stats100b_hw1

Takao

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1

```r
knitr::include_graphics("/Users/takaooba/Desktop/Screen Shot 2022-11-28 at 4.21.36 PM.png")
```

1. Let $\bar{X}$ be the average of a sample of 16 independent normal random variables with mean 0 and variance 4. Determine c such that
$P(|\bar{X}| \leq c) = 0.90$

From what is given in the problem, we know that

$$X \sim N(\mu = 0, \sigma^2 = 4)$$

Additionally, We want to find the mean and variance for Xbar, we will do so by,

$$\bar{X} \sim N(\mu_{\bar{X}} = \mu, \sigma^2_{\bar{X}} = \frac{\sigma^2}{n}) = N(0, \frac{4}{16}) = N(0, \frac{1}{4})$$

Moving on with the problem, we have

$$P(|\bar{X}| \leq c) = 0.9 = P(-c \leq \bar{X} \leq c) = P(\frac{-c - \mu_{\bar{X}}}{\sigma_{\bar{X}}} \leq \frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}} \leq \frac{c - \mu_{\bar{X}}}{\sigma_{\bar{X}}})$$

$$P(-2c \leq Z \leq 2c) = \Phi(2c) - \Phi(-2c) - [1 - \Phi(2c)] = 2\Phi(2c) - 1 = 0.9$$

Thus, simplifying the equation, we have

$$2\Phi(2c) = 1.95$$

$$\Phi(2c) = 0.95$$

Solving for c, we have

```r
cat("c = ", qnorm((0.95)) / 2)
```

```
## c =  0.8224268
```

2

Start with what is given through the problem,

$$X \sim N(\mu = 0, \sigma^2 = 4)$$

Next, we look at Xbar

$$\bar{X} \sim N(\mu_{\bar{X}} = \mu, \sigma^2_{\bar{X}} = \frac{\sigma^2}{n}) = N(0, \frac{4}{n})$$

Moving on to the problem, we have

$$P(|\bar{X}| \leq 0.1) = 0.9 = P(-0.1 \leq \bar{X} \leq 0.1) = P(\frac{-0.1 - \mu_{\bar{X}}}{\sigma_{\bar{X}}} \leq \frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}} \leq \frac{0.1 - \mu_{\bar{X}}}{\sigma_{\bar{X}}})$$

Plugging in the known values, we have,

$$P(\frac{-0.1 - 0}{2/\sqrt{n}} \leq Z \leq \frac{0.1 - 0}{2/\sqrt{n}}) = P(-0.05\sqrt{n} \leq Z \leq 0.05\sqrt{n}) = 2\Phi(0.05\sqrt{n}) - 1 = 0.9$$

Simplifying this equation, we get,

$$2\Phi(0.05\sqrt{n}) = 1.95$$
$$\Phi(0.05\sqrt{n}) = 0.95$$

Solving for n, we get

```
(qnorm(0.95) / 0.05)^2
```

```
## [1] 1082.217
```

However, since we want to find the smallest integer n such that th condition is met, we will get.

```
cat("smallest integer = ", 1083)
```

```
## smallest integer =  1083
```

3

We will be working first with what is given in the problem

$$\mu_X = 15, \sigma_X = 10$$

Since we are concerned with the 100 packages, let

$$S = X_1 + X_2 + \cdots + X_{100}$$

We want to find the mean and variance of S, and thus we will say

$$S \overset{approx}{\sim} N(\mu_S = n\mu_X, \sigma^2_S = n\sigma^2_X) = N(100 * 15, 100 * 10^2) = N(1500, 10000)$$

We can directly utilize this information to find the z-score

$$P(S \leq 1700) = P(Z \leq \frac{1700 - 1500}{100}) = P(Z \leq 2)$$

Thus, we must simply solve for this probability

```
cat("Probability = ", 1 - pnorm(2))
```

```
## Probability =  0.02275013
```

4

Since this is a fair game, the probability of winning is 1/2 and the probability of losing is 1/2 as well.

Additionally, since this is a fair game, the expected value of X is

$$E(X) = 5 \times 0.5 + (-5) \times 0.5 = 0$$

And since

$$E(X) = \mu_X = 0$$

To find the variance,

$$\sigma_X^2 = Var(X) = E(X^2) - E(X)^2 = 5^2 \times 0.5 + (-5)^2 \times 0.5 = 25$$

Similar to the prior problem, we will now be looking at the 50 games S.

$$S = X_1 + X_2 + \cdots + X_{50}$$

To approximate the mean and variance of S, we have

$$S \overset{approx}{\sim} N(\mu_S = n\mu_X, \sigma_S^2 = n\sigma_X^2) = N(0 * 50, 25 * 50) = N(0, 1250)$$

We attempt to find the probablity of S is less than or equal to -75 in terms of Z score

$$P(S \leq -75) = P(Z \leq \frac{-75 - 0}{\sqrt{1250}})$$

Now, we simply solve for the probability

```
cat("Probability = ", pnorm(-75/sqrt(1250)))
```

```
## Probability =  0.01694743
```

5

We are given through the problem that the distribution of the life time of the device follows an exponential distribution.

The new model is equipped with a total of three components

Thus let,

$$S = X_1 + X_2 + X - 3$$

We intend to find the moment generating function for S utilizing

$$M_S(t) = E(e^{tS}) \stackrel{ind}{=} E(e^{tX_1})E(e^{tX_2})E(e^{tX_3})$$

Since we know that X is exponentially distributed, we input that for f(x) when creating the moment generating function.

$$E(e^{tX}) = \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx = \lambda \int_0^\infty e^{-(\lambda-t)x}dx$$

Utilizing integration by parts and setting

$$u = (\lambda - t)x$$

and thus

$$dx = \frac{1}{\lambda - t}du$$

Completing the substitution we will get of u for x (and also remembering to account for the limits as well).

$$\int_{u=0}^{u=\infty} e^{-u}\frac{1}{\lambda - t}du = \frac{1}{\lambda - t}(0+1) = \frac{1}{\lambda - t}$$

Going back to creating the moment generating function, we must multiply the lambda to the solved integral. Thus

$$M_X(t) = \lambda \times \frac{1}{\lambda - t} = \frac{\lambda}{\lambda - t}$$

Since it is given that the device typically lasts for 100 hours,

$$E(X_1) = \frac{1}{\lambda} = 100$$

Solving this equation, we get that lambda = 0.01 and since the conditions for the three components are the same.

Moving on to the moment generating function of x, we have

$$M_S(t) = E(e^{tX_1})E(e^{tX-2})E(e^{tX_3}) = \frac{\lambda}{\lambda - t} \times \frac{\lambda}{\lambda - t} \times \frac{\lambda}{\lambda - t} = (\frac{\lambda}{\lambda - t})^3$$

When we look at this moment generating function, we can see that this is very similar to the gamma distribution.

Thus, we can say that the distribution of the life time for the new model follows a gamma distribution with alpha = 3 and lambda = 0.01. The sum of independent random variables follow gamma distributions also follows a gamma distribution.

$$S \sim Gamma(\alpha = 3, \lambda = 0.01)$$

6

Since the probability of drawing coin A is 0.5 and drawing coin B is 0.5 the probability of getting a head is

$$p = 0.5 \times p_1 + 0.5 \times p_2$$

Now referring back to the problem, we have that X is the total number of heads obtained when repeated n times in total, thus X follows a Binomial distribution, since there are two possible outcomes, where

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Now that we found the distribution of X, we intend to find a good approximation of the distribution of X for a large value of n, thus when n approaches infinity we have

$$X \sim N(np, np(1-p))$$

Additionally we can make this a standard normal distribution through a litle manipulation of

$$\frac{X - np}{\sqrt{np(1-p)}} \sim N(0,1)$$

7

Similar to the previous problem, we have that the probability of selecting coin A is 1/2 and the probability of selecting coin B is 1/2 as well.

First, we will look at the number of heads obtained in 2 tosses using coin A through X_A

The probability of getting two heads is

$$P(X_A = 2) = p_1^2$$

Next, the probability of getting one head is

$$P(X_A = 1) = 2 \times p_1 \times (1 - p_1)$$

Finally, the probability of getting no head is

$$P(X_A = 0) = (1 - p_1)^2$$

This applies for coin B as well where p1 becomes p2.

Next, we denote X as the number of heads obtained through two tosses. This will come out to

$$E(X) = 0.5 \times [2 \times p_1^2 + 1 \times 2p_1(1-p_1) + 0 \times (1-p_1)^2] + 0.5 \times [2 \times p_2^2 + 1 \times 2p_2(1-p_2) + 0 \times (1-p_2)^2]$$

Solving the equation, we get that the

$$E(X) = p_1 + p_2$$

Next we aim to find the variance and to do so we must find the expected value of x squared

$$E(X^2) = 0.5 \times [2^2 \times p_1^2 + 1^2 \times 2p_1(1-p_1)] + 0.5 \times [2^2 \times p_2^2 + 1^2 \times 2p_2(1-p_2)]$$

Simplifying this equation gives

$$E(X^2) = p_1(1 + p_1) + p_2(1 + p_2)$$

Now that we got all the necessary information, we attempt to find the variance

$$Var(X) = E(X^2) - E(X)^2 = p_1(1+p_1) + p_2(1+p_2) - (p_1+p_2)^2 = p_1(1-p_2) + p_2(1-p_1)$$

Now we consider Y which is the total number of heads obtained out of the n tosses, thus

$$E(Y) = \sum_{i=1}^{n/2} E(X_i) = \frac{n}{2}(p_1+p_2)$$

and similarly the variance will be

$$Var(Y) = \sum_{i=1}^{n/2} Var(X_i) = \frac{n}{2}[p_1(1-p_2) + p_2(1-p_1)]$$

Now, we see how the distribution of Y becomes as n approaches infinity which can be denoted as

$$Y \sim N(\frac{n}{2}(p_1+p_2), \frac{n}{2}[p_1(1-p_2) + p_2(1-p_1)])$$

And this can be written in standard normal form of

$$\frac{Y - E(Y)}{\sqrt{Var(Y)}} \sim N(0,1)$$

8

TO find the moment generation function of the given condition, we must simply utilize property and equations.

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

We input the given variables

$$M(t) = \frac{1}{2} \int_{5}^{7} e^{tx} dx$$

Solving this integral, we have

$$M(t) = \frac{1}{2t} \times (e^{7t} - e^{5t})$$

9

From the problem, we have that Z = 2X + 4Y, where X and Y are both Gamma variables

To find the moment generation function for Z and to simplify we must use properties.

$$M_Z(t) = E(e^{tZ}) = E(e^{2tX} e^{4tY}) \stackrel{ind}{=} E(e^{2tX}) \times E(e^{4tY})$$

We will initially try to solve the left hand side by

$$E(e^{2tX}) = \int_0^\infty e^{2tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx$$

To solve this integral, we will move the constants outside the integral and use integration by parts.

$$u = (\lambda - 2t)x, dx = \frac{1}{\lambda - 2t} du$$

Doing the substitutions and also for the limits as well, we have

$$\int_{x=0}^{x=\infty} x^{\alpha-1} e^{-(\lambda-2t)x} dx = \int_{u=0}^{u=\infty} (\frac{1}{\lambda - 2t} u)^{\alpha-1} e^{-u} \frac{1}{\lambda - 2t} du = \int_{u=0}^{u=\infty} (\frac{1}{\lambda - 2t})^{\alpha-1} u^{\alpha-1} e^{-u} \frac{1}{\lambda - 2t} du$$

Simplifying the integral, we have

$$E(e^{2tX}) = \frac{\lambda^\alpha}{\Gamma(\alpha)} (\frac{1}{\lambda - 2t})^\alpha \int_{u=0}^\infty u^{\alpha-1} e^{-u} du$$

We notice that the inside the integral will be the gamma distribution of alpha. Thus, the integral simplifies to

$$E(e^{2tX}) = (\frac{\lambda}{\lambda - 2t})^\alpha$$

Inputting the given values from the problem, we have that

$$E(e^{2tX}) = (\frac{1}{1-t})^2$$

Moving onto Y, we have

$$E(e^{4tY}) = \int_0^\infty e^{4ty} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda y} dy$$

To solve this integral, we will move the constants outside the integral and use integration by parts.

$$u = (\lambda - 4t)y, dy = \frac{1}{\lambda - 4t} du$$

Doing the substitutions and also for the limits as well, we have

$$\int_{y=0}^{y=\infty} x^{\alpha-1} e^{-(\lambda-4t)y} dy = \int_{u=0}^{u=\infty} (\frac{1}{\lambda - 4t} u)^{\alpha-1} e^{-u} \frac{1}{\lambda - 4t} du = \int_{u=0}^{u=\infty} (\frac{1}{\lambda - 4t})^{\alpha-1} u^{\alpha-1} e^{-u} \frac{1}{\lambda - 4t} du$$

Simplifying the integral, we have

$$E(e^{2tY}) = \frac{\lambda^\alpha}{\Gamma(\alpha)} (\frac{1}{\lambda - 4t})^\alpha \int_{u=0}^\infty u^{\alpha-1} e^{-u} du$$

As before we notice that the inside of the integral is gamma variables, thus, this simplifies to

$$E(e^{4tY}) = (\frac{4}{4-4t})^2 = (\frac{1}{1-t})^2$$

Going back to the original moment generating function,

$$M_Z(t) = E(e^{tZ}) = E(e^{2tX}) \times E(e^{4tY}) = (\frac{1}{1-t})^4$$

Thus, we can conclude that Z follows a gamma distribution with alpha = 4 and lambda = 1 or

$$Z \sim Gamma(\alpha = 4, \lambda = 1)$$

10

(a)

$$m(t) = 1 + 3t + 6t^2 + \cdots$$
$$m'(t) = 3 + 12t + \cdots$$
$$m''(t) = 12 + \cdots$$

To find the mean, we have

$$E(X) = m'(0) = 3$$

Additionally the formula to find the variance will be

$$Var(X) = E(X^2) - E(X)^2$$
$$E(X^2) = m''(0) = 12$$

Thus, the variance will be

$$12 - 3^2 = 3$$

The mean is 3 and the variance is 12.

(b)

The first equation cannot be a moment generating function. This is because

$$m_1(0) = 0.5 + 0 + \cdots = 0.5 \neq 1$$

This breaks the conditions to be a moment generating function, thus this function is not a moment generating function.

Next, the second equation cannot be a moment generating function. This is because when finding the variance of this function, we see that

$$E(X) = 3, E(X^2) = 6$$

And when utilizing the formula to find the variance, we see that

$$Var(X) = E(X^2) - E(X)^2 = 6 - 3^2 = -3$$

Variance cannot be negative, thus this cannot be a moment generating function.

The third function can be a moment generating function.

Thus, 1st and 2nd functions cannot be moment generating functions.