exist positive constants } c_1, c_2 \text{ and } n_0 \text{ such that } 0 \leq c_1 * g(n) \leq f(n) \leq c_2 * g(n) \text{ for all } n > n_0 \}$



$f(n) = \Theta(g(n))$
(a)

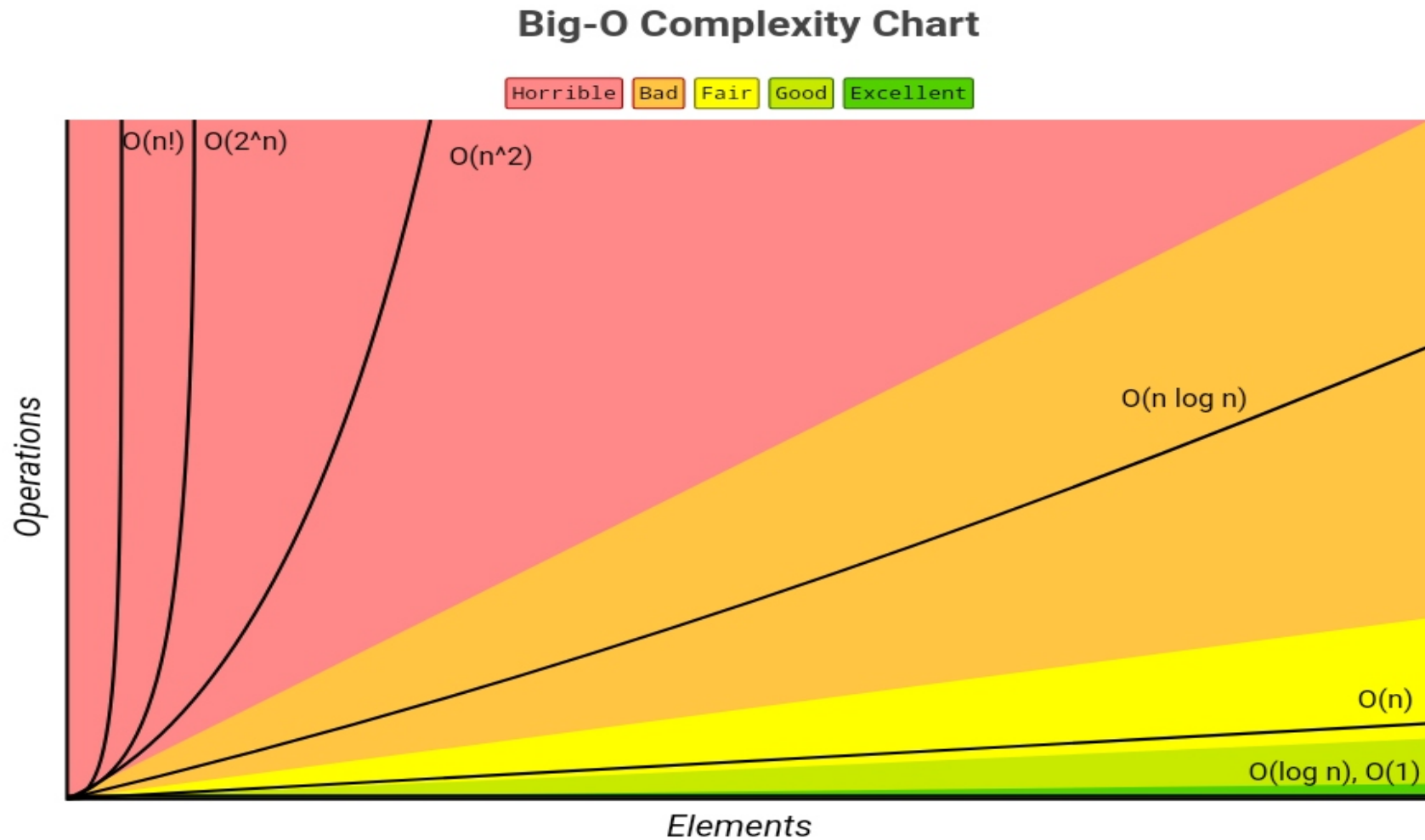


$f(n) = O(g(n))$
(b)



$f(n) = \Omega(g(n))$
(c)

N vs. Growth of Complexity:



Complexity $O(1)$:

Time complexity of a function (or set of statements) is considered as $O(1)$ if it doesn't contain loop, recursion and call to any other non-constant time function.

- For example swap() function has $O(1)$ time complexity.
- A loop or recursion that runs a constant number of times is also considered as $O(1)$. E.g. the following loop is $O(1)$.

```
for (int i = 1; i <= c; i++) {           // Here c is a constant  
    // some O(1) expressions  
}
```

Complexity $O(n)$:

Time Complexity of a loop is considered as $O(n)$ if the loop variable is incremented / decremented by a constant amount. For example following functions have $O(n)$ time complexity.

```
for (int i = 1; i <= n; i += c) { // Here c is a +ve integer constant  
    // some O(1) expressions  
}  
  
for (int i = n; i > 0; i -= c) {  
    // some O(1) expressions  
}
```

Complexity $O(n^c)$:

Time complexity of nested loops is equal to the number of times the innermost statement is executed. For example the following sample loops have $O(n^2)$ time complexity

```
for (int i = 1; i <= n; i += c) {  
    for (int j = 1; j <= n; j += c) {  
        // some  $O(1)$  expressions  
    }  
}
```

```
for (int i = n; i > 0; i -= c) {  
    for (int j = i+1; j <= n; j += c) {  
        // some  $O(1)$  expressions  
    }  
}
```

For example Selection sort and Insertion Sort have $O(n^2)$ time complexity.

Complexity $O(\log n)$:

Time Complexity of a loop is considered as $O(\log_c n)$ if the loop variables is divided / multiplied by a constant amount.

```
for (int i = 1; i <= n; i *= c) {  
    // some  $O(1)$  expressions  
}
```

```
for (int i = n; i > 0; i /= c) {  
    // some  $O(1)$  expressions  
}
```

For example Binary Search has $O(\log_2 n)$ time complexity.

Complexity $O(\text{LogLog}n)$:

Time Complexity of a loop is considered as $O(\text{LogLog}_c n)$ if the loop variables are reduced / increased exponentially by a const amount.

```
for (int i = 2; i <= n; i = pow(i, c)) { // c is a const greater than 1  
    // some O(1) expressions  
    }  
  
// Here fun is sqrt or cuberoot or any other constant root  
for (int i = n; i > 1; i = fun(i)) {  
    // some O(1) expressions  
    }
```

Time complexities of consecutive loops:

Time Complexity When there are consecutive loops, we calculate time complexity as sum of time complexities of individual loops.

```
for (int i = 1; i <= m; i += c) {  
    // some O(1) expressions  
}
```

```
for (int i = 1; i <= n; i += c) {  
    // some O(1) expressions  
}
```

Time complexity of the code is $O(m) + O(n)$ which is $O(m + n)$

If $m == n$, the time complexity becomes $O(2n)$ which is $O(n)$.

Quiz!


1. What is the complexity of the following snippet ? (n can be any integer)

```
for (int i = 0; i < n; i += 1) {  
    for (int j = 0; j < m; j += 1) {  
        print "Hello"  
    }  
}
```

- A. $O(n)$
- B. $O(m)$
- C. $O(n+m)$
- D. $O(nm)$

1. What is the complexity of the following snippet ? (n can be any integer)

```
for (int i = 0; i < n; i += 1) {  
    for (int j = 0; j < m; j += 1) {  
        print "Hello"  
    }  
}
```

- A. $O(n)$
- B. $O(m)$
- C. $O(n+m)$
-  D. $O(nm)$


2. What is the complexity of the following snippet ? (n can be any integer)

```
for (int j = 0; j < n; j += 1) {  
    for (int i = 0; i < n; i += 1)  
        print "First loop"  
}  
for (int i = 0; i < n; i +=1) {  
    print "Second loop"  
}  
for (int i = 0; i < n; i += 1) {  
    print "Third loop"  
}
```

- A. $O(n)$
- B. $O(n^2)$
- C. Can't be determined
- D. None of these

2. What is the complexity of the following snippet ? (n can be any integer)

```
for (int j = 0; j < n; j += 1) {  
    for (int i = 0; i < n; i += 1)  
        print "First loop"  
}  
for (int i = 0; i < n; i +=1) {  
    print "Second loop"  
}  
for (int i = 0; i < n; i += 1) {  
    print "Third loop"  
}
```

- A. $O(n)$
-  B. $O(n^2)$
- C. Can't be determined
- D. None of these

3. What is the complexity of the following snippet ? (n can be any integer)


```
while (var < n){  
    cout << pie << endl;  
    for (int j = 1; j < n; j+=2){  
        sum += 1  
    }  
    var *= 3  
}
```

- A. $O(n)$
- B. $O(\log n)$
- C. $O(n \log_3 n)$
- D. $O(n^2)$

3. What is the complexity of the following snippet ? (n can be any integer)

```
while (var < n){  
    cout << pie << endl;  
    for (int j = 1; j < n; j+=2){  
        sum += 1  
    }  
    var *= 3  
}
```

In the outer loop we have $\text{var} *= 3$, therefore outer loop will have $\log_3(n)$ complexity but the inner loop proceeds 1 to n, therefore ultimately the whole code will have complexity $n \log_3 n$.

- A. $O(n)$
- B. $O(\log n)$
-  C. $O(n \log_3 n)$
- D. $O(n^2)$


4. What is the complexity of the following snippet ? (n can be any integer)

```
for (int i = 0; i < n; i += 3) {  
    for (int j = 0; j < m; j += 2) {  
        print "Hello"  
    }  
}
```

- A. $O(n)$
- B. $O(m)$
- C. $O(n+m)$
- D. $O(nm)$

4. What is the complexity of the following snippet ? (n can be any integer)

```
for (int i = 0; i < n; i += 3) {  
    for (int j = 0; j < m; j += 2) {  
        print "Hello"  
    }  
}
```

- A. $O(n)$
- B. $O(m)$
- C. $O(n+m)$
-  D. $O(nm)$

5. What is the complexity of the following snippet (n can be any integer)?

```
for (int i = n/2; i <= n; i += 1) {  
    for (int j = 1; j < m; j *= 2) {  
        print "Hello"  
    }  
}
```

- A. $O(n \log_2 m)$
- B. $O(nm)$
- C. $O(\log_2 n * \log_2 m)$
- D. None of these

5. What is the complexity of the following snippet (n can be any integer)?

```
for (int i = n/2; i <= n; i += 1) {  
    for (int j = 1; j < m; j *= 2) {  
        print "Hello"  
    }  
}
```

- ✓ A. $O(n \log_2 m)$
- B. $O(nm)$
- C. $O(\log_2 n * \log_2 m)$
- D. None of these


6. What is the complexity of the following snippet ? (n can be any integer)

```
int i = 1, j = 1
for (; i < n; i += 1) {
    while(j < i) {
        print "Hello"
        j *= 2
    }
    j = 1
}
```

- A. $O(\log_2 n)$
- B. $O(n)$
- C. $O(n \log_2 n)$
- D. None of the above

6. What is the complexity of the following snippet ? (n can be any integer)

```
int i = 1, j = 1
for (; i < n; i += 1) {
    while(j < i) {
        print "Hello"
        j *= 2
    }
    j = 1
}
```

- A. $O(\log_2 n)$
- B. $O(n)$
-  C. $O(n \log_2 n)$
- D. None of the above


7. What is the complexity of the following snippet ? (n can be any integer)

```
for (int i = 1; i < n; i += i) {  
    for (int j = 0; j < n; j += 1) {  
        print "Hello"  
    }  
}
```

- A. $O(\log n)$
- B. $O(n \log n)$
- C. $O(n)$
- D. $O(n^2)$

7. What is the complexity of the following snippet ? (n can be any integer)

```
for (int i = 1; i < n; i += i) {  
    for (int j = 0; j < n; j += 1) {  
        print "Hello"  
    }  
}
```

- A. $O(\log n)$
-  B. $O(n \log n)$
- C. $O(n)$
- D. $O(n^2)$

8. What is the complexity of the following snippet ? (n can be any integer)

```
for (int i = 1; i < n; i *= 3) {  
    for (int j = 1; j < n; j += 2) {  
        print "Hello"  
    }  
}
```

- A. $O(\log_3 n)$
- B. $O(n \log_3 n)$
- C. $O(n)$
- D. $O(n^2)$

8. What is the complexity of the following snippet ? (n can be any integer)

```
for (int i = 1; i < n; i *= 3) {  
    for (int j = 1; j < n; j += 2) {  
        print "Hello"  
    }  
}
```

- A. $O(\log_3 n)$
- ☒ B. $O(n \log_3 n)$
- C. $O(n)$
- D. $O(n^2)$


9. What is the complexity of the following snippet ? (n can be any integer)

```
int i = 1 , j = 1
for (; i < n; i +=1) {
    while(j < i) {
        print "Hello"
        j *= 2
    }
}
```

- A. $O(\log n)$
- B. $O(n)$
- C. $O(n \log n)$
- D. None of the above

9. What is the complexity of the following snippet ? (n can be any integer)

```
int i = 1 , j = 1
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        j *= 2
    }
}
```

- A. $O(\log n)$
-  B. $O(n)$
- C. $O(n \log n)$
- D. None of the above


10. What is the complexity of the following snippet ? (n can be any integer)

```
for (int i = 1; i < n; i += 2) {  
    for (int j = 1; j < n; j *= 3) {  
        print "Hello"  
    }  
}
```

- A. $O(\log_3 n)$
- B. $O(n \log_3 n)$
- C. $O(n)$
- D. $O(n^2)$

10. What is the complexity of the following snippet ? (n can be any integer)

```
for (int i = 1; i < n; i += 2) {  
    for (int j = 1; j < n; j *= 3) {  
        print "Hello"  
    }  
}
```

- A. $O(\log_3 n)$
-  B. $O(n \log_3 n)$
- C. $O(n)$
- D. $O(n^2)$