Experimental Validation of Thermal Noise in Passive Circuits

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In passive circuits with finite resistances, thermal noise was experimentally confirmed by measuring potential differences across resistors. Using a known amplifier in a controlled environment, thermal noise aligns with the theoretical model $\bar{v_n^2} = 4k_BTR$. The one-sided power spectral density, reflecting voltage variance per hertz of bandwidth, is directly proportional to resistance, temperature, and Boltzmann's constant k_B. The experimental constant found agrees with the accepted value.

In modern electronics, the ability to detect extremely small signals is vital. This is exemplified across various applications, from particle physics experiments to sensitive medical imaging and telecommunication systems. This paper investigates the impact of thermal fluctuations in electrical circuits.

Electrons moving within conductors exhibit motion that adheres to thermodynamic and statistical principles. This results in a non-zero average voltage variance between the conductor's terminals due to their differing potentials. The theory of this thermal noise was originally developed using first principles [1]. The primary finding demonstrates that thermal fluctuations within a conductor with non-zero resistance give rise to a voltage variance (mean square) per hertz of bandwidth, expressed as

$$\bar{v_n^2} = 4k_BTR \tag{1}$$

The voltage variance is proportional to the resistance and temperature of the conductor, as well as a constant factor which includes the Boltzmann constant, k_B.

Previous to this, discussion on noise sources assumed that shot noise dominated thermal noise [2]. Thus, little experimental investigation has been conducted to explore the latter in detail.

This paper presents the experimental methods and results that provide insights into thermal noise and its impact on electrical circuits. The data collected in this paper yields a value for the Boltzmann constant (k_B) that aligns with the accepted value.

The understanding of the effect will be further developed in this paper with the general circuit schematic of Fig. 1.

The resistor under study was shielded in a copper enclosure with a BNC output. A pre-calibrated thermocouple was connected from the outside of this copper enclosure to a digital multimeter (DMM) to help study the temperature dependence of the effect. Another shielded enclosure contained 2 parallel and identical filtering amplifier circuits that each output a signal that was measured with a sound card.

The sound card used for data collection was calibrated by supplying it with known AC voltages in increments using a function generator.

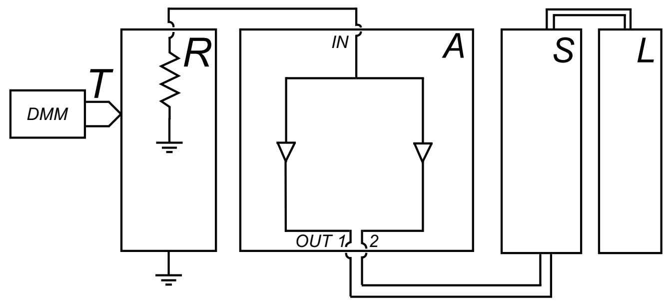

FIG. 1. A simplified schematic of the experimental setup. A Resistor (R) in a copper enclosure with a thermocouple (T) attached from the enclosure to a digital multimeter (DMM). The noise potential from the resistor is sent through a powered amplifier (A) via 2 broad-band amplifying circuits. The output 1 and 2 lines are measured via a soundcard (S) and logged on a lab computer (L).

The gain frequency response of the amplifier circuit was experimentally calibrated over a range of frequencies. The function generator was used to send constant amplitude AC voltages through a voltage divider circuit with known attenuation. This supplied small, known, constant amplitude inputs for the amplifier. The output of the amplifier circuit was measured while varying the function generator frequencies. Throughout the chosen range (100 Hz to 10 kHz) the function generator was able to output a relatively constant amplitude signal and the sound card (sampling at 44.1kHz) was able to measure without any aliasing issues. The gain at different frequencies was then given by the ratio of the output amplitude to the known function generator amplitude, times a multiplier derived from the voltage divider circuit. A theoretical curve was additionally calculated based on the amplifier circuit components. The results are compared in Fig. 2.

Specialized setups were utilized to maintain resistor temperatures independent of ambient room conditions. For lower temperatures, the resistor enclosure was positioned within a box filled with solid carbon dioxide, while higher temperatures were achieved by directing a heat gun's steady airflow at the resistor enclosure, elevated at a fixed height above the heat source. Temperature measurements relied on K-type thermocouples, with the

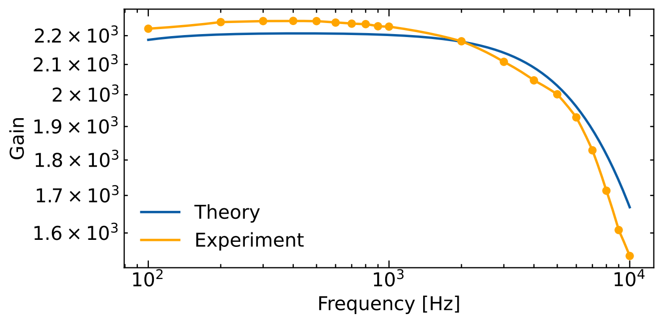

FIG. 2. Amplifier gain as a function of frequency was calculated both theoretically and experimentally. The drop-off corresponds to the amplifier high pass filtering.

laboratory room thermometer serving as an additional tool for temperature verification in cases where the temperature was held fixed.

Using these experimental tools, measurements of unpowered resistors were performed. Resistors were securely clamped between the signal line and a chassis ground within the copper enclosure. The resistances were consistently measured with a digital multimeter, maintaining the same resistance within the measured uncertainty across various temperatures. A BNC connection was established from the grounded resistor to the powered amplifier, and the sound card recorded the two amplifier outputs.

Additional sources of noise within the experiment were modelled. The so-called Shot-Effect, or leakage current, from the top gate of the transistors within the operational amplifier, is a current that produces a power spectral density (PSD) that is proportional to the size of the resistance squared that is used. At large resistance, this current noise is theorized to become dominant. Additionally, amplifier voltage noise creates some voltage variance irrespective of resistance. Thus we can theorize a toy model of the experimental measured voltage variance per Hz of the bandwidth with;

$$\bar{v_n^2} = S_I R^2 + 4k_B T R + S_V \tag{2}$$

where S_I and S_V are the one-sided PSDs of the current noise and the voltage noise respectively.

The voltage noise is dominant at small resistance values and unique to each amplifier circuit used. Thus, the two amplifier circuits were used in tandem to take measurements of the same thermal noise but different voltage noise. Using the principles of cross-correlation, the PSD from the thermal noise alone is extracted from the simultaneous measurements.

A CSD was calculated from concurrent noise measurements performed across different resistance values at a steady temperature. The CSDs were truncated to fit the bandwidth of known values from the calibrated gain frequency response that was experimentally measured. The CSD was normalized by dividing the values by the

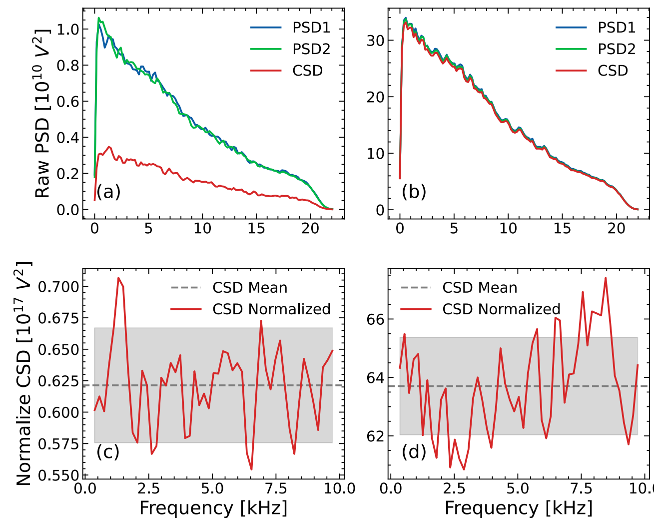

FIG. 3. Raw PSDs from the simultaneous amplifier lines are compared to the resulting CSD for 397Ω low (a) and 40kΩ mid-range (b) resistances. The CSD is taken with the two signals to cross-correlate them and output the PSD coming strictly from the correlated part, the thermal noise. At low resistance the voltage noise in the circuit is dominant, thus the CSD has a correspondingly large effect. The CSD is then truncated and normalized using the experimental gain function. The results are windowed for clarity and are shown on the bottom plots, as a relatively flat band again for the same low (c) and mid-range (d) resistances. The mean and standard deviation of the CSD are shown in grey.

squared gain curve. This allows for a one-to-one comparison of power at different frequencies, originating from the thermal noise.

Consequently this results in a flat white noise-like CSD. Fig. 3 provides an illustrative example of a low and high resistance measurement. The mean of this normalized CSD across the entire bandwidth is computed and saved for all resistance values. The error was additionally computed from the standard deviation from each of the measurements.

The data from the full range of resistances measured is plotted in Fig. 4. The toy model, $\bar{v_n^2} = S_I R^2 + 4k_B T R + S_V$ performs poorly with the data and fits a negative current noise voltage variance.

This is interpreted as an experimental artifact from parasitic capacitance in the resistors used, which dominates over the theorized current noise effect. More specifically it can be shown that if there is a shunt capacity across the resistor, the white thermal noise introduces a capacitive reactance. This creates an overall impedance on the system of the form $Z = (R^{-1} + i\omega C)^{-1}$. This impedance reaches a threshold and then drops off as the resistance is increased, this corresponds to a decrease in thermal noise power. Additionally, measurements at large resistance show non-flat thermal noise PSDs, which additionally supports the hypothesis of impedance in the experimental setup that increases at large frequency.

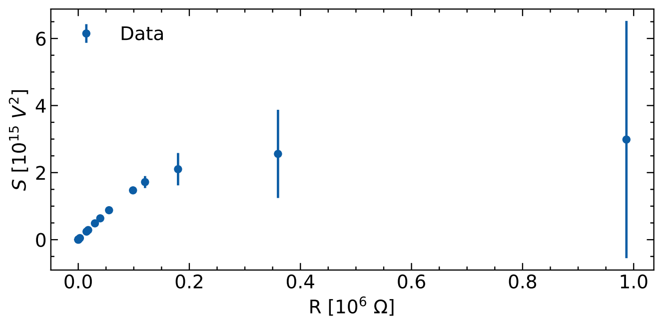

FIG. 4. The mean voltage variance is plotted against a large range of resistors and shows a non-linear effect at large resistances. The error bars correspond to the standard deviation from the mean.

Thus, the parasitic capacitance helps explain the feature at large resistance.

Further presentation and analysis explore the lower resistance regime with CSDs. The model used to fit then becomes $\bar{v_n^2} = 4k_BTR$. The data points at low resistance, plotted with log-log scales, show a nearly perfectly linear trend, signifying a resistance dependence of a singular and unique power of 1.

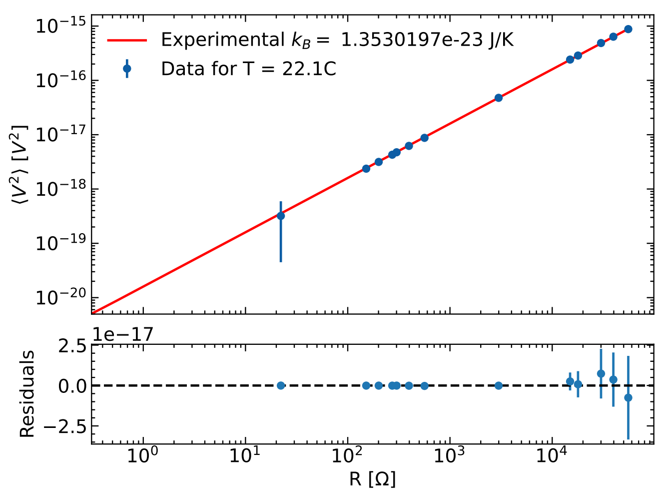

FIG. 5. The mean voltage variance is plotted against resistance and is fit with a linear model. A corresponding residual subplot is plotted below. The temperature was held fix in lab conditions at T=22.1 C. This data fits a value for the boltzmann constant of 1.353×10^{-23}J/K.

The impact of resistance element temperature was investigated using a comparative approach, similar to the method employed for varied resistance. Experiments covered a temperature range of roughly -70°C (using solid carbon dioxide) to 70°C (using a heat gun), with intermediate temperature measurements taken during cooling. Despite in situ measurements, resistance values showed negligible change. The voltage variance is found to be directly proportional to the absolute temperature of the resistance element, as illustrated in Fig. 6.

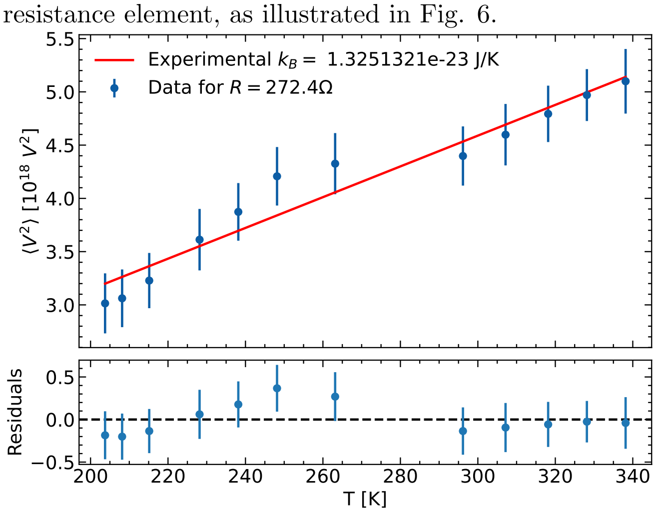

FIG. 6. The linear fit of the voltage variance as the temperature is changed with a fixed resistance of 272.4Ω. A corresponding residual plot is shown below. The fitting finds a value for the Boltzmann constant of 1.325×10^{-23}J/K.

A value for k_B is determined, closely approaching the accepted known value when employing the CSD.

In conclusion, using basic electronic lab equipment it is experimentally shown that there is a thermal noise that follows statistical and thermodynamical theories. With this, a value for the Boltzmann constant (k_B) was found that agrees with the theoretical value within an expected range based on the experimental uncertainties. This paves the way forward for significant improvement in all industries that use small signals.

[1] H. Nyquist, Thermal agitation of electric charge in conductors, Phys. Rev. **32**, 110 (1928).

[2] W. Schottky, Über spontane stromschwankungen in verschiedenen elektrizitätsleitern, Annalen der Physik **362**, 541 (1918).