Matrices used in worksheet 3

I this worksheet we have used affine transformation to obtain the desired perspectives of the cubes. First we present the 3 rotation matrices that enables us to rotate objects around the x, y and z axis, where theta is the angle that the object will be rotated around the given axis. Rotation matrices have the property that they don't scale or shear the linear space, which also means their determinant is zero:

$$x_{Rot} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad y_{Rot} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad z_{Rot} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To move the object around the canvas, scale it or shear it the following translation matrices will need to be used:

$$Translation = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad scale = \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad Sh = \begin{bmatrix} 1 & sh_x^y & sh_x^z & 0 \\ sh_y^x & 1 & sh_y^z & 0 \\ sh_z^x & sh_z^y & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where " sh_x^y " means that you shear the y-axis in relation to the x-axis. To obtain a perspective of the object on the canvas, you will need to use perspective matrices. Perspective matrices transforms the points of the object in specific ways to give the desired effect. In this worksheet we have used the following perspective matrices:

$$ortho = \begin{bmatrix} \frac{2}{right - left} & 0 & 0 & \frac{-(left + right)}{right - left} \\ 0 & \frac{2}{top - bottom} & 0 & \frac{-(top + bottom)}{top - bottom} \\ 0 & 0 & -\frac{2}{far - near} & \frac{-(far + near)}{far - near} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix gives an orthogonal perspective by scaling all axis equally and translating the object.

$$a = \arcsin\left(\tan\left(\frac{\pi}{6}\right)\right) \qquad b = \frac{\pi}{4}$$

$$Isometric = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(a) & \sin(a) & 0 \\ 0 & -\sin(a) & \cos(a) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(b) & 0 & -\sin(b) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(b) & 0 & \cos(b) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{6}} & 0 & \frac{-\sqrt{3}}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ \frac{\sqrt{2}}{\sqrt{6}} & \frac{-\sqrt{2}}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix gives an isometric perspective by changing the angel between any of the axis to a given value (here). This gives an object drawn in 2D an impression of 3D by making the lines that make up the object meet in the horizon. This is the same principle that is used in one-point, two-point and three-point perspective matrix, which looks like this:

$$perspective = \begin{bmatrix} \frac{f}{aspect} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{-(near + far)}{d} & \frac{-2 \cdot near \cdot far}{d} \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad f = \frac{1}{\tan\left(\frac{fovy}{2}\right)} : \qquad d = far - near :$$

This matrix transforms the points of an object by making the lines that makes up the object meet in the horizon. f gives the field of view, which gives the effect of a pinhole camera. The difference between one-point, two-point and three-point perspective is a matter of how you look at the object in relation to the three axis, if you look straight at the object from one axis you get one-point perspective, if you look at the object from the plane between two axis you get two-point perspective and finally if you look at the object in the space between 3 axis you get three-point perspective. The angel with which you look at the object can be changed by using the following matrix:

$$lookAt = \begin{bmatrix} v[1] & v[2] & v[3] & -(n \cdot Eye) \\ n[1] & n[2] & n[3] & -(u \cdot Eye) \\ u[1] & u[2] & u[3] & -(v \cdot Eye) \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad v = \frac{At - Eye}{\sqrt{(At - Eye).(At - Eye)}} \qquad n = \frac{v \times up}{\sqrt{(v \times up).(v \times up)}} \qquad u = \frac{n \times v}{\sqrt{(n \times v).(n \times v)}}$$

Where "Eye" is a vector indicating where you are looking from and "At" is a vector indicating in which direction you are looking at. "up" is simply a vector stating which direction in the space that is up.

How matrices were concatenated in worksheet 3

When you need to make multiple transformation to your object to make it look as specified the different effect that these matrices bring can be made into one matrix by multiplying the different matrices together. So you can get a matrix that can rotate around the x-axis and apply the isometric view by doing the following:

$$ROTX\_ISO = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) & 0 \\ 0 & \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{6}} & 0 & \frac{-\sqrt{3}}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ \frac{\sqrt{2}}{\sqrt{6}} & \frac{-\sqrt{2}}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}\sqrt{6}}{6} & 0 & -\frac{\sqrt{3}\sqrt{6}}{6} & 0 \\ \frac{\cos(\alpha)\sqrt{6}}{6} - \frac{\sin(\alpha)\sqrt{2}\sqrt{6}}{6} & \frac{\cos(\alpha)\sqrt{6}}{3} + \frac{\sin(\alpha)\sqrt{2}\sqrt{6}}{6} & \frac{\cos(\alpha)\sqrt{6}}{6} - \frac{\sin(\alpha)\sqrt{2}\sqrt{6}}{6} & 0 \\ \frac{\sin(\alpha)\sqrt{6}}{6} + \frac{\cos(\alpha)\sqrt{2}\sqrt{6}}{6} & \frac{\sin(\alpha)\sqrt{6}}{3} - \frac{\cos(\alpha)\sqrt{2}\sqrt{6}}{6} & \frac{\sin(\alpha)\sqrt{6}}{6} + \frac{\cos(\alpha)\sqrt{2}\sqrt{6}}{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In the first part the matrices were concatenated like this:

$$ctm = lookAt \cdot Isometric = \begin{bmatrix} v[1] & v[2] & v[3] & -(n \cdot Eye) \\ n[1] & n[2] & n[3] & -(u \cdot Eye) \\ u[1] & u[2] & u[3] & -(v \cdot Eye) \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{6}} & 0 & \frac{-\sqrt{3}}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ \frac{\sqrt{2}}{\sqrt{6}} & \frac{-\sqrt{2}}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{v_1\sqrt{3}\sqrt{6}}{6} + \frac{v_2\sqrt{6}}{6} + \frac{v_3\sqrt{2}\sqrt{6}}{6} & \frac{v_2\sqrt{6}}{3} - \frac{v_3\sqrt{2}\sqrt{6}}{6} & -\frac{v_1\sqrt{3}\sqrt{6}}{6} + \frac{v_2\sqrt{6}}{6} + \frac{v_3\sqrt{2}\sqrt{6}}{6} & -n \cdot Eye \\ \frac{n_1\sqrt{3}\sqrt{6}}{6} + \frac{n_2\sqrt{6}}{6} + \frac{n_3\sqrt{2}\sqrt{6}}{6} & \frac{n_2\sqrt{6}}{3} - \frac{n_3\sqrt{2}\sqrt{6}}{6} & -\frac{n_1\sqrt{3}\sqrt{6}}{6} + \frac{n_2\sqrt{6}}{6} + \frac{n_3\sqrt{2}\sqrt{6}}{6} & -u \cdot Eye \\ \frac{u_1\sqrt{3}\sqrt{6}}{6} + \frac{u_2\sqrt{6}}{6} + \frac{u_3\sqrt{2}\sqrt{6}}{6} & \frac{u_2\sqrt{6}}{3} - \frac{u_3\sqrt{2}\sqrt{6}}{6} & -\frac{u_1\sqrt{3}\sqrt{6}}{6} + \frac{u_2\sqrt{6}}{6} + \frac{u_3\sqrt{2}\sqrt{6}}{6} & -v \cdot Eye \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For part 2 we had to use the "lookAt" matrix with 3 different "Eye" vectors to achieve the three different point perspectives. For the point perspective we use a 45 degrees pinhole camera combining all this and you get the following:

$$Eye[1] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad Eye[2] = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad Eye[3] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$f = \frac{1}{\tan\left(\frac{fovy}{2}\right)} \qquad fovy = 45^o$$

$$d = far - near:$$

$$ctm[i] = lookAt[i] \cdot Perspective = \begin{bmatrix} v[1] & v[2] & v[3] & -(n \cdot Eye[i]) \\ n[1] & n[2] & n[3] & -(u \cdot Eye[i]) \\ u[1] & u[2] & u[3] & -(v \cdot Eye[i]) \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{f}{aspect} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & \frac{-(near+far)}{d} & \frac{-2 \cdot near \cdot far}{d} \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad i \in \{1, 2, 3\}$$

$$= \begin{bmatrix} \frac{v_1 f}{aspect} & v_2 f & \frac{v_3(-near - far)}{d} + n \cdot Eye_i & -\frac{2 v_3 \, near \, far}{d} \\ \frac{n_1 f}{aspect} & n_2 f & \frac{n_3(-near - far)}{d} + u \cdot Eye_i & -\frac{2 n_3 \, near \, far}{d} \\ \frac{u_1 f}{aspect} & u_2 f & \frac{u_3(-near - far)}{d} + v \cdot Eye_i & -\frac{2 u_3 \, near \, far}{d} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

There were all the concatenated matrices used in worksheet 3.