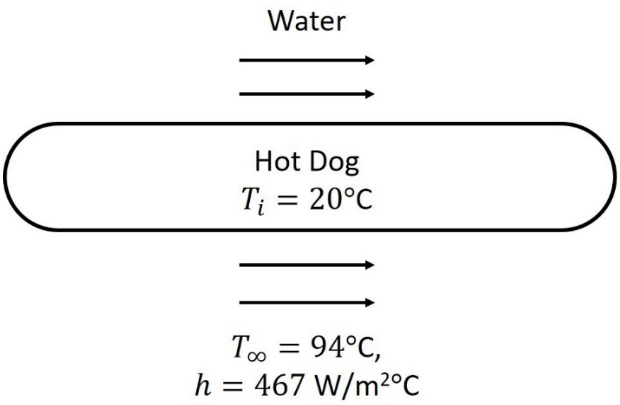
WORKED EXAMPLE

ANALYTICAL METHODS

A hot dog, initially at a uniform temperature of 20°C, is dropped into a pan of boiling water, which is at a temperature of 94°C. The convection heat transfer coefficient between the hot dog and the water is $h = 467 \left[\frac{\mathrm{W}}{\mathrm{m^2 °C}}\right]$.

The hot dog is 12.5 cm long and 2.2 cm in diameter with the following properties:

- Specific heat capacity, $C_\mathrm{p} = 3{,}900 \left[\frac{\mathrm{J}}{\mathrm{kg°C}}\right]$
- Density, $\rho = 980 \left[\frac{\mathrm{kg}}{\mathrm{m^3}}\right]$
- Thermal conductivity, $k = 0.771 \left[\frac{\mathrm{W}}{\mathrm{m°C}}\right]$



(1) Determine the centre and the surface temperatures of the hot dog 4 minutes after the start of the cooking.

(2) Determine the amount of heat transferred to the hot dog.

Solution

Step 1, check if we can use the lumped capacitance approach, starting with an estimation of L_c, the characteristic length:

$$L_c = \frac{V}{A_s} = \frac{\pi R^2 L}{2\pi R L} = \frac{R}{2} = \frac{0.011}{2} = 5.5 \times 10^{-3} [\mathrm{m}]$$

The Biot number can then be calculated as:

$$\therefore Bi = \frac{hL_C}{k} = \frac{467 \times 5.5 \times 10^{-3}}{0.771} = 3.33$$

Since $Bi \gg 0.1$ the lumped capacitance method cannot be used.

Next, since the length of the hot dog is more than 10 times its radius, it is reasonable to assume that heat transfer through the geometry can be modelled as 1-dimensional and we can therefore approach a solution to the problem using one of the analytical methods. We have two analytical methods to choose from.

<u>Method 1 – using the 1-term approximation</u>

The starting point for this approach is the expression for the 1-term solution to the heat equation (equation 24 from the notes):

$$\theta_{\text{cyl}} = A_1 e^{-\lambda_1^2 \tau} J_0\left(\lambda_1 \frac{R}{R_o}\right)$$

Where A_1 and λ_1 are coefficients (dependent only on the Biot number*) to be determined using look up tables, τ is the Fourier number, J_0 is a zeroth order Bessel function, which is also to be determined using a different set of look up tables. The argument of the Bessel function includes λ_1, the radial distance from the centre of the cylinder, R and the radius of the cylinder, R_o.

(*Note, the Biot number in this situation is not the Biot number used in the lumped capacitance approach)

The Biot number for the A_1 and λ_1 coefficients is given by:

$$Bi = \frac{hR_o}{k} = \frac{467 \times 0.011}{0.771} = 6.66$$

Now we can go to the look up tables for a cylinder (Table 4-1, Appendix A, lecture notes). However, since our value of Biot number is not listed in the table then we need to interpolate between the two surrounding known values (i.e. at $Bi_x = 6.0$ and $Bi_y = 7.0$.

$$\begin{aligned}
\lambda_1 &= \lambda_x + (Bi - Bi_x)\frac{\lambda_y - \lambda_x}{Bi_y - Bi_x} \\
&= 2.0490 + (6.66 - 6.0)\frac{(2.0937 - 2.0490)}{(7.0 - 6.0)} \\
&= 2.0785
\end{aligned}$$

Similarly, for A_1, we can use interpolation to get:

$$A_1 = A_x + (Bi - Bi_x)\frac{A_y - A_x}{Bi_y - Bi_x}$$

$$= 1.5253 + (6.66 - 6.0)\frac{(1.5411 - 1.5253)}{(7.0 - 6.0)}$$
$$= 1.5357$$

For $J_0(\lambda_1)$, we need to use Table 4-2 (Appendix A, lecture notes), and using linear interpolation, between known values of $\lambda_x = 2.0$ and $\lambda_y = 2.1$, we get:

$$J_0(\lambda_1) = J_0(\lambda_x) + (\lambda_1 - \lambda_x)\frac{\left(J_0(\lambda_y) - J_0(\lambda_x)\right)}{(\lambda_y - \lambda_x)}$$
$$= 0.2239 + (2.0785 - 2.0)\frac{(0.1666 - 0.2239)}{(2.1 - 2.0)}$$
$$= 0.1789$$

Next we need to estimate the Fourier number, τ:

$$\tau = \frac{\alpha t}{R^2}$$

where $t = 4 \times 60 = 240$ seconds is the time, and α is the thermal diffusivity

$$\alpha = \frac{k}{\rho C_p} = \frac{0.771}{980 \times 3900} = 2.017 \times 10^{-7}$$

$$\therefore \tau = \frac{2.017 \times 10^{-7} \times 240}{0.011^2} = 0.4$$

Note, since $\tau = 0.4 > 0.2$, the approximation of only considering the leading term in the solution of the heat equation should have minimal errors on the final result.

Now returning to our original equation for θ_{cyl}, we have, <u>at the surface of the cylinder</u>, after four minutes of cooking, the temperature will be:

$$\theta_{\text{cyl}} = \frac{T(R_0, t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} J_0(\lambda_1)$$

Rearranging for the unknown temperature:

$$T(R_0, t) = T_\infty + A_1 e^{-\lambda_1^2 \tau} J_0(\lambda_1)(T_i - T_\infty)$$
$$= 94 + 1.5357 \times e^{(-2.0785^2 \times 0.4)} \times 0.1789 \times (20 - 94)$$
$$= 90.39℃$$

Similarly, <u>at the centre of the cylinder</u>, after four minutes of cooking, the temperature will be:

$$\theta_{\text{cyl}} = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

Rearranging for the unknown temperature:

$$\begin{aligned} T_0 &= T_\infty + (T_i - T_\infty) A_1 e^{-\lambda_1^2 \tau} \\ &= 94 + (20 - 94) \times 1.5357 \times e^{(-2.0785^2 \times 0.4)} \\ &= 73.8℃ \end{aligned}$$

Next, we have to determine the amount of heat transferred to the hot dog, and for that we can refer to equations (27) from the lecture notes:

$$\frac{Q}{Q_{\max}} = 1 - 2\theta_0 \frac{J_1(\lambda_1)}{\lambda_1}$$

We can use Table 4-2 (Appendix A, lecture notes) to get a value for $J_1(2.0785)$, and by linear interpolation arrive at a value for $J_1(2.0785)$ of 0.5704. The parameter θ_0 can be estimated from:

$$\theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty} = \frac{73.8 - 94}{20 - 94} = 0.273$$

Then, the maximum heat transfer, $Q_{\max}$, is:

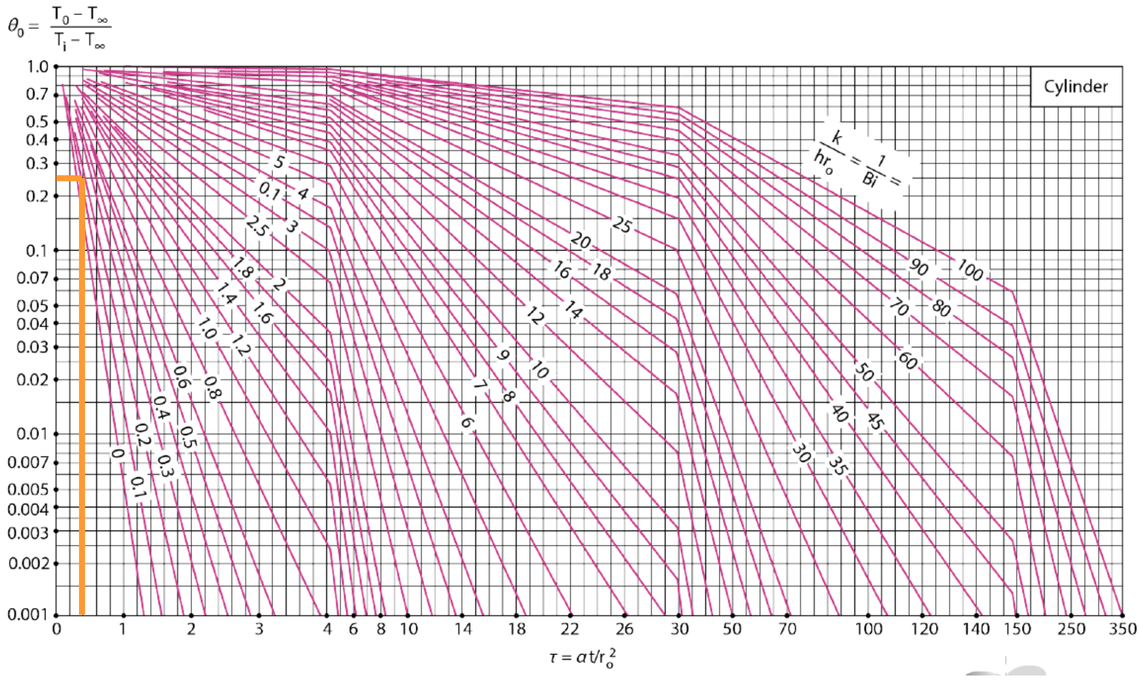
$$\begin{aligned} Q_{\max} &= m C_{\text{p}} (T_\infty - T_i) \\ &= \rho V C_{\text{p}} (T_\infty - T_i) \\ &= \rho (\pi R^2 L) C_{\text{p}} (T_\infty - T_i) \\ &= 980 \times \pi \times 0.011^2 \times 0.125 \times 3900 \times (94 - 20) \\ &= 13{,}439 \text{ [J]} \end{aligned}$$

Next, rearranging for the unknown heat transfer Q, we get:

$$\begin{aligned} Q &= \left[1 - 2\theta_0 \frac{J_1(\lambda_1)}{\lambda_1}\right] Q_{\max} \\ &= \left[1 - 2 \times 0.273 \times \frac{0.5704}{2.0785}\right] \times 13349 \\ &= 0.85 \times 13439 \\ &= 11.43 \text{ [kJ]} \end{aligned}$$

Method 2 – using the Heisler and Gröber charts.

First, to get the centreline temperature, we can refer to figure 4-16 (a) in Appendix A in the lecture notes, with known values of $\tau = 0.4$ and $1/Bi = \frac{1}{6.66} = 0.15$

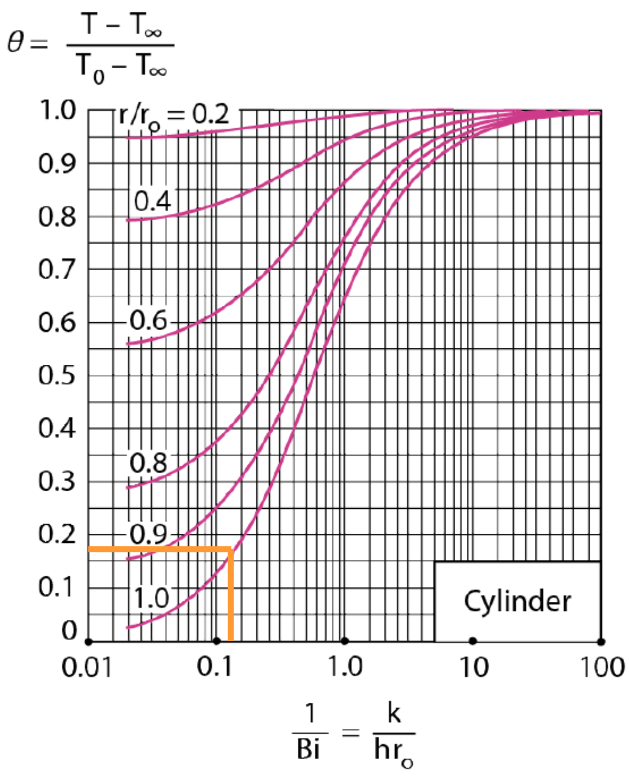


Reading the graph, we can get a value for $\theta_0 \sim 0.25$. We can then rearrange the equation for θ_0 to get the unknown centreline temperature:

$$\theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty} \Longrightarrow T_0 = 0.25 \times (20 - 94) + 94 = 75.5\text{°C}$$

For comparison, from method 1, $T_0 = 73.8\text{°C}$, which gives a discrepancy of approximately 2% with the result from the Heisler chart.

Next, to get the surface temperature, we can refer to the second of the Heisler charts, that is Figure 4-16 (b), in Appendix A of the lecture notes. For $1/Bi = 0.15$ and $R/R_0 = 1$, we can use the graph to get an estimate of θ:



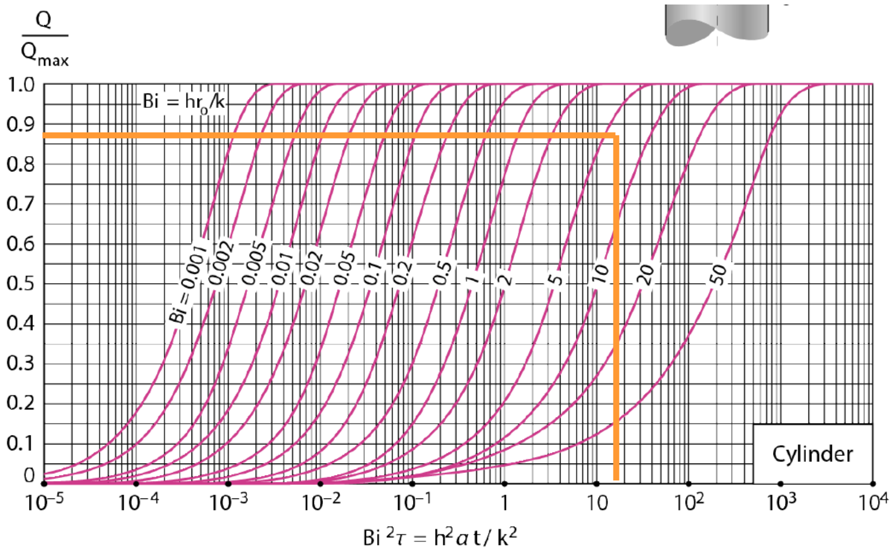
From the graph, we get $\theta \sim 0.175$. We can then rearrange the equation for θ to get the unknown surface temperature:

$$\theta = \frac{T - T_\infty}{T_0 - T_\infty} \Longrightarrow T = 0.175 \times (75.5 - 94) + 94 = 90.76℃$$

For comparison, from method 1, $T = 90.39℃$, which gives a discrepancy of less than 1% with the result from the Heisler chart.

Last of all, to get the unknown heat transfer, we can refer to the Gröber chart (Figure 4-16 (c), Appendix A, lecture notes). For this chart, we need an estimate of the parameter $Bi^2\tau$:

$$Bi^2\tau = 6.66^2 \times 0.4 = 17.7$$



From the chart, we can read that $Q/Q_{\max}{\sim}0.875$ and with our prior calculation of $Q_{\max} = 13.4$ kJ, we can get:

$$Q = 0.875 \times 13.4 = 11.73 \text{ [kJ]}$$

This can be compared against an estimate of 11.42 kJ derived from method 1.