Mathematical Analysis and Numerical Simulations of General Lotka-Volterra Systems

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Outline

Hare and Lynx Population Dynamics

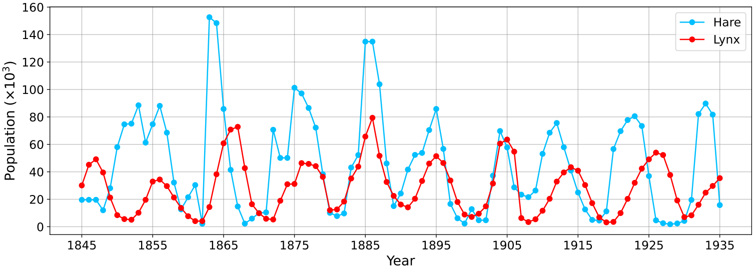

Figure: Development of the population sizes of lynx and hares in the northern boreal forests of North America.

Outline

General Lotka-Volterra Equations

Definition (General Lotka-Volterra Equations)

The general Lotka-Volterra equations for a n-species model is the system

$$F_i(x) = \frac{\mathrm{d}x_i}{\mathrm{d}t} = x_i \left(r_i + \sum_{j=1}^{n} a_{ij} x_j \right) \tag{1}$$

with $i = 1, 2, \ldots, n$. The matrix $A = (a_{ij})_{i,j=1,2,\ldots,n}$ is called the interaction matrix of the system. The vector $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ contains the growth rates of the i-th species [**10.25365/thesis.45530**, **Volterra1931**].

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Remark

- *The state space of system (1) is the nonnegative orthant* $\mathbb{R}^n$.
- *If* $a_{ii} = 0$ *for all* $i = 1, 2, \ldots, n$ *the system is called simple.*

2-Species Predator-Prey Model

Definition (2-dimensional Predator-Prey Model)

The system of equations

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1(r_1 - a_{12}x_2)\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_2(-r_2 + a_{21}x_2)\end{aligned} \tag{2}$$

for $r_1, r_2, a_{12}, a_{21} > 0$ is called simple predator-prey model and is a special case of system (1).

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- The biotope is completely isolated from external influences [**theoryvsempiry**].

Numerical Results

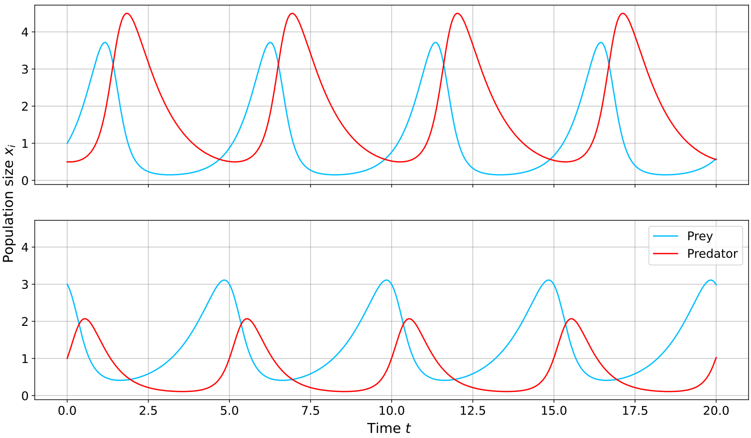

Figure: Numerical solutions of the 2-species predator-prey model (2) introduced by Lotka and Volterra. The first system was solved for the initial condition $x_1(0) = 1, x_2(0) = 0.5$, the second set of equations for $x_1(0) = 3, x_2(0) = 1$ respectively.

Slope Field and Phase Space Representation

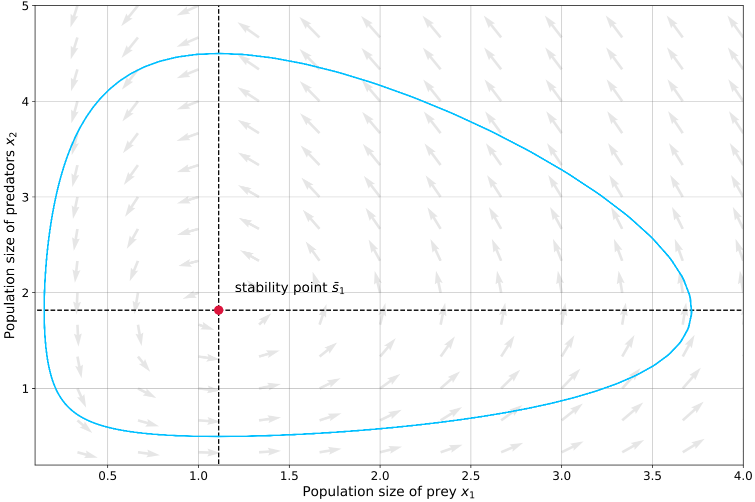

Figure: Slope field of system (2) and phase trajectory of the solution with initial value $x_1(0) = 1, x_2(0) = 0.5$.

Mathematical Properties

Remark (Stability)

The 2-species predator-prey model has the fixed points

$$\bar{s}_0 = (0,0) \qquad \text{unstable saddle point}$$
$$\bar{s}_1 = \left(\frac{r_2}{a_{21}}, \frac{r_1}{a_{12}}\right) \text{stable center.}$$

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Remark (Periodicity)

The linearized system implies periodic solutions around $\bar{s}_1$ with natural frequency $\omega = \sqrt{r_1 r_2}$ and phase shift $\varphi = \frac{\pi}{2}$ [**stability_2d_predator_prey_model**, **10.25365/thesis.45530**].

Mathematical Properties

Remark (Population Mean)

The mean values of the population sizes are

$$\bar{x}_1 = \frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)\,\mathrm{d}t = \frac{r_2}{a_{21}}$$

$$\bar{x}_2 = \frac{1}{T}\int_{t_0}^{t_0+T} x_2(t)\,\mathrm{d}t = \frac{r_1}{a_{12}}$$

for $T = \frac{2\pi}{\omega}$. *Therefore the means are independent of the initial values* $x_1(0)$ *and* $x_2(0)$ *[**10.25365/thesis.45530**].*

Predator-Prey Model with external Forcing

Definition (2-dimensional Predator-Pest Model)

The system of equations

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1(r_1 - a_{12}x_2) - \alpha \triangle(t-T) \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_2(-r_2 + a_{21}x_2) - \beta \triangle(t-T) \end{aligned} \tag{3}$$

for $r_1, r_2, a_{12}, a_{21}, \alpha, \beta > 0$ and

$$\triangle(t-T) = \begin{cases} \frac{1}{\epsilon}, & t \in [T, T+\epsilon] \\ 0, & \text{otherwise} \end{cases}$$

is called predator-pest model [**pesticide_paradox**].

Pest Simulations

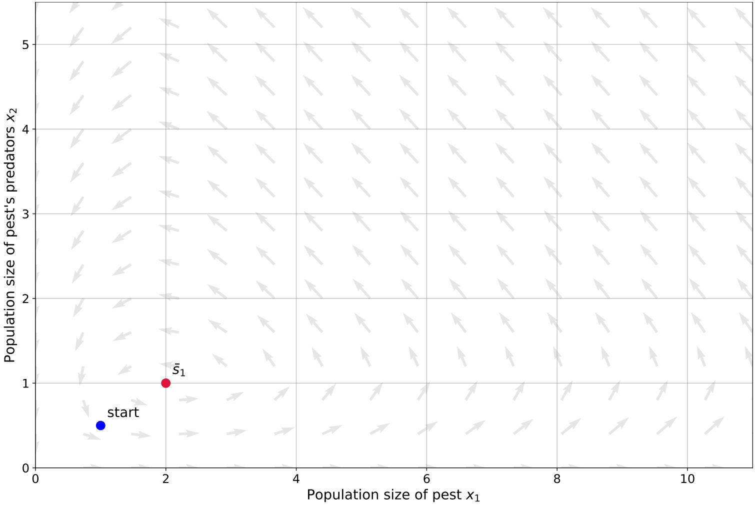

Figure: Trajectories of pesticide forced orbits in the 2-dimensional predator-pest-Model for different application times.

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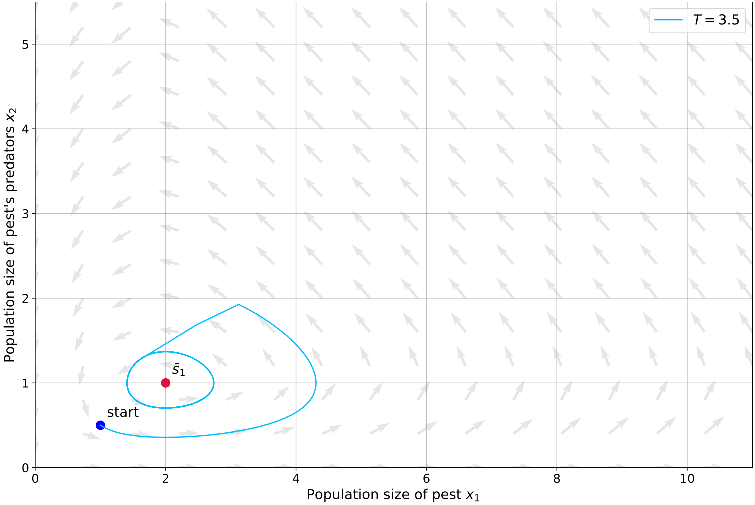

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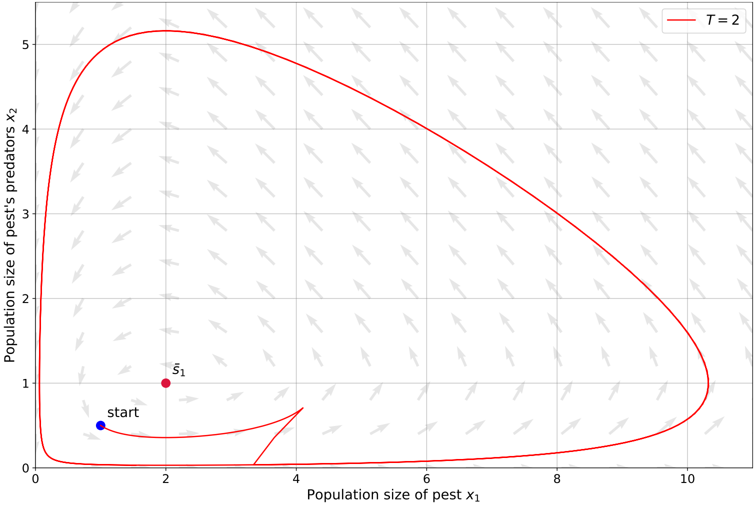

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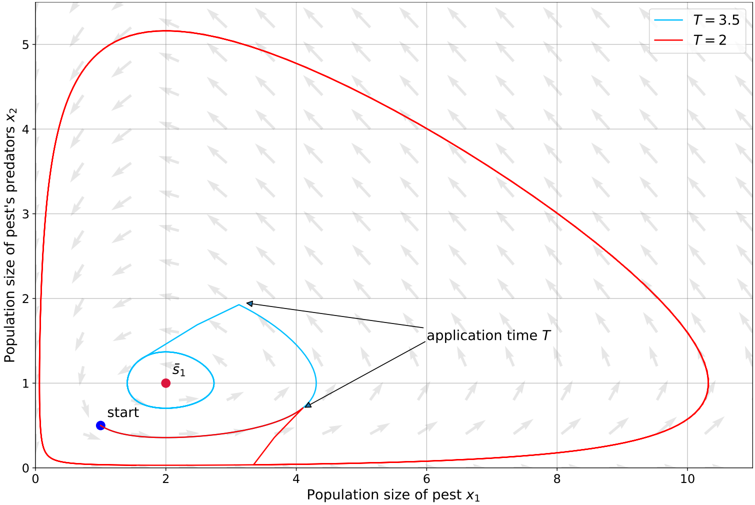

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Lotka-Volterra's Competition Model

Definition (Competitive Lotka-Volterra Equations)

The competitive Lotka-Volterra equations for a n-species model is the system

$$F_i(x) = \frac{\mathrm{d}x_i}{\mathrm{d}t} = x_i \left(r_i - \sum_{j=1}^{n} a_{ij} x_j \right) \tag{4}$$

with $i = 1, 2, \ldots, n$ and $a_{ij}, r_i > 0$ for all $i, j = 1, 2, \ldots, n$ [**stability_2d_predator_prey_model**].

Lotka-Volterra's Competition Model

Remark (Logistic Growth)

The restriction to the i-th equation in system (4) gives

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = x_i(r_i - a_{ii}x_i) = x_i r_i \left(1 - \frac{x_i}{K_i}\right)$$

with $K_i = \frac{r_i}{a_{ii}}$*, which is called the i-th carrying capacity [**theoryvsempiry**].*

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Numerical Results

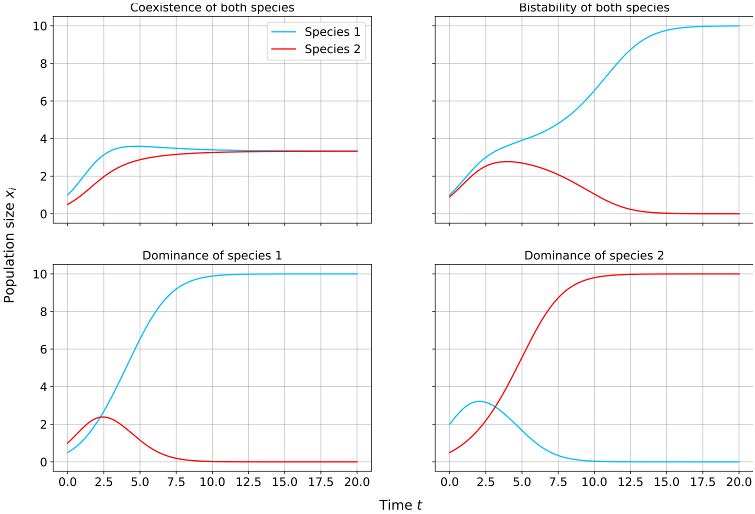

Figure: Classification of the 2-dimensional Lotka-Volterra competition model.

Slope Field and Phase Space Representation

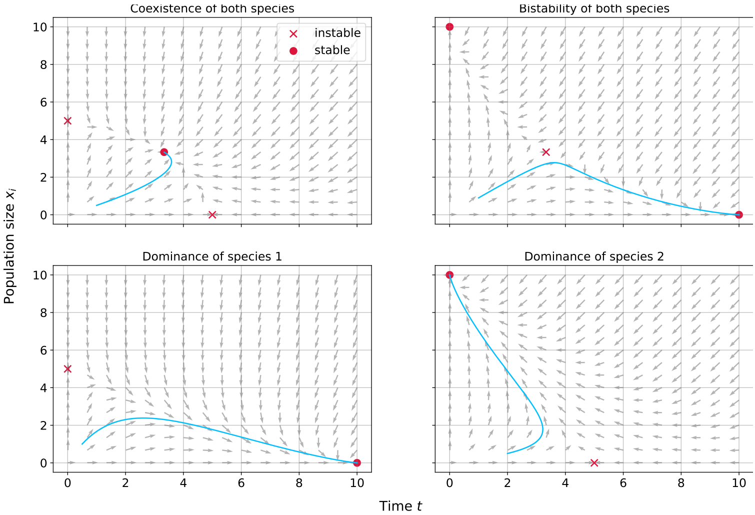

Figure: Phase-space representation of the solutions of the 2-dimensional competition model.

Competitive Exclusion Principle

Definition (Gause's Law (1934))

Two species or populations cannot inhabit the same niche: one will consistently out-compete the other [**gauses_law**, **gauses_law2**].

Figure: Image of paramecium aurelia.

Outline

Seasonal Effects

Remark (Lotka-Volterra Model with seasonal Fluctuations)

The Lotka-Volterra equations for a n-species model with seasonal term reads

$$F_i(x) = \frac{\mathrm{d}x_i}{\mathrm{d}t} = x_i \left(r_i + \mu_i \sin(\nu t) + \sum_{j=1}^{n} a_{ij} x_j \right) \tag{5}$$

with $i = 1, 2, \ldots, n$. *The parameter* μ_i *represents the impact of the environmental change for the* i*-th species [***predatorpreymodel***].*

Delayed Interaction Effects

Remark (Lotka-Volterra Model with Delay Kernels)

The Lotka-Volterra equations for a n-species model with changing rate of interaction is the system

$$F_i(x) = \frac{\mathrm{d}x_i}{\mathrm{d}t} = x_i \left(r_i + \sum_{j=1}^{n} \int_{-\infty}^{t} F_{ij}(t-\tau)x_j(\tau)\ \mathrm{d}\tau \right) \tag{6}$$

*with $i = 1, 2, \ldots, n$. The functions F_{ij} are non-negative continuous delay kernels representing the changing rate of interaction between species i and j with respect to the past [**predatorpreymodel**].*