Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

On Topological Quantum Field Theories

A Conversation Between Physics & Mathematics

Wissenschaftliche Hausarbeit
zur Erlangung des akademischen Grades
Master of Science (M.Sc.)
im Studiengang

Mathematical Physics

der Universität Hamburg

vorgelegt von

Tobias Hochreiter

Hamburg 2023

Erstgutachter: Prof. Dr. Timo Weigand
Zweitgutachter: Dr. Craig Lawrie

Eidesstattliche Versicherung

Hiermit versichere ich Eides statt, dass ich die vorliegende Arbeit im Masterstudiengang "Mathematical Physics" selbständig verfasst und keine anderen als die angegebenen Hilfsmittel - insbesondere keine im Quellenverzeichnis nicht benannten Internet-Quellen - benutzt habe. Alle Stellen, die wörtlich oder sinngemäß aus Veröffentlichungen entnommen wurden, sind als solche kenntlich gemacht. Ich versichere weiterhin, dass ich die Arbeit vorher nicht in einem anderen Prüfungsverfahren eingereicht habe und die eingereichte schriftliche Fassung der auf dem elektronischen Speichermedium entspricht.

Declaration

Through my signature, I declare upon oath that I have written the present thesis independently without usage of any AI-driven chatbots and the like, or further resources and aids other than those stated in the thesis. I have not submitted this thesis to an academic higher education institution that had been already accepted or evaluated as insufficient in an earlier examination procedure. Both forms of the thesis submitted for evaluation (i.e. the thesis in hard copy and the thesis submitted in electronic form) are identical.

The signature is sensitive data,
present only in the original version.

Datum Unterschrift

Abstract

We highlight various physical and mathematical aspects of TQFTs. Starting with the foundation, Chern-Simons theory, and its appearence in form of anomalies, we continue to build up towards intricate mathematical constructions that proved to encapsulate the physical results. However, from the mathematical perspective, various generalizations are possible, and we present one thereof: The state-sum. Finally, we describe physical usage of the constructions such as topological descriptions of RCFT correlators.

Contents

1 Introduction

Translation between two languages is in general not an easy task. Recall that Martin Luther considered this process worthy of his time. I can only conjecture that he not simply longed for some time-killing hobby but saw a fruitful harvest at the end of it - the translation of the Bible from Old Hebrew and Aramaic to German. Translation furnishes a bridge over which information can flow from the side that cannot solve a fixed problem to the side that perhaps can. Luther considered the side of German-speaking people powerful enough to solve the problems he had with Roman Catholicism so he had to transport relevant information, i.e. the Bible, to their side.

History shows that this was not a stupid plan and the present thesis aims to imitate this strategy by transporting problems from the physics of topological quantum field theories[1] to the mathematical side that calls them *functors*. The process of translation of phys. TQFTs to mathematical *category theory* has begun in Atiyah's first definitions [Ati88], has gained considerable thrust by Witten [Wit89], was rigorously constructed by Turaev and Reshetikhin [Tur10], and is to this day carried on by many more.

The aim is more of a to-and-fro movement over the bridge than a mere one-way journey as in Luther's case. I.e., we would like to work on both sides, profiting from their respective advantages, and finally deduce *physical results*. If this work has to decide of its nature, it would choose the physical side.

A phys. QFT (be it topological or not) is rarely squeezed into one compact definition. An ad-hoc definition of what it is and how it looks exactly would force on us resolutions of all problems that the QFT carries, already at page one (starting from infinite vacuum energy). But if all problematic diagrams are known to the person defining a QFT, she probably knows about the values of *all* diagrams. To a physicist it may seem useless to begin the work by demanding

[1] Physical **TQFT**s are defined vaguely as QFTs with almost no more structure than topological data of the space-time manifold; cf. e.g. the introduction of [Wit89].

scattering amplitudes to be known *a priori* (no predictive power) but recall for example the theory of Lie algebras: One assumes similarly that the Lie algebra with its structure is given but aims to find results like classification etc. top-down. In analogy, we demand all of a TQFT's output to be present in defining:

8× **Def. 1** (Atiyah [Ati88])**.** A d-**TQFT** is a symmetric monoidal functor[2]

$$Z : \mathbf{Cob}(d) \to \mathbf{Vect}_{\mathrm{fd}}(\mathbb{C})$$

from the category of d-dimensional cobordisms (defined in appendix A) to the category of finite dimensional $\mathbb{C}$-vectorspaces.[3]

This definition's elegance is a prime source of yet-increasing interest in the field, including my own.

Throughout the whole thesis we try to streamline the presentation by fixing the following notation:

5× **Not. 2.** The following denote oriented smooth d-dimensional manifolds:

- B_d ... possibly has Boundary, $\partial B_d \neq \emptyset$.
- C_d ... has no boundary and is compact, i.e. Closed, $\partial C_d = \emptyset$.

The assignments encoded in Def. 1 can be interpreted as[4]

$$\begin{aligned} Z : C_{d-1} &\longmapsto \text{ a vector space, the phys. Hilbert space} \\ [B_d] &\longmapsto \text{ a linear map, the propagator} \\ C_d &\longmapsto \text{ a number, the partition function} \\ \overline{C_{d-1}} &\longmapsto \text{ space of bra-states isomorphic to } Z(C_{d-1})^* \end{aligned}$$

What about functoriality? It is a requirement on the prescribed propagators which is, for a physicist, as basic as it gets, namely commutativity of

$$\begin{tikzcd} \mathcal{H} \arrow[rr, "Z(B''_d)"] \arrow[dr, "Z(B_d)"'] & & \mathcal{H}'' \\ & \mathcal{H}' \arrow[ur, "Z(B'_d)"'] & \end{tikzcd}$$

for B_d and B'_d composable, i.e. that parts of a long propagation is given by the composition of two propagations that share the start and end, respectively.

In order to not get overly excited, firstly we emphasize that the assigned vector spaces *have to be finite dimensional.*[5] Secondly, we quickly demonstrate

[2]The qualifier *symmetric* fixes the braiding's image in a canonical way. "Monoidal" and "functor" are assumed to be known qualifiers.

[3]Later in the thesis (section 5.3) we will weaken this demand. Recent literature includes various ways to modify this definition.

[4]The first two assignments (not the interpretation) follow because functors map objects to objects and morphisms to morphisms. Line three is an endomorphism of the monoidal unit on both sides and the last line follows from monoidality of Z and uniqueness of duals. Overline denotes opposite orientation.

[5]Again, essentially due to rigidity of $\mathbf{Cob}(d)$ and monoidality of Z.

that simple $(0+1)$-QM with Hamiltonian H and space of states $\mathcal{H}_{\text{phys}}$ is for example not topological under the canonical assignments

$$
\begin{aligned}
* &\longmapsto \mathcal{H}_{\text{phys}} \\
[0,\tau] &\longmapsto \text{ propagator } e^{-iH\tau}
\end{aligned}
$$

because we have used a *length* of the cobordism $* \xrightarrow{[0,\tau]} *$.[6]

So why should a physicist invest in learning these new category theoretic terms? Let me answer this by exemplifying with the example of symmetry transformations: There, the question would be translated into why it is useful to observe the mathematical structure of a Lie group on the set of symmetry transformations (of, say, a Lagrangian). It has proven very fruitful to strip off all additional information of the transformations apart from the well-known structure of a Lie group and profit from the mathematicians' work on this structure. Now, observing that the language of category theory manages to encode big parts of what a phys. TQFT is (scattering amplitudes and physical Hilbert spaces) in its central piece of structure - a functor - strongly motivates that it is worth stripping off the extra clutter of information from a phys. TQFT, as in the Lie group example, and profit from the results of the category theorists' work on such functors and their (co-)domains.

As a prime example, let me quote without dwelling on the details (they can be found in [Lur09]):

1× **Thm. 3** (Cobordism Hypothesis)**.** *For Z a framed fully-extended TQFT, the following assignment has an inverse on the level of isomorphism classes:*

$$Z \longmapsto Z(*).$$

From this we learn that (isomorphism classes of) special types of TQFTs are fully classified by their value on the point![7]

The **main goal** is to get a feel for TQFTs, aimed mainly at physicists. Mathematical abstraction might cloud the advantage of TQFTs as a solid framework to compute in, or to define in. Examples include correlators in CFT and defects from higher-form symmetries. These advantages aside, it can even be *satisfactory* to work in a TQFT setup. Similar to why we find pleasure in telling people that a donut and a cup are *topologically* the same.

Section 2 gives a detailed account on rigorous classical Chern-Simons theory in the principal bundle setup due to [Fre92]. There will be detailed discussions and complete calculations of gauge-dependency and the level of the theory. As is well-known, this is the field theory that first proved to produce 3-manifold invariants as described in [Wit89]. We aim to convey the message that a solid topological foundation is cemented into its definition. For readers unfamiliar with the CS-WZW bulk-boundary correspondence, the account for the WZW part might not be sufficient, but still one gets acquainted with the idea. The

[6] Contrary to the notation in appendix A, here $[0,\tau] \subset \mathbb{R}$ is simply the interval, not some equivalence class.

[7] An, at this stage, accessible demonstration of Thm. 3 can be found in Example 1.1.9 in [Lur09] or in the respective talk [Lur22].

final theorems then explicitly prove that the exponentiated Chern-Simons action obeys the sewing/gluing law. In TQFT parlance, this law is merely functoriality.

Section 3 decouples, at first glance, completely from Section 2: We present anomalies of QED and non-abelian Yang-Mills theory from a physicist's point of view, namely [Ber96]'s. Our remarkable observation, made explicit in (3.8), is that the anomaly's value is the same topological quantity used to define CS-theory in Def. 15.[8] But there is more! We continue to find resemblances of anomalies and TQFTs by viewing anomalies in higher dimensional background hence giving them a much better *raison-d'être*. This is dubbed *anomaly inflow* and was first explained in [CH85], which we summarize.

Section 4 is aimed to gently build up to subsection 5.3. We exemplify a TQFT's character as a representation of topological data. For this we fix $\mathbb{R}^3$ and *ribbon graphs* as the data to be represented. Ribbons, or bands, seem to be a completely arbitrary choice here. But a quick look into [Wit89] tells us that the observables of CS-theory are *framed knots*. And a framed knot is exactly the same as a closed (orientable[9]) ribbon as explained in appendix D. So, hopefully, a physicist willing to learn about [Wit89], is able to appreciate the introduced category theoretic structures, that, after all, only try to snuggle up to the desired framework of framed Wilson lines. We end by presenting this thesis's zenith of categorical structure: Modular Categories. These will appear at various points in subsequent sections. Albeit their frequent appearance, we will not test their axioms to the core.

Section 5 is all about constructing 3-TQFTs: The state-sum and the Reshetikhin-Turaev TQFT. Emphasise was put on the core ideas of the constructions, not on generality or proofs. We quickly talk about the relationship between the two. It is inspiring to see the categorical structures defined in section 4 working together with various topological tricks, in order to put Witten's ideas (for the Reshetikhin-Turaev construction, not the state-sum) on solid feet. The section hopes to bring some of this appreciation across.

Section 6 is a scratch on the surface of the applications of TQFTs. Here we choose to work with the Reshetikhin-Turaev TQFT, not the state-sum, and pick the description of RCFT correlators as cobordisms as application. We give a quick dictionary between CFT-quantities (OPE, vacuum field, etc.) and algebraic structures (Frobenius algebras) to which the TQFT furnishes the bridge.[10] This is by far not an exhaustive list. Other applications include: Higher-form/Non-invertible symmetries [CDI19], fusion of defects [CLSWY19], coupling of TQFTs to QFTs [FMT22], etc. These applications enjoy great scientific interest by the time this line is typed. For a recent paper on an application of the state-sum in the physics of anomaly-inflow, cf. e.g. [TW19].

[8] Note that knowledge about anomalies' topologicality has been well-established at least since APS's index theorem.

[9] We add orientable here, to emphasize that we do not allow for half-integer twists of the ribbon, as in a Möbius band.

[10] A category encodes lots of algebraic structure, and the passing from 2-CFT to 3-TQFT is the bridge over which this dictionary needs to walk.

1.1 A Note On LaTeX

To increase the speed of the flux of information, some equations and theorem-like environments come with a little counter in gray.[11] For example to the left of Def. 1. The counter quickly tells the reader how often this statement is `\ref`'ed (i.e. how often its number appears elsewhere in the thesis). This way, the reader might find it easier to decide on which results to channel available concentration.

1.2 Questions to be Touched or Answered

The reader interested in one of the following questions will find partial or even complete answers in the present work.

What is a mathematical TQFT?

The definition is Def. 1, a complete 3D example is the state-sum partially constructed in section 5.2. 1- and 2-TQFTs are quite easily classified - see around (5.1) for the latter. 4-TQFTs or even higher dimensional examples will not appear in this thesis.

Why does a TQFT produce topological manifold invariants?

In short: Because it is well-defined on a topological equivalence class of manifolds. A little more on this is written up in appendix A.

Is Chern-Simons theory topological?

In a mathematical sense, the structure it encodes is the one defined by Reshetikhin and Turaev, discussed in section 5.3. They do call it a TQFT.
In a physical sense, it is indeed, simply because its Lagrangian and observables are defined without a metric. However, using no metric is not a big challenge, but writing down a *gauge-invariant* term without a metric is not so easy.

Is Chern-Simons theory a gauge theory?

Not on the nose, but modulo 1 it is, which is proven in Prop. 21. This is captured by the "level" of Chern-Simons theory. It is remarkable, as the Lagrangian $\mathrm{tr}(A\,\mathrm{d}A + \frac{2}{3}A^3)$ looks extremely gauge-dependent. Cf. Lem. 16's proof to understand why this numerical factor of $\frac{2}{3}$ is crucial. Cf. Thm. 12 and Def. 15 for the topological reason of these factors.

Does Chern-Simons theory appear in Nature?

Yes, in at least two ways: The first one is via chiral Yang-Mills theories which are part of the Standard Model, namely through neutrinos. This connection is given via their anomalies discussed in section 3.1, more precisely at (3.8).
The second way is through (framed) knots in $\mathbb{R}^3$, which exist e.g. on sailing boats (if the ropes' ends are connected) or in math museums. That this connection exists is the prime result of [Wit89], and not repeated here.

[11] The package to load is hopefully published on https://ctan.org/ by the time this thesis is read and should go under the name `counttherefs`.

What does a 3D TQFT have to do with knots (1D manifolds)?

Nothing immediate if the TQFT is taken in Atiyah's sense of Def. 1. However, Reshetikhin-Turaev-Witten TQFTs discussed in section 5.3 are defined as a generalization of the latter in order to encode extra pieces of structure like framed knots.

Are TQFTs tools or results?

This answer will be rather subjective.
For math, a result; for physics, a tool. The latter is demonstrated briefly in section 6.1. To see this tool-property in action, the short [FFRS04] is a good starting point if one is interested in Kramers-Wannier duality. If instead, one likes higher-form/non-invertible symmetries, e.g. [FFRS07] explains in a rigorous, yet technical, language how to implement them in the setup described in section 5.3 here. As a third option, if one's interests lie in the grand picture of what a coupling of a d-TQFT to a $(d-1)$-QFT might be able to describe, see the general, but mathematically heavy [Fre22] & [FMT22].

Acknowledgements

I thank both my advisors, Timo Weigand and Craig Lawrie, and all my other professors, most notably Christoph Schweigert who never turned down my questions about his work, no matter the hour. My first advisor Timo for his contagious passion for high energy physics that has kept me -together with Paul V.'s influence- from becoming a cold-blooded mathematician. And my *de facto* main advisor Craig for proposing topics and directions so nicely suited to both our expertise. The interest in what I had newly understood, and had developed new passion for, has never ceased on Craig's side despite the rather high-frequency correspondence that we kept up.
On the other side, I want to thank my friends, the "CoCups" (denoted $\cap$), Ullrich H., and most importantly Leocardia, for providing everything needed to keep one going.

2 Cutting in Classical Chern-Simons Theory

From a differential geometric point of view, this section aims to convey topologicality of a non-trivial classical field theory: Classical Chern-Simons (CS) theory in $d = 2 + 1$. The famous Chern-Weil homomorphism is the first shimmer of topological light on CS's Lagrangian. This homomorphism does not care about empty or non-empty boundary of the *compact* base manifold B_3 at all.[12] In order to establish the "cutting property" (i.e. "sewing property" or simply functoriality), we will have to separate our definitions and theorems into $\partial C_3 = \emptyset$ and $\partial B_3 \neq \emptyset$. Both cases need to be treated because a TQFT allows for cutting which produces non-trivial boundaries in both cases. That a TQFT "allows cutting" stems from $\mathbf{Cob}(d)$ having *gluing* of cobordisms (d-dimensional manifolds) along objects ($d-1$-dimensional manifolds) as composition. And

[12] B_3 is an oriented 3-manifold with, possibly, boundary as described in Not. 2. Throughout this whole section it will additionally be compact.

then functoriality of a functor Z out of it gives something like[13]

$$Z\left(B'_d \coprod_{C_{d-1}} B_d\right) = Z(B'_d) \circ Z(B_d)$$

where C_{d-1} is the correct half of either boundaries ($\partial B'_d \cap \partial B_d = C_{d-1}$). In words, a phys. TQFT should obey

$$\boxed{\left.\begin{array}{c}\text{Do not cut and compute}\\ \text{the fully glued invariant}\end{array}\right\} = \left\{\begin{array}{c}\text{Cut, compute the two disjoint invariants,}\\ \text{combine the two invariants}\\ \text{(in a fixed way).}\end{array}\right.}$$

It is this equality that is absolutely crucial for a TQFT and this section aims to rigorously introduce CS theory and exhibit this property. Following first Nakahara [Nak03] and Bertlmann [Ber96], and then Freed [Fre92].

2.1 Preliminaries and Notation

Classical CS theory is a gauge field theory. Classical gauge theories of a group G are well described in the language of connections on principal G-bundles. [Nak03] or [Ber96]'s expositions of fibre bundles and connections on principal bundles including their curvature serves well as a detailed first acquaintance with these terms. Alternatively, the reader can have a look at the short appendix B.

Fix a Lie group G and denote the affine space of connections on a G-bundle $P \xrightarrow{G} B_3$ over a compact B_3 by $\mathcal{A}_P \ni \Theta$. A *gauge transformation* in the physical sense translates to switching the section along which Θ is pulled back down to B_3 for another section. But there is a slightly more elegant (equivalent - proven below) way to talk about gauge transformations, captured by the following definition.

3× **Def. 4.** (P, P' are principal G-bundles over B_3, B'_3, respectively)

- A **bundle map** $\varphi : P' \longrightarrow P$ is a smooth map of manifolds that intertwines the natural right G-actions defined in both bundles ($\varphi(p'.g) = \varphi(p').g$ where $p' \in P, g \in G$). It will then induce its **base map** $\overline{\varphi}$ via commutativity of

$$\begin{tikzcd} P' \arrow[r, "\varphi"] \arrow[d, "\pi'"'] & P \arrow[d, "\pi"] \\ B'_3 \arrow[r, "\overline{\varphi}"'] & B_3. \end{tikzcd}$$

- A **bundle iso** φ between P' and P, denoted $\varphi : P' \xrightarrow{\sim} P$, is a bundle map where $B'_3 = B_3$ and $\overline{\varphi} = id_{B_3}$. (This implies that φ^{-1} exists and is a bundle iso itself.) P' and P are then **isomorphic bundles**, $P' \cong P$. We spell out this last sentence as some subtleties will arise.

[13]Here, "$\coprod_{C_{d-1}}$" designates gluing along C_{d-1} which is composition in $\mathbf{Cob}(d)$ - cf. appendix A - and "$\circ$" simply means composition of linear maps.

- A **gauge trafo** is a bundle iso with $P' = P$, i.e. $\varphi : P \xrightarrow{\sim} P$. Gauge trafos of P carry a natural group structure, denoted $\mathcal{G}_P \ni \varphi$. By freeness and transitivity of the right G-action on P, one can equivalently consider the **induced map**

$$g_\varphi : P \longrightarrow G \quad \text{defined by} \quad \varphi(p) = p.g_\varphi(p) \quad (p \in P).$$

Let us be pedantic and elaborate on the equivalence between this notion of gauge trafo and a gauge transformation in the "section-switching" sense (the physical sense) mentioned above:

3× **Lem. 5.** *Two sections $\sigma, \tau \in \Gamma(P)$ of a principal G-bundle P define a unique gauge trafo $\varphi \in \mathcal{G}_P$. The converse, meaning that $\varphi \in \mathcal{G}_P$ and $\sigma \in \Gamma(P)$ define a unique $\tau \in \Gamma(P)$, holds as well.*

Proof. Given two $\sigma, \tau \in \Gamma(P)$, one starts by defining $\varphi : P \longrightarrow P$ on σ's image by

$$\tau = \varphi \circ \sigma.$$

Then, killing both the "φ needs to intertwine with the G-action"-bird and the "φ needs to be defined on the whole fibre"-bird with one stone, we extend this definition by

$$\varphi(p.g) := \varphi(p).g \quad \text{where} \quad p \in \mathrm{im}(\sigma), g \in G \setminus \{e\}.$$

(Recall transitivity and freeness of the right G-action.) The base map is of course the identity as required, which is implied by τ being a section:

$$\overline{\varphi} : B_3 \longrightarrow B_3, x \longmapsto \pi \circ \tau(x) = x.$$

The converse is immediate by defining $\tau = \varphi \circ \sigma$. □

Next, we aim to give a precise definition of the co-domain of the CS action on the base manifold B_3

$$S_{B_3} : \overline{\mathcal{C}_{B_3}} \longrightarrow \dots.$$

To this end, we will be happy to know:

Lem. 6. *$\varphi^*\Theta \in \mathcal{A}_P$ ($\varphi \in \mathcal{G}_P$, $\Theta \in \mathcal{A}_P$).*

Proof. (Sketch) Right invariance is implied by φ intertwining the G-action. That the pullback $\varphi^*\Theta$ gives the Maurer-Cartan form when itself pulled back along the inclusion of a fibre, is immediate as $\overline{\varphi} = id_{B_3}$, i.e. $\varphi : P \to P$ never mixes fibres. □

This lemma illuminates the following notion.

2× **Def. 7.** Two connections are **isomorphic** if there exists a bundle iso sending one to the other, i.e. for $\Theta \in \mathcal{A}_P, \Theta' \in \mathcal{A}_{P'}$

$$\exists \varphi : P' \xrightarrow{\sim} P \text{ with } \varphi^*(\Theta) = \Theta' \implies \Theta \cong \Theta'.$$

We will now be very explicit in the construction of the co-domain $\overline{\mathcal{C}_{B_3}}$ because the notation is going to be recycled throughout this section.

1× **Def. 8.** • A groupoid[14] category $\mathcal{C}^G_{B_3} =: \mathcal{C}_{B_3}$ is defined by

objects: Each bundle connection pair (P, Θ) with $P \xrightarrow{G} B_3$ and $\Theta \in \mathcal{A}_P$.

morphisms: Exactly all maps φ from Def. 7, i.e. the connection isomorphisms.

- This we use to define the space of gauge fields by

$$\overline{\mathcal{C}_{B_3}} := \pi_0\left(\mathcal{C}_{B_3}\right) := \mathrm{ob}(\mathcal{C}_{B_3})/\text{isomorphisms}.$$

I.e. in here we identify all isomorphic (bundle, connection) pairs.[15] This is the canonical domain of a gauge theory as commented on a little in appendix C.

2.2 Defining the Action S_{C_3} on $\partial C_3 = \emptyset$

S_{C_3} will be defined for any choice of invariant polynomial on $\mathfrak{g} := \mathrm{Lie}(G)$.

1× **Def. 9.** • A G-**invariant polynomial** is an n-linear map $\mathcal{P} : \mathfrak{g}^{\times n} \longrightarrow \mathbb{R}$ that is totally symmetric and (G-Ad-)invariant. The latter meaning

$$\mathcal{P}(\mathrm{Ad}_g\, x, \mathrm{Ad}_g\, y, \dots) = \mathcal{P}(x, y, \dots) \qquad (x, y, \dots \in \mathfrak{g}, g \in G)$$

where $\mathrm{Ad}_g \in \mathfrak{gl}(\mathfrak{g})$ is the pushforward of conjugation by g, $G \ni h \mapsto ghg^{-1}$.

- Inserting a $\mathfrak{g}$-valued differential form in a slot of $\mathcal{P}$ will mean the most natural thing. E.g. for two 1-forms $\omega = \omega_\mu\, \mathrm{d}x^\mu, \tau = \tau_\nu\, \mathrm{d}y^\nu \in \Omega^1(P) \otimes \mathfrak{g}$ valued[16] in $\mathfrak{g}$:

$$\mathcal{P}(\omega, \tau) := \mathcal{P}(\omega_\mu, \tau_\nu)\, \mathrm{d}x^\mu \wedge \mathrm{d}y^\nu$$

The set of G-invariant polynomials is called $I^n(G)$.

For convenience, we will translate our notation to customary physical notation and deduce some seemingly simple results implied by Def. 9.

1× **Not. 10.** • In our space-time dimension $d = 3$, we will be concerned with $\mathcal{P} \in I^2(G)$ by virtue of Thm. 12. Physicists mainly work with $\mathcal{P} = \mathrm{tr}$, the (symmetrized) trace, with a certain normalization. In fact, $\{A \mapsto \mathrm{tr}(A^k)\}_k$ spans $I^n(G)$ as a certain algebra, according to [Nak03] - in this sense, tr is the *only* invariant polynomial, giving physics use of it a touch of generality.

- Due to Cartan's structure equation and Thm. 12 we will need the following dense, yet conventional and reasonable notation: For two 1-forms $\omega = \omega_a T^a, \tau = \tau_b T^b \in \Omega^1(P) \otimes \mathfrak{g}$ valued in the Lie algebra, we have two notions of a product, $[-,-]_\mathfrak{g}$ & $\wedge$, and we set

$$[\omega \wedge \tau] := \omega_a \wedge \tau_b [T^a, T^b]_\mathfrak{g} \in \Omega^2(P) \otimes \mathfrak{g}$$

as their combination.[17] (Mathematical) physicists like [Ber96] or [Nak03] tend to also define the symbol $\omega \wedge \tau$ as

$$\omega \wedge \tau := \frac{1}{2}[\omega \wedge \tau] =: \omega\tau.$$

[14] Groupoid means that all morhpisms are invertible.
[15] "Isomorphic" in the sense of (a.).
[16] A little more on this well-known terminology can be found in footnote 101.
[17] Note how this definition implies $[\omega \wedge \tau] = [\tau \wedge \omega]$ but also $[\omega \wedge [\omega \wedge \omega]] = 0$ which might feel weird at first sight. These relations result from the interplay of $\wedge$-, and $[-,-]$-minuses.

Technically, this is allowed because $\mathfrak{g}$-valued forms do not have a $\wedge$-product defined. Semantically, it makes sense too if the braket is read as a commutator. Physically, the latter notation is popular, like "$\frac{2}{3}A^3$", and by virtue of the form-degree, this simply means matrix multiplication, i.e. luckily works out to have the most natural meaning. We will stick to the rigorous notation in this section.

Exp. 11. • The archetypical example of an invariant polynomial is the *symmetrized trace*, already mentioned above,

$$\mathrm{str}(A_1, \ldots, A_n) := \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{tr}(A_{\sigma(1)} \ldots A_{\sigma(n)})$$

where the "tr" gives invariance and the sum over permutations gives the total symmetry, so $\mathrm{str} \in I^n(G)$.

- For $\mathcal{P} \in I^2(G)$ and $\omega, \tau, o \in \Omega^1(P) \otimes \mathfrak{g}$ we have the following important implications:

$$\mathcal{P}(\omega, \tau) = -\mathcal{P}(\tau, \omega) \tag{2.1}$$

$$\mathcal{P}([\tau \wedge o], \omega) = +\mathcal{P}(\omega, [\tau \wedge o]) = +\mathcal{P}([\omega \wedge \tau], o) \tag{2.2}$$

  where the first two follow from $\mathcal{P}$'s symmetry and form degree, and the last one additionally from invariance.[18]

Chern-Weil Theory

Thm. 12 (Chern-Weil Homomorphism)**.** *There exists a well defined map*[19]

$$\begin{aligned} \chi : I^n(G) &\longrightarrow H^{2n}_{dR}(P) \\ \mathcal{P} &\longmapsto [\mathcal{P}(\Omega, \ldots, \Omega)]_{dR} \end{aligned} \tag{2.3}$$

where $\Omega := D\Theta$ is the curvature of some connection $\Theta \in \mathcal{A}_P$ on P.

The big surprise should be the "some" in the definition of χ. The difference of two choices hence needs to be (and is) an exact form, namely the Chern-Simons form (details later).

Proof. (Sketch; cf. e.g. [Nak03] or [Ber96]) Set $\mathcal{P}_n(\Omega) := \mathcal{P}(\Omega, \ldots, \Omega)$.

Firstly, we need to show that χ lands in its claimed image, i.e. show closedness of $\chi(\mathcal{P})$, i.e. $\mathrm{d}\mathcal{P}_n(\Omega) = 0$. Remembering Bianchi's identity, stating that a curvature's covariant derivative vanishes, $D\Omega = 0$, we get the claim from

$$\mathrm{d}\mathcal{P}_n(\Omega) = \sum_{i=1}^{n} \mathcal{P}(\underset{1}{\Omega}, \ldots, \Omega, \underset{i}{D\Omega}, \Omega, \ldots, \underset{n}{\Omega}). \tag{2.4}$$

That this equation holds true strongly relies on $\mathcal{P}$'s G-invariance. We write out the step in which this property enters the proof of (2.4): Use invariance along

[18] Similar to (2.5) but with a sign due to form degree. See [Ber96, §7.1] for details and generality.

[19] Both source and target carry natural algebra structures (cf. [Nak03]) of which χ can be shown to be a homomorphism of, illuminating the name **Chern-Weil homomorphism**.

the form-valued exponential curve $t \mapsto g_t$ in $G \otimes \Omega^1(\mathcal{P})$ with tangent vector Θ ($D\Theta \equiv \Omega$) at $t = 0$ by differentiating the equation of invariance,

$$\mathcal{P}_n(\mathrm{Ad}_{g_t}\,\Omega) = \mathcal{P}_n(\Omega),$$

at $t = 0$, to get

$$\sum_{i=1}^{n} \mathcal{P}_n(\underset{1}{\Omega}, \ldots, \Omega, \underset{i}{[\Omega \wedge \Theta]}, \Omega, \ldots, \underset{n}{\Omega}) = 0. \tag{2.5}$$

This 0 is to be added to d's action in $\mathrm{d}\mathcal{P}_n(\Omega)$ on the differential forms Ω, namely

$$\mathrm{d}\mathcal{P}_n(\Omega) = \sum_{i=1}^{n} \mathcal{P}(\underset{1}{\Omega}, \ldots, \Omega, \underset{i}{\mathrm{d}\Omega}, \Omega, \ldots, \underset{n}{\Omega})$$

where no signs appear as Ω has even form-degree. Now recall $D = \mathrm{d} + \frac{1}{2}[- \wedge \Theta]$.

Secondly, it has to be shown that insertion of two unrelated curvatures $D_1\Theta_1 = \Omega_1, D_0\Theta_0 = \Omega_0 \in D\mathcal{A}_P$ into $\mathcal{P}_n$ differs at most by an *exact* form. (A covariant derivative D with index means it is taken w.r.t. the connection with this index.)

The geometrical idea is to find a path $\Omega_{(-)} : t \mapsto \Omega_t$ in the space of curvatures on P, connecting Ω_0 and Ω_1, s.th. every infinitesimal step along $\mathcal{P}_n \circ \Omega_{(-)}$ is exact. This means

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}_n(\Omega_t) = \mathrm{d}\big[\ldots\big] \qquad (t \in [0,1]) \tag{2.6}$$

and the pleasant coincidence is that choosing the path $t \mapsto D_t\Theta_t \equiv \Omega_t$ induced by the simplest possible path

$$t \mapsto \Theta_t := \Theta_0 + t(\Theta_1 - \Theta_0) \in \mathcal{A}_P$$

from Θ_0 to Θ_1 is successful. After some steps, where $\mathcal{P}_n$'s invariance and Ω_t's Bianchi identity cruicially enter again, we get the important result

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}_n(\Omega_t) = \mathrm{d}\big[n\mathcal{P}(\Theta_1 - \Theta_0, \Omega_t, \ldots, \Omega_t)\big] \qquad (t \in [0,1]). \tag{2.7}$$

Integrating from $t = 0$ to 1 (and switching d and $\int$) gives the claim. $\square$

Assumption 13. *The principal G-bundle P is henceforth assumed trivial.*[20]

Rem. 14. Note that Thm. 12 can be pulled back along $\sigma \in \Gamma(P)$ as well,[21] i.e. it remains true if one replaces $\Omega \mapsto \sigma^*\Omega =: F$, $\Theta \mapsto \sigma^*\Theta =: A$ and $P \mapsto C_3$ (or even B_3). This is crucial as ultimately, fields will live on the base C_3, not P.

As promised, we will now give the name *Chern-Simons form* to the exact piece (2.7) from the last proof. As the action S_{C_3} (and hence the CS-form) should only take *one* field $\Theta := \Theta_1$ at a time, as opposed to (2.7), Θ_0 is set 0 in defining the form. This is commented on below.

[20] As quoted in [Fre92], if G is connected and simply-connected, every $P \xrightarrow{G} C_3$ will be trivial, $P \cong G \times C_3$. We demand triviality because we now want global sections, $\Gamma(P) \neq \emptyset$.

[21] Because its proof goes through on the base in complete analogy to the presented version.

2× **Def. 15** (Original Definition [CS74]). The **Chern-Simons form** in arbitrary dimensions d is defined for any $\mathcal{P} \in I^n(G)$ and $2n-1 \leq d$ on $\Theta \in \mathcal{A}_P$ as[22]

$$Q_{2n-1}(\Theta) := n \int_0^1 \mathcal{P}(\Theta, \underbrace{\Omega_t, \dots, \Omega_t}_{n-1 \text{ times}}) \, \mathrm{d}t \in \Omega^{2n-1}(P)$$

where $\Omega_t := D_t\Theta_t = t\,\mathrm{d}\Theta + t^2 \Theta \wedge \Theta$ and $\Theta_t := t\Theta$, reducing in $d = 2+1$ and for $n = 2$ to the form of our main interest,

$$\boxed{\alpha(\Theta) := Q_3(\Theta) = 2\int_0^1 \mathcal{P}(\Theta, \Omega_t) \, \mathrm{d}t = \mathcal{P}(\Theta, \Omega) - \frac{1}{6}\mathcal{P}(\Theta, [\Theta \wedge \Theta]).}$$

In verifying the last equality, a more familiar form of $\alpha(\Theta)$ will flash up:

$$\begin{aligned} \alpha(\Theta) &= 2\int_0^1 \mathcal{P}(\Theta, t\,\mathrm{d}\Theta + t^2 \Theta \wedge \Theta) \\ &= \mathcal{P}(\Theta, \mathrm{d}\Theta) + \frac{2}{3}\mathcal{P}(\Theta, \Theta \wedge \Theta) \qquad \text{... familiar form} \\ &= \mathcal{P}(\Theta, \Omega) - \frac{1}{6}\mathcal{P}(\Theta, [\Theta \wedge \Theta]) \end{aligned} \tag{2.8}$$ ×2

where the last step uses Cartan's structure equation $\Omega = D\Theta = \mathrm{d}\Theta + \frac{1}{2}[\Theta \wedge \Theta]$.

Regarding interpretation, one might have observed that setting $\Theta_0 = 0 \notin \mathcal{A}_P$ will spoil the geometric idea behind (2.7) of α. However, a trivial bundle admits a global section $\sigma \in \Gamma(P)$ along which α will be pulled back in all applications. Now note that 0 is a valid global gauge potential[23] in a trivial bundle, also called the *pure gauge* potential. Hence, interpretation is restored for $\sigma^*\alpha(\Theta) = \alpha(A)$.[24] The restored interpretation is that any *(Yang-Mills) field strength* $F := \sigma^*\Omega = D\sigma^*\Theta =: DA$ under any $\mathcal{P}_2$ lies in the same de-Rham cohomology class as the pure gauge field strength 0 under $\mathcal{P}_2$,

$$[\mathcal{P}_2(DA)]_{\mathrm{dR}} = [\mathcal{P}_2(0)]_{\mathrm{dR}}$$

where the exact form they differ by is defined to be $\mathrm{d}\sigma^*\alpha(\Theta) = \mathrm{d}\alpha(A)$.

Gauge Dependency

Interestingly enough, the CS form is not gauge invariant.

4× **Lem. 16.** *Let $\varphi \in \mathcal{G}_P$ be a gauge transformation with associated map (cf. Def. 4) $g_\varphi : P \to G$ and $\Theta \in \mathcal{A}_P$. Then the gauge dependency reads*

$$\varphi^*\big(\alpha(\Theta)\big) - \alpha(\Theta) = \mathrm{d}\,\mathcal{P}\left(\mathrm{Ad}_{g_\varphi^{-1}} \Theta, g_\varphi^*\vartheta\right) \underbrace{-\frac{1}{6} g_\varphi^* \mathcal{P}\left(\vartheta, [\vartheta \wedge \vartheta]\right)}_{g_\varphi^*(\textbf{\textit{WZW-term}})}$$

where ϑ is the Maurer-Cartan form of G, a.k.a. "pure gauge", a.k.a. $g_\varphi^{-1}\,\mathrm{d}g_\varphi$.

[22] Recall Not. 10.

[23] "Gauge potential" = "pulled back connection"

[24] This equality holds by definition of insertion of differential forms into invariant polynomials and $\sigma^* D_\Theta \Theta = D_A \sigma^*\Theta$. The latter, in turn, holds by $D = d + \frac{1}{2}[-, \Theta]_{\mathfrak{g}}$ and $\sigma^*\,\mathrm{d}\Theta = \mathrm{d}\sigma^*\Theta$.

Proof. We perform this calculation because it shows that the numerical factors in (2.8) cannot be chosen arbitrarily in order to produce a topological -i.e. *gauge* in this case- theory. Also - for the reader familiar with the calculation in coordinates, maybe from physics - this derivation is comparatively short.
Abbreviate $g_\varphi =: g$ and $\mathrm{Ad}_{g_\varphi^{-1}} =: \mathrm{Ad}$, and list well-known facts that we need:

1. $\mathrm{d}\mathcal{P}(\omega,\tau) = \mathcal{P}(\mathrm{d}\omega,\tau) - \mathcal{P}(\omega,\mathrm{d}\tau)$ for $\omega,\tau \in \Omega^1(P)\otimes\mathfrak{g}$, by definition of forms in $\mathcal{P}$ and ω's form degree. Cf. [Ber96, §7.1].

2. Under the gauge trafo φ one can check in [Fre92], or books on Yang-Mills theories, that

$$\Theta \xrightarrow{\varphi} \varphi^*\Theta = \mathrm{Ad}\,\Theta + g^*\vartheta = g^{-1}\Theta g + g^{-1}\,\mathrm{d}g$$
$$\Omega = D\Theta \xrightarrow{\varphi} \varphi^*\Omega = \mathrm{Ad}\,\Omega = g^{-1}\Omega g$$

   where the right-most is physics notation applicable for matrix Lie groups.

3. The *Maurer-Cartan equation*: $0 = \mathrm{d}\vartheta + \frac{1}{2}[\vartheta\wedge\vartheta]$.

4. $\mathrm{Ad}\,\mathrm{d}\Theta - \mathrm{d}(\mathrm{Ad}\,\Theta) = [\mathrm{Ad}\,\Theta\wedge g^*\vartheta]$ which follows for matrix groups, i.e. for physics notation, simply by the product rule. In the more general setup, one gets the equality by equating $\varphi^* D\Theta \overset{2.}{=} \mathrm{Ad}\,D\Theta$.[25]

The rest is a matter of book-keeping together with $\mathcal{P}$'s flexibility (2.2), giving the two "$3\cdot$", in

$$\begin{aligned}
\varphi^*\alpha(\Theta) &= \mathcal{P}(\varphi^*\Theta,\varphi^*\Omega) - \frac{1}{6}\mathcal{P}(\varphi^*\Theta,[\varphi^*\Theta\wedge\varphi^*\Theta])\\
&\overset{2.}{=} \mathcal{P}(\mathrm{Ad}\,\Theta + g^*\vartheta, \mathrm{Ad}\,D\Theta) - \frac{1}{6}\mathcal{P}(\mathrm{Ad}\,\Theta + g^*\vartheta,[\mathrm{Ad}\,\Theta + g^*\vartheta\wedge\mathrm{Ad}\,\Theta + g^*\vartheta])\\
&= \alpha(\Theta) + g^*(\textbf{WZW-term}) +\\
+\mathcal{P}(g^*\vartheta,\mathrm{Ad}\,D\Theta) &- \frac{1}{6}\Big(3\cdot\mathcal{P}(g^*\vartheta,[\mathrm{Ad}\,\Theta\wedge\mathrm{Ad}\,\Theta]) + 3\cdot\mathcal{P}\left(\mathrm{Ad}\,\Theta,[g^*\vartheta\wedge g^*\vartheta]\right)\Big)
\end{aligned}$$

where we made frequent use of $\mathcal{P}$'s Ad-invariance to get the $\alpha(\Theta)$ term. That we can pull in φ^* in the first step is by definition of forms in $\mathcal{P}$ and is the mathematical version of "a gauge trafo of a Lagrangian is given by the prescribed gauge trafos of its constituents".
The very last line is now exactly equal to

$$\begin{aligned}
\mathrm{d}\,\mathcal{P}\left(\mathrm{Ad}\,\Theta, g^*\vartheta\right) &\overset{1.}{=} \mathcal{P}(\mathrm{d}(\mathrm{Ad}\,\Theta), g^*\vartheta) - \mathcal{P}(\mathrm{Ad}\,\Theta,\mathrm{d}g^*\vartheta)\\
&\overset{4,3}{=} \mathcal{P}(\mathrm{Ad}\,\mathrm{d}\Theta - [\mathrm{Ad}\,\Theta\wedge g^*\vartheta], g^*\vartheta) + \frac{1}{2}\mathcal{P}(\mathrm{Ad}\,\Theta,[g^*\vartheta\wedge g^*\vartheta])\\
&= \mathcal{P}(g^*\vartheta,\mathrm{Ad}\,\mathrm{d}\Theta) - \frac{1}{2}\mathcal{P}(\mathrm{Ad}\,\Theta,[g^*\vartheta\wedge g^*\vartheta])
\end{aligned}$$

proving the Lemma. Equality with above is quickly seen, as $D\Theta = \mathrm{d}\Theta + \frac{1}{2}[\Theta\wedge\Theta]$ and $\mathrm{Ad}[\Theta\wedge\Theta] = [\mathrm{Ad}\,\Theta\wedge\mathrm{Ad}\,\Theta]$ because Ad is a Lie algebra map. ☐

[25] $\varphi^*D\Theta = \mathrm{d}\varphi^*\Theta + \frac{1}{2}[\varphi^*\Theta\wedge\varphi^*\Theta] \overset{2,3}{=} \mathrm{d}(\mathrm{Ad}\,\Theta) + \frac{1}{2}([\mathrm{Ad}\,\Theta\wedge\mathrm{Ad}\,\Theta] + [\mathrm{Ad}\,\Theta\wedge g^*\vartheta] + [g^*\vartheta\wedge\mathrm{Ad}\,\Theta])$ and now recall footnote 17, use that Ad is a Lie algebra map and equate as advised.

1× **Def. 17.** The **gauge-dependent action** is defined for $\sigma \in \Gamma(P), \Theta \in \mathcal{A}_P$ as

$$S_{C_3}(\sigma, \Theta) := \int_{C_3} \sigma^* \alpha(\Theta) \in \mathbb{R}.$$

The gauge dependency, under $\varphi \in \mathcal{G}_P$, as implied by Lem. 16, reads

$$\begin{aligned} S_{C_3}(\varphi\sigma, \Theta) - S_{C_3}(\sigma, \Theta) &= \int_{C_3} g^*(\textbf{WZW-term}) \\ &\equiv -\frac{1}{6} \int_{C_3} g^* \mathcal{P}(\vartheta, [\vartheta \wedge \vartheta]) \qquad (2.9)\times 3 \\ &\sim \int_{C_3} \operatorname{tr}\left(g^{-1}(\mathrm{d}g) g^{-1}(\mathrm{d}g) g^{-1}(\mathrm{d}g)\right) \end{aligned}$$

where $g := g_\varphi \circ \sigma \colon C_3 \to G$ and the last line is how it appears in physics with $\mathcal{P} \sim \operatorname{tr}$. Note how $\partial C_3 = \emptyset$ crucially simplifies (2.9). So only (**WZW-term**) is left standing between us and gauge invariance.

1× **Lem. 18.** *The form* (***WZW-term***) $\equiv -\frac{1}{6}\mathcal{P}(\vartheta, [\vartheta \wedge \vartheta]) \in \Omega^3(G)$ *is closed.*

Proof. Ignoring the (from this point onwards) irrelevant factor of $-1/6$, we find

$$\begin{aligned} \mathrm{d}\mathcal{P}(\vartheta, [\vartheta \wedge \vartheta]) &= \mathcal{P}(\mathrm{d}\vartheta, [\vartheta \wedge \vartheta]) - \mathcal{P}(\vartheta, \overbrace{\mathrm{d}[\vartheta \wedge \vartheta]}^{=0}) \\ &= -\frac{1}{2}\mathcal{P}([\vartheta \wedge \vartheta], [\vartheta \wedge \vartheta]) \\ &= +\frac{1}{2}\mathcal{P}([\vartheta \wedge [\vartheta \wedge \vartheta]], \vartheta) = 0 \end{aligned}$$

where we used some results stated in Lem. 16's proof together with what was mentioned in footnote 17. □

By Poincaré's Lemma, this implies that *locally*, $S_{C_3}(\sigma, \Theta)$ is indepedent of σ. It is the *global* topology of G and the bundle P that may have a chance to spoil gauge invariance.

Rise of the "Level"

We will now argue how a normalization of $\mathcal{P}$ can restrict gauge dependency of S_{C_3} to a physically irrelevant extent:[26]

Denote the de Rham cohomology class of (**WZW-term**) as $[\mathcal{P}] \in H^3_{\mathrm{dR}}(G)$ and consider its image under the de Rham isomorphism of abelian groups:

$$\begin{aligned} \mathrm{dR}^{-1} \colon H^3_{\mathrm{dR}}(G) &\xrightarrow{\sim} \operatorname{Hom}(H_3(G), \mathbb{R}) = H^3(G; \mathbb{R}) \\ [\mathcal{P}] &\longmapsto \left([s_G] \mapsto \int_{s_G(\Delta^3)} (\textbf{WZW-term})\right) \qquad (2.10)\times 1 \end{aligned}$$

[26] Because we need to use de Rham's theorem, language of cohomology has to be used. The reader unfamiliar with the concepts may wish to skip this part.

where $s_G : \Delta^3 \xrightarrow{\text{cont.}} G$ is a 3-cycle[27] and the equality follows from the Universial Coefficients Theorem together with $\mathbb{R}$ being more than a group, namely a field. To formulate the assumption of integrality on $\mathcal{P}$, observe that the canonical inclusion $\mathrm{Hom}(G;\mathbb{Z}) \hookrightarrow \mathrm{Hom}(G;\mathbb{R})$ induces an arrow $\mathrm{inc}\colon H^3(G;\mathbb{Z}) \to H^3(G;\mathbb{R})$.

3× **Assumption 19.** *We assume $\mathcal{P}$ to be re-scaled such that $\exists [P]_{\mathbb{Z}} \in H^3(G;\mathbb{Z})$:* [28]

$$\begin{aligned} \mathrm{dR} \circ \mathrm{inc}\colon H^3(G;\mathbb{Z}) &\longrightarrow H^3_{dR}(G) \\ [\mathcal{P}]_{\mathbb{Z}} &\longmapsto [\mathcal{P}] \end{aligned}$$

1× **Assumption 20.** *Assume G connected and compact.*

6× **Prop. 21.** *Under Assumptions 19 & 20, we have for any $\sigma \in \Gamma(P)$ and $\varphi \in \mathcal{G}_P$*

$$S_{C_3}(\varphi\sigma, \Theta) - S_{C_3}(\sigma, \Theta) \in \mathbb{Z}.$$

This integer depends on the choice of $[\mathcal{P}]_{\mathbb{Z}}$ but also on φ.

Proof. By virtue of (2.9), we need to show ($g \equiv g_\varphi \circ \sigma$)

$$\int_{C_3} g^*(\textbf{WZW-term}) \in \mathbb{Z}.$$

To this end we pick[29] a triangulation $T := \{s_{C_3}\colon \Delta^3 \xrightarrow{\text{cont.}} C_3 | \delta s_{C_3} = 0\}$ of our closed and compact C_3 into finitely many 3-cycles, i.e.

$$\bigcup_{s_{C_3} \in T} s_{C_3}(\Delta^3) = C_3 \quad \& \quad \coprod_{s_{C_3} \in T} s_{C_3}\left(\Delta^3 \setminus \partial\Delta^3\right) \subset C_3,$$

which lets us write the integral as a finite sum,

$$\int_{C_3} g^*(\textbf{WZW-term}) = \sum_{s_{C_3} \in T} \int_{s_{C_3}(\Delta^3)} g^*(\textbf{WZW-term}).$$

To prove integrality of the last integral, we first quote the implication[30]

$$\int_{g(s(\Delta^3))} (\textbf{WZW-term}) \in \mathbb{Z} \implies \int_{s(\Delta^3)} g^*(\textbf{WZW-term}) \in \mathbb{Z},$$

found in [(ht] based on [BT95], and now show the above LHS: Any continuous map like $g : C_3 \to G$ induces a map $g^\bullet : H^3(G;\mathbb{G}) \to H^3(C_3;\mathbb{G})$, for any abelian group $\mathbb{G}$, by post-composition. So we have the commutative diagram

$$\begin{tikzcd} & H^3(G;\mathbb{R}) \arrow[r, "g^\bullet_\varphi"] & H^3(C_3;\mathbb{R}) \arrow[dr, "\mathrm{dR}"] & \\ H^3_{\mathrm{dR}}(G) \arrow[ur, "\mathrm{dR}^{-1}"] & H^3(G;\mathbb{Z}) \arrow[u, "\mathrm{inc}"] \arrow[r, "g^\bullet_\varphi"] & H^3(C_3;\mathbb{Z}) \arrow[u, "\mathrm{inc}"'] & H^3_{\mathrm{dR}}(C_3) \\ {[\mathcal{P}]} \arrow[r, mapsto] & {[\mathcal{P}]_{\mathbb{Z}}} & & \end{tikzcd}$$

[27] So it is a 3-simplex with vanishing signed boundary operator $0 = \delta s_G \equiv \sum_{i \in \text{faces}} (-)^i \partial_i s_G$ familiar from algebraic topology.

[28] "Re-scaling": $\lambda\mathcal{P}$ for $\lambda \in \mathbb{R}^\times$, and its meaning is best seen in Prop. 21's statement.

[29] Triangulations exist for smooth manifolds of any dimension according to [Man16].

[30] More precisely $\int_{s_{C_3}(\Delta^3)} g^*(\textbf{WZW-term}) = \deg(g) \int_{g(s_{C_3}(\Delta^3))} (\textbf{WZW-term})$, and $\deg(g) \in \mathbb{Z}$. Here we need G to be compact and connected.

where $\mathrm{inc}\,(g^\bullet[\mathcal{P}]_\mathbb{Z}) \in H^3(C_3;\mathbb{R}) = \mathrm{Hom}(H_3(C_3),\mathbb{R})$ is the map

$$[s_{C_3}] \mapsto (\mathrm{dR}(\mathcal{P}))\,(g\circ s_{C_3}) = \int_{g\circ s_{C_3}(\Delta^3)} (\textbf{WZW-term})$$

which is integral because it comes from $H^3(C_3;\mathbb{Z})$, finishing the proof. $\square$

This proof shows that closed forms like (**WZW-term**) are "topological", in the sense of lying discrete.[31] The remarkable insight is that the presumably uncountable set of possible gauge transformations $\mathcal{G}_P$ can only produce discrete changes of $S_{C_3}(\sigma,\Theta)$, i.e. gets organized in a countable set of gauge differences, by virtue of Lemma 18. Note that this is crucial for all that follows.

Def. 22. Define the action of Chern-Simons theory on C_3 with $\partial C_3 = \emptyset$ as

$$\begin{aligned} S_{C_3} : \mathcal{C}_{C_3} &\longrightarrow \mathbb{R}/\mathbb{Z} \\ [(P,\Theta)] &\longmapsto S_{C_3}(\sigma,\Theta) \quad \text{for some } \sigma\in\Gamma(P). \end{aligned}$$

It is a well defined functional, and hence the action of a gauge theory, because the contributions of gauge transformations (i.e. different choices of σ) are divided out in the co-domain. This slight modification of the co-domain poses no problems in physical applications as these are concerned with

$$\begin{aligned} \exp(2\pi i S_{C_3}) : \overline{\mathcal{C}_{C_3}} &\longrightarrow U(1)\subset\mathbb{C} \\ [(P,\Theta)] &\longmapsto e^{2\pi i S_{C_3}(\Theta)} \end{aligned} \tag{2.11}$$

as the weight in path integrals.

2.3 Defining the Action S_{B_3} on $\partial B_3 \neq \emptyset$

After last section's intense warm-up, we now build towards our aim: The gluing law, Thm. 30. As mentioned earlier, CS-theory can only have a chance in defining a TQFT if after cutting B_3 there is notion of S_{B_3} on the two halves. But those necessarily have boundary even if $\partial B_3 = \emptyset$.

Last subsection's construction gloriously fails for this case as a gauge transformation now truly messes up our action. To see this, we first write out the gauge dependence as implied by Lem. 16 for convenience: Under a gauge transformation $\varphi\in\mathcal{G}_P$ from $\sigma\in\Gamma(P)$ ($g\equiv g_\varphi\circ\sigma\colon B_3\to G$), the gauge-dependent action changes as

$$S_{B_3}(\varphi\sigma,\Theta) - S_{B_3}(\sigma,\Theta) = \mathrm{WZW}_{\partial B_3}(g) + \int_{B_3} \mathrm{d}\,\sigma^*\mathcal{P}\left(\mathrm{Ad}_{g_\varphi^{-1}}\Theta, g_\varphi^*\vartheta\right) \tag{2.12}$$

where the **Wess-Zumino-Witten action** was introduced,[32]

$$\begin{aligned} \mathrm{WZW}_{\partial B_3}(g) &:= \int_{B_3} g^*(\textbf{WZW-term}) \\ &\equiv \int_{B_3} -\frac{1}{6}\mathcal{P}(\phi_g,[\phi_g\wedge\phi_g]) \\ &\overset{(2.9)}{\sim} \int_{B_3} \mathrm{tr}(g^{-1}(\mathrm{d}g)g^{-1}(\mathrm{d}g)g^{-1}(\mathrm{d}g)). \end{aligned} \tag{2.13}$$

[31] Please appreciate that discreteness was given even before Assumption 19 - this only scaled the discreteness to integers, i.e. has set the topological obstruction to $1\in\mathbb{Z}$.

[32] Notation may seem off here. Instead it is quite reasonable, according to Lem. 23.

We are now facing two obstructing terms, i.e. one more than in the closed case:

1. $\int_{B_3} \mathrm{d}\,\sigma^* \mathcal{P}(\mathrm{Ad}\dots) = \int_{\partial B_3} \sigma^* \mathcal{P}(\mathrm{Ad}\dots) \neq 0$
2. $\mathrm{WZW}_{\partial B_3}(g)$ cannot be rescued modulo 1 as in Prop. 21. Of course, (**WZW-term**) is still a closed form, but Prop. 21's proof fails because de Rham's iso (2.10) integrates over *cycles*. The problem is that a triangulation T of B_3 would contain at least one simplex with non-zero boundary -if $\partial B_3 \neq \emptyset$- and then discreteness is lost. Put differently, we loose the certainty that Assumption 19 will set $\mathrm{WZW}_{\partial B_3}(g) \in \mathbb{Z}$ for all $g \in \mathcal{G}_P$.

The first helpful observation is that point 1. is only half as bad, because the gauge dependence of this term reduces manifestly to a mere boundary-value dependence of the integrated functions.
There is reason to expect an analogous reduction to a mere boundary dependence for point 2. as Prop. 21 fails only because of non-empty boundary (at least mod 1):

4× **Lem. 23.** $\mathrm{WZW}_{\partial B_3}(g) \mod 1$ *only depends on* $g|_{\partial B_3} : \partial B_3 \to G$.

Proof. (Sketch) Assume $B_3, {B_3}'$ satisfy $\partial B_3 \cong \partial {B_3}'$ via an orientation reversing isomorphism ι, and any two functions $g : B_3 \to G, g' : {B_3}' \to G$ with $g|_{\partial B_3} \circ \iota = g'|_{\partial {B_3}'}$ on them. Then, gluing them like $C_3 := B_3 \coprod_\iota {B_3}'$ produces a 3-*cycle* $\tilde{g} : C_3 \to G$ and Prop. 21 gives

$$\mathrm{WZW}_{\partial B_3}(g) - \mathrm{WZW}_{\partial B_3}(g') = \mathrm{WZW}_{\partial C_3 = \emptyset}(\tilde{g}) \in \mathbb{Z}.$$

□

1× **Cor. 24.** *Modulo 1, the WZW-action (2.13) extends to any closed 2-manifold* C_2*, denoted* $\mathrm{WZW}_{C_2} : g \mapsto \mathrm{WZW}_{C_2}(g) \in \mathbb{R}$ *for* $g : C_2 \to G$.

Proof. Every such closed (i.e. compact and boundary-free) C_2 bounds some B_3 simply by "filling".[33] So we can, by Lem. 23 and modulo 1, well-define

$$\mathrm{WZW}_{C_2}(g) := \mathrm{WZW}_{\partial B_3}(g^{\mathrm{ext}})$$

with $g^{\mathrm{ext}} : B_3 \to G$ any extension of g to $B_3 \supset C_2$. □

To summarize, both problematic terms 1. and 2. only depend on the 2-dimensional boundary ∂B_3 of B_3 and values of section, gauge trafo and connection on it.

Definition of the Chern-Simons Action

In the upcoming definition[34] of the *Chern-Simons Line Bundle* -by virtue of the above simplifications- sections, gauge trafos and connections will be restricted to boundary, written like $\sigma|_\partial$ for $\sigma \in \Gamma(P)$, $\varphi|_\partial$ for $\varphi \in \mathcal{G}_P$, $\Theta|_\partial$ for $\Theta \in \mathcal{A}_P$.[35]
The fibre of the above mentioned CS-bundle will be the complex vector space of invariant sections of a non-complicated functor between the two categories

[33] E.g. the 2-sphere $C_2 = S^2$ bounds the 3-ball, the torus bounds the solid torus, etc.

[34] This possibly overwhelming definition directly follows [Fre92] and uses category theoretic language to get a grip on the quoted gauge dependency. Parts of it may be skipped.

[35] Please note how, contrary to [Fre92], we will never consider boundary bundles and sections over them in the following. This is not necessary - and otherwise questions about extensions pop up - as finally we only aim to define S_{B_3} on $\overline{\mathcal{C}_{B_3}}$ which consists of connections on bundles over the *3*-manifold B_3.

1. For a G-bundle $P \xrightarrow{\pi} B_3$ and $C_2 \subset \partial B_3$ a closed 2-submanifold, the category $\Gamma(C_2, P)$ with

objects: For any $\sigma \in \Gamma(P)$, the restriction $\sigma|_{C_2} : C_2 \to P$ is an object.

morphisms: For any $\varphi \in \mathcal{G}_P$ with $\tau = \varphi \circ \sigma$, the restriction $\varphi|_{\pi^{-1}(C_2)}$ is a morphism $\sigma|_{C_2} \xrightarrow{\varphi|_{\pi^{-1}(C_2)}} \tau|_{C_2}$.
(By Lem. 5 each such Hom-set has exactly 1 element.)

   Note that an assignment $\sigma|_{\partial B_3} \mapsto \sigma$ were not well defined. Lem. 23 will rescue arguments that would need it. Similarly for the morphisms.

2. the category $\mathcal{L}$ with

objects: Only one object, namely $\mathbb{C}$ equipped with the standard metric.

morphisms: The one Hom-set are the unitary isomorphisms, i.e. $\mathrm{U}(1)$.

composition: Complex multiplication.

We now define the aforementioned functor to encode our problematic gauge terms (2.12):[36]

Def. 25. For $\Theta \in \mathcal{A}_P$, $C_2 \subset \partial B_3$, let $\mathcal{F}_{\Theta|_{C_2}} : \Gamma(C_2, P) \to \mathcal{L}$ be the functor

$$\mathcal{F}_{\Theta|_{C_2}} : \varphi|_{\pi^{-1}(C_2)} \mapsto \exp 2\pi i \left(\mathrm{WZW}_{C_2}(g_\partial) + \int_{C_2} \mathcal{P}(\mathrm{Ad}_{g_\partial^{-1}} \sigma^* \Theta, g_\partial^* \vartheta) \right)$$

where $\sigma|_{C_2}$ is the source object of $\varphi|_{\pi^{-1}(C_2)}$, and $g_\partial := g_{\varphi|_\partial} \circ \sigma|_{C_2} : C_2 \longrightarrow G$.

We skip the proof of functoriality (some comments can be found in [Fre92]).

1× **Def. 26.** (Claims in here are proven below.)

- An **invariant section** s of a functor $\mathcal{F} : \mathcal{C} \to \mathcal{L}$ from any connected[37] groupoid category $\mathcal{C}$ into $\mathcal{L}$ is a choice of values

$$s = \{ s(C) \in \mathcal{F}(C) = \mathbb{C} \mid C \in \mathrm{ob}(\mathcal{C}) \},$$

  also called a **section of** $\mathcal{F}$, that is invariant under pushing forward any $\psi : C \xrightarrow{\sim} D$ in $\mathcal{C}$ along $\mathcal{F}$, i.e.

$$\mathcal{F}(\psi)(s(C)) = s(D) \quad \forall \psi \in \mathcal{C}(C, D). \tag{2.14}$$ ×5

  The inner product vector space of $\mathcal{F}$'s invariant sections is denoted $\mathrm{S}_{\mathcal{F}}$.

- For $\Theta \in \mathcal{A}_P$, $C_2 \subset \partial B_3$ a closed 2-submanifold, set

$$L_{\Theta|_{C_2}} := S_{\mathcal{F}_{\Theta|_{C_2}}} \quad \ldots \quad \textbf{Chern-Simons line}.$$

  Set $L_{\Theta|_\emptyset} := \mathbb{C}$ for consistency with last subsection.

[36] Note how there is no choice about what it does on objects, and that we need Cor. 24.

[37] *Connected* means that any two objects have a morphism between them (in the groupoid case).

Proof. 1. $S_{\mathcal{F}}$ carries a complex vectorspace structure inherited by the one on $\mathbb{C}$ because $\mathcal{F}(\psi) \in \mathbb{C}$ renders (2.14) a $\mathbb{C}$-linear condition.

2. $S_{\mathcal{F}}$ additionally carries a natural metric inherited by the standard inner product $\langle -, - \rangle_{\mathbb{C}}$ on $\mathbb{C}$, namely for $s, t \in S_{\mathcal{F}}$

$$\langle s, t \rangle := \langle s(C), t(C) \rangle \quad \text{for any } C \in \mathrm{ob}(\mathcal{C})$$

which is well defined by the unitarity requirement of $\mathcal{F}(\psi)$ in (2.14) w.r.t. $\langle -, - \rangle_{\mathbb{C}}$.

3. $\dim_{\mathbb{C}}(S_{\mathcal{F}}) \in \{0, 1\}$ owing to connectedness of $\mathcal{C}$. The first case is true iff there exists some $\psi \in \mathcal{C}(C, C)$ with $\mathcal{F}(\psi) \neq 1 \in \mathrm{U}(1)$. Because then (2.14) forces $s(C) = 0$ for every $s \in S_{\mathcal{F}}$ and connectedness transports this to every $C \in \mathrm{ob}(\mathcal{C})$. In such a case, the functor $\mathcal{F}$ is said to **have holonomy**.

   That the non-holonomous case gives precisely $\dim_{\mathbb{C}}(S_{\mathcal{F}}) = 1$ is seen by choosing for one $C \in \mathrm{ob}(\mathcal{C})$ any value $s(C) \in \mathbb{C}$ and defining $s \in S_{\mathcal{F}}$ on the rest of $\mathrm{ob}(\mathcal{C})$ by (2.14). $\mathcal{F}$'s functoriality crucially enters.

4. $\Gamma(C_2, P)$ is a connected (Lem. 5) groupoid, so $S_{\mathcal{F}_{\Theta|_{C_2}}} \equiv L_{\Theta|_{C_2}}$ is senseful.

5. $L_{\Theta|_{C_2}}$ is an actual *line*, i.e. 1D, by step 3 and $\mathcal{F}_{\Theta|_{C_2}}$ having no holonomy. (Which is true simply because there is no non-identity endo-gauge trafo to possibly give $\mathcal{F}_{\Theta|_{C_2}}$ holonomy.)

$\square$

To compensate for the high abstraction, let me paint parts of the bigger picture already here: Problems 1. and 2. from far above, the two summands that $S_{B_3}(\sigma, \Theta)$ picks up after a gauge transformation, make it essentially impossible to associate a well defined complex number $\exp(2\pi i S_{B_3}(\Theta))$ to just a connection Θ and no σ. This number depends on σ, no matter how one puts it. The villain here is definitely ∂B_3.
The physical origin of the demand that $\exp(2\pi i S_{B_3}(\Theta)) \in \mathbb{C}$ stems from Euler-Lagrange equations needing a notion of differentiation, i.e. taking the "difference" in S_{B_3}'s co-domain. But why not relax $\mathbb{C}$ to any metric space where a difference is similarly well defined? This is the way that has been paved by the last lemma. Instead of $\mathbb{C}$, the exponentiated action will land in the metrized lines $L_{\Theta|_{C_2}} \cong \mathbb{C}$. Notions of differentiation can be defined in there, so Euler-Lagrange equations have a chance to stay meaningful. The next lemma encapsulates some claims just made and defines which (manifestly gauge-invariant) element of the CS-line $L_{\Theta|_{C_2}}$ will be meant by $e^{2\pi i S_{B_3}(\Theta)}$.

Lem. 27.
- *For any $\Theta \in \mathcal{A}_P$ we have $L_{\Theta|_{C_2}} \cong \mathbb{C}$ non-canonically via a choice of $\sigma \in \Gamma(P)$ like $s \xmapsto{\sim} s(\sigma|_\partial) \in \mathbb{C}$.*
- *Any smooth manifold given by a family* $\mathrm{Fam} := \{\Theta|_{C_2}(u) \mid \Theta(u) \in \mathcal{A}_P\}_u$ *furnishes a smooth hermitian complex line bundle*

$$L_{P_\partial} := \{L_{\Theta|_{C_2}(u)}\}_u \xrightarrow{\pi} \mathrm{Fam} \subset \mathcal{A}_P$$

locally trivialized by any section $\sigma \in \Gamma(P)$.

- *Now only $C_2 = \partial B_3$. For any $\Theta \in \mathcal{A}_P$, the following is in $L_{\Theta|_{\partial B_3}}$:*

$$e^{2\pi i S_{B_3}(\Theta)} := \left\{ s(\sigma|_{\partial B_3}) := \exp(2\pi i \underbrace{S_{B_3}(\sigma,\Theta)}_{cf.\ Def.\ 17}) \right\}_{\sigma|_{\partial B_3} \in \mathrm{ob}(\Gamma(\partial B_3, P))} \tag{2.15}$$

  It is the (exponentiated) ***Chern-Simons action*** *of $(P,\Theta) \in \mathcal{C}_{B_3}$ and has unit norm.*

For $B_3 = \emptyset$, set $e^{2\pi i S_\emptyset(\Theta)} := 0$.

Proof. 1. Smoothness of the bundle L_{P_∂} means that all local trivializations ϕ_σ, induced by any $\sigma \in \Gamma(P)$, are diffeomorphisms. They are given by

$$\begin{aligned} \phi_\sigma : L_{P_\partial} &\xrightarrow{\sim} \mathrm{Fam} \times \mathbb{C} \\ L_{\Theta|_{C_2}(u)} \ni s_u &\longmapsto (\Theta|_{C_2}(u), s_u(\sigma|_{C_2})), \end{aligned}$$

invertible by (2.14), smooth in the first slot by assumption, smooth in the second slot as $u \mapsto c_{C_2}(\sigma^*\Theta(u), g_\partial)$ is manifestly smooth, and the inverse is smooth by definition of L_{P_∂}'s smooth manifold structure.

2. That the definition of the symbol $e^{2\pi i S_{B_3}(\Theta)}$ is indeed an *invariant* section was the whole motivation to define $\mathcal{F}_{\Theta|_{C_2}}$ on morphisms the way we did. Invariance means precisely that performing a gauge trafo on $\exp(2\pi i S_{B_3}(\sigma,\Theta))$ is the same as multiplying by the image of that gauge trafo under $\mathcal{F}_{\Theta|_{C_2}}$. That $s(\sigma|_{\partial B_3})$ is well defined in the given way is provided by $C_2 = \partial B_3$ and Lem. 23. Unit norm is obvious.

□

Note how a total space P was fixed in the bundle L_{P_∂}. Smoothness holds true (without proof, see [Fre92]) for varying P, giving the final CS-action on all of the topological space $\overline{\mathcal{C}_{B_3}}$. For well-definedness, [Fre92] proves functoriality of

$$\mathcal{C}_{B_3} \ni (P,\Theta) \longmapsto e^{2\pi i S_{B_3}(\Theta)}.$$

Gluing Law

To finally state the main result of cutting and gluing, we need to explain first the natural behaviour of the CS-line $L_{\Theta|_{\partial B_3}}$ under reversing orientation like $L_{\Theta|_{\overline{\partial B_3}}}$, and under disjoint unions like $L_{\Theta|_{C_2 \coprod C_2'}}$. In the latter case, B_3's connectedness is arbitrary with $\partial B_3 = C_2 \coprod C_2'$.

5× **Prop. 28.** *(Notation as above.)*

- *Orientation: There is a (natural) isometry and identity*

$$L_{\Theta|_{\overline{\partial B_3}}} \cong \overline{L_{\Theta|_{\partial B_3}}} \quad \& \quad e^{2\pi i S_{\overline{B_3}}(\Theta)} = \overline{e^{2\pi i S_{B_3}(\Theta)}}.$$

- *Disjoint union: There is a (natural) isometry and identity*

$$\mathrm{tear} : L_{\Theta|_{C_2 \coprod C_2'}} \xrightarrow{\sim} L_{\Theta|_{C_2}} \otimes L_{\Theta|_{C_2'}}$$
$$\&$$
$$e^{2\pi i S_{B_3 \coprod B_3'}(\Theta \coprod \Theta')} = e^{2\pi i S_{B_3}(\Theta)} \otimes e^{2\pi i S_{B_3'}(\Theta')}.$$

Proof. (No new ideas, but good for familiarization.) As the case $B_3 = \emptyset$ is trivial, we take $|\pi_0(B_3)| \geq 1$. Assume $\partial B_3 = C_2 \coprod C_2'$ and restrictions $\Theta|_{C_2}, \Theta|_{C_2'}$ of a connection over them. They can also be seen as just one of each, via disjoint juxtaposition. So the three symbols

$$L_{\Theta|_{C_2 \coprod C_2'}}, L_{\Theta|_{C_2}}, L_{\Theta|_{C_2'}}$$

have definite meaning.[38] The canonical isometry is now given by a canonical decomposition of a section $\sigma \cup \sigma' \in \Gamma(P \cup P')$ into $\sigma \in \Gamma(P)$ and $\sigma' \in \Gamma(P')$ as

$$\begin{aligned} \text{tear} : L_{\Theta|_{C_2 \coprod C_2'}} &\xrightarrow{\sim} L_{\Theta|_{C_2}} \otimes L_{\Theta|_{C_2'}} \\ \{s((\sigma \cup \sigma')|_\partial)\}_{\sigma\cup\sigma'} &\longmapsto \{t(\sigma|_\partial)\}_\sigma \otimes \{t'(\sigma'|_\partial)\}_{\sigma'} = \{t(\sigma|_\partial) \otimes t'(\sigma'|_\partial)\}_{\sigma,\sigma'} \end{aligned}$$

s.th. $s((\sigma \cup \sigma')|_\partial) = t(\sigma|_\partial)t'(\sigma'|_\partial) \in \mathbb{C}$.[39] It gives an *invariant* section as every gauge trafo on $P \cup P'$ can equally be decomposed into gauge trafos on P and P'. Further, the claimed identity reads

$$\begin{aligned} e^{2\pi i S_{B_3 \coprod B_3'}(\Theta \coprod \Theta')} &= \left\{\exp\left(2\pi i S_{B_3 \coprod B_3'}\left(\sigma \coprod \sigma', \Theta \coprod \Theta'\right)\right)\right\}_{\sigma \coprod \sigma'} \\ &= \{\exp(2\pi i S_{B_3}(\sigma, \Theta)) \otimes \exp(2\pi i S_{B_3'}(\sigma', \Theta'))\}_{\sigma,\sigma'} \\ &= e^{2\pi i S_{B_3}(\Theta)} \otimes e^{2\pi i S_{B_3'}(\Theta')} \end{aligned}$$

where $e^{\int_{\partial B_3 \coprod \partial B_3'}} = e^{\int_{\partial B_3}} \cdot e^{\int_{\partial B_3'}}$ was used. The orientation calculation is similar, using $e^{\int_{\overline{\partial B_3}}} = e^{-\int_{\partial B_3}}$ instead. $\square$

Rem. 29. The case where B_3 is connected but $\partial B_3 = C_2 \coprod C_2'$ is only covered in the above isometry, not the disjoint identity, as we wrote "$B_3 \coprod B_3{}'$". Problem is our restrictive notation, which we will stick to. But for cutting, we will need to tear apart the invariant $e^{2\pi i S_{B_3}(\Theta)}$ by sending it along the isometry:

$$\text{tear}\left(e^{2\pi i S_{B_3}(\Theta)}\right) \in L_{\Theta|_{C_2}} \otimes L_{\Theta|_{C_2'}}. \tag{2.16}$$

Now for the desired results of cutting and gluing. Let $C_2 \hookrightarrow B_3$ be a closed oriented co-dimension 1 submanifold along which we cut. The upcoming result will relate CS-actions on B_3 with the ones on the cut manifold B_3^{cut} which has $\partial B_3^{\text{cut}} = C_2 \coprod \overline{C_2} \coprod \partial B_3$. The bundle over B_3^{cut} is taken as the pullback bundle $P^{\text{cut}} := \text{glue}^* P$ along $\text{glue} : B_3^{\text{cut}} \to B_3$.[40] Pullback bundle means it sits in the limitting diagram

$$\begin{tikzcd} \text{glue}^* P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B_3^{\text{cut}} \arrow[d, "\text{glue}"] \\ P \arrow[r, "\pi"'] & B_3. \end{tikzcd} \tag{2.17}$$

[38]Notation should now be distinguished for the two cases $|\pi_0(B_3)| = 1$ and $|\pi_0(B_3)| > 1$. For brevity we treat both cases simultaneously by writing $\cup$. Then if $|\pi_0(B_3)| = 1$ there can only be one bundle, so $P \cup P'$ means P (i.e. $P' = P$). Similarly for sections.

[39]For $|\pi_0(B_3)| = 1$ where $\sigma \cup \sigma' = \sigma$, the assignment looks like

$$\{s(\sigma|_{C_2 \coprod C_2'})\}_\sigma \longmapsto \{t(\sigma|_{C_2})\}_{\sigma \in \text{ob}(\Gamma(C_2,P))} \otimes \{t'(\sigma|_{C_2'})\}_{\sigma \in \text{ob}(\Gamma(C_2',P))}$$

s.th. $s(\sigma|_{C_2 \coprod C_2'}) = t(\sigma|_{C_2})t'(\sigma|_{C_2'})$.

[40]The map glue simply is inverse to the given cutting map: $B_3 \to B_3^{\text{cut}}$.

Thm. 30 (cutting). *(Notation as above.)*
Cut and un-cut CS-actions are related via the ***contraction map***[41]

$$\begin{array}{rcl} \mathrm{Ctr}_{C_2} : L_{\Theta^{cut}|_{\partial B_3^{cut}}} \xrightarrow[\sim]{P28} L_{\Theta|_{C_2}} \otimes \overline{L_{\Theta|_{C_2}}} \otimes L_{\Theta|_{\partial B_3}} & \longrightarrow & L_{\Theta|_{\partial B_3}} \\ s \otimes \overline{s'} \otimes t & \longmapsto & \underbrace{\langle s, s' \rangle}_{\text{Def. } 26}\, t \end{array}$$

for any $\Theta \in \mathcal{A}_P$ *like*

$$e^{2\pi i S_{B_3}(\Theta)} = \mathrm{Ctr}_{C_2}\left(e^{2\pi i S_{B_3^{cut}}(\Theta^{cut})}\right) \tag{2.18}$$

where $\Theta^{cut} \in \mathcal{A}_{P^{cut}}$ *is canonically induced by* Θ*.*

This manifests our motivational equality of

$$\boxed{\left.\begin{array}{c}\text{Compute the uncut}\\ \text{CS-action on } B_3\end{array}\right\} = \left\{\begin{array}{c}\text{First cut } B_3 \text{ into two pieces } B_3^{\text{cut}},\\ \text{compute the two pieces separately,}\\ \text{then combine them via } \mathrm{Ctr}\,.\end{array}\right.}$$

Proof. Set $B_3^{\text{cut}} = \widetilde{B_3} \coprod B_3'$, and simplify to $\partial\widetilde{B_3} = C_2$ and $\partial B_3' = \overline{C_2} \coprod \partial B_3$. Writing $\widetilde{\Theta} := \Theta|_{\widetilde{B_3}}$ and $\Theta' := \Theta|_{B_3'}$, Ctr_{C_2} consists of the assignments

$$\begin{aligned} e^{2\pi i S_{B_3\text{cut}}(\Theta^{\text{cut}})} &\xmapsto{\text{tear}} e^{2\pi i S_{\widetilde{B_3}}(\widetilde{\Theta})} \otimes e^{2\pi i S_{B_3'}(\Theta')} \\ &\xmapsto{\text{id} \otimes \text{tear}} e^{2\pi i S_{\widetilde{B_3}}(\widetilde{\Theta})} \otimes \text{tear}\left(e^{2\pi i S_{B_3'}(\Theta')}\right) \\ &\equiv \{e^{2\pi i S_{\widetilde{B_3}}(\widetilde{\sigma},\widetilde{\Theta})}\}_{\widetilde{\sigma}} \otimes \text{tear}\left(\{e^{2\pi i S_{B_3'}(\sigma',\Theta')}\}_{\sigma'}\right) \\ &= \{e^{2\pi i S_{\widetilde{B_3}}(\widetilde{\sigma},\widetilde{\Theta})}\}_{\widetilde{\sigma}} \otimes \underbrace{\{t(\sigma'|_{\overline{C_2}})\}_{\sigma'|_{\overline{C_2}}} \otimes \{t'(\sigma'|_{\partial B_3})\}_{\sigma'|_{\partial B_3}}}_{\text{s.th. } e^{2\pi i S_{B_3'}(\sigma',\Theta')} = t(\sigma'|_{\overline{C_2}})t'(\sigma'|_{\partial B_3})} \end{aligned}$$

together with a pairing of the first two tensorands above via the hermitian metric on $L_{\Theta|_{C_2}}$. To this end, we choose[42]

$$t(\sigma'|_{\overline{C_2}}) = e^{-2\pi i S_{\widetilde{B_3}}(\widetilde{\sigma},\widetilde{\Theta})} \quad \text{whenever} \quad \sigma'|_{C_2} = \widetilde{\sigma}|_{C_2}$$

which defines the whole invariant section - because $\sigma|_{C_2} = \sigma|_{\overline{C_2}}$ for $\sigma \in \Gamma(P^{\text{cut}})$ - giving

$$t'(\sigma'|_{\partial B_3}) = e^{2\pi i S_{B_3'}(\sigma',\Theta')} e^{2\pi i S_{\widetilde{B_3}}(\widetilde{\sigma},\widetilde{\Theta})} = e^{2\pi i S_{B_3}(\sigma,\Theta)}$$

owing to $\int_{B_3\text{cut}} = \int_{B_3}$. By construction, the pairing mentioned before gives unity and we find that (2.18)'s r.h.s. reads

$$\{t'(\sigma'|_{\partial B_3})\}_{\sigma' \in \mathrm{ob}(\Gamma(\partial B_3, P))} \overset{\text{Def}}{=} e^{2\pi i S_{B_3}(\Theta)}.$$

□

[41] The first arrow consists of two tearings from Prop. 28 and then the orientation isometry from Prop. 28.

[42] Note the minus sign, by which we take care of the orientation isometry from Prop. 28.

Because of the cumbersome notation, we emphasize an important point: Fixing $\Theta \in \mathcal{A}_P$, it is impossible to define the CS-action only for disjoint *parts* of B_3's boundary. This has not been done in the last proof. We only teared it apart, but always carried along *both* parts of it in a tensor product. The sesquilinear form $\langle -, - \rangle$ descends to this tensor product, so we could apply it.

For gluing, we need of course impose equality of connections along the (oppositely oriented) diffeomorphic cut-boundaries. But it is not clear that equality of connections implies existence of a glued connection that is also smooth. Because a TQFT is mainly about cutting, i.e. assuming $\Theta \in \mathcal{A}_P$ that trivially induces $\Theta^{\text{cut}} \in \mathcal{A}_{P^{\text{cut}}}$ (as in Thm. 30), we will not prove the following gluing result. Call $g : \text{glue}^* P \to P$ the leftmost map in (2.17) that every pullback bundle comes with.

Prop. 31 (gluing)**.** *(Notation as in Thm. 30, its proof, and above.)*
If $\Theta^{cut}|_{C_2 \coprod \overline{C_2}} \in g^*(\mathcal{A}_{\pi^{-1}(C_2)})$ *then*

$$\exists \Theta \in \mathcal{A}_P : g^*(\Theta) \cong \Theta^{cut}.$$

3 Anomalies of QFTs

Quantisation of classical field theories - the procedure rendering them QFTs - is full of subtleties and difficulties where intuition often fails to grip. One such example is the intricate occurence of so-called **(quantum) anomalies**. Coloquially, one calls a (quantum) field theory anomalous whenever a classical symmetry's quantisation either is impossible or breaks the symmetry. This distinguishes two types of anomalies: t'Hooft and chiral anomalies, respectively. We will discuss only the second type.

3.1 Chiral Anomalies

The symmetry that is lost upon quantisation in this case is chiral symmetry. We follow [Ber96] in presenting the archetypical anomaly of massless QED.

So we fix the theory of massless QED - abelian for simplicity - via[43]

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}(i\partial\!\!\!/ - eA\!\!\!/)\psi$$

and observe that the well-known chiral action

$$\psi \to e^{i\alpha\gamma^5}\psi \quad \text{where} \quad \gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$$

is a symmetry of $\mathcal{L}$ as $[\gamma^0\gamma^\mu, \gamma^5] = 0$ because of $\{\gamma^\mu, \gamma^5\} = 0$ for any $\mu = 0, \dots 3$. Finally, Noether's theorem supplies the conserved current

$$j^5_\mu := \overline{\psi}\gamma_\mu\gamma^5\psi \quad \text{with} \quad \partial^\mu j^5_\mu = 0. \tag{3.1}$$

These have been classical considerations. Let us now consider higher-loop effects. Calculations will show that the particular choice of (Fourier-transformed) 3-point function

$$T_{\mu\nu\lambda}(k_1, k_2, q) := i \int \mathrm{d}^4(x, y, z) e^{i(k_1 x + k_2 y - qz)} \langle 0 | T j_\mu(x) j_\nu(y) j^5_\lambda(z) | 0 \rangle \,,$$

[43] We use fairly standard notation like $\overline{\psi} := \psi^\dagger \gamma^0$.

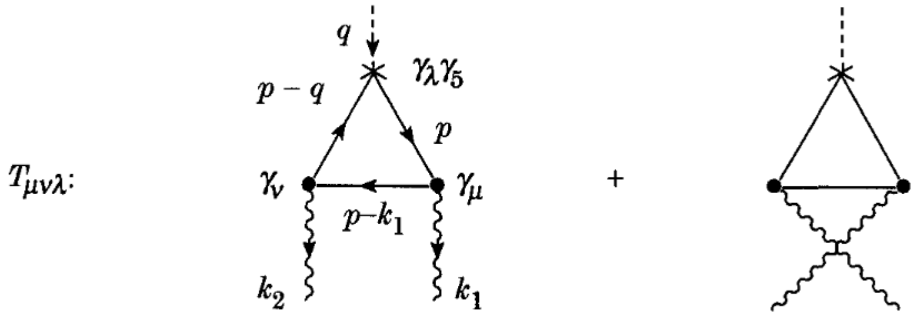


Figure 1: The two 1-loop contributions. $T_{\mu\nu\lambda}(k_1, k_2)$ amounts to summing the amputated (i.e. no external propagators) diagrams. From [Ber96].

the *triangle diagram*, comes with anomalous contributions at one-loop order (and only at one-loop order). Before sketching the calculations, we show what the result would look like, were massless QED free of anomalies.

Naïve Calculation: This is an immediate application of the Ward-Takahashi identity corresponding to (3.1) or quickly derived from

$$\begin{aligned} q^\lambda T_{\mu\nu\lambda}(k_1, k_2, q) &\overset{\text{ibP}}{=} \int d^4(x,y,z) e^{i(k_1 x + k_2 y - qz)} \partial_z^\lambda \langle 0 | T j_\mu(x) j_\nu(y) j_\lambda^5(z) | 0 \rangle \\ &= \int d^4(x,y,z) e^{i(k_1 x + k_2 y - qz)} \langle 0 | T j_\mu(x) j_\nu(y) \partial_z^\lambda j_\lambda^5(z) | 0 \rangle \overset{(3.1)}{=} 0 \end{aligned}$$

where we integrated by parts in the first step and suppressed possible contact terms (proportional to Dirac deltas) in the second step.

Correct Calculations: Interestingly, it is not the second step but the integration by parts that is the anomaly's origin. As we will sketch below, performing a pedestrian calculation in *some* regularization scheme, one shows that actually

$$q^\lambda T_{\mu\nu\lambda}(k_1, k_2, q) = -\frac{1}{2\pi^2} \epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta, \tag{3.2}$$

which is surely non-vanishing for some momenta and apparently not a contact term. So the former derivation had to be *wrong*! Surface terms were neglected, and it turns out they do not vanish. Note carefully that it is not the flawed derivation that is dubbed *anomaly*, but the fact that the Ward-Takahashi identity $q^\lambda T_{\mu\nu\lambda} = 0+$ (contact terms) fails to hold. In other words, the path integral measure fails to admit the symmetry, hence the lifted conservation equation (to higher loop orders) breaks down, which is anomalous behaviour.

Direct Calculation - Triangle Graph Regularization

Continuing to follow [Ber96], we calculate $T_{\mu\nu\lambda}(k_1, k_2)$ (q-dependence is trivial due to 4-momentum conservation, $q = k_1 + k_2$) at 1-loop via computing the two Feynman diagrams in Figure 1. We will not spell out all steps as they are well presented in [Ber96], and instead focus on the critical subtleties where the Ward identity starts crumbling. Upon using the QED Feynman rules, we start by

$$\begin{aligned} T_{\mu\nu\lambda} =& i \int \frac{d^4p}{(2\pi)^4} (-1) \operatorname{tr} \frac{i}{\not{p}} \gamma_\lambda \gamma^5 \frac{i}{\not{p} - \not{q}} \gamma_\nu \frac{i}{\not{p} - \not{k}_1} \gamma_\mu \\ &+ \{k_1 \leftrightarrow k_2, \mu \leftrightarrow \nu\} \end{aligned}$$

where the trace comes from the right γ_μ vertex, i.e. the fact that it is a loop which we traversed counter-clockwise. This equality has to be taken with a grain of salt as it is neither clear nor true that the value of $T_{\mu\nu\lambda}(k_1, k_2)$ is independent of the choice of route that p takes. In fact, and as is commented on quite explicitly in [Ber96], shifting the integration variable like $p \mapsto p + a$ changes $T_{\mu\nu\lambda}$. We will notationally suppress this dependence on a and choose the customary value in the end.[44] From $\not{q}\gamma^5 = \gamma^5(\not{p} - \not{q}) + \not{p}\gamma^5$ we get

$$\begin{aligned} q^\lambda T_{\mu\nu\lambda} = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \operatorname{tr} \left(\frac{1}{\not{p} - \not{k}_2} \gamma^5 \gamma_\nu \frac{1}{\not{p} - \not{q}} \gamma_\mu - \frac{1}{\not{p}} \gamma^5 \gamma_\nu \frac{1}{\not{p} - \not{k}_1} \gamma_\mu \right) \\ + \{k_1 \leftrightarrow k_2, \mu \leftrightarrow \nu\}. \end{aligned}$$

Recalling $q = k_1 + k_2$, we observe that a substitution $p \mapsto p + k_2$ in the first summand would make the two trace-summands cancel each other, giving a non-anomalous Ward-Takahashi identity. Rephrasing this, for suitable f, one can write

$$q^\lambda T_{\mu\nu\lambda} = \int f(p - k_2) - f(p) \, \mathrm{d}^4p + \{k_1 \leftrightarrow k_2\}.$$

It should not come as a surprise that the divergent quantities $\int f(p - k_2) \, \mathrm{d}^4p$, $\int f(p) \, \mathrm{d}^4p$ do not subtract to zero. It is customary to regularize the sum, here we proceed by momentum-cutoff. Bertlmann [Ber96] presents various schemes. Taylor expanding like

$$f(p - k_2) - f(p) = -k_2^\mu \partial_\mu f(p) + \dots$$

and using Gauss' theorem yields after some gamma/trace gymnastics the above claimed result

$$q^\lambda T_{\mu\nu\lambda} = -\frac{1}{2\pi^2} \epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta. \tag{3.3}$$ ×2

It is important to remark that this result, also known as the *ABJ anomaly* (after Adler, Bell, and Jackiw), is valid at a particular choice of the above a. The whole calculation revolves around non-vanishing surface terms in integral substitutions like $p \mapsto p + a$ so it is necessary to specify *which* route the fermion in Figure 1 takes. The closest-to-canonical choice is the value at which the Ward-Takahashi identity for the U(1) current $j_\mu := \overline{\psi}\gamma_\mu\psi$ is non-anomalous. It is a remarkable fact of QED anomalies that the U(1) and chiral anomalies' sizes are mutually dependent. Luckily, there exist the two sweet spots where one of them vanishes. This sweet spot is the customary value of the other non-vanishing anomaly and fixed our numerical factor in (3.3).

[44] To relax the discussion, be informed that there exists no momentum a such that massless QED fulfills all its Ward-Takahashi identities, i.e. is non-anomalous.

Massless QED Anomaly and Chern-Simons Theory

An equivalent expression for the anomaly is the operator equation

$$\partial^\mu j^5_\mu = \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}. \tag{3.4}$$

Proof. We need to show that both sides act equally. It is by virtue of ABJ's theorem that the only contributions to the anomalous operator $\partial^\mu j^5_\mu$ are the diagrams depicted in Fig. 1. Hence, as a corollary, if we prove[45]

$$\langle k_1, k_2|q^\mu \tilde{j}^5_\mu(q)|0\rangle = \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\langle k_1, k_2|\tilde{F}_{\mu\nu}(k_1)\tilde{F}_{\alpha\beta}(k_2)|0\rangle$$

for any k_1, k_2 and $q = k_1 + k_2$ the claim follows, as the r.h.s. of (3.4) surely acts by producing two photons from the vacuum. Again by ABJ and (3.3), for the l.h.s. we get

$$\begin{aligned}\langle k_1, k_2|q^\mu \tilde{j}^5_\mu(q)|0\rangle &= q^\lambda T_{\mu\nu\lambda}\epsilon(k_1)^{*\mu}\epsilon(k_2)^{*\nu}\\ &= -\frac{1}{2\pi^2}\epsilon_{\mu\nu\alpha\beta}\epsilon(k_1)^{*\mu}\epsilon(k_2)^{*\nu}k_1^\alpha k_2^\beta.\end{aligned}$$

For the r.h.s. we observe

$$\begin{aligned}\epsilon^{\mu\nu\alpha\beta}\tilde{F}_{\mu\nu}(k_1)\tilde{F}_{\alpha\beta}(k_2) &= 4\epsilon^{\mu\nu\alpha\beta}k_{1\mu}\tilde{A}(k_1)_\nu k_{2\alpha}\tilde{A}(k_2)_\beta\\ &= -4\epsilon^{\mu\nu\alpha\beta}\tilde{A}(k_1)_\mu\tilde{A}(k_2)_\nu k_{1\alpha}k_{2\beta}\end{aligned}$$

by definition of F and because $\{2,4,1,3\}$ is an odd permutation of $\{1,2,3,4\}$. $\square$

Rewriting this coordinate expression in differential forms, gives

$$\mathrm{d}{*j^5} = \frac{1}{4\pi^2}F \wedge F$$

where standard conventions like $*j^5 = *j^5_\mu\,\mathrm{d}x^\mu := \frac{1}{3!}j^{5\mu}\epsilon_{\mu\nu\alpha\beta}\,\mathrm{d}x^\nu\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta$ and $F = F_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ were used. The "$F \wedge F$" looks already very similar to a degree 2 invariant polynomial $\mathcal{P}_2$ - evaluated at $F = \sigma^*\Omega$ - as was free to be chosen in the definition of the Chern-Simons form (2.8). In fact, the bilinear map

$$\mathcal{P} : (\omega_1, \omega_2) \mapsto \omega_1 \wedge \omega_2$$

is a form-valued U(1)-invariant polynomial, i.e. $\mathcal{P} \in I^2(\mathrm{U}(1))$. So we find the tight connection between the anomalous chiral current conservation of massless QED and the abelian (pulled-back) Chern-Simons form $\sigma^*\alpha_{\mathrm{ab}}(\Theta) = \alpha_{\mathrm{ab}}(A)$:

$$\mathrm{d}{*j^5} = \frac{1}{4\pi^2}\,\mathrm{d}\sigma^*\alpha_{\mathrm{ab}}(\Theta) = \frac{1}{4\pi^2}\sigma^*\mathcal{P}(\Omega,\Omega) \tag{3.5}$$

where $\Theta \in \Omega^1(P)\otimes\mathfrak{u}(1)$ and $\sigma \in \Gamma(P)$ are connection and section in a principal bundle P fixed upon defining the QFT.

[45] Tilde means Fourier transformation.

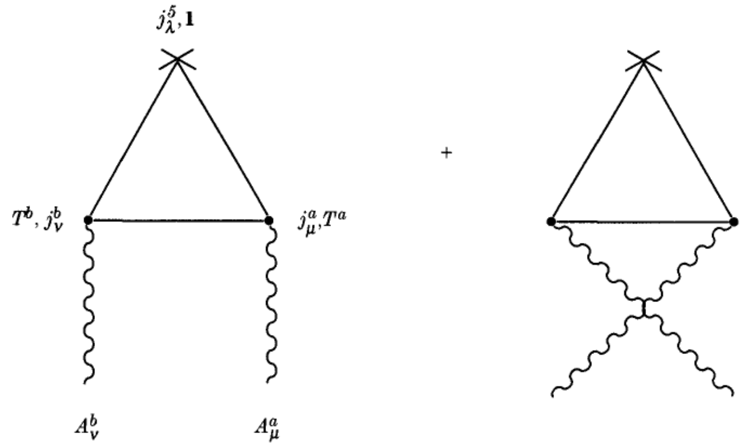


Figure 2: Singlet triangle contribution to $\partial^\lambda j^5_\lambda$. So j^5_λ is abelian. From [Ber96].

Generalisation to Yang-Mills Theories

Of course, physicists will not be satisfied with the restriction to abelian theory and would like to know whether this anomaly-CS connection survives the generalisation to non-abelian Yang-Mills theories. The answer is: In the *singlet case*, yes.

Singlet means that the chiral vertex has no Lie algebra generator attached to it.[46] The singlet triangle contribution, which is *not* the only one in the non-abelian case as will be clear in a moment, is depicted in Figure 2. Evaluating the amputated diagrams of Figure 2 as before (with outgoing boson momenta k_1, k_2 and incoming momentum $q = k_1 + k_2$) amounts to

$$T^{ab}_{\mu\nu\lambda}(k_1, k_2) := i \int \mathrm{d}^4(x, y, z) e^{i(k_1 x + k_2 y - qz)} \langle 0| T j^a_\mu(x) j^b_\nu(y) j^5_\lambda(z) |0\rangle$$

and can be quickly generalized from the prior triangle calculation

$$T^{ab}_{\mu\nu\lambda}(k_1, k_2) = T_{\mu\nu\lambda}(k_1, k_2) \frac{1}{2} \operatorname{tr}\{T^a, T^b\} =: T_{\mu\nu\lambda}(k_1, k_2) c^{ab} \tag{3.6}$$

where the trace comes, again, from the diagram being loopy, the anticommutator stems from the two diagrams and the $1/2$ from overcounting, as $T_{\mu\nu\lambda}$ already contains both summands. Following a parallel reasoning as before, the next step is to find an operator equation that encapsulates (3.6). The educated guess would be to use the same form

$$\partial^\mu j^5_\mu = \frac{c^{ab}}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^b_{\alpha\beta} \tag{3.7}$$

only including the factor c^{ab} and the non-abelian field strengths $F^a_{\mu\nu}$. Bertlmann [Ber96] argues, that this is the only singlet operator we can write down. Working with that, we conclude that now also a quadrangle diagram (the $\partial_\mu A_\nu A_\alpha A_\beta$ term in (3.8)) depicted in Figure 3 contributes. A pentrangle diagram's con-

[46] I.e. we again take (3.1) as an abelian current, and denote this as $\mathbb{1}$.

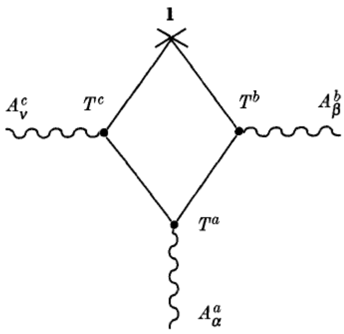


Figure 3: To the singlet contributing quadrangle diagram stemming from the non-linear terms in non-abelian field strengths. From [Ber96].

tribution (the $A_\mu A_\nu A_\alpha A_\beta$ term in (3.8)) vanishes due to the trace and anti-commuting 1-forms, i.e. $\epsilon^{\mu\nu\alpha\beta}$.
Now, the Chern-Simons form is about to emerge:

$$\begin{aligned} \partial^\mu j^5_\mu &= \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}_{\mathfrak{g}}\,F_{\mu\nu}F_{\alpha\beta} \\ \leftrightsquigarrow d * j^5 &= \frac{1}{4\pi^2}\,\mathrm{tr}_{\mathfrak{g}}\,F\wedge F \\ &=: \frac{1}{4\pi^2}\mathcal{P}(F,F) \\ &= \frac{1}{4\pi^2}\,\mathrm{d}\,\alpha(A) \end{aligned} \tag{3.8}$$

Note on index theorems: It would be a shame not to mention the Atiyah-Patodi-Singer index theorem in this context, even though further development of the present work will wander off in a different direction. Denote by $\not{D}_+$ the chiral Dirac operator, i.e.

$$\not{D}_+ := P_+\not{D} \equiv \frac{1}{2}(1+\gamma^5)(\not{\partial} + ie\not{A})$$

which is non-self-adjoint, and hence has an interesting (Fredholm) index. The APS index theorem states that this analytic index is dependent only on topological data and only from the base manifold - for example it does not depend on the connection A one chooses. Calculation shows, intriguingly, that

$$\begin{aligned} \mathrm{ind}\,\not{D}_+ &= d * j^5 \\ &\quad (= n_+ - n_-) \end{aligned}$$

where n_+, n_- are $\not{D}$'s zero mode counts of positive and negative chirality, respectively. Being pedantic, this equality is not bringing new big insights, as we already knew that the terms in (3.8) were topological.[47] Still, it is remarkable to find this bridge from Chern-Weil theory to index theory in chiral QED anomalies.

[47] The Chern-Weil homomorphism (2.3) and Rem. 14 taught us $\mathcal{P}(F,F) \in H^4_{\mathrm{dR}}(B_3)$.

Note on the choice of invariant polyonmial $\mathcal{P} \in I^2(G)$: Recall that our definition of α in section 2 left the choice of $\mathcal{P}$ open. However, there exists a customary choice, inspired by simplicity and probably Witten's Jones polynomial paper [Wit89], namely the symmetrised trace $\mathcal{P} = \text{str}$, a.k.a. the first Chern class. So our choice here puts us in the realm of customary Chern-Simons theory (of physics).

We saw that chiral QED anomalies intertwine non-trivially with theories of topology. This is a first motive to include their discussion in a thesis that is mainly about topological field theories. But as remarked earlier, TQFTs admit another characteristic - the jumps between dimensions (from cutting and gluing, respectively from the different levels of data in a category). Surprisingly, anomalies do not fall short on this characteristic either. In a sense, it is an even more central property of anomalies. The following subsection introduces anomaly inflow which should illuminate the above statements.

3.2 Anomalies Indicate Higher-Dimensional Ambience

Yet another tight bond between anomalies and TQFTs has been explained in Callan and Harvey's original paper on **anomaly inflow** [CH85]. The message we want to take from their work is, in essence, that anomalous theories can emerge from higher dimensional perfectly fine non-anomalous QFTs that are only looked at on a codimension-1 (or 2, etc.) slice. Picturing e.g. a line on a surface, it comes as no surprise that Noether-currents have non-vanishing divergences (i.e. charges are not conserved) when one forgets adding the transversal directional derivative. The seeming triviality of this thought-experiment stems mainly from already having the information about a consistent higher-dimensional QFT (of which the anomalous theory is a restriction in some sense) at hand.

Anomaly Inflow in 4D QED

For simplicity, we will reduce the generality of [CH85] to 4D with a (co)dimension-2 defect and only gauge anomalies. By analogy with 3D, this is dubbed a *string configuration* - counting from the top dimension downwards. Of course, we need to design the 4D theory such that there exists a sensible 2D QFT within. To this end, we choose chiral complex scalar QED described by the Lagrangian

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ + \Phi_1 + i\gamma^5\Phi_2)\psi \tag{3.9}$$

where ψ is a regular 4D fermionic field, $\gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3$ which is slightly different to our previous definition in order to follow the conventions of [CH85], and $\Phi = \Phi_1 + i\Phi_2$ is a background complex scalar field. *Background field* is the well-known term referring to the field being non-dynamical (i.e. not integrated over in the path integral). We take and need it to realize the (2D) stringy setup by imposing on it

$$\Phi(x) = f(\rho)e^{i\varphi} \quad (x_2 = \rho\cos\varphi, x_3 = \rho\sin\varphi) \tag{3.10}$$

i.e. $\partial_0\Phi = \partial_1\Phi = 0$ which implements symmetry under translation along the string directions x_0, x_1 in our 4D setup. Additionally demanding $f(0) = 0$ and $\lim_{\rho\to\infty} f(\rho) = \nu$ with $f'(\rho) > 0$ in between, gives the fermions a spatially

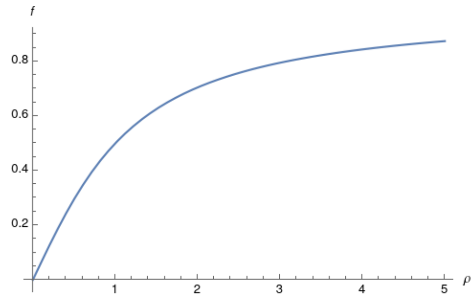


Figure 4: Possible $f(\rho)$, the limit is $\nu = 1$ in this case.

dependent mass (seen from (3.9)). Picturing a scalar field as depicted in Figure 4, it is no surprise that low-energy fermions are confined to the string ($\rho = 0$).

Concretely, the e.o.m. for $\overline{\psi}$ is now simply $(i\partial\!\!\!/ + \Phi_1 + i\gamma^5\Phi_2)\psi = 0$. Due to higher dimensional cylindrical symmetry the derivative in φ direction has to vanish. Using this simplification, the presented solution of negative 4D chirality (i.e. $\gamma^5\psi_- = -\psi_-$) reads

$$\psi_-(x) = \eta(x_0, x_1) \exp\left(-\int_0^\rho f(r)\,\mathrm{d}r\right) \tag{3.11}$$

where, crucially, $i(\gamma^0\partial_0 + \gamma^1\partial_1)\eta =: i\partial\!\!\!/^{\mathrm{str}}\eta = 0$. Accepting also that $\gamma^{\mathrm{str}}\eta := -i\gamma^0\gamma^1\eta = -\eta$, we summarize that the 4D solutions additionally solve the 2D massless Dirac equation, i.e. on string ("str"), and have definite 2D chirality.

We are now free to talk about a 4D and 2D theory. The latter is described by a massless 2D fermion, which we can couple to U(1). It turns out, however, that this renders the 2D theory essentially useless as it admits a brutal kind of anomaly - a gauge anomaly

$$\partial^a j_a = \frac{e}{4\pi}\epsilon^{ab}F_{ab} \quad a, b = 0, 1. \tag{3.12}$$

So, contrary to subsection 3.1, this theory looses its *gauge* symmetry upon quantization. Recalling that (for example in the BRST-formalism) gauge symmetry is a crucial ingredient in order to make sense of a quantum gauge theory, this anomaly has quite an impact. Apparently, the 2D theory cannot keep its total electrical charge constant, there exists non-trivial charge drift-off.

We now zoom out to the 4D theory of a U(1) fermion coupled additionally to an axion string Φ which, as discussed above, realizes a separate low-energy stringy 2D theory. In the adiabatic approximation where only the light fermions in vicinity of the string ($\rho \approx 0$) are allowed to move non-adiabatically, the electrical current expectation value from [CH85] is

$$\partial^\mu j_\mu = \frac{e}{4\pi}\epsilon^{ab}F_{ab}\delta(x_2)\delta(x_3) \tag{3.13}$$

where only directions $a, b = 0, 1$ contribute as the electric field was chosen to point along the string directions! Off the string ($x_2 \neq 0 \neq x_3$) the gauge current is conserved, but on-string the lack of gauge charge in the 4D theory exactly matches the anomaly in the 2D theory.

One has interpreted this non-vanishing 4-divergence as a 2D domain wall defect which flips the fermions' masses that traverse it - see e.g. [Jan96].

4 TQFT From Modular Categories

A TQFT $Z : \mathbf{Cob} \to \mathbf{Vect}$ can be viewed as a linear representation of topological structure, just like a linear group representation $\rho : G \to \mathrm{GL}(V)$ represents group structure. The following subsection aims to find a concrete example of topological representation, namely of ribbon graphs[48] inside flat $\mathbb{R}^3$ by abstract categories, leading towards and motivating the definition of a *modular* category in the subsection thereafter.

4.1 Representing Ribbons - An $\mathbb{R}^3$-Functor

We will not be precise in the definition of a ribbon graph. Refer to [Tur10, §I.2.1] for details. A directed ribbon graph consists of isotopy classes of:

- Directed ribbons or bands - oriented, smooth embeddings $\mathrm{rib} : [0,1]^2 \hookrightarrow \mathbb{R}^3$ with an arrow along the second $[0,1]$ factor giving the direction, and a distinguished side (called and drawn *up*, i.e. towards the reader) giving orientation (after fixing e.g. the right-handed orientation of $\mathbb{R}^3$) - drawn like dangerous streets in the desert from a bird's view.

- Plaquets or coupons - again smooth embeddings $\mathrm{plaq} : [0,1]^2 \hookrightarrow \mathbb{R}^3$ where one remembers the so-called *lower* base $\mathrm{plaq}(\{0\} \times [0,1])$ as a substitute for direction and orients them in the same fashion as for the ribbons - drawn like perfect rectangles.

Looking at Figure 5 it is easy to accept that the two building blocks above patch together into a whole *(directed) ribbon graph* iff the following rules are satisfied:

- The two ends $\mathrm{rib}([0,1] \times \{i\})$ for $i = 0,1$ of a ribbon either end at lower/upper borders of graph/plaquets or end at each other (producing a ribbon-knot as in Figure 5b).

- Every ribbon needs to face *up* at its ends.

- No intersections except for the gluing of ribbon to coupon.

- Two ribbon graphs are equal if they differ by an isotopy![49]

These hence defined directed ribbon graphs, being an isotopy class of embeddings in $\mathbb{R}^3$, provide a set of topological (and orientational and directional) data that we want to categorify and represent by a functor *out* of it. We will now define its target category (with more rigour as this definition will appear more often).

5× **Def. 32.** A strict monoidal category $\mathcal{V}$ is a **ribbon category** if it has

[48] To be precise, the ribbon graphs need to be equipped with an *orientation*, a *direction* and a *categorical colour* in order for us to represent them. I allow myself not to repeatedly emphasize this because these structures do not fall into what people refer to as *non-topological* (e.g. metrics, parametrizations). They lie somewhat much closer to topological, than to non-topological structure.

[49] This *taking up to isotopy* will inherently be the reason why our functor will produce *in*variants.

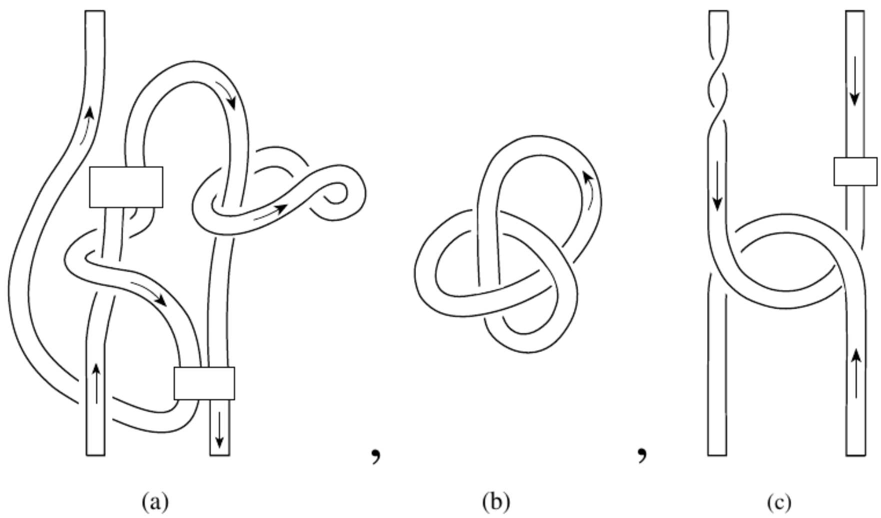


Figure 5: Three examples of directed ribbon graphs, taken from [Tur10]. (a) is a (2,1)-ribbon graph with 2 coupons and 2 disjoint components. (b) is a (0,0)-ribbon graph without coupons. The upper ribbon (c) contains a (full) twist. All of the ribbons are directed and oriented.

- a binatural isomorphism $c = \{c_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V\}_{V,W}$ called **braiding**
- a natural isomorphism $\theta = \{\theta_V : V \xrightarrow{\sim} V\}_V$ called **twist**
- **duality**, i.e. an assignment

$$\mathrm{ob}(\mathcal{V}) \ni V \mapsto \Big(V^* \in \mathrm{ob}(\mathcal{V}), \mathrm{b}_V : \mathbb{1} \to V \otimes V^*, \mathrm{d}_V : V^* \otimes V \to \mathbb{1}\Big)$$

meaning that for any $U, V, W \in \mathrm{ob}(\mathcal{V})$ they obey the consistency conditions[50]

a. $c_{U,V\otimes W} = (1_V \otimes c_{U,W})(c_{U,V} \otimes 1_W)$ and $c_{U\otimes V,W} = (c_{U,W} \otimes 1_V)(1_U \otimes c_{V,W})$

b. $\theta_{V\otimes W} = c_{W,V}c_{V,W}(\theta_V \otimes \theta_W)$

c. $(1_V \otimes \mathrm{d}_V)(\mathrm{b}_V \otimes 1_V) = 1_V$ and $(\mathrm{d}_V \otimes 1_{V^*})(1_{V^*} \otimes \mathrm{b}_V) = 1_{V^*}$

d. $(\theta_V \otimes 1_{V^*})\,\mathrm{b}_V = (1_V \otimes \theta_{V^*})\,\mathrm{b}_V$.

Any such $\mathcal{V}$ will be our representing functor's co-domain, so we will transport very rich topological structure (namely 3-ribbons) into $\mathcal{V}$ giving this list of axioms a strong foundation.

Because the co-domain $\mathcal{V}$ is freely chosen among strict ribbon categories, also (the categorification of) our topological domain of directed ribbon graphs

[50]This unhandy list of relations is spelled out *before* introducing graphical calculus to emphasize that a ribbon category has no *a priori* connection to topology whatsoever. I.e. we did not define it by saying something like "graphical calculus has to be well defined". However, this will essentially be the outcome, to be understandable after Thm. 34.

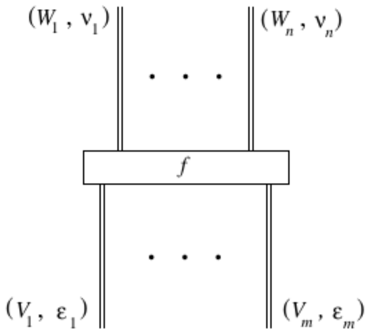


Figure 6: A generic plaquet as part of a directed ribbon graph. $\varepsilon_i, \nu_i \in \{\uparrow, \downarrow\}$ indicate the ribbons' directions. From [Tur10].

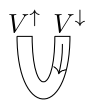
will depend on it. This is dubbed *colouring*: Given any monoidal category $\mathcal{V}$ with duality (no consistency in this case), one can $\mathcal{V}$-**colour** a directed ribbon graph simply by labeling/colouring every ribbon with any $C \in \mathrm{ob}(\mathcal{C})$ and every plaquet with an *allowed* morphism of $\mathcal{V}$. "Allowed" is probably intuitively clear but not so easily written down: A generic plaquet, which we want to colour by the morphism f, is depicted in Figure 6. This is possible iff

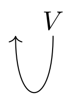
$$f \in \mathcal{V}\left(V_1^{(\varepsilon_1)} \otimes \cdots \otimes V_m^{(\varepsilon_m)}, W_1^{(\nu_1)} \otimes \cdots \otimes W_n^{(\nu_n)}\right)$$
$$\text{where } V_i^{(\varepsilon_i)} := \begin{cases} V_i & \text{if } \varepsilon_i = \uparrow \\ V_i^* & \text{if } \varepsilon_i = \downarrow, \end{cases} \quad \text{(similarly for } W_i\text{)}.$$

Def. 33. Let $\mathcal{V}$ be strict monoidal with duality. The strict monoidal category $\mathbf{Rib}_\mathcal{V}$ is defined to have

objects: Each $(V_1^{(\varepsilon_1)}, \ldots, V_n^{(\varepsilon_n)})$ where $n \in \mathbb{N}_0, V_i \in \mathrm{ob}(\mathcal{V})$ and $\varepsilon_i = \uparrow, \downarrow$.

morphisms: All $\mathcal{V}$-coloured directed ribbon graphs (i.e. isotopy classes) that "fit between" source and target. For example for $V \in \mathrm{ob}(\mathbf{Rib}_\mathcal{V})$



$=:$


$\in \mathbf{Rib}_\mathcal{V}\left(\emptyset, (V^\uparrow, V^\downarrow)\right)$.

$\otimes$: Juxtaposition for both objects and morphisms.

$\mathbb{1}$: The monoidal unit is taken as $\emptyset \in \mathrm{ob}(\mathbf{Rib}_\mathcal{V})$.

As is often after categorification, the important structure is encoded in the Hom-sets. Now for the central result.

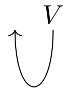
6× **Thm. 34** (Graph-To-Morphism)**.** *Let $\mathcal{V}$ be a strict ribbon category. There exists precisely one monoidal functor*[51] $\mathrm{G2M}_\mathcal{V} : \mathbf{Rib}_\mathcal{V} \to \mathcal{V}$ *assigning for* $V, W \in \mathrm{ob}(\mathcal{V})$

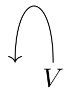
[51] The notation in [Tur10] for this functor is "$F_\mathcal{V}$".

a. $V^{(\uparrow)} \mapsto V$ *&* $V^{(\downarrow)} \mapsto V^*$

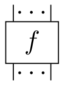
b. 

$\mapsto c_{V,W}$ *&* 

$\mapsto \theta_V$ *&* 

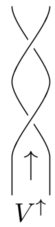
$\mapsto \mathrm{b}_V$ *&* 

$\mapsto \mathrm{d}_V$

c. 

$\mapsto f.$

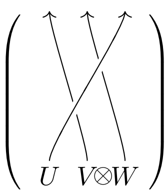
Where, for LaTeX-convenience and readability, we have represented (isotopy classes of) directed ribbons by drawing only their *cores*. In this depiction, the absence of twists is understood. Of course, twists are still a fundamental building block of directed ribbon graphs - so they should be included in the above list as uniqueness was claimed. They are in fact included as the directed ribbon that is being mapped to θ_V, the flip. This is due to the following equality of isotopy classes,

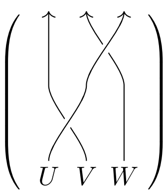
$=$


, (4.1)×1

of which one can convince oneself using an actual ribbon, for example the ones used as bookmarks. The proof and a long list of corollaries of the theorem can be found in [Tur10]. Let me list some:

- The list of consistency axioms in Def. 32 has now gained strong topological ground. Were there another axiom, not implied by the presented ones plus monoidality, this functor would not exist in general. Were the list of axioms shorter, the functor would not be unique.
- Every image of morphism η of $\mathbf{Rib}_{\mathcal{V}}$ under $\mathrm{G2M}_{\mathcal{V}}$ is an *invariant* of η as the image is well defined among the equivalence class given by isotopy. This makes precise our meaning of a TQFT producing *invariants*.
- In the co-domain, it is now safe to introduce graphical calculus.[52] We exemplify this by showing that the first consistency axiom in Def. 32, $c_{U,V\otimes W} \overset{(*)}{=} (1_V \otimes c_{U,W})(c_{U,V} \otimes 1_W)$, follows from the theorem alone[53]

$c_{U,V\otimes W} = \mathrm{G2M}$ 

$= \mathrm{G2M}$ 

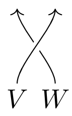
$= (1_V \otimes c_{U,W})(c_{U,V} \otimes 1_W)$

[52]Note how we do not need any further properties from the monoidal functor than stated. E.g. fully-faithful or the like. The power is that it *exists*. See Rem. 41.2 for further discussion.

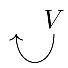
[53]Of course, one *assumes* $(*)$ in order for the theorem to hold in the first place - this is a mere demonstration of graphical calculus.

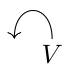
where the first equality is by Thm. 34 only and the last equality additionally by G2M$_{\mathcal{V}}$'s monoidality and functoriality. The key middle equality uses that morphisms in $\mathbf{Rib}_{\mathcal{V}}$ are defined as isotopy classes.

We will henceforth calculate in a ribbon category $\mathcal{V}$ by drawing the building blocks $(c, \theta, \mathrm{b}, \mathrm{d})$ exactly like the morphisms of $\mathbf{Rib}_{\mathcal{V}}$ that G2M$_{\mathcal{V}}$ maps to them,



$=: c_{V,W}$ &


$=: \theta_V$ &


$=: \mathrm{b}_V$ &


$=: \mathrm{d}_V$,

"birth" "death"

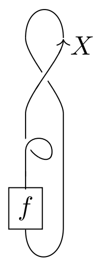
will compose in $\mathcal{V}$ by "putting on top", $\otimes$-multiply by juxtaposition, write morphisms of $\mathcal{V}$ as coupons and will consider isotopic changes as invariances, which is allowed by arguing as above.

4.2 Modular Categories - Beyond Ribbons

Now that representation of topological input in form of ribbons in $\mathbb{R}^3$ has been established, we aim to find the necessary categorical structure to extend to ribbons in arbitrary B_3. As the latter includes the former, it will not come as a surprise that this structure, a *modular* category, is, in particular, a ribbon category. We present the definition from [Tur10], but restrict to existence of $\oplus$ and $\mathbb{C}$ as ground ring.

We prepare by presenting the notion of a trace in an abstract category:

Def. 35. For a $\mathbb{C}$-linear ribbon category $\mathcal{M}$, we define the **trace** of any endomorphism $f \in \mathrm{Hom}_{\mathcal{M}}(X, X)$ as

$$\mathrm{tr}(f) := \mathrm{d}_X\, c_{X,X^*}((\theta_X f) \otimes \mathrm{id}_{X^*})\, \mathrm{b}_X =$$


$\in \mathbb{C}$

Note how we used all of the structure a ribbon category comes with.

2× **Def. 36** (modular category)**.** A $\mathbb{C}$-**modular category** consists of

- a ribbon $\mathbb{C}$-linear Ab-category $\mathcal{M}$ and[54]
- a finite family $\{S_i\}_{i \in I}$ of simple objects

underlying the following consistency axioms:

a. Normalization: $0 \in I$ with $S_0 = \mathbb{1}$, the tensor unit.

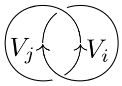
b. Duality: $i \in I \Rightarrow i^* \in I$ with $S_{i^*} \cong (S_i)^*$.

[54]"ribbon" was defined in Def. 32, "$\mathbb{C}$-linear" means that any morphism space is enriched over $\mathbf{vect}(\mathbb{C})$, and "Ab-category" that they are also enriched over the category of abelian groups, s.th. $\otimes$ and $\circ$ are bilinear.

c. Domination: All $X \in \mathrm{ob}(\mathcal{M})$ are semisimple w.r.t. $\oplus$.

d. **Non-Degeneracy**: The following matrix is invertible,

$$S := \left(\mathrm{tr}\left(c_{V_i,V_j} c_{V_j,V_i}\right) \in \mathbb{C}\right)_{i,j\in I} = \Bigg($$



$$\in \mathbb{C} \Bigg)_{i,j\in I}$$

where the equality is a non-trivial implication of Thm. 34.

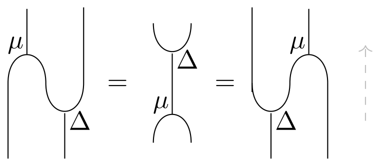
One might be wondering *why* exactly these axioms. For example non-degeneracy has a strong motivation and is the reason why these categories will end up producing projective representations of some mapping class group, which in the case of the torus is the modular group [Tur10, §IV.5], explaining the name. The rest of the axioms seems rather direct, but this is probably personal taste. We will not see crucial dependence of the constructions on these axioms because hardly any proofs are given in the upcoming section 5.

5 One-and-a-Half 3-TQFTs

2D TQFTs - being the highest dimension that is easily accessible to a 3D-thinking brain - have been classified for a while, see e.g. [Abr96]. It is a particularly satisfying line of reasoning[55] leading towards the answer that *every* 2-TQFT

$$Z_2 : \mathbf{Cob}(2) \to \mathbf{Vect}_{\mathbb{C}}$$

assigns to $S^1 \in \mathrm{ob}\,\mathbf{Cob}(2)$ a vector space that gets endowed - always - with the structure of a commutative **Frobenius algebra**. This section will eventually be on 3D TQFTs. But as a warm-up, and to convey the message that intuition is powerful, we exemplify some 2D TQFT reasoning. E.g. why the eponymous Frobenius relation



(5.1)×4

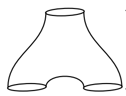
which is the customary depiction of this axiom on any Frobenius algebra (F, μ, Δ), equivalently

$$\mu \otimes \mathrm{id}_F \circ \mathrm{id}_F \otimes \Delta = \Delta \circ \mu = \mathrm{id}_F \otimes \mu \circ \Delta \otimes \mathrm{id}_F, \tag{5.2}$$

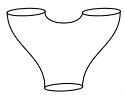
×1

necessarily holds for the vector space $Z_2(S^1) =: F$ provided one sets

$$\mu := Z_2\Bigg($$



$$\Bigg), \qquad \Delta := Z_2\Bigg($$



$$\Bigg).$$

Proof. Blowing up the tangle diagram (5.1) such that lines become tubes and 3-fold junctions become such (reverse) pair of pants, we can easily imagine that

[55] Modulo the steps to show that the list of generators of 2-surfaces are actual generators.

the so-constructed 2-surface version of (5.1) topologically holds (i.e. the three cobordisms lie in the same diffeomorphism class). Now using that the functor Z_2 transports equalities of diffeomorphism classes of 2-cobordisms to equalities of linear maps we immediately get (5.2) for our choice of μ, Δ. □

3D TQFTs are much more complicated. Try imagining a closed (compact & bounded) 3-manifold, for example. Building non-trivial 3-TQFTs will be subject to the next subsections. It is this dimension in which Chern-Simons theory was formulated so it will be the construction making Witten's CS-theory usage rigorous. Furthermore, it will pave the way to physical applications.

5.1 Motivation, Summary & Relations

It may seem exaggeration to present *two* such constructions:

1. **Turaev and Viros' state sum**[56] Z_{TV} (original publication: [TV92]; more details: [Tur10] or [TV17]; the paper we follow here: [KB10]) and
2. **Reshetikhin and Turaevs' surgery construction**[57] Z_{RT} (original publication: [RT91]; what we follow: [Tur10]),

- and partly it is unquestionably due to personal taste - but there exist physical applications of both functors to be discussed in subsequent sections. Regarding the mentioned personal taste, it is fascinating to come up with two different strategies to associate a non-trivial number $Z(C_3)$ to each C_3 (oriented closed 3-manifold) such that the set $\{Z(C_3)\}_{C_3}$ is *consistent*: functorial/gluing-law & topological.
The latter means that two C_3's need to give the same invariant even if they are only "topologically the same" (cf. Rem. 41.2). Hence, the constructions of the Z's both need to have some topological foundation - i.e. steps that must not distinguish two slightly deformed manifolds, decorated by wavy underlines in the upcoming list. These respective core ideas can be summarized as:

1. Z_{TV} first takes the 2-Poincaré dual of a linearized version of C_3. Then it interprets the resulting graph (of edges and vertices) as a morphism in a category. This interpretation as morphism is in *the spirit* of Thm. 34's functor $\text{G2M}_{\mathcal{V}}$.
2. Z_{RT} first writes C_3 as the result of Dehn-surgery on $S^3 = \mathbb{R}^3 \cup \{\infty\}$ prescribed by "the" link[58] L_{C_3}. Here one relies on, schematically,

$$\{\text{closed 3-manifolds}\}/\,\text{iso} \overset{\sim}{\longleftrightarrow} \{\text{links in } S^3\}/\text{Kirby moves}$$
$$C_3 \longmapsto L_{C_3}$$
$$\text{Dehn}(L) \longleftarrow L.$$

   Now, a framed link is very much viewable as a ribbon graph, so we need only give our link some $\mathcal{V}$-colour and push it through Thm. 34's $\text{G2M}_{\mathcal{V}}$ to see it as a linear $\mathbb{C} \longrightarrow \mathbb{C}$, i.e. a number.

[56] The "One" actual TQFT in the sense of Def. 1.
[57] The "and-a-Half" TQFT. It does not furnish a full TQFT in the sense of Def. 1. Nevertheless, it gets very close, as elaborated on later.
[58] Alarm bells ringing! A **link** - multiple non-intersecting knots - inherently needs braiding. So Z_{RT} will have to impose this structure on its base category.

The latter is understood to be the mathematical construction of Witten's framed Wilson lines in his seminal work on Chern-Simons theory [Wit89].

We record here the exciting relations between the two TQFTs. Incidentally, the two procedures are not independent. Still, they cannot be directly equal or isomorphic. This is quickly seen from the structural differences of the two **base categories** (the input categories, that need to be fixed in both cases):

1. $Z_{\mathrm{TV}} = Z_{\mathrm{TV}}^{\mathcal{A}}$ needs a *fusion* category $\mathcal{A}$ that is *spherical*. In particular, the state-sum can be constructed *without* the structural input of a braiding[59]!

2. $Z_{\mathrm{RT}} = Z_{\mathrm{RT}}^{\mathcal{M}}$ needs a *modular* category $\mathcal{M}$. That's a lot of structure, including fusion and braiding, and some extra properties as outlined in Def. 36.

The relation in question feeds off of [Mue01]'s result that $\mathcal{A}$'s Drinfeld center $\mathcal{Z}(\mathcal{A})$ *is* modular (under the above assumptions). So we can compare the TQFTs by passing to Z_{TV}'s base category's center - and indeed the most immediate comparison turns out to be true! Turaev and Virelizier (original: [TV13]; book: [TV17]) or Balsam and Kirillov Jr. [KB10] showed:

$$Z_{\mathrm{TV}}^{\mathcal{A}}(C_3) = Z_{\mathrm{RT}}^{\mathcal{Z}(\mathcal{A})}(C_3) \in \mathbb{C}$$

under the above assumptions on $\mathcal{A}$ (spherical and fusion). One gets around the need for a Drinfeld center $\mathcal{Z}$ upon using a unitary modular category $\mathcal{M}$ as base instead [TV13]:

$$Z_{\mathrm{TV}}^{\mathcal{M}}(C_3) = |Z_{\mathrm{RT}}^{\mathcal{M}}(C_3)|^2 \in \mathbb{R}.$$

The subsequent two subsections are essentially mutually independent.

5.2 State-Sum Construction $Z_{\mathbf{TV}}$ (sketch)

We try to convey the methodology of how one can - to *any* choice of spherical $\mathbb{C}$-linear fusion category $\mathcal{A}$ - associate a non-trivial 3-TQFT $Z_{\mathrm{TV}} : \mathbf{Cob}(3) \to \mathbf{Vect}_{\mathbb{C}}$. It is sufficient to be aware that

- "$\mathbb{C}$-linear" precisely guarantees that $\mathrm{Hom}_{\mathcal{A}}$-sets form $\mathbb{C}$-vector spaces,
- a **fusion category** is a monoidal category $(\mathcal{A}, \otimes, \mathbb{1} = \mathbb{C})$ together with a direct sum $\oplus$ under which we have a distinguished finite basis of **simple objects** $\{S_i\}_{i\in I}$ (with $\mathrm{card}(I) < \infty$), meaning that any object $X \in \mathrm{ob}\,\mathcal{A}$ can be decomposed like $A = \bigoplus S_j$ (S_j can appear more than once) which hence applies also to the **fusion** of two simples

$$S_i \otimes S_j = \bigoplus_k N_{ij}^k S_k$$

  where $N_{ij}^k \in \mathbb{N}_0$ is called **fusion coefficient**,
- **spherical** is a technical axiom on the structure of pivotality that we will pinpoint at upon usage.

[59]Commutativity isos $c_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V$ defined in Def. 32 without the need for twist or duality (hence only with condition Def. 32a).

Fix a spherical fusion category $\mathcal{A}$. We follow the "PL"-strategy (and notation) of Balsam and Kirillov Jr. [KB10] which is dual to the one of Turaev and Virelizier, the "skeleton"-strategy [TV17]. PL refers to a decomposition (in a cell-decomposition sense) of 3-manifolds - both with and without boundary. Without dwelling on those details, let me summarize by claiming that tetrahedra and generalizations with higher vertex count, are enough to assign invariants to in order to cover all of $\mathbf{Cob}(3)$ in a certain sense. So, having understood the concept of assigning a state sum $Z_{\mathrm{TV}}(T)$ to a regular tetrahedron $T \in \mathrm{Hom}_{\mathbf{Cob}(3)}$ covers an essential amount of the full construction presented in [KB10], in my opinion.

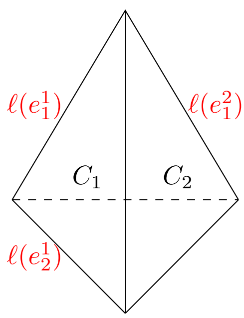
Sketch's Recipe

1. Choose a **simple colouring** / **labeling** ℓ of the *edges* (i.e. 1-cells) of T, which is no more than any map

   $$\ell : \{\text{edges}\} \to \{S_i\}_{i \in I}.$$

   So we deal with the following edge-coloured 3D object:

   $T =$

   $\in \mathrm{Hom}_{\mathbf{Cob}(3)}$

   where I included some simple colourings, and names of faces $C_\cdot$.
   To get rid of this choice in the end, we will "sum-it-out" like

   $$\bigoplus_{\text{simple labelings } \ell} \cdots$$

   where the sum is understood over iso-classes of labellings. Finiteness of the basis of simples guarantees these sums to be finite.

2. Construct a non-invariant $H(C_f, \ell) \in \mathrm{ob}\,\mathbf{Vect}_{\mathbb{C}}$ for each 2-cell of ∂T, i.e. $f = 1, \ldots \#\,\text{faces}(T)$. The vector spaces $\bigoplus_\ell H(C_f, \ell) =: H(C_f)$ will comprise all so-called **ground states** plus some **excited states**. The latter being the reason why $H(C_f)$ is not yet an invariant. This does not have to make sense right now.

3. The conceptual zenith of the construction comes now in assigning a linear map to T. Viewing T like $T \in \mathbf{Cob}(3)(\emptyset, \partial T)$, we can translate this to finding a *vector*

   $$H(T, \ell) \in H(\partial T, \ell) := \bigotimes_{\text{faces } C_f} H(C_f, \ell)$$

inside the tensor product space of the non-invariant spaces assigned to the faces C_f of T.
Note how this step will "use up" the whole category $\mathcal{A}$ without producing any infinite sums or similar problems!

4. The last step will, merely conceptually, globalize the above steps to a complete piecewise-linear ("PL" from above) manifold -glued together from such T- by some natural operations. This is done mainly to write down the final *state-sum* formula because we will need to tweak it later on in order to bridge to physics.

Step 1

Nothing more to do.

Step 2

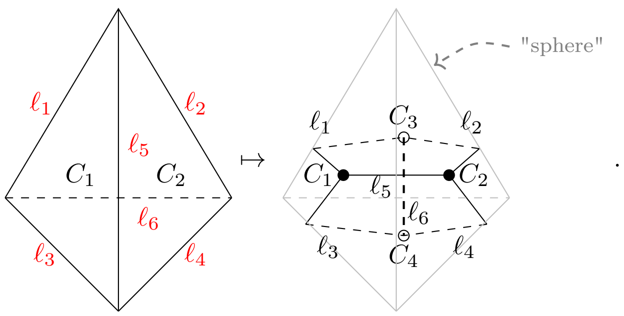
Set

$$H(C_f, \ell) := \mathcal{A}(\mathbb{C}, \ell_1 \otimes \ell_2 \otimes \ell_3)$$

where $\ell_1 := \ell(e_1^f), \ldots$ are ordered by inducing the given orientation of T on the edges $e_1^f, \ldots, e_3^f$ of the respective face $C_f \in \partial T$.[60]

Step 3: Glorious step

Simplifying notation, we write $\ell_1 = \ell(e_1^1), \ldots$ and mean arbitrary simple objects. First we take the 2-Poincaré dual of T, which produces a graph on a sphere like



In this way, such linear 3-manifolds always produce *non-intersecting*[61] graphs on S^2.

We aim to view this graph on S^2 as an element of $\mathcal{A}(\mathbb{C}, \mathbb{C})$. To this end, getting rid of the face labels as they were merely auxiliary, we planarize to a

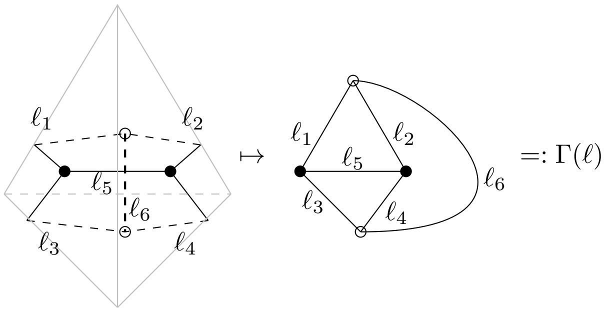
[60] You may object by noting that $1, 2, 3$ is a linear order but a 3-orientation can only induce a cyclic order on edges. Indeed, so here the first of two essential properties of the sphericality of $\mathcal{A}$ makes an appearance. It guarantees that there exist canonical (given by the chosen spherical structure and no further choices) linear isomorphisms

$$\mathcal{A}(\mathbb{C}, \ell_1 \otimes \ell_2 \otimes \ell_3) = \mathcal{A}(\mathbb{C}, \ell_3 \otimes \ell_1 \otimes \ell_2) = \mathcal{A}(\mathbb{C}, \ell_2 \otimes \ell_3 \otimes \ell_1) \in \mathrm{ob}\, \mathbf{Vect}_{\mathbb{C}}$$

which we take as identities.

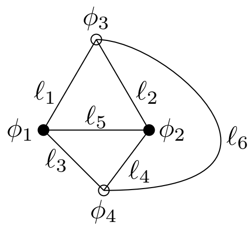
[61] Were there intersections, one would have to impose on $\mathcal{A}$ that it be *braided* - this can hardly be seen before (5.3).

graph on $\mathbb{R}^2$ like



Here, we chose to pull the ℓ_6 edge to the right which was arbitrary and will be commented on shortly. The aim is now to use all data from $\mathcal{A}$ (the more, the finer our invariants will be) in the last step where we interpret $\Gamma(\ell)$ as a morphism of $\mathcal{A}(\mathbb{C}, \mathbb{C})$. We insist on this particular interpretation because we intend to parallelize the construction from the ribbon chapter 4.1 that claimed that an assignment like {coloured isotopy classes of ribbon graphs} $\to$ {category morphisms} can *exist* (referring to well-definedness which is the property that translates into the *in*variants property). This connection can be seen as a big part of the proof of the state-sum's TQFT property because we now colour our vertices by morphisms of $\mathcal{A}$ like we did for the coupons back in section 4.1. But which morphisms exactly? If there is no canonical choice, *all* will always be distinguished. Intriguingly, we can by-pass the uncountable sum issue by virtue of the non-degenerate dual space pairing in finally fixing $H(F, \ell) \in H(\partial F, \ell) \overset{\text{Def}}{=} \bigotimes_{C_f} \mathcal{A}(\mathbb{C}, \ell_. \otimes \ell_. \otimes \ell_.)$ uniquely by

$$\mathbb{C} \ni \left\langle \underbrace{H(T, \ell)}_{\text{so defined}}, \phi_1 \otimes \cdots \otimes \phi_4 \right\rangle := \mathrm{ev}_{\mathcal{A}} \left(\quad \right) \tag{5.3}$$



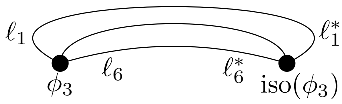
for all $\phi_.$ in the correct Hom-sets of $\mathcal{A}$. For example,

$$\phi_3 \in H(C_3, \ell)^* \overset{\text{Def}}{=} \mathcal{A}(\mathbb{C}, \ell_1 \otimes \ell_2 \otimes \ell_6)^* \cong \mathcal{A}(\mathbb{C}, \ell_6^* \otimes \ell_2^* \otimes \ell_1^*).$$

Notable subtleties:

- We have identified $\mathbb{C} = \mathcal{A}(\mathbb{C}, \mathbb{C})$.
- The orientation of $T \in \mathrm{Hom}_{\mathbf{Cob}}$ induces an orientation on T's edges, i.e. a direction, which induces a direction on the dual edges, which fixes our objects as in- or outgoing for the morphisms $\phi_.$. Depending on the direction and convention, this determines whether in the upper example we can actually write ℓ_1 or have to write ℓ_1^*. Similar subtleties arose in the ribbon section 4.1.

- The side to which we pulled over ℓ_6 in order to not create intersections[62] was arbitrary and sphericality of $\mathcal{A}$ guarantees -almost by definition- that "$\mathrm{ev}_{\mathcal{A}}$" is invariant under the two choices.
- The last iso in the above example can be seen graphically:



  where we might have to put an extra $*$ on all lines (and use $\ell_.^{**} = \ell_.$ b.m.o. pivotality) but the important message is that one morphism/vertex has the line as in-going and the other one as outgoing.

Step 4: Globalizing

We, like [KB10], are only concerned with PL manifolds $B_3 \in \mathrm{Hom}_{\mathbf{Cob}(3)}$. So B_3 is glued together from (linearly generalized) tetrahedra prescribed by a decomposition

$$\Delta := \underbrace{\Delta_3}_{\text{linear 3-cells}} \supset \underbrace{\Delta_2}_{\text{linear faces}} \supset \underbrace{\Delta_1}_{\text{linear edges}} .$$

Denote also by $\partial\Delta$ the subset of Δ_2 comprising ∂B_3. Step 3 provided a linear map/vector $H(T,\ell) \in H(\partial T, \ell)$ for $T \in \Delta_3$ where T might be a literal tetrahedra T as before or some linear generalization with more vertices. In such decompositions, every face $C \not\subset \partial\Delta$ has an adjacent partner $\overline{C}$ that it gets glued to and that carries the opposite orientation. Recalling $H(C,\ell) \cong H(\overline{C},\ell^*)^* \in \mathrm{ob}\,\mathbf{Vect}_{\mathbb{C}}$, the obvious extension of $H(T,\ell)$ to $H(B_3,\ell)$ will pair these mutually dual vector spaces. The faces of $\partial\Delta$ will be tensored over (to make the final functor monoidal). We set

$$H(B_3,\Delta,\ell) := \underset{\Delta_2\cap\mathrm{int}(B_3)}{\mathrm{pair}} \left(\bigotimes_{T\in\Delta_3} H(T,\ell) \right) \in \bigotimes_{C\in\partial\Delta} H(C,\ell) =: H(\partial B_3,\Delta,\ell).$$

It should not come as a surprise that $H(B_3,\Delta,\ell)$ is not yet the value of a TQFT as it depends on the neither topological nor orientational choice ℓ. We need to "sum-ℓ-out". This includes some well chosen weight factors which exist only to guarantee its invariance under choices of different partitions Δ: The following is called the **Turaev-Viro state sum**[63] [KB10]

$$Z_{\mathrm{TV}}(B_3,\Delta) := \mathcal{D}^{-2v(\Delta)} \sum_{\ell} H(B_3,\Delta,\ell) \prod_{e\in\Delta_1} \dim_{\mathcal{A}}(\ell(e))^{n_e}. \tag{5.4}$$
×3

[62] Remember that we have not assumed a braiding, so intersections would be rather difficult to interpret in "$\mathrm{ev}_{\mathcal{A}}$".

[63] Here, ℓ runs over all non-isomorphic simple labelings as mentioned before, $\mathcal{D} := \sqrt{\sum_{i\in I} \dim_{\mathcal{A}}(S_i)^2} \in \mathbb{C}^\times$ is $\mathcal{A}$'s *global dimension* and $n_e = 1$ if $e \in \mathrm{int}\,B_3$ or $n_e = 1/2$ if $e \in \partial B_3$. That "$\dim_{\mathcal{A}}$" exists, is by virtue of sphericality and all logarithms for the complex square roots have to be chosen and kept fixed.

It *is* a topological (and orientational) invariant linear map[64] between vector spaces modulo a tiny grain of salt: The vector spaces sitting at the boundaries of the B_3's are yet too big, they contain more than only ground states. In rigorous terms, for Δ another partition of B_3 one can prove [KB10],

$$\partial\Delta = \partial\Delta' \implies Z_{\mathrm{TV}}(B_3, \Delta) = Z_{\mathrm{TV}}(B_3, \Delta')$$

which is remarkable. Different decompositions of int B_3 give the same $Z_{\mathrm{TV}}(B_3, \Delta)$. Hence, we write $Z_{\mathrm{TV}}(B_3, \partial\Delta)$ for it. In state parlance, the above result could be re-translated like so:

$$\text{boundaries contain same states} \implies \text{propagator is full invariant.}$$

For a linear map such as the propagator, this is as close to topologicality as we can get for a non-topological domain. We now search for the correct domain. The aim is to find the maximal set of states that only depend on the topology of C_2 - the *ground states.* We perform a cheap but elegant trick to remove all excited states from $Z_{\mathrm{TV}}(C_2, \Delta) \in \mathrm{ob}\,\mathbf{Vect}_{\mathbb{C}}$ in setting

$$Z_{\mathrm{TV}}(C_2) := \mathrm{im}(Z_{\mathrm{TV}}(C_2 \times I, \partial\Delta)) \in \mathrm{ob}\,\mathbf{Vect}_{\mathbb{C}}$$

as the space of **ground states on** $C_2 \in \mathrm{ob}\,\mathbf{Cob}(3)$. Different boundary decompositions induce canonical isomorphisms [KB10] on this vector space. By virtue of this isomorphism result, we interpret our propagator as a killer of excited states. His hostility against excited states stems from his own topologicality - either project away the excited states or be non-topological.

The Δ-invariance of the ground states now lifts to an invariance of propagators simply by restricting the state-sum $Z_{\mathrm{TV}}(B_3, \Delta) = Z_{\mathrm{TV}}(B_3, \partial\Delta)$ to the ground states. (Physically this is immediate as the Hamiltonian commutes with itself, but if one prefers to treat "ground states" as a mere definition, a quick check is needed.) Finally - referring to [KB10] for proofs of further properties such as the gluing axiom a.k.a. functoriality - we summarize:

Thm. 37 (State-Sum 3-TQFT)**.** *The following well-defined assignments comprise a 3D* $\mathbb{C}$*-linear TQFT,*

$$\begin{aligned} Z_{TV} : \mathbf{Cob}(3) &\longrightarrow \mathbf{Vect}_{\mathbb{C}} \\ C_2 &\longmapsto \mathrm{im}\left(Z_{TV}(C_2 \times I, \partial\Delta)\right) \overset{Def}{=} Z_{TV}(C_2) \\ \left(C_2 \overset{B_3}{\to} C_2'\right) &\longmapsto Z_{TV}(B_3, \partial\Delta)|_{Z_{TV}(C_2)} =: Z_{TV}(B_3) \end{aligned}$$

where $Z_{TV}(B_3, \Delta) = Z_{TV}(B_3, \partial\Delta)$ *is the state-sum defined in (5.4) for which one has to choose a PL decomposition* Δ *of* B_3*.*

[64] Here, we refer to (5.4) - although it is merely a vector - as a linear map between vector spaces of dimensions possibly bigger than 1. This is by virtue of taking the following canonical linear isomorphism as an equality:

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, V_{C_2} \otimes V^*_{C_2'}) &\cong \mathrm{Hom}_{\mathbb{C}}(V_{C_2'}, V_{C_2}) \\ \phi &\mapsto (v \mapsto \phi(1)(v))\,. \end{aligned}$$

Here V_M is an arbitrary vector space and the indices should remind of boundary decomposition like $\partial B_3 = \overline{C_2'} \coprod C_2'$.

5.3 Surgery Construction $Z_{\mathbf{RT}}$ (sketch)

Fix a modular $\mathbb{C}$-linear category $\mathcal{M}$ (cf. Def. 36) and be sure to skim appendix D.
The power of the Z_{RT} construction is that it is Witten's generalization of 3D CS-theory in the bulk and 2D WZW-theory on the boundary. To include Witten's results [Wit89], and as already mentioned there, a mere topological theory in the sense of Atiyah, i.e. Def. 1, is *not* flexible enough. Here is a list of reasons:

- A CFT on some C_2 as space-time needs *point-like* insertions. A non-extended 3-TQFT, however, does not include values on any points. Even if we take it fully extended, there is only one value on a point for each connected component of C_2. There is no way to encode, e.g., non-trivial OPEs in a bulk TQFT of Atiyah's sense.
- The Jones Polynomial of a link $L = \coprod_{i=1}^{r} K_i$ on r knot components is obtained in [Wit89] by path-integrating over r framed Wilson-line insertions $W_{R_i}(K_i)$ inside some B_3: These are coloured by prescribed linear representations R_i of CS-theory's gauge group G and are 1D "particles" inserted along their knot $K_i \subset L$. So they produce an extra ribbon graph *inside* of our cobordism B_3, as explained in appendix D. Atiyah's TQFT knows *nothing* about such extra structure within cobordisms!

The present exposition is indebted to Turaev's overwhelmingly detailed book [Tur10]. Because it would be superfluous to repeat his work, here we aim to find a much more direct path in order to appreciate the construction, the new structures and their physical relevance. Throughout, we will point to the locations in [Tur10] for the reader that desires higher precision. If notations differ, hopefully some footnote will guide the way.

Notation: As usual we use C_3, C_2, B_3 as defined in Not. 2. And although the upcoming construction will *not* produce a TQFT in Atiyah's sense (Def. 1), we will keep notation "Z" and the name "TQFT", as is customary. The precise meaning of it can be found in [Tur10, §III], and the shortcomings are elaborated on at the end of this section.

Step 0: Dehn surgery in a nutshell

Usage of (Dehn) surgery was already put forward in [Wit89] to tackle the problem of finding a map[65]

$$\{\text{closed conn. oriented 3-MF}\} \xrightarrow{?} \{\text{framed links}\}$$
$$C_3 \longmapsto (L_{C_3}, \overline{n}) \equiv \coprod_i (K_i, n_i).$$

The reverse direction is the following construction of **Dehn surgery**:

- Fix a framed link $(L, \overline{n}) = (K, n)$ -for simplicity only one knot component- and the ambient space, to be done surgery *on*, to be $\mathbb{R}^3 \cup \{\infty\} = S^3$.

[65] Recall from appendix D that a framing of each knot component is precisely given by one integer n, the winding number. $\overline{n}$ denotes a multi-index.

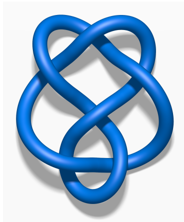


Figure 7: A tubular thickened knot, i.e. the image of a continuous embedding of the solid torus $D^2 \times S^1$.

- We denote by $\operatorname{tube}(K)$ the **tubular thickening** of K. Just like a solid torus is a tubular thickened circle. Figure 7 shows an example of a blue tubular thickened knot.
- Now drill out $\operatorname{tube}(K)$ from S^3 (lively writing for considering $S^3 \setminus \operatorname{tube}(K)$) and glue it back in. This reads like a Sisyphus-move, but there is a slight subtlety which is simultaneously the core idea: *How* to glue it back in? I.e. along which homeomorphism

  $$\mathrm{glue}\colon \partial \operatorname{tube}(K) \xrightarrow{\sim} \partial\left(S^3 \setminus \operatorname{tube}(K)\right) \quad ?$$

  The answer has been fixed in the beginning: It is prescribed by n. The precise prescription depends on conventions. Let it suffice here to think of n as the number of times that the one generator of $\pi_1(\partial \operatorname{tube}(K))$ that trivializes in the solid $\operatorname{tube}(K)$ (i.e. the "meridinal" one) is wrapped around the corresponding generator in $\partial(S^3 \setminus \operatorname{tube}(K))$ during the gluing process.[66]
- We have now described the surgery mapping[67]

  $$\{\text{framed links}\} \longrightarrow \{\text{closed conn. oriented 3-MF}\}.$$

The TQFT needs the inverse which does not exist. In fact, we only have surjectivity (up to isos) as is long known due to Lickorish and Wallace. Injectivity fails, however, as various different framed links can yield the same (orientation-preserving) homeomorphism class of C_3's. By definition, such links are **Kirby-equivalent**. It has been shown by Kirby [Kir78] that there exists a set of **Kirby-moves** s.th. there is a bijection

$$\begin{aligned} \mathrm{surg}^{-1}\colon \{\text{closed } C_3 \in \operatorname{ob}\mathbf{Cob}(3)\} &\xrightarrow{\sim} \{\text{framed links}\}/\langle\text{Kirby-moves}\rangle \\ C_3 &\longmapsto [L_{C_3}, \overline{n}] \equiv \left[\coprod_i (K_i, n_i)\right]. \end{aligned} \tag*{(5.5)$_{\times 2}$}$$

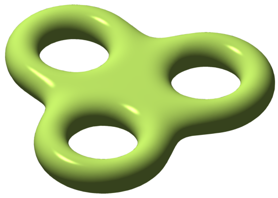
Here we emphasize that we merely get a *family* of links. It is the work of Reshetikhin and Turaev [Tur10, §II.3] to show that the TQFT construction is independent of the choice of family member, i.e. Kirby-invariant. In fact, this is one of the biggest chunks of proof in their work.

[66] E.g. [Wik]: Denote $\pi_1(\partial(S^3 \setminus \operatorname{tube}(K)))$'s meridinal and longitudinal generators by γ_m and γ_l, respectively. Now, using only n, we choose $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, and map the meridinal generator of $\pi_1(\partial \operatorname{tube}(K))$ to $\gamma_m^a \gamma_l^b$.

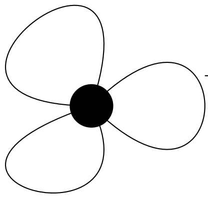
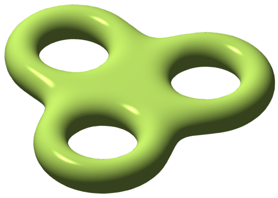
[67] For a different explanation, see [Tur10, §II.2.1].

Step 1A: $Z_{\mathbf{RT}}$ on C_2 without insertions

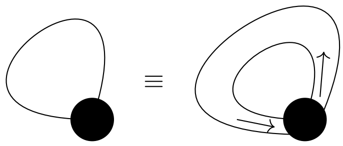
If one is merely interested in the space of states, one can leap directly to (5.7). The rest is for Step 2A. We construct a **standard handlebody** $H_{(g,\emptyset)}$ of genus $g = g(C_2)$, the surface C_2's genus. E.g. $H_{(3,\emptyset)}$ consists of the solid 3-manifold



-oriented arbitrarily[68] , yet fixed- together with an embedded -sort of canonical-directed ribbon graph of one coupon/vertex, i.e.:

$H_{(3,\emptyset)} := \mathrm{im}\Big($  $\xrightarrow[\text{inside}]{\text{embed}}$  $\Big)$ (5.6)×3

where the 3 loops mean annuli directed from right to left and showing its white side (the up side); e.g. the top left one:



Now, if $g(C_2) = 3$ and connected, we define its space of states as all possible simple colourings of the vertex $\bullet$ in (5.6):

$$Z_{\mathrm{RT}}(C_2) := \bigoplus_{i_k \text{ simple}} \mathrm{Hom}_{\mathcal{M}}\left(\mathbb{C}, \bigotimes_{k=1}^{3} i_k \otimes i_k^*\right). \tag{5.7}$$

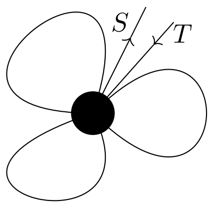
Step 1B: $Z_{\mathbf{RT}}$ on (C_2, ins) with insertions ins

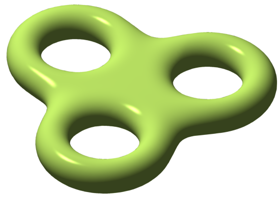
If one is merely interested in the space of states, one can leap directly to (5.10). The rest is for **Step 2B**. CFT bulk fields carry representations of the conformal algebra. We implement this property by colouring. Also, we get a sign for each insertion depending on whether the bulk field transforms in the holomorphic or anti-holomorphic algebra (left- or right-mover, respectively). So if ins consists of $\#\,\mathrm{ins} \in \mathbb{N}$ **insertions**, this means

$$\mathrm{ins} \in (\{\text{simples of } \mathcal{M}\} \times \{\pm 1\})^{\times(\#\,\mathrm{ins})}. \tag{5.8}$$

Now we need a handlebody that is able to carry this extra information, $H_{(g,\mathrm{ins})}$. E.g. $H_{(3,((S,-1),(T,+1)))}$ for $S, T \in \mathrm{ob}\,\mathcal{M}$ simples is defined as

[68] Turaev chooses the right-handed orientation in [Tur10, §IV.1.2]

im $\Bigg($ 

$\xrightarrow{\text{embed}}$  $\Bigg)$ (5.9)×2

where, as in (5.6), lines are to be read as untwisted ribbons, directed as depicted for S and T and as in (5.6) for the loops. Instead of "embed inside", we now only "embed" because the 2 open ends of the S and T lines should meet the green boundary, and hence **mark** its bounding surface by 2 $\mathcal{M}$-coloured arcs (at some, yet fixed, position). "Arcs" instead of points because, remember, lines represent (untwisted) ribbons. Set[69]

$$Z_{\mathrm{RT}}(C_2, \underbrace{((S,-),(T,+))}_{\text{ins}}) := \bigoplus_{i_k \text{ simple}} \mathrm{Hom}_{\mathcal{M}}\left(\mathbb{C}, S^* \otimes T \otimes \bigotimes_{k=1}^{3} i_k \otimes i_k^*\right). \tag{5.10}$$

×1

Step 2A: $Z_{\mathbf{RT}}$ on C_3 without extra ribbon graph inside

The composition of the following chain of assignments is defined to be Z_{RT} on closed cobordisms, i.e. their partition function.

$$C_3 \xmapsto[(5.5)]{\mathrm{surg}^{-1}} [L_{C_3}, \overline{n}] \equiv \left[\coprod_{i=1}^{r} (K_i, n_i)\right] \tag{5.11}$$

$$\xmapsto[(D.1)]{\text{can.}} \mathrm{Rib}(L_{C_3}, \overline{n}) =: R_{C_3} \tag{5.12}$$

where we chose[70] some $(L_{C_3}, \overline{n}) \in [L_{C_3}, \overline{n}]$. Now, if we were to choose a simple labeling $\ell : \{K_i\}_i \to \{i\}_{i \text{ simple}}$ to colour R_{C_3} by, we could directly use Thm. 34's functor $\mathrm{G2M}_{\mathcal{M}}$ like $\mathrm{col}_\ell(R_{C_3}) \mapsto \mathrm{G2M}_{\mathcal{M}}(\mathrm{col}_\ell(R_{C_3})) \in \mathbb{C}$. But we don't want to choose some ℓ (not topological), so we again *sum-it-out* with some weight factor $\dim(\ell) := \prod_{i=1}^{r} \dim_{\mathcal{M}}(\ell(K_i))$:[71]

$$R_{C_3} \longmapsto \sum_{\ell} \dim(\ell) \cdot \mathrm{G2M}_{\mathcal{M}}(\mathrm{col}_\ell\, R_{C_3}) =: \{L_{C_3}\} \tag{5.13}$$

×1

$$\longmapsto \Delta^{\sigma(L_{C_3})} \mathcal{D}^{-\sigma(L_{C_3})-r-1} \{L_{C_3}\} =: Z_{\mathrm{RT}}(C_3) \in \mathbb{C} \tag{5.14}$$

×1

The following steps will all essentially modify their setup to recycle this step.

[69] Translating to "Ψ_t" or "$\mathcal{T}(C_2)$" in [Tur10, §IV.1.4]. The handlebody is called "Σ_t" in [Tur10, §IV.1.2]. His presentation is different, so one might profit from a glance into the book.

[70] See end of **Step 0** if this troubles you.

[71] $\mathcal{D} = \mathcal{D}_{\mathcal{M}} \in \mathbb{C}^\times$ was defined already in (5.4), $\Delta = \Delta_{\mathcal{M}} \in \mathbb{C}^\times$ is a "framed version" of $\mathcal{D}$ defined in [Tur10, §II.0.6], σ : {framed link} $\to \{\pm 1\}$ is some signature defined in [Tur10, §II.2.1] and r the number of knot components.

Step 2B: $Z_{\mathbf{RT}}$ on (C_3, Ω) with ribbon graph Ω inside

Now we prescribe a value that is absolutely non-Atiyah's-TQFT-like. It is diligent work motivated by Witten's results [Wit89]. Using the Rib map (D.1), a bunch of mutually entangled framed Wilson lines $\coprod_{i=1}^{r}(W_{R_i}(K_i), n_i)$, that Witten path-integrated over, are easily translated into a coloured directed (0,0)-ribbon graph

$$\Omega =: \mathrm{col}_{\ell_\Omega}(\dot{\Omega})$$

with simple colouring ℓ_Ω. Now we smuggle in the extra ribbon graph Ω and proceed as in **Step 2A** in order to define $Z_{\mathrm{RT}}(C_3, \Omega) \in \mathbb{C}$:

$$\begin{aligned} &(C_3, \Omega) \xmapsto{(5.12)\circ(5.11)\times\mathrm{id}} (R_{C_3}, \Omega) \xmapsto{\text{untangled union}} R_{C_3} \coprod \Omega \\ &\quad\mapsto \Delta^{\sigma(L_{C_3})}\mathcal{D}^{-\sigma(L_{C_3})-r-1} \sum_{\ell} \dim(\ell)\, \mathrm{G2M}_{\mathcal{M}} \left(\mathrm{col}_{\ell \coprod \ell_\Omega} R_{C_3} \coprod \Omega\right) \end{aligned} \tag{5.15}$$

where the last step looks clumsy but is in analogy to (5.14) $\circ$ (5.13).

Step 3A: $Z_{\mathbf{RT}}$ on B_3 with no insertions on ∂B_3 or ribbons inside

Take ∂B_3 connected for simplicity, and choose it as in-going.
The aim is to glue in $H_{(g,\emptyset)}$ with $g := g(\partial B_3)$ from **Step 1A** to obtain something closed with ribbon graph inside, and then use (5.15) from **Step 2B**.

To glue, we *have* to choose a **parametrization**, i.e. an orientation-preserving homeomorphism

$$\mathrm{par}\colon \partial H_{(g(\partial B_3),\emptyset)} \xrightarrow{\sim} \partial B_3$$

along which we glue domain and co-domain together to form a closed 3-manifold. Recall that the handlebody comes with an uncoloured ribbon graph, denote it $\dot{\Omega}_g$. Set[72]

$$\begin{aligned} Z_{\mathrm{RT}}(B_3; \mathrm{par})\colon Z_{\mathrm{RT}}(\partial B_3) &\longrightarrow \mathbb{C}, \\ \ell &\longmapsto Z_{\mathrm{RT}}\left(B_3 \coprod_{\mathrm{par}} H_{(g,\emptyset)}, \mathrm{col}_\ell\left(\dot{\Omega}_g\right)\right) \overset{\mathbf{2B}}{\in} \mathbb{C} \end{aligned}$$

as $\mathbb{C}$-linear map, the *propagator*. It depends non-trivially on the chosen par.

Step 3B: $Z_{\mathbf{RT}}$ on $(B_3, \mathrm{ins}, \Omega)$ with both insertions and ribbon graph

Take, again, ∂B_3 connected for simplicity, and choose it as in-going.
The insertion data ins is as before in (5.8). As probably expected, we now use the handlebody $H_{(g(\partial B_3),\mathrm{ins})}$. Our new kind of parametrization

$$\mathrm{par}_{\partial\Omega}\colon H_{(g(\partial B_3),\mathrm{ins})} \xrightarrow{\sim} \partial B_3$$

generates an exciting subtlety. Note how this homeomorphism determines the *space time location* plus tangential direction of each insertion w.r.t. our reference surface $\partial H_{(\dots)}$.[73] It does this because $H_{(g(\partial B_3),\mathrm{ins})}$ came with $\#\,\mathrm{ins} \in \mathbb{N}$

[72] Recall that a state $\ell \in Z_{\mathrm{RT}}(\partial B_3, \mathrm{par}) \equiv \bigoplus_{i_k} \mathrm{Hom}_{\mathcal{M}}\left(\mathbb{C}, \bigotimes_{k=1}^{g} i_k \otimes i_k^*\right)$ is precisely one full simple colouring of $\dot{\Omega}_g$ according to (5.7).

[73] Additionally, and only noted because it is emphasized at various points in [Tur10], the total ordering of the arcs on $H_{(\dots,\mathrm{ins})}$ -induced by the canonical total order of (5.8)- induces an *order* on our insertions. Physically this has no apparent interpretation, so is a "technicality". In the "extended" setup, as discussed later, Turaev gets rid of this order.

marked arcs on its surface. Those free ends of ribbons in $H_{(\dots)}$ were created to now be attached to free ends in our ribbon graph Ω. So, contrary to **Step 2B**, Ω is a coloured, directed[74] $(\#\,\mathrm{ins},0)$-ribbon graph, i.e. it has to have the same number of free ends as we specify insertions. Hence the extra index "$\partial\Omega$" in $\mathrm{par}_{\partial\Omega}$. This reminds us that the homeomorphism has to send the arcs on $\partial H_{(g(\partial B_3),\mathrm{ins})}$ orientation-preserving homeomorphically to the free ends of Ω. (The "0" stems from our simplifying assumption.)
Recall our notation for the given $\Omega \equiv \mathrm{col}_{\ell_\Omega}(\dot{\Omega})$ and remember that the uncoloured ribbon graph Ω_g inside the handlebody can be glued to Ω along $\mathrm{par}_{\partial\Omega}$ to give a $(0,0)$-ribbon graph - and indeed, those we know how to handle:[75]

$$\begin{aligned} Z_{\mathrm{RT}}(B_3,\mathrm{ins},\Omega;\mathrm{par}_{\partial\Omega}) : Z_{\mathrm{RT}}(\partial B_3,\mathrm{ins}) &\longrightarrow \mathbb{C}, \\ \ell \longmapsto Z_{\mathrm{RT}}\left(B_3 \coprod_{\mathrm{par}_{\partial\Omega}} H_{(g,\mathrm{ins})}, \mathrm{col}_{\ell \coprod \ell_\Omega}\left(\Omega_g \coprod_{\mathrm{par}_{\partial\Omega}} \dot{\Omega}\right)\right) &\overset{\mathbf{2B}}{\in} \mathbb{C} \end{aligned}$$

Final comments

- It is very peculiar to demand such parametrizations $\mathrm{par} : \partial H_{(\dots)} \xrightarrow{\sim} \partial B_3$ as a TQFT.[76] However, one cannot immediately infer that this is a *shortcoming* of Z_{RT}. Recall that we will desire to specify locations of insertions, for example to read-off OPEs. Still, it is weird, and one would probably be very keen to know *how* much Z_{RT} depends on it, i.e.

$$Z_{\mathrm{RT}}(B_3,\mathrm{ins},\Omega;\mathrm{par}_{\partial\Omega}) \overset{?}{\neq} Z_{\mathrm{RT}}(B_3,\mathrm{ins},\Omega;\overline{\mathrm{par}}_{\partial\Omega}). \tag{5.16}$$

  Turaev has found that a choice of *Lagrangian subspace* $\lambda \subset H_1(\partial B_3;\mathbb{R})$ of the first homology, which is much weaker structure -and much more topological- is sufficient. This means that (5.16) becomes[77] an equality if $(\mathrm{par}_{\partial\Omega})_*(\lambda) \overset{(*)}{=} (\overline{\mathrm{par}}_{\partial\Omega})_*(\lambda)$ using the degenerate mapping

$$\begin{aligned} \{\text{parametrization } \mathrm{par}_{\partial\Omega}\} &\longrightarrow \{\text{Lagrangian subspace}\} \\ \mathrm{par}_{\partial\Omega} &\longmapsto (\mathrm{par}_{\partial\Omega})_*(\lambda) \end{aligned}$$

  that pushes a canonical Lagrangian subspace on the handlebody[78] forward along the parametrization. In particular, homotopic parametrizations leave Z_{RT} invariant.
  He then defined a different TQFT[79] taking such λ's instead of par's.

[74] The direction and colouring of each free end needs to match the handlebody's.
[75] This definition is the "operator invariant" τ in [Tur10, §IV.1.8]. Insertions and parametrization are encoded in his "decorated 3-manifolds", defined in [Tur10, §IV.1.6].
[76] Parametrizations are quite un-*topological*. Maybe even the exact opposite.
[77] Although not explicitly specified, this can be deduced from [Tur10, Lemma IV.6.5.1]: Translating to his notation, we make the parametrization in $\check{M} =: (M,\mathrm{par}_{\partial\Omega})$ notationally explicit to compare to $(M,\overline{\mathrm{par}}_{\partial\Omega})$. We show $\tau(M,\mathrm{par}_{\partial\Omega}) = \tau(M,\overline{\mathrm{par}}_{\partial\Omega})$. Using the mentioned Lemma, we get

$$k(M,\mathrm{par}_{\partial\Omega}) \cdot (\mathrm{par}_{\partial\Omega})_\# \circ \tau(M,\mathrm{par}_{\partial\Omega}) = k(M,\overline{\mathrm{par}}_{\partial\Omega}) \cdot (\overline{\mathrm{par}}_{\partial\Omega})_\# \circ \tau(M,\overline{\mathrm{par}}_{\partial\Omega})$$

and now the k's agree (and are non-zero) due to our assumption (*). Also, by definition in [Tur10, §IV.6.3], the isos $(\overline{\mathrm{par}}_{\partial\Omega})_\# = (\mathrm{par}_{\partial\Omega})_\#$ tautologically without the need for (*).
[78] Defined in [Tur10, §IV.4.2] as spanned by the generator loops on the handlebody's surface that trivialize in the solid handlebody.
[79] See [Tur10, §IV.6] where he defines this "renormalized" TQFT $(\mathcal{T}^e,\tau^e)$ and the corresponding structures on the surfaces, called "extended".

- Both our Z_{RT} and the mentioned $(\mathcal{T}^e, \tau^e)$ do *not obey the gluing law*, i.e. functoriality. Instead, if B_3 is cut along a par'ed C_2 into B_3^+ and B_3^-, then in general

$$Z_{\mathrm{RT}}(B_3) = k \cdot Z_{\mathrm{RT}}(B_3^+) \circ Z_{\mathrm{RT}}(B_3^-)$$

holds for some $k \in \mathbb{C}^\times$ but we might find $k \neq 1$ which is then dubbed a **gluing anomaly**. These anomalies get killed by another "renormalization" in [Tur10, §IV.9], but this is not of fundamental importance. One reason is that k above can quickly be computed, as shown in [Tur10, §IV.4].

- Judging by the above points, Z_{RT} is definitely not an Atiyah-like TQFT. Nevertheless, lots of good properties stay put. For example, only the boundary of B_3 has to be parametrized. Hence, if B_3 and B_3' are homeomorphic with the same par'ed boundary, then the **naturality axiom** of Turaev's TQFTs from [Tur10, §III.1.4][80] implies

$$Z_{\mathrm{RT}}(B_3; \mathrm{par}) = Z_{\mathrm{RT}}(B_3'; \mathrm{par}).$$

In particular

$$C_3 \cong C_3' \implies Z_{\mathrm{RT}}(C_3) = Z_{\mathrm{RT}}(C_3').$$

So we do get *in*variants! For further properties, cf. also [Tur10, §III.2].

6 Z_{RT} in the Physics of Rational CFTs

Having been constructed *in order to* mimic physics of CFT, it comes as no shock that CFT results and calculations can be translated into the rigorous Z_{RT} framework. However, it is remarkable that this TQFT can encode *full* 2D CFTs, meaning a set of conformal blocks necessary to build all possible correlators. Thorough mathematical research has established this result for CFTs with central charges $c, \bar{c} \in \mathbb{Q}$ in a sequence of papers: [FRS02a; FRS02b; FRS04a; FRS04b; FRS05; FFRS06]. We will not summarize the cited work. Instead, the following list expresses the intentions of the present section:

- Construct $(B_3, \mathrm{ins}, \Omega)$ and[81] $(\partial B_3, \mathrm{ins})$ that produce prescribed correlators upon feeding them into Z_{RT}.

- List the algebraic structures and operations that are sufficient to model physical objects and operations.

6.1 Constructing the Cobordism of a Given Correlator

The following only aims to give an idea about how to use Z_{RT} in physics, because it is exciting to build an n-point function with ribbon graphs in 3-manifolds. If desired, high precision can be found in the cited papers.
Before starting, we try to answer the following urgent question such that it does not block the physically inclined mind: *Why* are these constructions giving the claimed physical objects? Well, how is a "physical" object defined? Usually -if

[80] Proven in [Tur10, §IV.2] for the presented Z_{RT} called "$(\mathcal{T}, \tau)$".
[81] The notation should remind of subsection 5.3.

not always- it just is not, only its mathematical model is. Sometimes even the latter has no precise meaning, like some path integrals. However, one usually has at least partial rigorous mathematical description including certain rigorous implications. In CFT for example, we demand that the modes of a CFT's energy-momentum tensor represent a Virasoro algebra. Some of these properties are put into the "TFT construction of RCFT" by hand. Those that are not, have to be *shown* to hold under the rules of Z_{RT}.[82] Examples include:

- Partition functions furnish "NIM-representations". [FRS02b]
- Partition functions are modular invariant. [FRS02b]
- Correlators sit in the trivial representation of the relative modular group. [FFRS06]
- Bulk (and boundary) fields obey factorization constraints". [FFRS06]

Conformal blocks

When thinking of a correlator/n-point function, one usually has a function of n space-time points in mind, like

$$\langle \phi_1(x)\phi_2(y)\rangle = \log|x-y|$$

but the astonishing restrictions of 2D conformal symmetry enable us to encode such information in a vector of the vector space of conformal blocks.[83] We now build a B_3 with ribbon graph inside, which can be fed into Z_{RT}. Then, we will without proof take the word of Fuchs, Runkel, Schweigert et al. that this gives us precisely a specified n-point function of primaries in the form of this vector from above.
To this end, we summarize what information a physicist provides when asking for such n-point functions:

a. A chiral algebra[84] $\mathcal{V}$ underlying the type[85] of CFT A.

b. An oriented connected space time manifold C_2 together with n marked arcs at the locations of insertions. Each arc is specified by one point and a tangential direction.[86]

c. The insertion data for each marked arc, given by a conformal weight h, an anti-conformal weight $\overline{h}$, and the specific kind[87] of field.

[82] Note that the cited authors do not work with $Z_{\mathrm{RT}} \equiv (\mathcal{T}, \tau)$, but the "extended" version "$(\mathcal{T}^e, \tau^e)$". It is my own work and decision to translate their setup to our Z_{RT}. There will be footnotes arguing why certain choices are in line with their choices. However, getting all signs right was no easy endeavour. Please do not rely completely on them, but focus on the coarse idea.

[83] See e.g. [FRS05, §5] for a translation between the blocks and functions of space-time.

[84] E.g. in a WZW-model "chiral algebra" means the affine Lie algebra. In general, the vertex operator algebra is meant, which colloquially speaking consists of the conformal algebra together with other symmetries of the theory that do not spoil the conformal algebra.

[85] E.g., see [DFMS97, §17.7] for some "types" of CFTs in WZW models.

[86] The direction is a reminiscence of the local coordinates and can be motivated by the need to choose a complex logarithm around insertion points because conformal weights may be rational. We need these directions in order to frame our insertion-knots.

[87] E.g. the vacuum field 1, or for the Ising model maybe the spin-field σ.

This we translate 1:1 into

a. The base modular[88] category plus an object:

$$\boxed{\mathcal{M} := \mathrm{Rep}(\mathcal{V}) \quad \& \quad A \in \mathrm{ob}\,\mathcal{M}.} \tag{6.1}$$

The OPEs, the vacuum and some properties of the given CFT type provide several morphisms $\mu, \eta, \Delta, \epsilon$ on (tensor products of) A. For now it is sufficient to remember that these are fully specified by the physicist.

b. An isomorphism (= orientation preserving homeomorphism)

$$\mathrm{par}_{1/2} \colon \partial H_{(g(C_2),n)} \xrightarrow{\sim} C_2$$

that we *choose* subject to two conditions:

- Each marked arc needs to isomorphically get sent to one marked arc.
- Longitudinal fundamental loops (the loops parallel to the ribbon loops in (5.9)) need to get mapped to longitudinals again.[89] We demand this for consistency with Turaev's and FRS's setup.[90]

c. The insertion data[91]

$$\begin{aligned} \mathrm{ins} &:= \mathrm{ins}_{1/2} \coprod \overline{\mathrm{ins}_{1/2}} \qquad \text{for} \\ \mathrm{ins}_{1/2} &:= ((U_{h_i}, +1))_{i=1}^{n} \\ \overline{\mathrm{ins}_{1/2}} &:= ((U_{\overline{h_i}}, +1))_{i=1}^{n} \end{aligned}$$

where U_h is an irreducible highest weight h representation of $\mathcal{V}$, i.e. a simple object of $\mathcal{M}$. The "kind of field" of index i translates into one morphism in $\mathrm{Hom}_{\mathcal{M}}(U_{h_i} \otimes A, U^{\vee}_{\overline{h_i}})$.

We now present the construction from [FRS02a] in the simplified setup of connected C_2 (and $\partial C_2 = \emptyset$) and without using the "extended" TQFT $(\mathcal{T}^e, \tau^e)$, but the less elegant (but more accessible) TQFT Z_{RT} from subsection 5.3. The advantage is that we do not have to introduce *Lagrangian subspaces*, the disadvantage is that we had to choose a concrete $\mathrm{par}_{1/2}$, most of which are equivalent but Z_{RT} demands one such choice. Be informed that there exists another construction by Kapustin and Saulina [KS10] relying on surface defects instead of a surface partition.

Def. 38. Given data (a-c) from above, we define B_3 with ribbon graph Ω:

1. As 3-manifold B_3 we take $C_2 \times [-1, 1]$ with canonical orientation. So $\partial B_3 = C_2 \coprod \overline{C_2}$ of which both copies come with the "same" arcs.

[88] Here, we simply assume modularity of $\mathcal{M}$. Note that sufficient conditions on VOAs are known such that this modularity of their representation categories follows. See the cited works for more details.

[89] This already implies that meridinal generators are mapped to meridinals. Those are the rest, namely the loops that trivialize upon solidifying.

[90] In [Tur10, §IV.4.2], the "distinguished Lagrangian space" $\lambda(C_2)$ of C_2, defined by $\left(\mathrm{par}_{1/2}\right)_* (\lambda(\partial H_{(g(C_2),n)}))$, needs to be -like $\lambda(\partial H_{(g(C_2),n)})$- given precisely by the meridinal loops in order to comply with [FFFS00, §3.2]'s conventions. Would $\mathrm{par}_{1/2}$ mix meridinal and longitudinal generators, this were exactly not the case.

[91] The signs are taken from [FRS05, §3.8].

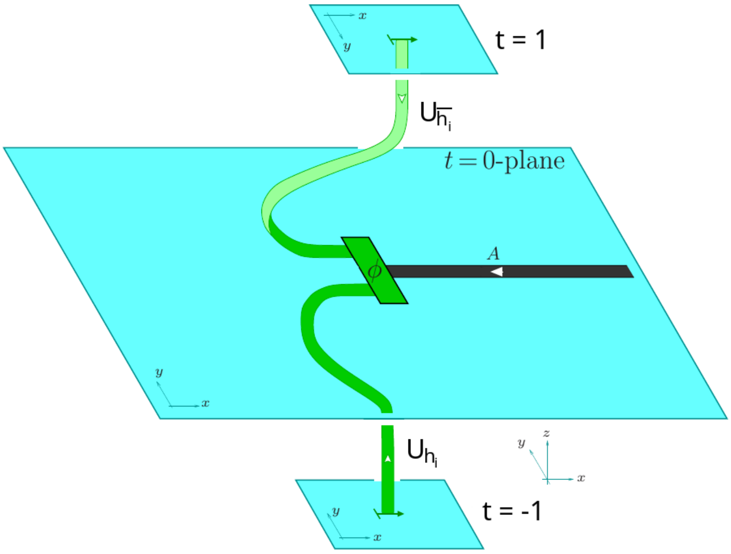


Figure 8: How to picture the bulk insertion i. The morphism $\phi \in \mathrm{Hom}_{\mathcal{M}}(U_{h_i} \otimes A, U^{\vee}_{\overline{h_i}})$ is prescribed by the physicist (datum (c)). Taken from [FRS05, §3.3] with permission, but modified in order to stay within the conventions of [FRS02a] (which are not the same as [FRS05], as noted e.g. in [FRS05, §3.1]).

2. For each insertion $i = 1, \ldots, n$ we add to Ω one straight ribbon R_i along $[-1, 1]$ attached to the "same" arcs. R_i is decorated later.
3. Now we *choose* a partition of $C_2 \times \{0\} \subset B_3$ into polygonal faces:
   - Each vertex is 3-fold and no face intersects more than one R_i.
   - By inserting a suitable[92] morphism $A \to A^{\vee}$ in between any two such vertices, all edges are directed away from the vertex.
   - The edges and vertices of this partition are added to Ω: The edges as untwisted ribbons with colouring A, the vertices with some canonical morphism $A^{\vee} \to A \otimes A$.[93]
4. Next we connect each arc $R_i \cap (C_2 \times \{0\})$ with *some* edge of its polygonal face by an A-coloured untwisted ribbon directed towards R_i. The vertex on the partitions side is coloured as all partition vertices. The other vertex between R_i and A-connection is coloured in the next step.
5. Now for the treatment of R_i. The upper half $R_i \cap (C_2 \times (0, 1])$ we colour by $U_{\overline{h_i}}$, and the lower half by U_{h_i}. Both are directed towards $C_2 \times \{0\}$. The vertex connecting the two to the partition is coloured as prescribed by data (c). See Figure 8.

[92]One uses $\Phi_1 := (\epsilon \circ \mu \otimes \mathrm{id}_{A^{\vee}}) \circ (\mathrm{id}_A \otimes b_A)$ where b_A is the birth morphism existing by $\mathcal{M}$'s ribbon property.

[93]One uses $\Delta \circ \Phi_1^{-1} \in \mathrm{Hom}_{\mathcal{M}}(A^{\vee}, A \otimes A)$.

6. Finally we define $\mathrm{par}_{\partial\Omega}$ by lifting $\mathrm{par}_{1/2}$ to all of ∂B_3: For $C_2 \times \{-1\} \subset \partial B_3$ we use the given $\mathrm{par}_{1/2}$. For $C_2 \times \{1\} \cong \overline{C_2}$ we need also an orientation preserving homeomorphism onto some standard handlebody.[94] As in [Tur10, §IV.1.7], we use a handlebody $\mathrm{mir}(H_{(g(C_2),n)})$ mirrored along a plane sitting above (5.9), parallel to all 3 genus radii, with all ribbon directions reversed, but same (right-handed) orientation. As mentioned in [Tur10, §IV.1.7], this mirror map reduces to an orientation reversing map

$$\partial\,\mathrm{mir}(H_{(g(C_2),n)}) \longrightarrow \partial H_{(g(C_2),n)}$$

which we compose with $\mathrm{par}_{1/2}$ for our desired "anti-holomorphic lid" iso.[95]

This way we have defined $(B_3, \mathrm{ins}, \Omega; \mathrm{par}_{\partial\Omega})$ where $\partial B_3 = C_2 \coprod \overline{C_2}$ and Ω is a coloured directed (n, n)-ribbon graph. There are various ways to feed this datum into Z_{RT}, and we choose to do it by viewing B_3 as a cobordism like $\emptyset \longrightarrow C_2 \coprod \overline{C_2}$. Hence

$$Z_{\mathrm{RT}}(B_3, \mathrm{ins}, \Omega; \mathrm{par}) \colon \mathbb{C} \longrightarrow Z_{\mathrm{RT}}\left(C_2 \coprod \overline{C_2}, \mathrm{ins}\right)$$

and evaluating this on $1 \in \mathbb{C}$ results in precisely one vector: Our correlator.

6.2 Algebraic Structures

This exciting, yet colloquial, paragraph can be considered the bridge from CFT to Algebra. For further (physical) background and details, see [FRS02b, §3.1].

The colouring of vertices forced on us existence of about 6 morphisms:

$$\begin{aligned} \mu \colon A \otimes A &\longrightarrow A \\ \eta \colon \mathbb{C} &\longrightarrow A \\ \Delta \colon A &\longrightarrow A \otimes A \\ \epsilon \colon A &\longrightarrow \mathbb{C} \\ (\epsilon \circ \mu \otimes \mathrm{id}_{A^\vee}) \circ (\mathrm{id}_A \otimes b_A) \equiv \Phi_1 \colon A &\longrightarrow A^\vee \quad \text{with inverse} \end{aligned}$$

The first two, μ & η, have immediate CFT interpretations, which we quickly take as *definition*:

$$\boxed{\mu = \mu_A \text{ is the Operator Product}}$$

which is indeed bilinear (so we can write it with domain $A \otimes A$).

$$\boxed{\eta(1) \in A \text{ is the vacuum primary field}}$$

which we assume to be *unique*. Physics supplies us with two properties:

- μ is an associative multiplication,
- (μ, η) are unital: $\mu \circ (\mathrm{id}_A \otimes \eta) = \mathrm{id}_A = \mu \circ (\eta \otimes \mathrm{id}_A)$.

[94] Note that subsection 5.3 had the simplifying assumption $\pi_0(\partial B_3) = 1$, so here we generalize the construction given there.

[95] The reversal of ribbon directions in $\mathrm{mir}(H_{(\dots)})$, as prescribed by [Tur10, §IV.1.7], implies that $(\mathrm{par}_{\partial\Omega})_* (\lambda(\partial H_{(\dots)} \coprod \partial\,\mathrm{mir}(H_{(\dots)}))) = \langle a \oplus \overline{a} \rangle \in H_1(C_2; \mathbb{R}) \oplus H_1(\overline{C_2}; \mathbb{R})$ where a runs over the meridinal loops of C_2. This is as desired in [FFFS00, §3.2].

To now motivate that the other morphisms assemble into a Frobenius structure on A, we quote [FRS02b, Theorem 3.6]:

Thm. 39. *If A comes from a boundary condition preserving the chiral algebra of a rational CFT with unique vacuum, then Δ, ϵ (and hence Φ_1) exist and make A into a Frobenius algebra (that is symmetric and[96] special).*

Proof. (Merely a roadmap following [FRS02b, §3.2]) The proof depends strongly on the physical fact that to every boundary field Ψ, there is some other boundary field Ψ' s.th.

$$\langle \Psi(x)\Psi'(y) \rangle \neq 0$$

which is the kick-off into establishing a "κ-Frobenius structure" using $\kappa := \epsilon \circ \mu$. This is one of several definitions of a Frobenius structure provided $\kappa \colon A \otimes A \to \mathbb{C}$ is non-degenerate (implied by the above) and invariant (implied by associativity of μ). $\square$

We summarize

$$\boxed{\text{Primary fields furnish a symmetric special Frobenius algebra } A}$$

and collect some exciting implications that translate between the three sides of physics, algebra, and TQFT ([FRS02a]):

$$\boxed{\text{CFT's crossing symmetry} \implies \text{"Frobenius" axiom (5.1)}}$$

$$\boxed{\left.\begin{array}{c}\text{"Frobenius" property} \\ \text{special property}\end{array}\right\} \implies \left\{\begin{array}{c}\text{Independence of } Z_{\mathrm{RT}}(B_3, \mathrm{ins}, \Omega; \mathrm{par}) \\ \text{on choice of partition in Def. 38.}\end{array}\right.}$$

$$\boxed{A, B \text{ Morita equivalent s.s. Frobenius algebras} \implies \textsc{Cft}_A = \textsc{Cft}_B}$$

Here, $\textsc{Cft}_A$ is a CFT built from A, and Morita equivalent coarsely means that the representation theories are isomorphic, but see [FRS02a, §6] for definitions and details.

Conclusion

TQFTs are one of the rare QFTs with a rigorous mathematical definition that actually show up in Nature. The axial QED anomaly, for example - a Chern-Simons-type object - has solved the π^0 decay rate dilemma [Ber96]. Interestingly enough, they have not been discovered by mathematicians alone, at least not the Reshetikhin-Turaev TQFT, but by physicists, using non-rigorous path integral methods. It comes as no surprise that a topic as divided between math and physics attracted the interest of many scientists over the last decades.

In this work, we have tried to convey these different aspects of TQFTs. History shows how fruitful the to-and-fro movement - as was anticipated in the introduction - between math and physics can be. Chern and Simons' motivation to study the form now bearing their name, was purely of differential geometric

[96] **Symmetric** means $(\epsilon \circ \mu \otimes \mathrm{id}_{A^\vee}) \circ (\mathrm{id}_A \otimes b_A) \equiv \Phi_1 = \Phi_2 := (\mathrm{id}_{^\vee A} \otimes \epsilon \circ \mu) \circ (\tilde{b}_A \otimes \mathrm{id}_A)$ and **special** is a Frobenius algebra if $\mu \circ \Delta = \mathrm{id}_A$ ("bubble = line") together with $\epsilon \circ \eta = \dim(A)\, \mathrm{id}_{\mathbb{C}}$.

nature [CS74]. The physicists' exploit of this new kind of insertions - one dimensional - was revolutionary and has shown to produce powerful constructions, such as the Reshetikhin-Turaev TQFT. We did not focus on the usage of these constructions, but tried to acquaint the reader with it in section 6. From there, popular topics like *non-invertible symmetries* lie rather close, for example in [FFRS07]. A quick account on some of these aspects have been summarized in appendix E, which is not directly connected to the rest of the topics, but might serve well as a pre-introduction to these new symmetries.

A Cob(d) and How It Tests For Invariants

The following definition of the monoidal, symmetrically braided category $\mathbf{Cob}(d)$ for $d \in \mathbb{N}$ is standard and can e.g. be found in [Lur09]:

Def. 40. The category $\mathbf{Cob}(d)$ is defined by means of *orientation preserving diffeomorphisms that preserve boundaries*[97] which we write as $\cong$ here:

objects: All C_{d-1} from Not. 2.

morphisms: $[B_d]\colon C_{d-1} \longrightarrow C'_{d-1}$ is a morphism iff B_d is from Not. 2 and its boundary can be decomposed as $\partial B_d \cong \overline{C_{d-1}} \coprod C'_{d-1}$. We package a lot of different such B_d into one (category theoretic) morphism by setting $[B_d] := \{B'_d \cong B_d\}$.

composition: $C_{d-1} \xrightarrow{[B'_d]} X$ and $X \xrightarrow{[B_d]} C'_{d-1}$ have as composition $[B_d] \circ [B'_d] := [B'_d \coprod_X B_d]$, where $\coprod_X$ means gluing left to right along X.[98]

identity: We set $\mathrm{id}_{C_{d-1}} := [C_{d-1} \times [0,1]]$, with $[0,1] \subset \mathbb{R}$ the interval.

One choice of monoidal structure, which happens to be strict and rigid, is:

$\otimes$: $C_{d-1} \otimes C'_{d-1} := C_{d-1} \coprod C'_{d-1}$ and $[B_d] \otimes [B'_d] := [B_d \coprod B'_d]$, i.e. disjoint union on objects and morphisms.

$\mathbb{1}$: The monoidal unit is $\emptyset$.[99]

One choice of braiding, which happens to be symmetric, is:

braiding: We set the natural braiding iso $c_{C_{d-1},C'_{d-1}}\colon C_{d-1} \otimes C'_{d-1} \xrightarrow{\sim} C'_{d-1} \otimes C_{d-1}$ to be given by the class of the obvious cobordism that eXchanges the two disjoint (as $\otimes = \coprod$) manifolds, like an X.

2× *Rem.* 41. 1. Note how we wrote "can be decomposed". Of course there are multiple ways to decompose into two disjoint unions. E.g. if $\partial B_d =$

[97] "Preserve boundaries" means the obvious here: an iso needs to isomorphically send boundary to boundary.

[98] More precisely, it means gluing along the two isos from the bordisms to X. There could be multiple ways to do that, so $B'_d \coprod_X B_d$ could mean multiple non-equal manifolds. However, according to [Lur09], all of them lie in one class, i.e. $[B'_d \coprod_X B_d]$ is uniquely defined.

[99] Being pedantic, one should add here that $\emptyset \in \mathrm{ob}\,\mathbf{Cob}(d)$.

Notation	diff. geo. name	physics name
B_3	base manifold	space-time / fields' configuration space
G	structure group	gauge group / symmetry group
$P = B_3 \times G$	total space	-

Table 1: Dictionary principal bundles - Yang-Mills gauge theory.

$\overline{C_{d-1}} \coprod C'_{d-1}$ then we have the following morphisms in $\mathbf{Cob}(d)$:

$$C_{d-1} \xrightarrow{[B_d]} C'_{d-1}$$
$$C_{d-1} \coprod \overline{C'_{d-1}} \xrightarrow{[B_d]} \emptyset$$
$$\emptyset \xrightarrow{[B_d]} \overline{C_{d-1}} \coprod C'_{d-1}$$
$$\overline{C'_{d-1}} \xrightarrow{[B_d]} \overline{C_{d-1}}$$

This 4-fold degenerate interpretation is also the key idea in proofing rigidity (existence of left and right duals and dual data), and the message to take from it is merely:

$$\text{given}\,[B_d] \implies\!\!\!\!\!\!/\;\; \text{which Hom-space it is in.}$$

2. Any function F of topological data B_d, which simultaneously is a function of some equivalence class $[B_d]$, produces invariants because this means

$$B_d \cong B'_d \implies F(B_d) = F(B'_d)$$

even if $B_d \neq B'_d$. I.e. F is *invariant* under "equivalent" changes in its domain and $F(B_d)$ is called the **invariant**. In particular, every functor out of $\mathbf{Cob}(d)$ includes such a function, i.e. a separate invariant for each oriented smooth d-dimensional manifold, possibly with boundary.

B Gauge Theories Modeled as Principal Bundles

This non-rigorous appendix aims to bridge two gaps:

- For physicists: How and Why differential forms and principal bundles beautifully encapsulate all central notions of Yang-Mills gauge theories.
- For mathematicians: A (far from sufficient) introduction to some terms from QFT that may ring some bells from past physics knowledge, or spark interest sufficient to dig deeper oneself.

A principal bundle $P \xrightarrow{G} B_3$, or $G \hookrightarrow P \xrightarrow{\pi} B_3$, is a package of consistent data, summarized in Table 1. Every principal bundle can be equipped with a particular further structure: A connection 1-form Θ ($\rightsquigarrow$ spin-1 gauge field). But Θ can only be defined (i.e. can only take tangent vector fields) on the total space P, i.e. $\Theta \in \Omega^1(P) \otimes \mathfrak{g}$![100] (Ignore the "$\otimes\mathfrak{g}$" for now.[101]) This

[100] This is similar to how left-invariant vector fields can only be defined if the manifold is equipped with the extra structure of a Lie group.

[101] $\Omega^1(P) \otimes \mathfrak{g}$ is *not* exactly the conventional tensor product of vector spaces. By a slight abuse of notation, it attaches a $\otimes\mathfrak{g}$ *to each* point of P. I.e. $\Omega^1(P) \equiv \Omega^1(P) \otimes \mathbb{R}$ because a $\omega_\mu \, \mathrm{d}x^\mu \in \Omega^1(P)$ has $\omega_\mu(p) \in \mathbb{R}$ for all $p \in P$. So $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ is a $\mathfrak{g}$-*valued* form.

may be the prime motivation to talk about principal bundles when trying to "mathematize" Yang-Mills gauge theory. Of course, gauge fields from physics "live" (i.e. take values) on the base B_3. We need to pull back Θ from P down to earth, so to speak. This can only be done through a *choice* of section $\sigma \in \Gamma(P)$. Which section? Unfortunately, no section can be canonically distinguished. Hence, track needs to be kept of all choices of sections, creating possibly a huge redundancy of information. Lem. 5 states that this redundancy is precisely encoded by the set of all G-gauge transformations. A gauge field A is then defined as $\sigma^*\Theta \in \Omega^1(B_3) \otimes \mathfrak{g}$.

For the further interested reader, let me comment on how coupling a Yang-Mills gauge field to matter can also be quite naturally described in this framework - namely by the associated vector bundle. (Of course, were it not able to describe matter coupling, physicists would have probably refrained from wading through this chapter of mathematics.) Matter fields are separate fields ϕ of arbitrary spin in a QFT. One usually starts to investigate a QFT-encoding *Lagrangian* $\mathcal{L}$, a vaguely motivated function of ϕ and derivatives (and not yet A), observes that a non-trivial G-action ρ_{glob} on ϕ, where the whole ϕ is considered as *one* vector acted upon, leaves $\mathcal{L}$ invariant, calls this action a *symmetry* of $\mathcal{L}$, physically motivates that the G-action symmetry ρ_{glob} should stay a G-symmetry ρ_{loc} even if $\phi(x)$ is acted on by a different element of G than $\phi(y)$ for $x \neq y$ and observes that the derivatives of ϕ in $\mathcal{L}$ mentioned above make this demand, dubbed *gauging the symmetry*, very challenging. Yang and Mills showed that a connection 1-form field A_ϕ is necessarily created if one demands to gauge a G-symmetry of $\mathcal{L}$ in ϕ. One then calls ϕ *charged under* G via A_ϕ. To advertise Yang-Mills theories, here are three prime applications of the last sentence in the standard model:

- Electrons are charged under U(1) via photons.
- Neutrinos are charged under SU(2) via W/Z-bosons.
- Quarks are charged under SU(3) via gluons.

The action $\rho_{\text{loc}} \equiv \rho : G \to \{\text{fields } \phi \text{ as vectors at every point } x \text{ of space-time}\}$ mentioned above, from after gauging, will now be central in the construction of the associated vector bundle. As motivated before, this gauge action ρ is a *redundancy* of the QFT, a reminiscence of impossibility to define a connection 1-form on B_3 directly. A useful description of coupled matter should divide out those redundant DOFs, i.e. keep track only of those field values unrelated by a gauge transformation. This is precisely achieved by the (to the principal G-bundle P) *associated vector bundle*

$$P \times_\rho V := (P \times V)/G := (P \times V)/\langle (p.g, v) \sim (p, g.v) \rangle$$

where the right G-action on P is a property of every principal bundle and the left G-action on V is ρ_{loc} for one point x of space-time, i.e. ρ_{glob}. Hence, V is the vector space of matter fields ϕ evaluated at x.

C On the Physical Use of $\overline{\mathcal{C}_{B_3}}$ as Action Domain

As Freed [Fre92] elaborates on in more detail, the (topological) space $\overline{\mathcal{C}_{B_3}}$ from Def. 8 gives a slightly nicer co-domain for the action functional as it is more canonical than the usual choice. Let us first agree on the meaning of this word...

Agreement 42. A choice is **canonical** if it is singled out from (almost) all other possible choices, by some sort of "naturality". E.g. the principle of relativity, axiomatizing the fact that no system of inertia should be singled out, can be rephrased and motivated by saying that there exists no canonical choice of system of intertia, so none should be prioritized.

So how did the old, non-canonical way of fixing a co-domain of S_{B_3} go? Well, people simply picked a set S of representatives of equivalence classes of isomorphic (in the sense of Def. 4) G-bundles over B_3, i.e.

$$S := \{P_i \mid i \neq j \Rightarrow P_i \ncong P_j\} \text{ s.th. any } P \xrightarrow{G} B_3 \text{ has } P \cong P_i \text{ for some } i.$$

This way can indeed be bijectively translated into our new setting, via the homeomorphism

$$\overline{\mathcal{C}_{B_3}} \cong \coprod_i \frac{\mathcal{A}_{P_i}}{\mathcal{G}_{P_i}} \tag{C.1}$$

of physical fields (new: l.h.s.; old: r.h.s.), but this bijection is *not canonical*. On the r.h.s. we had to *choose* representatives P_i, but

$$P_i \cong P_j \nRightarrow \forall \Theta_i \in \mathcal{A}_{P_i} \exists \Theta_j \in \mathcal{A}_{P_j} : \ (P_i, \Theta_i) \cong (P_j, \Theta_j)$$

where, the left $\cong$ is a bundle iso symbol and the right $\cong$ an iso symbol of $\mathcal{C}_{B_3}$. But even if a choice S were already fixed, the bijection (C.1) would *still* not be canonical, as one would have to choose a connection $\Theta_i \in \mathcal{A}_{P_i}$ on the r.h.s. for a pair (P_j, Θ_j) of the l.h.s. and there might not be a single "obvious"/canonical choice, i.e. a connection $\Theta_i \in \mathcal{A}_{P_i}$ isomorphic to Θ_j, even if $P_i \cong P_j$.
Rephrasing the above, immediately identifying isomorphic bundles (meaning Def. 4) puts the space of connections in an awkward situation. Bundles being isomorphic does not imply that connections on them need to be isomorphic (meaning Def. 7). So better keep track of all pairs $(P, \Theta) \in \mathrm{ob}(\mathcal{C}_{B_3})$ and identify isomorphic *pairs*, as in $\overline{\mathcal{C}_{B_3}}$. The topology on this space is best understood via the homeomorphism (C.1).

D Framing of Knots and Links

An (**oriented**) **knot** is the (oriented) image of any continuous, injective function $f\colon S^1 \to B_3$ with $S^1 = [0,1]/(0 \sim 1)$ as domain (equipped with the standard orientation). Some examples are depicted in Fig. 9.
A **link** with r components is the image of any continuous, injective function $f_r : (S^1)^{\coprod r} \to B_3$. Each of the r connected components is called a *knot component*. An oriented example of a link with 2 tangled components is depicted in Fig. 10.

Def. 43.
- A **framing** of a knot K is given by a choice of directed, oriented ribbon that has K as its oriented core.[102] The **framed knot** is exactly this ribbon.

[102] Directed, oriented ribbons have been explained in section 4.1. The "core" is the middle line of the ribbon - e.g. the arrows in Fig. 5 lie precisely on, and along the direction of the core of their respective ribbon.

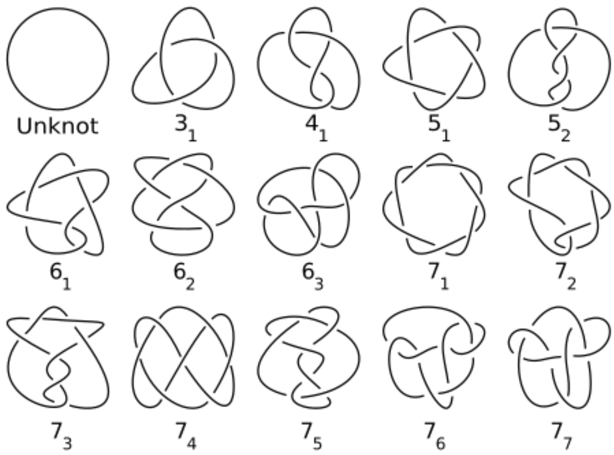


Figure 9: Some knots.

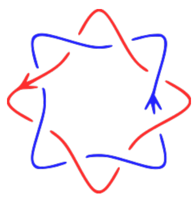


Figure 10: A tangled link on 2 components with orientation.

- A **framed link** is a link with all of its knot components framed.

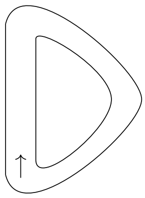
Because TQFTs will never distinguish between knots or ribbons in the same isotopy classes, we will now always pass to the isotopy classes of knots, and framed knots (denoted by "iso"). There are two essential aspects about framings: The physical reason, and that the choices of framings are naturally given by the integers. The latter is the quite canonical bijection

$$\{\text{knots}\}/\text{iso} \times \mathbb{Z} \xrightarrow{\sim} \{\text{framed knots}\}/\text{iso}$$

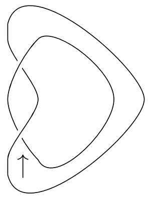
where the number $n \in \mathbb{Z}$ is called **winding number** because it counts how many twists one has in the framed knot.[103] This can be done for each knot component independently, giving

$$\text{Rib}\colon \bigcup_{r=0}^{\infty} (\{\text{links with } r \text{ components}\}/\text{iso} \times \mathbb{Z}) \xrightarrow{\sim} \{\text{framed links}\}/\text{iso} \qquad \text{(D.1)}$$ ×6

2× **Exp. 44.** The following two ribbons are framed unknots,



&



and the right one has winding number one different than the left one under (D.1)'s Rib^{-1}.

The **physical origin** for the need of a framing comes from [Wit89, §2.1]'s calculation that the correlator of a link with r components in abelian Chern-Simons theory is the exponential of the *linking number* of the link, essentially. A link's **linking number** sums over the linking number of *any* two pairs of knot components.[104] In particular, the correlator includes the sum of the linking number of one knot component *with itself* - dubbed **self-linking number** - which, of course, has no obvious meaning. The conclusion Witten draws is that this correlator *does not exist*, and CS-theory can only compute the correlator of *framed* links. A framing gives an easy way out because it provides a canonical self-linking number in the following way:

We take the two boundary lines of the framed knot component (cf. e.g. Exp. 44) as two knots and forget about the interior "ribbon material". Both inherit an orientation and do not intersect, so there is a well-defined linking number between the two boundaries, which is taken as the framed knot's self-linking number. E.g. in Exp. 44, the first framed knot has trivial self-linking number, whereas the second one has a non-trivial one.

[103] Precisely one twist is depicted in (4.1). There is no "half-twist" as e.g. in the Möbius band because we agreed to take only oriented ribbons. We do not fix a zero winding number - i.e. we do not calibrate the winding number scale - as there is no need for this in order to describe the constructions in the text.

[104] The linking number between two knots is an integer essentially telling how much they are tangled. For example the Hopf link or Fig. 10 have non-trivial linking number.

E Higher Form Symmetries

The concept of a *1-form symmetry* is introduced - a special case of *higher form symmetries.* They generalize the (to a physicist) well-known 0-form symmetries, which have a corresponding charge obtained as the space-integral over a Noether current, like in $d = 3 + 1$

$$\text{Symmetry charge } = \int_{\mathbb{R}^3} \text{Noether current } d^3x. \tag{E.1}$$

The "form" comes from the language of differential forms, in which the defining equation of a conserved Noether current j, $\partial_\mu j^\mu = 0$ we set to take the form

$$\mathrm{d} * j = 0. \tag{E.2}$$

This implies that these everyday j's need to be 1-forms, so we indicate the degree like

$$j =: j^{(1)}.$$

Def. 45. The symmetry associated to a 1-form Noether current (via Noether's thm.) is called a **0-form symmetry**.

Introducing the conventions of C. Córdova et al. [CDI19], (E.1) can be rewritten as

$$Q_A(\Sigma_3) = \int_{\Sigma_3} *j_A^{(1)}$$

with A the name of the associated quantum field A_μ that would appear in the Lagrangian and Σ_3 a 3-cycle (language of chain complexes where the boundary operator ∂ of smooth manifolds is taken as the differential/boundary operator). I.e. Σ_3 is a closed 3-manifold, i.e. C_3 in Not. 2's conventions.
Already here we can bridge the gap to physical TQFTs, as the conservation (E.2) of such charges translate into a topological property which they enjoy:

Lem. 46. *"Topological changes" (cf. proof) between two oriented codimension-1 submanifold Σ_{d-1} to Σ'_{d-1} s.th. $\partial\Sigma_{d-1} = \partial\Sigma'_{d-1}$ leave the charge invariant,*

$$Q(\Sigma_{d-1}) \equiv \int_{\Sigma_{d-1}} *j^{(1)} = \int_{\Sigma'_{d-1}} *j^{(1)},$$

provided (E.2) holds everywhere.

Proof. A topological change usually refers to a smooth deformation. Owing to the assumption $\partial\Sigma_{d-1} = \partial\Sigma'_{d-1}$, we take it to mean that the difference of such change is a smooth manifold (and do not dwell on subtleties). I.e. there is a smooth manifold B_d s.th. $\partial B_d = \Sigma_{d-1} \cup \overline{\Sigma'_{d-1}}$ (where we do not mind which orientation to choose at $\partial\Sigma_{d-1}$ as the integral does not mind). Now Stoke's theorem will close the proof:

$$\int_{\Sigma_{d-1}} *j^{(1)} - \int_{\Sigma'_{d-1}} *j^{(1)} = \int_{B_d} \mathrm{d} * j^{(1)} \overset{(E.2)}{=} 0.$$

□

Rem. 47. Note that this is a simple variant of Noether's fundamental proof of charge conservation

$$\partial_t Q \equiv \partial_t \int *j = 0$$

where the missing dimension in Σ_{d-1} is time t. However, Noether could not assume $\partial\Sigma_{d-1} = \partial\Sigma'_{d-1}$ as this would have implied $t = t'$ and rendered the theorem trivial. Rather, she had to assume that the extra boundary $\partial B_d \setminus (\Sigma_{d-1} \cup \overline{\Sigma'_{d-1}})$ carries zero current (because it lies at spatial infinity).

Higher form symmetries allow for mixed gauge transformations as explained in [CDI19]. The mathematical tool that can describe such mixed symmetries are *2-groups*.

E.1 2-Groups

Intriguingly, this will again be a categorical concept, tightening the bond between abstract nonsense and modern physics. We follow J. Baez and A. Lauda [BL03].

5× **Def. 48.** A **strict 2-group** is a strict monoidal category $(\mathcal{C}, \otimes, 1_\otimes)$ in which all morphisms and objects are invertible. An object x being **invertible** iff there exists an object x^{-1}, s.th. $x \otimes x^{-1} = 1_\otimes$.

For physicists to keep track, think of an object x in such a 2-group $\mathcal{C}$ as a symmetry transformation. Or even better, note that the tuple $(\mathrm{ob}(\mathcal{C}), \otimes)$ furnishes a group - our symmetry group. So the extra structure lies precisely within the morphisms. But after a second of meditation, one notes that there is no canonical way to also package all the morphisms into one group. They are invertible and associative by definition (of a *strict* monoidal category) but what would the group operation be? It for sure cannot be composition. Rather think of the morphisms as two-way connections between symmetry transformations. To assimilate the concept, an equivalent definition that additionally enforces the justification of the naming, would be for example:

1× **Prop. 49.** *$(\mathcal{C}, \otimes, 1_\otimes)$ is a strict 2-group iff the 2-category given by*

- *ob.: $*$*
- *1-mor.: the objects of $\mathcal{C}$ with $\otimes = \circ$*
- *2-mor.: the (1-)morphisms of $\mathcal{C}$ with respective source and target*

has only invertible 1- and 2-morphisms.

(The idea of the proof should be apparent.) It is convenient to switch between the two avatars of a 2-group, Def. 48 and Prop. 49, when trying to unravel categorical confusion.
Relaxing strictness in Def. 48 will furnish the physically desired structure:

1× **Def. 50.** In the realm of Def. 48, a **2-group** $(\mathcal{C}, \otimes, 1_\otimes)$ is the relaxation of "=" to "≅" in Def. 48, i.e. a monoidal category with only invertible morphisms and only weakly invertible objects (for x we find $\bar{x}$, a fixed weak inverse, s.th. $x \otimes \bar{x} \cong 1_\otimes$ for some fixed isomorphism).

Note how we fix the weak inverse as there could be multiple. In the same spirit, we fix an associator:

Def. 51. The fixed isomorphism

$$a_{x,y,z} : (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z)$$

in a 2-group is called **associator** and is denoted as depicted.

All this data contained in Def. 48 is tedious to specify. A classification result of 2-groups should now be in order:

Thm. 52. *Up to equivalences of categories, a 2-group is specified by a quatruple* $(G, H, \alpha, [a])$*, where*

- G *is a group,*
- H *is an abelian group,*
- $\alpha : G \longrightarrow \mathrm{Aut}(H)$ *is an action,*
- $[a] \in H^3_{Grp,\alpha}(G, H)$ *is a group-cohomology class with coefficients in* H*.*

We will discuss important points of the proof. The in-dept proof is found in [BL03, §8.3] where a 2-group is called *coherent 2-group*. The result goes back probably to Sinh's doctoral thesis with Groethendieck from 1975, see [BL03] for some history.

Compact guide to group cohomology used in the proof [BCH19]:

The differential we will see is *twisted* by a given action α. On $a : G^\ell \to H$, where G^ℓ is the ℓ-fold Cartesian product of a group G, it acts as

$$\begin{aligned} d_\alpha a(g_0, \ldots, g_\ell) = {} & \underbrace{g_0.a(g_1, \ldots, g_\ell)}_{\alpha\text{-action}} + \\ & + \sum_{k=1}^{\ell} (-1)^k a(g_0, \ldots, g_{k-2}, \underbrace{g_{k-1}\, g_k}_{G\text{-mult.}}, g_{k+1}, \ldots, g_\ell) + \\ & + (-1)^{\ell+1} a(g_0, \ldots, g_{\ell-1}) \end{aligned} \tag{E.3}$$

and defines, as $d_\alpha^2 = 0$, in the usual algebraic topological sense, cohomology groups denoted $H^*_{\mathrm{Grp},\alpha}(G, H)$.

Proof. Roadmap to obtain the quadruple $(G, H, \alpha, [a])$ from a 2-group $\mathcal{C}$:

1. As a category, $\mathcal{C}$ is equivalent to a special type of 2-group $\mathcal{C}_{\mathrm{sp}}$:

   a. All structural isomorphisms in Def. 50 (left/right unit, left/right inverses, associator) are again strict, *on the nose*, identities except maybe for the associator $a_{x,y,z}$.

   b. As a category, $\mathcal{C}_{\mathrm{sp}}$ is skeletal. I.e. if there exists any isomorphism between two objects, the objects are the same. (Note that this and (a) do *not* imply that $a_{x,y,z} = 1_{x \otimes y \otimes z}$, the identity morphism.)

2. $G := (\mathrm{ob}(\mathcal{C}_{\mathrm{sp}}), \otimes)$, the symmetry group.

3. $H := \mathrm{Aut}_{\mathcal{C}_{\mathrm{sp}}}(1_\otimes) = \mathrm{End}_{\mathcal{C}_{\mathrm{sp}}}(1_\otimes)$, the abelian group.

4. $g.h := \alpha(g,h) := (1_g \otimes h) \otimes 1_{\bar{g}}$, the action.
Note that one could drop the parentheses in the last term by step 1b.

5. The associator gives rise to a map

$$\begin{aligned} a : G \times G \times G &\longrightarrow H \equiv \mathrm{Aut}(1_\otimes) \\ (g_1, g_2, g_3) &\longmapsto a_{g_1,g_2,g_3} \otimes \overline{g_1 \otimes g_2 \otimes g_3} \end{aligned} \tag{E.4}$$

which we defined to take values in H (because $g \otimes \bar{g} = 1_\otimes$ for e.g. $g = g_1 \otimes g_2 \otimes g_3$ by (1b)) and has to obey the pentagon identity from the axioms of a monoidal category, which takes the form

$$\begin{aligned} g_0.a(g_1,g_2,g_3) - a(g_0g_1,g_2,g_3)+ \\ + a(g_0,g_1g_2,g_3) - a(g_0,g_1,g_2g_3) + a(g_0,g_1,g_2) = 0. \end{aligned} \tag{E.5}$$

We observe that this equality is equivalently giving $a \in Z^3_{\mathrm{Grp},\alpha}$ by (E.3), i.e. $[a] \in H^3_{\mathrm{Grp},\alpha}(G,H)$. Our desired cohomology class.

Step 2 should be clear considering step 1b.
The *Eckmann-Hilton argument* gives that H in step 3 is abelian. For this one needs to observe that $\otimes = \circ$ as $1_\otimes \otimes 1_\otimes = 1_\otimes$ by virtue of step 1. A similar observation proofs that $g.h$ in step 4 actually lies within H.
Step 5 has already been spelled out, so the important (but not so physically intriguing) step is number 1.
From step 1a we can already infer that the associator alone carries the difference of information between a non-strict and a strict 2-group which is hence encoded in α and $[a]$. The details can be found in [BL03].

Conversely, given $(G, H, \alpha, [a])$ we can define the objects and a multiplication $\otimes$ of a skeletal category via step 2 and the morphisms via step 3 (as every morphism is an isomorphism and the category skeletal, all morphisms are endomorphisms). We can interpret the "." in (E.5) as the action of α and then take $a \in [a]$ satisfying the pentagon axiom. We should find that any choice of such a gives an equivalent monoidal category upon interpreting a as the associator. $\square$

References

[Wit89] Edward Witten. "Quantum Field Theory and the Jones Polynomial". In: *Commun. Math. Phys.* 121 (1989). Ed. by Asoke N. Mitra, pp. 351–399.

[Ati88] Michael F. Atiyah. "Topological quantum field theory". en. In: *Publications Mathematiques de l'IHÉS* 68 (1988), pp. 175–186. URL: http://www.numdam.org/item/PMIHES_1988__68__175_0/.

[Tur10] V.G. Turaev. *Quantum Invariants of Knots and 3-manifolds*. De Gruyter studies in mathematics. New York, 2010. URL: https://books.google.de/books?id=w7dActmezxQC.

[Lur09] Jacob Lurie. *On the Classification of Topological Field Theories*. 2009. URL: https://arxiv.org/abs/0905.0465.

[Lur22] Jacob Lurie. *The Classification of Extended Topological Field Theories -Jacob Lurie*. 2022. URL: https://youtu.be/svsidHyNk1k?t=5089.

[Fre92] Daniel S. Freed. "Classical Chern-Simons theory, Part 1". In: (1992). URL: https://arxiv.org/abs/hep-th/9206021.

[Ber96] R. A. Bertlmann. *Anomalies in quantum field theory*. 1996.

[CH85] Curtis G. Callan Jr. and Jeffrey A. Harvey. "Anomalies and Fermion Zero Modes on Strings and Domain Walls". In: *Nucl. Phys. B* 250 (1985), pp. 427–436.

[CDI19] Clay Córdova, Thomas T. Dumitrescu, and Kenneth Intriligator. "Exploring 2-group global symmetries". In: *Journal of High Energy Physics* 2019.2 (2019). URL: https://doi.org/10.1007%2Fjhep02%282019%29184.

[CLSWY19] Chi-Ming Chang, Ying-Hsuan Lin, Shu-Heng Shao, Yifan Wang, and Xi Yin. "Topological defect lines and renormalization group flows in two dimensions". In: *Journal of High Energy Physics* 2019.1 (2019). URL: https://doi.org/10.1007%2Fjhep01%282019%29026.

[FMT22] Daniel S. Freed, Gregory W. Moore, and Constantin Teleman. *Topological symmetry in quantum field theory*. 2022. URL: https://arxiv.org/abs/2209.07471.

[TW19] Ryan Thorngren and Yifan Wang. *Fusion Category Symmetry I: Anomaly In-Flow and Gapped Phases*. 2019. arXiv: 1912.02817 [hep-th].

[FFRS04] Jürg Fröhlich, Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. "Kramers-Wannier Duality from Conformal Defects". In: *Physical Review Letters* 93.7 (2004). URL: https://doi.org/10.1103\%2Fphysrevlett.93.070601.

[FFRS07] Jürg Fröhlich, Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. "Duality and defects in rational conformal field theory". In: *Nuclear Physics B* 763.3 (2007), pp. 354–430. URL: https://doi.org/10.1016\%2Fj.nuclphysb.2006.11.017.

[Fre22] Daniel S. Freed. *Introduction to topological symmetry in QFT*. 2022. URL: https://arxiv.org/abs/2212.00195.

[Nak03] M. Nakahara. *Geometry, topology and physics*. 2003.

[CS74] Shiing-Shen Chern and James Simons. "Characteristic Forms and Geometric Invariants". In: *Annals of Mathematics* 99.1 (1974), pp. 48–69. URL: http://www.jstor.org/stable/1971013 (visited on 04/21/2023).

[Man16] Ciprian Manolescu. *Lectures on the triangulation conjecture*. 2016. arXiv: 1607.08163 [math.GT].

[(ht] levap (https://math.stackexchange.com/users/32262/levap). *Pullback of a differential form by a local diffeomorphism.* Mathematics Stack Exchange. (version: 2018-11-28). URL: https://math.stackexchange.com/q/2113990.

[BT95] R. Bott and L.W. Tu. *Differential Forms in Algebraic Topology.* Graduate Texts in Mathematics. Springer New York, 1995. URL: https://books.google.de/books?id=S6Ve0KXyDj8C.

[Jan96] K Jansen. "Domain wall fermions and chiral gauge theories". In: *Physics Reports* 273.1 (1996), pp. 1–54. URL: https://doi.org/10.1016%2F0370-1573%2895%2900081-x.

[Abr96] Leonard S. Abrams. "Two-dimensional topological quantum field theories and Frobenius algebras". In: *J. Knot Theor. Ramifications* 5 (1996), pp. 569–587.

[TV92] V.G. Turaev and O.Y. Viro. "State sum invariants of 3-manifolds and quantum 6j-symbols". In: *Topology* 31.4 (1992), pp. 865–902. URL: https://www.sciencedirect.com/science/article/pii/004093839290015A.

[TV17] V. Turaev and A. Virelizier. *Monoidal Categories and Topological Field Theory.* Progress in Mathematics. Springer International Publishing, 2017. URL: https://books.google.de/books?id=wf0RvgAACAAJ.

[KB10] Alexander Kirillov and Benjamin Balsam. *Turaev-Viro invariants as an extended TQFT.* 2010. URL: https://arxiv.org/abs/1004.1533.

[RT91] N. Reshetikhin and V.G. Turaev. "Invariants of 3-manifolds via link polynomials and quantum groups." In: *Inventiones mathematicae* 103.3 (1991), pp. 547–598. URL: http://eudml.org/doc/143870.

[Mue01] Michael Mueger. *From Subfactors to Categories and Topology II. The quantum double of tensor categories and subfactors.* 2001. arXiv: math/0111205 [math.CT].

[TV13] Vladimir Turaev and Alexis Virelizier. *On two approaches to 3-dimensional TQFTs.* 2013. arXiv: 1006.3501 [math.GT].

[Wik] Wikipedia. *Dehn-Chirurgie.* (version: 2023-06-07). URL: https://de.wikipedia.org/wiki/Dehn-Chirurgie.

[Kir78] Robion Kirby. "A Calculus for Framed Links in S3." In: *Inventiones mathematicae* 45 (1978), pp. 35–56. URL: http://eudml.org/doc/142535.

[FRS02a] Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. "Conformal correlation functions, Frobenius algebras and triangulations". In: *Nucl. Phys. B* 624 (2002), pp. 452–468. arXiv: hep-th/0110133.

[FRS02b] Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. "TFT construction of RCFT correlators 1. Partition functions". In: *Nucl. Phys. B* 646 (2002), pp. 353–497. arXiv: hep-th/0204148.

[FRS04a] Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. "TFT construction of RCFT correlators. 2. Unoriented world sheets". In: *Nucl. Phys. B* 678 (2004), pp. 511–637. arXiv: hep-th/0306164.

[FRS04b] Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. "TFT construction of RCFT correlators. 3. Simple currents". In: *Nucl. Phys. B* 694 (2004), pp. 277–353. arXiv: hep-th/0403157.

[FRS05] Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. "TFT construction of RCFT correlators IV: Structure constants and correlation functions". In: *Nucl. Phys. B* 715 (2005), pp. 539–638. arXiv: hep-th/0412290.

[FFRS06] Jens Fjelstad, Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. "TFT construction of RCFT correlators. V. Proof of modular invariance and factorisation". In: *Theor. Appl. Categor.* 16 (2006), pp. 342–433. arXiv: hep-th/0503194.

[DFMS97] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. New York: Springer-Verlag, 1997.

[FFFS00] Giovanni Felder, Jürg Fröhlich, Jürgen Fuchs, and Christoph Schweigert. *Correlation functions and boundary conditions in RCFT and three-dimensional topology*. 2000. arXiv: hep-th/9912239 [hep-th].

[KS10] Anton Kapustin and Natalia Saulina. *Surface operators in 3d Topological Field Theory and 2d Rational Conformal Field Theory*. 2010. arXiv: 1012.0911 [hep-th].

[BL03] John C. Baez and Aaron D. Lauda. "Higher-Dimensional Algebra V: 2-Groups". In: (2003). URL: https://arxiv.org/abs/math/0307200.

[BCH19] Francesco Benini, Clay Córdova, and Po-Shen Hsin. "On 2-group global symmetries and their anomalies". In: *Journal of High Energy Physics* 2019.3 (2019). URL: https://doi.org/10.1007%2Fjhep03%282019%29118.