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Physical and Geometrical Viewpoints of the Korteweg de Vries Equation

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Abstract

In 1834 John Scott Russell, a Scottish civil engineer, discovered some beautiful physics of which the importance was not recognised until more than one hundred years later. He watched a wave, created by an abruptly stopped boat in a shallow water channel, which propagated, bound by the walls, with such speed that he needed to make use of his horse to keep up with it. Although he followed it for quite some time, the wave showed almost no significant change in shape. This thesis' aim is to study the physics behind this phenomena and will try to bring with it some mathematical intuition originated from more or less basic differential geometry.

Introduction

Arriving at same results whilst taking different routes is one of the many beauties of physics and mathematics. A well-known example is classical mechanics where one can choose out of a variety of different techniques to arrive at the sought equations of motion. A remarkable fact is that during the course of a Physics bachelor the students are usually taught more than just *the best* method. This can easily be explained when one accepts that there exits no best method. Newton is usually taught in the first semester, as

$$\boldsymbol{F} = \dot{\boldsymbol{p}}$$

is more digestible than

$$\frac{\partial L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)}{\partial \dot{q}_i}$$

or

$$\frac{\partial H(\boldsymbol{q}, \boldsymbol{p}, t)}{\partial p_i} = \dot{q}_i \qquad \frac{\partial H(\boldsymbol{q}, \boldsymbol{p}, t)}{\partial q_i} = -\dot{p}_i$$

whereas it is not obviously given that any of the three are easier to apply or understand. Depending on the problem either can provide a better way of interpretation, a more natural and therefore more intuitive process of calculation or simply a smaller amount of pages needed to finish.

If this thesis works out, one can profit of it by gaining insight into two absolutely different origins of a single partial differential equation (PDE), namely the *Korteweg - de Vries equation (KdV)* which has not left the interest of mathematics and physics ever since its publication in 1895 in [1]. Physicists cannot turn their backs as it often comes up in the description of various systems (see chapter 3) and mathematicians because it has the property of describing a Hamiltonic system or *integrable* system, which can be defined as the possibility that one can find enough (in some cases infinitely many) first integrals to solve the PDE exactly (given the initial condition). What is so interesting about integrable systems? Some of the solutions are *solitonic*, which means that they keep their shape over a good amount of time even with small perturbation in the game which was exactly the property that Russell was so intrigued by and the reason why he dedicated much time experimenting with those waves.

That KdV can describe solitonic solutions was first discovered numerically [2] by Zabusky and Kruskal 70 years after the publication of Korteweg and de Vries. Section 4.4 will make use of an alternative, mathematical derivation of KdV by Ulrich Pinkall [3] to give some insight into this behaviour. Pinkall's derivation will be based on the theory of symplectic manifolds of which standard phase space is an example. Those symplectic manifolds (smooth manifold with a 2-form that has some properties) can predict the flow (i.e. evolution) of an element of the manifold (i.e. a point in phase space) by one scalar function, the Hamiltonian. It can therefore be viewed as a generalization of the Hamilton-formalism in physics to arbitrary smooth manifolds.

To sum up, as in the comparison above of Newton's and Lagrange's/Hamiltonian's formalism, the following will be a confrontation of the derivation by Korteweg and de

Vries which could most naturally be compared to the *Newtonian way* (extensive use of hydrodynamical laws) and the derivation by Pinkall which very directly represents a generalization of the *Hamiltonian way*.

1 The Newtonian Way (Physics)

In 1895 Prof. D. J. Korteweg and Dr. G. de Vries, in their paper "*On the Change of Form of Long Waves advancing in a Rectangular Canal, and on a New Type of Long Stationary Waves.*"[1] sought insight in the equation which describes the phenomena that Russell had observed. This was where their famous equation originated, of which the derivation will be sketched in the following.

1.1 Derivation by Korteweg and de Vries [1]

In order to find the desired PDE, the conditions will have to be specified. A perfect liquid in shallow water with waves of small amplitude but big wavelength will be assumed. Shallow is necessary because in order to obtain a soliton as solution, which is stable in time, the system needs an effect to cancel dispersion, which will be friction between water and bottom of the tank. This will be discussed more deeply in section 2.3. Amplitude and wavelength have to be restricted such that one can neglect terms of high order.

The wave will propagate in a tank with approximate height l as seen in fig. 1.1. Korteweg and de Vries began with an expansion of the speed of the wave $\boldsymbol{v}(x, y, t) = v_x(x, y, t)\,\hat{\boldsymbol{e}}_x + v_y(x, y, t)\,\hat{\boldsymbol{e}}_y \in \mathbb{R}^2$ at (x, y, t) in terms of the height y

$$\begin{aligned} v_x(x, y, t) &= \sum_{i \in \mathbb{N}_0} y^i f_i(x, t) \\ v_y(x, y, t) &= \sum_{j \in \mathbb{N}} y^j g_j(x, t) \end{aligned} \tag{1.1}$$

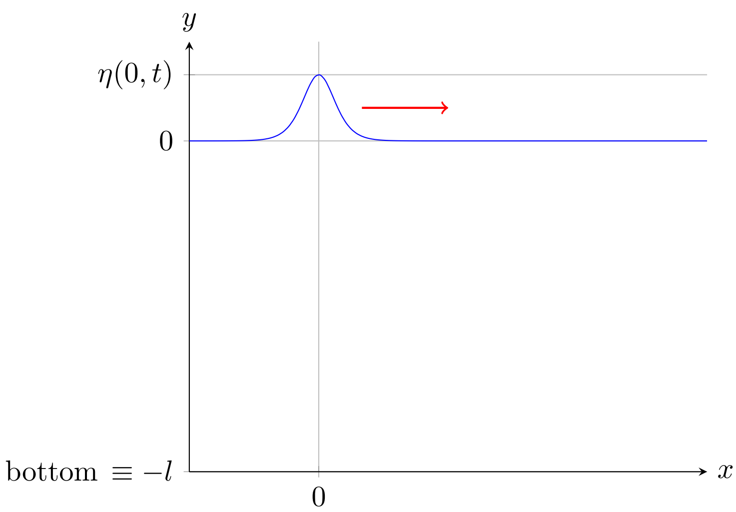

fig. 1.1: Description of variables.

of which the validity i.e. quick convergence emerges from the assumptions which will be seen later. The laws of hydrodynamics and the assumption of the perfect fluid will reduce these infinitely many degrees of freedom, namely the functions f_i and g_i, to just one free function $f_0 =: f$.

Firstly, the continuity equation of hydrodynamics $\dot{\rho} + \nabla \cdot \rho \boldsymbol{v} = 0$ reduces with the properties of a perfect fluid, $\dot{\rho} = 0$, $\nabla \rho = \mathbf{0}$, to

$$\partial_x v_x + \partial_y v_y = 0. \tag{1.2}$$

Secondly, it will be assumed that the speedfield arises from a scalar field, i.e.

$$\exists \phi \in C^2(\mathbb{R}) : \boldsymbol{v} = \nabla \phi \Longrightarrow \mathrm{rot}(\boldsymbol{v}) = \partial_x v_y - \partial_y v_x = 0. \tag{1.3}$$

Thirdly, conservation of momentum will be retained by fulfilling the Euler equation

$$(\partial_t + \boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \frac{1}{\rho}\nabla P = \frac{\boldsymbol{f}}{\rho} = -g\,\hat{\boldsymbol{e}}_y \tag{1.4}$$

with the pressure inside the wave $P(x, y, t)$ and an external forcedensity $\boldsymbol{f}$ which in this case is associated with the gravitational pull. Rewriting $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \frac{1}{2}\nabla(\boldsymbol{v}^2) - \boldsymbol{v} \times (\nabla \times \boldsymbol{v}) = \nabla\left(\frac{1}{2}\boldsymbol{v}^2\right)$ and $-g\,\hat{\boldsymbol{e}}_y = \nabla(-gy)$ renders (1.4) equivalent to

$$\nabla\left(\partial_t \phi + \frac{1}{2}\boldsymbol{v}^2 + \frac{P}{\rho} + g(y + l)\right) = 0 \Longrightarrow \exists \chi(t) : \partial_t \phi + \frac{1}{2}\boldsymbol{v}^2 + \frac{P}{\rho} + g(y + l) = \chi. \tag{1.5}$$

Also, the following definition of a surface tension T

$$\left(P = P_0 - T\partial_x^2 y\right)_{\text{surface}} \tag{1.6}$$

equated at the surface of the liquid with P_0 the atmospheric pressure will be used. The final condition connects the definition of the height of a surface particle

$$y|_{\text{surface}}(x, t) =: \eta(x, t) \tag{1.7}$$

(see fig. 1.1) with the speed of the surface

$$\begin{aligned} v_y\big|_{\text{surface}} &= \dot{y}\big|_{\text{surface}} = \partial_t \eta + \dot{x}\big|_{\text{surface}} \partial_x \eta \\ \Longleftrightarrow -(\partial_x \phi)_{\text{surface}} \partial_x \eta &+ (\partial_y \phi)_{\text{surface}} - \partial_t \eta = 0. \end{aligned} \tag{1.8}$$

Equations (1.2), (1.3), (1.5), (1.6) and (1.8) are the necessary conditions to restrict (1.1) to reproduce the KdV-equation in the surfaceheight variable $\eta(x, t)$. The following will be proceeded, skipping some steps in calculation for the sake of lucidity - see [1] for more precision in calculation.

Equations (1.2) and (1.3) provide a recursive formula apiece, rendering the scalarpotential $\phi(x, y, t)$ to be

$$\phi = \int f \mathrm{d}x + \sum_{n=1}^{\infty} (-1)^n y^{2n} \frac{\partial_x^{2n-1} f}{(2n)!} \cong \int f \mathrm{d}x - \frac{y^2}{2}\partial_x f + \frac{y^4}{24}\partial_x^3 f. \tag{1.9}$$

Note that $f = f(x,t)$ carries no index as it was defined above as $f \equiv f_0$ from expansion (1.1). From this, one can by means of (1.3) easily calculate the speed of a water particle at (x,y,t) by

$$\boldsymbol{v} \cong \left(f - \frac{y^2}{2}\partial_x^2 f + \frac{y^4}{24}\partial_x^4 f, -y\partial_x f + \frac{y^3}{6}\partial_x^3 f \right). \tag{1.10}$$

Note that the first estimations has been made in the last terms. These and the many following will be based on the *long wavelength* restriction, meaning that the speed of the surface varies weakly w.r.t. spacial x. In maths,

$$\partial_x \phi = v_x \gg y\partial_x v_x = y\partial_x^2 \phi \gg y^2 \partial_x^2 v_x = y^2 \partial_x^3 \phi.$$

As ϕ is now only constructed of one function that depends on x, namely $f(x,t)$, the $\partial_x^n f$ shrink quickly with increasing n. Together with the small $y|_{\text{surface}}$, one obtains an extremely fast converging series for ϕ and $\boldsymbol{v}$ in the monomials $(y\partial_x^{-1} f)^{2n}$ on the surface of the wave.

Proceeding with calculations, ϕ in (1.8) is eliminated in favour of f by (1.9) which gives

$$-\left(f - \frac{\eta^2}{2}\partial_x^2 f + \frac{\eta^4}{24}\partial_x^4 f \right)\partial_x \eta + \left(-\eta\partial_x f + \frac{\eta^3}{6}\partial_x^3 f \right) - \partial_t \eta = 0 \tag{1.11}$$

where (1.7) and (1.10) was used. One can now eliminate the pressure P from (1.6) in (1.5), equating (1.5) also at the surface of course:

$$\partial_t \phi|_{\text{surface}} + \frac{1}{2}\boldsymbol{v}^2|_{\text{surface}} + \frac{P_0 - T\partial_x^2 \eta}{\rho} + g(\eta + l) = \chi. \tag{1.12}$$

Here one inserts (1.9) for ϕ and (1.10) for $\boldsymbol{v}$. As especially the term from $\boldsymbol{v}^2$ gets rather tedious, the calculations will here be skipped, but the practice will be explained. Having eliminated $\boldsymbol{v}$ and ϕ in (1.12), one differentiates it w.r.t. x to get rid of $\chi(t)$. Thus, one obtains together with (1.11) the two PDEs that now only depend on f and η (and T, ρ, g, l but they are assumed to be known). Keep in mind to equate everything at the surface, which replaces by definition y with η.

Two PDEs for two unknowns are promising, but there is still one small step to make, namely a first order approximation. Korteweg and de Vries now set $f(x,t) =: q + \beta(x,t)$ and find in (too) low order approximation (orders of $\partial_x^n \partial_t^m \eta^o$, $n, m, o \in \mathbb{N}_0$) of the two PDEs the solutions

$$\dot{\eta} = \dot{\beta} = 0 \qquad \beta(x,t) = -q\frac{\eta(x,t) + \alpha}{l} \qquad \alpha \in \mathbb{R} \qquad q = \sqrt{gl}.$$

Obviously too low order because $\dot{\eta} = 0$ describes no wave one would follow, on horseback, for multiple miles, as Russell did. Yet, it provides insight into a better ansatz, namely

$$f(x,t) =: q\left(1 - \frac{\eta(x,t) + \alpha + \gamma(x,t)}{l} \right)$$

with $\gamma(x,t)$ the new, even smaller perturbation. Note that γ does not introduce a new unknown as it will be substituted for f.

Introducing the modern notation for the partial derivation w.r.t. the variable v of a function F

$$F_v \coloneqq \partial_v F, \tag{1.13}$$

putting the ansatz for f into the two PDEs and eliminating $\gamma(x,t)$ with one of them, renders the second one to read

$$\eta_t - \frac{3q}{2l}\left[\eta_x\eta + \frac{2\alpha}{3}\eta_x + \left(\frac{l^3}{9} - \frac{Tl}{3\rho g}\right)\eta_{xxx}\right] = \mathcal{O}\left(\gamma\eta_x, \gamma_x\eta, \eta_x\eta_{xx}, (\eta_x)^3, \eta_{txx}, \eta\eta_{xxx}\right).$$

Applying the final assumptions and inserting $q = \sqrt{gl}$, one obtains the (to be investigated) *Korteweg - de Vries - equation* (KdV):

$$\eta_t - \frac{3}{2}\sqrt{\frac{g}{l}}\left[\eta_x\eta + \frac{2\alpha}{3}\eta_x + \left(\frac{l^3}{9} - \frac{Tl}{3\rho g}\right)\eta_{xxx}\right] = 0 \tag{1.14}$$

with $\alpha \in \mathbb{R}$ arbitrary.

2 Properties and Solution

In this chapter, the flexibility of constants in KdV will be demonstrated and the most trivial (non-constant) solution will be presented. Afterwards, the behaviour will be investigated and the interesting property of the soliton qualitatively explained.

2.1 Coefficients of KdV

The KdV-equation (1.14) appears in many topics and different fields. Yet, rarely ever in the form of (1.14). If one gives up the physical meaning of the coordinates and η, KdV can come in (nearly) whatever form one wishes. Beginning by setting all coefficients to 1, KdV for an arbitrary function $V(X,T)$ reads

$$V_T - V_X - V_X V + V_{XXX} = 0. \tag{2.1}$$

Introducing now the transformations ($a \in \mathbb{R}\backslash\{0\}, b \in \mathbb{R}, \tilde{c} > 0, d > 0$)

$$\begin{aligned}
&V \to u(V,X,T) = \frac{1}{\sqrt{\tilde{c}}}V + \frac{\sqrt{\tilde{c}} - b}{\tilde{c}} \Longleftrightarrow V = \sqrt{\tilde{c}}\,u - 1 + \frac{b}{\sqrt{\tilde{c}}} \\
&X \to \xi(X) = \sqrt{d}\,\tilde{c}^{-1/4}X \Longrightarrow \partial_X = \sqrt{d}\,\tilde{c}^{-1/4}\partial_\xi \\
&T \to \tau(T) = a\sqrt{d}\,\tilde{c}^{-3/4}\,T \Longrightarrow V_T = \sqrt{d}\,\tilde{c}^{-3/4}V_\tau
\end{aligned} \tag{2.2}$$

transforms (2.1) with the abbreviation $\alpha := \sqrt{d}\,\tilde{c}^{-1/4}$ into

$$\begin{aligned}
a\alpha u_\tau - b\alpha u_\xi - \tilde{c}\alpha u_\xi u + d\alpha u_{\xi\xi\xi} = 0 \Longleftrightarrow \\
a\,u_\tau - b\,u_\xi - \tilde{c}\,u_\xi u + d\,u_{\xi\xi\xi} = 0
\end{aligned} \tag{2.3}$$

where the constants can be chosen freely with the restriction that a, d cannot be 0 because that would render τ, ξ into very bad coordinates and $\tilde{c} \neq 0$ because it appears in many denominators. Also, $\tilde{c}, d$ have to be positive in order to stay within the real numbers. Yet, it can easily be seen that $b = 0$ produces no problems, it is therefore allowed (note that b is proportional to α in (1.14)).

From now on, KdV will be used in the form the most appropriate, where ξ and τ are somehow stretched variables of which the absolutes still grow as spacial x and time t grow. Therefore, for improved possibility of interpretation, the following changes in notation are made:

$$\begin{aligned}
\tau &\longrightarrow t \\
\xi &\longrightarrow x.
\end{aligned} \tag{2.4}$$

Note that t and x do not carry units of time and space. Note also that the interpretation of u as the offset of the surface of a wave from sea-level (namely η) might be lost, depending on c and b, because the transformation (2.2) of $V(X,T)$ contains a constant offset which might destroy the ability to neglect terms of order $\eta\eta_{xxx}$ or $\gamma_x\eta$ which was necessary in (1.14) in order to obtain KdV.

2.2 Solution with wave symmetry

As the original KdV originated from the physics of water waves, searching for a solution with the symmetry $u(x,t) = u(x-ct) =: u(z)$, with $c > 0$ the speed of a wave moving to the right, seems appropriate. Keeping the possibility of modifications of KdV in mind, as discussed in chapter 2.1, the version

$$u_t - \chi u_x - \lambda u_x u + \gamma u_{xxx} = 0 \tag{2.5}$$

with $\lambda, \gamma \neq 0$ will be solved. Plugging in $z \equiv x - ct$ and $\bar{c} := c + \chi$, yields

$$\gamma u'''(z) - \lambda u'(z)u(z) - \bar{c}u'(z) = 0$$

which can easily be integrated if one uses $u'u = (u^2)'/2$. The boundary conditions [4]

$$\lim_{z\to\pm\infty} \Big(u(z), u'(z), u''(z)\Big) = 0 \tag{2.6}$$

are introduced, which set all integration constants to zero.

Integrating and multiplying by u' gives

$$\gamma u''u' - \frac{\lambda}{2}u^2u' - \bar{c}u'u = \frac{\gamma}{2}(u'^2)' - \frac{\lambda}{6}(u^3)' - \frac{\bar{c}}{2}(u^2)' = 0$$

which can be integrated again and the variables separated into

$$\mathrm{d}z = \frac{\mathrm{d}u}{u\sqrt{\frac{\lambda}{3\gamma}u + \frac{\bar{c}}{\gamma}}}$$

which is integrable by substituting $\sqrt{\frac{\lambda}{3\gamma}u + \frac{\bar{c}}{\gamma}}$ and expanding into two partial fractions. The preliminary result reads

$$z - z_0 = \sqrt{\frac{\gamma}{\bar{c}}}\ln\left|\frac{\sqrt{1+\frac{\lambda u}{3\bar{c}}}-1}{\sqrt{1+\frac{\lambda u}{3\bar{c}}}+1}\right| = -2\sqrt{\frac{\gamma}{\bar{c}}}\operatorname{artanh}\left(\sqrt{1+\frac{\lambda u}{3\bar{c}}}\right)$$

where $\ln\left(\frac{1+x}{1-x}\right) = 2\operatorname{artanh}(x)$ for $|x| < 1$ was used. Inserting $u =: \frac{3\bar{c}}{\lambda}\left(\tanh^2(\zeta) - 1\right)$ gives $z - z_0 = -2\sqrt{\frac{\gamma}{\bar{c}}}\zeta$, therefore

$$\begin{aligned} u(z) &= \frac{3\bar{c}}{\lambda}\left[\tanh^2\left(\sqrt{\frac{\bar{c}}{\gamma}}\frac{z_0 - z}{2}\right) - 1\right] \\ &= -3\frac{c+\chi}{\lambda}\cosh^{-2}\left(\sqrt{\frac{c+\chi}{\gamma}}\frac{z-z_0}{2}\right) \qquad z \equiv x - ct \end{aligned} \tag{2.7}$$

where $\cosh(-x) = \cosh(x)$ was used. A beautiful connection between the amplitude and the speed of the wave c can either be seen in the result or in fig. 2.1.

Some interesting connections between this general solution and the original KdV (1.14) can be seen: Comparing coefficients, one obtains

$$\begin{aligned} \lambda &= \frac{3}{2}\sqrt{\frac{g}{l}} \\ \gamma &= \frac{1}{6}\sqrt{\frac{g}{l}}\left(3\frac{Tl}{\rho g} - \frac{l^3}{3}\right). \end{aligned} \tag{2.8}$$

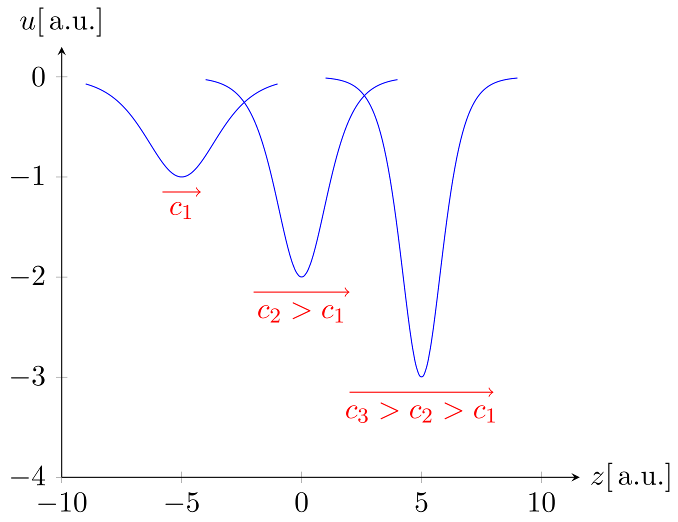

fig. 2.1: Solutions with different speeds and therefore amplitudes. The negative amplitude was chosen to undermine the degree of freedom that originated in the arbitrary variable α in (1.14) and manifests itself in the free parameter χ. The phase shifts were chosen to aid visibility.

Sign of the amplitude: The signs of the coefficients have to fulfil the condition

$$\operatorname{sign}\gamma \stackrel{!}{=} \operatorname{sign}(c+\chi) \tag{2.9}$$

in order to stay within the real numbers. It is concluded that

$$\operatorname{sign} u(z) = -\operatorname{sign}\gamma \cdot \operatorname{sign}\lambda \stackrel{(2.8)}{=} \operatorname{sign}\left(l^2 - 3\frac{T}{\rho g}\right). \tag{2.10}$$

To investigate the sign of the solution further, one has to insert experimental values. Approximating for pure water ($T \cong 70 \times 10^{-3}\,\mathrm{N/m}$, $\rho \cong 10^3\,\mathrm{kg/m^3}$ and $g \cong 10\,\mathrm{m/s^2}$) gives that

$$\operatorname{sign} u(z) \cong \operatorname{sign}(l - 5\,\mathrm{mm})$$

which will very likely be positive for all experiments with waves even in low tanks.

Regarding the speed c, neither restrictions nor relations to λ or γ have come up. It can therefore be concluded that it is arbitrary and solely determined by the initial condition (i.e. the speed of Russel's boat right before it stopped abruptly). This also holds for the integration constant z_0.

2.3 Dispersion vs. non-linear effects

In his story, Russell observed a wave which was stable over miles, meaning that it showed no or very little dispersion, just like solutions of the 3D-wave equation

$$\frac{1}{c^2}\boldsymbol{u}_{tt}(\boldsymbol{r},t) = \Delta\boldsymbol{u}(\boldsymbol{r},t) \tag{2.11}$$

which could emerge from the Maxwell-equations in vacuum ($\boldsymbol{u} \in \{\boldsymbol{E}, \boldsymbol{B}\}$), where the wave keeps its shape due to the fact that an electromagnetic wave experiences no dispersion in vacuum. One very qualitative way to understand this property is that the formula takes same orders of derivatives in space and time, thus, the amount of change in time is equal to the amount of change in space, balancing each other out and therefore producing stable solutions (the factor $1/c^2$ does not break this interpretation, which will be seen later). A more rigorous definition for lack of dispersion will be expressed as

$$\text{no dispersion} :\Longleftrightarrow v^{\text{ph}} := \frac{\omega}{k} = \omega_k =: v^g \tag{2.12}$$

for a solution $\exp\left[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\right]$, $k := |\boldsymbol{k}|$ of (2.11).

The question comes up why KdV shows the same qualities, given that it shows different highest orders of derivatives in space and time. The astonishing answer is that the non-linearity cancels it [4], which shall be sketched in the following.

2.3.1 Dispersion

Neglecting the non-linear term and choosing easy factors, leaves

$$u_t + u_{xxx} = 0 \tag{2.13}$$

to investigate. Using again the argument that searching for wave properties gives rise to an ansatz of the form

$$u(x,t) = u(kx-\omega t) =: u(z) \qquad k,\omega > 0 \tag{2.14}$$

producing the ODE

$$k^3 u'''(z) - \omega u'(z) = 0$$

which can be integrated (with the integration constant set to 0 as in (2.6)) to look just like a harmonic oscillator $u''(z) = \frac{\omega}{k^3}u(z)$ with an element from solution space

$$u(kx-\omega t) = \exp\left(\sqrt{\frac{\omega}{k^3}}(kx-\omega t)\right). \tag{2.15}$$

By definition (2.12) one can see that this (and all other solutions) show dispersion:

$$u_t = -\left(\frac{\omega}{k}\right)^{3/2} u \overset{(2.13)}{=} -u_{xxx} = -\sqrt{\omega k^3}u$$

from which

$$\begin{aligned} \omega(k) = \pm k^3 \Longrightarrow v^{\text{ph}} \equiv \frac{\omega}{k} = \pm k^2 \neq \omega_k = \pm 3k^2 \equiv v^g \\ \Longleftrightarrow v^g = 3\, v^{\text{ph}} \end{aligned} \tag{2.16}$$

follows. It is interesting to investigate the factor from which the group velocity v_g deviates from the phase velocity v_{ph}, which is 3 in the case of (2.13). If one carries out the same calculation for a PDE of the form of a Schrödinger equation $u_t + u_{xx} = 0$, the factor will be 2. In that case, dispersion is slower (weaker), giving away the trend that bigger differences in orders of derivatives in space and time result in faster dispersion, which gives the interpretation right before (2.12) more plausibility. The mentioned persistence of this argument even when the coefficients of $\boldsymbol{u}_{tt}$ and $\Delta\boldsymbol{u}$ differ (as they do in the standard wave equation by $1/c^2$), follows from linearity of the partial derivative in the comparison of ω_k and ω/k.

2.3.2 Non-linearity

In order to understand how the rather fast dispersion which emerges from the u_{xxx} term can be compensated, the PDE

$$u_t = u_x u \tag{2.17}$$

will be investigated. Zabusky and Kruskal proposed

$$u(x,t) = \cos[x + u(x,t)\, t] \tag{2.18}$$

as implicit solution [2]. The validity can easily be checked by differentiating both sides

$$u_x = -\sin(x+ut)(+u_x t) \qquad u_t = -\sin(x+ut)(u_t t + u)$$

$$\iff u_x = -\frac{\sin(x+ut)}{1-t\sin(x+ut)} \qquad u_t = -\frac{u\sin(x+ut)}{1-t\sin(x+ut)} = u_x u.$$

Because of the fact that no explicit solution can be given, one cannot simply calculate v^{ph} or v^{g} and find that these solutions show the opposite trend in dispersion as in subsection 2.3.1. This might have also been the reason that solitons from KdV were first discovered numerically [2]. Therefore, only qualitative interpretation will be given in this part.

Firstly, note that the signs in (2.17) match those in the original KdV (1.14). The speed of one piece of surface at (x,t) is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{u_t}{u_x} = u(x,t) \tag{2.19}$$

which depends on the amplitude. In other words, the speed of different parts of one wave grow with their height. This quality, when no dispersive processes are present, will be observed as *breaking* of the wave. Water waves in sea tend to break only when they approach a coast, where the only difference to open sea is that the water is shallower (which was assumed in the derivation of (1.14), otherwise terms of order $l^3\eta\eta_{xxx}$ would not have been negligible). Therefore, friction between the propagating water and the bottom seems like the most promising interpretation.

To sum up, KdV describes a wave that is effected on the one hand by a relatively fast dispersion which broadens the wave packet and on the other hand by the property that water tends to climb up the wave because lower parts are decelerated and act like a ramp for the water from behind which sharpens the wave (see fig. 2.2), keeping the form of it stable in time.

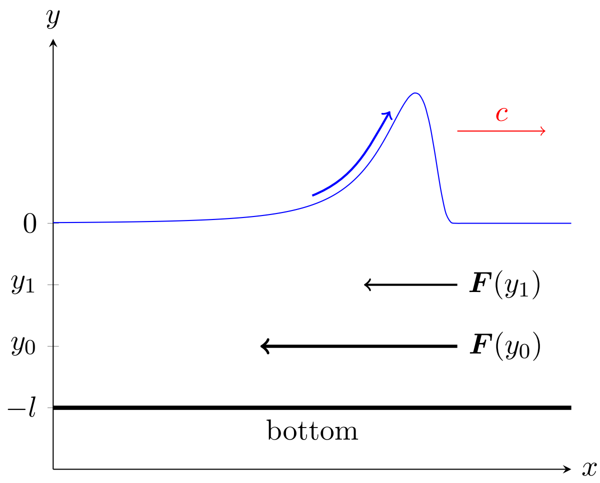

fig. 2.2: Visualization of how the non-linear term manages to sharpen the wave and thereby cancels in good approximation dispersive effects. The frictional force between the moving water and the bottom $\boldsymbol{F}(y)$ decreases with increasing distance to the bottom which enables greater speeds at higher values of spacial y. The thick blue arrow shows the trend of a particle on the surface which will tend to flow upwards as it gets faster the higher it climbs up the ramp of decelerated water. Note that this form of the wave does not solve KdV, it is merely an imagination of how the wave would look with no dispersion present shortly before it breaks.

3 Other Physical Occurrences

It has already been mentioned that KdV arises in systems other than shallow water waves. In fact, if one relaxes the exact form of KdV in favour of the important two qualities of this PDE (i.e. system), namely

1. non-linear term ($\eta_x \eta$ in (1.14))
2. dispersive term (η_{xxx} in (1.14))

one finds different names for slightly different equations, e.g. mKdV (modified-KdV) that introduces a non-linear term of cubic form $\eta_x \eta^2$, or BKdV (Burgers-KdV) that has $\eta_{xx} + \eta_{xxx}$ as dispersive term (the first summand is also called the dissipative term as it occurs in non-loss-free systems). These PDEs are again integrable and show solitonic properties just like KdV and are found in many fields [5], including the following.

3.1 The Great Red Spot (GRS)

An important example was the study of GRS in the Jovian atmosphere which is an enormous, timely persistent, anticyclonic storm, redly coloured (see fig. 3.1). As explained in [5] it is an application of the PDE as the amplitude of the perturbation of the streamfunction obeys mKdV.

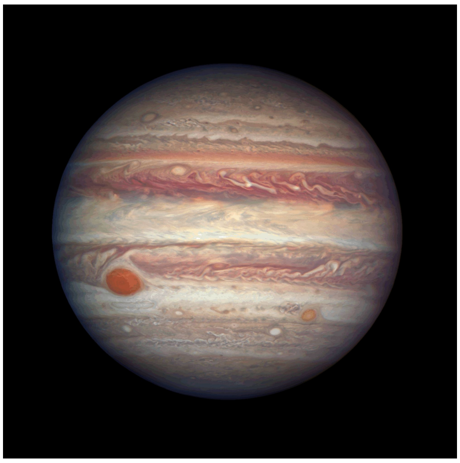

fig. 3.1: GRS on the left.[6]

3.2 Acoustics of liquids with different phases

Here, the opposite direction of arriving at a KdV-like system will be presented. As given in [5], in a system with the two phases *water* and *gas* (i.e. water and bubbles) with a concentration

$$\alpha = \frac{V^w}{V^g} = 0.1$$

of the volumes V^w, V^g for water and gas, respectively, an extremely low-lying speed of sound

$$c = 40\,\mathrm{m/s}$$

has been measured, which is one order lower than the smaller speed of sound of the two pure phases ($c^g = 340\,\mathrm{m/s}$ and $c^w = 1500\,\mathrm{m/s}$). Insight can be found if one inspects the determining quantities of longitudinal speed of sound [7]

$$c := \sqrt{\frac{C}{\rho}} \tag{3.1}$$

with the elastic modulus C of the corresponding phase and ρ its density. The results of monophase speeds c^w, c^g clearly state that there has to exist non-linear interaction i.e. mixing of the phases

$$c \propto \sqrt{\frac{C^g}{\rho^w}} \tag{3.2}$$

in order to arrive at such slow speeds. It was used that the interaction and therefore C^g in gas is relatively weak whereas the water's density ρ^w is large.

The second ingredient, dispersion, has been measured in [8]. At a concentration of $\alpha = 2 \cdot 10^{-4}$ van Wijngaarden found speeds c between $500\,\mathrm{m/s}$ and $2800\,\mathrm{m/s}$ for bubbles with radius $R = (1.2 \pm 0.4) \times 10^{-4}\,\mathrm{m}$. Figure 2 in [8] shows a parabolic form $c(f)$ for a frequency f far below the resonant frequency of the gas.

In this regime, KdV or BKdV (depending on whether the model includes dissipative effects) for the pressure can be expected. Interestingly, this PDE was found experimentally and explained theoretically for example in [9]. The central result is a BKdV for dimensionless difference in pressure p with and without perturbation:

$$p_\tau + pp_\xi - \frac{p_{\xi\xi}}{Re} + \frac{p_{\xi\xi\xi}}{\sigma^2} = 0. \tag{3.3}$$

Here τ and ξ are dimensionless time and space variables ($\tau := tu/\ell$ and $\xi := x/\ell$), $Re = u\ell/\eta$ the Reynolds number of the liquid and $\sigma := \ell\sqrt{u/\beta}$ the dispersion parameter. ℓ and u are two characteristic quantities for length and speed, respectively and $\beta := \frac{R^2 c}{6\alpha(1-\alpha)}$ with c some low-lying constant speed of sound.

3.3 Internal solitons in the ocean

This time, not the envelope but deeper layers in water will be found to behave solitonic and to obey, in certain models, KdV. An absolute necessity for this phenomenon is either continuously or unsteadily *varying* density $\rho(z)$ w.r.t. the height z. Assuming now a very simplistic model where $\rho(z)$ only takes two values

$$\rho(z) = \begin{cases} \rho^a & -h^a \leq z \leq 0 \\ \rho^b & -(h^a + h^b) \leq z < -h^a \end{cases} \tag{3.4}$$

for the two thicknesses of the layers **a**bove and **b**elow the abrupt change in density, $h^a, h^b > 0$, and applying the *rigid lid approximation* where one assumes that the topmost water keeps its height constant, one finds a KdV in the variable of the difference of the height of an isopycnal surface (surface with constant density) from its equilibrium height $\eta(x,t)$

$$\eta_t + a\eta_x + \frac{3a}{2}\frac{h^a - h^b}{h^a h^b}\eta_x\eta + \frac{a}{6}h^a h^b \eta_{xxx} \cong 0 \tag{3.5}$$

with the additional approximation that $\delta\rho := |\rho^a - \rho^b|$ is small [10]. The constant $a > 0$ is proportional to $\delta\rho$, therefore the approximation restricts the system to a regime with weak non-linearity and dispersion. Equation (2.7) shows the correlation between the sign of the amplitude and the coefficient of the non-linear term. The amplitude reads

$$\eta(x,t) = \underbrace{2\frac{c-a}{a}}_{>0 \text{ as } a \ll c} \frac{h^a h^b}{h^a - h^b}\cosh^{-2}(\dots) \tag{3.6}$$

for the speed c. Therefore, internal waves in the lower half evolute as a standard elevating wave, whereas those at the top of the sea where $h^a < h^b$ will show negative amplitudes. Those waves are called *depressions* and look like the plots in fig. 2.1. The dependence on the sign of the coefficient of the linearity can alternatively be understood with (2.19).

3.4 Acoustics in cold plasma

Another famous example are ion-acoustic waves in a plasma of cold ions. This regime was the first reoccurrence of KdV after 1895. As described in [5], in a simplified system (no ion pressure, perfect gas law for the electrons) one arrives at four equations

$$n_t + (nv)_x = 0 \qquad \text{(continuity)}, \tag{3.7}$$

$$v_t + vv_x = E \qquad \text{(conservation of momentum)}, \tag{3.8}$$

$$\rho E + \rho_x = 0 \qquad \text{(force balance for the electrons)}, \tag{3.9}$$

$$E_x = n - \rho \qquad \text{(Maxwell 1)} \tag{3.10}$$

for the number density n and velocity v of the ions (with charge $+e$), the electron density ρ and the electrical field E in non-dimensional quantities. Like (1.2)-(1.8) these equations suffice to reproduce KdV in n.

4 The Hamiltonian Way (Maths)

The following presents a derivation of KdV from an absolutely different point of view. No physical laws will be used. The strategy in order to get more insight into KdV will be to accept the methods used and the assumptions made and try to interpret the result later on.

4.1 Derivation by U. Pinkall [3] part I

Pinkall begins by restricting his workspace to a smooth, periodic curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ which has to be *starshaped*, meaning that

$$\det(\gamma, \gamma_s) \neq 0 \Leftrightarrow \gamma, \gamma_s \text{ not parallel} \tag{4.1}$$

where $\gamma_s = \gamma'(s)$ as in (1.13) and $\det(\gamma, \gamma_s)$ is the determinant of a 2×2 matrix with the two components of the two vectors γ, γ_s as entries (vectors as lines or rows, those are equivalent). See fig. 4.1 for some examples of (non)starshaped curves. Note that this definition for starshaped differs from the usual one where a set $S \subseteq \mathbb{R}^2$ is starshaped if and only if $\exists\, x_0 \in S \quad \forall x \in S$: the straight line from x_0 to x fully lies in S, because in Pinkall's definition x_0 has to be the origin of $\mathbb{R}^2$.

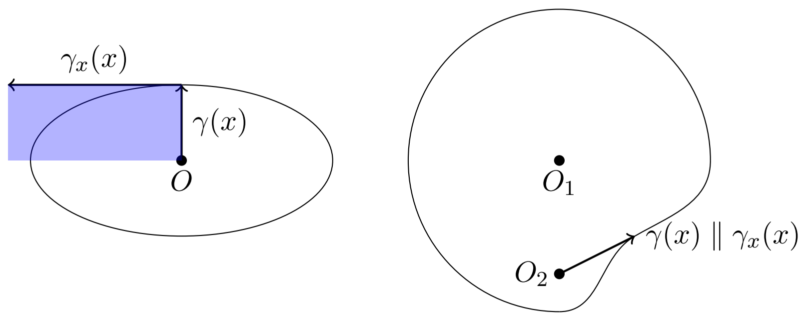

fig. 4.1: Visualization of the *starshaped* property. The ellipse is starshaped for all origins O that lie inside of it. The area of the blue rectangle has the value of $\det(\gamma, \gamma_s)$. Whether the planar curve on the right is starshaped depends on the position of the origin. If it is chosen to be O_1 it fulfils the property but if O_2 is the origin one finds a point on the curve $\gamma(s)$ for which $\exists \alpha \in \mathbb{R} : \gamma(s) = \alpha\gamma_s(s)$. This violates condition (4.1) because the coordinate transformation $s \to x$ (4.2) would be impossible for that point as the determinant vanishes.

This condition provides the possibility to transform s into x such that

$$\det(\gamma, \gamma_x) = 1 \tag{4.2}$$

by $\frac{\mathrm{d}s}{\mathrm{d}x} = \det(\gamma, \gamma_s)^{-1}$ [11]. x is called the *central affine arc length.* Note that it is in general not equal to *arc length*, so $||\gamma_x|| \neq 1$. Equation (4.2) shall frequently be used in the course of this derivation. It is best thought of as a simplification of calculation rather than a deeper connection to the physical water waves. Continuing by differentiating w.r.t. x ($\det(\gamma, \gamma_x)_x = \det(\gamma_x, \gamma_x) + \det(\gamma, \gamma_{xx})$) to obtain

$$\det(\gamma, \gamma_{xx}) = 0 \Leftrightarrow \exists p : \mathbb{R} \to \mathbb{R} \quad \text{such that} \quad \gamma_{xx} = -p\gamma. \tag{4.3}$$

The scalar function p is now called the *central affine curvature of* γ and will be the function that fulfils KdV in the end. Given γ, p can quickly be calculated by

$$\det(\gamma_x, \gamma_{xx}) \equiv \sum_{j,k\in\{1,2\}} \epsilon_{jk}(\gamma_j)_x(\gamma_k)_{xx} \overset{(4.3)}{=} p \sum_{jk} \epsilon_{kj}(\gamma_j)_x\gamma_k \equiv p \det(\gamma, \gamma_x) \overset{(4.2)}{=} p \tag{4.4}$$

with $\epsilon_{jk} = -\epsilon_{kj}$ and $\epsilon_{12} = 1$ the standard Levi-Civita-tensor with two indices. Note the difference to conventional planar curvature κ which is usually defined as

$$\kappa = \frac{\det(\gamma_x, \gamma_{xx})}{||\gamma_x||^3} = \frac{p}{||\gamma_x||^3}. \tag{4.5}$$

Formula (4.4) explains the other name of p: *centroaffine speed of* γ_x. To elaborate this name a bit further, consider that if both γ_x and γ_{xx} are lifted to vectors in $\mathbb{R}^3$ with 0 as z-component then

$$p = ||\gamma_x \times \gamma_{xx}|| \tag{4.6}$$

with the usual vector product. So p measures the portion of γ_{xx} (the speed of γ_x) which is orthogonal to γ_x which, by (4.2) cannot point towards the centre.

This interpretation gives rise to the following definition of the *volume enclosed by* γ_x

$$H(\gamma) = \frac{1}{2} \oint \mathrm{d}x\, p(x) \tag{4.7}$$

where one integrates over a period. This scalar function was named H as it will be the Hamiltonian which will determine a flow that evolutes p by KdV.

Gathering all γ produces a manifold

$$M = \{\gamma : \mathbb{R} \to \mathbb{R}^2 |\ \text{periodic, smooth,}\ \det(\gamma, \gamma_x) = 1\} \tag{4.8}$$

which for two tangent vectors $X, Y \in T_\gamma M$ at γ comes with a 2-form

$$\omega(X, Y) = \oint \det(X, Y) \tag{4.9}$$

which alternates because

$$\omega(Y, X) = -\oint \det(X, Y) = -\omega(X, Y). \tag{4.10}$$

Note that γ and γ_x span $\mathbb{R}^2$ because of condition (4.1), meaning that X can be linearly combined by

$$X = g\gamma - h\gamma_x \tag{4.11}$$

for some two smooth functions $g, h : \mathbb{R} \to \mathbb{R}$.

In order for X to lie in $T_\gamma M$, Pinkall now introduces a smooth variational field $\bar{\gamma}(x,t)$ of $\gamma(x)$ meaning that $\bar{\gamma}(x,0) = \gamma(x)\,\forall x \in \mathbb{R}$. Considering that

$$\bar{\gamma}_t\Big|_{t=0} = X \tag{4.12}$$

and that a tangential vector to the manifold M at γ has to satisfy

$$\det(\bar{\gamma}, \bar{\gamma}_x)_t\Big|_{t=0} = 0 \tag{4.13}$$

one obtains with

$$\bar{\gamma}_t\Big|_{t=0} = g\gamma - h\gamma_x \Rightarrow \bar{\gamma}_{tx}\Big|_{t=0} \overset{(4.3)}{=} (g_x + hp)\gamma + (g - h_x)\gamma_x$$

that (4.13) is equivalent to

$$0 = [\det(\bar{\gamma}_t, \gamma_x) + \det(\gamma, \bar{\gamma}_{xt})]_{t=0} = \det(g\gamma, \gamma_x) + \det\left(\gamma, (g - h_x)\gamma_x\right) \overset{(4.2)}{=} 2g - h_x$$

which produces the nice result that all tangent vectors X at γ have to be of the form

$$X = \frac{h_x}{2}\gamma - h\gamma_x \tag{4.14}$$

for an arbitrary smooth function h. This identifies $T_\gamma M$ as $C^\infty(\mathbb{R}, \mathbb{R})$ [11]. This result will now be used to show that ω is non-degenerate:

Let $Y \in T_\gamma M$ be of the form

$$Y = \frac{f_x}{2}\gamma - f\gamma_x$$

for a smooth $f(x) \in \mathbb{R}$. Then by (4.9) Pinkall finds

$$\begin{aligned} \omega(X,Y) &= \oint \mathrm{d}x\, \det\left(\frac{h_x}{2}\gamma - h\gamma_x, \frac{f_x}{2}\gamma - f\gamma_x\right) \overset{(4.2)}{=} \oint \mathrm{d}x\, \left(-\frac{h_x}{2}f + h\frac{f_x}{2}\right) \\ &= \oint \mathrm{d}x\, \left(-\frac{1}{2}(hf)_x + hf_x\right) = \oint \mathrm{d}x\, hf_x \end{aligned} \tag{4.15}$$

where

$$\oint \mathrm{d}x\, \xi_x = 0 \quad \xi \in C^\infty(\mathbb{R}) \text{ arbitrary} \tag{4.16}$$

was used in the last step.

It therefore follows that for an X for which $\omega(X,Y) = 0\ \forall Y \in T_\gamma M$ that by

$$\oint \mathrm{d}x\, hf_x = 0 \Rightarrow h = 0 \Rightarrow X = 0 \tag{4.17}$$

which is the definition of a 2-form which is non-degenerate. Combining the smooth manifold M with the closed, non-degenerate 2-form ω gives a *symplectic manifold*. The derivation will be continued after a short section which aims to connect the results at this juncture with the desired physical interpretation.

4.2 Short derivation break

A symplectic manifold is a generalisation of the standard physical *phase space* which can most easily be seen by Darboux's theorem that each 2-form can locally be represented by some coordinate pair (q, p) such that

$$\omega = \mathrm{d}q \wedge \mathrm{d}p \tag{4.18}$$

with q and p the canonical coordinates.

Usually, one seeks the evolution of one point of phase space in time. In physics this is determined by Hamilton's equations. In maths, the Hamiltonian should be a function $H : M \to \mathbb{R}, \gamma \mapsto H(\gamma)$ that generates the flow of the system by a unique vectorfield, the *Hamiltonian vectorfield* V, which is generated by the condition

$$\mathrm{d}H(\cdot) = \bar{\omega}(V, \cdot) \tag{4.19}$$

with the *differential* $dH(\cdot) : TM \to C^\infty(\mathbb{R})$ and $\bar{\omega}$ a 2-form that needs to fulfil two important properties:

- The Hamiltonian has to stay constant if one follows the flow of the system: $0 = dH(V) = \bar{\omega}(V, V)$ (conservation of energy) meaning that $\bar{\omega}$ has to **alternate**.
- For each Hamiltonian function H and Hamiltonian vector field V there can only exist one $\bar{\omega}$ (otherwise there could exist two different time evolutions for the same system) which follows from the requirement of **non-degeneracy** of $\bar{\omega}$.

The first condition was proven for ω in (4.10) and the second in (4.17). We can therefore conclude that ω is the 2-form for some Hamiltonian. And the flow it generates will produce the KdV-equation for the central affine curvature p of γ.

Note that the first condition gives that for a time-independent Hamiltionian, V gives the direction of infinitesimal change of a point in phase space such that the total energy stays constant. So, given that a curve γ maps into phase space, one can obtain the evolution of this phase element under this Hamiltonian by

$$V = \gamma_t \tag{4.20}$$

so in ordinary physical phase space $\bar{M}$ with the standard canonical coordinates (q, p), V has two famous entries as components:

$$T\bar{M} \ni V = (q_t, p_t) = (H_p, -H_q). \tag{4.21}$$

4.3 Derivation by U. Pinkall [3] part II

This part will be about the (choosable) Hamiltonian and the Hamiltonian vectorfield V which is then determined and can therefore be calculated (recall that ω was already fixed). Pinkall chooses the volume enclosed by γ_x defined in (4.7) as $H(\gamma)$ as the Hamiltonian function $H : M \to \mathbb{R}$.

Using the definition of V (4.19) one evaluates for an arbitrary $Y \in T_\gamma M$:

$$dH(Y) = YH = H_t \equiv \frac{1}{2} \oint \mathrm{d}x \, p_t \tag{4.22}$$

where the time derivative, as in (4.12), means the derivative of a variational field at $t = 0$. For this variation of Y, meaning that

$$\bar{\gamma}_t\Big|_{t=0} = Y \equiv \frac{f_x\gamma}{2} - f\gamma_x \tag{4.23}$$

(f arbitrary smooth function), one calculates (at t=0)

$$\begin{aligned}\bar{\gamma}_{tx} &\overset{(4.2)}{=} \left(\frac{f_{xx}}{2} + pf\right)\gamma - \frac{f_x}{2}\gamma_x \\ \bar{\gamma}_{txx} &\overset{(4.2)}{=} \left[\left(\frac{f_{xx}}{2} + pf\right)_x + \frac{pf_x}{2}\right]\gamma + pf\gamma_x\end{aligned} \tag{4.24}$$

which (again at $t = 0$) helps to find an expression for

$$p_t \overset{(4.4)}{=} \det(\gamma_{xt}, \gamma_{xx}) + \det(\gamma_x, \gamma_{xxt}) \overset{(4.3),(4.2)}{=} -\frac{f_x}{2}p - \left[\left(\frac{f_{xx}}{2} + pf\right)_x + \frac{pf_x}{2}\right] \tag{4.25}$$

in order to evaluate (4.22) further:

$$\mathrm{d}H(Y) \overset{(4.16)}{=} -\frac{1}{2}\oint \mathrm{d}x f_x p \overset{(4.16)}{=} \frac{1}{2}\oint \mathrm{d}x f p_x \overset{(4.15)}{=} \frac{1}{2}\omega(Y, -2V) = \omega(V, Y) \tag{4.26}$$

with the hereby calculated Hamiltonian vectorfield

$$V = \frac{p}{2}\gamma_x - \frac{p_x}{4}\gamma \tag{4.27}$$

which was obtained by setting $h = -p/2$ in (4.14) (which follows from comparison of the third expression in (4.26) with the last expression in (4.15)). Having found the flow of H, one can, as in (4.20), write down the evolution equation for γ

$$V = \gamma_t = \frac{p}{2}\gamma_x - \frac{p_x}{4}\gamma \tag{4.28}$$

and as γ fixes p uniquely by (4.4), this flow defines a unique evolution of the central affine curvature p of γ by setting $f = -p/2$ in (4.25) (just like above with $h = -p/2$):

$$p_t = \frac{p_{xxx}}{4} + \frac{3}{2}p_x p \tag{4.29}$$

which is a version of KdV.

4.4 Interpretation

Pinkall has shown that a periodic, starshaped curve in the plane, when described by the Hamiltonian of the volume enclosed by its first spacial derivative, evolutes in time such that its central affine curvature obeys KdV.

In other words, if one requires

$$2H \equiv \oint \mathrm{d}x\, p(x,t) = \oint \mathrm{d}x \det\left[\gamma_x(x,t), \gamma_{xx}(x,t)\right] \overset{!}{=} \text{const.}(t) \tag{4.30}$$

then $p(x,t)$ obeys KdV.

In order to physically interpret this result, the two quantities H and p have to be investigated. Those two reduce to just p, as H can easily be understood as the sum of p's of one period. After a coordinate transformation into x, the *central affine arc length*, p was defined by the equation

$$\gamma_{xx} = -p\gamma$$

which can in this coordinate be expressed as

$$p = \det(\gamma_x, \gamma_{xx}) = \kappa||\gamma_x||^3 \propto \kappa \tag{4.31}$$

with κ the standard *curvature* for a planar curve (cf. (4.5)). Note the factor of $||\gamma_x||^3$ which comes from the fact that the chosen parametrization was by *central affine* arc length and not standard arc length where $||\gamma_x||$ would equal 1.

With this proportionality in mind, bricks to a rather wobbly bridge can be laid out: The key result of section 2.3 was the solitonic property of a KdV-solution - it keeps its shape over time. The shape or envelope of the wave is uniquely given by its curvature function. This means that $\kappa(x,t)$ and $||\gamma_x(x,t)||$ have to be constant in time. From the solitonic property therefore follows that

$$\int \mathrm{d}x\, \kappa(x,t) = \mathrm{const.}(t) \tag{4.32}$$

or better:

$$H = \mathrm{const.}(t) \Leftrightarrow dH = 0.$$

Note that interpretations connecting water waves with planar periodic curves that are starshaped have to be dealt with very carefully as the assumptions or starting points of the two worlds are far from each other. For example, the envelope of the wave, described by $\eta(x,t)$, cannot be identified with $\gamma(x,t)$ as η maps into $\mathbb{R}$ and has no periodicity T with $T < \infty$. This is actually not bad news, as η satisfies KdV but not γ! Just the central affine curvature p of γ satisfies it and there does not seem to be an obvious connection between p and η apart from KdV itself.

Nevertheless, some interpretations have been made, which were based solely on the conservation of the Hamiltonian that sums over p of one period. So this starshaped curve has to conserve its sum of central affine curvature just like the water wave conserves its form. Because of the fact that those worlds lie so far apart, it is quite difficult to translate individual properties from one to another. As both worlds arrive at KdV in their own, consistent way, the main advantage of having two different approaches arises in further investigation of KdV. One can choose the world where one does research on KdV freely.

Conclusion

The latter has been a presentation of two fundamentally different techniques of how to derive KdV together with some qualitative properties of the equation. Those stunning qualities play a big role in the success of this PDE. But they are not the sole reasons for modern science to deal with it. Non-linear, integrable systems unite mathematics and physics (as seen), combining rigorousness of mathematics with intuition and using in realistic problems of physics. They find application in many fields, such as ordinary differential equations, algebraic geometry, Lie group theory and differential geometry on the mathematical hand and quantum field theory, string and conformal field theory, quantum gravity and classical general relatively on the physical hand [5]. Concrete physical examples include condensed matter, semiconductor physics, nonlinear optics, laser physics, hydrodynamics, meteorology, plasma physics, protein systems and neurophysiology [5].

Later on, the simplest solution with wave symmetry has been calculated and the connection to original KdV has been made. From there it has been seen quite elegantly in (**??**) that the original KdV can only be valid for shallow water. The physical interpretation of this condition has been made in section 2.3 where dispersion faced non-linear effects (namely fraction with the bottom).

Chapter 3 then presented a small subset of applications of the PDE. One could have continued this presentation for an astonishingly large amount of pages.

Finally, a field of differential geometry, namely the study of symplectic manifolds, which are generalizations of physical phase space, has been made use of to see a non-physical approach to KdV whilst still using a very physical method, namely the flow of points in phase space determined by a Hamiltonian.

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