Physics 509 Theory of Measurements Course Notes

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Introduction:
Notes written at UBC 2025W1 with Dr. Colin Gay.

Contents

1 Foundations of Probability Theory

This lecture covered the course structure and grading.

1.1 Motivation: Stochastic Nature of Experimental Data

- Stochastic processes:
  - muon decay
  - inherent stochasticity
  - quantum mechanics
- Mostly concerned with measurement devices — how do we measure?
- Example: a muon lifetime experiment
  - Take a cosmic muon, detect light, and discriminate.
  - Muon decays into an electron and neutrinos, and the electron produces light.
  - Measure the time between light pulses.
  - Many factors cause noise in the data — results change even if the same mechanism occurs twice.

1.2 Probabilistic Interpretation of Experimental Results

- Experiments are repeated trials.
- Probability (probabilistic interpretation):
  - Results are interpreted as the long-term average of repeating an experiment many times.
  - Example: coin flip

$$P(H) = \lim_{N\to\infty} \frac{n_H}{N}$$

$$n(H) = \text{number of heads in } N \text{ trials}$$

1.3 Sample Spaces and Stochastic Variables

- In modern probability theory:
  - 3 axioms (Kolmogorov)
  - Let X be a stochastic variable.
  - Define sample space S (Ω):

$$S = \{x_1, x_2, ...\}$$

  - Examples:
    1. Coin flip:

$$S = \{H, T\}$$

    2. Roll a die:

$$S = \{1, 2, 3, 4, 5, 6\}$$

    3. Grade in this class:

$$S = \{0, 1, 2, ..., 100\}$$

    4. Decay time of a radioactive atom:

$$S = [0, \infty)$$

  - S can be finite (Binomial), countable (Poisson), or infinite (Gaussian, Uniform).

1.4 Events and Set Operations

- Definition: An event E is a subset of S.
- Example: one die roll

$$S = \{1, 2, 3, 4, 5, 6\}$$
$$E = \text{rolling an even number} = \{2, 4, 6\}$$

- Example: E = atom decayed by time t_0

$$S = [0, t_0]$$

- Operations on events:
  - Union (OR) and Intersection (AND)
  - Let A, B be events in S:

$$E = A \cup B = \{e : e \in A \text{ or } e \in B \text{ (or both)}\}$$

  - Example: flip a coin twice

$$S = \{HH, HT, TH, TT\}$$
$$A = \text{1st flip is H} = \{HH, HT\}$$
$$B = \text{2nd flip is H} = \{HH, TH\}$$
$$A \cup B = \{HH, HT, TH\}$$
$$A \cap B = \{e \mid e \in A \text{ and } e \in B\} = \{HH\}$$
$$AB = A \cap B$$
$$A^c = \{e \mid e \in S \text{ and } e \notin A\} = \{TH, TT\}$$

- Properties:
  - Commutative:

$$A \cup B = B \cup A, \quad AB = BA$$

  - Associative:

$$A \cup (B \cup C) = (A \cup B) \cup C, \quad (AB)C = A(BC)$$

  - Distributive:

$$(A \cup B)C = AC \cup BC, \quad A(B \cup C) = AB \cup AC$$

  - De Morgan's Laws:

$$(A \cup B)^c = A^c B^c, \quad (AB)^c = A^c \cup B^c$$

1.5 Kolmogorov's Axioms of Probability

- A function P on S is a probability measure if it satisfies:
  1. $P(S) = 1$
  2. $P(\emptyset) = 0$
  3. For any countable sequence of disjoint events $E_1, E_2, \ldots$ in S:

$$E_i E_j = \emptyset \text{ for } i \neq j$$
$$P\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

1.6 Consequences of the Probability Axioms

- $$P(\emptyset) = 0$$

  Let

  $$E_1 = S, \quad E_2 = \emptyset$$
  $$E_1 E_2 = \emptyset$$
  $$P(S \cup \emptyset) = P(S) + P(\emptyset) = 1 + P(\emptyset)$$
  $$P(S) = 1,\ P(\emptyset) = 0$$

- $$P(E^c) = 1 - P(E)$$
  $$1 = P(S) = P(E \cup E^c) = P(E) + P(E^c)$$

- If $B \subset A$, then:

  $$P(B) \leq P(A)$$
  $$A = B \cup (B^c A)$$
  $$P(A) = P(B \cup (B^c A))$$
  $$P(B) = P(A) - P(B^c A) \leq P(A)$$

- $$P(A \cup B) = P(A) + P(B) - P(AB)$$

  If we let the areas of the Venn diagram be 1 (A), 2 (A+B), 3 (B), then:

  $$A \cup B = 1 \cup 2 \cup 3$$
  $$P(A \cup B) = P(1 \cup 2 \cup 3) = P(1) + P(2) + P(3)$$
  $$P(A) = P(1) + P(2), \quad P(B) = P(2) + P(3)$$
  $$P(A) + P(B) - P(2) = P(1) + P(2) + P(3) = P(A \cup B)$$
  $$\text{equivalently } P(A) + P(B) - P(AB) = P(A) + P(B) - P(AB)$$

1.7 Uniform Probability on Finite Sample Spaces

- $$E_i = S_i \text{ for } i = 1, 2, ...n$$
  $$E_i E_j = \emptyset \text{ for } i \neq j$$
  $$S = \cup_{i=1}^{n} E_i$$
  $$P(S) = 1 = P(\cup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i)$$
  $$P(E_i) = P(E_j) \quad \text{all equally likely}$$
  $$1 = \sum_{i=1}^{N} P(E_i) = NP(E_i)$$
  $$P(E_i) = \frac{1}{N} = P(E_j)$$
  $$N = |S| = \text{number of elements in (cardinality of) } S$$
  $$F \text{ be any event (set) in } S \text{ with } k \text{ elements} \quad |F| = k$$
  $$P(F) = P(\cup_{S_i \in F} \{E_i\}) = \sum_{i=1}^{k} P(E_j) = \sum_{i=1}^{k} \frac{1}{N} = \frac{k}{N} = \frac{|F|}{|S|}$$

1.8 Example: Probability of a Straight in Poker

- Example: 5-card poker hand forming a straight

$$S = \{(AC, 2C, 3C, 4C, 5C), (2C, 3C, 4C, 5C, 6C), ...\}$$

$$S = \binom{52}{5} = \frac{52!}{5!\,47!} = 2,598,960$$

- Event = straight = 5 consecutive cards, not of the same suit, any starting card.

$$10(4^5 - 4) = 10200$$

- Starting cards: Ace (A,2,3,4,5), 2 (2,3,4,5,6), ..., 10 (10,J,Q,K,A)
- Not all the same suit: $4^5 - 4$ (exclude all same suit)

$$P(\text{straight}) = \frac{10(4^5 - 4)}{\binom{52}{5}} = 0.00392465$$

1.9 Conditional Probability

- Given 2 events E, F, sample space S:

$$P(E) = \text{probability of a trial from } S \text{ in } E$$

$$P(F) = \text{probability of a trial from } S \text{ in } F$$

- Conditional probability of E given F has occurred:

$$P(E|F) = \text{probability of a trial from } S \text{ in } E, \text{ given the trial is in } F$$

- Note: $P(EF)$ is the probability of a trial from S in both E and F.
- Need to normalize by $P(F)$, so we define:

$$P(E|F) = \frac{P(EF)}{P(F)} \quad \text{if } P(F) > 0$$

$$P(EF) = P(E|F)P(F)$$

- Example: flip a coin 2 times

$$S = \{HH, HT, TH, TT\}$$

  Conditional probability of $HH \equiv A$ given:

  - First flip $= H \equiv B = \{HH, HT\}$
  - Either flip is $H \equiv C = \{HH, HT, TH\}$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(\{HH\})}{P(\{HH, HT\})} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

$$P(A|C) = \frac{P(AC)}{P(C)} = \frac{P(\{HH\})}{P(\{HH, HT, TH\})} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

2 Bayesian Inference and Random Variables

2.1 Bayes' Formula

- Let E, F be events:

$$E = EF \cup EF^c$$
$$P(E) = P(EF) + P(EF^c)$$
$$P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)$$
$$P(E) = P(E|F)P(F) + P(E|F^c)(1 - P(F))$$

- **Example:** Suppose a blood test is 95% effective in detecting a disease if the person has it. It also has a 1% false positive rate. Suppose 0.5% of the population has the disease.

$$D = \text{person has disease}$$
$$E = \text{test is positive}$$

- We want:

$$P(D|E) = \frac{P(ED)}{P(E)}$$

$$\begin{aligned} P(D|E) &= \frac{P(E|D)P(D)}{P(E|D)P(D) + P(E|D^c)(1 - P(D))} \\ &= \frac{0.95 \times 0.005}{0.95 \times 0.005 + 0.01 \times 0.995} = 0.32 \end{aligned}$$

- So even with a positive test, there is only a 32% chance of having the disease.

2.2 Law of Total Probability

- Let $\{F_i\}$ be mutually exclusive events such that:

$$\cup_{i=1}^{n} F_i = S$$

Then for any event E:

$$E = E \cap \left(\cup_{i=1}^{n} F_i\right) = \cup_{i=1}^{n}(EF_i)$$
$$P(E) = P\left(\cup EF_i\right) = \sum_{i=1}^{n} P(EF_i) = \sum_{i=1}^{n} P(E|F_i)P(F_i)$$

2.3 Independent Events

- Generally, $P(E|F) \neq P(E)$.
- If knowing F does not change the probability of E:

$$P(E|F) = \frac{P(EF)}{P(F)} = P(E)$$

$$\boxed{P(EF) = P(E)P(F)}$$

2.3.1 Example: Rolling Two Dice

- Let:

$$E_1 \equiv \text{sum} = 6$$
$$F \equiv \text{first die} = 4$$
$$E_1 : \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$$
$$F : \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}$$
$$E_1F = \{(4,2)\}$$
$$P(E_1F) = \frac{1}{36}$$
$$P(E_1) = \frac{5}{36}$$
$$P(F) = \frac{6}{36} = \frac{1}{6}$$
$$P(E_1)P(F) = \frac{5}{36} \times \frac{1}{6} = \frac{5}{216} \neq P(E_1F)$$

- Let:

$$E_2 \equiv \text{sum} = 7$$
$$E_2 : \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$
$$E_2F = \{(4,3)\}$$
$$P(E_2) = \frac{6}{36} = \frac{1}{6}$$
$$P(F) = \frac{1}{6}$$
$$P(E_2F) = \frac{1}{36}$$

2.4 Random Variables and Probability Distributions

- $S = \{\text{all possible outcomes of stochastic process } X\}$

$$x = \text{random variable}$$

$$S = \text{finite or countable infinite: discrete random variable}$$
$$S = \text{uncountable infinite: continuous random variable}$$

- Continuous case:

$$P(x_0, x_0 + dx) = p(x)dx$$

where $p(x)$ is the probability density function (pdf).

- Discrete case:

$$S = S_i$$
$$p_i = \text{probability of } S_i \quad \text{(probability mass function, pmf)}$$

$$0 \leq P(S_i) \leq 1$$
$$1 = P(S)$$
$$0 \leq p(x)$$
$$\int_{-\infty}^{\infty} p(x)dx = 1$$

2.5 Describing a Distribution

- To describe $p(x)$ in general we specify:
  - **Mode —** peak value of $p(x)$
  - **Median —** 50% cumulative value
  - **Mean —** average value of x weighted by $p(x)$

2.6 Cumulative Distribution Function (CDF)

- $$F(x) = \int_{-\infty}^{x} p(x')dx' = P(X \leq x)$$
  $$F(-\infty) = 0, \quad F(\infty) = 1$$

2.7 Expectation Values

- Expectation of any function $f(x)$ over $p(x)$:

$$E(f) = \int_{\Omega} f(x)p(x)dx$$

E is a linear operator: $E(af + bg) = aE(f) + bE(g)$

- Expectation of powers of x:

$$E(x^0) = E(1) = \int 1 \cdot p(x)dx = 1$$

$$E(x^1) = \int xp(x)dx \equiv \mu = \text{mean value of } x$$

$$E(x^2) = \int x^2 p(x)dx \equiv \sigma^2 = \text{variance of } x$$

2.8 Characteristic Function

- The characteristic function of $p(x)$:

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} p(x)dx = E(e^{itx})$$

$$\varphi(t) = E\left(1 + itx + \frac{(itx)^2}{2!} + \dots\right)$$

$$= 1 + itE(x) + \frac{(it)^2}{2!}E(x^2) + \dots$$

$$= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \mu_{k'}$$

- Moments from $\varphi(t)$:

$$\left.\frac{d^n \varphi(t)}{dt^n}\right|_{t=0} = i^n \mu_{n'}$$

2.9 Central Moments

- $$E((x-\mu)^n) = \int (x-\mu)^n p(x)dx \equiv \mu_n$$
  $$\mu = E(x)$$
- 1st central moment:
  $$E((x-\mu)^1) = E(x) - E(\mu) = \mu - \mu = 0$$
- 2nd central moment (variance):
  $$E((x-\mu)^2) \equiv V(x) = \sigma^2$$
- 3rd central moment (skewness):
  $$\text{skewness} = \frac{E((x-\mu)^3)}{\sigma^3}$$
- 4th central moment (kurtosis):
  $$\text{kurtosis} = \frac{E((x-\mu)^4)}{\sigma^4} - 3$$
  (The -3 ensures that the kurtosis of a normal distribution is 0.)

3 Bayesian Reasoning and Probability Distributions

Tuesday, September 16th 2025

3.1 Bayes Theorem and Its Applications

- Bayes Theorem: for events A and B, we have

$$\boxed{P(AB) = P(A|B)P(B) = P(B|A)P(A) = P(BA)}$$

- Usually it is given in this form:
$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$
- People argued about when you are allowed to use this theorem.

3.2 The Monty Hall Problem: A Bayesian Analysis

- Example: Monty Hall Problem (Game show with host named Monty Hall)
  - There are 3 doors; behind one is a car, behind the other two are goats.
  - You select a door; if the car is behind it, you win.
  - Twist: after you select a door, Monty opens one of the other 2 doors to reveal a goat.
  - Question: stay or switch?
  - Solution: use Bayes theorem.
  - Sample space: $S = \{C_1 = \text{cgg}, C_2 = \text{gcg}, C_3 = \text{ggc}\}$
  - Event 2 = MH opens door 2.
  - Event 3 = MH opens door 3.
  - Number such that your choice is door 1.
  - Take case E_2, then we want to know $P(C_1|E_2)$.
$$P(C_1|E_2) = \frac{P(E_2|C_1)P(C_1)}{P(E_2)}$$
  - $P(C_1) = \frac{1}{3}$
  - $P(E_2|C_1) = \frac{1}{2}$ because if the car is behind door 1, Monty can open either door 2 or 3.
  - $P(E_2) = \frac{1}{2}$
  - Law of total probability:
$$P(E_2) = P(E_2|C_1)P(C_1) + P(E_2|C_2)P(C_2) + P(E_2|C_3)P(C_3) = \frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = \frac{1}{2}$$
  - $P(C_1|E_2) = \frac{P(E_2|C_1)P(C_1)}{P(E_2)} = \frac{\frac{1}{2}\frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}$
  - $P(C_1|E_2) = \frac{1}{3}$
  - $P(C_2|E_2) = 0$
  - $P(C_3|E_2) = \frac{P(E_2|C_3)P(C_3)}{P(E_2)} = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$

3.3 Alternate Monty Hall Formulations

- Alternate version: $E =$ MH shows you a goat from $\{2, 3\}$.
  - We want to find $P(C_1|E)$.
  - $P(C_1|E) = \frac{P(E|C_1)P(C_1)}{P(E)}$
  - $P(C_1) = \frac{1}{3}$
  - $P(E|C_1) = 1$ because if the car is behind door 1, Monty can open either door 2 or 3.
  - $P(E) = 1$ by law of total probability:
  $$P(E) = P(E|C_1)P(C_1) + P(E|C_2)P(C_2) + P(E|C_3)P(C_3) = 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = 1$$
  - $P(C_1|E) = \frac{1 \cdot \frac{1}{3}}{1} = \frac{1}{3}$
- Another version: What if MH does *not* know where the car is?
  - $E =$ MH opens $\{2, 3\}$ and reveals a goat.
  - We want to find $P(C_1|E)$.
  - $P(C_1|E) = \frac{P(E|C_1)P(C_1)}{P(E)}$
  - $P(C_1) = \frac{1}{3}$ because we picked door 1.
  - $P(E|C_1) = \frac{1}{2}$ because if the car is behind door 1, Monty can open either door 2 or 3 since he does not know where the car is.
  - By law of total probability:
  $$P(E) = P(E|C_1)P(C_1) + P(E|C_2)P(C_2) + P(E|C_3)P(C_3) = 1 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{2}{3}$$
  - $P(C_1|E) = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$

3.4 Monty Hall Generalized to n Doors

- Now back to the standard version of the problem but with n doors.
  - You pick door 1, MH opens any door with a goat behind it from 2 to n ($n - 1$ options).
  - $P(C_1|E) = \frac{P(E|C_1)P(C_1)}{P(E)}$
  - $P(E) = 1$ because he can always choose a door with a goat behind it (many options and he knows the answers).
  - $P(C_1) = \frac{1}{n}$
  - $P(E|C_1) = 1$ because if the car is behind door 1, Monty can open any of the other doors.

3.5 Continuous Probability Distributions and Moments

- Continuous probability distribution $p(x)$:
- Moments:

$$E(x^n) = \int_{-\infty}^{\infty} x^n p(x)\, dx$$

mean:	$\mu = E(x)$
variance:	$V(x) = \sigma^2 = E((x-\mu)^2) = E(x^2) - \mu^2$
std dev:	$\sigma = \sqrt{\sigma^2}$

- Central moments:

$$E(x-\mu) = E(x) - \mu = 0$$
$$E((x-\mu)^2) = \sigma^2$$
$$E((x-\mu)^3) = \text{skewness}$$
$$E((x-\mu)^4) = \text{kurtosis}$$

- Characteristic function:

$$\Phi(t) = E(e^{itx}) = \int_{-\infty}^{\infty} e^{itx} p(x)\, dx \tag{3.1}$$
$$= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \mu_k \tag{3.2}$$

$$\Phi_\mu(t) = E(e^{it(x-\mu)}) = E(e^{itx})e^{-it\mu} = \Phi(t)e^{-it\mu}$$

$$V(x) = E((x-\mu)^2) \tag{3.3}$$
$$= E(x^2 - 2\mu x + \mu^2) \tag{3.4}$$
$$= E(x^2) - 2\mu E(x) + \mu^2 E(1) \tag{3.5}$$
$$= E(x^2) - 2\mu^2 + \mu^2 \tag{3.6}$$
$$= E(x^2) - \mu^2 = E(x^2) - (E(x))^2 \tag{3.7}$$

3.6 Discrete Probability Distributions

- The discrete case (e.g., rolling a die, picking a card) uses a probability mass function.
- Usually denote outcomes as r:
- p_r = probability of outcome r.
- $\sum_r p_r = 1$
- $E(r) = \sum_r p_r r$ = mean μ
- Variance: $V(r) = \sum_r (r-\mu)^2 p_r = E(r^2) - \mu^2$
- Coin flip example: $S = \{H, T\}$.
- Often map to 0 or 1: $H = 0$, $T = 1$.
- But in theory you can pick any two numbers a and b to map outcomes, just so you can calculate mean and variance.

$$E(r) = ap_H + bp_T$$

3.7 Cumulative Distribution Functions

- For continuous case:
$$F(x) = \int_{-\infty}^{x} f(x')\,dx'$$
- For discrete case:
$$F(r) = \sum_{r' \leq r} p_{r'}$$
- $F(x)$ is the cumulative distribution function (CDF).
- $F(x)$ is non-decreasing, $F(-\infty) = 0$, $F(\infty) = 1$.

3.8 Multivariate Distributions and Covariance

- Distribution of multiple variables:
- Elements belong to real vector space $\mathbb{R}^n$.
- $P(AB) \ldots P(A, B)$
- $p(x_1, x_2, \ldots, x_n) \geq 0$ is the joint probability distribution function (PDF).
- $\int_\Omega p(\vec{x}) d^n x = 1$
- $E(f(\vec{x})) = \int_\Omega f(\vec{x}) p(\vec{x}) d^n x$
- $\mu_i = \int x_i p(\vec{x}) d^n x$
- $V(x_i) = \sigma_i^2 = \int (x_i - \mu_i)^2 p(\vec{x}) d^n x$
- Covariance:
- $V_{i,j} = E((x_i - \mu_i)(x_j - \mu_j))$
- $V_{i,i} = \sigma_i^2 = E((x_i - \mu_i)^2)$ (variance)
- $V_{i,j} = V_{j,i}$ (symmetry)

4 Joint Distributions, Correlations, and Variable Transformations

Thursday, September 18th 2025

4.1 Conditional Probability: A Simple Example

- For fun, example that depends on cultural assumptions: A king comes from a family with two kids. What is the probability that the king's sibling is a sister?
- $S = \{(m, m), (m, f), (f, m), (f, f)\}$
- $P(S|K) = \frac{P(SK)}{P(K)} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$

4.2 Distributions of Multiple Random Variables

- $p(x_1, x_2, \ldots, x_n)$
- $S = \mathbb{R}^n$
- $\int p(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2\, \ldots\, dx_n = 1$
- For any function $f(\vec{x})$:
$$E(f) = \int f(\vec{x}) p(\vec{x}) d\vec{x}$$
- $E(x_1) = \int x_1 p(\vec{x}) d\vec{x} = \mu_1$
- $E(x_i) = \mu_i$
- $V(x_i) \equiv \sigma_i^2 = \int (x_i - \mu_i)^2 p(\vec{x}) d\vec{x}$

4.3 Covariance Matrix and Correlation Coefficient

- Define covariance:
$$V_{ij} = E((x_i - \mu_i)(x_j - \mu_j))$$
- $V_{ii} = \sigma_i^2$ (variance)
- $V_{ij} = V_{ji}$ (symmetry)
- $V_{ij} = 0$ for independent variables
- Expanding the covariance matrix:
$$\begin{aligned} V_{ij}(\vec{x}) &= E((x_i - \mu_i)(x_j - \mu_j)) \\ &= E(x_i x_j - \mu_i x_j - \mu_j x_i + \mu_i \mu_j) \\ &= E(x_i x_j) - \mu_i E(x_j) - \mu_j E(x_i) + \mu_i \mu_j \\ &= E(x_i x_j) - \mu_i \mu_j - \mu_j \mu_i + \mu_i \mu_j \\ &= E(x_i x_j) - \mu_i \mu_j \end{aligned}$$
- So we can say that $V_{ij} \geq 0$
- V_{ij} can be negative, zero, or positive

- Define the correlation coefficient:

$$\rho(x_i, x_j) = \rho_{ij} = \frac{V_{ij}}{\sqrt{V_{ii}}\sqrt{V_{jj}}} = \frac{V_{ij}}{\sigma_i \sigma_j}$$

- We find that $-1 \leq \rho_{ij} \leq 1$

4.4 Independence and Uncorrelated Variables

- Random variables $x_i, \ldots, x_n$ are independent if the joint pdf factorizes:

$$p(x_1, \ldots, x_n) = p_1(x_1)p_2(x_2)\ldots p_n(x_n)$$

- Independent variables are uncorrelated:

$$\begin{aligned} E(x_i x_j) &= \int x_i x_j p(\vec{x}) d\vec{x} \\ &= \int x_i x_j p_1(x_1)\ldots p_n(x_n) dx_1 \ldots dx_n \\ &= \int x_i p_i(x_i) dx_i \int x_j p_j(x_j) dx_j \int p_2(x_2) dx_2 \ldots \int p_n(x_n) dx_n = \mu_i \mu_j \end{aligned}$$

$$V_{ij} = E(x_i x_j) - \mu_i \mu_j$$

In the case of independent variables:

$$V_{ij} = \mu_i \mu_j - \mu_i \mu_j = 0$$

- Independent variables are uncorrelated, but uncorrelated variables are not necessarily independent.

4.5 Examples of Correlated and Uncorrelated Variables

- 100% correlation example:
- $x = \text{Uniform}[-1, 1]$, plot distribution from -1 to 1.
- $y = x$:
- $V_{ij} = E(xy) - E(x)E(y) = E(x^2) = \int_{-1}^{1} x^2 \frac{1}{2} dx = \frac{1}{3} \neq 0$
- $y = |x|$:
- $E(xy) = \int_{-1}^{0} x(-x)p(x)dx + \int_0^1 xxp(x)dx$
- $E(xy) = \int_0^1 x^2 \frac{1}{2} dx - \int_{-1}^{0} x^2 \frac{1}{2} dx = \frac{1}{6} - \frac{1}{6} = 0$

4.6 Marginal Distributions

- For a joint pdf $p(x_1, x_2, \ldots, x_n)$, the marginal probability density functions are:

$$f_1(x_1) = \int p(x_1, x_2, \ldots, x_n) dx_2 dx_3 \ldots dx_n$$

- If variables are independent:

$$\begin{aligned} f_1(x_1) &= \int p(x_1, x_2, \ldots, x_n) dx_2 \ldots dx_n \\ &= p_1(x_1) \int p_2(x_2) dx_2 \int p_3(x_3) dx_3 \ldots \int p_n(x_n) dx_n \\ &= p_1(x_1) \cdot 1 \cdot 1 \cdot \ldots \cdot 1 = p_1(x_1) \end{aligned}$$

4.7 Change of Variables in Probability Densities

- Something we need to know, because we do it all the time:
  - Change of variables of P
  - Calculate new V_{ij} under new variables
- Let x be a random variable with pdf $f(x)$ and let y be some function.
- First: y is one-to-one with f

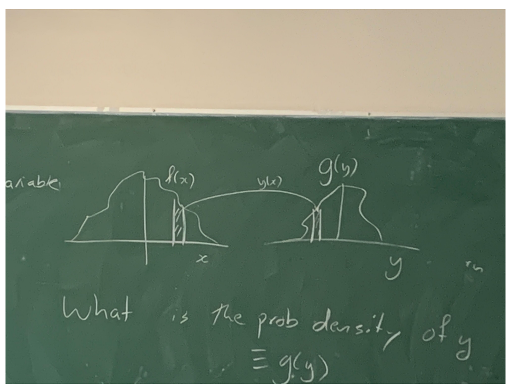


Figure 4.1: 1-to-1 function

- What is the probability density of y, denoted $g(y)$?
- Conservation of probability:
- $f(x)dx = g(y)dy$
- $g(y) = f(x) \left| \frac{dx}{dy} \right|$

$$\boxed{f(x) \left| \frac{dx}{dy} \right| = g(y)}$$

4.8 Change of Variables: Non One-to-One Case

- If y is not one-to-one: sum over all segments that map to the same y.
- Example: $f(x)$ uniform on $[0,1]$, $f(x) = 1$
- Let $y(x) = \frac{-1}{\lambda}\ln(x)$
- $\frac{dy}{dx} = \frac{-1}{\lambda x}$
- $\frac{dx}{dy} = -\lambda x$
- $-\lambda x = \ln x$
- $e^{-\lambda y} = x$
- $\lambda > 0 \Rightarrow \frac{dx}{dy} = -\lambda x = -\lambda e^{-\lambda y}$
- $g(y) = f(x)\left|\frac{dx}{dy}\right| = 1 \cdot \lambda e^{-\lambda y} = \lambda e^{-\lambda y}$

4.9 Multivariate Transformations and the Jacobian

- If we have variables $\{x_i\}$ and transform to new variables $\{y_i\}$:
- Region $\mathbb{R}$ in x-space maps to region $\mathbb{R}'$ in y-space.

$$\int_{\mathbb{R}} f(\vec{x})d\vec{x} = \int_{\mathbb{R}'} f(\vec{x}(\vec{y}))(\vec{y})\left|\frac{\partial \vec{x}}{\partial \vec{y}}\right| d\vec{y}$$

$$g(\vec{y}) = f(\vec{x}(\vec{y}))\,|J|$$

- Where $\left|\frac{\partial \vec{x}}{\partial \vec{y}}\right|$ is the Jacobian determinant of the transformation.
- Jacobian matrix J:

$$J = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{bmatrix}$$

4.10 Example: Cartesian to Polar Transformation

- Change to polar coordinates:
- $x = r\cos\theta$
- $y = r\sin\theta$
- $P'(r,\theta) =? = p(x,y)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|$
- $\frac{\partial x}{\partial r} = \cos\theta$
- $\frac{\partial y}{\partial r} = \sin\theta$

- $\frac{\partial x}{\partial \theta} = -r \sin \theta$
- $\frac{\partial y}{\partial \theta} = r \cos \theta$
- $J = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$
- $J = r \cos^2 \theta + r \sin^2 \theta = r$
- $p'(r, \theta) = \frac{r}{\pi} dr d\theta$

5 Propagation of Uncertainty in Measurements

Tuesday, September 20th 2025

5.1 Propagation of Errors for a Single Variable

- Given $f(x)$ pdf, $\mu \equiv E(x)$, $\sigma^2 \equiv V(x) = E(x^2) - \mu^2$
- Know $f(x) \to g(y)$, given $y(x)$.
- Taylor expand $y(x)$ about mean μ:

$$y(x) = y(\mu) + y'(\mu)(x-\mu) + \frac{1}{2!}y''(\mu)(x-\mu)^2 + \dots$$

$$E(y(x))? \equiv \mu_y$$

$$\begin{aligned} E(y(x)) &= E(y(\mu)) + y'(\mu)E(x-\mu) + \frac{1}{2!}y''(\mu)E((x-\mu)^2) + \dots \\ &= y(\mu) + y'(\mu)\cdot 0 + \frac{1}{2!}y''(\mu)V(x) + \dots \end{aligned}$$

- To the 1st order:

$$\mu_y = E(y(x)) = y(\mu) = y(E(x))$$

5.2 Variance Propagation for a Single Variable

- Variance of y:

$$\begin{aligned} V(y) &= E((y(x) - E(y(x)))^2) && (5.1) \\ &= E((y(x) - \mu_y)^2) && (5.2) \\ &= E((y'(\mu)(x-\mu) + \frac{1}{2!}y''(\mu)(x-\mu)^2 + \dots)^2) && (5.3) \\ &= E(y'(\mu)^2(x-\mu)^2 + y'(\mu)y''(\mu)(x-\mu)^3 + O((x-\mu)^4)) && (5.4) \\ &= y'(\mu)^2 V(x) + \dots && (5.5) \end{aligned}$$

- Some relations:

$$\begin{aligned} E(x) &\equiv \mu_x \\ V(x) &\equiv \sigma_x^2 \\ y &= y(x) \\ E(y) &\equiv \mu_y = y(\mu_x) \\ V(y) &\equiv \sigma_y^2 = (y'(\mu_x))^2\sigma_x^2 \end{aligned}$$

$$\sigma_y = |y'(\mu_x)|\sigma_x$$

- Example: $y = \frac{1}{x}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2}$

$$\sigma_y^2 = \frac{1}{\mu_x^4}\sigma_x^2$$

5.3 Propagation of Errors for Multiple Variables

- Let us suppose we have n variables $\{x_i\}$, with pdf $f(\vec{x})$.
- Let $y_j = 1, 2, \ldots, m$ be m functions of x_i.
- $y_j = y_j(x_1, x_2, \ldots, x_n)$
- $V_{ij}(x)_{n\times n}(\vec{x}) =$ covariance matrix of $\{x_i\}$
- $V_{ij}(\vec{x}) = E((x_i - \mu_i)(x_j - \mu_j))$
- Taylor expand each y_j: $\vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$
- $y_j(\vec{x}) = y_j(\vec{\mu}) + \sum_i \frac{\partial y_j}{\partial x_i}\Big|_{\vec{\mu}} (x_i - \mu_i) + \frac{1}{2!}\sum_{i,k} \frac{\partial^2 y_j}{\partial x_i \partial x_k}\Big|_{\vec{\mu}} (x_i - \mu_i)(x_k - \mu_k) + \ldots$
- $E(y_j(\vec{x})) = E(y_j(\vec{\mu})) + \sum \frac{\partial y_j}{\partial x_i} E(x_i - \mu_i) + \ldots = y_j(\vec{\mu})$

5.4 Covariance Propagation for Functions of Multiple Variables

- Covariance between y_k and y_l:

$$
\begin{aligned}
&E((y_k - \mu_{y_k})(y_l - \mu_{y_l})) \\
&= E((y_k - y_k(\mu))(y_l - y_l(\mu))) \\
&= E\left(\sum_i \frac{\partial y_k}{\partial x_i}\Big|_{\mu} (x_i - \mu_i) \sum_j \frac{\partial y_l}{\partial x_j}\Big|_{\mu} (x_j - \mu_j)\right) \\
&= \sum_{i,j} \frac{\partial y_k}{\partial x_i}\Big|_{\mu} \frac{\partial y_l}{\partial x_j}\Big|_{\mu} E((x_i - \mu_i)(x_j - \mu_j))
\end{aligned}
$$

$$
\boxed{V_{kl}(\vec{y})_{m\times m} = \sum_{i,j} \frac{\partial y_k}{\partial x_i}\Big|_{\vec{\mu}} \frac{\partial y_l}{\partial x_j}\Big|_{\vec{\mu}} V_{ij}(\vec{x})_{n\times n}}
$$

- Example: x, y random variables,

$$
V(x,y) = \begin{bmatrix} V_{xx} & V_{xy} \\ V_{yx} & V_{yy} \end{bmatrix} = \begin{bmatrix} \sigma_x^2 & \rho_{xy}\sigma_x\sigma_y \\ \rho_{xy}\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}
$$

- $z = x + y$
- $V(z) = \sigma_z^2 = \left(\frac{\partial z}{\partial x}\right)^2 V_{xx} + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} V_{xy} + \left(\frac{\partial z}{\partial y}\right)^2 V_{yy}$
- $= \sigma_x^2 + 2\rho_{xy}\sigma_x\sigma_y + \sigma_y^2$
- If x_i are uncorrelated,

$$
V_{ij} = \sigma_{i,j}\sigma_i^2 = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}
$$

$$
V_{kl}(\vec{y}) = \sum_i \frac{\partial y_k}{\partial x_i}\Big|_{\mu} \frac{\partial y_l}{\partial x_i}\Big|_{\mu} V_{ii}(\vec{x})
$$

$$
\text{variance } V_{kk} = \sum_i \left(\frac{\partial y_k}{\partial x_i}\right)^2 \sigma_i^2
$$

5.5 Examples of Error Propagation in Measurements

- Example: Measuring resistances. x_i independent, $z = x + y$, $x = R_1$ resistor value, $y = R_2$ resistor value, $z = R_{\text{tot}}$ total resistance.
- $R_1 \pm \sigma_{R_1}$
- Convention is to use $\sqrt{V(R)}$ as uncertainty.
- For a good measuring device, $E(R) = R_{\text{true}} \leftarrow$ unbiased.
- $V(R) = \text{small}$
- $R_1 \pm \sigma_{R_1}$, $R_2 \pm \sigma_{R_2}$, then $\sigma_{R_{\text{tot}}} = \sqrt{\sigma_{R_1}^2 + \sigma_{R_2}^2}$
- $R = R_{\text{tot}} = R_1 + R_2$
- $z = xy$, like I, R
- $\sigma_z^2 = \left(\frac{\partial z}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial z}{\partial y}\right)^2 \sigma_y^2 = y^2\sigma_x^2 + x^2\sigma_y^2$

$$\boxed{\left(\frac{\sigma_z}{z}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$$

5.6 Matrix Formulation of Linear Error Propagation

- Formula is exact if transformation of variables is linear.
- $\vec{y} = A\vec{x}$, A is $m \times n$ matrix, $\vec{x}$ is $n \times 1$, $\vec{y}$ is $m \times 1$.
- $\frac{\partial y_k}{\partial x_i} = \text{constant} \Rightarrow$ higher order terms in Taylor expansion are 0
- $V_{kl}(\vec{y}) = \sum_{i,j} \frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_j} V_{ij}(\vec{x})$
- Matrix notation:
- $V_{kl}(\vec{y}) = \sum_{i,j} A_{ki} A_{lj} V_{ij}(\vec{x})$
- $= \sum_{i,j} A_{ki} V_{ij}(\vec{x}) A_{lj}$
- $= \sum_{i,j} A_{ki} V_{ij}(A^T)_{jl}$
- $= (AV(\vec{x})A^T)_{kl}$

$$\boxed{V(\vec{y})_{m\times m} = A_{m\times n} V(\vec{x})_{n\times n} A^T_{n\times m}}$$

5.7 Variance of the Arithmetic Mean

- Example: Arithmetic mean. Let $x_i = n$ identical independent variables with $V(x_i) = \sigma_x^2$
- Set $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$
- Recall that $V(ax) = a^2 V(x)$
- $V(\bar{x}) = V\left(\frac{1}{n}\sum_{i=1}^n x_i\right) = \frac{1}{n^2} V\left(\sum_{i=1}^n x_i\right) = \frac{1}{n^2}\sum_{i=1}^n V(x_i) = \frac{1}{n^2} n\sigma_x^2 = \frac{\sigma_x^2}{n}$

- If variables are different σ_i^2: n measurements
- $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$
- $V(\bar{x}) = \frac{1}{n^2}\sum_{i=1}^{n} V(x_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2$
- $\sigma_{\bar{x}} = \frac{1}{n}\sqrt{\sum_{i=1}^{n}\sigma_i^2}$

5.8 Example: Measuring the Period of a Sine Wave

- Example: Measure period of sine wave on scope.
- $T = \Delta t = t_2 - t_1$
- $\sigma_T^2 = \left(\frac{\partial \Delta t}{\partial t_1}\right)^2 \sigma_t^2 + \left(\frac{\partial \Delta t}{\partial t_2}\right)^2 \sigma_t^2 = \sigma_t^2 + \sigma_t^2 = 2\sigma_t^2$
- Measure N cycles, $T = \frac{1}{N}\Delta t$
- $\sigma_{T^2} = \frac{1}{N^2}\sigma_{\Delta t}^2 = \frac{2}{N^2}\sigma_t^2$

6 Covariance Transformations and the Binomial Distribution

Tuesday, September 25th 2025

6.1 Covariance Transformation Under Linear Transformations

- Linear transformation:

$$\vec{y} = A\vec{x}$$

$$V_{kl}(\vec{y}) = \sum_{i,j} \frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_j} V_{ij}(\vec{x})$$

- Linear $y_k = \sum A_{kj} x_j$
- then

$$V_{kl}(\vec{y}) = \sum_{i,j} A_{ki} A_{lj} V_{ij}(\vec{x})$$

- or in matrix form

$$V(\vec{y}) = \left(AV(\vec{x})A^T\right)_{kl}$$

Diagonalization via Eigenvectors

- If $\hat{e}_i$ are the eigenvectors of V, then

$$V(\vec{x})\hat{e}_i = \lambda_i \hat{e}_i$$

- Form:

$$A = \begin{pmatrix} \hat{e}_1 \\ \dots \\ \hat{e}_n \end{pmatrix} = \begin{pmatrix} \hat{e}_{11} & \hat{e}_{12} & \dots & \hat{e}_{1n} \\ \dots & \dots & \dots & \dots \\ \hat{e}_{n1} & \hat{e}_{n2} & \dots & \hat{e}_{nn} \end{pmatrix}$$

- then

$$A^T A = I$$

- then:

$$VA^T = V \begin{pmatrix} \hat{e}_1 & \dots \\ \dots & \dots \\ \hat{e}_n & \dots \end{pmatrix} = \begin{pmatrix} \lambda_1 \hat{e}_{11} & \dots & \lambda_n \hat{e}_{n1} \\ \dots & \dots & \dots \\ \lambda_1 \hat{e}_{1n} & \dots & \lambda_n \hat{e}_{nn} \end{pmatrix}$$

- Then:

$$AVA^T = \begin{pmatrix} \hat{e}_{11} & \dots & \hat{e}_{1n} \\ \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} \lambda_1 \hat{e}_{11} & \dots & \\ \dots & \dots & \dots \\ \lambda_1 \hat{e}_{1n} & \dots & \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

- Then:

$$AVA^T = V(\vec{y}) = \begin{pmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sigma_n^2 \end{pmatrix}$$

6.2 The Binomial Distribution

- Consider an experiment with two outcomes.
- E.g. coin flips, selecting a ball with 2 possible colours, etc.
- One trial is called a Bernoulli trial.

Bernoulli Trials and Sampling Methods

- Example – Method 1: You have an urn filled with N balls. Some are red (R), some are blue (B).
- (0) What is your estimate of n_R, n_B, or $f = n_R/N$ or p of drawing R?
- (1) You pick a ball: R. Q: estimate of $p = n_R/N$?
- (2) You pick another without replacing 1st ball: get R.
- (3) R
- (4) Get B
- This is a question about this ONE urn.
- Now Method 2: you draw red, and you PUT IT BACK. You repeat this several times.
- Now Method 3: We have an infinite source of balls with fraction p red and $(1-p)$ blue.

$$P(R) = p$$
$$P(B) = 1 - p$$

Derivation of the Binomial Probability

- Make infinite number of urns all with N balls, with fraction p red and $(1-p)$ blue.
- Open all, count n_R red balls, n_B blue balls.
- In our case we have N balls, prob $p = R$ and $1 - p = q = B$.
- Prob of getting sequence RRB is:

$$P(RRB) = p \cdot p \cdot (1-p) = p^2(1-p)$$

- If we don't care about order, then:

$$P(RRB) = P(RBR) = P(BRR) = p^2(1-p)$$

- There are 3 ways of ordering RRB, so total probability is:

$$P(2R, 1B) = 3p^2(1-p) = 3p^2q$$

- Number of ways to choose r items from N is:

$$\binom{N}{r} = \frac{N!}{r!(N-r)!}$$

- Probability of getting exactly r R out of N:

$$P_r = \binom{N}{r} p^r (1-p)^{N-r} = B(r; N, p)$$

- This is called the Binomial distribution and applies to anything where there are 2 outcomes $(A, \bar{A})$.

Mean and Variance of the Binomial Distribution

- Want mean, σ

$$\begin{aligned}
E(r) &= \sum_{r=0}^{n} r P_r = \sum_{r=0}^{n} r \binom{n}{r} p^r (1-p)^{n-r} \\
&= \sum_{r=0}^{n} r \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r} \\
&= \sum_{r=1}^{n} \frac{n!}{(r-1)!(n-r)!} p^r (1-p)^{n-r} \\
&= np \sum_{r=1}^{n} \frac{(n-1)!}{(r-1)!(n-r)!} p^{r-1} (1-p)^{n-r}
\end{aligned}$$

- Change sum $r' = r - 1 \rightarrow n' = n - 1$

$$\begin{aligned}
E(r) &= np \sum_{r'=0}^{n-1} \frac{(n-1)!}{r'!(n-1-r')!} p^{r'} (1-p)^{(n-1)-r'} \\
&= np \sum_{r'=0}^{n-1} \binom{n-1}{r'} p^{r'} (1-p)^{n'-r'} \\
&= np \cdot 1 = np
\end{aligned}$$

- from:

$$(p+q)^n = \sum_{r=0}^{n} \binom{n}{r} p^r q^{n-r}$$

$$(p+1-q)^n = 1^n = 1$$

$$E(r) = np$$

- This is what we want!
- Now:

$$V(r) = \sum r^2 p_r - E(r)^2 = \sum r^2 p_r - n^2 p^2$$

- Slightly easier to calculate:

$$\sum r(r-1) p_r = \sum_{r=0}^{n} r(r-1) \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r} = \sum_{r=2}^{n} \frac{n!}{(r-2)!(n-r)!} p^r q^{n-r}$$

$$= n(n-1)p^2 \sum_{r=2}^{n} \frac{(n-2)!}{(r-2)!(n-r)!} p^{r-2} q^{n-r}$$

- Sub $r' = r - 2$

$$\begin{aligned}
&= n(n-1)p^2 \sum_{r'=0}^{n-2} \binom{n-2}{r'} p^{r'} q^{(n-2)-r'} \\
&= n(n-1)p^2 \cdot 1 = n(n-1)p^2 \\
&= n^2 p^2 - np^2
\end{aligned}$$

- Such that:

$$\boxed{V(r) = \sum r^2 p_r - n^2 p^2} = np(1-p) = npq$$

Applications to Histograms and Counting Statistics

- Why is this important? Histograms are often Binomially distributed.
- Data either falls A: falls in bin, or $\bar{A}$: does not fall in bin.
- $p =$ probability of falling in ith bin.
- $+n$ entries, e.g. students in class, histogram = grades.
- Expected number of entries is np.
- Plot of taking distribution several times and checking how many fall in bin i and then plotting that distribution is Binomial.
- Usually you have 1 histogram.
- Look at entries in bin i – n_i/n = fraction of entries in bin i.
- Estimator $p = n_i/n$.
- Expect if you repeated $\Rightarrow$ n_i would follow Binomial distribution with mean $\sigma_i = \sqrt{npq}$ and $V_i = \sigma_i^2 = np(1-p)$.
- $$p = \frac{n_i}{n}$$
- $$\boxed{\sigma_i = \sqrt{n_i\left(1 - \frac{n_i}{n}\right)}} \approx \sqrt{n_i} \text{ if } n \gg n_i$$
- Notes: we do know the total number n, how often is it in bin i.
- HW: given the distribution, how many times n do I need to do it to get that.
- r fixed n, vs. n fixed r.

7 Lecture 7

Thursday, October 2nd 2025

7.1 Review of the Binomial Distribution and Its Properties

- Recall last time:
$$B(r, n, p) = \binom{n}{r} p^r (1-p)^{n-r}$$
$$E(r) = np = \mu$$
$$V(r) = np(1-p) = \sigma^2$$
$$r = \sqrt{np(1-p)}$$
$$p = \frac{\mu}{n}$$
- $\epsilon = \frac{r}{n}$
- Number of detections (people often forget the $(1-p)$ term):
$$n\epsilon \pm \sqrt{n\epsilon(1-\epsilon)}$$
- $\sigma_\epsilon = \frac{1}{n}\sigma_r = \frac{1}{n}\sqrt{r\left(1-\frac{r}{n}\right)} = \frac{1}{\sqrt{n}}\sqrt{\epsilon(1-\epsilon)}$
- Standardized skewness:
$$E\left[\left(\frac{x-\mu}{\sigma}\right)^3\right] = \frac{1-2p}{\sqrt{np(1-p)}}$$
- Excess kurtosis:
$$\frac{1-6p(1-p)}{np(1-p)}$$

7.2 Bernoulli Distribution as a Special Case of the Binomial

- Bernoulli Distribution: Binomial with $n = 1$.
$$B(r, n=1, p) = P_r = \binom{1}{r} p^r (1-p)^{1-r} = p^r (1-p)^{1-r}$$
$$P_0 = 1-p$$
$$P_1 = p$$
$$\mu = E(r) = np = p$$
$$V = \sigma^2 = np(1-p) = p(1-p)$$
$$E(r^k) = \sum_{r=0}^{1} r^k P_r = 0^k(1-p) + 1^k p = p$$
- Central moments: $E[(r-p)^k]$

7.3 Negative Binomial and Geometric Distributions

- Negative Binomial: How many n to get r successes.
- Geometric distribution: negative binomial with $r = 1$ (number of trials to get first success).

$$G(n, p) = p(1-p)^{n-1}$$

- Example: Roll a die with $p(i) = 1/6$ (success = get 4)

$$E(n) = \frac{1}{p} = 6$$

- $P(n \leq 5) = 0.598$
- $P(n \geq 7) = 0.335$

7.4 Samples and the Concept of an Ensemble

- Samples: A set of N draws/trials from a pdf $p(x)$, $\{P_r\}$, is called a **sample** of size N: $\{x_i\}_{i=1}^{N}$.
- Orthodox statistics: your sample is one of many possible, and we can answer questions about the **ensemble** of samples.
- Samples:
  - 1: $\{x_{1i}\}_{i=1}^{N}$ from pdf $p(x)$
  - 2: $\{x_{2i}\}_{i=1}^{N}$ from pdf $p(x)$
  - ...
  - M
- $p(x|\mu, \sigma)$
- p = probability of H (heads), $q = 1 - p$ = probability of T (tails)
- These are limits: $p = \lim_{n\to\infty} \frac{n_H}{n}$
- Flip 10 times: exactly $2^{10} = 1024$ possible outcomes

$$S_1 = \{HHHHHHHHHH\}$$
$$S_2 = \{HTHHHHHHHH\}$$
$$\dots$$

- Given true p, calculate probability of any S_i:

$$P_i = \lim_{N\to\infty} \frac{n_i}{N}$$

- From the pdf probability, select k samples from a sample space of size F.
- Science/statistics infers from sample space $\mathcal{R}$ to get $p(x)$.
- Not purely deductive: ∞ number of pdfs map to one sample.
- Inductive: If sample pdf is more likely.
- Sample HHTHT ... known, from it we try to infer $p = p_H$.

7.5 Poisson Distribution as a Limit of the Binomial

- Poisson Distribution: limit of binomial for large n, small p.
-
$$B(r, n, p) = \binom{n}{r} p^r (1-p)^{n-r}$$
- Let $n \to \infty$, $p \to 0$ such that $np = \mu$ is constant.
- Stirling approximation:
$$n! \approx \sqrt{2\pi n} n^n e^{-n}$$
- Then:
$$B(r, n, p) = \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r}$$
- r is finite, $n \to \infty$, $n - r \to \infty$

$$\begin{aligned}
B(r, n, p) &= \frac{1}{r!} \frac{\sqrt{2\pi n}}{\sqrt{2\pi(n-r)}} \frac{n^n e^{-n}}{(n-r)^{n-r} e^{-(n-r)}} \left(\frac{\mu}{n}\right)^r \left(1 - \frac{\mu}{n}\right)^{n-r} \\
&= \frac{1}{r!} \sqrt{\frac{n}{n-r}} \left(\frac{n}{n-r}\right)^{n-r} \frac{\mu^r}{e^r} \frac{\left(1 - \frac{\mu}{n}\right)^n}{\left(1 - \frac{\mu}{n}\right)^r} \\
&= \frac{1}{r!} \sqrt{\frac{n}{n-r}} \frac{(1 - \frac{r}{n})^r}{(1 - \frac{r}{n})^n} \frac{\mu^r}{e^r} \frac{\left(1 - \frac{\mu}{n}\right)^n}{\left(1 - \frac{\mu}{n}\right)^r}
\end{aligned}$$

- $n \to \infty$, $\sqrt{\frac{n}{n-r}} \to 1$
- $\left(1 - \frac{\mu}{n}\right)^n \to e^{-\mu}$
- $\left(1 - \frac{\mu}{n}\right)^r \to 1$
- Then:
$$\lim_{n \to \infty, np = \mu} B(r, n, p) = \frac{1}{r!} \mu^r e^{-\mu} = P(r|\mu) = \text{Poisson}$$

7.6 Poisson Process and Radioactive Decay

- Consider radioactive decay of some atoms:
  1. Any time interval $[t, t + dt]$ contains at most one decay.
  2. Probability of a decay occurring in this interval is proportional to dt.
  3. Whether or not an atom decays in the interval is independent of any other non-overlapping interval.
- From (1) and (2):
$$P_d(dt) = \lambda dt$$

- Probability of no decay in interval:

$$P_0(dt) = 1 - \lambda dt$$

- Probability of no decay by time $t + dt$:

$$\begin{aligned}
P_0(t+dt) &= P_0(t)P_0(dt) \\
&= P_0(t)(1 - \lambda dt) \\
P_0(t+dt) - P_0(t) &= -\lambda P_0(t) dt \\
\frac{dP_0(t)}{dt} &= -\lambda P_0(t) \\
P_0(t) &= P_0(0)e^{-\lambda t} = e^{-\lambda t}
\end{aligned}$$

- Probability of getting r decays in time $t + dt$:

$$\begin{aligned}
P_r(t+dt) &= P_r(t)P_0(dt) + P_{r-1}(t)P_d(dt) \\
&= P_r(t)(1 - \lambda dt) + P_{r-1}(t)\lambda dt \\
\frac{dP_r(t)}{dt} &= -\lambda P_r(t) + \lambda P_{r-1}(t)
\end{aligned}$$

- Solution to PDE:

$$P_r(t) = \frac{1}{r!}(\lambda t)^r e^{-\lambda t}$$

- Poisson distribution with $\mu = \lambda t$

7.7 Moments and Variance of the Poisson Distribution

- Properties:

$$\begin{aligned}
E(r) &= \sum_{r=0}^{\infty} r P(r, \mu) \\
&= \sum_{r=0}^{\infty} r \frac{\mu^r}{r!} e^{-\mu} \\
&= \mu e^{-\mu} \sum_{r=1}^{\infty} \frac{\mu^{r-1}}{(r-1)!} \\
&= \mu e^{-\mu} e^{\mu} \\
&= \mu
\end{aligned}$$

- $V(r) = E(r^2) - \mu^2$

$$\begin{aligned}
E[r(r-1)] &= E(r^2) - \mu \\
&= \sum_{r=2}^{\infty} r(r-1) \frac{\mu^r}{r!} e^{-\mu} \\
&= \mu^2 e^{-\mu} \sum_{r=2}^{\infty} \frac{\mu^{r-2}}{(r-2)!} \\
&= \mu^2 e^{-\mu} e^{\mu} = \mu^2
\end{aligned}$$

- $E(r^2) - \mu = \mu^2$
- $V(r) = E(r^2) - \mu^2 = \mu$
- Binomial: $\mu = np$, $V(r) = np(1-p) = \mu$ for $p \to 0$ and $n \to \infty$.

8 Lecture 8

Tuesday, October 7th, 2025

8.1 The Gaussian (Normal) Distribution

$$G(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

8.2 The Standard Normal Distribution

$$N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

8.3 Example: Circular Symmetry (Darts on a Board)

Distribution of darts:

$$f(x,y) = h(x)k(y)$$

Transform to polar coordinates:

$$g(r,\theta) \approx f(x,y) = h(x)k(y), \qquad g(r,\theta) = g(r)$$

$$\frac{\partial g}{\partial \theta} = 0 = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

Coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta = -y, \qquad \frac{\partial y}{\partial \theta} = r\cos\theta = x$$

Condition:

$$0 = h'(x)k(y)(-y) + h(x)k'(y)x$$

$$\frac{h'(x)}{xh(x)} = \frac{k'(y)}{yk(y)} = a \quad \text{(constant)}$$

Solutions:

$$h(x) = ce^{ax^2}, \qquad k(y) = de^{ay^2}$$

$$f(x,y) = Ae^{a(x^2+y^2)} = Ae^{ar^2} \approx Ae^{-r^2}$$

8.4 Expectation Value of a Gaussian

$$E(x) = \mu = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} xe^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \mu$$

Useful identity:

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

8.5 Moments of the Gaussian

The nth central moment:

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)^n e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$$

- All odd moments vanish (symmetry about μ).
Define

$$I_0(a) = \int_{-\infty}^{\infty} e^{-ay^2}dy = \sqrt{\frac{\pi}{a}}$$

Differentiation rule:

$$\frac{\mathrm{d}^n I_0(a)}{\mathrm{d}a^n} = (-1)^n \frac{(2n)!}{n!}\frac{I_0(a)}{(2a)^n}$$

8.6 Variance of the Gaussian

Let $y = x - \mu$. Then

$$V(y) = V(x) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} y^2 e^{-\frac{y^2}{2\sigma^2}}dy$$

$$= -\left.\frac{\mathrm{d}I_0(a)}{\mathrm{d}a}\right|_{a=\frac{1}{2\sigma^2}} = \sigma^2$$

8.7 Kurtosis of the Gaussian

Fourth central moment:

$$\int y^4 e^{-ay^2}dy = \frac{d^2 I_0(a)}{da^2}\frac{1}{4a^2}$$

Evaluates to:

$$E\left[\left(\frac{x-\mu}{\sigma}\right)^4\right] = 3$$

Thus the Gaussian kurtosis = 3. - Excess kurtosis = 0. - Distributions with > 3 have "fat tails."

8.8 Poisson Distribution and Gaussian Limit

$$P(r|\lambda) = \frac{1}{r!}\lambda^r e^{-\lambda}$$

For large r, Stirling approximation:

$$r! \approx \sqrt{2\pi r}\left(\frac{r}{e}\right)^r$$

$$\log P(r|\lambda) = -\log(r!) + r\log\lambda - \lambda$$

Expanding around $r \approx \lambda$ leads to Gaussian limit with variance λ:

$$P(r|\lambda) \approx \frac{1}{\sqrt{2\pi\lambda}}e^{-\frac{(r-\lambda)^2}{2\lambda}}$$

8.9 Central Limit Theorem (CLT)

Let $x_1, \ldots, x_n$ be independent random variables with mean μ, variance σ^2. Define sample mean

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

Then as $n \to \infty$:

$$\bar{x} \sim \mathcal{N}\left(\mu, \, \frac{\sigma^2}{n}\right)$$

8.10 Cumulative Distribution Function of a Gaussian

$$F(x) = \int_{-\infty}^{x} G(y|\mu, \sigma) dy$$

Define error function:

$$\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-y^2} dy, \qquad \mathrm{erfc}(t) = 1 - \mathrm{erf}(t) = \frac{2}{\sqrt{\pi}} \int_t^{\infty} e^{-y^2} dy$$

8.11 Gaussian Confidence Intervals

- 1σ: 68.27%
- 2σ: 95.45%
- 3σ: 99.73%
- 5σ: 99.99994%

8.12 Estimators

Given a sample of size n, an *estimator* is any function designed to estimate a property of the true pdf from which the samples were drawn.

9 Lecture 9

Thursday, October 9th 2025

9.1 Properties of Estimators

- Consistent: $\lim_{n\to\infty} \hat{a} = a$
- Unbiased: $\mathbb{E}[a(\hat{x})] = a$
- Efficient: smallest variance of all unbiased estimators

9.2 Example: Measurements and Models

- Let $x_1, x_2, \ldots, x_n$ be n measurement points.
- Example applications: number of elements in a histogram bin, position of hits in a detector.
- y_i are the measured values at each x_i, with variances $V(y_i) = \sigma_i^2$.
- Suppose we suspect a model for the histogram shape (e.g. linear background + Gaussian signal):

$$\text{Number of entries} = mx_i + b + Ae^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

- More generally, assume a function $f(x, \vec{\theta})$ with parameters $\vec{\theta}$.

9.3 Least Squares Estimation

- The best estimate for $\vec{\theta}$ is the value that minimizes the chi-squared:

$$\chi^2 = \sum_{i=1}^{n} \frac{(y_i - f(x_i, \vec{\theta}))^2}{\sigma_i^2}.$$

- Condition for minimization:

$$\frac{\partial \chi^2}{\partial \theta_j} = 0.$$

- Equivalent system of equations:

$$\sum_{i=1}^{n} \frac{(y_i - f(x_i, \vec{\theta}))}{\sigma_i^2} \frac{\partial f(x_i, \vec{\theta})}{\partial \theta_j} = 0.$$

9.4 Straight Line Fit

- For $f(x_i, \vec{\theta}) = mx_i + b$, with $\vec{\theta} = (m, b)$:

$$\chi^2 = \sum_{i=1}^{n} \frac{(y_i - (mx_i + b))^2}{\sigma_i^2}.$$

- Normal equations from minimization:

$$\frac{\partial \chi^2}{\partial m} = -2\sum_{i=1}^{n} \frac{(y_i - (mx_i + b))}{\sigma_i^2} x_i = 0,$$
$$\frac{\partial \chi^2}{\partial b} = -2\sum_{i=1}^{n} \frac{(y_i - (mx_i + b))}{\sigma_i^2} = 0.$$

- Two equations, two unknowns. Can be written in matrix form:

$$\begin{bmatrix} S_{xx} & S_x \\ S_x & S_1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} S_{xy} \\ S_y \end{bmatrix},$$

where

$$S_{xx} = \sum_i \frac{x_i^2}{\sigma_i^2}, \quad S_x = \sum_i \frac{x_i}{\sigma_i^2}, \quad S_1 = \sum_i \frac{1}{\sigma_i^2},$$
$$S_{xy} = \sum_i \frac{x_i y_i}{\sigma_i^2}, \quad S_y = \sum_i \frac{y_i}{\sigma_i^2}.$$

9.5 Generalized Least Squares with Covariance Matrix

- In general, for non-diagonal covariance matrix V of y_i:

$$\chi^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} (y_i - f(x_i, \vec{\theta}))\, E_{ij}\, (y_j - f(x_j, \vec{\theta})),$$

where $E = V^{-1}$ is the inverse covariance matrix.

- Linear case: if $f(x_i, \vec{\theta})$ is linear in θ:

$$\vec{f} = A\vec{\theta}.$$

- Then

$$\chi^2 = (\vec{y} - A\vec{\theta})^T V^{-1} (\vec{y} - A\vec{\theta}).$$

- Minimization gives:

$$(A^T V^{-1} A)\vec{\theta} = A^T V^{-1} \vec{y},$$
$$\Rightarrow \vec{\theta} = (A^T V^{-1} A)^{-1} A^T V^{-1} \vec{y}.$$

9.6 Covariance of the Estimated Parameters

- Propagation of covariance:

$$V(\vec{y}) = BV(\vec{x})B^T.$$

- For parameter estimates:

$$V(\vec{\theta}) = (A^T V^{-1} A)^{-1}.$$

9.7 Goodness of Fit

- The chi-squared statistic

$$\chi^2 = \sum_{i=1}^{n} \frac{(y_i - f(x_i, \vec{\theta}))^2}{\sigma_i^2}$$

is distributed as χ^2 with n degrees of freedom if the y_i are Gaussian.

- If the model is good, $\chi^2/\text{dof} \sim 1$; if the model is bad, $\chi^2/\text{dof} \gg 1$.

10 Lecture 10

Tuesday, October 14th 2025

In this lecture, I start to not take all notes, instead I write down only the key points.

10.1 Chi-Squared for Uncorrelated and Correlated Measurements

- Recall difference in chi-squared formula for uncorrelated and correlated measurements.
- Uncorrelated:

$$\chi^2 = \sum_{i=1}^{n} \frac{(y_i - \mu_i)^2}{\sigma_i^2}$$

- Correlated (general case):

$$\chi^2 = \sum_{i,j}^{n} (y_i - \mu_i) V_{ij}^{-1} (y_j - \mu_j)$$

  where V^{-1} is the inverse of the covariance matrix V.

10.2 Covariance Matrix and Linear Transformations

- The covariance matrix is defined as

$$V_{ij} = \mathrm{Cov}(y_i, y_j) = \langle (y_i - \mu_i)(y_j - \mu_j) \rangle.$$

- Suppose we apply a linear transformation B to $\vec{y}$, where B is an $n \times n$ matrix of eigenvectors that diagonalizes V.
- Define transformed variables:

$$\vec{z} = B\vec{y}, \quad V_z = BVB^T, \quad \mu_z = B\mu_y.$$

- Then:

$$\boxed{\chi^2(z) = \left(\vec{z} - \vec{\mu_z}\right)^T V_z^{-1} \left(\vec{z} - \vec{\mu_z}\right)}$$

10.3 Modeling Data with Parameters

- Suppose we have a model

$$y_i = f(x_i, \vec{\theta}),$$

  where x_i are independent variables and $\vec{\theta}$ are model parameters.
- Example: $y_i = mx_i + b$, with $\sigma_i \neq \sigma_j$ (heteroscedastic errors).
- We want to find the best estimate of $\vec{\theta}$.

10.4 Least Squares Estimation

- Define:

$$\chi^2(\vec{\theta}) = \sum_{i=1}^{n} \frac{(y_i - f(x_i, \vec{\theta}))^2}{\sigma_i^2}.$$

- The best estimators for μ_i are $f(x_i|\vec{\theta})$ and

$$\chi^2(\hat{\theta}) = \chi^2_{\text{min}} = \sum_i \left(\frac{y_i - f(x_i|\hat{\theta})}{\sigma_i} \right)^2.$$

- Principle of least squares:

$$\hat{\theta} = \text{arg min}\ \chi^2(\vec{\theta})$$

i.e. the value of $\vec{\theta}$ that minimizes $\chi^2(\vec{\theta})$.

- Solution satisfies:

$$\frac{\partial \chi^2}{\partial \theta_j} = 0.$$

10.5 Distribution of Parameter Estimates

- Note: $\hat{\theta}$ is itself a random variable, with its own probability distribution.
- A different sample $\{x_i, y_i\}$ will lead to a different $\hat{\theta}$.

10.6 Quadratic Expansion of chi-squared and Error Estimates

- For polynomial (linear in parameters) fits:

$$f(x_i|\vec{\theta}) = (A\vec{\theta})_i.$$

- Then:

$$\chi^2 = (\vec{y} - A\vec{\theta})^T V^{-1} (\vec{y} - A\vec{\theta})$$
$$\chi^2 = (y_i - A_{im}\theta_m) V_{ij}^{-1} (y_j - A_{jn}\theta_n).$$

- First derivative:

$$\frac{\partial \chi^2}{\partial \theta_k} = -2(y_i - A_{im}\theta_m) V_{ij}^{-1} A_{jk}.$$

- Second derivative:

$$\frac{\partial^2 \chi^2}{\partial \theta_k \partial \theta_l} = 2 A_{il} V_{ij}^{-1} A_{jk} = 2(A^T V^{-1} A)_{kl}.$$

10.7 Covariance of Parameter Estimates

- For linear fits:

$$V(\hat{\theta}) = (A^T V^{-1} A)^{-1}.$$

- For non-linear fits:

$$V^{-1}(\hat{\theta}) = \frac{1}{2} \frac{\partial^2 \chi^2}{\partial \theta_j \partial \theta_k} \bigg|_{\theta = \hat{\theta}}.$$

11 Lecture 11

Thursday, October 16th 2025. I was gone this day, so I copied notes from J. Liang.

11.1 Chi-Squared Minimization and Degrees of Freedom

- Degrees of Freedom (dof):

$$\chi^2(\theta) = (y - A\vec{\theta})^T V^{-1}(y - A\vec{\theta})$$

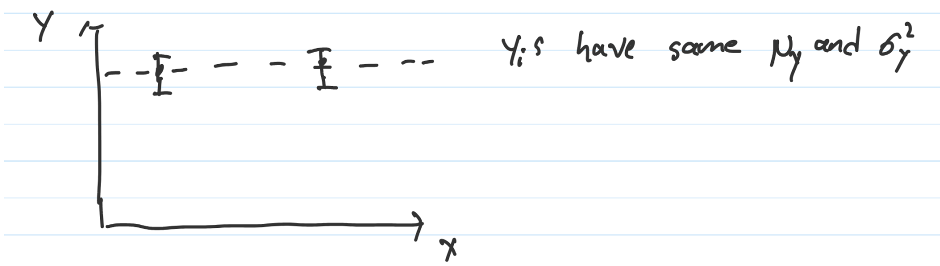
- At $\hat{\theta}$, χ^2 is minimized:

$$\left.\frac{\partial \chi^2}{\partial \theta}\right|_{\theta=\hat{\theta}} = 0 = F(\vec{y}|\vec{\theta},\vec{x}) = \begin{matrix} F(\vec{y}|\theta_1,\vec{x}) \\ F(\vec{y}|\theta_2,\vec{x}) \\ \vdots \\ F(\vec{y}|\theta_k,\vec{x}) \end{matrix}$$

- i.e. for a linear fit $\vec{y} = A\vec{\theta}$ with k equations:

$$\hat{\theta} = (A^T V^{-1} A)^{-1}(A^T V^{-1})\vec{y}$$

- Think about it like this: If I know $n-k$ of the y_i's, the remaining k y_i's are fixed. Their relations might be complex but they are fixed since we have k equations.



y_i's have some N_y and σ_y^2.

11.2 Two-Measurement Example and Correlated Variables

- Best estimator of the true N is $\hat{y} = \frac{y_1+y_2}{2}$.

$$\chi^2 = \left(\frac{y_1 - \hat{N}}{\sigma_y}\right)^2 + \left(\frac{y_2 - \hat{N}}{\sigma_y}\right)^2$$

- Note:

$$y_1 - \hat{N} = \frac{1}{2}(y_1 - y_2)$$
$$y_2 - \hat{N} = \frac{1}{2}(y_2 - y_1)$$

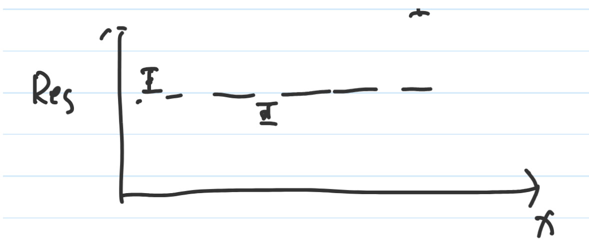
- Where $z_1 = y_1 - y_2$, and $z_2 = y_2 - y_1$ such that $z_2 = -z_1$.

$$V(z_1, z_2) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

- Determinant of V is 0, so it is not invertible.
- Thus we effectively have only one independent variable.

11.3 Residuals and Goodness of Fit

- Residual: $r_i = y_i - f_i(\vec{\theta})$.
- When talking about goodness of fit, people usually divide χ^2 by the number of degrees of freedom (dof). If $\chi^2/\text{dof} \approx 1$, it is a good fit.
- What about residuals?

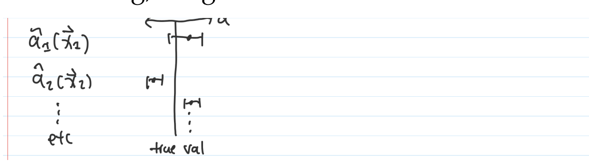


Sometimes one weird data point can throw off the whole χ^2 in unexpected ways, so it is important to check the residuals as well.

- If residuals are randomly scattered around 0, it indicates a good fit.

11.4 Distribution of Estimators

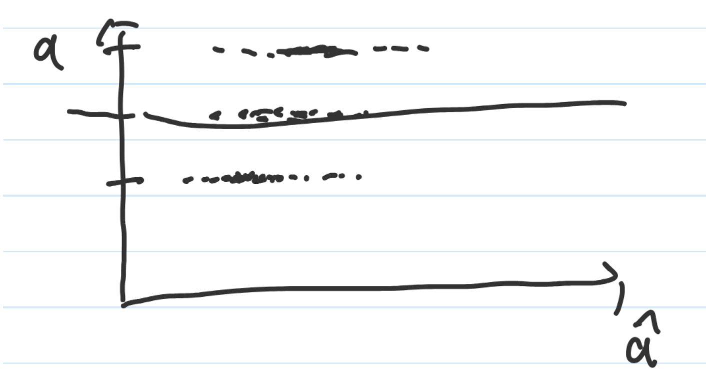
- We have been talking about estimators, but we want values of the parameters.
- Suppose we perform a fit on a parameter with true value a.
- From fitting, we get different estimator values from different data sets.



- These estimates can be scattered over a range of values.

11.5 Toy Monte Carlo Simulations for Estimator Distributions

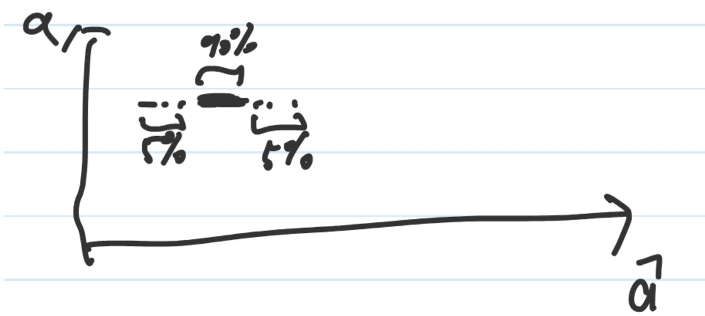
- Toy Monte Carlo:



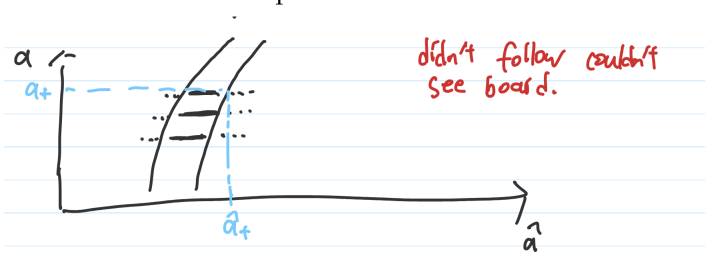
- Often, the model is too complicated to get an analytic form of the estimator distribution.

1. Pick a parameter a.
2. Generate many data sets according to the model with parameter a.
3. For each data set, compute the estimator $\hat{a}$.
4. Plot a histogram of $\hat{a}$.
5. Repeat steps 2–4 n times.
6. Repeat step 1 m times.

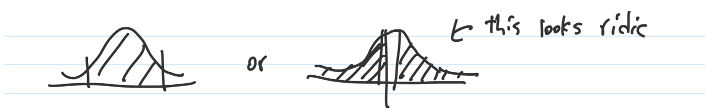
- For each a value, we can determine e.g. 5%, 90%, and 5% quantiles.



- We then connect these points.



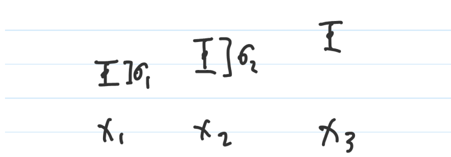
- $P(a_- \le a \le a_+) = 0.90$. Coverage = fraction of time your prescription for the estimator interval contains the true value a.
- In other words, we find or choose $\hat{a}_-$ and $\hat{a}_+$ such that 90% of the time, the true a lies in our interval (corresponding to a_- and a_+).
- The bias in all this is that we can really pick intervals however we want.



11.6 Typical Applications of Least Squares Fitting

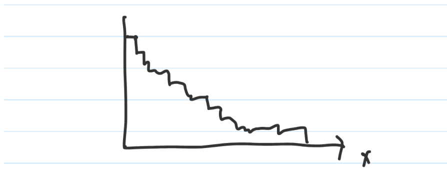
- Typical usage of Least Squares (LS):

1. x_i, y_i, with known σ_x, and known σ_y for y_i.

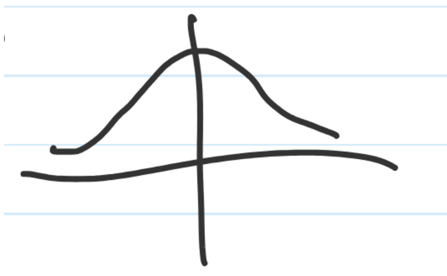


If y_i's are measured by a detector, we can determine σ_i by analyzing the detector.

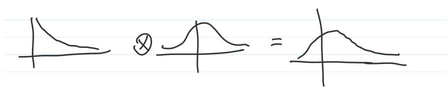
2. Histograms: i.e., measure x on a coordinate, which can have resolution effects, etc., where $y =$ statistics of the number of events at the same x.



Say the resolution has:



so what you see is actually a convolution of the true distribution with the resolution function.



So the fit should actually be an exponential $\circledast$ Gaussian in this case.

$$\chi^2 = \sum_i \frac{(n_i - f_i(\theta))^2}{\sigma_i^2}$$

Suppose we expect Poisson statistics:

$$\sigma_i^2 = n_i$$

Then the data are weighted by n_i:

$$\text{Neyman } \chi^2 \equiv \sum_i \frac{(n_i - f_i(\theta))^2}{n_i}$$

which is a modified least squares form.
Alternatively, it is sensible to say that the expected entries are given by our model:

$$\Rightarrow \text{ use } f_i(\vec{\theta}) \text{ as the mean.}$$

Therefore:

$$\text{Pearson } \chi^2 \equiv \sum_i \frac{(n_i - f_i(\theta))^2}{f_i(\theta)}$$

For Neyman χ^2, if a bin is empty ($n_i = 0$) then it diverges. Both Neyman and Pearson forms are biased in opposite ways. One could use Neyman and Pearson together, but is it worth the effort?

12 Lecture 12

Tuesday, October 21st 2025

12.1 Least-Squares Fits

- Usually we have data $(x_i, y_i \pm \sigma_i)$. We want to fit a model $y = f(x; \theta)$ to the data.

$$\chi^2 = \sum_i \frac{(y_i - f(x_i; \theta))^2}{\sigma_i^2}$$

- Sometimes we are given $y = f(x)$ and sometimes we are given $x = g(y)$.

$$\chi^2 = \sum_i \frac{(x_i - g(y_i; \theta))^2}{\sigma_{x,i}^2}$$

$$\sigma_x = \left|\frac{dg}{dy}\right| \sigma_y$$

- Now that is the end of least-squared fits for a while.

12.2 Unbinned Data and Likelihood Functions

- Now we move on to unbinned data.
- Idea is that you have some data drawn from some probability distribution $P(x; a)$

$$P(t) \sim \frac{1}{\tau} e^{-t/\tau}$$

- Data sample of size n: $\{x_1, x_2, \ldots, x_n\}$
- Form likelihood function:

$$\mathcal{L}(x_1, x_2, \ldots, x_n; a) = \prod_{i=1}^{n} P(x_i; a)$$

- This is equivalent to the probability of getting the data given the parameter a: $P(\vec{x}|a)$.
- Not a probability distribution in a! It is a function of a.
- Suppose we have an estimator $\hat{a}$ for a. Then the expectation value of the estimator is:

$$E[\hat{a}] = \int \hat{a}(\vec{x}) P(\vec{x}; a) d\vec{x}$$

- The maximum likelihood principle states that the best estimate for a is the value $\hat{a}$ that maximizes $\mathcal{L}(\vec{x}; a)$:

$$\left.\frac{\partial \mathcal{L}(\vec{x}; a)}{\partial a}\right|_{a=\hat{a}} = 0$$

- Often easier to maximize $\ln \mathcal{L}$ since ln is monotonic:

$$\ln \mathcal{L}(\vec{x}; a) = \ln \prod_{i=1}^{n} P(x_i; a) = \sum_{i=1}^{n} \ln P(x_i; a)$$

- Then we will find:
  1. max for $\ln \mathcal{L}(\vec{x};\, a)$
  2. min for $-\ln \mathcal{L}(\vec{x};\, a)$
- ML estimators tend to be not unbiased, but consistent, often efficient.

$$\left.\frac{\partial \ln \mathcal{L}(\vec{x};\, a)}{\partial a}\right|_{a=\hat{a}} = 0$$

12.3 Example: Exponential Distribution

- Example: Exponential distribution

$$P(t;\, \tau) = \frac{1}{\tau} e^{-t/\tau}$$

$$\mathcal{L}(t_1, t_2, \ldots, t_n;\, \tau) = \prod_{i=1}^{n} \frac{1}{\tau} e^{-t_i/\tau} = \sum_{i=1}^{n} \ln\left(\frac{1}{\tau} e^{-t_i/\tau}\right) = -n \ln \tau - \frac{1}{\tau} \sum_{i=1}^{n} t_i$$

$$\frac{\partial \ln \mathcal{L}}{\partial \tau} = 0 = -\frac{n}{\tau} + \frac{1}{\tau^2} \sum_{i=1}^{n} t_i$$

$$\hat{\tau} = \frac{1}{n} \sum_{i=1}^{n} t_i = \bar{t}$$

  This is unbiased.

- $$E(\hat{\tau}) = \frac{1}{n} E(\sum_{i=1}^{n} t_i) = \frac{1}{n} \sum_{i=1}^{n} E(t_i) = \frac{1}{n} n\tau = \tau$$

12.4 Example: Lifetime with Cutoff T

- (Not normalized P:)

$$P(t|\tau) = \begin{cases} \frac{1}{\tau} e^{-t/\tau} / e^{-T/\tau} & 0 \geq t \geq T \\ 0 & \text{otherwise} \end{cases}$$

- Now must normalize! $\sum_{t=T}^{\infty} P(t|\tau) = 1$

$$P(t|\tau) = \frac{1}{(1 - e^{-T/\tau})\tau} e^{-t/\tau}$$

- Log-likelihood:

$$\ln \mathcal{L} = \sum_{i=1}^{n} \left[\ln\left((1 - e^{-T/\tau})\tau\right)\right] - \frac{1}{\tau} \sum_{i=1}^{n} t_i$$

$$= -n \ln\Big(1 - e^{-T/\tau}\Big) - n \ln \tau - \frac{1}{\tau} \sum_{i=1}^{n} t_i$$

- Set derivative to zero:

$$\frac{\partial \ln \mathcal{L}}{\partial \tau} = 0$$

$$\frac{\partial \ln \mathcal{L}}{\partial \tau} = \frac{-n(-e^{-T/\tau})(\frac{T}{\tau^2})}{(1 - e^{-T/\tau})} - \frac{n}{\tau} + \frac{1}{\tau^2} \sum_{i=1}^{n} t_i = 0$$

12.5 Example: Multiple Gaussian Measurements

- Multiple measurements of some quantity:

$$P(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

$$\ln \mathcal{L} = -\sum_i \ln\left(\sqrt{2\pi}\right) - \sum_i \ln \sigma - \frac{1}{2}\sum_i \frac{(x_i - \mu)^2}{\sigma^2}$$

$$\frac{\partial \ln \mathcal{L}}{\partial \mu} = 0 = \frac{-1}{2}(-2)\sum_i \frac{(x_i - \mu)}{\sigma^2}$$

$$= \frac{1}{\sigma^2}\sum_i (x_i - \hat{\mu}) = 0$$

$$\sum_i x_i = n\hat{\mu} \Rightarrow \hat{\mu} = \frac{1}{n}\sum_i x_i = \bar{x}$$

- Now for σ:

$$\left.\frac{\partial \ln \mathcal{L}}{\partial \sigma}\right|_{\hat{\sigma},\hat{\mu}} = 0 = -\frac{n}{\sigma} - \frac{1}{2}\left(\frac{-2}{\sigma^3}\right)\sum_i (x_i - \mu)^2$$

$$-n\hat{\sigma^2} + \sum_i (x_i - \hat{\mu})^2 = 0$$

$$\hat{\sigma^2} = \frac{1}{n}\sum_i (x_i - \hat{\mu})^2 = \frac{1}{n}\sum_i (x_i - \bar{x})^2$$

- This is a biased estimator for σ^2. Unbiased is with $1/(n-1)$.

12.6 Properties of the Maximum-Likelihood Estimator

- This next stuff is not really testable but can be interesting to see where it comes from.
- To avoid confusion call a_0 the true value of a.
- We have $\mathcal{L}(\vec{x};\, a)$ and we want to know how well $\hat{a}$ estimates a_0.

$$\left.\frac{\partial \ln \mathcal{L}(\vec{x};\, a)}{\partial a}\right|_{a=\hat{a}} = 0$$

- Taylor expand around a_0:

$$f(\hat{a}) = f(a_0) + f'(a_0)(\hat{a} - a_0) + \frac{1}{2}f''(a_0)(\hat{a} - a_0)^2 + \dots$$

- So we have:

$$\left.\frac{\partial \ln \mathcal{L}(\vec{x};\, a)}{\partial a}\right|_{a=\hat{a}} = \left.\frac{\partial \ln \mathcal{L}(\vec{x};\, a)}{\partial a}\right|_{a=a_0} + (\hat{a} - a_0)\left.\frac{\partial^2 \ln \mathcal{L}(\vec{x};\, a)}{\partial a^2}\right|_{a=a_0} + \dots = 0$$

- For $n \to \infty$, $\hat{a} \to a_0$ (consistent estimator), so we can neglect higher order terms.

$$\left.\frac{\partial \ln \mathcal{L}(\vec{x};\, a)}{\partial a}\right|_{a=a_0} \to \left.\frac{\partial \ln \mathcal{L}(\vec{x};\, a)}{\partial a}\right|_{a=\hat{a}} = 0$$

13 Lecture 13

Thursday, October 23rd 2025

13.1 Definition of the Likelihood Function

- Likelihood

$$\mathcal{L}(\vec{x}|\theta) = \text{probability of x and not theta}$$

- Model $\vec{\theta}$ with $p(\vec{x}|\vec{\theta})$

$$\mathcal{L} = \prod_{i=1}^{N} p(x_i|\vec{\theta})$$

$$\ln \mathcal{L} = \sum_{i=1}^{N} \ln p(x_i|\vec{\theta})$$

13.2 Maximum Likelihood Estimation (MLE)

- Maximize likelihood to get best estimate of $\vec{\theta}$. Choose $\hat{\theta}$ such that:

$$\hat{\theta} = \text{argmax}_{\vec{\theta}} \mathcal{L}(\vec{x}|\vec{\theta})$$

$$\left.\frac{\partial \ln \mathcal{L}(\vec{\subseteq})}{\partial \theta}\right|_{\hat{\theta}} = 0$$

13.3 Quadratic Approximation of the Log-Likelihood

- Shape of $\vec{a}$ distribution: $\ln \mathcal{L}(a)$ around $\hat{a}$ is approximately quadratic.
- True $a = \hat{a}$; expand about a_0:
- Taylor expansion:

$$f(a) = f(a_0) + (\hat{a} - a_0) f'(a_\star) \qquad \text{where } a_\star \text{ is between } \hat{a} \text{ and } a_0$$

- So, for $f = \frac{\partial \ln \mathcal{L}(a)}{\partial a}$:

$$0 = \left.\frac{\partial \ln \mathcal{L}(a)}{\partial a}\right|_{a_0} + (\hat{a} - a_0) \left.\frac{\partial^2 \ln \mathcal{L}(a)}{\partial a^2}\right|_{a_\star}$$

13.4 Asymptotic Limit and Expectation Relation

- Large n for consistent $\hat{a} \to a_0$:

$$\lim_{n\to\infty} \left.\frac{\partial^2 \ln \mathcal{L}(a)}{\partial a^2}\right|_{a_\star} = \lim_{n\to\infty} \sum_i \left.\frac{\partial^2 \ln p(x_i|a)}{\partial a^2}\right|_{a_\star} \approx \lim_n n \int p(x|a) \left.\frac{\partial^2 \ln p(x|a)}{\partial a^2}\right|_{a_\star} dx$$

- Sum over samples x_i drawn from $p(x|a)$:

$$= \lim_{n\to\infty} nE \left(\left.\frac{\partial^2 \ln p(x|a)}{\partial a^2}\right|_{a_\star} \right)$$

$$= E\left(\left.\frac{\partial^2 \ln \mathcal{L}}{\partial a^2}\right|_{a_\star}\right)$$

$$\hat{a} - a_0 = -\frac{\left.\frac{\partial \ln \mathcal{L}(a)}{\partial a}\right|_{a_0}}{E\left(\left.\frac{\partial^2 \ln \mathcal{L}}{\partial a^2}\right|_{a_\star}\right)}$$

$$0 = \left.\frac{\partial \ln \mathcal{L}(a)}{\partial a}\right|_{a_0} + (\hat{a} - a_0)\left.\frac{\partial^2 \ln \mathcal{L}(a)}{\partial a^2}\right|_{a_\star}$$

13.5 Normalization of the Likelihood Function

- The likelihood is normalized:

$$\int \mathcal{L}(\vec{x}|a)d\vec{x} = 1$$

$$\Rightarrow \int \frac{\partial \mathcal{L}(\vec{x}|a)}{\partial a}d\vec{x} = 0$$

- Relation between $\mathcal{L}$ and $\ln \mathcal{L}$:

$$\frac{\partial \mathcal{L}}{\partial a} = \frac{\partial \ln \mathcal{L}}{\partial a}\mathcal{L}$$

$$\frac{\partial \ln \mathcal{L}}{\partial a} = \sum_{i=1}^{n} \frac{\partial \ln p(x|a)}{\partial a}$$

13.6 Gaussian Approximation via the Central Limit Theorem

- The sum of n variables with zero mean:
- By the Central Limit Theorem, for large n, $\frac{\partial \ln \mathcal{L}}{\partial a}$ is Gaussian with mean 0.

$$\boxed{E\left(\frac{\partial^2 \ln \mathcal{L}}{\partial a^2}\right) = -E\left(\left(\frac{\partial \ln \mathcal{L}}{\partial a}\right)^2\right)}$$

13.7 Variance of the Estimator and the Fisher Information

- Variance of $\hat{a} - a_0$:

$$\mathrm{Var}(\hat{a} - a_0) = \frac{\mathrm{Var}\left(\left.\frac{\partial \ln \mathcal{L}(a)}{\partial a}\right|_{a_0}\right)}{\left(E\left(\left.\frac{\partial^2 \ln \mathcal{L}}{\partial a^2}\right|_{a_\star}\right)\right)^2}$$

$$= \frac{E\left(\left(\left.\frac{\partial \ln \mathcal{L}(a)}{\partial a}\right|_{a_0}\right)^2\right)}{\left(E\left(\left.\frac{\partial^2 \ln \mathcal{L}}{\partial a^2}\right|_{a_\star}\right)\right)^2}$$

$$= -\frac{E\left(\left.\frac{\partial^2 \ln \mathcal{L}}{\partial a^2}\right|_{a_0}\right)}{\left(E\left(\left.\frac{\partial^2 \ln \mathcal{L}}{\partial a^2}\right|_{a_\star}\right)\right)^2}$$

- When $n \to \infty$, $a_\star \to a_0$ and $\hat{a} \to a_0$:

$$\boxed{\mathrm{Var}(\hat{a}) = -\frac{1}{E\left(\left.\frac{\partial^2 \ln \mathcal{L}}{\partial a^2}\right|_{a_0}\right)}}$$

- Fisher Information matrix:

$$E\left(\frac{\partial^2 \ln \mathcal{L}}{\partial a^2}\right)$$

- For large n, $\hat{a} \to a_0$. Estimate $E(\cdot)$ by the observed value:

$$\left.\frac{\partial^2 \ln \mathcal{L}}{\partial a^2}\right|_{\hat{a}}$$

- So the estimate of variance of $\hat{a} - a_0$ is:

$$\boxed{\mathrm{Var}(\hat{a} - a_0) = -\frac{1}{\left.\frac{\partial^2 \ln \mathcal{L}}{\partial a^2}\right|_{\hat{a}}}}$$

13.8 Taylor Expansion Near the Maximum Likelihood Estimate

- Taylor expansion again:

$$\frac{\partial \ln \mathcal{L}(a)}{\partial a} = \cancelto{0}{\left.\frac{\partial \ln \mathcal{L}(a)}{\partial a}\right|_{\hat{a}}} + (a - \hat{a})\left.\frac{\partial^2 \ln \mathcal{L}(a)}{\partial a^2}\right|_{\hat{a}} + \ldots$$

where $\left.\frac{\partial^2 \ln \mathcal{L}(a)}{\partial a^2}\right|_{\hat{a}} = -\frac{1}{V(\hat{a})}$

$$\frac{-(a - \hat{a})}{V(\hat{a})} + \ldots$$

- Note that $V(\hat{a} - a_0) = V(\hat{a})$ because a_0 is constant and does not change the variance (it just shifts the distribution).

- So,

$$\ln \mathcal{L}(a) = \ldots \text{ missed this part}$$

13.9 Goodness of Fit and the Kolmogorov–Smirnov Test

- Note that the value you get from the maximum likelihood does not give information on how good the fit is—it is just relative to other values of the parameters.
- Kolmogorov–Smirnov test for goodness of fit (KS):
  1. Order data points $\{t_i\}$ such that $t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_N$
  2. Form an accumulator F (same model CDF C).
- Metric:

$$\max |F(t_i) - C(t_i)|$$