

UNIVERSITY OF ABERDEEN

ACADEMIC YEAR 2024–25

# EX3030/EM4012

Degree Examination in EX3030/EM4012 Heat, Mass, & Momentum Transfer

16<sup>th</sup> July 2025

Time: 2 pm – 5 pm

## PLEASE NOTE THE FOLLOWING

- (i) You **must not** have in your possession any material other than that expressly permitted in the rules appropriate to this examination. Where this is permitted, such material **must not** be amended, annotated or modified in any way.
- (ii) You **must not** have in your possession any material that could be determined as giving you an advantage in the examination.
- (iii) You **must not** attempt to communicate with any candidate during the exam, either orally or by passing written material, or by showing material to another candidate, nor must you attempt to view another candidate's work.
- (iv) You **must not** take to your examination desk any electronic devices such as mobile phones or other "smart" devices. The only exception to this rule is an approved calculator.

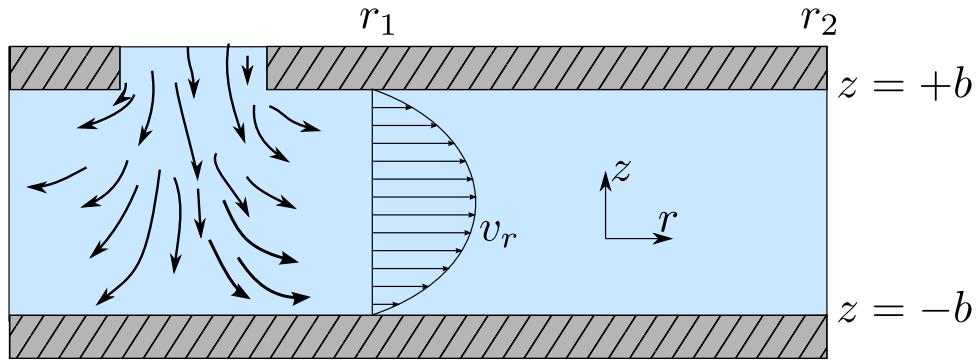
## FURTHER REQUIREMENTS SPECIFIC TO THE SCHOOL OF ENGINEERING

- a) Candidates **ARE** permitted to use only an approved calculator.
- b) Candidates **ARE NOT** permitted to use the Engineering Mathematics Examinations Handbook.
- c) Candidates **ARE NOT** permitted to use GREEN or RED pen in their exam booklet.
- d) Data sheets are attached to the paper.
- e) All question papers must be submitted with the exam booklet.

**Candidates should attempt all questions.** This exam contains 5 questions, each worth 20 marks. This exam will be marked using Engineering Percentages.

**Question 1**

Oil is used to lubricate two horizontal parallel circular plates by injecting it and allowing it to flow radially outwards from the point of injection (see Fig. 1). The fluid is flowing radially as there is a pressure difference of  $P_1 - P_2$  between the inner and outer radii,  $r_1$  and  $r_2$  respectively. It can be assumed that the flow becomes fully developed at the radial position  $r_1$ .



**Figure 1:** Radial flow between two plates.

- a) Simplify the continuity equation to demonstrate that  $r v_r$  is a function of  $z$  only. [5 marks]
- b) Derive the following equation from the momentum balance equation: [10 marks]

$$\rho v_r \frac{\partial v_r}{\partial r} = \mu \left( 2 \frac{\partial^2 v_r}{\partial r^2} + \frac{2}{r} \frac{\partial v_r}{\partial r} - \frac{2 v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2} \right) - \frac{\partial p}{\partial r}$$

**Note:** You should assume the fluid is Newtonian, and you must be careful during your derivation and make sure you expand each term of  $\tau$  with Newton's law before cancellation.

- c) The momentum balance equation from the previous question has been solved for you using the creeping flow assumption to give the following result [5 marks]:

$$v_r = -r^{-1} \frac{\Delta P}{2 \mu \ln(r_2/r_1)} (z^2 + C_1 z + C_2)$$

Determine the integration constants  $C_1$  and  $C_2$  using the boundary conditions, and give the final expression for the velocity profile.

**[Question total: 20 marks]**

**Question 2**

- a) Carry out a dimensional analysis on the forced convection heat transfer coefficient,  $h$ , to determine which are the fundamental dimensionless numbers involved. You may assume the following general dependence

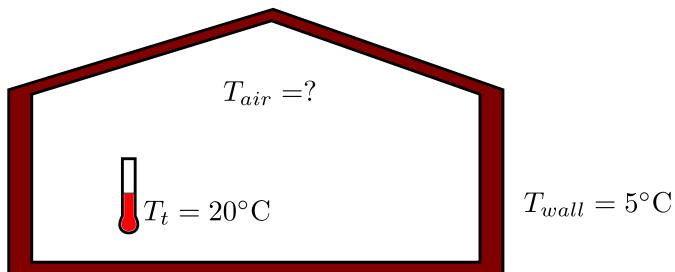
$$h = f(d, \mu, k, \langle v \rangle, \rho, C_p)$$

where  $d$  is the channel diameter (m),  $\mu$  is the viscosity (Pa s),  $k$  is the thermal conductivity ( $\text{W m}^{-1} \text{K}^{-1}$ ),  $\langle v \rangle$  is the mean flow velocity ( $\text{m s}^{-1}$ ),  $\rho$  is the mass density ( $\text{kg m}^{-3}$ ), and  $C_p$  is the specific heat capacity at constant pressure ( $\text{kJ kg}^{-1} \text{K}^{-1}$ ).

**[10 marks]**

- b) Consider an unshielded thermometer placed in a room (see Fig. 2). The walls of the house are poorly insulated and the internal surfaces are at a temperature of  $5^\circ\text{C}$ . If the thermometer reads  $20^\circ\text{C}$  and all surfaces have an emissivity of 0.9, what is the real temperature of the air? You may assume a rough estimate of the natural convective coefficient as  $h \approx 10 \text{ W m}^{-2} \text{ K}^{-1}$ .

**[10 marks]**



**Figure 2:** An unshielded thermometer in a room with cold walls.

**[Question total: 20 marks]**

**Question 3**

a) In a manufacturing facility, 10 cm diameter brass balls initially at  $121^{\circ}\text{C}$  are quenched in a water bath at  $49^{\circ}\text{C}$  for a period of 2 min and at a rate of 120 balls per minute. If the average convective heat transfer coefficient is  $238 \text{ W m}^{-2} \text{ C}^{-1}$ , determine:

- i) Temperature of the balls after quenching, and; **[5 marks]**
- ii) Rate (in kW) at which heat needs to be removed from the water in order to keep its temperature constant at  $49^{\circ}\text{C}$ . **[5 marks]**

Given properties of brass:  $k = 111 \text{ W m}^{-1} \text{ K}^{-1}$ ,  $\rho = 8552 \text{ kg m}^{-3}$  and  $C_p = 285 \text{ J kg}^{-1} \text{ K}^{-1}$

b) Consider an oil-to-oil double-pipe heat exchanger whose flow arrangement is not known. The temperature measurements indicate that the cold oil enters at  $20^{\circ}\text{C}$  and leaves at  $55^{\circ}\text{C}$ , while the hot oil enters at  $80^{\circ}\text{C}$  and leaves at  $45^{\circ}\text{C}$ .

- i) Is this parallel-flow or counter-flow heat exchanger? What makes it only possible to be one type? **[5 marks]**
- ii) Assuming the mass flow rates of both fluids to be identical, determine the effectiveness of this heat exchanger. **[5 marks]**

**[Question total: 20 marks]**

**Question 4**

A non-Newtonian fluid flows through a 20 m length pipe with a diameter of 25 mm. Its apparent viscosity is  $0.1 \text{ N s m}^{-2}$  at a shear rate of  $1000 \text{ s}^{-1}$  and its density is estimated to be  $1600 \text{ kg m}^{-3}$ .

- a) If the flow index  $n$  is 0.33, show that the consistency  $k$  is 10 if the Power Law model applies. Give the rheological equation for the fluid, state what type of fluid is this and how will it respond to increasing rates of shear. **[5 marks]**
- b) If a flow-rate of  $1 \text{ m}^3 \text{ hr}^{-1}$  is required, show that the flow would be laminar and calculate the pressure drop. **[10 marks]**

**Note:** The definition of the Metzner-Reed Reynolds number for Power-Law fluids in pipes is given by

$$\text{Re}_{MR} = 8 \left( \frac{n}{6n+2} \right)^n \frac{\rho \langle v \rangle^{2-n} D_H^n}{k}$$

- c) Roughly sketch the flow profile for this fluid comparing it to the sketch of a Newtonian fluid and a Bingham-plastic fluid. Explain the differences between the profiles. **[5 marks]**

**[Question total: 20 marks]**

**Question 5**

Consider a spherical aggregate (or ball) of bacterial cells (assumed to be homogeneous) of radius  $R$ . Under certain circumstances, the oxygen metabolism rate of the bacterial cells is an almost constant reaction (zero-order) with respect to the oxygen concentration  $\sigma_{O_2} = -k_{O_2}$ . The diffusion of oxygen within the ball may be described by Fick's law with an effective pseudobinary diffusivity for oxygen in the bacterial medium of  $D_{O_2-M}$ . Neglect transient and convection effects because the oxygen solubility is very low in the system. Let  $C_{O_2}^{(R)}$  be the oxygen mass concentration at the aggregate surface:

- a) Show all of your working and state all assumptions made while demonstrating that the oxygen balance for the system,

$$\frac{\partial C_{O_2}}{\partial t} = -\nabla \cdot \mathbf{N}_{O_2} + \sigma_{O_2},$$

simplifies to the following expression,

$$\frac{\partial}{\partial r} (r^2 N_{O_2, r}) = -k_{O_2} r^2.$$

**[4 marks]**

- b) Demonstrate that the oxygen flux obeys the following relationship:

$$N_{O_2, r} = -k_{O_2} R^2 \left( \frac{r}{3R^2} + \frac{C_1 R}{6r^2} \right)$$

where  $C_1$  is an unknown constant.

**[4 marks]**

- c) Demonstrate that the concentration profile obeys the following form in the limit that the  $O_2$  concentration is small:

$$C_{O_2} = \frac{k_{O_2} R^2}{6 D_{O_2-M}} \left( \frac{r^2}{R^2} - C_1 \frac{R}{r} \right) + C_2$$

**[4 marks]**

- d) Using the only available boundary condition, determine  $C_2$  and demonstrate that the final expression for the concentration is:

$$C_{O_2} = \frac{k_{O_2} R^2}{6 D_{O_2-M}} \left( \frac{r^2}{R^2} + C_1 \left( 1 - \frac{R}{r} \right) - 1 \right) + C_{O_2}^{(R)}$$

**[4 marks]**

- e) It is possible that the spherical bacterial ball has an oxygen-free core ( $C_{O_2} = 0$  for  $r \leq r_{core}$ ). Prove that this only happens for:

$$\frac{k_{O_2} R^2}{D_{O_2-M} C_{O_2}^{(R)}} \geq 6$$

**Hints:** As the concentration and diffusive flux are continuous, they both must go to zero at the core radius  $r_{core}$ . Use this to solve for  $C_1$ , then solve for  $r_{core}$  and consider what is required if  $r_{core} \geq 0$ .

**[4 marks]**

**[Question total: 20 marks]**

**END OF PAPER**

## DATASHEET

### General balance equations:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{v} \quad (\text{Mass/Continuity}) \quad (1)$$

$$\frac{\partial C_A}{\partial t} = -\nabla \cdot \mathbf{N}_A + \sigma_A \quad (\text{Species}) \quad (2)$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\rho \mathbf{v} \cdot \nabla \mathbf{v} - \nabla \cdot \boldsymbol{\tau} - \nabla p + \rho \mathbf{g} \quad (\text{Momentum}) \quad (3)$$

$$\rho C_p \frac{\partial T}{\partial t} = -\rho C_p \mathbf{v} \cdot \nabla T - \nabla \cdot \mathbf{q} - \boldsymbol{\tau} : \nabla \mathbf{v} - p \nabla \cdot \mathbf{v} + \sigma_{energy} \quad (\text{Heat/Energy}) \quad (4)$$

In Cartesian coordinate systems,  $\nabla$  can be treated as a vector of derivatives. In curvilinear coordinate systems, the directions  $\hat{r}$ ,  $\hat{\theta}$ , and  $\hat{\phi}$  depend on the position. For convenience in these systems, look-up tables are provided for common terms involving  $\nabla$ .

### Cartesian coordinates (with index notation examples)

where  $s$  is a scalar,  $\mathbf{v}$  is a vector, and  $\boldsymbol{\tau}$  is a tensor.

$$\begin{aligned} \nabla s &= \nabla_i s = \left[ \frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}, \frac{\partial s}{\partial z} \right] \\ \nabla^2 s &= \nabla_i \nabla_i s = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} \\ \nabla \cdot \mathbf{v} &= \nabla_i v_i = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \\ \nabla \cdot \boldsymbol{\tau} &= \nabla_i \tau_{ij} \\ [\nabla \cdot \boldsymbol{\tau}]_x &= \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \\ [\nabla \cdot \boldsymbol{\tau}]_y &= \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \\ [\nabla \cdot \boldsymbol{\tau}]_z &= \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \\ \mathbf{v} \cdot \nabla \mathbf{v} &= v_i \nabla_i v_j \\ [\mathbf{v} \cdot \nabla \mathbf{v}]_x &= v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \\ [\mathbf{v} \cdot \nabla \mathbf{v}]_y &= v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \\ [\mathbf{v} \cdot \nabla \mathbf{v}]_z &= v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \end{aligned}$$

### Cylindrical coordinates

where  $s$  is a scalar,  $\mathbf{v}$  is a vector, and  $\tau$  is a tensor. All expressions involving  $\tau$  are for symmetrical  $\tau$  only.

$$\begin{aligned}\nabla s &= \left[ \frac{\partial s}{\partial r}, \frac{1}{r} \frac{\partial s}{\partial \theta}, \frac{\partial s}{\partial z} \right] \\ \nabla^2 s &= \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial s}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 s}{\partial \theta^2} + \frac{\partial^2 s}{\partial z^2} \\ \nabla \cdot \mathbf{v} &= \frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \\ [\nabla \cdot \tau]_r &= \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rr}) + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{1}{r} \tau_{\theta\theta} + \frac{\partial \tau_{rz}}{\partial z} \\ [\nabla \cdot \tau]_\theta &= \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2}{r} \tau_{r\theta} + \frac{\partial \tau_{\theta z}}{\partial z} \\ [\nabla \cdot \tau]_z &= \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rz}) + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} \\ [\mathbf{v} \cdot \nabla \mathbf{v}]_r &= v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \\ [\mathbf{v} \cdot \nabla \mathbf{v}]_\theta &= v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \\ [\mathbf{v} \cdot \nabla \mathbf{v}]_z &= v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}\end{aligned}$$

### Spherical coordinates

where  $s$  is a scalar,  $\mathbf{v}$  is a vector, and  $\tau$  is a tensor. All expressions involving  $\tau$  are for symmetrical  $\tau$  only.

$$\begin{aligned}\nabla s &= \left[ \frac{\partial s}{\partial r}, \frac{1}{r} \frac{\partial s}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial s}{\partial \phi} \right] \\ \nabla^2 s &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial s}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial s}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 s}{\partial \phi^2} \\ \nabla \cdot \mathbf{v} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} \\ [\nabla \cdot \tau]_r &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{rr}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{r\theta} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial \tau_{r\phi}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r} \\ [\nabla \cdot \tau]_\theta &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\theta}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{\theta\theta} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial \tau_{\theta\phi}}{\partial \phi} + \frac{\tau_{r\theta}}{r} - \frac{\cot \theta}{r} \tau_{\phi\phi} \\ [\nabla \cdot \tau]_\phi &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\phi}) + \frac{1}{r} \frac{\partial \tau_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{\tau_{r\theta}}{r} + \frac{2 \cot \theta}{r} \tau_{\theta\phi} \\ [\mathbf{v} \cdot \nabla \mathbf{v}]_r &= v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \\ [\mathbf{v} \cdot \nabla \mathbf{v}]_\theta &= v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2 \cot \theta}{r} \\ [\mathbf{v} \cdot \nabla \mathbf{v}]_\phi &= v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi + v_\theta v_\phi \cot \theta}{r}\end{aligned}$$

Rectangular		Cylindrical		Spherical	
$q_x$	$-k \frac{\partial T}{\partial x}$	$q_r$	$-k \frac{\partial T}{\partial r}$	$q_r$	$-k \frac{\partial T}{\partial r}$
$q_y$	$-k \frac{\partial T}{\partial y}$	$q_\theta$	$-k \frac{1}{r} \frac{\partial T}{\partial \theta}$	$q_\theta$	$-k \frac{1}{r} \frac{\partial T}{\partial \theta}$
$q_z$	$-k \frac{\partial T}{\partial z}$	$q_z$	$-k \frac{\partial T}{\partial z}$	$q_\phi$	$-k \frac{1}{r \sin \theta} \frac{\partial T}{\partial \phi}$
$\tau_{xx}$	$-2 \mu \frac{\partial v_x}{\partial x} + \mu^B \nabla \cdot \mathbf{v}$	$\tau_{rr}$	$-2 \mu \frac{\partial v_r}{\partial r} + \mu^B \nabla \cdot \mathbf{v}$	$\tau_{rr}$	$-2 \mu \frac{\partial v_r}{\partial r} + \mu^B \nabla \cdot \mathbf{v}$
$\tau_{yy}$	$-2 \mu \frac{\partial v_y}{\partial y} + \mu^B \nabla \cdot \mathbf{v}$	$\tau_{\theta\theta}$	$-2 \mu \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) + \mu^B \nabla \cdot \mathbf{v}$	$\tau_{\theta\theta}$	$-2 \mu \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) + \mu^B \nabla \cdot \mathbf{v}$
$\tau_{zz}$	$-2 \mu \frac{\partial v_z}{\partial z} + \mu^B \nabla \cdot \mathbf{v}$	$\tau_{zz}$	$-2 \mu \frac{\partial v_z}{\partial z} + \mu^B \nabla \cdot \mathbf{v}$	$\tau_{\phi\phi}$	$-2 \mu \left( \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r + v_\theta \cot \theta}{r} \right) + \mu^B \nabla \cdot \mathbf{v}$
$\tau_{xy}$	$-\mu \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)$	$\tau_{r\theta}$	$-\mu \left( r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right)$	$\tau_{r\theta}$	$-\mu \left( r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right)$
$\tau_{yz}$	$-\mu \left( \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right)$	$\tau_{\theta z}$	$-\mu \left( \frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \right)$	$\tau_{\theta\phi}$	$-\mu \left( \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left( \frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right)$
$\tau_{xz}$	$-\mu \left( \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right)$	$\tau_{zr}$	$-\mu \left( \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right)$	$\tau_{\phi r}$	$-\mu \left( \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left( \frac{v_\phi}{r} \right) \right)$

**Table 1:** Fourier's law for the heat flux and Newton's law for the stress in several coordinate systems. Please remember that the stress is symmetric, so  $\tau_{ij} = \tau_{ji}$ .

### Viscous models:

Power-Law Fluid:

$$|\tau_{xy}| = k \left| \frac{\partial v_x}{\partial y} \right|^n \quad (5)$$

Bingham-Plastic Fluid:

$$\frac{\partial v_x}{\partial y} = \begin{cases} -\mu^{-1} (\tau_{xy} - \tau_0) & \text{if } \tau_{xy} > \tau_0 \\ 0 & \text{if } \tau_{xy} \leq \tau_0 \end{cases}$$

### Dimensionless Numbers

$$\text{Re} = \frac{\rho \langle v \rangle D}{\mu} \quad \text{Re}_H = \frac{\rho \langle v \rangle D_H}{\mu} \quad \text{Re}_{MR} = -\frac{16 L \rho \langle v \rangle^2}{R \Delta p} \quad (6)$$

The hydraulic diameter is defined as  $D_H = 4 A / P_w$ .

### Single phase pressure drop calculations in pipes:

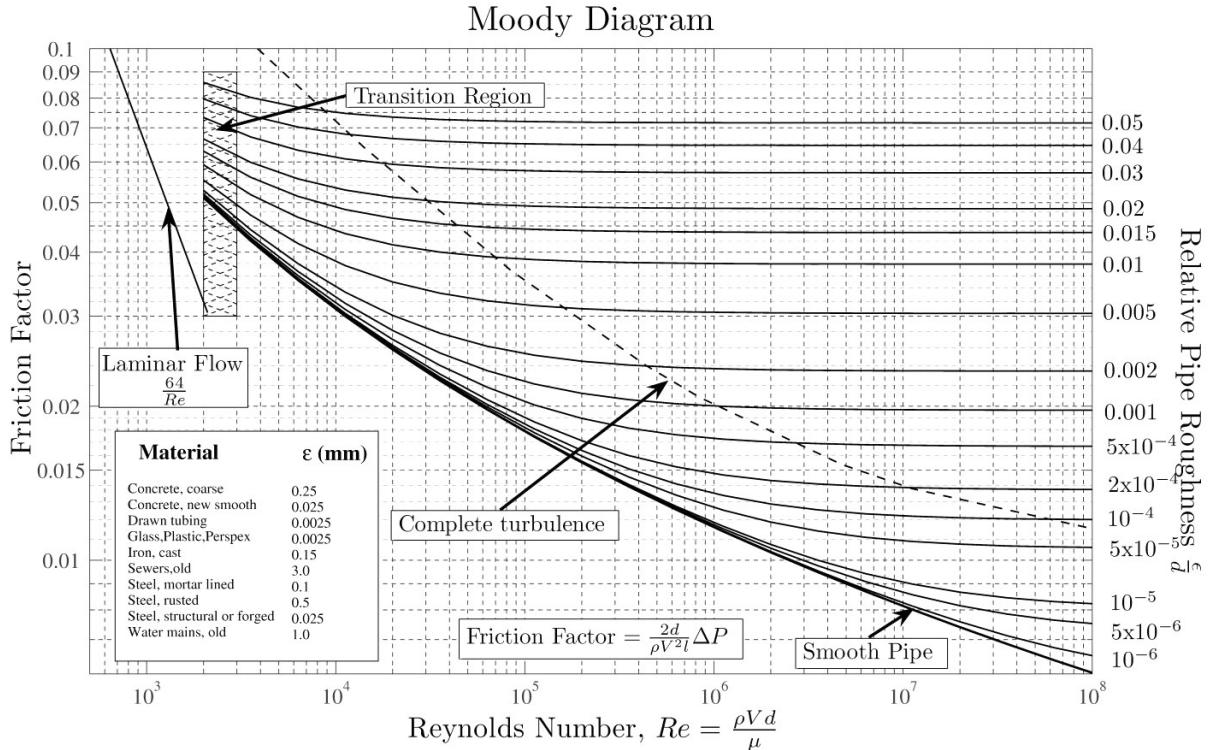
Darcy-Weisbach equation:

$$\frac{\Delta p}{L} = -\frac{C_f \rho \langle v \rangle^2}{R} \quad (7)$$

where  $C_f = 16/Re$  for laminar Newtonian flow. For turbulent flow of Newtonian fluids in smooth pipes, we have the Blasius correlation:

$$C_f = 0.079 Re^{-1/4} \quad \text{for } 2.5 \times 10^3 < Re < 10^5 \text{ and smooth pipes.}$$

Otherwise, you may refer to the Moody diagram.



**Figure 3:** The Moody diagram for flow in pipes.

Laminar Power-Law fluid:

$$\dot{V} = \frac{n \pi R^3}{3 n + 1} \left( \frac{R}{2k} \right)^{\frac{1}{n}} \left( -\frac{\Delta p}{L} \right)^{\frac{1}{n}}$$

### Two-Phase Flow:

Lockhart-Martinelli parameter:

$$X^2 = \frac{\Delta p_{liq.-only}}{\Delta p_{gas-only}}$$

Pressure drop calculation:

$$\Delta p_{two-phase} = \Phi_{liq.}^2 \Delta p_{liq.-only} = \Phi_{gas}^2 \Delta p_{gas-only}$$

Chisholm's relation:

$$\Phi_{gas}^2 = 1 + c X + X^2$$

$$\Phi_{liq.}^2 = 1 + \frac{c}{X} + \frac{1}{X^2}$$

$$c = \begin{cases} 20 & \text{turbulent liquid \& turbulent gas} \\ 12 & \text{laminar liquid \& turbulent gas} \\ 10 & \text{turbulent liquid \& laminar gas} \\ 5 & \text{laminar liquid \& laminar gas} \end{cases}$$

Farooqi and Richardson expression for liquid hold-up in co-current flows of Newtonian fluids and air in horizontal pipes:

$$h = \begin{cases} 0.186 + 0.0191 X & 1 < X < 5 \\ 0.143 X^{0.42} & 5 < X < 50 \\ 1 / (0.97 + 19/X) & 50 < X < 500 \end{cases}$$

**Heat Transfer Dimensionless numbers:**

$$Nu = \frac{hL}{k} \quad Pr = \frac{\mu C_p}{k} \quad Gr = \frac{g \beta \rho^2 (T_w - T_\infty) L^3}{\mu^2}$$

where  $\beta = V^{-1}(\partial V/\partial T)$ .

**Heat transfer: Resistances**

$$Q = U_T A_T \Delta T = R_T^{-1} \Delta T$$

	Conduction Shell Resistances			Radiation
	Rect.	Cyl.	Sph.	
$R$	$\frac{X}{kA}$	$\frac{\ln(R_{outer}/R_{inner})}{2\pi L k}$	$\frac{R_{inner}^{-1} - R_{outer}^{-1}}{4\pi k}$	$\left[A\varepsilon\sigma(T_j^2 + T_i^2)(T_j + T_i)\right]^{-1}$

**Radiation Heat Transfer:**

Stefan-Boltzmann constant  $\sigma = 5.6703 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$ .

Summation relationship,  $\sum_j F_{i \rightarrow j} = 1$ , and reciprocity relationship,  $F_{i \rightarrow j} A_i = F_{j \rightarrow i} A_j$ . Radiation shielding factor  $1/(N+1)$ .

$$Q_{rad.,i \rightarrow j} = \sigma \varepsilon F_{i \rightarrow j} A_i (T_j^4 - T_i^4) = h_{rad.} A (T_\infty - T_w)$$

**Natural Convection**

$\text{Ra} = \text{Gr Pr}$	$C$	$m$
$< 10^4$	1.36	1/5
$10^4\text{--}10^9$	0.59	1/4
$> 10^9$	0.13	1/3

**Table 2:** Natural convection coefficients for isothermal vertical plates in the empirical relation  $Nu \approx C (\text{Gr Pr})^m$ .

For isothermal vertical cylinders, the above expressions for isothermal vertical plates may be used but must be scaled by a factor,  $F$  (i.e.,  $\text{Nu}_{v.cyl.} = F \text{Nu}_{v.plate}$ ):

$$F = \begin{cases} 1 & \text{for } (D/H) \geq 35 \text{Gr}_H^{-1/4} \\ 1.3 \left[ H D^{-1} \text{Gr}_D^{-1} \right]^{1/4} + 1 & \text{for } (D/H) < 35 \text{Gr}_H^{-1/4} \end{cases} \quad (8)$$

where  $D$  is the diameter and  $H$  is the height of the cylinder. The subscript on Gr indicates which length is to be used as the critical length to calculate the Grashof number.

Churchill and Chu expression for natural convection from a horizontal pipe:

$$\text{Nu}^{1/2} = 0.6 + 0.387 \left\{ \frac{\text{Gr Pr}}{\left[ 1 + (0.559/\text{Pr})^{9/16} \right]^{16/9}} \right\}^{1/6} \quad \text{for } 10^{-5} < \text{Gr Pr} < 10^{12} \quad (9)$$

### Forced Convection:

Laminar flows:

$$\text{Nu} \approx 0.332 \text{Re}^{1/2} \text{Pr}^{1/3} \quad (10)$$

Well-Developed turbulent flows in smooth pipes:

$$\text{Nu} \approx 0.023 \text{Re}_D^{4/5} \text{Pr}^n \quad (11)$$

where  $n = 0.4$  if the fluid is being heated, and  $n = 0.3$  if the fluid is being cooled.

### Boiling:

Forster-Zuber pool-boiling coefficient:

$$h_{nb} = 0.00122 \frac{k_L^{0.79} C_{p,L}^{0.45} \rho_L^{0.49}}{\gamma^{0.5} \mu_L^{0.29} h_{fg}^{0.24} \rho_G^{0.24}} (T_w - T_{sat})^{0.24} (p_w - p_{sat})^{0.75} \quad (12)$$

Mostinski correlations:

$$h_{nb} = 0.104 p_c^{0.69} q^{0.7} \left[ 1.8 \left( \frac{p}{p_c} \right)^{0.17} + 4 \left( \frac{p}{p_c} \right)^{1.2} + 10 \left( \frac{p}{p_c} \right)^{10} \right] \quad (13)$$

$$q_c = 3.67 \times 10^4 p_c \left( \frac{p}{p_c} \right)^{0.35} \left[ 1 - \frac{p}{p_c} \right]^{0.9} \quad (14)$$

(Note: for the Mostinski correlations, the pressures are in units of bar)

### Condensing:

Horizontal pipes

$$h = 0.72 \left( \frac{k^3 \rho^2 g_x E_{latent}}{D \mu (T_w - T_\infty)} \right)^{1/4} \quad (15)$$

### Lumped capacitance method:

$$\text{Bi} = \frac{h L_c}{\kappa}$$

$$L_c = V/A \quad \text{for Bi} < 0.1$$

$$\frac{T(t) - T_\infty}{T_0 - T_\infty} = e^{-bt} \quad b = \frac{h A_s}{\rho V C_p}$$

### 1-D Transient Heat Conduction:

$$Fo = \frac{\alpha \Delta t}{(\Delta x)^2} = \tau, \quad \alpha = \kappa (\rho C_p)^{-1}$$

$$\theta_{\text{wall}} = \frac{T(x, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \cos \left( \frac{\lambda_1 x}{L} \right), \quad \theta_{\text{cyl}} = \frac{T(r, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \mathbf{J}_0 \left( \frac{\lambda_1 r}{r_0} \right)$$

$$\theta_{\text{sph}} = \frac{T(r, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} \frac{\sin \left( \frac{\lambda_1 r}{r_0} \right)}{\frac{\lambda_1 r}{r_0}}$$

$$\theta_{0,\text{wall}} = \theta_{0,\text{cyl}} = \theta_{0,\text{sph}} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

$$\left( \frac{\mathcal{Q}}{\mathcal{Q}_{\max}} \right)_{\text{wall}} = 1 - \theta_{0,\text{wall}} \frac{\sin \lambda_1}{\lambda_1}, \quad \left( \frac{\mathcal{Q}}{\mathcal{Q}_{\max}} \right)_{\text{cyl}} = 1 - 2\theta_{0,\text{cyl}} \frac{\mathbf{J}_1(\lambda_1)}{\lambda_1}$$

$$\left( \frac{\mathcal{Q}}{\mathcal{Q}_{\max}} \right)_{\text{sph}} = 1 - 3\theta_{0,\text{sph}} \frac{\sin \lambda_1 - \lambda_1 \cos \lambda_1}{\lambda_1^3}$$

**Finite-Difference Method:**

$$\frac{\partial}{\partial t}(\rho\phi) + \frac{\partial}{\partial x}(\rho\mathbf{v}\phi) = \nabla \cdot (\Gamma\nabla\phi) + \mathcal{S} \quad (1D \text{ transport equation})$$

$$\left(\frac{d\phi}{dx}\right)_i = \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} \quad \text{and} \quad \left(\frac{d^2\phi}{dx^2}\right)_i = \frac{\phi_{i-1} + \phi_{i+1} - 2\phi_i}{(\Delta x)^2}$$

$$T_i^{j+1} = (1 - 2\tau) T_i^j + \tau \left( T_{i+1}^j + T_{i-1}^j \right) + \frac{\tau (\Delta x)^2}{\kappa} \mathcal{S}_i^j$$

**Overall Heat Transfer Coefficient:**

$$\dot{Q} = \frac{\Delta T}{\mathcal{R}} = UA\Delta T = U_i A_i \Delta T = U_o A_o \Delta T$$

$$\mathcal{R} = R_i + R_{\text{wall}} + R_o = \frac{1}{h_i A_i} + \frac{\ln D_o/D_i}{2\pi\kappa L} + \frac{1}{h_o A_o}$$

**Fouling Factor:**

$$\mathcal{R} = \frac{1}{h_i A_i} + \frac{R_{f,i}}{A_i} + R_{\text{wall}} + \frac{R_{f,o}}{A_o} + \frac{1}{h_o A_o}$$

**LMTD Method:**

$$\dot{Q} = UA_s \Delta T_{\text{lm}} \quad \text{with} \quad \Delta T_{\text{lm}} = \frac{\Delta T_2 - \Delta T_1}{\ln \frac{\Delta T_2}{\Delta T_1}} = \frac{\Delta T_1 - \Delta T_2}{\ln \frac{\Delta T_1}{\Delta T_2}}$$

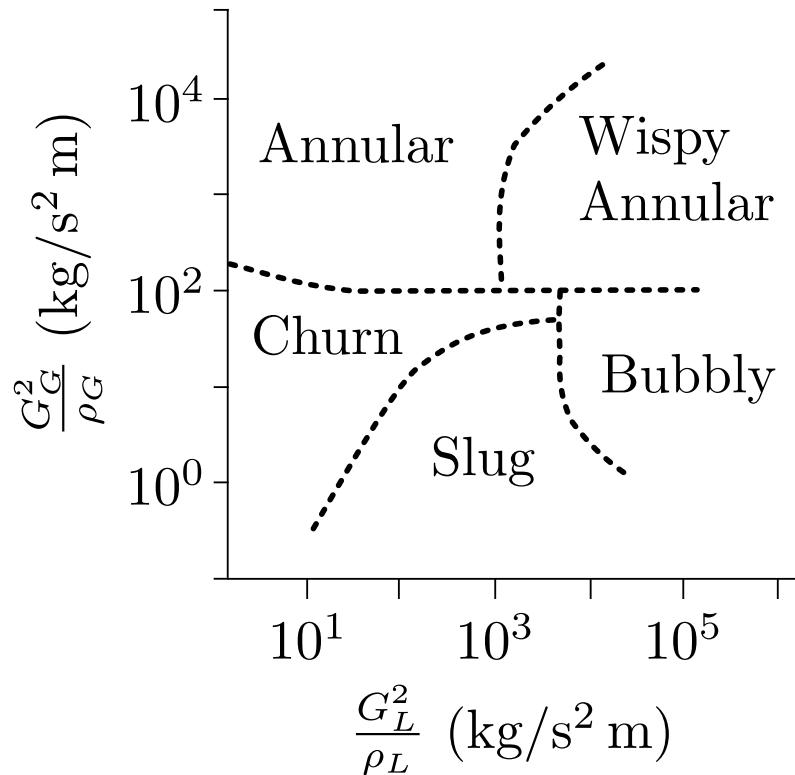
$$\text{Parallel flows: } \begin{cases} \Delta T_1 = T_{\text{hot,in}} - T_{\text{cold,in}} \\ \Delta T_2 = T_{\text{hot,out}} - T_{\text{cold,out}} \end{cases}$$

$$\text{Counter flows: } \begin{cases} \Delta T_1 = T_{\text{hot,in}} - T_{\text{cold,out}} \\ \Delta T_2 = T_{\text{hot,out}} - T_{\text{cold,in}} \end{cases}$$

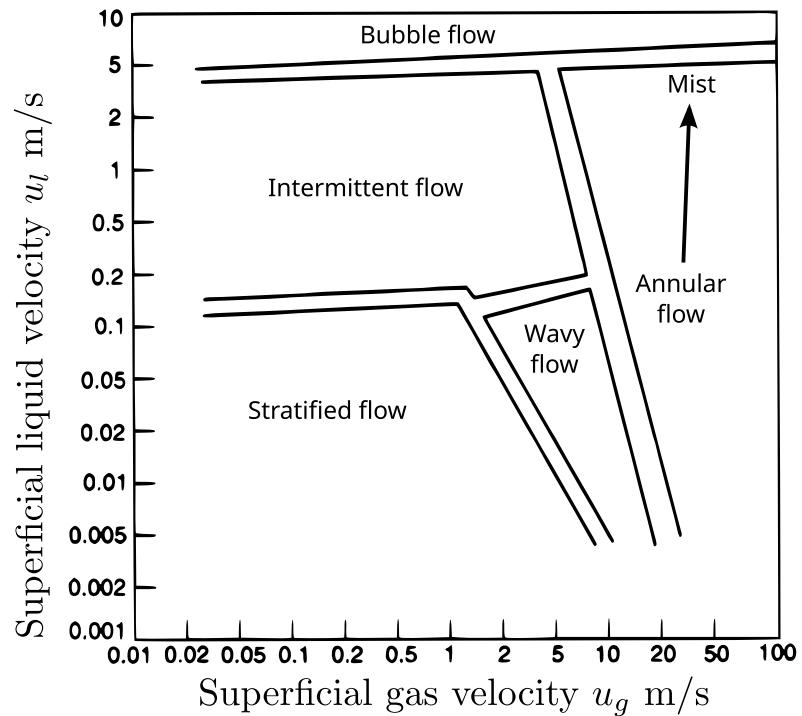
 **$\epsilon$ -NTU Method:**

$$\epsilon = \frac{\dot{Q}}{\dot{Q}_{\max}}, \quad \text{with} \quad \dot{Q}_{\max} = C_{\min} (T_{\text{hot,in}} - T_{\text{cold,in}}) \quad \text{and} \quad C_{\min} = \text{Min} \{ \dot{m}_{\text{hot}} C_{p,\text{hot}}, \dot{m}_{\text{cold}} C_{p,\text{cold}} \}$$

$$\text{NTU} = \frac{UA_s}{C_{\min}}$$



**Figure 4:** Hewitt-Taylor flow pattern map for multiphase flows in vertical pipes.



**Figure 5:** Chhabra and Richardson flow pattern map for horizontal pipes.

**TABLE 4–2**

Coefficients used in the one-term approximate solution of transient one-dimensional heat conduction in plane walls, cylinders, and spheres ( $\text{Bi} = hL/k$  for a plane wall of thickness  $2L$ , and  $\text{Bi} = hr_o/k$  for a cylinder or sphere of radius  $r_o$ )

Bi	Plane Wall		Cylinder		Sphere	
	$\lambda_1$	$A_1$	$\lambda_1$	$A_1$	$\lambda_1$	$A_1$
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.1	0.3111	1.0161	0.4417	1.0246	0.5423	1.0298
0.2	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.3	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.4	0.5932	1.0580	0.8516	1.0931	1.0528	1.1164
0.5	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.6	0.7051	1.0814	1.0184	1.1345	1.2644	1.1713
0.7	0.7506	1.0918	1.0873	1.1539	1.3525	1.1978
0.8	0.7910	1.1016	1.1490	1.1724	1.4320	1.2236
0.9	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1785	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2899	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7202
5.0	1.3138	1.2403	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8673
8.0	1.3978	1.2570	2.1286	1.5526	2.7654	1.8920
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2880	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
$\infty$	1.5708	1.2732	2.4048	1.6021	3.1416	2.0000

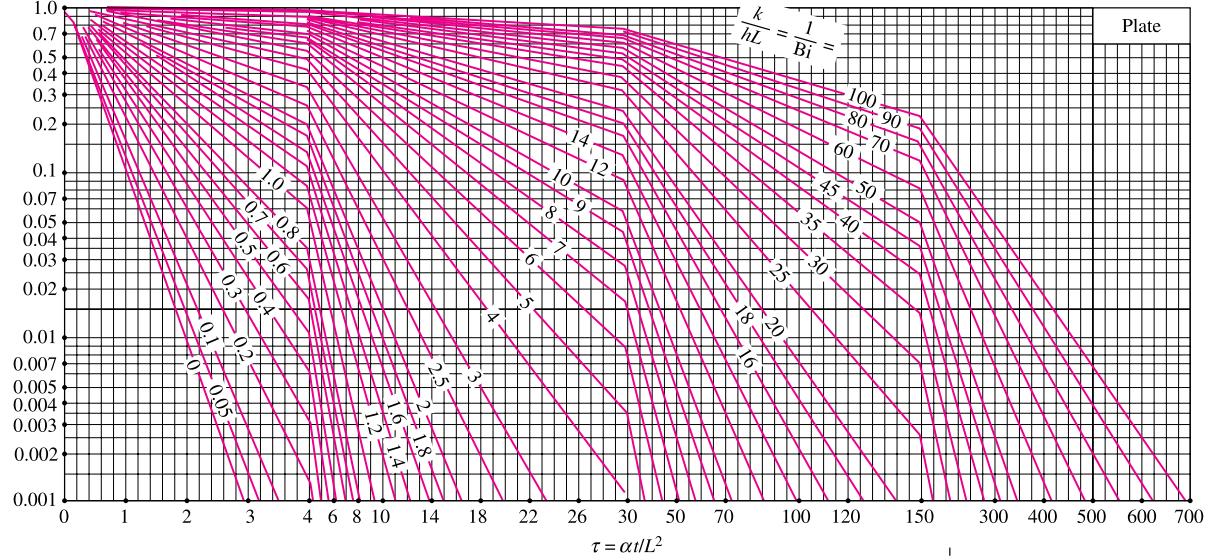
**TABLE 4–3**

The zeroth- and first-order Bessel functions of the first kind

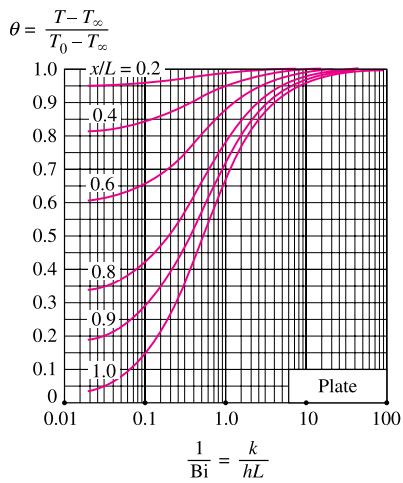
$\eta$	$J_0(\eta)$	$J_1(\eta)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419
1.5	0.5118	0.5579
1.6	0.4554	0.5699
1.7	0.3980	0.5778
1.8	0.3400	0.5815
1.9	0.2818	0.5812
2.0	0.2239	0.5767
2.1	0.1666	0.5683
2.2	0.1104	0.5560
2.3	0.0555	0.5399
2.4	0.0025	0.5202
2.6	-0.0968	-0.4708
2.8	-0.1850	-0.4097
3.0	-0.2601	-0.3391
3.2	-0.3202	-0.2613

**Figure 6:** Coefficients for the 1D transient equations.

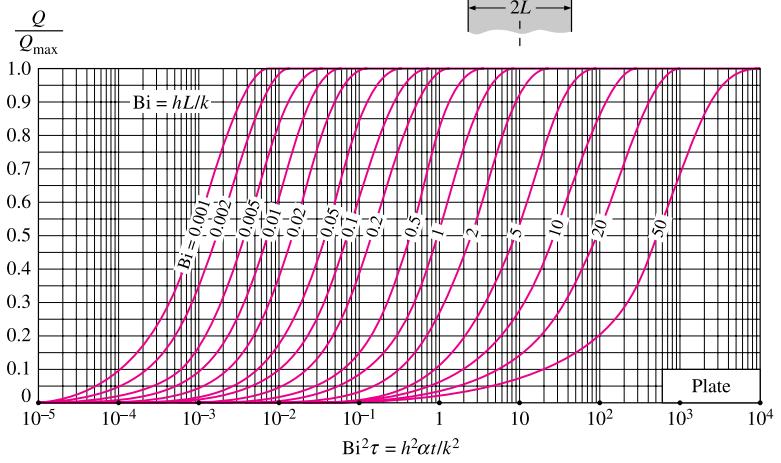
$$\theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty}$$



(a) Midplane temperature (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME* 69, 1947, pp. 227–36. Reprinted by permission of ASME International.)



(b) Temperature distribution (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME* 69, 1947, pp. 227–36. Reprinted by permission of ASME International.)

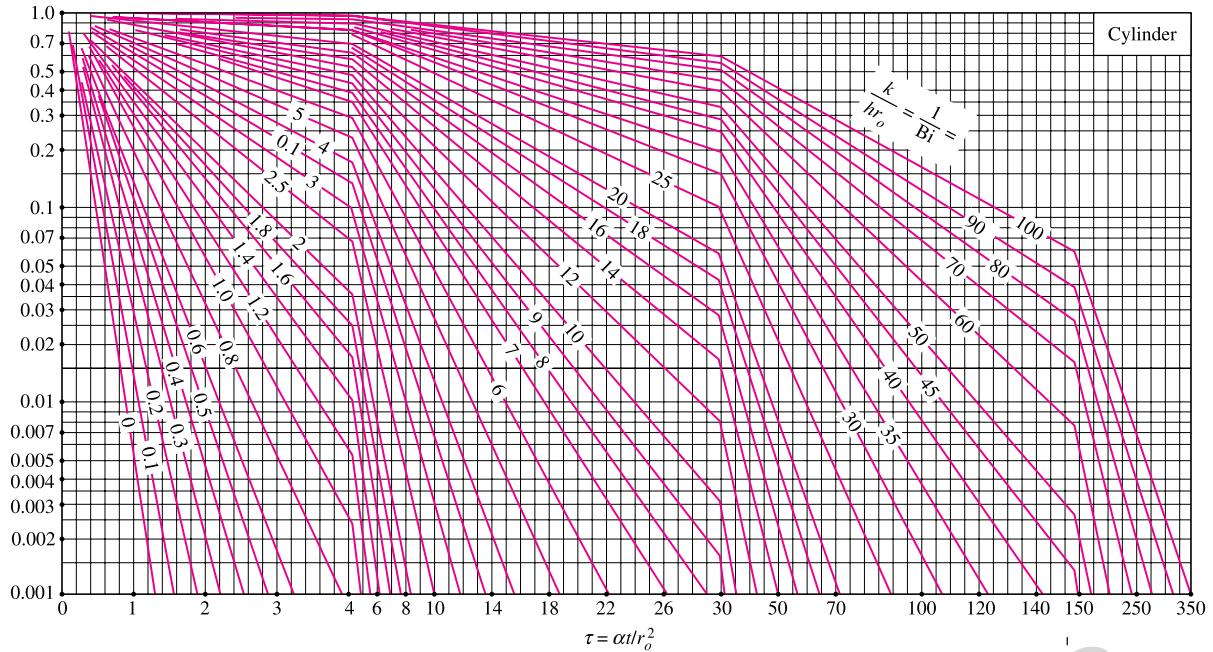


(c) Heat transfer (from H. Gröber et al.)

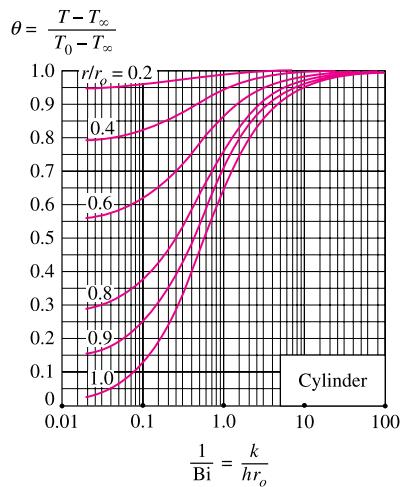
Transient temperature and heat transfer charts for a plane wall of thickness  $2L$  initially at a uniform temperature  $T_i$  subjected to convection from both sides to an environment at temperature  $T_\infty$  with a convection coefficient of  $h$ .

**Figure 7:**

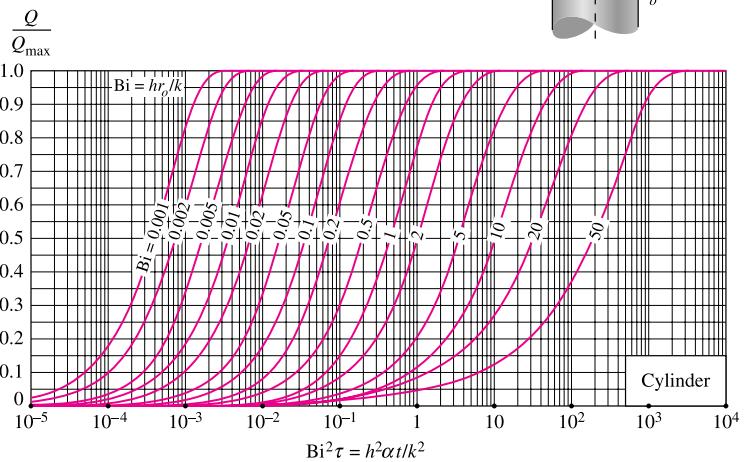
$$\theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty}$$



(a) Centerline temperature (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME* 69, 1947, pp. 227–36. Reprinted by permission of ASME International.)



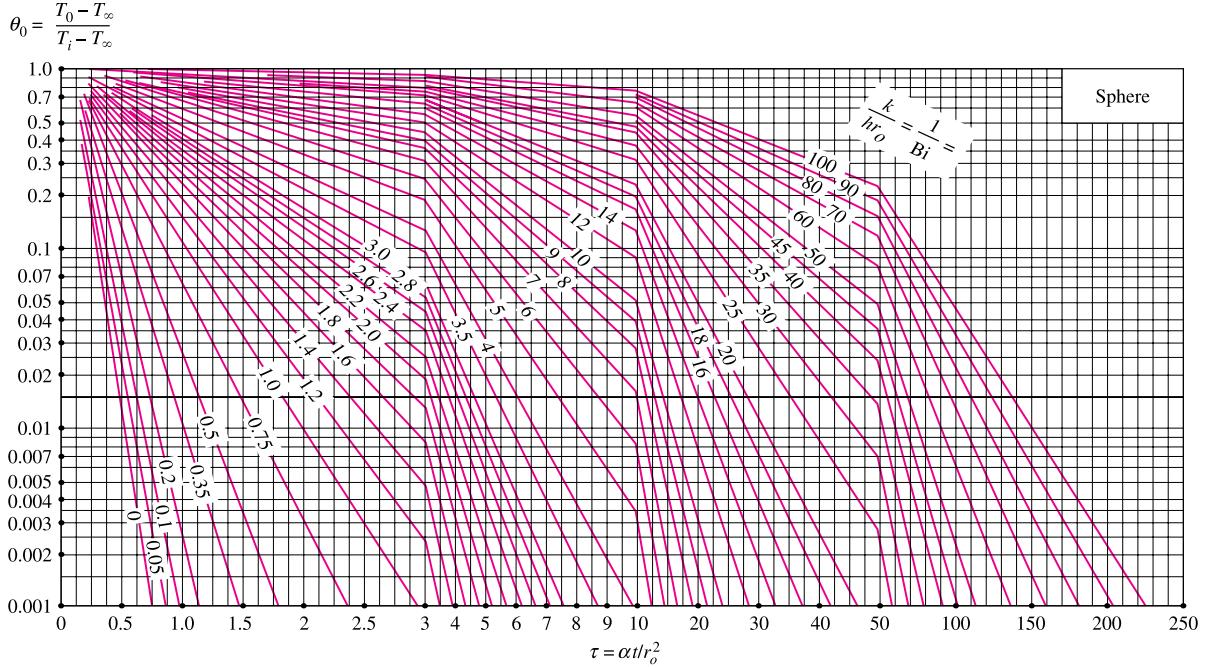
(b) Temperature distribution (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME* 69, 1947, pp. 227–36. Reprinted by permission of ASME International.)



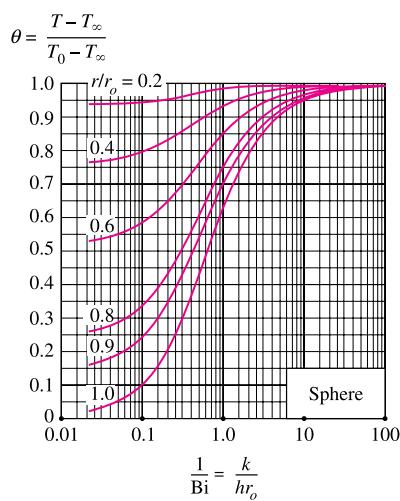
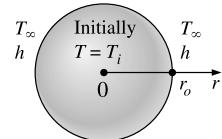
(c) Heat transfer (from H. Gröber et al.)

Transient temperature and heat transfer charts for a long cylinder of radius  $r_o$  initially at a uniform temperature  $T_i$  subjected to convection from all sides to an environment at temperature  $T_\infty$  with a convection coefficient of  $h$ .

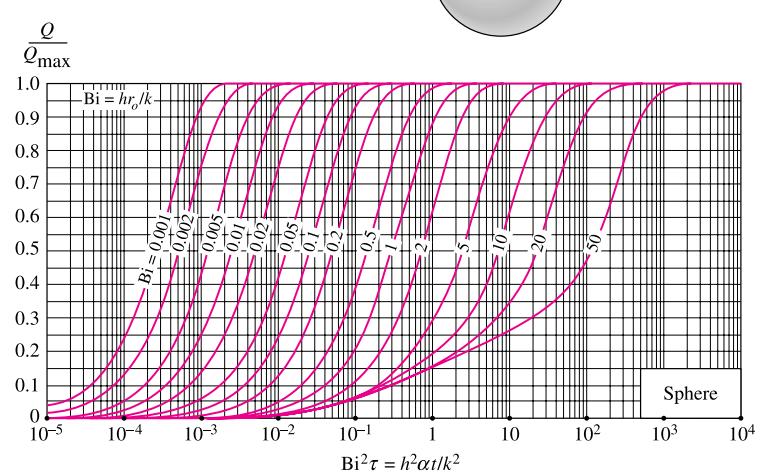
**Figure 8:**



(a) Midpoint temperature (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME* 69, 1947, pp. 227–36. Reprinted by permission of ASME International.)



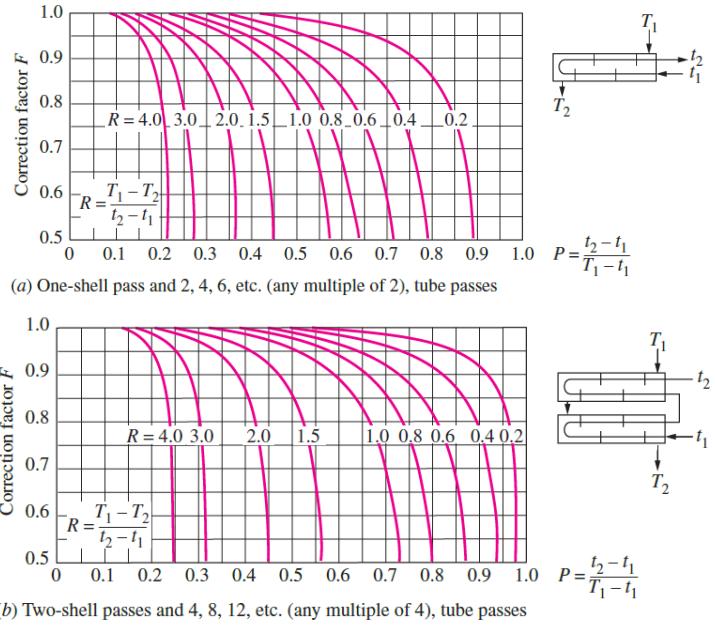
(b) Temperature distribution (from M. P. Heisler, "Temperature Charts for Induction and Constant Temperature Heating," *Trans. ASME* 69, 1947,



(c) Heat transfer (from H. Gröber et al.)

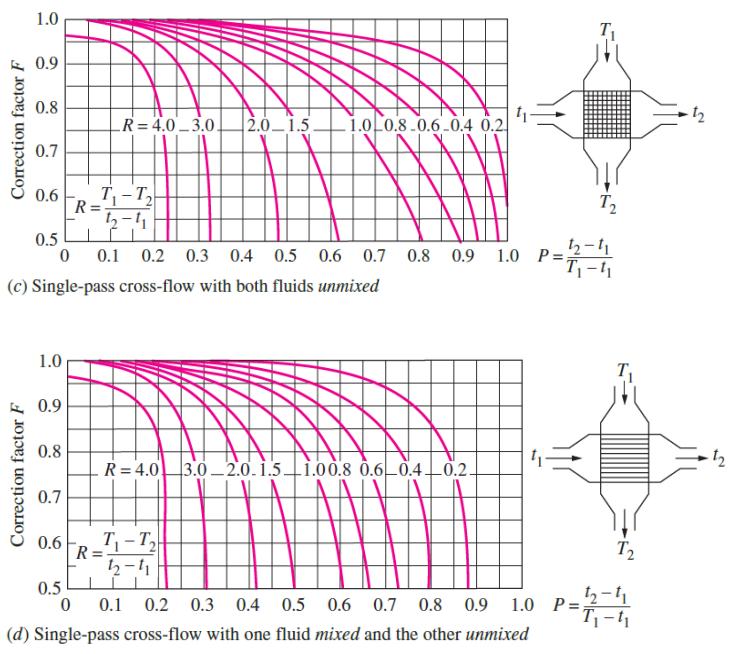
Transient temperature and heat transfer charts for a sphere of radius  $r_o$  initially at a uniform temperature  $T_i$  subjected to convection from all sides to an environment at temperature  $T_\infty$  with a convection coefficient of  $h$ .

**Figure 9:**



**Figure 10.8**  
Correction factor  $F$  charts  
for common shell-and-tube and  
cross-flow heat exchangers (from  
Bowman, Mueller, and Nagle, Ref. 2).

**Figure 10:** Correction-factors for LMTD Method, extracted from Y. A. Cengel, "Heat transfer:A practical approach", 2nd Ed.



**Figure 10.8**  
Correction factor  $F$  charts  
for common shell-and-tube and  
cross-flow heat exchangers (from  
Bowman, Mueller, and Nagle, Ref. 2).

**Figure 11:** Correction-factors for LMTD Method, extracted from Y. A. Cengel, "Heat transfer:A practical approach", 2nd Ed.

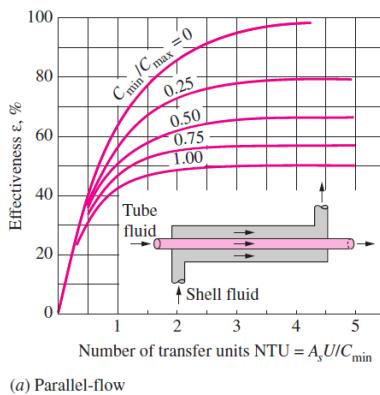
Effectiveness relations for heat exchangers:  $NTU = UA_s/C_{min}$ , and  $c = C_{min}/C_{max} = (\dot{m}C_p)_{min}/(\dot{m}C_p)_{max}$  (Kays and London, Ref. 5.)

Heat exchanger type	Effectiveness relation
1 Double pipe:	
Parallel-flow	$\epsilon = \frac{1 - \exp[-NTU(1 + c)]}{1 + c}$
Counter-flow	$\epsilon = \frac{1 - \exp[-NTU(1 - c)]}{1 - c \exp[-NTU(1 - c)]}$
2 Shell and tube:	
One-shell pass	$\epsilon = 2 \left( 1 + c + \sqrt{1 + c^2} \frac{1 + \exp[-NTU\sqrt{1 + c^2}]}{1 - \exp[-NTU\sqrt{1 + c^2}]} \right)^{-1}$
2, 4, ... tube passes	
3 Cross-flow (single-pass)	
Both fluids unmixed	$\epsilon = 1 - \exp \left\{ \frac{NTU^{0.22}}{c} [\exp(-c NTU^{0.78}) - 1] \right\}$
$C_{max}$ mixed, $C_{min}$ unmixed	$\epsilon = \frac{1}{c}(1 - \exp[1 - c[1 - \exp(-NTU)])]$
$C_{min}$ mixed, $C_{max}$ unmixed	$\epsilon = 1 - \exp \left\{ -\frac{1}{c}[1 - \exp(-c NTU)] \right\}$
4 All heat exchangers with $c = 0$	$\epsilon = 1 - \exp(-NTU)$

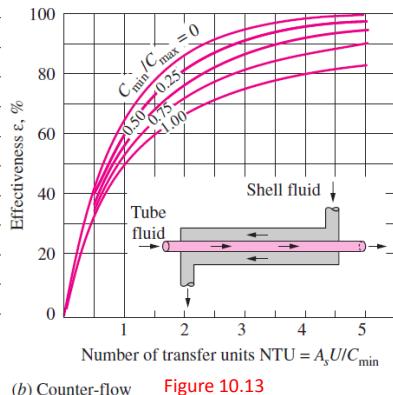
NTU relations for heat exchangers  $NTU = UA_s/C_{min}$ , and  $c = C_{min}/C_{max} = (\dot{m}C_p)_{min}/(\dot{m}C_p)_{max}$  (Kays and London, Ref. 5.)

Heat exchanger type	NTU relation
1 Double-pipe: Parallel-flow	$NTU = -\frac{\ln[1 - \epsilon(1 + c)]}{1 + c}$
Counter-flow	$NTU = \frac{1}{c - 1} \ln \left( \frac{\epsilon - 1}{\epsilon c - 1} \right)$
2 Shell and tube: One-shell pass 2, 4, ... tube passes	$NTU = -\frac{1}{\sqrt{1 + c^2}} \ln \left( \frac{2/\epsilon - 1 - c - \sqrt{1 + c^2}}{2/\epsilon - 1 - c + \sqrt{1 + c^2}} \right)$
3 Cross-flow (single-pass) $C_{max}$ mixed, $C_{min}$ unmixed	$NTU = -\ln \left[ 1 + \frac{\ln(1 - \epsilon c)}{c} \right]$
$C_{min}$ mixed, $C_{max}$ unmixed	$NTU = -\frac{\ln[c \ln(1 - \epsilon) + 1]}{c}$
4 All heat exchangers with $c = 0$	$NTU = -\ln(1 - \epsilon)$

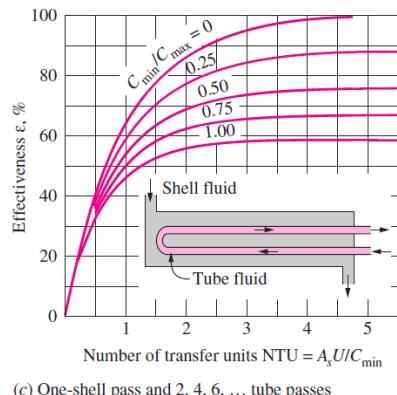
**Figure 12:** NTU relations extracted from Y. A. Cengel, “Heat transfer:A practical approach”, 2nd Ed.



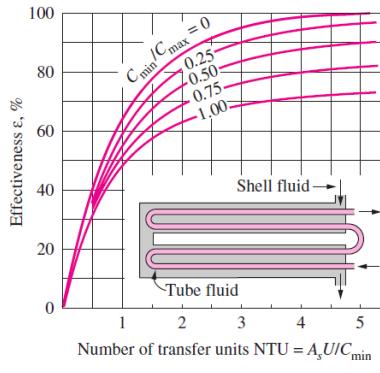
(a) Parallel-flow



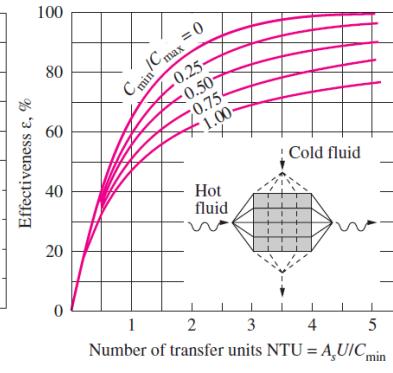
(b) Counter-flow



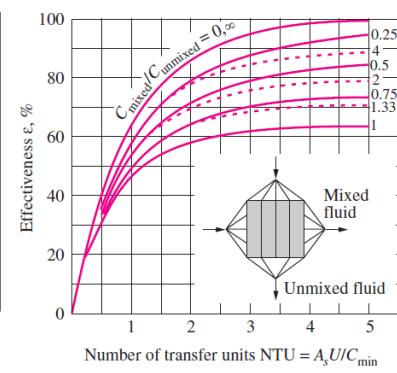
(c) One-shell pass and 2, 4, 6, ... tube passes



(d) Two-shell passes and 4, 8, 12, ... tube passes



(e) Cross-flow with both fluids unmixed



(f) Cross-flow with one fluid mixed and the other unmixed

Extracted from Y.A. Cengel, “Heat Transfer: A Practical Approach”, 2<sup>nd</sup> Edition.

**Figure 13:** NTU plots extracted from Y. A. Cengel, “Heat transfer:A practical approach”, 2nd Ed.

## Diffusion Dimensionless Numbers

$$\text{Sc} = \frac{\mu}{\rho D_{AB}}$$

$$\text{Le} = \frac{k}{\rho C_p D_{AB}}$$

### Diffusion

General expression for the flux:

$$\mathbf{N}_A = \mathbf{J}_A + x_A \sum_B \mathbf{N}_B$$

Fick's law:

$$\mathbf{J}_A = -D_{AB} \nabla C_A$$

Stefan's law:

$$N_{s,r} = -D \frac{c}{1-x} \frac{\partial x}{\partial r}$$

### Ideal Gas

$$P V = n R T$$

$$R \approx 8.314598 \text{ J K}^{-1} \text{ mol}^{-1}$$

### Geometry

$$P_{\text{circle}} = 2 \pi r \quad A_{\text{circle}} = \pi r^2 \quad A_{\text{sphere}} = 4 \pi r^2 \quad V_{\text{sphere}} = \frac{4}{3} \pi r^3$$

$$A_{\text{cylinder}} = P_{\text{circle}} L \quad V_{\text{cylinder}} = A_{\text{circle}} L$$