Matematikk 2 - Øving 4

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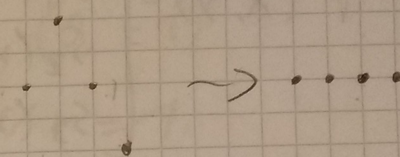
Øving 4

1a) $x = [0, 1, 0, -1]$



$n = 4$

Med fjerde enhetsrot $\omega = e^{-i2\pi/4}$
$= e^{-i\pi/2}$

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \frac{1}{\sqrt{n}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^4 \\ \omega^0 & \omega^2 & \omega^3 & \omega^6 \\ \omega^0 & \omega^4 & \omega^6 & \omega^9 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2} \\ -\frac{i}{2} - \frac{i}{2} \\ -\frac{1}{2} + \frac{1}{2} \\ \frac{i}{2} + \frac{i}{2} \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0 \\ -i \\ 0 \\ i \end{bmatrix}}}$$

$\omega^1 = e^{-i\pi/2} = -i$

$\omega^2 = e^{-i\pi} = -1$

$\omega^3 = e^{-i\frac{3\pi}{2}}, \ldots$

$\omega^3 = (-i)^3 = (-1)^3(-1)^{3/2} = (-1)^{3+1}(-1)^{1/2} = i$

$\omega^4 = (-i)^4 = (-1)^4(-1)^{4/2} = 1$

$\omega^6 = (-1)^6(-1)^{6/2} = (-1)^9 = -1$

$\omega^9 = (-1)^9(-1)^{9/2} = (-1)^{13}(-1)^{1/2} = -i$

3c) $y = [1, -i, 1, i] \Rightarrow n = 4$
as usual

$$F_4^{-1} y = \frac{1}{\sqrt{4}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^{-1} & \omega^{-2} & \omega^{-3} \\ \omega^0 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ \omega^0 & \omega^{-3} & \omega^{-6} & \omega^{-9} \end{bmatrix} \begin{bmatrix} 1 \\ -i \\ 1 \\ i \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1/i & -1 & 1/i \\ 1 & -1 & 1 & -1 \\ 1 & 1/i & -1 & -1/i \end{bmatrix} \begin{bmatrix} 1 \\ -i \\ 1 \\ i \end{bmatrix} = \begin{bmatrix} 2 \cdot \frac{1}{2} + 0i \\ 0 + \frac{i}{2i} + \frac{i}{2i} \\ 1 + \frac{i}{2} - \frac{i}{2} \\ 0 - \frac{1}{2} - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

10.2

1a) $t = [0 \ \frac{1}{4} \ \frac{1}{2} \ \frac{3}{4}]^T \quad x = [0 \ 1 \ 0 \ -1]^T$

DFT fast:

$$y = F_4 x = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2} \\ -i/2 - i/2 \\ -1/2 + 1/2 \\ i/2 + i/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -i \\ 0 \\ i \end{bmatrix}$$

Så bruker vi korollar 8: (intervall $[c, d] = [0, 1]$)

$$P_4(t) = \frac{a_0}{\sqrt{4}} + \frac{2}{\sqrt{4}} \sum_{k=1}^{\frac{4}{2}-1} \left(a_k \cos \frac{2k\pi(t-0)}{1-0} - b_k \sin \frac{2k\pi(t-0)}{1-0} \right)$$
$$+ \frac{a_{4/2}}{\sqrt{4}} \cos \frac{4\pi(t-0)}{1-0}$$

og at $y_k = a_k + b_k i$

~~forts.~~ 10.2.1a)

$p_4(t) = \cancel{\frac{0}{2}} - (-1)\sin(2\cdot 1\cdot \pi t) + \cancel{\frac{0/2}{2}\cdot \cos 4\pi t}$

$\llcorner a(b_1 = -1)$

(siden summen gikk fra $k=1$ til $k=\frac{n}{2}-1=1\ldots$)

$p_4(t) = \sin(2\pi t)$ wow. så enkel.

12.1

5a) 3×3-matrise A
egenverdier $\{3,1,4\}$

Største egenverdi er 4, det er den potensiterasjon vil konvergere mot, se teorem 12.2.

Raten: $S = \left|\frac{\lambda_2}{\lambda_3}\right| = \left|\frac{3}{4}\right| = \frac{3}{4} = 0{,}75$

Sortert:
$\lambda_1 = 4 > \lambda_2 = 3 > \lambda_3 = 1$

5b) $\lambda_1 = -4$, $S = |3/(-4)| = \frac{3}{4}$

5c) $\lambda_1 = 4$, $S = |2/4| = \frac{1}{2}$

5d) $\lambda_1 = 10$, $S = |9/10| = \frac{9}{10}$

7a) $s = 0 \Rightarrow A - sI = A, \quad \lambda_i \in \{3, 1, 4\}$

I følge lemma 12.3 vil egenverdien til A^{-1} m/ størst abs.verdi være $\frac{1}{\lambda_3}$ til A.

$\lambda_3 = 1$ og $\frac{1}{\lambda_3} = \frac{1}{1} = \underline{\underline{1}}$

er egenverdien $(A - sI)^{-1}$ vil konvergere mot.

$$s = \left| \frac{\frac{1}{\lambda_2}}{\frac{1}{\lambda_3}} \right| = \left| \frac{\frac{1}{3}}{\frac{1}{1}} \right| = \underline{\underline{\frac{1}{3} = 0,\overline{3}}}$$

7b) $\frac{1}{\lambda_3} = 1, \quad s = \left| \frac{1/3}{1/1} \right| = \frac{1}{3}$

7c) $\frac{1}{\lambda_3} = \frac{1}{-1} = -1, \quad s = \left| \frac{1/2}{1/(-1)} \right| = \frac{1}{2}$

7d) $\lambda - s$ og $s = 6 \Rightarrow \{-5, 3, 4\}$

$\lambda - s$ synkende: $-5, 4, 3$

og $\frac{1}{\lambda_3 - 5} = \frac{1}{3}, \quad s = \left| \frac{1/4}{1/3} \right| = \frac{3}{4}$

↳ altså egenverdien $\lambda_3 = 9$ opprinnelig

8a) $\{3, 1, 4\}$ sortert $\lambda_1 = 4, \lambda_2 = 3, \lambda_3 = 1$

$\lambda - s = \lambda - 5 \Rightarrow \{-2, -4, -1\}$

Invers potensiterasjon sortert $-4, -2, -1$
$\lambda_1 - s, \lambda_2 - s, \lambda_3 - s$

konv. mot $\frac{1}{\lambda_1 - s} = \frac{1}{-1} = \underline{\underline{-1}} = \lambda_1 - s$

og har $S = \left| \frac{1/(\lambda_2 - s)}{1/(\lambda_3 - s)} \right| = \left| \frac{1/-2}{1/-1} \right| = \underline{\underline{\frac{1}{2}}}$

Egenverdien for A:

$\lambda_1 - s = -1 \Leftrightarrow \lambda_1 = -1 + s = -1 + 5 = \underline{\underline{4}}$

9a)

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

Skal finne λ så $Ax = \lambda x$

$$Ax - \lambda x = 0$$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)(3-\lambda) - 2 \cdot 4$$
$$= 3 - \lambda - 3\lambda + \lambda^2 - 8 =$$

$$\det(A - \lambda I) = 0$$

Karakteristisk likning: $\lambda^2 - 4\lambda - 5 = 0$

$$\lambda = \frac{-(-4) \pm \sqrt{16 - 4 \cdot 1 \cdot (-5)}}{2 \cdot 1} = \frac{4 \pm \sqrt{36}}{2}$$

forts

$$= \frac{4 \pm 6}{2} = 2 \pm 3 \Rightarrow \underline{\underline{\lambda_1 = 5, \lambda_2 = -1}}$$

9a) forts., fikk $\lambda_1 = 5$ og $\lambda_2 = -1$

Setter inn i $(A - \lambda I)\vec{x} = 0$:

$\lambda_1 = 5$:

$$\begin{bmatrix} 1-5 & 2 \\ 4 & 3-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right] \overset{(1)}{\sim} \left[\begin{array}{cc|c} -4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$-4x_1 + 2x_2 = 0$$

$$2x_2 = 4x_1$$

$$x_2 = 2x_1$$

La $x_1 = s$, da er $x_2 = 2s$

$$\vec{x} = \begin{bmatrix} s \\ 2s \end{bmatrix} = s\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ med } s \in \mathbb{R} \setminus \{0\}$$

Med f.eks. $s = 1$: egenvektor $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda_2 = -1$:

$$\left[\begin{array}{cc|c} 1-(-1) & 2 & 0 \\ 4 & 3-(-1) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right] (-2)$$

$$\sim \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \cdot \tfrac{1}{2} \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1 + x_2 = 0 \iff x_1 = x_2$$

La $x_1 = s \implies \vec{x} = \begin{bmatrix} s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}, s \in \mathbb{R} \setminus \{0\}$

Med $s = 1$: egenvektor $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

10.3.1a)

Koeffisientene fra 10.2.1a:

$a_0 = b_0 = 0,\ a_1 = 0,\ b_1 = -1,\ a_2 = b_2 = 0,\ a_3 = 0,\ b_3 = 1$

kun $\underline{b_1 = -1}$ og $\underline{b_3 = 1}$ er $\neq 0$.

$$P_2(t) = \frac{a_0}{\sqrt{n}} + \frac{2}{\sqrt{n}} \sum_{k=1}^{\frac{m}{2}-1} \left(a_k \cos \frac{2k\pi(t-c)}{d-c} - b_k \sin \frac{2k\pi(t-c)}{d-c} \right) + \frac{2a_{m/2}}{\sqrt{n}} \cos \frac{n\pi(t-c)}{d-c}$$

(korollar 10.12)

siden $m = 2$

blir $\frac{m}{2} - 1 = 1 - 1 = 0$,

så hele summeleddet forsvinner.

Vi står igjen med:

$$P_2(t) = \frac{a_0}{\sqrt{4}} + \frac{2a_2}{\sqrt{4}} \cos \frac{4\pi(t-c)}{d-c}$$

som blir $\underline{\underline{0}}$ siden $a_0 = a_2 = 0$.

10.2 Trigonometric Interpolation

CP3

$$F_p^{-1}\sqrt{\frac{p}{n}}F_n x$$

de blå sirklene er datapunktene, den blå kurven er $f(t) = e^t$ og den oransje kurven er interpolasjonen.

```
range = (0:1/8:7/8)'
```

```
range = 8×1
         0
    0.1250
    0.2500
    0.3750
    0.5000
    0.6250
    0.7500
    0.8750
```

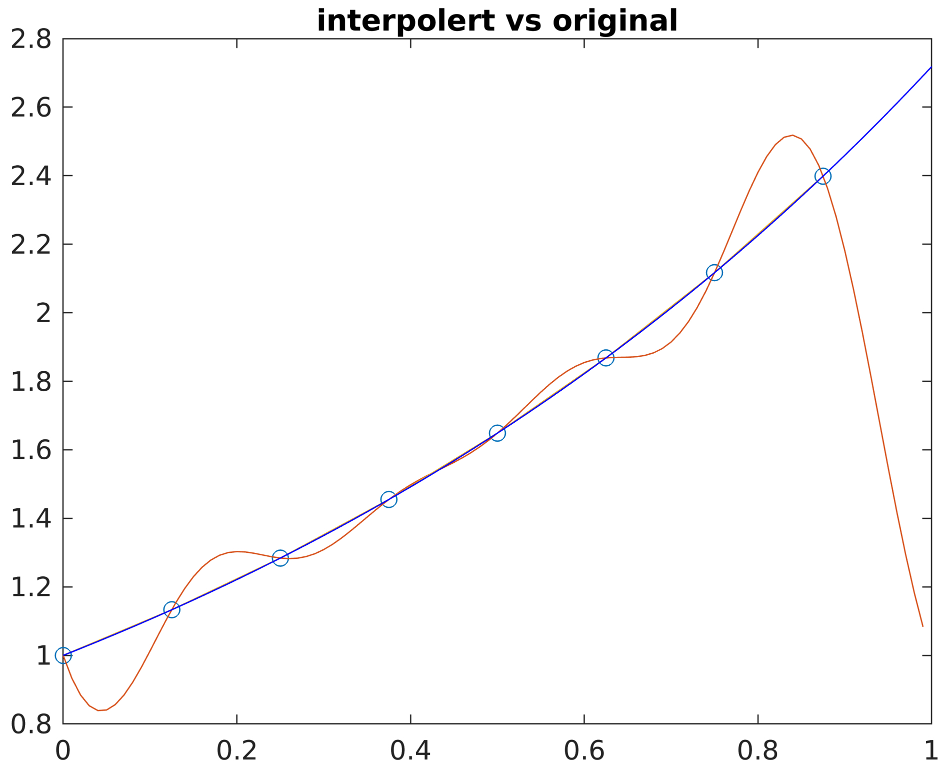
```
long_range = linspace(0,1,100)
```

```
long_range = 1×100
         0    0.0101    0.0202    0.0303    0.0404    0.0505    0.0606    0.0707 ⋯
```

```
x = arrayfun(@exp, range);
hold off;
dftinterp([0, 1], x, 8, 100)
```

```
ans = 100×1
    1.0000
    0.9334
    0.8844
    0.8532
    0.8390
    0.8407
    0.8565
    0.8842
    0.9213
    0.9651
      ⋮
```

```
hold on;
plot(range, x)
plot(long_range, arrayfun(@exp, long_range), 'color', 'blue')
title('interpolert vs original')
```




```
function xp=dftinterp(intervall,x,n,p)
c = intervall(1);
d = intervall(2);
t=c+(d-c)*(0:n-1)/n;
tp=c+(d-c)*(0:p-1)/p;
y=fft(x); % apply DFT
yp=zeros(p,1); % yp will hold coefficients for ifft
yp(1:n/2+1)=y(1:n/2+1); % move n frequencies from n to p
yp(p-n/2+2:p)=y(n/2+2:n); % same for upper tier
xp=real(ifft(yp))*(p/n); % invert fft to recover data
plot(t,x,'o',tp,xp) % plot data points and interpolant
end
```

12.1 Power Iteration

"You're stuck in a cycle, a repeating pattern. You want a way out."

"Know that this does not make you special—every living thing shares that same frustration. From the microbes in the processing strata to me, who am, if you excuse me, godlike in comparison."

CP1a

```
A = [10 -12 -6;
      5  -5 -4;
     -1   0  3];
[lambda, r] = powit(A, [5; 5; 5], 50)
```

```
lambda = 4.0000
r = 3×1
   -0.5774
   -0.5774
    0.5774
```

CP2a

```
[lambda, r] = invpowit(A, [0;1;0], 0, 50)
```

```
lambda = 1.0000
r = 3×1
    0.8165
    0.4082
    0.4082
```

Compared to exercise 7a's result of eigenvalue 1, this seems perfectly fine.

CP3a

With the shift $s = 5$:

```
[lambda, r] = invpowit(A, [0;1;0], 5, 50)
```

```
lambda = 4
r = 3×1
   -0.5774
   -0.5774
    0.5774
```

This matches with the result in 8a, where the shifted matrix converges towards the original matrix's eigenvalue 4.

CP4a

Rayleigh Quotient iteration – utilizing the Rayleigh Quotient as a self-updating shift towards an eigenvalue, since the quotient itself is an estimate of an eigenvalue. Shifiting by a number close to an eigenvaule will make inverse power iteration converge to that eigenvalue.

```
[lambda, r] = rayleigh(A, [-4;3;7], 6)
```

```
lambda = 4.0000
```

```
r = 3×1
   -0.5774
   -0.5774
    0.5774
```

Addendum

Notice the rather low number of iterations I set the code to run through. To achieve the same level of accuracy in CP1a I had to use 50 iterations.

```
% demonstration of < 50 iterations in CP1a
[lambda, r] = powit(A, [5; 5; 5], 49)
```

```
lambda = 4.0000
r = 3×1
   -0.5774
   -0.5774
    0.5773
```

One digit lost, see?

"At the end of time none of this will matter I suppose, but it would be nice if you took another way out."

```
function [lambda,u]=powit(A,x,k)
for j=1:k
    u = x/norm(x); % normalize
    x = A*u; % power step
    lambda = u'*x; % Rayleigh quotient
end
u = x/norm(x); % final normalization
end

function [lambda,u]=invpowit(A,x,s,k)
As=A-s*eye(size(A)); % shift matrix A
for j=1:k
    u = x/norm(x); % normalize
    x = As\u; % inverse power step
    lambda = u'*x; % Rayleigh Quotient
end
lambda = 1/lambda + s; % take reciprocal and unshift
u = x/norm(x);
end

function [lambda,u]=rayleigh(A,x,k)
I = eye(size(A));
for j=1:k
    u = x/norm(x); % normalize current  x
    lambda = u'*A*u; % Rayleigh quotient to shift by
    x = (A-lambda*I)\u; % inverse power iteration
end
u=x/norm(x);
lambda=u'*A*u;
end
```