stats100b_hw3

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4/25/2022

1

```r
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```

1. Suppose that X is a discrete random variable following a geometric distribution,
   $P(X = k) = p(1-p)^{k-1},\ for\ k = 1,2,3,.....$
   where $0 < p < 1$
   Suppose n observations are obtained independently from this distribution :
   $x_1, x_2, \cdots, x_i, \cdots, x_n$

(i) Write down the log joint likelihood function and find the MLE of p.
(ii) Suppose that n= 2, $x_1 = 3$ and $x_2 = 4$. Please plot the log likelihood function for the datapoint $x_1 = 3$, the log likelihood function for the datapoint $x_2 = 4$, and the log joint likelihood function. Find MLE and check that MLE is located at the peak of the graph of log joint likelihood.

i

The pdf of a geometric function is given as

$$L(p;x) = \prod_i p(1-p)^{x_i - 1}$$

We find the log joint likelihood

$$l(p;x) = \log L(p;x) = \sum_i \log[p(1-p)^{x_i-1}] = \sum_i \log(p) + \sum_i (x_i - 1)\log(1-p)$$

Moving on, we can simplify to

$$= n\log(p) + \log(1-p)\sum_i (x_i - 1)$$

Taking the derivative with respect to the unknown parameter, we have

$$\frac{d}{dp}l(p;x) = \frac{n}{p} - \frac{1}{1-p}\sum_i (x_i - 1) = 0$$

$$\frac{1}{1-p}\sum_i (x_i - 1) = \frac{n}{p}$$

$$p\sum_i (x_i - 1) = n - np$$

$$p\sum_i (x_i - 1) + np = n$$

Further simplifying, we have that

$$p[\sum_i (x_i - 1) + n] = n$$

Estimation of p is given through

$$\hat{p}_{MLE} = \frac{n}{\sum_i x_i}$$

ii

We are given two data points so we plug them into the answer from the prior question.

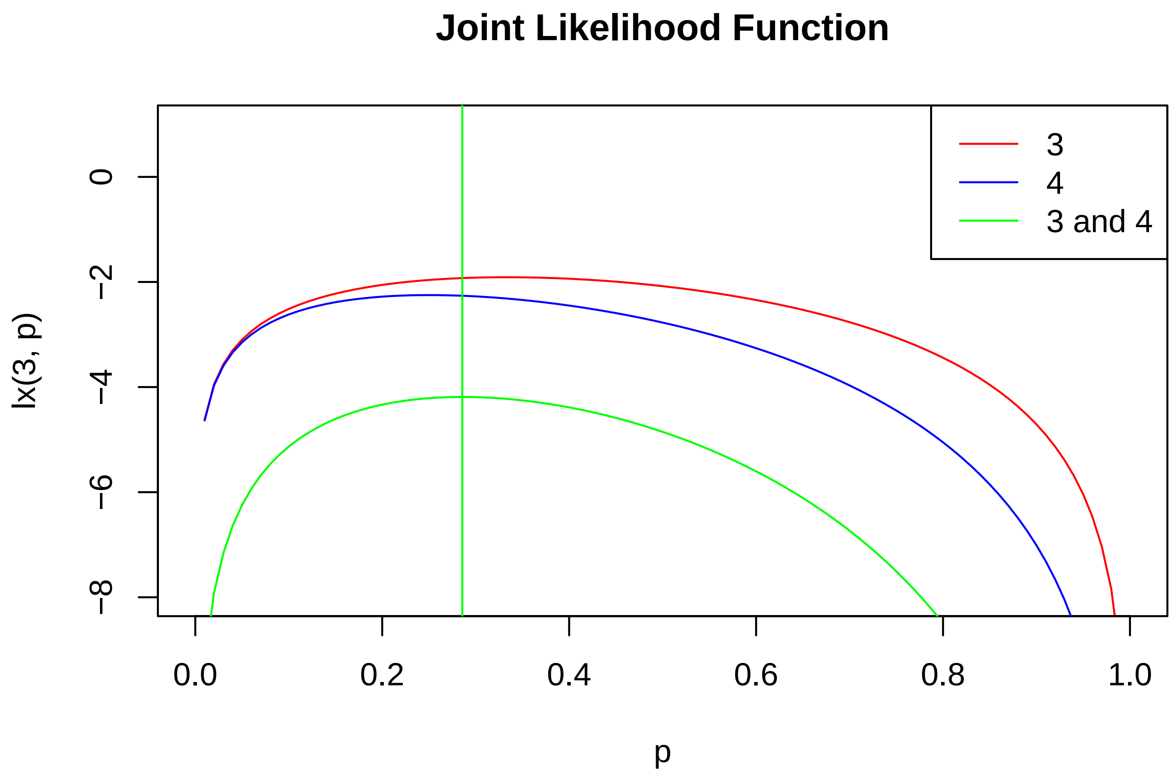
$$\hat{p}_{MLE} = \frac{2}{[(3-1)+(4-1)]+2}$$

Using r, we have

```
p <- seq(0, 1, 0.01)
lx <- function(xi,p) log(p) + log(1-p)*(xi - 1)
lxx <- function(x1, x2, p) 2*log(p) + log(1-p)*(x1-1+x2-1)

plot(p, lx(3, p), type = "l", ylim = c(-8,1), col = "red", main = "Joint Likelihood Function")
lines(p, lx(4,p),col = "blue")
lines(p, lxx(3,4,p), col = "green")
legend(legend = c("3", "4", "3 and 4"), col = c("red", "blue", "green"), "topright", lty = c(1,1,1))

MLE <- 2/(2+3+2)
abline(v = MLE, col = "green")
```



2

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```

2. The density function of X is $f(x\,|\,\theta) = (1+\theta)x^{\theta},\ 0 \le x \le 1$ where $\theta > -1$
Suppose we have a random sample of size n independently generated from this distribution, $x_1, \ldots x_n$.

(i) Suppose the sample size n = 3. Write down the joint likelihood function for $x_1 = 0.3,\ x_2 = 0.5,\ x_3 = 0.4$ and plot the joint likelihood function and the log joint likelihood function.
(ii) For any sample size n, write down the log joint likelihood function and find the formula of MLE.

i

By definition, we have that,

$$L(\theta; x_1, x_2, x_3) = \prod_{i=1}^{3} (1+\theta) x_i^{\theta} = (1+\theta)^3 x_1^{\theta} x_2^{\theta} x_3^{\theta}$$

Taking the log

$$l(\theta; x_1, x_2, x_3) = 3\log(1+\theta) + \sum_{i=1}^{3} \log(x_i^{\theta}) = 3\log(1+\theta) + \theta \sum_{i=1}^{3} \log(x_i)$$

The log joint likelihood is

$$3\log(1+\theta) + \theta[\log(x_1) + \log(x_2) + \log(x_3)]$$

Transfer above equation to R language.

```
theta <- seq(-1, 4, 0.01)
l <- function(theta, x1, x2, x3) (1+theta)^3*x1^theta*x2^theta*x3^theta

ll <- function(theta, x1, x2, x3) 3*log(1 + theta) + theta*(log(x1) + log(x2) + log(x3))

par(mfrow = c(1,2))
plot(theta, l(theta, 0.3, 0.5, 0.4), type = "l", col = "red", main = "Joint Likelihood Function")
plot(theta, ll(theta, 0.3, 0.5, 0.4), type = "l", col = "blue", cmain = "joint Log Likelihood Function")
```

```
## Warning in plot.window(...): "cmain" is not a graphical parameter

## Warning in plot.xy(xy, type, ...): "cmain" is not a graphical parameter

## Warning in axis(side = side, at = at, labels = labels, ...): "cmain" is not a
## graphical parameter

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## graphical parameter

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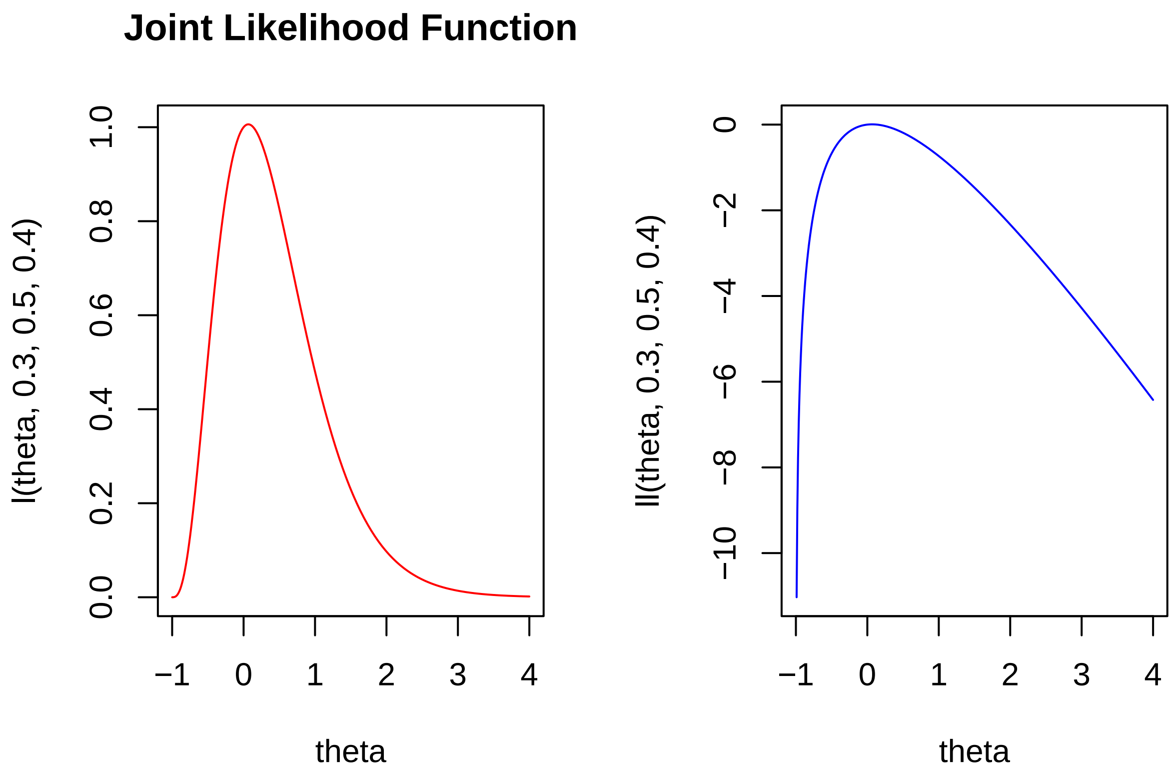
```



ii

Multiply for N copies of pdf

$$L(\theta; x) = \prod_i (1+\theta)x_i^{\theta} = (1+\theta)^n \prod_i x_i^{\theta}$$

Taking the log, we have,

$$l(\theta; x) = n\log(1+\theta) + \sum_i \log(x_i^{\theta}) = n\log(1+\theta) + \theta\sum_i \log(x_i)$$

Next, we take the derivative with respect to the unknown parameter,

$$\frac{d}{d\theta} l(\theta; x) = \frac{n}{1+\theta} + \sum_i \log(x_i) = 0$$

We aim to simplify by

$$1 + \theta = -\frac{n}{\sum_i \log(x_i)}$$

We can solve for MLE by

$$\hat{\theta}_{MLE} = -\frac{n}{\sum_i \log(x_i)} - 1$$

3

```r
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```

3. Suppose that X is a discrete random variable with

$$P(X=0) = \frac{2}{3}\theta,\ P(X=1) = \frac{1}{3}\theta,\ P(X=2) = 1-\theta$$

where the domain of the parameter is $0 < \theta < 1$. Suppose 5 independent observations are taken from this distribution, yielding the data : (1, 0, 0, 0,2).

(i) Write down the joint likelihood function and plot it.

(ii) Plot the log joint likelihood function and give the maximum likelihood estimate of θ.

(iv) For any sample size n, give a formula of MLE in terms of three numbers, y_0, y_1, y_2,

y_o = the number of times 0 is contained in the list of $x_1, \ldots x_n$.

y_1 = the number of times 1 is contained in the list of $x_1, \ldots x_n$

y_2 = the number of times 2 is contained in the list of $x_1, \ldots x_n$

This is a PMF

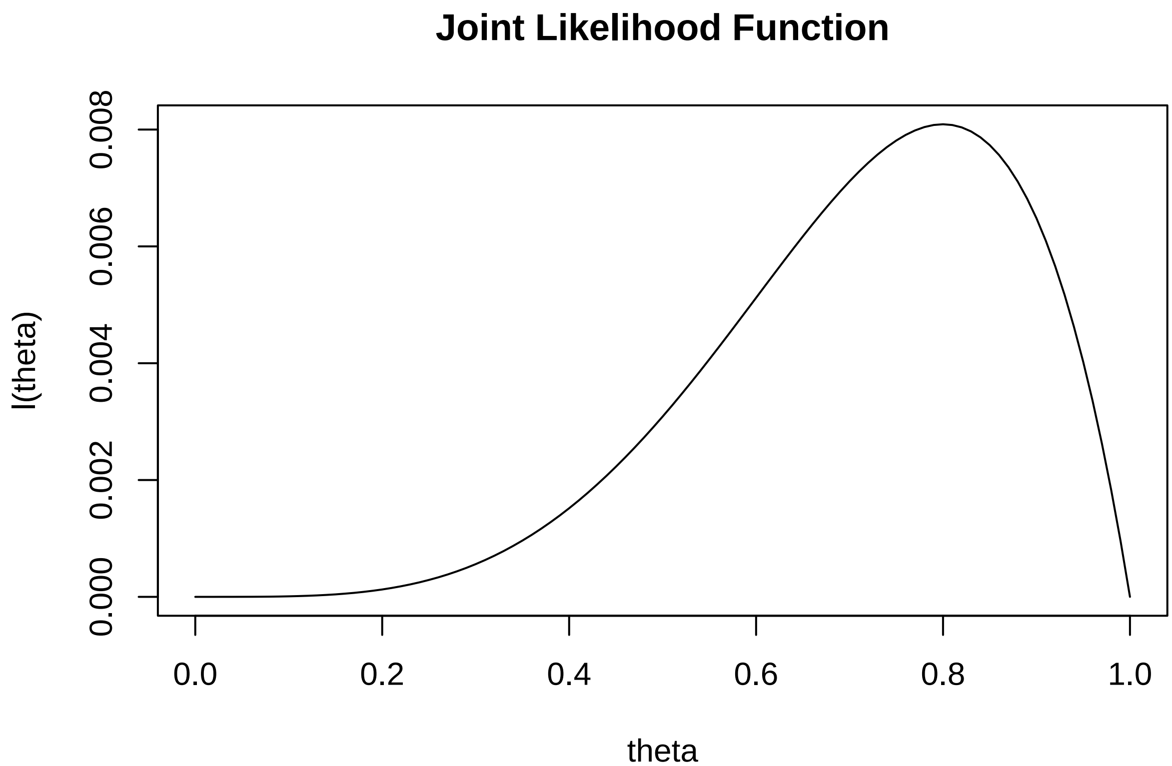
i

We are given through the problem the data points

$$L(\theta; x) = \frac{1}{3}\theta \times \frac{2}{3}\theta \times \frac{2}{3}\theta \times \frac{2}{3}\theta \times (1-\theta) = \frac{8}{81}\theta^4(1-\theta)$$

Plotting the joint likelihood function using R, we have that

```r
theta <- seq(0,1,0.01)
l <- function(theta) 8/81*theta^4*(1-theta)
plot(theta,l(theta), type = "l", main = "Joint Likelihood Function")
```



ii

We initially start with logging

$$l(\theta; x) = \log(\frac{8}{81}) + 4\log(\theta) + \log(1-\theta)$$

Taking the derivative, we have that

$$\frac{d}{d\theta}; (\theta; x) = \frac{4}{\theta} - \frac{1}{1-\theta} = 0$$

Simplifying we have that

$$\frac{4}{\theta} = \frac{1}{1-\theta}$$

Further,

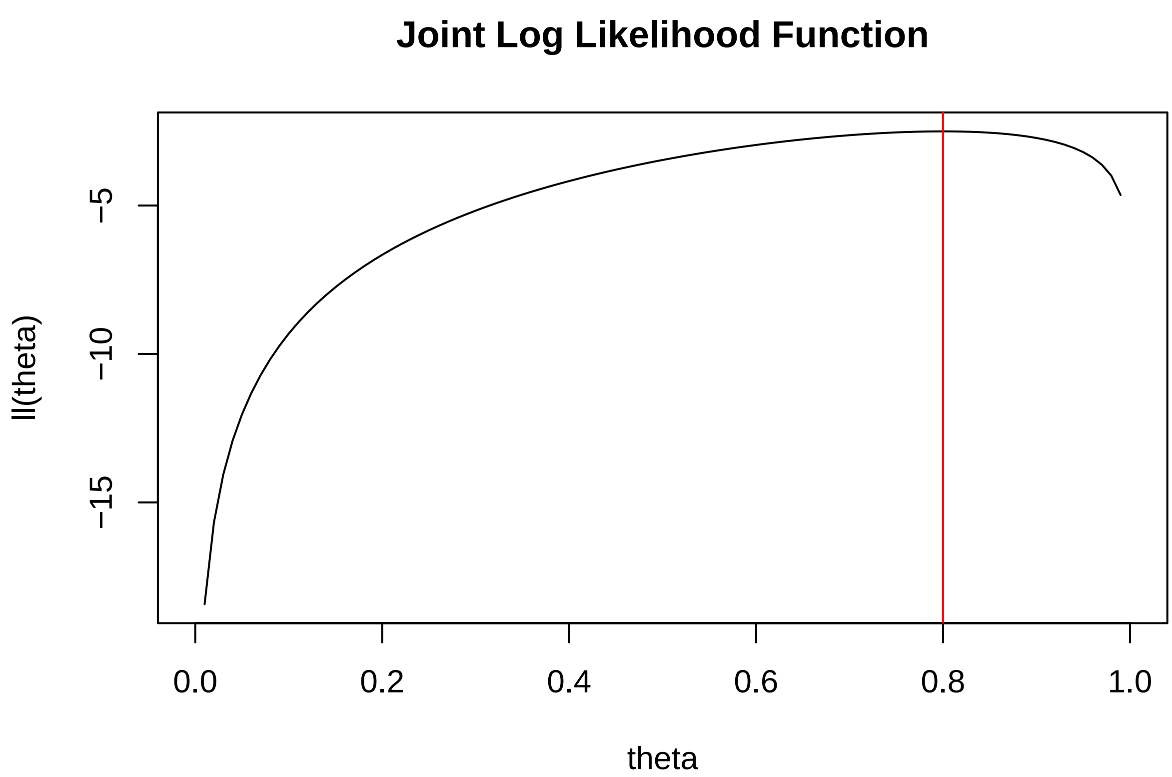
$$4 - 4\theta = \theta$$

Now, solving for MLE, we have

$$\hat{\theta}_{MLE} = \frac{4}{5}$$

Since the numeric values are given, we can solve using R

```
ll <- function(theta) 4*log(theta) + log(1-theta)
plot(theta, ll(theta), type = "l", main = "Joint Log Likelihood Function")
abline(v = 0.8, col = "red")
```



iii

Raise probablity to the power of their corresponding occurence

$$L(\theta; x) = (\frac{2}{3}\theta)^{y_0} \times (\frac{1}{3}\theta)^{y_1} \times (1-\theta)^{y_2}$$

Taking the log, we have that

$$l(\theta; x) = y_0 \log(\frac{2}{3}) + y_1 \log(\frac{1}{3}) + y_0 \log(\theta) + y_1 \log(\theta) + y_2 \log(1-\theta)$$

Moving on, we will take the derivative, which will be

$$\frac{d}{d\theta} l(\theta; x) = \frac{y_0 + y_1}{\theta} - \frac{y_2}{1-\theta} = 0$$

Solving, we have that

$$\frac{y_0 + y_1}{\theta} = \frac{y_2}{1-\theta}$$

Through simplification, we have that

$$y_0 + y_1 = (y_0 + y_1 + y_2)\theta$$

Additionally, we can solve MLE through,

$$\hat{\theta}_{MLE} = \frac{y_0 + y_1}{y_0 + y_1 + y_2}$$

4

```r
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```

4. Suppose that X has the density function

$$f(x\,|\,\theta) = e^{\theta - x}, for \quad x \geq \theta$$
$$= 0, \; for \quad x < \theta$$

Suppose n observations are obtained independently from this distribution :

$x_1, x_2, \cdots, x_i, \cdots, x_n$

(i) Suppose that the *i*th datapoint happens to take the value $x_i = 1.5$. Write down the the likelihood function that represents this particular datapoint and plot it. Is the likelihood function continuous ?

(ii) Write down the joint likelihood function and find the maximum likelihood estimate of θ.

[Tips: It helps if you study the lecture slide on the example of uniform distribution first. Before you engage in the formal derivation, it may help if you try out some concrete cases, say n = 3, the data are 0.5, 1.5, 0.1, for example]

There is condition for x

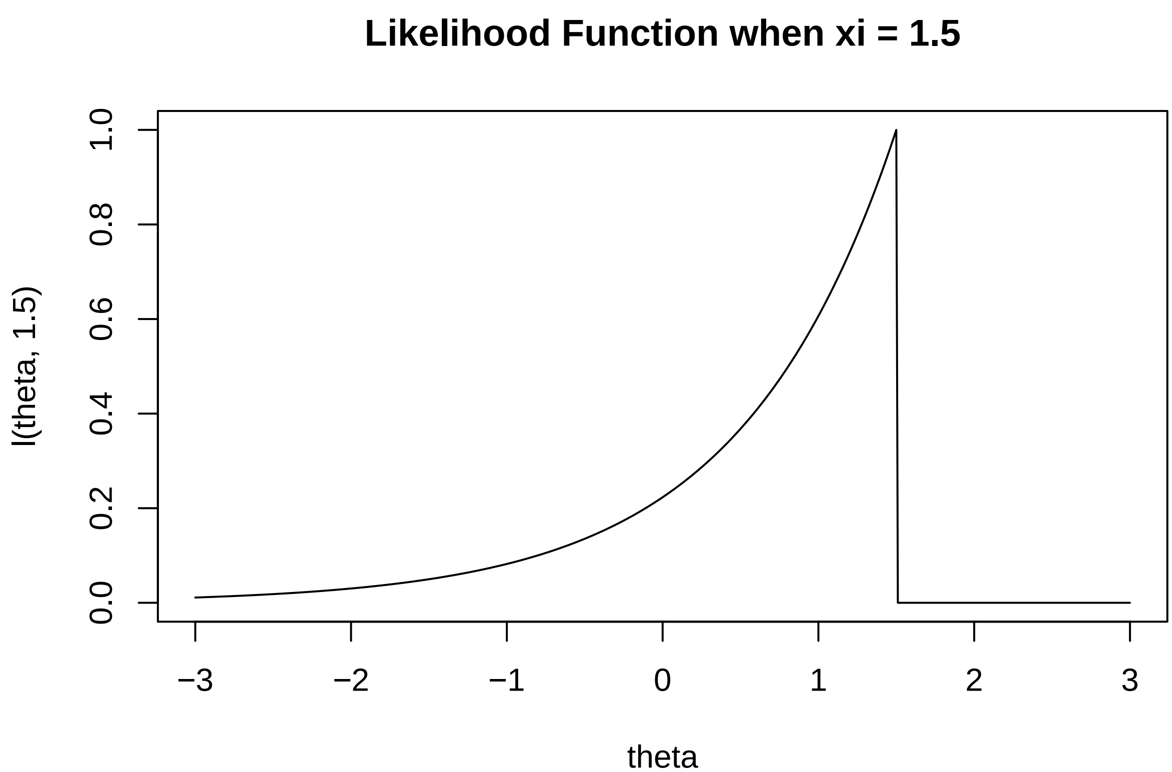
i

Thus, the likelihood function is not continuous

$$L(\theta; x_1) = I(x_1 \geq \theta) \times \exp(\theta - x_1)$$

Using R, we can solve that, also the graph drops at theta - 1.5 to 0.

```r
theta <- seq(-3, 3, 0.01)
l <- function(theta, x1) (x1 >= theta)*exp(theta-x1)
plot(theta, l(theta,1.5), type = "l", main = "Likelihood Function when xi = 1.5")
```

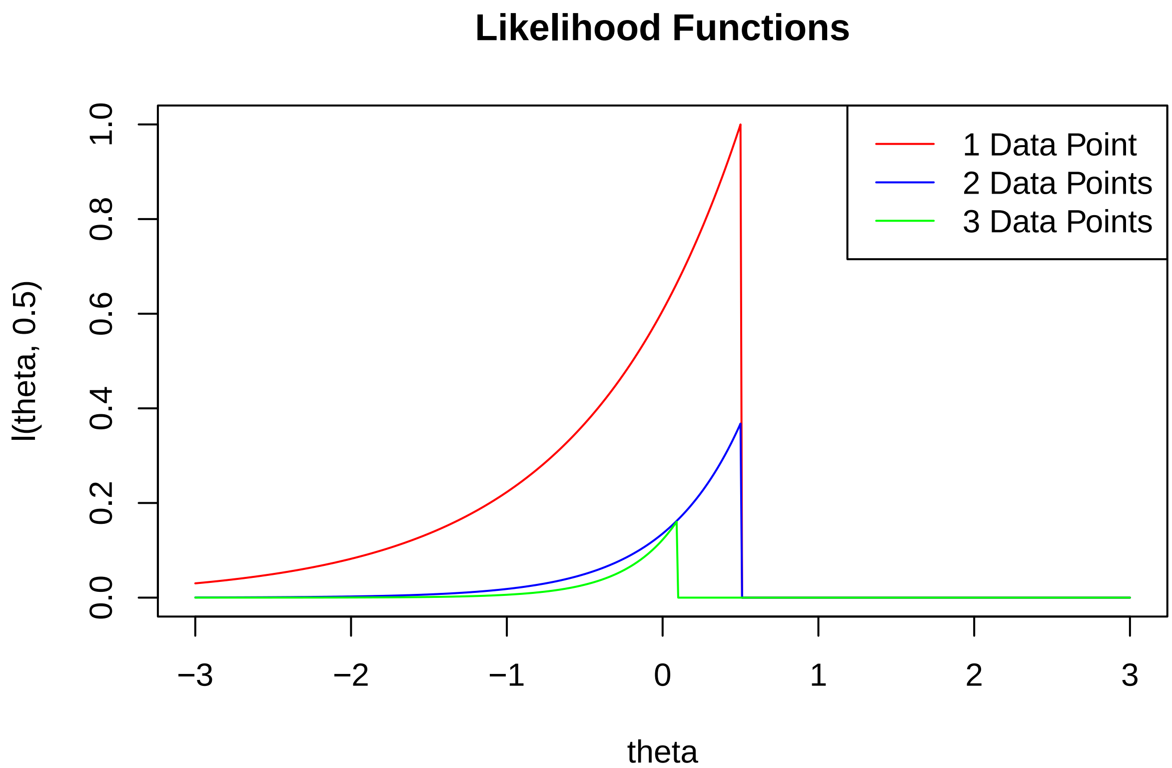


ii

We want to find the joint likelihood function and MLE

```r
l <- function(theta, x1) (x1 >= theta)*exp(theta - x1)
ll <- function(theta, x1, x2) (x1 >= theta)*(x2 >= theta)*exp(theta - x1)*exp(theta - x2)
lll <- function(theta, x1, x2, x3) {
  (x1 >= theta)*(x2 >= theta)*(x3 >= theta)*exp(theta - x1)*exp(theta - x2)*exp(theta - x3)
}

plot(theta, l(theta, 0.5), type = "l", main = "Likelihood Functions", col = "red")
lines(theta, ll(theta, 0.5, 1.5), col = "blue")
lines(theta, lll(theta, 0.5, 1.5,0.1), col = "green")
legend(legend = c("1 Data Point", "2 Data Points", "3 Data Points"), col = c("red", "blue", "green"), "
```



Judged from the graph above,

$$\hat{\theta}_{MLE} = min\{x_1, \cdots, x_n\}$$

5

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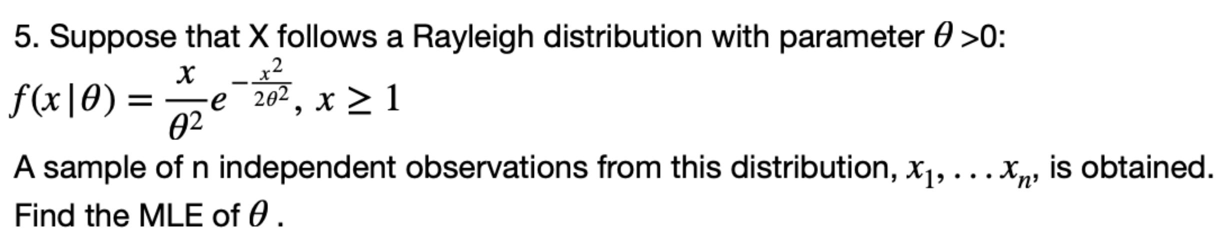

5. Suppose that X follows a Rayleigh distribution with parameter θ >0:

$$f(x\,|\,\theta) = \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}},\ x \geq 1$$

A sample of n independent observations from this distribution, $x_1, \ldots x_n$, is obtained. Find the MLE of θ .

Since independent, we can take the product and will thus be

$$L(\theta; x) = \prod_i \frac{x_i}{\theta^2} \exp(-\frac{x_i^2}{2\theta^2}$$

Taking the log, we have,

$$l(\theta; x) = \sum_i \log[\frac{x_i}{\theta^2} \exp(-\frac{x_i^2}{2\theta^2})] = \sum_i [\log \frac{x_i}{\theta^2} - \frac{x_i^2}{2\theta^2}] = \sum_i [\log x_i - \log \theta^2 - \frac{x_i^2}{2\theta^2}]$$

Distributing the summation sign, we have that

$$l(\theta; x) = \sum_i \log x_i - n \log \theta^2 - \frac{1}{2\theta^2} \sum_i x_i^2$$

Taking the derivative and setting it equal to 0, we have that

$$\frac{d}{d\theta} l(\theta; x) = -n \frac{1}{\theta^2} 2\theta - \frac{1}{2}(-2\theta^{-3}) \sum_i x_i^2 = -2n \frac{1}{\theta} + \theta^{-3} \sum_i x_i^2 = 0$$

Through simplification, we have that

$$-2n\theta^2 + \sum_i x_i^2 = 0$$

Further,

$$2n\theta^2 = \sum_i x_i^2$$

And we can isolate the theta by

$$\theta^2 = \frac{\sum_i + x_i^2}{2n}$$

We can now find the MLE through

$$\hat{\theta}_{MLE} = \sqrt{\frac{\sum_i x_i^2}{2n}}$$

The theta will be positive values and the negative will be ignored.

6

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6. Assuming that the density function of X is given by the equation $f(x \mid \theta)$ given in Problem 2.
(i) Find the Fisher information of this statistical model.

Suppose a sample of n independent observations are to be taken and the maximum likelihood estimator will be used to estimate the unknown parameter θ.
(ii) Find the asymptotic variance of the MLE and give a 95% confidence interval of θ approximately for large n.

i

Density function is given through

$$f(x;\theta) = (1+\theta)x^{\theta}$$

Further, taking the log, we have

$$l(\theta;x) = \log f(x;\theta) = \log(1+\theta) + \theta \log(x)$$

First order derivative is given through

$$S(\theta) = \frac{d}{d\theta} l(\theta;x) = \frac{1}{1+\theta} + \log(x)$$

The second order derivative can be denoted as

$$H(\theta) = \frac{d}{d\theta} S(\theta) = -\frac{1}{(1+\theta)^2}$$

Using the negative expectation, we can solve as

$$I(\theta) = -E[H(\theta)] = -E[-\frac{1}{(1+\theta)^2}] = \frac{1}{(1+\theta)^2}$$

ii

To calculate the confidence interval, we first recognize that when n becomes infinity, then we have normality This can be expressed through

$$\hat{\theta}_{MLE} \sim N(\theta, \frac{1}{nI(\theta)})$$

Further, the variance can be found through plugging in the given equation in part i

$$Var(\hat{\theta}_{MLE}) = \frac{1}{nI(\theta)} = \frac{(1+\theta)^2}{n}$$

We can now construct the 95% confidence interval through

$$[\hat{\theta}_{MLE} - Z_{0.975} \times \sqrt{\frac{(1+\hat{\theta}_{MLE})^2}{n}}, \hat{\theta}_{MLE} + Z_{0.975} \times \sqrt{\frac{(1+\hat{\theta}_{MLE})^2}{n}}]$$

7

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7. Consider the geometric distribution given in Problem 1.
(i)Find the Fisher information of this statistical model.

Suppose a sample of n independent observations are to be taken and the maximum likelihood estimator will be used to estimate the unknown parameter p.
(ii) Find the asymptotic variance of the MLE and give a 95% confidence interval of p approximately for large n.

i

We intend to find the fisher information through

$$f(x;p) = p(1-p)^{x-1}$$

Taking the log, we have

$$l(p;x) = \log f(x;p) = \log(p) + (x-1)\log(1-p)$$

Further taking the derivative, we have that

$$S(p) = \frac{d}{dp} l(p;x) = \frac{1}{p} + \frac{1-x}{1-p}$$

Finding the second order derivative, we have that

$$H(p) = \frac{d}{dp} S(p) = -\frac{1}{p^2} + \frac{1-x}{(1-p)^2}$$

Taking the negative expected value, we have that

$$I(p) = -E[H(p)] = \frac{1}{p^2} - \frac{1-E(x)}{(1-p)^2} = \frac{1}{p^2} - \frac{\frac{p}{p} - \frac{1}{p}}{(1-p)^2} = \frac{1}{p^2} + \frac{\frac{1-p}{p}}{(1-p)^2} = \frac{1}{p^2} + \frac{1}{p(1-p)}$$

ii

To find the confidence interval, again, we use the normality assumption as

$$\hat{p}_{MLE} \sim N(P, \frac{1}{nI(p)})$$

Plugging in the given information from the prior question, we have that

$$\hat{p}_{MLE} \sim N(p, \frac{1}{nI(p)})$$

We can then find the variance of this through

$$Var(\hat{p}_{MLE}) = \frac{1}{nI(\theta)} = [\frac{n}{p^2} + \frac{n}{p(1-p)}]^{-1}$$

We can then find the confidence interval through plugging values in,

$$[\hat{p}_{MLE} - Z_{0.975} \times \{\frac{n}{\hat{p}_{MLE}^2} + \frac{n}{\hat{p}_{MLE}(1-\hat{p}_{MLE})}\}^{-\frac{1}{2}}, \hat{p}_{MLE} + Z_{0.975} \times \{\frac{n}{\hat{p}_{MLE}^2} + \frac{n}{\hat{p}_{MLE}(1-\hat{p}_{MLE})}\}^{-\frac{1}{2}}$$

8

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8. Consider the Rayleigh distribution given in Problem 5.
(i) Find the Fisher information of this statistical model.

Suppose a sample of n independent observations are to be taken and the maximum likelihood estimator will be used to estimate the unknown parameter θ.
(ii) Find the asymptotic variance of the MLE and give a 95% confidence interval of p approximately for large n.

i

The distribution is given through

$$f(x;\theta) = \frac{x}{\theta^2}\exp(-\frac{x^2}{2\theta^2})$$

Additionally, taking the log, we have

$$l(\theta;x) = \log f(x;\theta) = \log(x) - \log(\theta^2) - \frac{x^2}{2\theta^2}$$

Taking the derivative, we have that

$$S(\theta) = \frac{d}{d\theta}l(\theta;x) = -\frac{2\theta}{\theta^2} - \frac{1}{2}x^2(-2\frac{1}{\theta^3}) = -\frac{2}{\theta} + \frac{x^2}{\theta^3}$$

Taking the seonc derivative, we have

$$H(\theta) = \frac{d}{d\theta}S(\theta;x) = \frac{2}{\theta^2} - \frac{3}{\theta^4}x^2$$

Taking the negative expectation, we have

$$I(\theta) = -E[H(\theta)] = -E(\frac{2}{\theta^2} - \frac{3x^2}{\theta^4}) = -\frac{2}{\theta^2} + \frac{3}{\theta^4}E(x^2) = -\frac{2}{\theta^2} + \frac{3}{\theta^4}[Var(x) + E(x)^2]$$

Solving the equation,

$$I(\theta) = -\frac{2}{\theta^2} + \frac{3}{\theta^4}[\frac{4-\pi}{2}\theta^2 + \frac{\pi}{2}\theta^2] = -\frac{2}{\theta^2} + \frac{3}{\theta^4}(2\theta^2) = \frac{4}{\theta^2}$$

ii

Using the normality assumption, we have that

$$\hat{\theta}_{MLE} \sim N(\theta, \frac{1}{nI(\theta)})$$

Additionally, the variance can be found by

$$Var(\hat{\theta}_{MLE}) = \frac{1}{nI(\theta)} = \frac{\theta^2}{4n}$$

The 95% confidence interval can be found through

$$[\hat{\theta}_{MLE} - Z_{0.975} \times \sqrt{\frac{\hat{\theta}^2_{MLE}}{4n}}, \hat{\theta}_{MLE} + Z_{0.975} \times \sqrt{\frac{\hat{\theta}^2_{MLE}}{4n}}]$$

9

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9. Consider the double exponential distribution given by
$$f(x\,|\,\theta) = \frac{1}{2}e^{-|x-\theta|}\ , \ -\infty < x < \infty$$
The unknown parameter θ is a real number. Suppose an i.i.d. sample of size n=2m+1 (an odd number) is observed. Show that the sample median is the MLE of θ.

Given the double exponential distribution, we start with the joint likelihood function

$$L(\theta; x) = \prod_i \frac{1}{2}\exp(-|x_i - \theta|)$$

Taking the log, we have

$$l(\theta; x) = \sum_i [log(\frac{1}{2}) - |x_i - \theta|] = n\log(\frac{1}{2}) - \sum_{i:x_i \geq \theta}(x_i - \theta) - \sum_{i:x_i < \theta}(\theta - x_i)$$

Taking the derivative, we have that

$$\frac{d}{d\theta}l(\theta; x) = \sum_{i:x_i \geq \theta} 1 + \sum_{i:x_i < \theta} -1 = 0$$

We must set this derivative to equal 0. We will thus have to have theta to be the median of n sample.

10

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knitr::include_graphics("/Users/takaooba/Desktop/Screen Shot 2022-11-28 at 4.37.53 PM.png")
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10. Suppose that X has a normal density function with mean θ and variance θ^2 , $\theta \neq 0$ Suppose that n independent observations are to be taken. Denote the data observed by $x_1, \ldots x_n$. Find MLE of θ.
[hint : To decide which root to use, you may use the law of large number for help]

We initially start again with

$$L(\theta; x) = \prod_i \frac{1}{\sqrt{2\pi}} \times \frac{1}{\theta} \times \exp[-\frac{1}{2}(\frac{x_i - \theta}{\theta})^2]$$

Taking the log, we have that

$$l(\theta; x) = \sum_i [\log(\frac{1}{\sqrt{2\pi}}) + \log(\frac{1}{\theta}) - \frac{1}{2}(\frac{x_i - \theta}{\theta})^2] = n\log(\frac{1}{\sqrt{2\pi}}) + n\log(\frac{1}{\theta}) - \frac{1}{2}\sum_i \frac{x_i^2 - 2\theta x_i + \theta^2}{\theta^2}$$

Taking the derivative, we have that

$$\frac{d}{d\theta} l(\theta; x) = -\frac{n}{\theta} - \frac{1}{2}\sum_i \frac{(-2x_i + 2\theta)\theta^2 - (2\theta)(x_i^2 - 2\theta x_i + \theta^2)}{\theta^4} = -\frac{n}{\theta} - \frac{1}{2}\sum_i \frac{-2\theta^2 x_i + 2\theta^3 - 2\theta x_i^2 + 4\theta^2 x_i - 2\theta^3}{\theta^4}$$

Solving and simplifying,

$$-\frac{n}{\theta} - \frac{1}{2}\sum_i \frac{2\theta^2 x_i - 2\theta x_i^2}{\theta^4} = 0$$

Further, we have that

$$-\frac{n}{\theta} - \sum_i \frac{\theta x_i - x_i^2}{\theta^3} = 0$$

As we can distribute the summation, we have that

$$-n\theta^2 - \theta \sum_i x_i + \sum_i x_i^2 = 0b$$

Dividing everything by -n and letting

$$m = \frac{1}{n}\sum_i x_i^2$$

This results in

$$\theta^2 + \theta\bar{x} - m = 0$$

We can now solve MLE through knowing what m is

$$\hat{\theta}_{MLE} = \frac{-\bar{x} \pm \sqrt{\bar{x}^2 + 4m}}{2}$$

As n approaches infinity, we have that his value approaches

$$\hat{\theta}_{MLE} \rightarrow \theta$$

Additionally,

$$\bar{x} \rightarrow E(X) = \theta$$

Moving on, we have that

$$m \rightarrow E(X^2) = Var(X) + E(X)^2 = \theta^2 + \theta^2 = 2\theta^2$$

We can simplify through

$$\theta = \frac{-\theta \pm \sqrt{\theta^2 + 4(2\theta^2)}}{2}$$

$$\theta = \pm|\theta|$$

This condition can happen in two ways:

when x bar is greater than 0,

$$\hat{\theta}_{MLE} = \frac{-\bar{x} + \sqrt{\bar{x}^2 + 4m}}{2}$$

When x bar is less than 0,

$$\hat{\theta}_{MLE} = \frac{-\bar{x} - \sqrt{\bar{x}^2 + 4m}}{2}$$