Problem Set 2

PHYS 509

Instructions: show your work. Justify all steps.

1. Let X, Y be independent random variables. The Conditional Expectation Formula is:

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Y))$$

To be clearer, we can explicitly denote which variable the expectation is with respect to as follows:

$$\mathbb{E}_X(X) = \mathbb{E}_Y(\mathbb{E}_X(X|Y))$$

   (a) Prove this for the case of discrete X, Y.

   (b) Suppose you are playing a computer game in which you are stuck in a room with 3 exits that are indistinguishable, which means you have equal probability of choosing exit 1,2 or 3. If you leave via exit 1, you get free after traversing 3 other rooms. If you leave via exit 2, you end up back at this same starting room after traversing 5 rooms. If you leave via exit 3, you end up back at this same starting room after traversing 7 rooms. Sadly, if you get back to the room, you are so disoriented that you can't tell which exit you chose last time, so you randomly pick again. How many rooms do you visit on average before getting free? (Bonus: how annoying is the game designer?)

2. Calculate the characteristic function of a Gaussian of variance σ^2 and show it is a Gaussian of variance $\frac{1}{\sigma^2}$.

3. (a) Show that, if the characteristic functions for two independent random variables X, Y are $\phi_X(t)$ and $\phi_Y(t)$, then the characteristic function for the random variable made from the sum, $Z = X + Y$, is just the product $\phi_Z(t) = \phi_X(t)\phi_Y(t)$.

   (b) If X_1, X_2 are two independent Gaussian random variables with means μ_1, μ_2 and variances σ_1^2, σ_2^2, prove that the random variable $Y = a_1X_1 + a_2X_2$ ($a_i \in \mathbb{R}$) is Gaussian with mean $\mu = a_1\mu_1 + a_2\mu_2$ and variance $\sigma^2 = a_1^2\sigma_2^2 + a_2^2\sigma_2^2$.

4. (a) Show that the characteristic function for a Poisson is

$$\phi(t) = \sum_{r=0}^{\infty} e^{itr} P(r; \mu) = \exp(\mu(e^{it} - 1))$$

   and check that you get the right mean and variance from the characteristic function.

   (b) Use this to show that the sum of two Poisson random variables with means μ_X, μ_Y is a Poisson with mean $\mu_X + \mu_Y$

   (c) Prove this the hard way, by computing this directly from the formulas for the Poissons. Note how much easier it was to prove using the characteristic functions!

5. Let $\{x_i\}, i = 1, n$ be a sample of size n drawn from a probability density function $p(x)$ with mean μ and variance σ^2. Define an estimator by the *sample mean*

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i \tag{1}$$

   (a) Show $\hat{\mu}$ is an unbiased estimator for μ.
   (b) Show that the variance of the sample, $\hat{s}^2 = \frac{1}{n} \sum_i (x_i - \hat{\mu})^2$ is a biased estimator for the true pdf (population) variance σ^2.
   (c) Show that $\hat{s}^2 = \frac{1}{n-1} \sum_i (x_i - \hat{\mu})^2$ is an unbiased estimator for the true pdf variance σ^2.

   (The way to remember this is that estimating the mean is one combination of the n data points, so you effectively only have $n - 1$ left to estimate the variance.)

6. Suppose you have a beam of particles with an average rate R per second passing through your detector.
   (a) What is the probability of getting n particles passing through in T seconds?
   (b) Suppose your detector has an efficiency of ϵ to successfully detect a particle. Derive the probability of getting r counts from the detector in T seconds.
   (c) Suppose $RT = 100$ and $\epsilon = 0.10$. What is the probability of having 110 particles in this time? What is the probability of observing 11 in your detector? What is the probability of having any number between 90 and 110 particles (inclusive) in this time? What is the probability of observing between 9 and 11 in your detector?

7. Suppose you have a market of many stocks. The price of a stock at year $i + 1$ is related to the price at the previous year as

$$P_{i+1} = P_i + (P_{i+1} - P_i) = P_i(1 + \frac{P_{i+1} - P_i}{P_i}) \equiv P_i(1 + r_i)$$

   where r_i is the *return* in year i. If you histogram r_i for many years for a given stock, it will have a distribution with some mean μ and variance σ^2. Often this is modeled as a normal distribution $N(\mu, \sigma^2)$. The variance of any given stock is quite high, typically $\sigma \approx 0.3$ (ie 30%). (In finance, this is almost always called the *risk*, which can be a bit confusing. But in that field, risk = volatility = σ.)
   Being an enterprising graduate student, you think "I know how to combine n random variables, take their mean, and drive the variance (=risk) down by increasing n". Let's model this (diversified) portfolio of stocks as spreading your money equally among n stocks (it's easy to extend to the case of different weights for each stock).

   Let stock $S_i, i = 1, n$ have a return that is a random variable X_i distributed as $N(\mu_i, \sigma_i^2)$. Form your portfolio by dividing your money into n equal parts invested in these n stocks. Each will contribute a return weighted by $1/n$, and so the sum will have returns which are randomly distributed as:

$$R = \frac{1}{n} \sum_{i=1}^{n} X_i$$

   We want to calculate σ_R = risk of this portfolio. However, you realize that the X_i are *not independent*. The price of stocks may have a component that affects most in the same way due to variables common to them all, e.g. interest rate changes, global pandemics, oil price shocks, etc. So the covariance matrix V for the n stocks is certainly not diagonal.

Show that you can break up σ_R into two parts, one based upon the variances of the individual stocks and the other based on the covariances. In particular, show that

$$\sigma_R^2 = \frac{1}{n}\bar{v} + \frac{n-1}{n}\bar{c}$$

where $\bar{v}$ = average value of the individual variances,

$$\bar{v} = \frac{1}{n}\sum_i \sigma_i^2$$

and $\bar{c}$ = average value of the covariances

$$\bar{c} = \frac{2}{n(n-1)}\sum_{i<j} V(i,j)$$

This shows that you can *diversify* away the idiosyncratic risk of individual stocks, as that component decreases with n, but you can't get rid of the *systematic/market* risk that affects the market as a whole. So, you don't get rewarded with higher expected returns for taking idiosyncratic risk that everyone knows how to minimize, just for the market risk that cannot be diversified away.

8. Suppose you have a photon detector that is made of sensor strips (rather than pixels as you might have on a camera), ie they are long in one dimension compared to the other. The detector electronics outputs only a simple binary 1/0 for registering a photon hit or not. Let the strip spacing be l, with strip boundaries at, say, $x = 0, l, 2l, ...$

   You have a source of photons which is uniform over the surface of your detector. Consider hits on a particular strip in location $[l_0, l_0 + l]$.

   (a) What is the probability density function for the x position of the hits in this strip?

   (b) What is $\mathbb{E}(x)$?

   (c) What is $V(x)$, or σ_x?. This is a quite commonly useful result to remember.

9. The χ^2 Distribution: Let $X_i, i = 1, n$ be n independent normal random variables with Gaussian p.d.f.s $N(\mu_i, \sigma_i^2)$

   Define

$$\chi^2 = \sum_{i=1}^{n}\left(\frac{x_i - \mu_i}{\sigma_i}\right)^2$$

   Then (you can take this as given) the pdf for χ^2 is

$$p(\chi^2; n) = \frac{1}{2^{\frac{n}{2}}\Gamma(n/2)}(\chi^2)^{\frac{n}{2}-1}e^{-\frac{1}{2}\chi^2}$$

   which is called the χ^2 distribution with n degrees of freedom

   (a) Make plots of this for n=1, 2, 3 and 10

   (b) Find the mode of this distribution as a function of n

(c) Calculate the mean as a function n

(d) Calculate the variance as a function of n

(Hint: Think of this as a sum of n independent variables, each distributed as $N(0,1)$. Leverage what you know about the mean and variance of sums of independent variables, and also what you know about the standard moments of a Gaussian. If you want to calculate these from the probability density itself, you may need relations such as $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}$)

10. The Exponential Distribution. The exponential probability density function is

$$E(x;\beta) = \frac{1}{\beta}e^{-\frac{x}{\beta}}, x \geq 0$$

(a) What is the mode of this distribution?

(b) Derive the mean

(c) Derive the variance

(d) Derive the median

(e) Calculate the third moment of x and use this and the previous results to derive the standard skewness $\mathbb{E}((\frac{x-\mu}{\sigma})^3)$