Maximum of the two-dimensional Gaussian free field

Probability and PDEs

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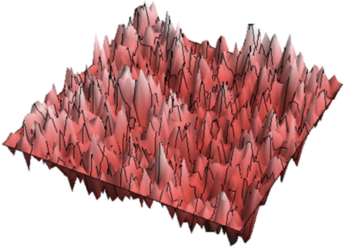
Part I

Introduction to the discrete Gaussian free field

Motivation

What is the Gaussian Free Field ?

- d-dimensional-time analog of Brownian motion



Why is it interesting to study?

- Useful for many constructions in quantum field theory
- Connections to the Schramm-Loewner evolution

General overview of the talk

- A review on the definition of DGFF and its basic properties
- Discuss the derivation of the behaviour of the maximum
- Size of the set of vertices where the field is close the max

Notation

Definition of average over the neighbourhood:

Let $f : \mathbb{Z}^d \to \mathbb{R}$,

$$\bar{f}(x) \coloneqq \frac{1}{2d} \sum_{y:y\sim x} f(y).$$

Definition of the discrete Laplacian Δf:

$$\Delta f(x) \coloneqq \bar{f}(x) - f(x).$$

Definition of the discrete boundary of $D \subset \mathbb{Z}^d$:

$$\partial D = \{x \in \mathbb{Z}^d \mid d(x, D) = 1\},$$

$$\bar{D} \coloneqq D \cup \partial D.$$

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Let

$$\mathcal{F}_{(D)} := \{f : \mathbb{Z}^d \to \mathbb{R} \mid f(x) = 0 \quad \forall\, x \notin D\}.$$

Let $E_{\bar{D}}$ be the set of all edges in $\mathbb{Z}^d$ such that at least one endpoint of the edge is in D.

For each $F \in \mathcal{F}_{(D)}$ and $e \in E_{\bar{D}}$, we define:

$$|\nabla F(e)| := |F(x) - F(y)|,$$

where x and y are the endpoints of edge e.

Finally, when D is finite, we define:

$$\mathcal{E}_D(F) := \sum_{e \in E_{\bar{D}}} |\nabla F(e)|^2.$$

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Definition of DGFF via the density function

The discrete GFF in D with Dirichlet boundary conditions (zero boundary conditions) on ∂D is the centered Gaussian vector $(\Phi(x))_{x\in D}$ whose density function on $\mathbb{R}^D$ at $(\phi_x)_{x\in D}$ is a constant multiple of

$$\exp\big(-\frac{1}{2}\frac{\mathcal{E}_D(\phi)}{2d}\big) = \exp\big(-\frac{1}{2}\frac{1}{2d}\sum_{e\in E_{\bar{D}}}|\nabla\phi(e)|^2\big),$$

where

$$|\nabla\phi(e)| = |\phi_x - \phi_y|,$$

with the convention that $\phi = 0$ on ∂D.

Note that $(\phi_x)_{x\in D} \mapsto \mathcal{E}_D(\phi)$ is positive definite.

Resampling procedure and consequences

Suppose that x is a given point in D. What is the conditional distribution of $\Phi(x)$ given $(\Phi(y))_{y\in D\setminus\{x\}}$?

It turns out the density of the conditional distribution of $\Phi(x)$ given that $\Phi(y) = h(y)$ for $y \in D \setminus \{x\}$ is proportional to:

$$\exp\big(-\frac{1}{2\times 2d}\sum_{y:\ y\sim x}|\phi_x - h(y)|^2\big).$$

Expanding this sum over y, we get:

$$\exp(-\frac{1}{2}(\phi_x - \bar{h}(x))^2).$$

Thus, the conditional law is that of the Gaussian distribution $\mathcal{N}(\bar{h}(x), 1)$

Observations

- The conditional distribution only depends on the values $h(y)$ at the neighbours of x
- The conditional law of $\Phi(x) - \bar{h}(x)$ is a standard normal Gaussian for all choices of h. This implies that $\Phi(x) - \bar{\Phi}(x)$ is a standard Gaussian RV independent of $(\Phi(y))_{y \in D \setminus \{x\}}$

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Consequences

1. Indicates the natural Markov chain (on the space of functions) for which the law of GFF is stationary. The Markovian step is the following: Given a function h in $\mathcal{F}_{(D)}$, choose a point $x \in D$ uniformly at random and replace the value $h(x)$ by $\bar{h}(x) + N$, when N is standard Gaussian RV.
2. Allows us to derive interesting properties about the covariance function of Φ.

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2. Allows us to derive interesting properties about the covariance function of Φ.

Property of the Covariance function of Φ

For all x and y in D, denote the covariance function by

$$\Sigma(x, y) = \Sigma_x(y) \coloneqq \mathbb{E}[\Phi(x)\Phi(y)].$$

$$\Delta\Sigma_x(y) = -\mathbb{1}_{\{y=x\}}.$$

Proof of the property

For each given x, $y \mapsto \Sigma_x(y)$ is a function in $\mathcal{F}_{(D)}$.

Then, for $x \neq y$ both in D,

$$
\begin{aligned}
\Sigma_x(y) &= \mathbb{E}[\Phi(x)\Phi(y)] = \mathbb{E}[\Phi(x)\bar{\Phi}(y)] + \mathbb{E}[\Phi(x)(\Phi(y) - \bar{\Phi}(y))] \\
&= \mathbb{E}[\Phi(x)\bar{\Phi}(y)] = \frac{1}{2d} \sum_{z\colon z \sim y} \mathbb{E}[\Phi(x)\Phi(z)] = \bar{\Sigma}_x(y).
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&= \frac{1}{2d} \sum_{z\colon z \sim x} \mathbb{E}[\Phi(x)\Phi(z)] + \mathbb{E}[(\Phi(x) - \bar{\Phi}(x))^2] \\
&\qquad\qquad + \mathbb{E}[(\Phi(x) - \bar{\Phi}(x))\bar{\Phi}(x)] \\
&= \bar{\Sigma}_x(x) + 1 + 0.
\end{aligned}
$$

Property of the Covariance function of Φ

Combining the results, we derive the following property for $\Sigma(x, y)$

$$\Delta\Sigma_x(y) = -\mathbb{1}_{\{y=x\}}.$$

The discrete Green's function

Let $(X_n)_{n\in\mathbb{N}}$ be a simple random walk on $\mathbb{Z}^d$ starting at x. Let

$$\tau = \tau_D := \inf\{n \geq 0 : X_n \notin D\}$$

be the first exit time of X_n from D.

Definition

The discrete Green's function on D is given by

$$G_D(x, y) := \mathbb{E}_x \left[\sum_{n=0}^{\tau-1} 1_{\{X_n = y\}} \right].$$

If either x or y is not in D, then $G_D(x, y) = 0$.

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The discrete Green's function

We can reexpress the Green's function as

$$\sum_{k\geq 0} \frac{1}{(2d)^k} \left|\{\text{paths of length } k \text{ from } x \text{ to } y \text{ contained in } D\}\right|.$$

We can see from this expression that the Green's function is symmetric, that is, $G_D(x,y) = G_D(y,x)$ for all $x, y \in \mathbb{Z}^d$. We can also see that if $D \subseteq D'$, then $G_D(x,y) \leq G_{D'}(x,y)$ for all $x, y \in \mathbb{Z}^d$.

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Second definition of the discrete GFF

Recall that we defined the discrete GFF as the centred Gaussian vector with density proportional to

$$\exp\left(-\frac{1}{2} \times \frac{\mathcal{E}_D(\phi)}{2d}\right)$$

where $\mathcal{E}_D(\phi) = \sum_{e \in E_{\overline{D}}} |\nabla\phi(e)|^2$ is a quadratic form.

Definition

The discrete GFF in D with Dirichlet boundary conditions on ∂D is the centred Gaussian vector $(\Phi(x))_{x \in D}$ whose covariance function is the discrete Green's function $G_D(x, y)$.

Second definition of the discrete GFF

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Equivalence between the definitions

We already saw that the covariance function of the discrete GFF satisfies

$$\Delta\Sigma_x(y) = -1_{\{y=x\}}.$$

Consider the function $g_x = G_D(\cdot, x)$. Let $y \in \mathbb{Z}^d$. Conditioning on the first step in the random walk, we get

$$\begin{aligned} g_x(y) &= \mathbb{E}_y\left[\sum_{n=0}^{\tau-1} 1_{\{X_n=x\}}\right] \\ &= 1_{\{y=x\}} + \frac{1}{2d}\sum_{z\sim y}\mathbb{E}_z\left[\sum_{n=1}^{\tau-1} 1_{\{X_n=x\}}\right] \\ &= 1_{\{y=x\}} + \overline{g_x}(y). \end{aligned}$$

Therefore, $\Delta g_x(y) = \overline{g_x}(y) - g_x(y) = -1_{\{y=x\}}$. It remains to show that Δ is injective.

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Injectivity of the discrete Laplacian

Lemma

Let D be a finite subset of $\mathbb{Z}^d$. Then, the discrete Laplacian $\Delta_D : \mathcal{F}_{(D)} \to \mathcal{F}_{(D)}$ is injective.

Proof.

$\square$

What's special about $d = 2$?

The case $d = 2$ is particularly interesting when we want to take the limit of the discrete GFF.

Lemma

Let $V_N = (0, N)^d \cap \mathbb{Z}^d$ *and for any* $\delta \in (0, 1/2)$*, denote* $V_N^\delta := (\delta N, (1-\delta)N)^d \cap \mathbb{Z}^d$*. Then, for any* $x \in V_N^\delta$*,*

$$G_{V_N}(x,x) \sim \begin{cases} N, & d = 1, \\ \log N, & d = 2, \\ 1, & d \geq 3. \end{cases}$$

Scaling limit when $d = 1$

When $d = 1$, fixing the domain and scaling the lattice down leads to the standard Brownian bridge.

Theorem

Suppose $d = 1$ and let Φ_{V_N} be the DGFF in $V_N = (0, N) \cap \mathbb{Z}$. Then

$$\left\{ \frac{1}{\sqrt{N}} \Phi_{V_N}(\lfloor tN \rfloor) : t \in [0,1] \right\} \to \left\{ \sqrt{2} W_t : t \in [0,1] \right\}$$

in law as $N \to \infty$, where W_t is the standard Brownian bridge.

Infinite volume limit when $d \geq 3$

When $d \geq 3$, fixing the lattice and scaling the domain up leads to the usual GFF on $\mathbb{Z}^d$.

Theorem

Suppose $d \geq 3$ and let $\tilde{V}_N := (-N/2, N/2)^d \cap \mathbb{Z}^d$. Then for any $x, y \in \mathbb{Z}^d$,

$$G_{\tilde{V}_N}(x, y) \to G_{\mathbb{Z}^d}(x, y)$$

as $N \to \infty$. In particular, $\Phi_{\tilde{V}_N} \to \mathcal{N}(0, G_{\mathbb{Z}^d})$, the full space DGFF.

Taking the limit with $d = 2$

What limit can we take when $d = 2$?

- Taking the infinite volume limit when $d = 2$ doesn't work since the Green function blows up by recurrence of the simple random walk.
- Taking the scaling limit doesn't work either if we normalize the variance by $\sqrt{\log N}$ since it converges to independent Gaussian variables indexed by $[0, 1]^2$, and therefore we lose all of the structure.
- We actually need to scale without normalizing. It doesn't converge pointwise, but it leads to the continuum GFF, which is defined on distributions.

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Discrete GFF with non-zero boundary

So far we have defined the discrete GFF in D to be zero on ∂D. More generally, given an arbitrary function

$$f : \partial D \to \mathbb{R},$$

we can define

Definition

The discrete GFF in D with boundary values f on ∂D is the Gaussian vector $(\Phi(x))_{x \in D}$ with density proportional to

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for ϕ such that $\phi\big|_{\partial D} = f$.

The expression for the density looks the same as for zero boundary conditions. The values of f are hidden in the expression for $\mathcal{E}_D(\phi)$.

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Markov Property, Version 1

Here is a Spatial Markov property. We will prove another version shortly.
Fix $O \subseteq D$ and a function $f : O^c \to \mathbb{R}$. Let Φ be a discrete GFF on D, with any given boundary values.

Proposition

The conditional law of $\Phi\big|_O$ given $\Phi\big|_{O^c} = f$ is the same as the law of a discrete GFF on O with boundary values $f\big|_{\partial O}$ on ∂O.

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Detour 1

Before *stating* the second Markov Property, we need to take a detour into discrete harmonic extensions.

Discrete Harmonic Extensions

Definition

Given a function
$$f : \partial D \to \mathbb{R}$$
the (discrete) harmonic extension of f in D is the unique function
$$F : \overline{D} \to \mathbb{R}$$
such that $F\big|_{\partial D} = f$ and $\Delta F\big|_{D} = 0$.

Existence and Uniqueness of Harmonic Extensions 1

Lemma

Every function $f : \partial D \to \mathbb{R}$ has a unique harmonic extension F to $\overline{D}$.

Similarly to proof of injectivity of Δ_D:

Proof of uniqueness.

Given F, G both extensions of f, consider $F - G$.

$$(F-G)\big|_{\partial D} = 0 \quad \text{and} \quad \Delta(F-G)\big|_{D} = 0.$$

By maximum principle, $F - G = 0$. □

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Existence and Uniqueness of Harmonic Extensions 2

Let $\mathbb{P}_x$ be law of simple random walk $(X_n)_{n\in\mathbb{N}}$ from x. Set

$$\tau = \tau_D := \inf\{n \geq 0 : X_n \notin D\}.$$

Proof of existence.

Define $F : \overline{D} \to \mathbb{R}$,

$$F(x) := \mathbb{E}_x \left[f(X_\tau) \right].$$

1. D is finite, so F is well-defined.
2. $F\big|_{\partial D} = f$.
3. $\Delta F\big|_D = 0$, since given every $x \in D$,

$$\mathbb{E}_x \left[f(X_\tau) \right] = \sum_{y\sim x} \frac{1}{2d} \mathbb{E}_y \left[f(X_\tau) \right]. \qquad \square$$

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Markov Property, Version 2

How can we relate the discrete GFFs with zero and non-zero boundary conditions with each other? Let Φ be a discrete GFF in D with zero boundary conditions on ∂D. Fix a function f on ∂D, and let F be its harmonic extension to D.

Lemma (Markov Property, Version 2)

$\Phi + F$ is a discrete GFF in D with f as boundary values.

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Detour 2

Before *proving* this Markov Property, we need to recall a fact about $\mathcal{E}_D$.

Adding Dirichlet Energies 1

$\mathcal{E}$ can be defined for all functions from $\mathbb{Z}^d$ of finite support. Here is the corresponding inner product, up to a constant factor.

Definition

Given $F_1, F_2 : \mathbb{Z}^d \to \mathbb{R}$ of finite support, we set

$$(F_1, F_2) := \frac{1}{2}\frac{1}{2d} \sum_{x \in \mathbb{Z}^d} \sum_{y \sim x} (F_1(y) - F_1(x))(F_2(y) - F_2(x)).$$

Adding Dirichlet Energies 2

This inner product can be written in another way:

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(F_1, F_2) &= \frac{1}{2}\frac{1}{2d}\sum_{x\in\mathbb{Z}^d}\sum_{y\sim x}(F_1(y)-F_1(x))(F_2(y)-F_2(x)) \\
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Adding Dirichlet Energies 3

From this formula, we know

Lemma

If F_1 is zero outside D, and F_2 is harmonic inside D, then $(F_1, F_2) = 0$. In particular,

$$(F_1, F_1) + (F_2, F_2) = (F_1 + F_2, F_1 + F_2).$$

Proof of Markov Property, Version 2

We are now ready to prove the second Markov Property.

Lemma (Markov Property, Version 2)

$\Phi + F$ is a discrete GFF in D with f as boundary values.

$\Phi + F$ has density proportional to

$$\exp\left(-\frac{1}{2}\frac{1}{2d}\mathcal{E}_D(\phi - F)\right) = \exp\left(-(\phi - F, \phi - F)\right),$$

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Proof of Markov Property, Version 2 (continued)

$\phi - F$ is zero outside D, and F is harmonic inside D. So

$$(\phi - F, \phi - F) + (F, F) = (\phi, \phi).$$

Hence the density of $\Phi + F$ is proportional to

$$\begin{aligned}\exp\left(-(\phi - F, \phi - F)\right) &\propto \exp\left(-(\phi, \phi)\right) \\ &= \exp\left(-\frac{1}{2}\frac{1}{2d}\mathcal{E}_D(\phi)\right).\end{aligned}$$

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Key Takeaways

To follow the rest of the presentation, the key facts to remember are

1. Definition of the discrete GFF via density
2. Definition of the discrete GFF via Green's functions
3. Spatial Markov Properties

Part II

Extremes of the discrete two-dimensional Gaussian free field

Introduction and Notation

Consider the square $V_N = \{1, \ldots, N\}^2$ for $N \geq 1$ and the discrete Gaussian free field $\Phi_N = (\phi_x)_{x \in V_N}$. We want to understand the maximum $\max_{x \in V_N} \phi_x$. We will establish that

$$\max_{x \in V_N} \phi_x \qquad \text{behaves like} \qquad 2\sqrt{g}\log(N)$$

for $g = \frac{2}{\pi}$.

We compare this result to the behaviour of independent Gaussian variables. Let $X_1, \ldots, X_{N^2}$ be independent $\mathcal{N}(0, \sigma^2)$-distributed variables. Then

$$\max_{1 \leq i \leq N^2} X_i \qquad \text{behaves like} \qquad \sigma\sqrt{2\log(N^2)} = 2\sigma\sqrt{\log(N)}.$$

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Behaviour of the Green's function

We denote $V_N^\delta = [\delta N, (1-\delta)N]^2 \cap \mathbb{Z}^2$.

Lemma

(a) For $g = \frac{2}{\pi}$ and c an absolute constant, it holds that

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$$\lim_{N\to\infty} \mathbb{P}_N[\max_{x\in V_N} \phi_x \geq 2\sqrt{g}\log(N)] = 0$$

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Recap: Domain-Markov Property

Notation: $\mathbb{E}_B$ denotes the expectation w.r.t. a DGFF $(\phi_x^B)_{x\in B}$ on $B \subset \mathbb{Z}^2$. Analogously for var_B. Also, $\mathcal{F}_{\partial B}$ denotes the sigma algebra generated by the boundary values $\phi_x^B, x \in \partial B$. If clear from context, we will drop the sub-/supercript B.

For $x \in B \subset C$, the conditional expectation of the field in B, given the boundary values, is the harmonic extension of these:

$$\mathbb{E}_C[\phi_x^C \mid \mathcal{F}_{\partial B}] = \sum_{y\in\partial B} \alpha_{\partial B}(x,y)\phi_y^C,$$

where $\{\alpha_{\partial B}(x,y)\}_{y\in\partial B}$ is the first-exit distribution of a simple random walk started in x.

Moreover, we can write

$$\phi_x^C = \mathbb{E}[\phi_x^C \mid \mathcal{F}_{\partial B}] + \phi_x^B,$$

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Consequences of the Domain-Markov Property (DMP)

Additivity of Fluctuations: For $x \in B \subset C$, we have:

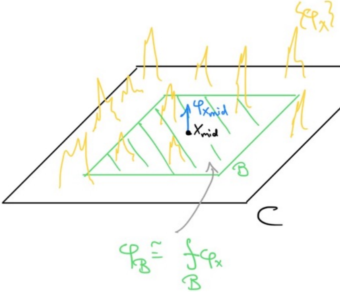
$$\mathrm{var}_C(\phi_x) = \mathrm{var}_C(\mathbb{E}_C[\phi_x | \mathcal{F}_{\partial B}]) + \mathrm{var}_B(\phi_x).$$

In particular, in the case where $B \subset C$ are two boxes with centre x_{mid}:

$$\boxed{\phi_B := \mathbb{E}_C[\phi_{x_{\mathrm{mid}}} | \mathcal{F}_{\partial B}] \quad \simeq \frac{1}{|B|} \sum_{x \in B} \phi_x},$$

and we obtain

$$\mathrm{var}_C(\phi_{x_{\mathrm{mid}}}) = \mathrm{var}_C(\phi_B) + \mathrm{var}_B(\phi_{x_{\mathrm{mid}}}).$$



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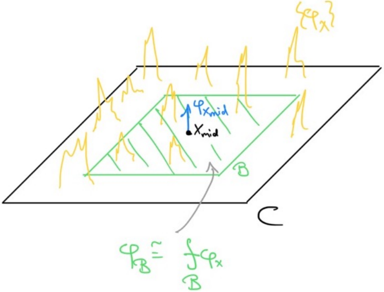

Scaling Behaviour of Fluctuations

Recall: We have control over the field's fluctuations in a box:

1. $\max_{x \in V_N} \mathrm{var}_{V_N}(\phi_x) \leq g \log(N) + c$
2. $\max_{x \in V_N^\delta} |\mathrm{var}_{V_N}(\phi_x) - g \log(N)| \leq c(\delta)$ for $\delta \in [0, 1/2)$

Suppose $B \subseteq C$ are boxes of side-length N^α and N^β, respectively, where $0 \leq \alpha < \beta \leq 1$. Also assume that $\mathrm{dist}(B, \partial C) \geq \frac{1}{4} N^\beta$. Then:

Scaling Behaviour

$$\mathrm{var}_C(\phi_B) = g(\beta - \alpha) \log(N) + O(1)$$

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Alpha Boxes

We are interested in intermediate length scales $N^\alpha, \alpha \in (0,1)$. We also fix $\delta = 1/2 - \bar{\delta} \in [0, 1/2)$.

To avoid irritating corrections we always assume that N^α is an odd integer and that $N^\alpha - 1$ divides $2\bar{\delta}N - 1$. For $i = (i_1, i_2)$, $1 \leq i_1, i_2 \leq \frac{2\bar{\delta}N-1}{N^\alpha - 1}$ we consider sub-boxes:

$$\begin{aligned} B_i^\alpha = &[(i_1 - 1)(N^\alpha - 1) + 1, i_1(N^\alpha - 1) + 1] \\ &\times [(i_2 - 1)(N^\alpha - 1) + 1, i_2(N^\alpha - 1) + 1]. \end{aligned}$$

We denote the set of α-boxes in V_N^δ by Π_α.

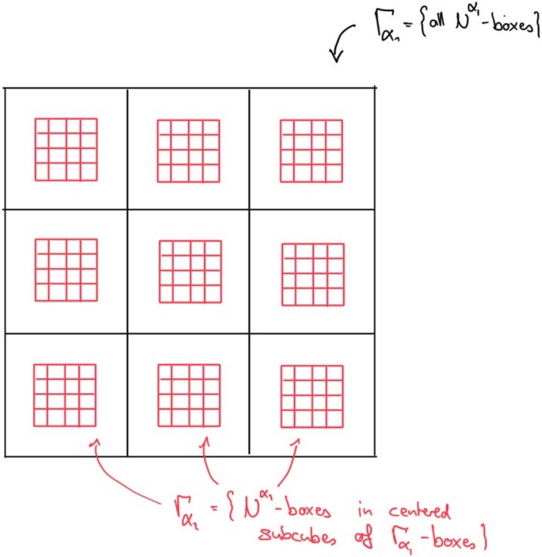
Alpha Boxes Cont.

Boundaries of neighbouring α-boxes intersect. We define:

$$\bigcup_i \partial B_i^\alpha = \Delta_N^\alpha := \left\{ k(N^\alpha - 1) + 1 : 0 \leq k \leq \frac{2\bar{\delta}N - 1}{N^\alpha - 1} \right\}^2 .$$

We denote by $\mathcal{F}_\alpha$ the σ-field generated by $\phi_x, x \in \Delta_n^\alpha$.

Arranging the Boxes. . .



- $K \in \mathbb{N}$
- $\alpha_i = \frac{K-i+1}{K}\alpha$
- $1 < i < K$
- $(N^{\alpha_i} - 1)/2$

Towards a Proof of Extremal Asymptotics

Theorem (Bolthausen, Deuschel, Giacomin)

(a)
$$\lim_{N\to\infty} \mathbb{P}_N[\max_{x\in V_N} \phi_x \geq 2\sqrt{g}\log(N)] = 0$$

(b) For $\eta > 0$ and any $\delta \in [0, \frac{1}{2})$ there exists $c = c(\delta, \eta) > 0$ such that

$$\mathbb{P}_N\left[\sup_{x\in V_N^\delta} \phi_x \leq (2\sqrt{g}-\eta)\log(N)\right] \leq \exp[-c(\log N)^2]$$

if N is large enough.

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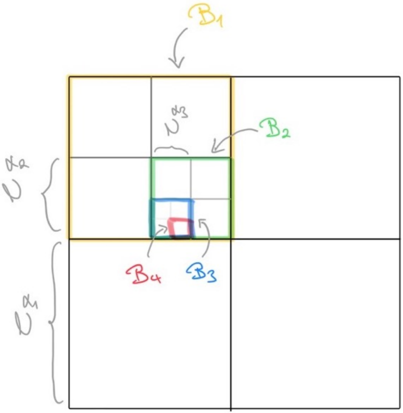

Recall: For $B_{k+1} \in \Gamma_{\alpha_{k+1}, B_k}$ we have

$$\begin{aligned} &\operatorname{var}_{\mathcal{F}_{\partial B_k}}(\phi_{B_{k+1}} - \phi_{B_k}) \\ &= \frac{\alpha}{K} g \log N + O(1). \end{aligned}$$

If the random variables $\{\phi_{B_{k+1}} - \phi_{B_k}\}$ with $B_{k+1} \in \Gamma_{\alpha_{k+1}, B_k}$ were independent, we would obtain

$$\begin{aligned} \phi_{B_{k+1}} &\cong \phi_{B_k} + 2\sqrt{\frac{\alpha}{K} g} \log N \\ &\geq \phi_{B_k} + \frac{\alpha}{K} 2\sqrt{g} \log N. \end{aligned}$$

Supposing $\phi_{B_1} \geq 0$, we could conclude...

Problem: For $B^* \in \Gamma_{\alpha_k}$ the averages ϕ_B over the boxes $B \in \Gamma_{B^*,\alpha_{k+1}}$ are not independent.

$\rightsquigarrow$ We need "many" candidates $B^* \in \Gamma_{\alpha_k}$.

Abundance of Growth Chains: We denote by $\underline{B}^{(k)} = (B_1, \dots, B_k)$ a sequence $B_1 \supseteq \cdots \supseteq B_k$ of nested boxes with $B_i \in \Gamma_{\alpha_i}$. Define the sequence of events ($k = 1, \dots, K$)

$$C_k := \Big\{ \#\{\underline{B}^{(k)} \mid \phi_{B_i} \geq (i-1)\tfrac{\alpha}{K} 2\sqrt{g}\log(N) \times (1 - \tfrac{1}{2\sqrt{g}K}) \text{ for } i = 1, \dots, k\} \geq N^{\kappa} \Big\},$$

with some (small) abundance parameter $\kappa > 0$. Then

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with $K = K(\eta)$ and $\alpha = \alpha(\delta, \eta)$ appropriately chosen.

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Bootstrap Procedure:

$$\boxed{\mathbb{P}[C_{k+1}^{\mathrm{c}}] \leq \mathbb{P}[C_k^{\mathrm{c}}] + \mathbb{E}\Big[\,\mathbb{P}[C_{k+1}^{\mathrm{c}}|\mathcal{F}_k]\,\Big|C_k\Big]}$$

Hence, to bound $\mathbb{P}_N[C_K^{\mathrm{c}}]$ we…

- …show "*abundance of positivity*":

  $$\mathbb{P}[C_1^{\mathrm{c}}] = \mathbb{P}\left[\#\{B \in \Gamma_{\alpha_1} \mid \phi_B \geq 0\} < N^\kappa\right] \leq \exp[-c(\log N)^2]$$

- …show that "*failing to grow if you're large is unlikely*":

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<u>Induction Base Case</u>: We have control over $\mathbb{P}[C_1^c]$ by…

Lemma (Abundance of Positivity)

Given $\alpha \in (\frac{1}{2}, 1)$, *there exist* $\kappa(\alpha), a(\delta, \alpha)$ *such that*

$$\mathbb{P}[\#\{B \in \Pi_\alpha : \phi_B \geq 0\} \leq N^\kappa] \leq \exp[-a(\log N)^2]$$

Induction Step: Conditioned on C_k, let $\{B_j\}_{j=1}^{N^\kappa}$ enumerate the (at least N^κ) "large" α_k-boxes. We can write

$$C_k \cap C_{k+1}^{\mathrm{c}} \subseteq C_k \cap \left\{ \sum_{j=1}^{N^\kappa} \zeta_j \leq 4N^{\kappa - \frac{2\alpha}{K}} \right\}, \qquad (\heartsuit)$$

where

$$\zeta_j = \frac{1}{\left|\Gamma_{B_j,\alpha_{k+1}}\right|} \sum_{B \in \Gamma_{B_j,\alpha_{k+1}}} \mathbb{1}\left\{ \phi_B - \phi_{B_j} \geq \frac{\alpha}{K} 2\sqrt{g} \log(N) \times \left(1 - \frac{1}{2\sqrt{g}K}\right) \right\},$$

is the fraction of α_{k+1}-boxes in B_j with "large fluctuations".

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Short Interlude:

Lemma (Large Deviation Estimates for bounded i.i.d. sums.)

Suppose $\zeta_1, \ldots, \zeta_n$ are i.i.d. with $\|\zeta_i\|_\infty \leq 1$. Write $\mu = \mathbb{E}[\zeta_i]$ and $\sigma^2 = \mathrm{var}[\zeta_i]$. Then, for any $t > 0$:

$$\mathbb{P}\left[\left|\sum_{i=1}^{n}(\zeta_i - \mu)\right| \geq n\sigma^2\, t\right] \leq 2\exp\left[-n\sigma^2 \frac{t^2}{1+t}\right]$$

Final Spurt: We can show for $K \gg 1$

$$\mathbb{E}[\zeta_j|\mathcal{F}_k] \geq N^{-\frac{2\alpha}{K}+\frac{\alpha}{\sqrt{g}K^2}}.$$

Therefore, conditional on C_k and $\mathcal{F}_k$, we have for $N \gg 1$

$$\left\{\sum_{j=1}^{N^\kappa} \zeta_j \leq 4N^{\kappa-\frac{2\alpha}{K}}\right\} \subseteq \left\{\left|\sum_{j=1}^{N^\kappa} \zeta_j - \mathbb{E}[\zeta_j|\mathcal{F}_k]\right| \geq \frac{1}{2}N^{\kappa-\frac{2\alpha}{K}+\frac{\alpha}{\sqrt{g}K^2}}\right\}$$

We also have $0 \leq \zeta_j \leq 1$, so by the large deviation estimate:

$$\mathbb{E}\Big[\,\mathbb{P}[C_{k+1}^{\mathrm{c}}|\mathcal{F}_k]\;\Big|C_k\Big] \leq 2\exp\left[-c\,N^{\kappa-\frac{2\alpha}{K}}\right],$$

for large enough K and α sufficiently close to 1.

Strapping the Boots: For any $\eta, \delta > 0$, we can choose $\alpha = \alpha(\eta, \delta)$ close to 1 and $K = K(\alpha)$ large, such that

$$\begin{aligned} \mathbb{P}_N[\max_{x \in V_N^\delta} \phi_x \leq (2\sqrt{g} - \eta) \log N] &\leq \mathbb{P}_N[C_K^{\mathrm{c}}] \\ &\leq \mathbb{P}_N[C_1^{\mathrm{c}}] + (K-1)\, 2 \exp[-c\, N^{\kappa - \frac{2\alpha}{K}}] \\ &\leq \exp[-c(\log N)^2], \end{aligned}$$

with a constant $c = c(\eta) > 0$. ■ Q.E.D.

Part III

Entropic repulsion and the maximum of the two dimensional harmonic crystal

Setting

We fix an arbitrary $\delta \in (0, 1/2)$ and define

$$V_N := \{1, \dots, N\}^2, \quad V_N^\delta := [\delta N, (1-\delta)N]^2 \cap \mathbb{Z}^2.$$

We have seen that

$$\lim_{N\to\infty} \mathbb{P}\Big[\sup_{x \in V_N} \phi_x \geq 2\sqrt{g}\log(N)\Big] = 0,$$

$$\mathbb{P}\Big[\sup_{x \in V_N^\delta} \phi_x \leq (2\sqrt{g} - \eta)\log(N)\Big] \leq e^{-c(\log(N))^2}, \quad \eta > 0.$$

Olivier Daviaud:

- How can we characterise level sets?
- How are extreme points related?

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Olivier Daviaud:

- How can we characterise level sets?
- How are extreme points related?

Notation

Low points and high points:

$$\mathcal{L}_N(\eta) := \big\{x \in V_N^\delta \colon \phi_x \le 2\sqrt{g}\log(N)\,(1-\eta)\big\},$$
$$\mathcal{H}_N(\eta) := \big\{x \in V_N^\delta \colon \phi_x \ge 2\sqrt{g}\log(N)\,\eta\big\}, \qquad \eta \in (0,1).$$

Downward and upward spikes:

$$D_N(\eta) := \sup\Big\{a \in \mathbb{N} \colon \exists\, x \in V_N^\delta \text{ s.t. } \max_{B(x,a)} \phi \le 2\sqrt{g}\log(N)\,(1-\eta)\Big\},$$
$$U_N(\eta) := \sup\Big\{a \in \mathbb{N} \colon \exists\, x \in V_N^\delta \text{ s.t. } \max_{B(x,a)} \phi \ge 2\sqrt{g}\log(N)\,\eta\Big\}.$$

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$$\Omega^+_{N,\delta} := \{\phi_x \ge 0, x \in V_N^\delta\}.$$

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High points and size of upward spikes

Theorem

Let $\eta \in (0,1)$, *then*

$$\lim_{N\to\infty} \frac{\log|\mathcal{H}_N(\eta)|}{\log(N)} = 2(1-\eta^2) \quad \textit{in probability.}$$

Moreover, for all $\varepsilon > 0$ *there exists a constant* $C > 0$ *such that*

$$\mathbb{P}\left[|\mathcal{H}_N(\eta)| \leq N^{2(1-\eta^2)-\varepsilon}\right] \leq e^{-C(\log(N))^2}.$$

Theorem

Let $\eta \in (-1,1)$, *then*

$$\lim_{N\to\infty} \frac{\log(U_N(\eta))}{\log(N)} = \frac{1-\eta}{2} \quad \textit{in probability.}$$

Low points and size of downward spikes

Theorem

Let $\eta \in (0,1)$, then

$$\lim_{N\to\infty} \frac{\log|\mathcal{L}_N(\eta)|}{\log(N)} = 2(1-\eta^2) \quad \textit{in probability under } \mathbb{P}(\,\cdot\,|\Omega^+_{N,\delta}).$$

Theorem

Let $\eta \in (0,1)$, then

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Clustering of high points

Consider the number of η-high points in the neighbourhood $D(x, N^{\beta})$ around a point x with $0 < \eta < \beta < 1$.

- evenly spread: $\mathcal{O}(N^{2\beta}N^{-2}N^{2-2\eta^2}) = \mathcal{O}(N^{2\beta-2\eta^2})$
- x is "ordinary" point: $\mathcal{O}(N^{2\beta-2\eta^2/\beta})$
- x is high point: $\mathcal{O}(N^{2\beta-2\eta^2\beta})$

Hence, high points of the DGFF appear in clusters.

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Size of set of high points

Recall: $\mathcal{H}_N(\eta) := \left\{x \in V_N^\delta \colon \phi_x \geq \eta\, 2\sqrt{g}\log(N)\right\}, \qquad \eta \in (0,1).$

Theorem (Daviaud)

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Recall

- There exist c and $c(\delta)$ such that for any N
  1. $\sup_{x \in V_N} \mathrm{Var}(\phi_x) \leq g \log N + c$,
  2. $\left| \sup_{x \in V_N^\delta} \mathrm{Var}(\phi_x) - g \log N \right| \leq c(\delta)$.

- If $X \sim \mathcal{N}(0, \sigma^2)$, then for any $a \geq 1$,

$$\mathbb{P}[|X| \geq a] \leq e^{-a^2/(2\sigma^2)}.$$

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Proof of upper bound

We want to show that $\forall \varepsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}\left[\frac{\log|\mathcal{H}_N(\eta)|}{\log(N)} > 2(1-\eta^2)+\varepsilon\right] = 0.$$

By Markov's inequality

$$\mathbb{P}[|\mathcal{H}_N(\eta)| \geq N^{2(1-\eta^2)+\varepsilon}] \leq N^{-2(1-\eta^2)-\varepsilon}\mathbb{E}[|\mathcal{H}_N(\eta)|].$$

And

$$\begin{aligned}\mathbb{E}[|\mathcal{H}_N(\eta)|] &= \mathbb{E}[\sum_{x\in V_N} \mathbb{1}\{\phi_x \geq 2\eta\sqrt{g}\log N\}] \\ &\leq N^2 \max_{x\in V_N} \mathbb{P}[\phi_x \geq 2\eta\sqrt{g}\log N] \\ &\leq N^2 \max_{x\in V_N} \exp\left(-\frac{4\eta^2 g(\log N)^2}{2\mathrm{Var}(\phi_x)}\right) \lesssim N^{2-2\eta^2}.\end{aligned}$$

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We want to show that $\forall \varepsilon > 0$

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Proof of lower bound

We want to show that $\forall\, \varepsilon > 0$,

$$\mathbb{P}\left[|\mathcal{H}_N(\eta)| < N^{2(1-\eta^2)-\varepsilon} \right] \leq \exp(-c(\log N)^2).$$

Notations:

- For $A \subset V_N$, let $\mathcal{F}_A := \sigma(\{\phi_x\}_{x\in A})$,
- For any box B, x_B is its center, and $\phi_B := \mathbb{E}[\phi_{x_B} \,|\, \mathcal{F}_{\partial B}]$,
- For $\beta \in [0,1)$, Π_β is the collection of boxes of edge length N^β in V_N^δ such that $V_N^\delta = \cup_{B\in\Pi_\beta} B$.

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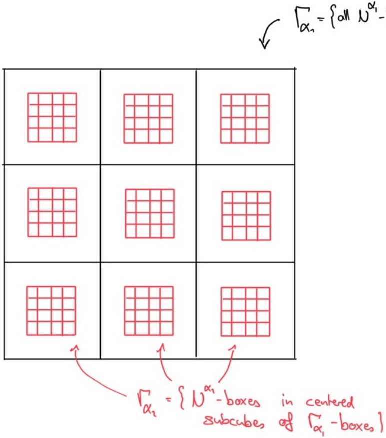
Setting



- $1/2 < \alpha < 1$, $K \in \mathbb{N}$.
- $\alpha_i := \frac{K-i}{K}\alpha, \qquad 0 \le i \le K.$
- The collections of α_i-boxes:

$$\Gamma_{\alpha_0} := \Pi_{\alpha_0},$$
$$\Gamma_{\alpha_{i+1}} := \bigcup_{B \in \Gamma_{\alpha_i}} \Gamma_{B,\alpha_{i+1}},$$

  where $\Gamma_{B,\alpha_{i+1}}$ contains the α_{i+1}-boxes that intersect a square of side length $N^{\alpha_i}/2$ inside B.
- $\underline{B}^{(k)} = (B_0, \dots, B_k)$ is a sequence of boxes $B_0 \supseteq \cdots \supseteq B_k$ with $B_i \in \Gamma_{\alpha_i}$.

Proof of lower bound

We will show that

$$\mathbb{P}\left[|\mathcal{H}_N(\eta\alpha(1-1/K))| < N^{\kappa+2\alpha(1-\eta^2)}\right] \leq \exp(-c(\log N)^2)$$

where κ is as in the extremal asymptotics proof.

The lower bound for $|\mathcal{H}_N(\eta)|$ follows by choosing α close to 1 and K large enough.

Proof of lower bound

Let

$$D_k := \{\underline{\mathrm{B}}^{(k)} : \phi_{B_i} \geq i\frac{\alpha}{K}(1-1/K)\eta\, 2\sqrt{g}\log N \quad \forall\, 0 \leq i \leq k\}.$$

If $\underline{\mathrm{B}}^{(K)} \in D_K$, then

- $\phi_{B_K} \geq \alpha(1-1/K)\eta\, 2\sqrt{g}\log N$.
- $\alpha_K = 0$, $B_K \in \Gamma_{\alpha_K} \implies B_K = \{x\}$ and $\phi_{B_K} = \phi_x$.

$\rightsquigarrow x \in \mathcal{H}_N(\alpha(1-1/K)\eta)$.

$$C_k := \{\# D_k \geq n_k\},$$

where

$$n_k := N^{\kappa + 2k\frac{\alpha}{K}(1-\eta^2)}.$$

We have

$$C_K \subset \left\{ |\mathcal{H}_N(\eta\alpha(1-1/K))| \geq n_K := N^{\kappa+2\alpha(1-\eta^2)} \right\}.$$

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Proof of lower bound

We have

$$C_k \cap C_{k+1}^{\mathrm{c}} \subset C_k \cap \left\{ \sum_{j=1}^{n_k} \zeta_j \le \frac{1}{\left|\Gamma_{B_1,\alpha_{k+1}}\right|} n_{k+1} \right\},$$

where

$$\zeta_j := \frac{1}{\left|\Gamma_{B_j,\alpha_{k+1}}\right|} \sum_{B \in \Gamma_{B_{j,\alpha_{k+1}}}} \mathbb{1}\{\phi_B - \phi_{B_j} \ge \frac{\alpha}{K}(1-1/K)\eta 2\sqrt{g}\log N\}.$$

with $B_j \in \Gamma_{\alpha_k}$ enumerating the (at least n_k) boxes of scale α_k from C_k.

- ζ_j are i.i.d. under $\mathbb{P}[\cdot \,|\, \mathcal{F}_k]$ (where $\mathcal{F}_k := \sigma(\cup_{B\in\Pi_{\alpha_k}} \partial B)$).
- We have $\mathrm{Var}_{\mathcal{F}_k}(\phi_B - \phi_{B_j}) \ge \frac{g\alpha}{K}\log N + \mathcal{O}(1)$.

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Proof of lower bound

We can show that

$$\mathbb{E}[\zeta_j \,|\, \mathcal{F}_k] \geq N^{-\frac{2\alpha}{K}(1-\frac{1}{K})\eta^2}.$$

By the large-deviation lemma we obtain

$$\begin{aligned}
&\mathbb{E}[\mathbb{P}[C_{k+1}^{\mathrm{c}} \mid \mathcal{F}_k]; C_k] \\
&\quad \leq 2 \exp\left[-c\, N^{\kappa+\frac{2\alpha}{K}(1-\eta^2)k-\frac{4\alpha}{K}(1-\frac{1}{K})\eta^2}\right].
\end{aligned}$$

Considering the decomposition

$$\begin{aligned}
\mathbb{P}[C_K^{\mathrm{c}}] &\leq \mathbb{P}[C_K^{\mathrm{c}} \cap C_{K-1}] + \mathbb{P}[C_{K-1}^{\mathrm{c}}] \\
&\leq \sum_{k=1}^{K} \mathbb{P}[C_k^{\mathrm{c}} \cap C_{k-1}] + \mathbb{P}[C_0^{\mathrm{c}}] \\
&= \sum_{k=1}^{K} \mathbb{E}[\mathbb{P}[C_k^{\mathrm{c}} \mid \mathcal{F}_{k-1}]; C_{k-1}] + \mathbb{P}[C_0^{\mathrm{c}}],
\end{aligned}$$

and using the fact that $\mathbb{P}[C_0^{\mathrm{c}}] \leq e^{-a(\log N)^2}$, we get the result.

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and using the fact that $\mathbb{P}[C_0^{\mathrm{c}}] \leq e^{-a(\log N)^2}$, we get the result.

High points in fixed neighborhood

Theorem

For $0 < \alpha < \beta < 1$, $\varepsilon > 0$,

$$\lim_{N\to\infty} \max_{x \in V_N^\delta} \mathbb{P}\left[\left|\frac{\log\left|\mathcal{H}_N(\alpha) \cap B(x, N^\beta)\right|}{\log(N)} - 2\beta\big(1 - \frac{\alpha^2}{\beta^2}\big)\right| > \varepsilon\right] = 0.$$

Sketch of proof. Let $B := B(x, 4N^\beta)$, $y \in B(x, N^\beta)$ we have

$$\mathbb{E}[\phi_y \,|\, \mathcal{F}_{\partial B}] \approx \mathbb{E}[\phi_x \,|\, \mathcal{F}_{\partial B}] =: \phi_B \approx 0.$$

Then

$$\phi_y \approx \phi_y - \mathbb{E}[\phi_y \,|\, \mathcal{F}_{\partial B}],$$

$$\{\phi_y \geq \alpha\sqrt{g}\log N\} \approx \{\phi_y - \mathbb{E}[\phi_y \,|\, \mathcal{F}_{\partial B}] \geq (\alpha/\beta)\sqrt{g}\log 4N^\beta\}.$$

Conditioning on $\mathcal{F}_{\partial B}$ the number of such points is

$$|\mathcal{H}_{N^\beta}(\alpha/\beta)| \approx (N^\beta)^{2(1-(\alpha/\beta)^2)}.$$

High points in fixed neighborhood

Theorem

For $0 < \alpha < \beta < 1$, $\varepsilon > 0$,

$$\lim_{N\to\infty} \max_{x \in V_N^\delta} \mathbb{P}\left[\left|\frac{\log\left|\mathcal{H}_N(\alpha) \cap B(x, N^\beta)\right|}{\log(N)} - 2\beta\big(1 - \frac{\alpha^2}{\beta^2}\big)\right| > \varepsilon\right] = 0.$$

Sketch of proof. Let $B := B(x, 4N^\beta)$, $y \in B(x, N^\beta)$ we have

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High points around a high point

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For $0 < \alpha, \beta < 1$ *and* $\varepsilon > 0$,

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References

M. Biskup.
Extrema of the two-dimensional discrete Gaussian free field.
In *PIMS-CRM Summer School in Probability*, pages 163–407. Springer, 2017.

E. Bolthausen, J.-D. Deuschel, and G. Giacomin.
Entropic repulsion and the maximum of the two-dimensional harmonic crystal.
The Annals of Probability, 29(4):1670–1692, 2001.

O. Daviaud.
Extremes of the discrete two-dimensional Gaussian free field.
The Annals of Probability, 34(3):962–986, 2006.

W. Werner and E. Powell.
Lecture notes on the Gaussian free field.
arXiv preprint arXiv:2004.04720, 2020.