

# Physics 509 Theory of Measurements Course Notes

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## Introduction:

Notes written at UBC 2025W1 with Dr. Colin Gay.

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# 1 Tuesday, September 9th 2025

This lecture covered the course structure and grading.

## 1.1 Motivation: Stochastic Nature of Experimental Data

- Stochastic processes:
  - muon decay
  - inherent stochasticity
  - quantum mechanics
- Mostly concerned with measurement devices — how do we measure?
- Example: a muon lifetime experiment
  - Take a cosmic muon, detect light, and discriminate.
  - Muon decays into an electron and neutrinos, and the electron produces light.
  - Measure the time between light pulses.
  - Many factors cause noise in the data — results change even if the same mechanism occurs twice.

## 1.2 Probabilistic Interpretation of Experimental Results

- Experiments are repeated trials.
- Probability (probabilistic interpretation):
  - Results are interpreted as the long-term average of repeating an experiment many times.
  - Example: coin flip

$$P(H) = \lim_{N \rightarrow \infty} \frac{n_H}{N}$$

$n(H)$  = number of heads in  $N$  trials

## 1.3 Sample Spaces and Stochastic Variables

- In modern probability theory:
  - 3 axioms (Kolmogorov)
  - Let  $X$  be a stochastic variable.
  - Define sample space  $S (\Omega)$ :

$$S = \{x_1, x_2, \dots\}$$

- Examples:

1. Coin flip:

$$S = \{H, T\}$$

2. Roll a die:

$$S = \{1, 2, 3, 4, 5, 6\}$$

3. Grade in this class:

$$S = \{0, 1, 2, \dots, 100\}$$

4. Decay time of a radioactive atom:

$$S = [0, \infty)$$

- $S$  can be finite (Binomial), countable (Poisson), or infinite (Gaussian, Uniform).

## 1.4 Events and Set Operations

- Definition: An event  $E$  is a subset of  $S$ .
- Example: one die roll

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$E = \text{rolling an even number} = \{2, 4, 6\}$$

- Example:  $E = \text{atom decayed by time } t_0$

$$S = [0, t_0]$$

- Operations on events:

– Union (OR) and Intersection (AND)

– Let  $A, B$  be events in  $S$ :

$$E = A \cup B = \{e : e \in A \text{ or } e \in B \text{ (or both)}\}$$

– Example: flip a coin twice

$$S = \{HH, HT, TH, TT\}$$

$$A = \text{1st flip is H} = \{HH, HT\}$$

$$B = \text{2nd flip is H} = \{HH, TH\}$$

$$A \cup B = \{HH, HT, TH\}$$

$$A \cap B = \{e \mid e \in A \text{ and } e \in B\} = \{HH\}$$

$$AB = A \cap B$$

$$A^c = \{e \mid e \in S \text{ and } e \notin A\} = \{TH, TT\}$$

- Properties:

– Commutative:

$$A \cup B = B \cup A, \quad AB = BA$$

– Associative:

$$A \cup (B \cup C) = (A \cup B) \cup C, \quad (AB)C = A(BC)$$

– Distributive:

$$(A \cup B)C = AC \cup BC, \quad A(B \cup C) = AB \cup AC$$

– De Morgan's Laws:

$$(A \cup B)^c = A^c B^c, \quad (AB)^c = A^c \cup B^c$$

## 1.5 Kolmogorov's Axioms of Probability

- A function  $P$  on  $S$  is a probability measure if it satisfies:

1.  $P(S) = 1$

2.  $P(\emptyset) = 0$

3. For any countable sequence of disjoint events  $E_1, E_2, \dots$  in  $S$ :

$$E_i E_j = \emptyset \text{ for } i \neq j$$

$$P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$$

## 1.6 Consequences of the Probability Axioms

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$$P(\emptyset) = 0$$

Let

$$E_1 = S, \quad E_2 = \emptyset$$

$$E_1 E_2 = \emptyset$$

$$P(S \cup \emptyset) = P(S) + P(\emptyset) = 1 + P(\emptyset)$$

$$P(S) = 1, P(\emptyset) = 0$$

- 

$$P(E^c) = 1 - P(E)$$

$$1 = P(S) = P(E \cup E^c) = P(E) + P(E^c)$$

- If  $B \subset A$ , then:

$$P(B) \leq P(A)$$

$$A = B \cup (B^c A)$$

$$P(A) = P(B \cup (B^c A))$$

$$P(B) = P(A) - P(B^c A) \leq P(A)$$

- 

$$P(A \cup B) = P(A) + P(B) - P(AB)$$

If we let the areas of the Venn diagram be 1 (A), 2 (A+B), 3 (B), then:

$$A \cup B = 1 \cup 2 \cup 3$$

$$P(A \cup B) = P(1 \cup 2 \cup 3) = P(1) + P(2) + P(3)$$

$$P(A) = P(1), \quad P(B) = P(2) + P(3)$$

$$P(A) + P(B) - P(2) = P(1) + P(2) + P(3) = P(A \cup B)$$

$$\text{equivalently } P(A) + P(B) - P(AB) = P(A) + P(B) - P(AB)$$

## 1.7 Uniform Probability on Finite Sample Spaces

- 

$$E_i = S_i \text{ for } i = 1, 2, \dots, n$$

$$E_i E_j = \emptyset \text{ for } i \neq j$$

$$S = \bigcup_{i=1}^n E_i$$

$$P(S) = 1 = P(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n P(E_i)$$

$$P(E_i) = P(E_j) \quad \text{all equally likely}$$

$$1 = \sum_{i=1}^N P(E_i) = NP(E_i)$$

$$P(E_i) = \frac{1}{N} = P(E_j)$$

$N = |S| = \text{number of elements in (cardinality of) } S$

$F$  be any event (set) in  $S$  with  $k$  elements  $|F| = k$

$$P(F) = P(\bigcup_{S_i \in F} \{E_i\}) = \sum_{i=1}^k P(E_j) = \sum_{i=1}^k \frac{1}{N} = \frac{k}{N} = \frac{|F|}{|S|}$$

## 1.8 Example: Probability of a Straight in Poker

- Example: 5-card poker hand forming a straight

$$S = \{(AC, 2C, 3C, 4C, 5C), (2C, 3C, 4C, 5C, 6C), \dots\}$$

$$S = \binom{52}{5} = \frac{52!}{5! 47!} = 2,598,960$$

- Event = straight = 5 consecutive cards, not of the same suit, any starting card.

$$10(4^5 - 4) = 10200$$

- Starting cards: Ace (A,2,3,4,5), 2 (2,3,4,5,6), ..., 10 (10,J,Q,K,A)
- Not all the same suit:  $4^5 - 4$  (exclude all same suit)

$$P(\text{straight}) = \frac{10(4^5 - 4)}{\binom{52}{5}} = 0.00392465$$

## 1.9 Conditional Probability

- Given 2 events  $E, F$ , sample space  $S$ :

$P(E)$  = probability of a trial from  $S$  in  $E$

$P(F)$  = probability of a trial from  $S$  in  $F$

- Conditional probability of  $E$  given  $F$  has occurred:

$P(E|F)$  = probability of a trial from  $S$  in  $E$ , given the trial is in  $F$

- Note:  $P(EF)$  is the probability of a trial from  $S$  in both  $E$  and  $F$ .
- Need to normalize by  $P(F)$ , so we define:

$$P(E|F) = \frac{P(EF)}{P(F)} \quad \text{if } P(F) > 0$$

$$P(EF) = P(E|F)P(F)$$

- Example: flip a coin 2 times

$$S = \{HH, HT, TH, TT\}$$

Conditional probability of  $HH \equiv A$  given:

- First flip =  $H \equiv B = \{HH, HT\}$
- Either flip is  $H \equiv C = \{HH, HT, TH\}$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(\{HH\})}{P(\{HH, HT\})} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

$$P(A|C) = \frac{P(AC)}{P(C)} = \frac{P(\{HH\})}{P(\{HH, HT, TH\})} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

## 2 Thursday, September 11th 2025

### 2.1 Bayes' Formula

- Let  $E, F$  be events:

$$E = EF \cup EF^c$$

$$P(E) = P(EF) + P(EF^c)$$

$$P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)$$

$$P(E) = P(E|F)P(F) + P(E|F^c)(1 - P(F))$$

- Example:** Suppose a blood test is 95% effective in detecting a disease if the person has it. It also has a 1% false positive rate. Suppose 0.5% of the population has the disease.

$D$  = person has disease

$E$  = test is positive

- We want:

$$P(D|E) = \frac{P(ED)}{P(E)}$$

$$\begin{aligned} P(D|E) &= \frac{P(E|D)P(D)}{P(E|D)P(D) + P(E|D^c)(1 - P(D))} \\ &= \frac{0.95 \times 0.005}{0.95 \times 0.005 + 0.01 \times 0.995} = 0.32 \end{aligned}$$

- So even with a positive test, there is only a 32% chance of having the disease.

### 2.2 Law of Total Probability

- Let  $\{F_i\}$  be mutually exclusive events such that:

$$\bigcup_{i=1}^n F_i = S$$

Then for any event  $E$ :

$$E = E \cap (\bigcup_{i=1}^n F_i) = \bigcup_{i=1}^n (EF_i)$$

$$P(E) = P(\bigcup_{i=1}^n EF_i) = \sum_{i=1}^n P(EF_i) = \sum_{i=1}^n P(E|F_i)P(F_i)$$

### 2.3 Independent Events

- Generally,  $P(E|F) \neq P(E)$ .
- If knowing  $F$  does not change the probability of  $E$ :

$$P(E|F) = \frac{P(EF)}{P(F)} = P(E)$$

$P(EF) = P(E)P(F)$

### 2.3.1 Example: Rolling Two Dice

- Let:

$$\begin{aligned}
 E_1 &\equiv \text{sum} = 6 \\
 F &\equiv \text{first die} = 4 \\
 E_1 : &\{(1,5), (2,4), (3,3), (4,2), (5,1)\} \\
 F : &\{(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)\} \\
 E_1F &= \{(4,2)\} \\
 P(E_1F) &= \frac{1}{36} \\
 P(E_1) &= \frac{5}{36} \\
 P(F) &= \frac{6}{36} = \frac{1}{6} \\
 P(E_1)P(F) &= \frac{5}{36} \times \frac{1}{6} = \frac{5}{216} \neq P(E_1F)
 \end{aligned}$$

- Let:

$$\begin{aligned}
 E_2 &\equiv \text{sum} = 7 \\
 E_2 : &\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\} \\
 E_2F &= \{(4,3)\} \\
 P(E_2) &= \frac{6}{36} = \frac{1}{6} \\
 P(F) &= \frac{1}{6} \\
 P(E_2F) &= \frac{1}{36}
 \end{aligned}$$

## 2.4 Random Variables and Probability Distributions

- $S = \{\text{all possible outcomes of stochastic process } X\}$

$x$  = random variable

$S$  = finite or countable infinite: discrete random variable  
 $S$  = uncountable infinite: continuous random variable

- Continuous case:

$$P(x_0, x_0 + dx) = p(x)dx$$

where  $p(x)$  is the probability density function (pdf).

- Discrete case:

$$S = S_i$$

$p_i$  = probability of  $S_i$  (probability mass function, pmf)

$$\begin{aligned}
 0 &\leq P(S_i) \leq 1 \\
 1 &= P(S) \\
 0 &\leq p(x) \\
 \int_{-\infty}^{\infty} p(x)dx &= 1
 \end{aligned}$$

## 2.5 Describing a Distribution

- To describe  $p(x)$  in general we specify:
  - **Mode** — peak value of  $p(x)$
  - **Median** — 50% cumulative value
  - **Mean** — average value of  $x$  weighted by  $p(x)$

## 2.6 Cumulative Distribution Function (CDF)

- $$F(x) = \int_{-\infty}^x p(x')dx' = P(X \leq x)$$

$$F(-\infty) = 0, \quad F(\infty) = 1$$

## 2.7 Expectation Values

- Expectation of any function  $f(x)$  over  $p(x)$ :

$$E(f) = \int_{\Omega} f(x)p(x)dx$$

$E$  is a linear operator:  $E(af + bg) = aE(f) + bE(g)$

- Expectation of powers of  $x$ :

$$\begin{aligned} E(x^0) &= E(1) = \int 1 \cdot p(x)dx = 1 \\ E(x^1) &= \int xp(x)dx \equiv \mu = \text{mean value of } x \\ E(x^2) &= \int x^2 p(x)dx \equiv \sigma^2 = \text{variance of } x \end{aligned}$$

## 2.8 Characteristic Function

- The characteristic function of  $p(x)$ :

$$\begin{aligned} \varphi(t) &= \int_{-\infty}^{\infty} e^{itx} p(x)dx = E(e^{itx}) \\ \varphi(t) &= E \left( 1 + itx + \frac{(itx)^2}{2!} + \dots \right) \\ &= 1 + itE(x) + \frac{(it)^2}{2!} E(x^2) + \dots \\ &= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \mu_k \end{aligned}$$

- Moments from  $\varphi(t)$ :

$$\frac{d^n \varphi(t)}{dt^n} \Big|_{t=0} = i^n \mu_n$$

## 2.9 Central Moments

- 

$$E((x - \mu)^n) = \int (x - \mu)^n p(x) dx \equiv \mu_n$$
$$\mu = E(x)$$

- 1st central moment:

$$E((x - \mu)^1) = E(x) - E(\mu) = \mu - \mu = 0$$

- 2nd central moment (variance):

$$E((x - \mu)^2) \equiv V(x) = \sigma^2$$

- 3rd central moment (skewness):

$$\text{skewness} = \frac{E((x - \mu)^3)}{\sigma^3}$$

- 4th central moment (kurtosis):

$$\text{kurtosis} = \frac{E((x - \mu)^4)}{\sigma^4} - 3$$

(The  $-3$  ensures that the kurtosis of a normal distribution is 0.)

### 3 Tuesday, September 16th 2025

#### 3.1 Bayes Theorem and Its Applications

- Bayes Theorem: for events  $A$  and  $B$ , we have

$$P(AB) = P(A|B)P(B) = P(B|A)P(A) = P(BA)$$

- Usually it is given in this form:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- People argued about when you are allowed to use this theorem.

#### 3.2 The Monty Hall Problem: A Bayesian Analysis

- Example: Monty Hall Problem (Game show with host named Monty Hall)
  - There are 3 doors; behind one is a car, behind the other two are goats.
  - You select a door; if the car is behind it, you win.
  - Twist: after you select a door, Monty opens one of the other 2 doors to reveal a goat.
  - Question: stay or switch?
  - Solution: use Bayes theorem.
  - Sample space:  $S = \{C_1 = \text{cgg}, C_2 = \text{gcg}, C_3 = \text{ggc}\}$
  - Event 2 = MH opens door 2.
  - Event 3 = MH opens door 3.
  - Number such that your choice is door 1.
  - Take case  $E_2$ , then we want to know  $P(C_1|E_2)$ .

$$P(C_1|E_2) = \frac{P(E_2|C_1)P(C_1)}{P(E_2)}$$

- $P(C_1) = \frac{1}{3}$
- $P(E_2|C_1) = \frac{1}{2}$  because if the car is behind door 1, Monty can open either door 2 or 3.
- $P(E_2) = \frac{1}{2}$
- Law of total probability:

$$P(E_2) = P(E_2|C_1)P(C_1) + P(E_2|C_2)P(C_2) + P(E_2|C_3)P(C_3) = \frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = \frac{1}{2}$$

- $P(C_1|E_2) = \frac{P(E_2|C_1)P(C_1)}{P(E_2)} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}$
- $P(C_1|E_2) = \frac{1}{3}$
- $P(C_2|E_2) = 0$
- $P(C_3|E_2) = \frac{P(E_2|C_3)P(C_3)}{P(E_2)} = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$

### 3.3 Alternate Monty Hall Formulations

- Alternate version:  $E = \text{MH}$  shows you a goat from  $\{2, 3\}$ .
  - We want to find  $P(C_1|E)$ .
  - $P(C_1|E) = \frac{P(E|C_1)P(C_1)}{P(E)}$
  - $P(C_1) = \frac{1}{3}$
  - $P(E|C_1) = 1$  because if the car is behind door 1, Monty can open either door 2 or 3.
  - $P(E) = 1$  by law of total probability:
$$P(E) = P(E|C_1)P(C_1) + P(E|C_2)P(C_2) + P(E|C_3)P(C_3) = 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = 1$$
  - $P(C_1|E) = \frac{1 \cdot \frac{1}{3}}{1} = \frac{1}{3}$

- Another version: What if MH does *not* know where the car is?

- $E = \text{MH}$  opens  $\{2, 3\}$  and reveals a goat.
- We want to find  $P(C_1|E)$ .
- $P(C_1|E) = \frac{P(E|C_1)P(C_1)}{P(E)}$
- $P(C_1) = \frac{1}{3}$  because we picked door 1.
- $P(E|C_1) = \frac{1}{2}$  because if the car is behind door 1, Monty can open either door 2 or 3 since he does not know where the car is.
- By law of total probability:

$$P(E) = P(E|C_1)P(C_1) + P(E|C_2)P(C_2) + P(E|C_3)P(C_3) = 1 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{2}{3}$$

- $P(C_1|E) = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$

### 3.4 Monty Hall Generalized to $n$ Doors

- Now back to the standard version of the problem but with  $n$  doors.
  - You pick door 1, MH opens any door with a goat behind it from 2 to  $n$  ( $n - 1$  options).
  - $P(C_1|E) = \frac{P(E|C_1)P(C_1)}{P(E)}$
  - $P(E) = 1$  because he can always choose a door with a goat behind it (many options and he knows the answers).
  - $P(C_1) = \frac{1}{n}$
  - $P(E|C_1) = 1$  because if the car is behind door 1, Monty can open any of the other doors.

### 3.5 Continuous Probability Distributions and Moments

- Continuous probability distribution  $p(x)$ :

- Moments:

$$E(x^n) = \int_{-\infty}^{\infty} x^n p(x) dx$$

mean:  $\mu = E(x)$

variance:  $V(x) = \sigma^2 = E((x - \mu)^2) = E(x^2) - \mu^2$

std dev:  $\sigma = \sqrt{\sigma^2}$

- Central moments:

$$E(x - \mu) = E(x) - \mu = 0$$

$$E((x - \mu)^2) = \sigma^2$$

$$E((x - \mu)^3) = \text{skewness}$$

$$E((x - \mu)^4) = \text{kurtosis}$$

- Characteristic function:

$$\Phi(t) = E(e^{itx}) = \int_{-\infty}^{\infty} e^{itx} p(x) dx \quad (3.1)$$

$$= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \mu_k \quad (3.2)$$

$$\Phi_{\mu}(t) = E(e^{it(x-\mu)}) = E(e^{itx})e^{-it\mu} = \Phi(t)e^{-it\mu}$$

$$V(x) = E((x - \mu)^2) \quad (3.3)$$

$$= E(x^2 - 2\mu x + \mu^2) \quad (3.4)$$

$$= E(x^2) - 2\mu E(x) + \mu^2 E(1) \quad (3.5)$$

$$= E(x^2) - 2\mu^2 + \mu^2 \quad (3.6)$$

$$= E(x^2) - \mu^2 = E(x^2) - (E(x))^2 \quad (3.7)$$

### 3.6 Discrete Probability Distributions

- The discrete case (e.g., rolling a die, picking a card) uses a probability mass function.
- Usually denote outcomes as  $r$ :
- $p_r$  = probability of outcome  $r$ .
- $\sum_r p_r = 1$
- $E(r) = \sum_r p_r r = \text{mean } \mu$
- Variance:  $V(r) = \sum_r (r - \mu)^2 p_r = E(r^2) - \mu^2$
- Coin flip example:  $S = \{H, T\}$ .
- Often map to 0 or 1:  $H = 0, T = 1$ .
- But in theory you can pick any two numbers  $a$  and  $b$  to map outcomes, just so you can calculate mean and variance.

$$E(r) = ap_H + bp_T$$

### 3.7 Cumulative Distribution Functions

- For continuous case:

$$F(x) = \int_{-\infty}^x f(x') dx'$$

- For discrete case:

$$F(r) = \sum_{r' \leq r} p_{r'}$$

- $F(x)$  is the cumulative distribution function (CDF).

- $F(x)$  is non-decreasing,  $F(-\infty) = 0$ ,  $F(\infty) = 1$ .

### 3.8 Multivariate Distributions and Covariance

- Distribution of multiple variables:

- Elements belong to real vector space  $\mathbb{R}^n$ .

- $P(AB) \dots P(A, B)$

- $p(x_1, x_2, \dots, x_n) \geq 0$  is the joint probability distribution function (PDF).

- $\int_{\Omega} p(\vec{x}) d^n x = 1$

- $E(f(\vec{x})) = \int_{\Omega} f(\vec{x}) p(\vec{x}) d^n x$

- $\mu_i = \int x_i p(\vec{x}) d^n x$

- $V(x_i) = \sigma_i^2 = \int (x_i - \mu_i)^2 p(\vec{x}) d^n x$

- Covariance:

- $V_{i,j} = E((x_i - \mu_i)(x_j - \mu_j))$

- $V_{i,i} = \sigma_i^2 = E((x_i - \mu_i)^2)$  (variance)

- $V_{i,j} = V_{j,i}$  (symmetry)

## 4 Thursday, September 18th 2025

### 4.1 Conditional Probability: A Simple Example

- For fun, example that depends on cultural assumptions: A king comes from a family with two kids. What is the probability that the king's sibling is a sister?
- $S = \{(m, m), (m, f), (f, m), (f, f)\}$
- $P(S|K) = \frac{P(SK)}{P(K)} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$

### 4.2 Distributions of Multiple Random Variables

- $p(x_1, x_2, \dots, x_n)$
- $S = \mathbb{R}^n$
- $\int p(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n = 1$
- For any function  $f(\vec{x})$ :

$$E(f) = \int f(\vec{x})p(\vec{x})d\vec{x}$$

- $E(x_1) = \int x_1 p(\vec{x})d\vec{x} = \mu_1$
- $E(x_i) = \mu_i$
- $V(x_i) \equiv \sigma_i^2 = \int (x_i - \mu_i)^2 p(\vec{x})d\vec{x}$

### 4.3 Covariance Matrix and Correlation Coefficient

- Define covariance:

$$V_{ij} = E((x_i - \mu_i)(x_j - \mu_j))$$

- $V_{ii} = \sigma_i^2$  (variance)
- $V_{ij} = V_{ji}$  (symmetry)
- $V_{ij} = 0$  for independent variables
- Expanding the covariance matrix:

$$\begin{aligned} V_{ij}(\vec{x}) &= E((x_i - \mu_i)(x_j - \mu_j)) \\ &= E(x_i x_j - \mu_i x_j - \mu_j x_i + \mu_i \mu_j) \\ &= E(x_i x_j) - \mu_i E(x_j) - \mu_j E(x_i) + \mu_i \mu_j \\ &= E(x_i x_j) - \mu_i \mu_j - \mu_j \mu_i + \mu_i \mu_j \\ &= E(x_i x_j) - \mu_i \mu_j \end{aligned}$$

- So we can say that  $V_{ij} \geq 0$
- $V_{ij}$  can be negative, zero, or positive
- Define the correlation coefficient:

$$\rho(x_i, x_j) = \rho_{ij} = \frac{V_{ij}}{\sqrt{V_{ii}} \sqrt{V_{jj}}} = \frac{V_{ij}}{\sigma_i \sigma_j}$$

- We find that  $-1 \leq \rho_{ij} \leq 1$

## 4.4 Independence and Uncorrelated Variables

- Random variables  $x_1, \dots, x_n$  are independent if the joint pdf factorizes:

$$p(x_1, \dots, x_n) = p_1(x_1)p_2(x_2) \dots p_n(x_n)$$

- Independent variables are uncorrelated:

$$\begin{aligned} E(x_i x_j) &= \int x_i x_j p(\vec{x}) d\vec{x} \\ &= \int x_i x_j p_1(x_1) \dots p_n(x_n) dx_1 \dots dx_n \\ &= \int x_i p_i(x_i) dx_i \int x_j p_j(x_j) dx_j \int p_2(x_2) dx_2 \dots \int p_n(x_n) dx_n = \mu_i \mu_j \end{aligned}$$

$$V_{ij} = E(x_i x_j) - \mu_i \mu_j$$

In the case of independent variables:

$$V_{ij} = \mu_i \mu_j - \mu_i \mu_j = 0$$

- Independent variables are uncorrelated, but uncorrelated variables are not necessarily independent.

## 4.5 Examples of Correlated and Uncorrelated Variables

- 100% correlation example:
- $x = \text{Uniform}[-1, 1]$ , plot distribution from  $-1$  to  $1$ .
- $y = x$ :
- $V_{ij} = E(xy) - E(x)E(y) = E(x^2) = \int_{-1}^1 x^2 \frac{1}{2} dx = \frac{1}{3} \neq 0$
- $y = |x|$ :
- $E(xy) = \int_{-1}^0 x(-x)p(x)dx + \int_0^1 x x p(x)dx$
- $E(xy) = \int_0^1 x^2 \frac{1}{2} dx - \int_{-1}^0 x^2 \frac{1}{2} dx = \frac{1}{6} - \frac{1}{6} = 0$

## 4.6 Marginal Distributions

- For a joint pdf  $p(x_1, x_2, \dots, x_n)$ , the marginal probability density functions are:

$$f_1(x_1) = \int p(x_1, x_2, \dots, x_n) dx_2 dx_3 \dots dx_n$$

- If variables are independent:

$$\begin{aligned} f_1(x_1) &= \int p(x_1, x_2, \dots, x_n) dx_2 \dots dx_n \\ &= p_1(x_1) \int p_2(x_2) dx_2 \int p_3(x_3) dx_3 \dots \int p_n(x_n) dx_n \\ &= p_1(x_1) \cdot 1 \cdot 1 \cdot \dots \cdot 1 = p_1(x_1) \end{aligned}$$

## 4.7 Change of Variables in Probability Densities

- Something we need to know, because we do it all the time:
  - Change of variables of  $P$
  - Calculate new  $V_{ij}$  under new variables
- Let  $x$  be a random variable with pdf  $f(x)$  and let  $y$  be some function.
- First:  $y$  is one-to-one with  $f$

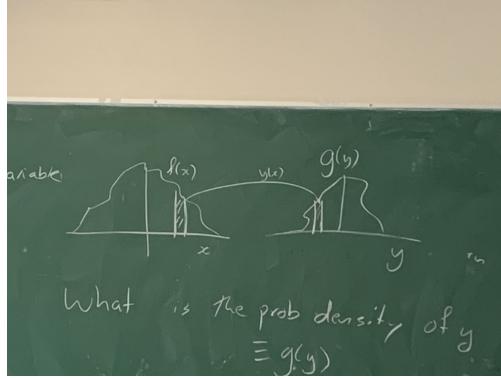


Figure 4.1: 1-to-1 function

- What is the probability density of  $y$ , denoted  $g(y)$ ?
- Conservation of probability:
- $f(x)dx = g(y)dy$
- $g(y) = f(x) \left| \frac{dx}{dy} \right|$

$$f(x) \left| \frac{dx}{dy} \right| = g(y)$$

## 4.8 Change of Variables: Non One-to-One Case

- If  $y$  is not one-to-one: sum over all segments that map to the same  $y$ .
- Example:  $f(x)$  uniform on  $[0, 1]$ ,  $f(x) = 1$
- Let  $y(x) = \frac{-1}{\lambda} \ln(x)$
- $\frac{dy}{dx} = \frac{-1}{\lambda x}$
- $\frac{dx}{dy} = -\lambda x$
- $-\lambda x = \ln x$
- $e^{-\lambda y} = x$
- $\lambda > 0 \Rightarrow \frac{dx}{dy} = -\lambda x = -\lambda e^{-\lambda y}$
- $g(y) = f(x) \left| \frac{dx}{dy} \right| = 1 \cdot \lambda e^{-\lambda y} = \lambda e^{-\lambda y}$

## 4.9 Multivariate Transformations and the Jacobian

- If we have variables  $\{x_i\}$  and transform to new variables  $\{y_i\}$ :
- Region  $\mathbb{R}$  in  $x$ -space maps to region  $\mathbb{R}'$  in  $y$ -space.

$$\int_{\mathbb{R}} f(\vec{x}) d\vec{x} = \int_{\mathbb{R}'} f(\vec{x}(\vec{y})) (\vec{y}) \left| \frac{\partial \vec{x}}{\partial \vec{y}} \right| d\vec{y}$$

$$g(\vec{y}) = f(\vec{x}(\vec{y})) |J|$$

- Where  $\left| \frac{\partial \vec{x}}{\partial \vec{y}} \right|$  is the Jacobian determinant of the transformation.

- Jacobian matrix  $J$ :

$$J = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{bmatrix}$$

## 4.10 Example: Cartesian to Polar Transformation

- Change to polar coordinates:

- $x = r \cos \theta$

- $y = r \sin \theta$

- $P'(r, \theta) = ? = p(x, y) \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right|$

- $\frac{\partial x}{\partial r} = \cos \theta$

- $\frac{\partial y}{\partial r} = \sin \theta$

- $\frac{\partial x}{\partial \theta} = -r \sin \theta$

- $\frac{\partial y}{\partial \theta} = r \cos \theta$

- $J = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$

- $|J| = r \cos^2 \theta + r \sin^2 \theta = r$

- $p'(r, \theta) = \frac{r}{\pi} dr d\theta$

## 5 Tuesday, September 20th 2025

### 5.1 Propagation of Errors for a Single Variable

- Given  $f(x)$  pdf,  $\mu \equiv E(x)$ ,  $\sigma^2 \equiv V(x) = E(x^2) - \mu^2$
- Know  $f(x) \rightarrow g(y)$ , given  $y(x)$ .
- Taylor expand  $y(x)$  about mean  $\mu$ :

$$y(x) = y(\mu) + y'(\mu)(x - \mu) + \frac{1}{2!}y''(\mu)(x - \mu)^2 + \dots$$

$$E(y(x)) \equiv \mu_y$$

$$\begin{aligned} E(y(x)) &= E(y(\mu)) + y'(\mu)E(x - \mu) + \frac{1}{2!}y''(\mu)E((x - \mu)^2) + \dots \\ &= y(\mu) + y'(\mu) \cdot 0 + \frac{1}{2!}y''(\mu)V(x) + \dots \end{aligned}$$

- To the 1st order:

$$\mu_y = E(y(x)) = y(\mu) = y(E(x))$$

### 5.2 Variance Propagation for a Single Variable

- Variance of  $y$ :

$$V(y) = E((y(x) - E(y(x)))^2) \quad (5.1)$$

$$= E((y(x) - \mu_y)^2) \quad (5.2)$$

$$= E((y'(\mu)(x - \mu) + \frac{1}{2!}y''(\mu)(x - \mu)^2 + \dots)^2) \quad (5.3)$$

$$= E(y'(\mu)^2(x - \mu)^2 + y'(\mu)y''(\mu)(x - \mu)^3 + O((x - \mu)^4)) \quad (5.4)$$

$$= y'(\mu)^2V(x) + \dots \quad (5.5)$$

- Some relations:

$$E(x) \equiv \mu_x$$

$$V(x) \equiv \sigma_x^2$$

$$y = y(x)$$

$$E(y) \equiv \mu_y = y(\mu_x)$$

$$V(y) \equiv \sigma_y^2 = (y'(\mu_x))^2\sigma_x^2$$

$$\sigma_y = |y'(\mu_x)|\sigma_x$$

- Example:  $y = \frac{1}{x}$ ,  $\frac{dy}{dx} = -\frac{1}{x^2}$

$$\sigma_y^2 = \frac{1}{\mu_x^4}\sigma_x^2$$

### 5.3 Propagation of Errors for Multiple Variables

- Let us suppose we have  $n$  variables  $\{x_i\}$ , with pdf  $f(\vec{x})$ .
- Let  $y_j = 1, 2, \dots, m$  be  $m$  functions of  $x_i$ .
- $y_j = y_j(x_1, x_2, \dots, x_n)$
- $V_{ij}(x)_{n \times n}(\vec{x}) = \text{covariance matrix of } \{x_i\}$
- $V_{ij}(\vec{x}) = E((x_i - \mu_i)(x_j - \mu_j))$
- Taylor expand each  $y_j$ :  $\vec{\mu} = (\mu_1, \mu_2, \dots, \mu_n)$
- $y_j(\vec{x}) = y_j(\vec{\mu}) + \sum_i \frac{\partial y_j}{\partial x_i} \Big|_{\vec{\mu}} (x_i - \mu_i) + \frac{1}{2!} \sum_{i,k} \frac{\partial^2 y_j}{\partial x_i \partial x_k} \Big|_{\vec{\mu}} (x_i - \mu_i)(x_k - \mu_k) + \dots$
- $E(y_j(\vec{x})) = E(y_j(\vec{\mu})) + \sum \frac{\partial y_j}{\partial x_i} E(x_i - \mu_i) + \dots = y_j(\vec{\mu})$

### 5.4 Covariance Propagation for Functions of Multiple Variables

- Covariance between  $y_k$  and  $y_l$ :

$$\begin{aligned}
 & E((y_k - \mu_{y_k})(y_l - \mu_{y_l})) \\
 &= E((y_k - y_k(\mu))(y_l - y_l(\mu))) \\
 &= E \left( \sum_i \frac{\partial y_k}{\partial x_i} \Big|_{\mu} (x_i - \mu_i) \sum_j \frac{\partial y_l}{\partial x_j} \Big|_{\mu} (x_j - \mu_j) \right) \\
 &= \sum_{i,j} \frac{\partial y_k}{\partial x_i} \Big|_{\mu} \frac{\partial y_l}{\partial x_j} \Big|_{\mu} E((x_i - \mu_i)(x_j - \mu_j)) \\
 & \boxed{V_{kl}(\vec{y})_{m \times m} = \sum_{i,j} \frac{\partial y_k}{\partial x_i} \Big|_{\vec{\mu}} \frac{\partial y_l}{\partial x_j} \Big|_{\vec{\mu}} V_{ij}(\vec{x})_{n \times n}}
 \end{aligned}$$

- Example:  $x, y$  random variables,

$$V(x, y) = \begin{bmatrix} V_{xx} & V_{xy} \\ V_{yx} & V_{yy} \end{bmatrix} = \begin{bmatrix} \sigma_x^2 & \rho_{xy}\sigma_x\sigma_y \\ \rho_{xy}\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$$

- $z = x + y$
- $V(z) = \sigma_z^2 = \left(\frac{\partial z}{\partial x}\right)^2 V_{xx} + 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} V_{xy} + \left(\frac{\partial z}{\partial y}\right)^2 V_{yy}$
- $= \sigma_x^2 + 2\rho_{xy}\sigma_x\sigma_y + \sigma_y^2$
- If  $x_i$  are uncorrelated,

$$\begin{aligned}
 V_{ij} &= \sigma_{i,j}\sigma_i^2 = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \\
 V_{kl}(\vec{y}) &= \sum_i \frac{\partial y_k}{\partial x_i} \Big|_{\mu} \frac{\partial y_l}{\partial x_i} \Big|_{\mu} V_{ii}(\vec{x}) \\
 \text{variance } V_{kk} &= \sum_i \left( \frac{\partial y_k}{\partial x_i} \right)^2 \sigma_i^2
 \end{aligned}$$

## 5.5 Examples of Error Propagation in Measurements

- Example: Measuring resistances.  $x_i$  independent,  $z = x + y$ ,  $x = R_1$  resistor value,  $y = R_2$  resistor value,  $z = R_{\text{tot}}$  total resistance.
- $R_1 \pm \sigma_{R_1}$
- Convention is to use  $\sqrt{V(R)}$  as uncertainty.
- For a good measuring device,  $E(R) = R_{\text{true}} \leftarrow$  unbiased.
- $V(R) = \text{small}$
- $R_1 \pm \sigma_{R_1}, R_2 \pm \sigma_{R_2}$ , then  $\sigma_{R_{\text{tot}}} = \sqrt{\sigma_{R_1}^2 + \sigma_{R_2}^2}$
- $R = R_{\text{tot}} = R_1 + R_2$
- $z = xy$ , like  $I, R$
- $\sigma_z^2 = \left(\frac{\partial z}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial z}{\partial y}\right)^2 \sigma_y^2 = y^2 \sigma_x^2 + x^2 \sigma_y^2$

$$\boxed{\left(\frac{\sigma_z}{z}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$$

## 5.6 Matrix Formulation of Linear Error Propagation

- Formula is exact if transformation of variables is linear.
- $\vec{y} = A\vec{x}$ ,  $A$  is  $m \times n$  matrix,  $\vec{x}$  is  $n \times 1$ ,  $\vec{y}$  is  $m \times 1$ .
- $\frac{\partial y_k}{\partial x_i} = \text{constant} \Rightarrow$  higher order terms in Taylor expansion are 0
- $V_{kl}(\vec{y}) = \sum_{i,j} \frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_j} V_{ij}(\vec{x})$
- Matrix notation:
- $V_{kl}(\vec{y}) = \sum_{i,j} A_{ki} A_{lj} V_{ij}(\vec{x})$
- $= \sum_{i,j} A_{ki} V_{ij}(\vec{x}) A_{lj}$
- $= \sum_{i,j} A_{ki} V_{ij} (A^T)_{jl}$
- $= (AV(\vec{x})A^T)_{kl}$

$$\boxed{V(\vec{y})_{m \times m} = A_{m \times n} V(\vec{x})_{n \times n} A_{n \times m}^T}$$

## 5.7 Variance of the Arithmetic Mean

- Example: Arithmetic mean. Let  $x_i = n$  identical independent variables with  $V(x_i) = \sigma_x^2$
- Set  $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$
- Recall that  $V(ax) = a^2 V(x)$
- $V(\bar{x}) = V\left(\frac{1}{n} \sum_{i=1}^n x_i\right) = \frac{1}{n^2} V\left(\sum_{i=1}^n x_i\right) = \frac{1}{n^2} \sum_{i=1}^n V(x_i) = \frac{1}{n^2} n \sigma_x^2 = \frac{\sigma_x^2}{n}$

- If variables are different  $\sigma_i^2$ :  $n$  measurements

- $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$

- $V(\bar{x}) = \frac{1}{n^2} \sum_{i=1}^n V(x_i) = \frac{1}{n^2} \sum_{i=1}^n \sigma_i^2$

- $\sigma_{\bar{x}} = \frac{1}{n} \sqrt{\sum_{i=1}^n \sigma_i^2}$

## 5.8 Example: Measuring the Period of a Sine Wave

- Example: Measure period of sine wave on scope.

- $T = \Delta t = t_2 - t_1$

- $\sigma_T^2 = \left( \frac{\partial \Delta t}{\partial t_1} \right)^2 \sigma_t^2 + \left( \frac{\partial \Delta t}{\partial t_2} \right)^2 \sigma_t^2 = \sigma_t^2 + \sigma_t^2 = 2\sigma_t^2$

- Measure  $N$  cycles,  $T = \frac{1}{N} \Delta t$

- $\sigma_{T^2} = \frac{1}{N^2} \sigma_{\Delta t}^2 = \frac{2}{N^2} \sigma_t^2$

## 6 Tuesday, September 25th 2025

### 6.1 Covariance Transformation Under Linear Transformations

- Linear transformation:

$$\vec{y} = A\vec{x}$$

$$V_{kl}(\vec{y}) = \sum_{i,j} \frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_j} V_{ij}(\vec{x})$$

- Linear  $y_k = \sum A_{kj}x_j$

- then

$$V_{kl}(\vec{y}) = \sum_{i,j} A_{ki} A_{lj} V_{ij}(\vec{x})$$

- or in matrix form

$$V(\vec{y}) = (AV(\vec{x})A^T)_{kl}$$

#### Diagonalization via Eigenvectors

- If  $\hat{e}_i$  are the eigenvectors of  $V$ , then

$$V(\vec{x})\hat{e}_i = \lambda_i \hat{e}_i$$

- Form:

$$A = \begin{pmatrix} \hat{e}_1 \\ \dots \\ \hat{e}_n \end{pmatrix} = \begin{pmatrix} \hat{e}_{11} & \hat{e}_{12} & \dots & \hat{e}_{1n} \\ \dots & \dots & \dots & \dots \\ \hat{e}_{n1} & \hat{e}_{n2} & \dots & \hat{e}_{nn} \end{pmatrix}$$

- then

$$A^T A = I$$

- then:

$$VA^T = V \begin{pmatrix} \hat{e}_1 & \dots \\ \dots & \dots \\ \hat{e}_n & \dots \end{pmatrix} = \begin{pmatrix} \lambda_1 \hat{e}_{11} & \dots & \lambda_n \hat{e}_{n1} \\ \dots & \dots & \dots \\ \lambda_1 \hat{e}_{1n} & \dots & \lambda_n \hat{e}_{nn} \end{pmatrix}$$

- Then:

$$AVA^T = \begin{pmatrix} \hat{e}_{11} & \dots & \hat{e}_{1n} \\ \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} \lambda_1 \hat{e}_{11} & \dots & \dots \\ \dots & \dots & \dots \\ \lambda_1 \hat{e}_{1n} & \dots & \dots \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

- Then:

$$AVA^T = V(\vec{y}) = \begin{pmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sigma_n^2 \end{pmatrix}$$

## 6.2 The Binomial Distribution

- Consider an experiment with two outcomes.
- E.g. coin flips, selecting a ball with 2 possible colours, etc.
- One trial is called a Bernoulli trial.

### Bernoulli Trials and Sampling Methods

- Example – Method 1: You have an urn filled with  $N$  balls. Some are red (R), some are blue (B).
- (0) What is your estimate of  $n_R$ ,  $n_B$ , or  $f = n_R/N$  or  $p$  of drawing R?
- (1) You pick a ball: R. Q: estimate of  $p = n_R/N$ ?
- (2) You pick another without replacing 1st ball: get R.
- (3) R
- (4) Get B
- This is a question about this ONE urn.
- Now Method 2: you draw red, and you PUT IT BACK. You repeat this several times.
- Now Method 3: We have an infinite source of balls with fraction  $p$  red and  $(1 - p)$  blue.

$$\begin{aligned} P(R) &= p \\ P(B) &= 1 - p \end{aligned}$$

### Derivation of the Binomial Probability

- Make infinite number of urns all with  $N$  balls, with fraction  $p$  red and  $(1 - p)$  blue.
- Open all, count  $n_R$  red balls,  $n_B$  blue balls.
- In our case we have  $N$  balls, prob  $p = R$  and  $1 - p = B$ .
- Prob of getting sequence RRB is:

$$P(RRB) = p \cdot p \cdot (1 - p) = p^2(1 - p)$$

- If we don't care about order, then:

$$P(RRB) = P(RBR) = P(BRR) = p^2(1 - p)$$

- There are 3 ways of ordering RRB, so total probability is:

$$P(2R, 1B) = 3p^2(1 - p) = 3p^2q$$

- Number of ways to choose  $r$  items from  $N$  is:

$$\binom{N}{r} = \frac{N!}{r!(N-r)!}$$

- Probability of getting exactly  $r$  R out of  $N$ :

$$P_r = \binom{N}{r} p^r (1 - p)^{N-r} = B(r; N, p)$$

- This is called the Binomial distribution and applies to anything where there are 2 outcomes ( $A, \bar{A}$ ).

## Mean and Variance of the Binomial Distribution

- Want mean,  $\sigma$

$$\begin{aligned}
E(r) &= \sum_{r=0}^n r P_r = \sum_{r=0}^n r \binom{n}{r} p^r (1-p)^{n-r} \\
&= \sum_{r=0}^n r \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r} \\
&= \sum_{r=1}^n \frac{n!}{(r-1)!(n-r)!} p^r (1-p)^{n-r} \\
&= np \sum_{r=1}^n \frac{(n-1)!}{(r-1)!(n-r)!} p^{r-1} (1-p)^{n-r}
\end{aligned}$$

- Change sum  $r' = r - 1 \rightarrow n' = n - 1$

$$\begin{aligned}
E(r) &= np \sum_{r'=0}^{n-1} \frac{(n-1)!}{r'!(n-1-r')!} p^{r'} (1-p)^{(n-1)-r'} \\
&= np \sum_{r'=0}^{n-1} \binom{n-1}{r'} p^{r'} (1-p)^{n'-r'} \\
&= np \cdot 1 = np
\end{aligned}$$

- from:

$$\begin{aligned}
(p+q)^n &= \sum_{r=0}^n \binom{n}{r} p^r q^{n-r} \\
(p+1-q)^n &= 1^n = 1 \\
E(r) &= np
\end{aligned}$$

- This is what we want!

- Now:

$$V(r) = \sum r^2 p_r - E(r)^2 = \sum r^2 p_r - n^2 p^2$$

- Slightly easier to calculate:

$$\begin{aligned}
\sum r(r-1)p_r &= \sum_{r=0}^n r(r-1) \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r} = \sum_{r=2}^n \frac{n!}{(r-2)!(n-r)!} p^r q^{n-r} \\
&= n(n-1)p^2 \sum_{r=2}^n \frac{(n-2)!}{(r-2)!(n-r)!} p^{r-2} q^{n-r}
\end{aligned}$$

- Sub  $r' = r - 2$

$$\begin{aligned}
&= n(n-1)p^2 \sum_{r'=0}^{n-2} \binom{n-2}{r'} p^{r'} q^{(n-2)-r'} \\
&= n(n-1)p^2 \cdot 1 = n(n-1)p^2 \\
&= n^2 p^2 - np^2
\end{aligned}$$

- Such that:

$$\boxed{V(r) = \sum r^2 p_r - n^2 p^2} = np(1-p) = npq$$

## Applications to Histograms and Counting Statistics

- Why is this important? Histograms are often Binomially distributed.
- Data either falls  $A$ : falls in bin, or  $\bar{A}$ : does not fall in bin.
- $p$  = probability of falling in  $i$ th bin.
- $+n$  entries, e.g. students in class, histogram = grades.
- Expected number of entries is  $np$ .
- Plot of taking distribution several times and checking how many fall in bin  $i$  and then plotting that distribution is Binomial.
- Usually you have 1 histogram.
- Look at entries in bin  $i$  –  $n_i/n$  = fraction of entries in bin  $i$ .
- Estimator  $p = n_i/n$ .
- Expect if you repeated  $\Rightarrow n_i$  would follow Binomial distribution with mean  $\sigma_i = \sqrt{npq}$  and  $V_i = \sigma_i^2 = np(1-p)$ .
- $$p = \frac{n_i}{n}$$
- $$\sigma_i = \sqrt{n_i \left(1 - \frac{n_i}{n}\right)} \approx \sqrt{n_i} \text{ if } n \gg n_i$$
- Notes: we do know the total number  $n$ , how often is it in bin  $i$ .
- HW: given the distribution, how many times  $n$  do I need to do it to get that.
- $r$  fixed  $n$ , vs.  $n$  fixed  $r$ .

## 7 Thursday, October 2nd 2025

### 7.1 Review of the Binomial Distribution and Its Properties

- Recall last time:

$$B(r, n, p) = \binom{n}{r} p^r (1-p)^{n-r}$$

$$E(r) = np = \mu$$

$$V(r) = np(1-p) = \sigma^2$$

$$r = \sqrt{np(1-p)}$$

$$p = \frac{\mu}{n}$$

$$\bullet \epsilon = \frac{r}{n}$$

- Number of detections (people often forget the  $(1-p)$  term):

$$n\epsilon \pm \sqrt{n\epsilon(1-\epsilon)}$$

$$\bullet \sigma_\epsilon = \frac{1}{n}\sigma_r = \frac{1}{n}\sqrt{r\left(1-\frac{r}{n}\right)} = \frac{1}{\sqrt{n}}\sqrt{\epsilon(1-\epsilon)}$$

- Standardized skewness:

$$E\left[\left(\frac{x-\mu}{\sigma}\right)^3\right] = \frac{1-2p}{\sqrt{np(1-p)}}$$

- Excess kurtosis:

$$\frac{1-6p(1-p)}{np(1-p)}$$

### 7.2 Bernoulli Distribution as a Special Case of the Binomial

- Bernoulli Distribution: Binomial with  $n = 1$ .

$$B(r, n=1, p) = P_r = \binom{1}{r} p^r (1-p)^{1-r} = p^r (1-p)^{1-r}$$

$$P_0 = 1 - p$$

$$P_1 = p$$

$$\mu = E(r) = np = p$$

$$V = \sigma^2 = np(1-p) = p(1-p)$$

$$E(r^k) = \sum_{r=0}^1 r^k P_r = 0^k (1-p) + 1^k p = p$$

- Central moments:  $E[(r - p)^k]$

### 7.3 Negative Binomial and Geometric Distributions

- Negative Binomial: How many  $n$  to get  $r$  successes.
- Geometric distribution: negative binomial with  $r = 1$  (number of trials to get first success).

$$G(n, p) = p(1 - p)^{n-1}$$

- Example: Roll a die with  $p(i) = 1/6$  (success = get 4)

$$E(n) = \frac{1}{p} = 6$$

- $P(n \leq 5) = 0.598$
- $P(n \geq 7) = 0.335$

### 7.4 Samples and the Concept of an Ensemble

- Samples: A set of  $N$  draws/trials from a pdf  $p(x)$ ,  $\{P_r\}$ , is called a **sample** of size  $N$ :  $\{x_i\}_{i=1}^N$ .
- Orthodox statistics: your sample is one of many possible, and we can answer questions about the **ensemble** of samples.
- Samples:
  - 1:  $\{x_{1i}\}_{i=1}^N$  from pdf  $p(x)$
  - 2:  $\{x_{2i}\}_{i=1}^N$  from pdf  $p(x)$
  - ...
  - M
- $p(x|\mu, \sigma)$
- $p =$  probability of  $H$  (heads),  $q = 1 - p =$  probability of  $T$  (tails)
- These are limits:  $p = \lim_{n \rightarrow \infty} \frac{n_H}{n}$
- Flip 10 times: exactly  $2^{10} = 1024$  possible outcomes

$$S_1 = \{HHHHHHHHHH\}$$

$$S_2 = \{HTHHHHHHHH\}$$

...

- Given true  $p$ , calculate probability of any  $S_i$ :

$$P_i = \lim_{N \rightarrow \infty} \frac{n_i}{N}$$

- From the pdf probability, select  $k$  samples from a sample space of size  $F$ .
- Science/statistics infers from sample space  $\mathcal{R}$  to get  $p(x)$ .
- Not purely deductive:  $\infty$  number of pdfs map to one sample.
- Inductive: If sample pdf is more likely.
- Sample HHTHT ... known, from it we try to infer  $p = p_H$ .

## 7.5 Poisson Distribution as a Limit of the Binomial

- Poisson Distribution: limit of binomial for large  $n$ , small  $p$ .

- 

$$B(r, n, p) = \binom{n}{r} p^r (1-p)^{n-r}$$

- Let  $n \rightarrow \infty, p \rightarrow 0$  such that  $np = \mu$  is constant.

- Stirling approximation:

$$n! \approx \sqrt{2\pi n} n^n e^{-n}$$

- Then:

$$B(r, n, p) = \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r}$$

- $r$  is finite,  $n \rightarrow \infty, n-r \rightarrow \infty$

$$\begin{aligned} B(r, n, p) &= \frac{1}{r!} \frac{\sqrt{2\pi n}}{\sqrt{2\pi(n-r)}} \frac{n^n e^{-n}}{(n-r)^{n-r} e^{-(n-r)}} \left(\frac{\mu}{n}\right)^r \left(1 - \frac{\mu}{n}\right)^{n-r} \\ &= \frac{1}{r!} \sqrt{\frac{n}{n-r}} \left(\frac{n}{n-r}\right)^{n-r} \frac{\mu^r}{e^r} \frac{\left(1 - \frac{\mu}{n}\right)^n}{\left(1 - \frac{\mu}{n}\right)^r} \\ &= \frac{1}{r!} \sqrt{\frac{n}{n-r}} \frac{\left(1 - \frac{\mu}{n}\right)^r}{\left(1 - \frac{\mu}{n}\right)^n} \frac{\mu^r}{e^r} \frac{\left(1 - \frac{\mu}{n}\right)^n}{\left(1 - \frac{\mu}{n}\right)^r} \end{aligned}$$

- $n \rightarrow \infty, \sqrt{\frac{n}{n-r}} \rightarrow 1$

- $\left(1 - \frac{\mu}{n}\right)^n \rightarrow e^{-\mu}$

- $\left(1 - \frac{\mu}{n}\right)^r \rightarrow 1$

- Then:

$$\lim_{n \rightarrow \infty, np = \mu} B(r, n, p) = \frac{1}{r!} \mu^r e^{-\mu} = P(r|\mu) = \text{Poisson}$$

## 7.6 Poisson Process and Radioactive Decay

- Consider radioactive decay of some atoms:

- Any time interval  $[t, t+dt]$  contains at most one decay.
- Probability of a decay occurring in this interval is proportional to  $dt$ .
- Whether or not an atom decays in the interval is independent of any other non-overlapping interval.

- From (1) and (2):

$$P_d(dt) = \lambda dt$$

- Probability of no decay in interval:

$$P_0(dt) = 1 - \lambda dt$$

- Probability of no decay by time  $t + dt$ :

$$\begin{aligned} P_0(t + dt) &= P_0(t)P_0(dt) \\ &= P_0(t)(1 - \lambda dt) \\ P_0(t + dt) - P_0(t) &= -\lambda P_0(t)dt \\ \frac{dP_0(t)}{dt} &= -\lambda P_0(t) \\ P_0(t) &= P_0(0)e^{-\lambda t} = e^{-\lambda t} \end{aligned}$$

- Probability of getting  $r$  decays in time  $t + dt$ :

$$\begin{aligned} P_r(t + dt) &= P_r(t)P_0(dt) + P_{r-1}(t)P_d(dt) \\ &= P_r(t)(1 - \lambda dt) + P_{r-1}(t)\lambda dt \\ \frac{dP_r(t)}{dt} &= -\lambda P_r(t) + \lambda P_{r-1}(t) \end{aligned}$$

- Solution to PDE:

$$P_r(t) = \frac{1}{r!}(\lambda t)^r e^{-\lambda t}$$

- Poisson distribution with  $\mu = \lambda t$

## 7.7 Moments and Variance of the Poisson Distribution

- Properties:

$$\begin{aligned} E(r) &= \sum_{r=0}^{\infty} rP(r, \mu) \\ &= \sum_{r=0}^{\infty} r \frac{\mu^r}{r!} e^{-\mu} \\ &= \mu e^{-\mu} \sum_{r=1}^{\infty} \frac{\mu^{r-1}}{(r-1)!} \\ &= \mu e^{-\mu} e^{\mu} \\ &= \mu \end{aligned}$$

- $V(r) = E(r^2) - \mu^2$

$$\begin{aligned} E[r(r-1)] &= E(r^2) - \mu \\ &= \sum_{r=2}^{\infty} r(r-1) \frac{\mu^r}{r!} e^{-\mu} \\ &= \mu^2 e^{-\mu} \sum_{r=2}^{\infty} \frac{\mu^{r-2}}{(r-2)!} \\ &= \mu^2 e^{-\mu} e^{\mu} = \mu^2 \end{aligned}$$

- $E(r^2) - \mu = \mu^2$
- $V(r) = E(r^2) - \mu^2 = \mu$
- Binomial:  $\mu = np$ ,  $V(r) = np(1-p) = \mu$  for  $p \rightarrow 0$  and  $n \rightarrow \infty$ .

## 8 Tuesday, October 7th, 2025

### 8.1 The Gaussian (Normal) Distribution

$$G(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

### 8.2 The Standard Normal Distribution

$$N(0, 1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

### 8.3 Example: Circular Symmetry (Darts on a Board)

Distribution of darts:

$$f(x, y) = h(x)k(y)$$

Transform to polar coordinates:

$$g(r, \theta) \approx f(x, y) = h(x)k(y), \quad g(r, \theta) = g(r)$$

$$\frac{\partial g}{\partial \theta} = 0 = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta}$$

Coordinates:

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$\frac{\partial x}{\partial \theta} = -r \sin \theta = -y, \quad \frac{\partial y}{\partial \theta} = r \cos \theta = x$$

Condition:

$$0 = h'(x)k(y)(-y) + h(x)k'(y)x$$

$$\frac{h'(x)}{xh(x)} = \frac{k'(y)}{yk(y)} = a \quad (\text{constant})$$

Solutions:

$$h(x) = ce^{ax^2}, \quad k(y) = de^{ay^2}$$

$$f(x, y) = Ae^{a(x^2+y^2)} = Ae^{ar^2} \approx Ae^{-r^2}$$

### 8.4 Expectation Value of a Gaussian

$$E(x) = \mu = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} xe^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \mu$$

Useful identity:

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

## 8.5 Moments of the Gaussian

The  $n$ th central moment:

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x - \mu)^n e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

- All odd moments vanish (symmetry about  $\mu$ ).

Define

$$I_0(a) = \int_{-\infty}^{\infty} e^{-ay^2} dy = \sqrt{\frac{\pi}{a}}$$

Differentiation rule:

$$\frac{d^n I_0(a)}{da^n} = (-1)^n \frac{(2n)!}{n!} \frac{I_0(a)}{(2a)^n}$$

## 8.6 Variance of the Gaussian

Let  $y = x - \mu$ . Then

$$\begin{aligned} V(y) &= V(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} y^2 e^{-\frac{y^2}{2\sigma^2}} dy \\ &= -\left. \frac{dI_0(a)}{da} \right|_{a=\frac{1}{2\sigma^2}} = \sigma^2 \end{aligned}$$

## 8.7 Kurtosis of the Gaussian

Fourth central moment:

$$\int y^4 e^{-ay^2} dy = \frac{d^2 I_0(a)}{da^2} \frac{1}{4a^2}$$

Evaluates to:

$$E\left[\left(\frac{x-\mu}{\sigma}\right)^4\right] = 3$$

Thus the Gaussian kurtosis = 3. - Excess kurtosis = 0. - Distributions with  $> 3$  have "fat tails."

## 8.8 Poisson Distribution and Gaussian Limit

$$P(r|\lambda) = \frac{1}{r!} \lambda^r e^{-\lambda}$$

For large  $r$ , Stirling approximation:

$$r! \approx \sqrt{2\pi r} \left(\frac{r}{e}\right)^r$$

$$\log P(r|\lambda) = -\log(r!) + r \log \lambda - \lambda$$

Expanding around  $r \approx \lambda$  leads to Gaussian limit with variance  $\lambda$ :

$$P(r|\lambda) \approx \frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{(r-\lambda)^2}{2\lambda}}$$

## 8.9 Central Limit Theorem (CLT)

Let  $x_1, \dots, x_n$  be independent random variables with mean  $\mu$ , variance  $\sigma^2$ . Define sample mean

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

Then as  $n \rightarrow \infty$ :

$$\bar{x} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

## 8.10 Cumulative Distribution Function of a Gaussian

$$F(x) = \int_{-\infty}^x G(y|\mu, \sigma) dy$$

Define error function:

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-y^2} dy, \quad \text{erfc}(t) = 1 - \text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-y^2} dy$$

## 8.11 Gaussian Confidence Intervals

- $1\sigma$ : 68.27%
- $2\sigma$ : 95.45%
- $3\sigma$ : 99.73%
- $5\sigma$ : 99.99994%

## 8.12 Estimators

Given a sample of size  $n$ , an *estimator* is any function designed to estimate a property of the true pdf from which the samples were drawn.

## 9 Thursday, October 9th 2025

### 9.1 Properties of Estimators

- Consistent:  $\lim_{n \rightarrow \infty} \hat{a} = a$
- Unbiased:  $\mathbb{E}[\hat{a}(x)] = a$
- Efficient: smallest variance of all unbiased estimators

### 9.2 Example: Measurements and Models

- Let  $x_1, x_2, \dots, x_n$  be  $n$  measurement points.
- Example applications: number of elements in a histogram bin, position of hits in a detector.
- $y_i$  are the measured values at each  $x_i$ , with variances  $V(y_i) = \sigma_i^2$ .
- Suppose we suspect a model for the histogram shape (e.g. linear background + Gaussian signal):

$$\text{Number of entries} = mx_i + b + Ae^{-\frac{(x_i - \mu)^2}{2\sigma^2}}$$

- More generally, assume a function  $f(x, \vec{\theta})$  with parameters  $\vec{\theta}$ .

### 9.3 Least Squares Estimation

- The best estimate for  $\vec{\theta}$  is the value that minimizes the chi-squared:

$$\chi^2 = \sum_{i=1}^n \frac{(y_i - f(x_i, \vec{\theta}))^2}{\sigma_i^2}.$$

- Condition for minimization:

$$\frac{\partial \chi^2}{\partial \theta_j} = 0.$$

- Equivalent system of equations:

$$\sum_{i=1}^n \frac{(y_i - f(x_i, \vec{\theta}))}{\sigma_i^2} \frac{\partial f(x_i, \vec{\theta})}{\partial \theta_j} = 0.$$

### 9.4 Straight Line Fit

- For  $f(x_i, \vec{\theta}) = mx_i + b$ , with  $\vec{\theta} = (m, b)$ :

$$\chi^2 = \sum_{i=1}^n \frac{(y_i - (mx_i + b))^2}{\sigma_i^2}.$$

- Normal equations from minimization:

$$\frac{\partial \chi^2}{\partial m} = -2 \sum_{i=1}^n \frac{(y_i - (mx_i + b))}{\sigma_i^2} x_i = 0,$$

$$\frac{\partial \chi^2}{\partial b} = -2 \sum_{i=1}^n \frac{(y_i - (mx_i + b))}{\sigma_i^2} = 0.$$

- Two equations, two unknowns. Can be written in matrix form:

$$\begin{bmatrix} S_{xx} & S_x \\ S_x & S_1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} S_{xy} \\ S_y \end{bmatrix},$$

where

$$\begin{aligned} S_{xx} &= \sum_i \frac{x_i^2}{\sigma_i^2}, & S_x &= \sum_i \frac{x_i}{\sigma_i^2}, & S_1 &= \sum_i \frac{1}{\sigma_i^2}, \\ S_{xy} &= \sum_i \frac{x_i y_i}{\sigma_i^2}, & S_y &= \sum_i \frac{y_i}{\sigma_i^2}. \end{aligned}$$

## 9.5 Generalized Least Squares with Covariance Matrix

- In general, for non-diagonal covariance matrix  $V$  of  $y_i$ :

$$\chi^2 = \sum_{i=1}^n \sum_{j=1}^n (y_i - f(x_i, \vec{\theta})) E_{ij} (y_j - f(x_j, \vec{\theta})),$$

where  $E = V^{-1}$  is the inverse covariance matrix.

- Linear case: if  $f(x_i, \vec{\theta})$  is linear in  $\theta$ :

$$\vec{f} = A\vec{\theta}.$$

- Then

$$\chi^2 = (\vec{y} - A\vec{\theta})^T V^{-1} (\vec{y} - A\vec{\theta}).$$

- Minimization gives:

$$\begin{aligned} (A^T V^{-1} A) \vec{\theta} &= A^T V^{-1} \vec{y}, \\ \Rightarrow \vec{\theta} &= (A^T V^{-1} A)^{-1} A^T V^{-1} \vec{y}. \end{aligned}$$

## 9.6 Covariance of the Estimated Parameters

- Propagation of covariance:

$$V(\vec{y}) = B V(\vec{x}) B^T.$$

- For parameter estimates:

$$V(\vec{\theta}) = (A^T V^{-1} A)^{-1}.$$

## 9.7 Goodness of Fit

- The chi-squared statistic

$$\chi^2 = \sum_{i=1}^n \frac{(y_i - f(x_i, \vec{\theta}))^2}{\sigma_i^2}$$

is distributed as  $\chi^2$  with  $n$  degrees of freedom if the  $y_i$  are Gaussian.

- If the model is good,  $\chi^2/\text{dof} \sim 1$ ; if the model is bad,  $\chi^2/\text{dof} \gg 1$ .

## 10 Tuesday, October 14th 2025

In this lecture, I start to not take all notes, instead I write down only the key points.

### 10.1 Chi-Squared for Uncorrelated and Correlated Measurements

- Recall difference in chi-squared formula for uncorrelated and correlated measurements.
- Uncorrelated:

$$\chi^2 = \sum_{i=1}^n \frac{(y_i - \mu_i)^2}{\sigma_i^2}$$

- Correlated (general case):

$$\chi^2 = \sum_{i,j}^n (y_i - \mu_i) V_{ij}^{-1} (y_j - \mu_j)$$

where  $V^{-1}$  is the inverse of the covariance matrix  $V$ .

### 10.2 Covariance Matrix and Linear Transformations

- The covariance matrix is defined as

$$V_{ij} = \text{Cov}(y_i, y_j) = \langle (y_i - \mu_i)(y_j - \mu_j) \rangle.$$

- Suppose we apply a linear transformation  $B$  to  $\vec{y}$ , where  $B$  is an  $n \times n$  matrix of eigenvectors that diagonalizes  $V$ .
- Define transformed variables:

$$\vec{z} = B\vec{y}, \quad V_z = BVB^T, \quad \mu_z = B\mu_y.$$

- Then:

$$\boxed{\chi^2(z) = (\vec{z} - \vec{\mu}_z)^T V_z^{-1} (\vec{z} - \vec{\mu}_z)}$$

### 10.3 Modeling Data with Parameters

- Suppose we have a model

$$y_i = f(x_i, \vec{\theta}),$$

where  $x_i$  are independent variables and  $\vec{\theta}$  are model parameters.

- Example:  $y_i = mx_i + b$ , with  $\sigma_i \neq \sigma_j$  (heteroscedastic errors).
- We want to find the best estimate of  $\vec{\theta}$ .

### 10.4 Least Squares Estimation

- Define:

$$\chi^2(\vec{\theta}) = \sum_{i=1}^n \frac{(y_i - f(x_i, \vec{\theta}))^2}{\sigma_i^2}.$$

- The best estimators for  $\mu_i$  are  $f(x_i|\vec{\theta})$  and

$$\chi^2(\hat{\theta}) = \chi_{\min}^2 = \sum_i \left( \frac{y_i - f(x_i|\hat{\theta})}{\sigma_i} \right)^2.$$

- Principle of least squares:

$$\hat{\theta} = \arg \min \chi^2(\vec{\theta})$$

i.e. the value of  $\vec{\theta}$  that minimizes  $\chi^2(\vec{\theta})$ .

- Solution satisfies:

$$\frac{\partial \chi^2}{\partial \theta_j} = 0.$$

## 10.5 Distribution of Parameter Estimates

- Note:  $\hat{\theta}$  is itself a random variable, with its own probability distribution.
- A different sample  $\{x_i, y_i\}$  will lead to a different  $\hat{\theta}$ .

## 10.6 Quadratic Expansion of chi-squared and Error Estimates

- For polynomial (linear in parameters) fits:

$$f(x_i|\vec{\theta}) = (A\vec{\theta})_i.$$

- Then:

$$\begin{aligned}\chi^2 &= (\vec{y} - A\vec{\theta})^T V^{-1} (\vec{y} - A\vec{\theta}) \\ \chi^2 &= (y_i - A_{im}\theta_m) V_{ij}^{-1} (y_j - A_{jn}\theta_n).\end{aligned}$$

- First derivative:

$$\frac{\partial \chi^2}{\partial \theta_k} = -2(y_i - A_{im}\theta_m) V_{ij}^{-1} A_{jk}.$$

- Second derivative:

$$\frac{\partial^2 \chi^2}{\partial \theta_k \partial \theta_l} = 2A_{il} V_{ij}^{-1} A_{jk} = 2(A^T V^{-1} A)_{kl}.$$

## 10.7 Covariance of Parameter Estimates

- For linear fits:

$$V(\hat{\theta}) = (A^T V^{-1} A)^{-1}.$$

- For non-linear fits:

$$V^{-1}(\hat{\theta}) = \frac{1}{2} \frac{\partial^2 \chi^2}{\partial \theta_j \partial \theta_k} \Big|_{\theta=\hat{\theta}}.$$

# 11 Thursday, October 16th 2025

I was gone this day, so I copied notes from J. Liang.

## 11.1 Chi-Squared Minimization and Degrees of Freedom

- Degrees of Freedom (dof):

$$\chi^2(\theta) = (\vec{y} - A\vec{\theta})^T V^{-1} (\vec{y} - A\vec{\theta})$$

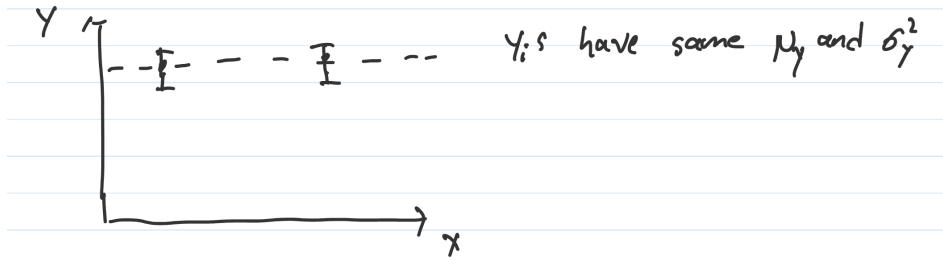
- At  $\hat{\theta}$ ,  $\chi^2$  is minimized:

$$\frac{\partial \chi^2}{\partial \theta} \Big|_{\theta=\hat{\theta}} = 0 = F(\vec{y}|\vec{\theta}, \vec{x}) = \begin{matrix} F(\vec{y}|\theta_1, \vec{x}) \\ F(\vec{y}|\theta_2, \vec{x}) \\ \vdots \\ F(\vec{y}|\theta_k, \vec{x}) \end{matrix}$$

- i.e. for a linear fit  $\vec{y} = A\vec{\theta}$  with  $k$  equations:

$$\hat{\theta} = (A^T V^{-1} A)^{-1} (A^T V^{-1}) \vec{y}$$

- Think about it like this: If I know  $n - k$  of the  $y_i$ 's, the remaining  $k$   $y_i$ 's are fixed. Their relations might be complex but they are fixed since we have  $k$  equations.



$y_i$ 's have some  $N_y$  and  $\sigma_y^2$ .

## 11.2 Two-Measurement Example and Correlated Variables

- Best estimator of the true  $N$  is  $\hat{y} = \frac{y_1 + y_2}{2}$ .

$$\chi^2 = \left( \frac{y_1 - \hat{N}}{\sigma_y} \right)^2 + \left( \frac{y_2 - \hat{N}}{\sigma_y} \right)^2$$

- Note:

$$y_1 - \hat{N} = \frac{1}{2}(y_1 - y_2)$$

$$y_2 - \hat{N} = \frac{1}{2}(y_2 - y_1)$$

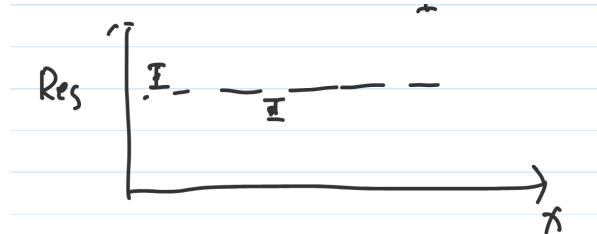
- Where  $z_1 = y_1 - y_2$ , and  $z_2 = y_2 - y_1$  such that  $z_2 = -z_1$ .

$$V(z_1, z_2) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

- Determinant of  $V$  is 0, so it is not invertible.
- Thus we effectively have only one independent variable.

### 11.3 Residuals and Goodness of Fit

- Residual:  $r_i = y_i - f_i(\vec{\theta})$ .
- When talking about goodness of fit, people usually divide  $\chi^2$  by the number of degrees of freedom (dof). If  $\chi^2/\text{dof} \approx 1$ , it is a good fit.
- What about residuals?

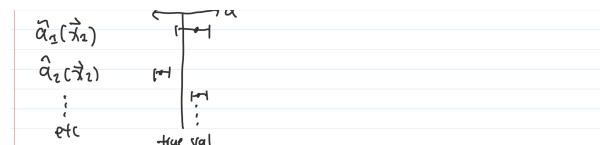


Sometimes one weird data point can throw off the whole  $\chi^2$  in unexpected ways, so it is important to check the residuals as well.

- If residuals are randomly scattered around 0, it indicates a good fit.

### 11.4 Distribution of Estimators

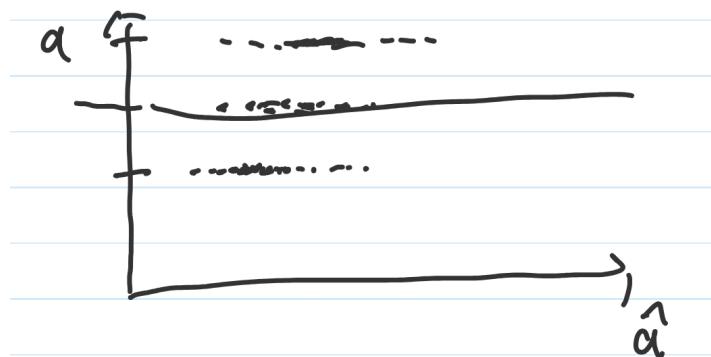
- We have been talking about estimators, but we want values of the parameters.
- Suppose we perform a fit on a parameter with true value  $a$ .
- From fitting, we get different estimator values from different data sets.



- These estimates can be scattered over a range of values.

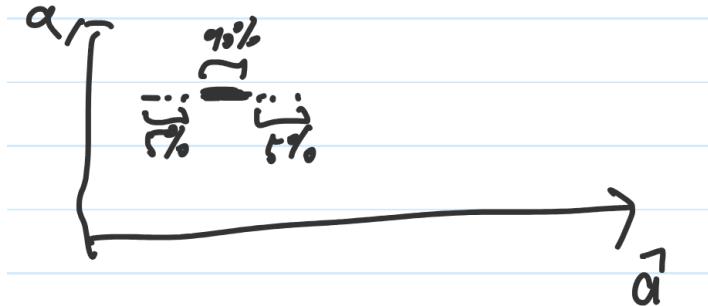
### 11.5 Toy Monte Carlo Simulations for Estimator Distributions

- Toy Monte Carlo:

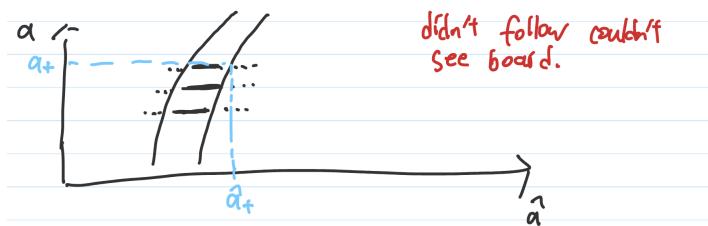


- Often, the model is too complicated to get an analytic form of the estimator distribution.

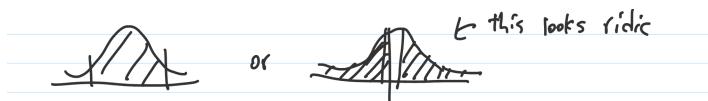
1. Pick a parameter  $a$ .
  2. Generate many data sets according to the model with parameter  $a$ .
  3. For each data set, compute the estimator  $\hat{a}$ .
  4. Plot a histogram of  $\hat{a}$ .
  5. Repeat steps 2–4  $n$  times.
  6. Repeat step 1  $m$  times.
- For each  $a$  value, we can determine e.g. 5%, 90%, and 5% quantiles.



- We then connect these points.

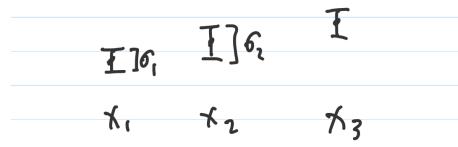


- $P(a_- \leq a \leq a_+) = 0.90$ . Coverage = fraction of time your prescription for the estimator interval contains the true value  $a$ .
- In other words, we find or choose  $\hat{a}_-$  and  $\hat{a}_+$  such that 90% of the time, the true  $a$  lies in our interval (corresponding to  $a_-$  and  $a_+$ ).
- The bias in all this is that we can really pick intervals however we want.



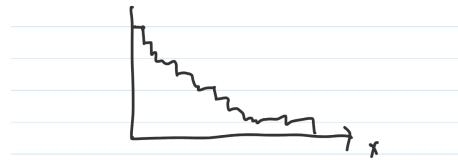
## 11.6 Typical Applications of Least Squares Fitting

- Typical usage of Least Squares (LS):
  1.  $x_i, y_i$ , with known  $\sigma_x$ , and known  $\sigma_y$  for  $y_i$ .

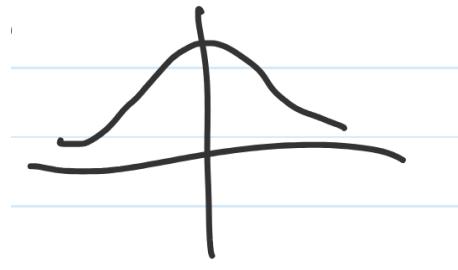


If  $y_i$ 's are measured by a detector, we can determine  $\sigma_i$  by analyzing the detector.

2. Histograms: i.e., measure  $x$  on a coordinate, which can have resolution effects, etc., where  $y =$  statistics of the number of events at the same  $x$ .



Say the resolution has:



so what you see is actually a convolution of the true distribution with the resolution function.



So the fit should actually be an exponential  $\otimes$  Gaussian in this case.

$$\chi^2 = \sum_i \frac{(n_i - f_i(\theta))^2}{\sigma_i^2}$$

Suppose we expect Poisson statistics:

$$\sigma_i^2 = n_i$$

Then the data are weighted by  $n_i$ :

$$\text{Neyman } \chi^2 \equiv \sum_i \frac{(n_i - f_i(\theta))^2}{n_i}$$

which is a modified least squares form.

Alternatively, it is sensible to say that the expected entries are given by our model:

$\Rightarrow$  use  $f_i(\vec{\theta})$  as the mean.

Therefore:

$$\text{Pearson } \chi^2 \equiv \sum_i \frac{(n_i - f_i(\theta))^2}{f_i(\theta)}$$

For Neyman  $\chi^2$ , if a bin is empty ( $n_i = 0$ ) then it diverges. Both Neyman and Pearson forms are biased in opposite ways. One could use Neyman and Pearson together, but is it worth the effort?

## 12 Tuesday, October 21st 2025

### 12.1 Least-Squares Fits

- Usually we have data  $(x_i, y_i \pm \sigma_i)$ . We want to fit a model  $y = f(x; \theta)$  to the data.

$$\chi^2 = \sum_i \frac{(y_i - f(x_i; \theta))^2}{\sigma_i^2}$$

- Sometimes we are given  $y = f(x)$  and sometimes we are given  $x = g(y)$ .

$$\chi^2 = \sum_i \frac{(x_i - g(y_i; \theta))^2}{\sigma_{x,i}^2}$$

$$\sigma_x = \left| \frac{dg}{dy} \right| \sigma_y$$

- Now that is the end of least-squared fits for a while.

### 12.2 Unbinned Data and Likelihood Functions

- Now we move on to unbinned data.
- Idea is that you have some data drawn from some probability distribution  $P(x; a)$

$$P(t) \sim \frac{1}{\tau} e^{-t/\tau}$$

- Data sample of size  $n$ :  $\{x_1, x_2, \dots, x_n\}$
- Form likelihood function:

$$\mathcal{L}(x_1, x_2, \dots, x_n; a) = \prod_{i=1}^n P(x_i; a)$$

- This is equivalent to the probability of getting the data given the parameter  $a$ :  $P(\vec{x}|a)$ .
- Not a probability distribution in  $a$ ! It is a function of  $a$ .
- Suppose we have an estimator  $\hat{a}$  for  $a$ . Then the expectation value of the estimator is:

$$E[\hat{a}] = \int \hat{a}(\vec{x}) P(\vec{x}; a) d\vec{x}$$

- The maximum likelihood principle states that the best estimate for  $a$  is the value  $\hat{a}$  that maximizes  $\mathcal{L}(\vec{x}; a)$ :

$$\left. \frac{\partial \mathcal{L}(\vec{x}; a)}{\partial a} \right|_{a=\hat{a}} = 0$$

- Often easier to maximize  $\ln \mathcal{L}$  since  $\ln$  is monotonic:

$$\ln \mathcal{L}(\vec{x}; a) = \ln \prod_{i=1}^n P(x_i; a) = \sum_{i=1}^n \ln P(x_i; a)$$

- Then we will find:

1. max for  $\ln \mathcal{L}(\vec{x}; a)$
2. min for  $-\ln \mathcal{L}(\vec{x}; a)$

- ML estimators tend to be not unbiased, but consistent, often efficient.

$$\frac{\partial \ln \mathcal{L}(\vec{x}; a)}{\partial a} \Big|_{a=\hat{a}} = 0$$

### 12.3 Example: Exponential Distribution

- Example: Exponential distribution

$$P(t; \tau) = \frac{1}{\tau} e^{-t/\tau}$$

$$\mathcal{L}(t_1, t_2, \dots, t_n; \tau) = \prod_{i=1}^n \frac{1}{\tau} e^{-t_i/\tau} = \sum_{i=1}^n \ln \left( \frac{1}{\tau} e^{-t_i/\tau} \right) = -n \ln \tau - \frac{1}{\tau} \sum_{i=1}^n t_i$$

$$\frac{\partial \ln \mathcal{L}}{\partial \tau} = 0 = -\frac{n}{\tau} + \frac{1}{\tau^2} \sum_{i=1}^n t_i$$

$$\hat{\tau} = \frac{1}{n} \sum_{i=1}^n t_i = \bar{t}$$

This is unbiased.

- 

$$E(\hat{\tau}) = \frac{1}{n} E\left(\sum_{i=1}^n t_i\right) = \frac{1}{n} \sum_{i=1}^n E(t_i) = \frac{1}{n} n\tau = \tau$$

### 12.4 Example: Lifetime with Cutoff $T$

- (Not normalized  $P$ ):

$$P(t|\tau) = \begin{cases} \frac{1}{\tau} e^{-t/\tau} / e^{-T/\tau} & 0 \geq t \geq T \\ 0 & \text{otherwise} \end{cases}$$

- Now must normalize!  $\sum_{t=T}^{\infty} P(t|\tau) = 1$

$$P(t|\tau) = \frac{1}{(1 - e^{-T/\tau})\tau} e^{-t/\tau}$$

- Log-likelihood:

$$\begin{aligned} \ln \mathcal{L} &= \sum_{i=1}^n \left[ \ln \left( (1 - e^{-T/\tau})\tau \right) \right] - \frac{1}{\tau} \sum_{i=1}^n t_i \\ &= -n \ln \left( 1 - e^{-T/\tau} \right) - n \ln \tau - \frac{1}{\tau} \sum_{i=1}^n t_i \end{aligned}$$

- Set derivative to zero:

$$\frac{\partial \ln \mathcal{L}}{\partial \tau} = 0$$

$$\frac{\partial \ln \mathcal{L}}{\partial \tau} = \frac{-n(-e^{-T/\tau})(\frac{T}{\tau^2})}{(1 - e^{-T/\tau})} - \frac{n}{\tau} + \frac{1}{\tau^2} \sum_{i=1}^n t_i = 0$$

## 12.5 Example: Multiple Gaussian Measurements

- Multiple measurements of some quantity:

$$\begin{aligned}
 P(x|\mu, \sigma) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \\
 \ln \mathcal{L} &= -\sum_i \ln(\sqrt{2\pi}) - \sum_i \ln \sigma - \frac{1}{2} \sum_i \frac{(x_i - \mu)^2}{\sigma^2} \\
 \frac{\partial \ln \mathcal{L}}{\partial \mu} &= 0 = \frac{-1}{2}(-2) \sum_i \frac{(x_i - \mu)}{\sigma^2} \\
 &= \frac{1}{\sigma^2} \sum_i (x_i - \hat{\mu}) = 0 \\
 \sum_i x_i &= n\hat{\mu} \Rightarrow \hat{\mu} = \frac{1}{n} \sum_i x_i = \bar{x}
 \end{aligned}$$

- Now for  $\sigma$ :

$$\begin{aligned}
 \left. \frac{\partial \ln \mathcal{L}}{\partial \sigma} \right|_{\hat{\sigma}, \hat{\mu}} &= 0 = -\frac{n}{\sigma} - \frac{1}{2} \left( \frac{-2}{\sigma^3} \right) \sum_i (x_i - \mu)^2 \\
 -n\hat{\sigma}^2 + \sum_i (x_i - \hat{\mu})^2 &= 0 \\
 \hat{\sigma}^2 &= \frac{1}{n} \sum_i (x_i - \hat{\mu})^2 = \frac{1}{n} \sum_i (x_i - \bar{x})^2
 \end{aligned}$$

- This is a biased estimator for  $\sigma^2$ . Unbiased is with  $1/(n - 1)$ .

## 12.6 Properties of the Maximum-Likelihood Estimator

- This next stuff is not really testable but can be interesting to see where it comes from.
- To avoid confusion call  $a_0$  the true value of  $a$ .
- We have  $\mathcal{L}(\vec{x}; a)$  and we want to know how well  $\hat{a}$  estimates  $a_0$ .

$$\left. \frac{\partial \ln \mathcal{L}(\vec{x}; a)}{\partial a} \right|_{a=\hat{a}} = 0$$

- Taylor expand around  $a_0$ :

$$f(\hat{a}) = f(a_0) + f'(a_0)(\hat{a} - a_0) + \frac{1}{2}f''(a_0)(\hat{a} - a_0)^2 + \dots$$

- So we have:

$$\left. \frac{\partial \ln \mathcal{L}(\vec{x}; a)}{\partial a} \right|_{a=\hat{a}} = \left. \frac{\partial \ln \mathcal{L}(\vec{x}; a)}{\partial a} \right|_{a=a_0} + (\hat{a} - a_0) \left. \frac{\partial^2 \ln \mathcal{L}(\vec{x}; a)}{\partial a^2} \right|_{a=a_0} + \dots = 0$$

- For  $n \rightarrow \infty, \hat{a} \rightarrow a_0$  (consistent estimator), so we can neglect higher order terms.

$$\left. \frac{\partial \ln \mathcal{L}(\vec{x}; a)}{\partial a} \right|_{a=a_0} \rightarrow \left. \frac{\partial \ln \mathcal{L}(\vec{x}; a)}{\partial a} \right|_{a=\hat{a}} = 0$$

# 13 Thursday, October 23rd 2025

## 13.1 Definition of the Likelihood Function

- Likelihood

$$\mathcal{L}(\vec{x}|\theta) = \text{probability of } x \text{ and not theta}$$

- Model  $\vec{\theta}$  with  $p(\vec{x}|\vec{\theta})$

$$\mathcal{L} = \prod_{i=1}^N p(x_i|\vec{\theta})$$

$$\ln \mathcal{L} = \sum_{i=1}^N \ln p(x_i|\vec{\theta})$$

## 13.2 Maximum Likelihood Estimation (MLE)

- Maximize likelihood to get best estimate of  $\vec{\theta}$ . Choose  $\hat{\theta}$  such that:

$$\hat{\theta} = \operatorname{argmax}_{\vec{\theta}} \mathcal{L}(\vec{x}|\vec{\theta})$$

$$\frac{\partial \ln \mathcal{L}(\vec{x})}{\partial \theta} \Big|_{\hat{\theta}} = 0$$

## 13.3 Quadratic Approximation of the Log-Likelihood

- Shape of  $\vec{a}$  distribution:  $\ln \mathcal{L}(a)$  around  $\hat{a}$  is approximately quadratic.
- True  $a = \hat{a}$ ; expand about  $a_0$ :
- Taylor expansion:

$$f(a) = f(a_0) + (\hat{a} - a_0)f'(a_*) \quad \text{where } a_* \text{ is between } \hat{a} \text{ and } a_0$$

- So, for  $f = \frac{\partial \ln \mathcal{L}(a)}{\partial a}$ :

$$0 = \frac{\partial \ln \mathcal{L}(a)}{\partial a} \Big|_{a_0} + (\hat{a} - a_0) \frac{\partial^2 \ln \mathcal{L}(a)}{\partial a^2} \Big|_{a_*}$$

## 13.4 Asymptotic Limit and Expectation Relation

- Large  $n$  for consistent  $\hat{a} \rightarrow a_0$ :

$$\lim_{n \rightarrow \infty} \frac{\partial^2 \ln \mathcal{L}(a)}{\partial a^2} \Big|_{a_*} = \lim_{n \rightarrow \infty} \sum_i \frac{\partial^2 \ln p(x_i|a)}{\partial a^2} \Big|_{a_*} \approx \lim_n n \int p(x|a) \frac{\partial^2 \ln p(x|a)}{\partial a^2} \Big|_{a_*} dx$$

- Sum over samples  $x_i$  drawn from  $p(x|a)$ :

$$\begin{aligned} &= \lim_{n \rightarrow \infty} n E \left( \frac{\partial^2 \ln p(x|a)}{\partial a^2} \Big|_{a_*} \right) \\ &= E \left( \frac{\partial^2 \ln \mathcal{L}}{\partial a^2} \Big|_{a_*} \right) \end{aligned}$$

$$\hat{a} - a_0 = - \frac{\frac{\partial \ln \mathcal{L}(a)}{\partial a} \Big|_{a_0}}{E \left( \frac{\partial^2 \ln \mathcal{L}}{\partial a^2} \Big|_{a_*} \right)}$$

$$0 = \frac{\partial \ln \mathcal{L}(a)}{\partial a} \Big|_{a_0} + (\hat{a} - a_0) \frac{\partial^2 \ln \mathcal{L}(a)}{\partial a^2} \Big|_{a_*}$$

### 13.5 Normalization of the Likelihood Function

- The likelihood is normalized:

$$\int \mathcal{L}(\vec{x}|a) d\vec{x} = 1$$

$$\Rightarrow \int \frac{\partial \mathcal{L}(\vec{x}|a)}{\partial a} d\vec{x} = 0$$

- Relation between  $\mathcal{L}$  and  $\ln \mathcal{L}$ :

$$\frac{\partial \mathcal{L}}{\partial a} = \frac{\partial \ln \mathcal{L}}{\partial a} \mathcal{L}$$

$$\frac{\partial \ln \mathcal{L}}{\partial a} = \sum_{i=1}^n \frac{\partial \ln p(x|a)}{\partial a}$$

### 13.6 Gaussian Approximation via the Central Limit Theorem

- The sum of  $n$  variables with zero mean:
- By the Central Limit Theorem, for large  $n$ ,  $\frac{\partial \ln \mathcal{L}}{\partial a}$  is Gaussian with mean 0.

$$E \left( \frac{\partial^2 \ln \mathcal{L}}{\partial a^2} \right) = -E \left( \left( \frac{\partial \ln \mathcal{L}}{\partial a} \right)^2 \right)$$

### 13.7 Variance of the Estimator and the Fisher Information

- Variance of  $\hat{a} - a_0$ :

$$\text{Var}(\hat{a} - a_0) = \frac{\text{Var} \left( \frac{\partial \ln \mathcal{L}(a)}{\partial a} \Big|_{a_0} \right)}{\left( E \left( \frac{\partial^2 \ln \mathcal{L}}{\partial a^2} \Big|_{a_*} \right) \right)^2}$$

$$= \frac{E \left( \left( \frac{\partial \ln \mathcal{L}(a)}{\partial a} \Big|_{a_0} \right)^2 \right)}{\left( E \left( \frac{\partial^2 \ln \mathcal{L}}{\partial a^2} \Big|_{a_*} \right) \right)^2}$$

$$= -\frac{E \left( \frac{\partial^2 \ln \mathcal{L}}{\partial a^2} \Big|_{a_0} \right)}{\left( E \left( \frac{\partial^2 \ln \mathcal{L}}{\partial a^2} \Big|_{a_*} \right) \right)^2}$$

- When  $n \rightarrow \infty$ ,  $a_* \rightarrow a_0$  and  $\hat{a} \rightarrow a_0$ :

$$\boxed{\text{Var}(\hat{a}) = -\frac{1}{E \left( \frac{\partial^2 \ln \mathcal{L}}{\partial a^2} \Big|_{a_0} \right)}}$$

- Fisher Information matrix:

$$E \left( \frac{\partial^2 \ln \mathcal{L}}{\partial a^2} \right)$$

- For large  $n$ ,  $\hat{a} \rightarrow a_0$ . Estimate  $E(\cdot)$  by the observed value:

$$\frac{\partial^2 \ln \mathcal{L}}{\partial a^2} \Big|_{\hat{a}}$$

- So the estimate of variance of  $\hat{a} - a_0$  is:

$$\boxed{\text{Var}(\hat{a} - a_0) = -\frac{1}{\frac{\partial^2 \ln \mathcal{L}}{\partial a^2} \Big|_{\hat{a}}}}$$

### 13.8 Taylor Expansion Near the Maximum Likelihood Estimate

- Taylor expansion again:

$$\frac{\partial \ln \mathcal{L}(a)}{\partial a} = \cancel{\frac{\partial \ln \mathcal{L}(a)}{\partial a} \Big|_{\hat{a}}}^0 + (a - \hat{a}) \frac{\partial^2 \ln \mathcal{L}(a)}{\partial a^2} \Big|_{\hat{a}} + \dots$$

where  $\frac{\partial^2 \ln \mathcal{L}(a)}{\partial a^2} \Big|_{\hat{a}} = -\frac{1}{V(\hat{a})}$

$$\frac{-(a - \hat{a})}{V(\hat{a})} + \dots$$

- Note that  $V(\hat{a} - a_0) = V(\hat{a})$  because  $a_0$  is constant and does not change the variance (it just shifts the distribution).

- So,

$$\begin{aligned} \ln \mathcal{L}(a) &= \ln \mathcal{L}(\hat{a}) - \frac{1}{2V(\hat{a})} (a - \hat{a})^2 \\ \mathcal{L} &= \mathcal{L}(\hat{a}) \exp \left( -\frac{1}{2} \left( \frac{a - \hat{a}}{\sigma_{\hat{a}}} \right)^2 \right) \end{aligned}$$

### 13.9 Goodness of Fit and the Kolmogorov–Smirnov Test

- Note that the value you get from the maximum likelihood does not give information on how good the fit is—it is just relative to other values of the parameters.
- Kolmogorov–Smirnov test for goodness of fit (KS):
  1. Order data points  $\{t_i\}$  such that  $t_0 \leq t_1 \leq t_2 \leq \dots \leq t_N$
  2. Form an accumulator  $F$  (same model CDF  $C$ ).
- Metric:

$$\max |F(t_i) - C(t_i)|$$

## 14 Tuesday, October 28th 2025

### 14.1 Taylor Expansion of the Log-Likelihood Around the Maximum

- Continuing from last time:

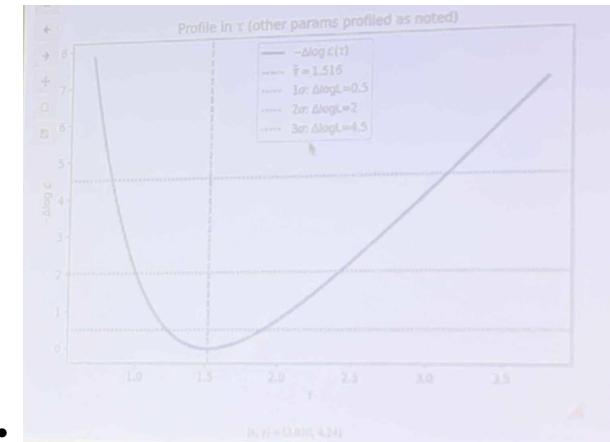
$$\begin{aligned}\ln \mathcal{L}(\vec{\theta}) &= \ln \mathcal{L}(\hat{\theta}) + \sum_i (\theta_i - \hat{\theta}_i) \frac{\partial \ln \mathcal{L}}{\partial \theta_i} \Big|_{\hat{\theta}} + \frac{1}{2} \sum_{i,j} (\theta_i - \hat{\theta}_i)(\theta_j - \hat{\theta}_j) \frac{\partial^2 \ln \mathcal{L}}{\partial \theta_i \partial \theta_j} \Big|_{\hat{\theta}} + \dots \\ &= \ln \mathcal{L}(\hat{\theta}) + (\vec{\theta} - \hat{\vec{\theta}})^T \nabla \ln \mathcal{L} \Big|_{\hat{\theta}} + \frac{1}{2} (\vec{\theta} - \hat{\vec{\theta}})^T H (\vec{\theta} - \hat{\vec{\theta}}) + \dots\end{aligned}$$

- This is the Hessian matrix  $H$ :

$$V^{-1} = -H$$

### 14.2 Parameter of Interest and Nuisance Parameters

- Often interested in only one parameter  $\tau$ . Pick  $\tau$ : what is  $\max \mathcal{L}$  if that were true.
- $\tau$  is what we want.  $\sigma$  is a nuisance parameter.



### 14.3 Likelihood versus Chi-Squared Interpretation

- Maximum Likelihood is only a function of the shape of your data and whether it matches your model.
- $\chi^2$  is a function of both the height and the shape of your data.

### 14.4 Normalization and Extended Likelihood

- We can incorporate the number of data events:

$$\int P(x|a) dx = 1 \quad \forall a$$

- Define  $Q$  such that:

$$\int Q(x|a)dx = L(a)$$

- Example: number of muons from cosmic rays passing through counters.

- Extend likelihood:

$$\mathcal{L} = \frac{e^{-\nu}\nu^n}{n!}$$

- This is called the extended likelihood and contains the Poisson term.

$$\ln \mathcal{L} = \sum_i \ln(p(x_i|a)) - \nu - n \ln \nu + \ln n!^0$$

## 14.5 Nuisance Parameters and Systematic Effects

- Nuisance parameters often represent systematic effects or resolution uncertainties:

$$\mathcal{L}(\theta) = \mathcal{L}_0 \frac{1}{\sqrt{2\pi\sigma_\theta}} \exp\left(-\frac{(\theta - \mu)^2}{2\sigma_\theta^2}\right)$$

- $\theta \sim \mathcal{N}(\mu, \sigma_\theta^2)$  is a nuisance parameter.

## 14.6 Gaussian Data Likelihood and Chi-Squared Minimization

- Suppose we have data points of the form  $\{x_i, y_i \pm \sigma_i\}$  and  $y_i$  are Gaussian.
- We think we know  $y(x) = f(x|a)$ .

$$p(y_i|a) = \frac{1}{\sqrt{2\pi\sigma_i}} \exp\left(-\frac{(y_i - f(x_i|a))^2}{2\sigma_i^2}\right)$$

$$\mathcal{L} = \prod_i p(y_i|a)$$

$$\ln \mathcal{L} = -\frac{1}{2} \sum_i \left( \frac{y_i - f(x_i|a)}{\sigma_i} \right)^2 + \text{const}$$

- Maximizing  $\ln \mathcal{L}$ :

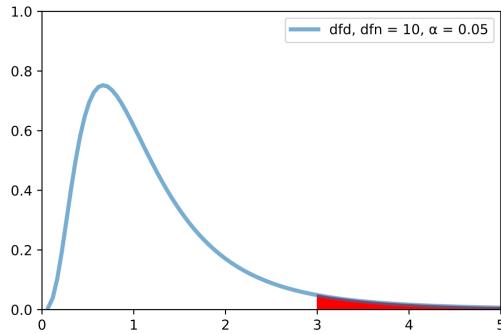
- Minimizing  $\chi^2$ :

$$\chi^2 = \sum_i \left( \frac{y_i - f(x_i|a)}{\sigma_i} \right)^2$$

- i.e. Maximizing likelihood is equivalent to minimizing  $\chi^2$  for Gaussian data.

## 14.7 Introduction to Hypothesis Testing

- Next: Hypothesis Testing.
- A simple hypothesis is one for which the test statistic has a completely specified pdf.
- Example: data come from Poisson process with mean  $\nu = 5-6$ .
- A composite hypothesis is one for which the test statistic has a pdf that depends on unknown parameters.
- Here is a similar example diagram from the internet:



## 14.8 Rejection Regions and Type I Error

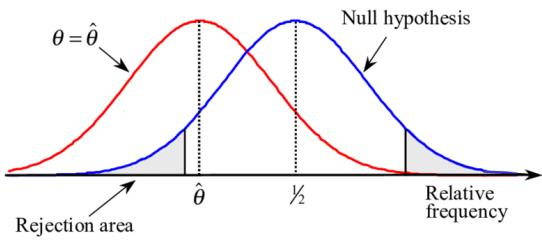
- Decide in advance a  $\sigma_i^2$  region of  $x$ . Reject  $H_0$  if  $x$  is in this region.

$$\int_{\text{region}} p(x|H_0)dx = \alpha = \text{significance of test}$$

- For  $x \geq x_0$ : “Reject”  $H_0$  when true occurs  $\alpha$  fraction of the time.
- Type I error: Reject  $H_0$  when true (probability  $\alpha$ ).
- Fisher.
- $H_0 = \text{Data is } \sim \mathcal{N}(\mu_0, \sigma_0^2)$ .
- Get pairs  $x_1, x_2$ .
- Your choice of possible regions is huge and can lead to very weird tests.

## 14.9 Neyman–Pearson Lemma and Type II Error

- Neyman–Pearson: fixed by adding second hypothesis  $H_A$ .



- $\beta$  = power of test.
- Only two choices:  $H_0$  or  $H_A$ .
- Type II error: Accept  $H_0$  when false (probability  $\beta$ ).

#### 14.10 Likelihood Ratio Test and Interpretation

- Law: Null hypothesis  $H_0$  is rejected in favor of alternative hypothesis  $H_A$  if

$$\Lambda(x) = \frac{\mathcal{L}(x|H_A)}{\mathcal{L}(x|H_0)} \geq k_\alpha$$

- In legal terms:  $H_0$  = innocent until proven guilty,  $H_A$  = guilty.
- In science:  $H_0$  = no new physics,  $H_A$  = new physics.

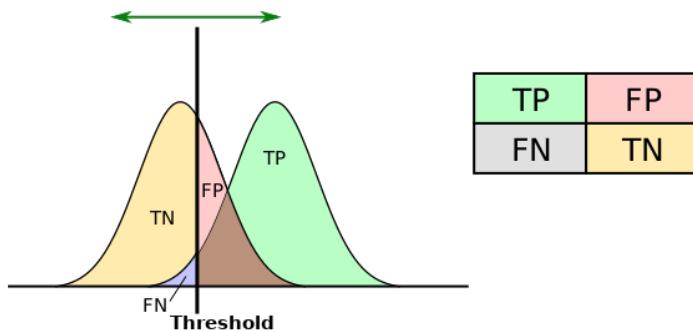
# 15 Thursday, October 30th 2025

## 15.1 Introduction to Hypothesis Testing

- Recall the Neyman-Pearson test that we ended last class talking about.
- Hypothesis testing: the test statistic is a function of data.
- Null hypothesis  $H_0$ , alternative hypothesis  $H_A$ .
- The main idea from Neyman and Pearson was to include a second hypothesis  $H_A$ .

	$H_0$ true	$H_A$ true
Accept $H_0$	Correct decision	Type II error
Reject $H_0$	Type I error	Correct decision

Table 1: Outcomes of hypothesis testing.



# Types of errors

		Truth	
		No diff	Diff
		$H_0$ to be not rejected	$H_0$ to be rejected ( $H_1$ )
<b>Decision based on the <math>p</math> value</b>	$H_0$ not rejected <b>No diff</b>	Right decision $1-\alpha$	$\beta$ <b>Type II error</b>
	$H_0$ rejected ( $H_1$ ) <b>Diff</b>	$\alpha$ <b>Type I error</b>	Right decision $1-\beta$

- $H_0$  is “true” but rejected: Type I or  $\alpha$  error
- $H_0$  is “false” but not rejected: Type II or  $\beta$  error

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- Null Hypothesis Significance Testing (NHST).

## 15.2 The Neyman-Pearson Lemma and Optimal Test Regions

- NP test: How to optimize the region  $R$  to minimize Type II error for a fixed Type I error rate  $\alpha$ .

$$1 - \beta = \int_R P_A(x) dx$$

- If we move a point  $x_1$  by  $\delta x_1$  from  $R$  to  $\tilde{R}$ , it must be replaced by another point  $x_2$  such that  $\delta_2 P_H(x_2) = \delta_1 P_H(x_1)$ . (Gain to loss condition.)

$$\int_R P_H = \alpha$$

- Then change in  $1 - \beta$  is  $\delta_2 P_A(x_2) - \delta_1 P_A(x_1)$  (gain to loss).
- Select  $R$  by picking  $x$  values that maximize  $P_A(x)/P_{H_0}(x)$  until  $\alpha$  is reached:

$$\frac{\delta_2 P_A(x_2)}{\delta_2 P_{H_0}(x_2)} \geq \frac{\delta_1 P_A(x_1)}{\delta_1 P_{H_0}(x_1)}$$

- Decision rule:

$$\frac{P_A(x)}{P_{H_0}(x)} \geq c_\alpha$$

- For some fixed  $\alpha$ , set the region by

$$\frac{P_A(x)}{P_{H_0}(x)} = c_\alpha$$

- i.e.

$$\frac{L(\vec{x}|H_A)}{L(\vec{x}|H_0)} \geq c_\alpha$$

### 15.3 Gaussian Example: Testing Mean Values

- Example: data Gaussian with  $\sigma = 1$ .

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}}$$

- Our hypotheses:

$$\begin{aligned} H_0 : \mu &= \mu_0 \\ H_A : \mu &= \mu_1 \end{aligned}$$

- $n$  measurements:

$$\mathcal{L}(\vec{x}|H_0) = \prod_{i=1}^n f(x_i) = \left( \frac{1}{\sqrt{2\pi}} \right)^n e^{-\sum_{i=1}^n \frac{(x_i - \mu)^2}{2}}$$

- Sample mean:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

- Variance of sample mean:

$$V(\bar{x}) = \frac{\sigma^2}{n} = \frac{1}{n}$$

- Sample variance:

$$S^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

- Expansion:

$$\sum (x_i - \mu_0)^2 = n \left( (\bar{x} - \mu_0)^2 + S^2 \right)$$

- Likelihood ratio:

$$\frac{\mathcal{L}(\vec{x}|H_1)}{\mathcal{L}(\vec{x}|H_0)} = e^{-\frac{n}{2}((\bar{x}-\mu_1)^2 - (\bar{x}-\mu_0)^2)} \geq c_\alpha$$

$$\frac{n}{2} (2\bar{x}(\mu_1 - \mu_0) + \mu_1^2 - \mu_0^2) \geq \ln(c_\alpha)$$

$$(\mu_1 - \mu_0)\bar{x} \geq \frac{2}{n} \ln(c_\alpha) + \frac{1}{2}(\mu_1^2 - \mu_0^2)$$

- Take the case  $\mu_1 > \mu_0$ :

$$\bar{x} \geq \frac{1}{\mu_1 - \mu_0} \left( \frac{2}{n} \ln(c_\alpha) + \frac{1}{2}(\mu_1^2 - \mu_0^2) \right)$$

- This is a Simple Statistic (and in our case it is a Sufficient Statistic as well).

- The distribution of  $\bar{x}$  is Gaussian about the true mean with width  $\sigma/\sqrt{n} = \frac{1}{\sqrt{n}}$ .

- So:

$$\int_R \mathcal{L}(\vec{x}|H_0) d\vec{x} = \int_{\bar{x}_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{n}{2}(\bar{x}-\mu_0)^2} d\bar{x} = \alpha$$

## 15.4 Applied Example: Mineral Density Classification

- Example: Mining.
- Opal density =  $2.2 \text{ g/cm}^3$ .
- Quartz density =  $2.6 \text{ g/cm}^3$ .
- Sample standard deviation  $\sigma = 0.2 \text{ g/cm}^3$ .
- Gaussian hypotheses:
  - $H_0 \sim N(2.2, \sigma^2)$
  - $H_A \sim N(2.6, \sigma^2)$
- Likelihood ratio:

$$\frac{L(p|H_A)}{L(p|H_0)} = \frac{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(p-2.6)^2}{2\sigma^2}}}{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(p-2.2)^2}{2\sigma^2}}} = e^{-\frac{1}{2\sigma^2}((p-2.6)^2 - (p-2.2)^2)} \geq c_\alpha$$

- Cut off at  $p \leq 2.56$  (which corresponds to  $1.64\sigma$ ) so we keep 95% of opals.
- But needlessly investigate 36% of useless mines.

## 15.5 Common Misconceptions in Hypothesis Testing

- A common misconception: if  $P(H_0|x)$  is small, then  $P(x|H_0)$  must be small. This is **not true**.
- Example:  $P(\text{American}|\text{walked on moon}) = 1$ , but  $P(\text{walked on moon}|\text{American})$  is very small.

# 16 Tuesday, November 4th 2025

## 16.1 Likelihood in Hypothesis Testing and Nuisance Parameters

- Likelihood in Hypothesis Testing: Use  $\mathcal{L}(\vec{x}|\vec{\theta})$  to select between  $H_0$  (null) and  $H_1$  (alternative). Parameters include both *parameters of interest* and *nuisance parameters*.
- Nuisance Parameters with Constraints: Nuisance parameters often have external measurements that provide probabilistic constraints, e.g.,  $\theta_{\text{nuisance}} \sim \mathcal{N}(\theta_0, \sigma^2)$ . These are incorporated as additional terms in the likelihood:

$$\mathcal{L}_{\text{total}} = \mathcal{L}(\vec{x}|\vec{\theta}) \cdot p(\theta_{\text{nuisance}})$$

## 16.2 Bayes' Theorem and Its Application to Hypothesis Testing

- Bayes' Theorem: Relates prior knowledge to observations via:

$$P(H|D) = \frac{P(D|H) P(H)}{P(D)}$$

where  $H$  = hypothesis,  $D$  = data. This allows us to go from  $\mathcal{L}(\vec{x}|H)$  to  $P(H|\vec{x})$ .

- Bayes Theorem approach to hypothesis testing:

$$P(H_0|x) = \frac{P(x|H_0)P(H_0)}{P(x)}$$

- Example: measure  $c$  = speed of light. In no sense is  $c$  a stochastic variable.
- We have  $\mathcal{L}(\vec{x}|H)$  and we want  $P(H|\vec{x})$ .
- Compare to exponential that when Fourier transformed gives Cauchy distribution.
- Bayes  $\Rightarrow$  *posterior* (blue), *likelihood* (red), *prior* (purple), *evidence* (orange).

$$P(H|D) = \frac{P(D|H)P(H)}{P(D)}$$

$$P(H|D) = \frac{\mathcal{L}(D|H)\text{prior}}{\mathcal{L}(D)}$$

$$p(\vec{\theta}|\vec{x}) = \frac{\mathcal{L}(\vec{x}|\vec{\theta})\pi(\vec{\theta})}{\mathcal{L}(\vec{x})}$$

$$\int \mathcal{L}(\vec{x}|\vec{\theta}) = 1$$

- Think conditional probability.

$$p(\vec{x}) = p(\vec{x}|H_0)P(H_0) + p(\vec{x}|H_1)P(H_1)$$

- As long as  $H_0$  and  $H_1$  span all possibilities.

$$p(\vec{x}) = \int p(\vec{x}|\vec{\theta})p(\vec{\theta})d\vec{\theta}$$

- Example: DNA evidence.  $H_0$  = innocent,  $H_1$  = guilty,  $D$  = we have DNA evidence at crime.

$$P(D) = P(D|H_0)P(H_0) + P(D|H_1)P(H_1)$$

### 16.3 Maximum A Posteriori (MAP) Estimation

- Given data of  $x$  what value of theta is the most probable: i.e. maximum a posteriori (MAP) estimate (similar to MLE).
- Posterior =  $p(\vec{\theta}|\vec{x})$
- Recall:  $p(\vec{x}|\vec{\theta}) = \mathcal{L}(\vec{x}|\vec{\theta})$
- Now we have a way to add in prior information about  $\vec{\theta}$

$$p(\vec{\theta}|\vec{x}) = \frac{p(\vec{x}|\vec{\theta})p(\vec{\theta})}{p(\vec{x})}$$

$$\ln(p(\vec{\theta}|\vec{x})) = \ln(p(\vec{x}|\vec{\theta})) + \ln(p(\vec{\theta})) - \ln(p(\vec{x}))$$

$$\frac{\partial}{\partial \vec{\theta}} \ln(p(\vec{\theta}|\vec{x})) = \frac{\partial}{\partial \vec{\theta}} \ln(p(\vec{x}|\vec{\theta})) + \frac{\partial}{\partial \vec{\theta}} \ln(p(\vec{\theta})) = 0$$

- $\hat{\theta}_{MAP}$ : Maximum A Posteriori estimate.
- If prior is flat then MAP = MLE.
- BUT my solution for  $\hat{\theta}_{MAP}$  will depend on my choice of prior! i.e.  $\neq$  your solution for  $\hat{\theta}_{MAP}$
- BUT as data get 'better, more' dependence  $\sim$  disappears.
- Example: 99% I am nice, 1% I am evil.
- We can see in demo that with enough data the prior influence goes away, but it takes different amounts of data to finally get to the same answer.

### 16.4 MAP Estimation for Normal Distribution with Gaussian Prior

- Example: estimate  $\mu$  of sample. Drawn from  $\mathcal{N}(\mu, \sigma^2)$  with  $\sigma^2$  known.
- Prior,  $\mu = \mu_0, \mathcal{N}(\mu_0, \sigma_0^2)$ .

$$\mathcal{L} = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right)$$

$$p(\vec{\theta}|\vec{x}) \sim \left( \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right) \right) \cdot \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left(-\frac{(\mu_0 - \mu)^2}{2\sigma_0^2}\right)$$

$$\log p(\mu|\vec{x}) = -\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 - \frac{1}{2\sigma_0^2} (\mu_0 - \mu)^2 + \underbrace{n \log \sqrt{2\pi}\sigma}_\text{const.} + \underbrace{\log \sqrt{2\pi}\sigma_0}_\text{const.}$$

$$\frac{d \log p(\mu|\vec{x})}{d\mu} \Big|_{\mu=0} = -\frac{1}{\sigma^2} (-2) \sum_{i=1}^n (x_i - \mu) - 2 \frac{1}{\sigma_0^2} (\hat{\mu} - \mu_0 + 0)$$

$$\frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu) - \frac{\hat{\mu} - \mu_0}{\sigma_0^2} = 0$$

$$\hat{\mu} \left( \frac{n^2}{\sigma^2} + \frac{1}{\sigma_0^2} \right) = \frac{1}{\sigma^2} \sum_{i=1}^n x_i + \frac{1}{\sigma_0^2} \mu_0$$

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

- Rearranging gives:

$$\hat{\mu}_{MAP} = \frac{\frac{1}{\sigma^2} \sum_{i=1}^n x_i + \frac{1}{\sigma_0^2} \mu_0}{\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}}$$

- This is a weighted average of the sample mean and the prior mean.
- As  $n \rightarrow \infty$ ,  $\hat{\mu}_{MAP} \rightarrow \hat{\mu}_{MLE}$ .

## 16.5 Conjugate Priors and the Beta Distribution

- Example:

$$P(\vec{\theta}|\vec{x}) = P(\vec{x}|\vec{\theta})P(\vec{\theta})$$

Coin flip:  $\theta$  = prob. of heads. Bernoulli  $\Rightarrow: x \in \{0, 1\}$

$$p(x) = \theta^x(1-\theta)^{1-x}$$

Conjugate priors.

- There used to be an old trick using Beta function.

$$p(\theta) = \text{Beta}(\theta|\alpha, \beta)$$

$$p(\theta) = \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha, \beta)}$$

- Now:

$$\log p(\vec{\theta}|\vec{x}) = \log \mathcal{L}(\vec{x}|\vec{\theta}) + \log p(\vec{\theta})$$

- $\hat{\theta}_{MLE}$  is just  $\bar{\theta}_{MAP}$  with uniform prior  $p(\vec{\theta})$  (i.e. Uninformed prior).

## 16.6 Choice of Prior: Practical Considerations and Pitfalls

- Example: Proposing Dark Matter (DM) experiment.
  - WIMP:  $\sim 10^{12}$  eV
  - Axion:  $\sim 10^{-22}$  eV
- Don't know so take flat prior, but then we end up with massive probability for WIMP and tiny for axion because axion is unbiased.

## 17 Thursday, November 6th 2025

- The following notes from this entire lecture are not testable.

### 17.1 Deductive and Inductive Reasoning; Weak Syllogism and Bayesian Motivation

- How do we reason in this realm of deductive vs inductive logic (following textbook from ET Jaynes, from math of Polya around 1450)?
- Aristotelian logic: deductive reasoning from true premises to true conclusions.

$$A \Rightarrow B$$

i.e. if  $A$  is true, then  $B$  is true.

- You can also say that  $B$  is false if  $A$  is false.
- But if  $B$  is true,  $A$  may be true or false.
- Simple Math Example:  $A : x > 7$ ,  $B : x > 5$ :
- If  $x > 7$  then  $x > 5$ . i.e. if  $A$  true then  $B$  true.
- But if  $A$  is false (i.e.  $x \leq 7$ ) then  $B$  may be true or false.
- or another example: A: I tried to do a “hard-flip” on my skateboard. B: I will get hurt. If not A then B is false.
- Inductive reasoning: from observations to general rules.
- Almost all of science (IRL) reasoning is inductive.
- “weak syllogism”: outcomes are not a certainty, only a likelihood: If  $B$  is true, then  $A$  becomes more plausible.
- This is the basis of Bayesian reasoning.
- Example: A: If it is raining, B: the ground is wet. Then if the ground is wet. Therefore it is more plausible that it is raining.
- Example: It will start to rain at 10am. B: it will get cloudy before 10am. If we see clouds at 9am, then it is more plausible that it will rain at 10am.
- Example: Police officer at night. A: Hears a burglar alarm. B: sees a jewelry store with broken windows. Sees person in mask climbing out window with a bag of jewelry.
- Science Example: A: If GR is true. B: then perihelion advance of Mercury. If we observe perihelion advance of Mercury, then GR becomes more plausible.
- Science Example: If simplistic SUSY breaking mechanism is true, B: particular type of dark matter.

## 17.2 Plausibility Notation and Consistency Requirements

- $A|B \equiv$  plausibility of A given B.
- $A|B \in \mathcal{R}$  we will insist on self-consistency of our operations.
- If  $A|C$  is “more plausible” than  $B|C$ , then we write:

$$A|C > B|C$$

- If conclusions can be reached multiple ways, these must give same plausibility.
- Equivalent states of knowledge must be represented by same plausibility.
- Also require Qualitative correspondence with common sense reasoning.
- $C + C'$  contains new information from C such that A becomes more plausible i.e.

$$A|C' > A|C$$

and plausibility of B given A is not changed:

$$B|AC' = B|AC$$

then plausibility of  $AB|C' \geq AB|C$ .

- Also if  $A|C' > A|C$  then  $\bar{A}|C' < \bar{A}|C$ .
- We then can do some heavy math and show that the only consistent way to assign plausibilities is to identify them with probabilities that follow the rules of probability theory.

## 17.3 Product Rule for Combined Plausibility

- We try to find a consistent rule for calculating plausibility of A B. We want  $AB|C$  (i.e. a product rule).
  - Decide  $B = \text{True}, B|C$
  - Given B true decide if A is true:  $A|BC$
- Decide  $A = \text{True}, A|C$ , given  $A = \text{True}$  we need to decide if  $B = T, B|AC$
- Want:  $AB|C = ?$
- With  $(AB|C)$  we might have functions:  $F(B|C, A|BC) = F(A|C, B|AC)$ .
- $A|C, B|C, A|BC, B|AC$
- By looking at edge cases:

$$B = \bar{A}$$

- If  $C \rightarrow C'$ , then  $B|C' > B|C$ . But plausibility for A given is unchanged:

$$A|BC' = A|BC$$

- Now calculate consistency for:  $ABC|D$

- Take triple product and expand into singles as products.

$$\begin{aligned}
ABC|D &= F(BC|D, A|BCD) \\
&= F(F(C|D, B|CD), A|BCD) \\
&= F(C|D, AB|CD) \\
&= F(C|D)F(B|CD, A|BCD)
\end{aligned}$$

- Note that for simplicity we can start using new parameters
- From this we get the functional equation:

$$F(F(x, y), z) = F(x, F(y, z))$$

- Associativity equation.

## 17.4 Solution of the Associativity Equation via Transform

- Can find most general solution:

$$u \equiv F(x, y), \quad v \equiv F(y, z)$$

with  $x, y, z$  independent.

$$F(x, v) = F(u, z)$$

- Take derivatives wrt  $x, y, z$ .

$$\frac{\partial F(x, v)}{\partial x} \equiv F(x, v) = \frac{\partial F}{\partial u} \frac{\partial u}{\partial x}$$

$$\bullet \quad F_1(x, y) = \frac{\partial F(x, y)}{\partial x}, \quad F_2(x, y) = \frac{\partial F(x, y)}{\partial y}$$

- Divide.

$$\frac{F_2(x, v)}{F_1(x, v)} F_1(y, z) = \frac{F_2(x, y)}{F_1(x, y)}$$

$$\bullet \quad \text{Set } G(x, y) = \frac{F_2(x, y)}{F_1(x, y)}$$

- Then:

1.  $u$ :

$$G(x, v) F_1(y, z) = G(x, y)$$

2.  $v$ :

$$G(x, v) F_2(y, z) = G(x, y) G(y, z)$$

This last term is not a function of  $y$ .

- So:

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial y}$$

$$G(x, v) F_{12}(y, z) = 0$$

$$\frac{\partial v}{\partial y} = G(x, v) F_{21}(y, z)$$

$G(x, y) G(y, z)$  independent of  $y$

- Most generic form of  $G(x, y) = H(x)/H(y)$

$$\bullet \quad w(x) = \exp \left( \int^x H(x') dx' \right)$$

$$\bullet \quad w(F(y, z)) = w(v) = w(y)w(z)$$

## 18 Thursday, November 13th 2025

- Continuation of the previous lecture on building up the idea of plausibility and converting it into a numerical measure.
- Again this is not testable material.

### 18.1 Plausibility as a Generalization of Logic

- Plausibility is a generalization of logic to situations where we do not have certainty.

$$(AB|C) \in \mathcal{R}$$

$$\begin{aligned} (AB|C) &= F[(B|C), (A|BC)] \\ &= F[(A|C), (B|AC)] \end{aligned}$$

### 18.2 Derivation of the Functional Form via Associativity

- Then  $(ABC|D)$ :

$$\begin{aligned} F[F(C|D, B|CD), A|BCD] &= F[C|D, F(B|CD, A|BCD)] \\ F[F(x, y), z] &= F[x, F(y, z)] \\ u = F(x, y), \quad v &= F(y, z) \\ F(u, z) &= F(x, v) \end{aligned}$$

- Take derivatives wrt x,y,z.

$$\begin{aligned} F_1(x, y) &= \frac{\partial F(x, y)}{\partial x}, \quad F_2(x, y) = \frac{\partial F(x, y)}{\partial y} \\ G(x, y) &= \frac{F_2(x, y)}{F_1(x, y)} \end{aligned}$$

- Then:

$$\begin{aligned} Q &= G(x, v)F_1(y, z) = G(x, y) \\ R &= G(x, v)F_2(y, z) = G(x, y)G(y, z) \end{aligned}$$

- Now use  $Q$  and  $R$ :

$$\begin{aligned} \frac{\partial Q}{\partial z} &= G_2(x, v)F_2(y, z)F_1(y, z) + G(x, v)F_{12}(y, z) = 0 \\ \frac{\partial R}{\partial y} &= G_2(x, v)F_1(y, z)F_2(y, z) + G(x, v)F_{21}(y, z) = 0 \end{aligned}$$

- This is used to show that  $G(x, y)G(y, z)$  is independent of y.

$$\frac{\partial Q}{\partial y} = 0 \Rightarrow G(x, v)F_{12}(y, z) = 0$$

$$\frac{\partial R}{\partial y} = 0 \Rightarrow G(x, y)G(y, z) \text{ is independent of y}$$

- General form is  $G(x, y) = r \cdot H(x)/H(y) \quad r \in \mathcal{R}$

- Pick  $y_0$  s.t.  $G(x, y_0) \neq 0, r = G(y_0, y_0)$ :

- Define:  $H(x) = G(x, y_0)$

$$G(x, y)G(y, z) = G(z, y_0)G(y_0, z)$$

$z = y_0$  gives:

$$G(x, y)G(y, y_0) = G(x, y_0)G(y_0, y_0)$$

$$G(x, y)H(y) = H(x)r$$

$$G(x, y) = r \frac{H(x)}{H(y)}$$

- using our first expression

$$G(x, v)F_1(y, z) = G(x, y)$$

- we have

$$r \frac{H(x)}{H(v)} F_1(y, z) = r \frac{H(x)}{H(y)}$$

$$F_1(y, z) = \frac{H(v)}{H(y)}$$

- then using our second expression

$$r \frac{H(x)}{H(v)} F_2(y, z) = r \frac{H(x)}{H(y)} \cdot r \frac{H(y)}{H(z)}$$

- we have

$$F_2(y, z) = \frac{rH(v)}{H(z)}$$

- Recall that  $v = F(y, z)$

$$dv = dF(y, z) = F_1(y, z)dy + F_2(y, z)dz$$

$$dv = \frac{H(v)}{H(y)}dy + \frac{rH(v)}{H(z)}dz$$

- Divide by  $H(v)$

$$\frac{1}{H(v)}dv = \frac{1}{H(y)}dy + \frac{r}{H(z)}dz$$

- $F$  was shown earlier to be monotonic, so the derivatives of the functions are non-zero. So  $G$  is positive, and  $H$  is  $G$ 's at some fixed point, so  $H$  is also non-zero.

- We can then integrate on both sides.

$$\int \frac{1}{H(v)}dv = \int \frac{1}{H(y)}dy + \int \frac{r}{H(z)}dz$$

- Let  $I(x) = \int \frac{1}{H(x')}dx'$

$$I(v) = I(y) + I(rz)$$

- We can exponentiate both sides.

$$e^{I(v)} = e^{I(y)} \cdot e^{I(rz)}$$

- Let  $w = e^I = \exp \int^x \frac{1}{H(x')} dx'$

$$w(F(y, z)) = w(y) \cdot w^r(z)$$

- Now recall,  $u = F(x, y)$ ,  $v = F(y, z)$ :

$$F(x, v) = F(u, z)$$

- So:

$$w(F(x, v)) = w(x)w^r(v) = w(F(u, z)) = w^r(z)w(u)$$

- So:

$$r^2 = r \Rightarrow r = 1$$

- So we have:

$$w(F(x, y)) = w(x) \cdot w(y)$$

$$F(x, y) = w^{-1}(w(x) \cdot w(y))$$

### 18.3 Product Rule for Plausibility

- Recall we wanted the product rule for plausibility.

$$w(AB|C) = w(A|C) \cdot w(B|C)$$

$$w(AB|C) = w(B|AC) \cdot w(A|C)$$

- Suppose A is certain if C is certain (i.e.  $C = \text{True}$ )

- Then:  $(AB|C) = (B|C)$

- Then:  $w(AB|C) = w(B|C)$

- Then:  $w(AB|C) = w(A|BC) \cdot w(B|C)$

- Then:  $w(AB|C) = w(A|C) \cdot w(B|C)$

- So:

$$w(A|C) = 1 \quad \text{if A is certain if C is given as True}$$

- If A is impossible given C is true:

$$(A|C) = \text{False}$$

$$(AB|C) = \text{False}$$

- Then:

$$w(AB|C) = w(A|BC) \cdot w(B|C)$$

$$w(A|C) = w(A|C) \cdot w(B|C)$$

- So case  $w(A|C) = 0$  (or  $w = \infty$ ), take  $w' = 1/w$ .

- $\exists w$  satisfies product rule.  $0 \leq w(x) \leq 1, \quad \forall$  plausibilities.

## 18.4 Sum Rule for Plausibility

- Sum rule, any proposition A:

$$A\bar{A} = F$$

$$A + \bar{A} = \text{True}$$

- Set  $u = w(A|B)$ ,  $v = w(\bar{A}|B)$

- Must exist  $S$ , s.t.

$$v = S(u)$$

- with constraints:

$$S(0) = 1, \quad S(1) = 0$$

- Can swap A and  $\bar{A}$ :

$$u \text{ must } = S(v)$$

- So:

$$u = S(S(u))$$

$$S^{-1}(u) = S(u)$$

- Skipping some steps, after lots of algebra we get:

$$S(x) = (1 - x^m)^{1/m}$$

- for  $0 \leq x \leq 1$

- for  $0 < m < \infty$

- So we have:

$$w(\bar{A}|B) = (1 - (w(A|B))^m)^{1/m}$$

- This is the sum rule for plausibilities.

$$w^m(\bar{A}|B) = 1 - w^m(A|B)$$

$$\boxed{w^m(A|B) + w^m(\bar{A}|B) = 1}$$

For some m.

$$w^m(AB|C) = w^m(A|BC) \cdot w^m(B|C)$$

- Then  $w' = w^m$  so take  $m = 1$  for simplicity.

## 18.5 Identification of Plausibility with Probability

- Finally  $\exists$  function  $w^m(x) \equiv p(x)$  s.t. :

$$\begin{aligned} p(AB|C) &= p(A|BC) \cdot p(B|C) \\ &= p(B|AC) \cdot p(A|C) \\ p(A|C) + p(\bar{A}|C) &= 1 \end{aligned}$$

where  $0 \leq p(x) \leq 1$  in monotonic in plausibility.

- Any logical function can be expressed in terms of AND, NOT.

## 18.6 Application to Deductive Logic

- Deductive logic:

$$A \Rightarrow B$$

- $A$  is True, so  $B$  is True.
- $B$  is False, so  $A$  is False.
- Define  $C = \text{prop that } A \Rightarrow B$

$$p(AB|C) = p(B|AC) \cdot p(A|C)$$

or

$$p(B|AC) = \frac{p(AB|C)}{p(A|C)}$$

and

$$p(A\bar{B}|C) = p(A|\bar{B}C) \cdot p(\bar{B}|C)$$

$\Rightarrow$

$$p(A|\bar{B}C) = \frac{p(A\bar{B}|C)}{p(\bar{B}|C)}$$

- But  $A \Rightarrow B$  means  $A\bar{B}$  is impossible.

$$p(A\bar{B}|C) = 0$$

$$p(A|\bar{B}C) = 0$$

- Now another case:

$$\begin{aligned} p(AB|C) &= p(A|BC) \cdot p(B|C) \\ &= p(B|AC) \cdot p(A|C) \end{aligned}$$

$\Rightarrow$

$$p(A|BC) = \frac{p(B|AC) \cdot p(A|C)}{p(B|C)}$$

$A \Rightarrow B$  means if  $B$  is True, then  $A$  is more plausible.

$$p(B|AC) = 1$$

$$p(B|C) \leq 1$$

$p(A|C) \leq \frac{p(A|C)}{p(B|C)} = p(A|BC)$

- Now  $A \Rightarrow B$  means if  $A$  is False, then  $B$  is less plausible.

$$p(B|\bar{A}C) = \frac{p(\bar{A}|BC)p(B|C)}{p(\bar{A}|C)}$$

$$p(A|BC) \geq p(A|C)$$

so  $p(\bar{A}|BC) \leq p(\bar{A}|C)$

$p(B|\bar{A}C) \leq \frac{p(\bar{A}|C)p(B|C)}{p(\bar{A}|C)} \leq p(B|C)$

## 18.7 Mutually Exclusive and Exhaustive Propositions

- So we will see next class:
- $A_1, A_2, \dots, A_n$  mutually exclusive, exhaustive propositions.
- We also have B proposition.
- $A_i =$  everything that can happen.
- $\Rightarrow p(A_i|B)$  B says nothing
- $A_i A_j = F$
- $p(A_i|B) = \frac{1}{n}$
- (i.e.  $p(\text{certain}|B) = 1$ )

$$p\left(\sum(A_i)|B\right) = 1$$