PHY509 – Problem Set 1

1. Prove that the conditional probability $P(\cdot|F)$, for any event F in S is indeed a probability (ie satisfies the axioms), where the dot represents the location of the function input. For example, one axiom is $P(S|F) = 1$

2. For any 3 events E, F, G express $P(E \cup F \cup G)$ in terms of the probabilities for E, F, G, the products EF, FG, EG and EFG.

3. The following data were reported in a study of a group of 1000 people: there were 312 professionals, 470 married persons, 525 college graduates, 42 professional college graduates, 147 married college graduates, 86 married professionals, and 25 married professional college graduates. Show that the numbers reported in the study must be incorrect. Hint: Use the results of the previous problem.

4. Let E, F, G be events. Find expressions of the following events

   (a) only E occurs
   (b) both E and G, but not F, occur
   (c) at least one of the events occurs
   (d) at least two of the events occur
   (e) all three events occur
   (f) none of the events occur
   (g) at most one of the events occurs
   (h) at most two of the events occur

5. A pair of dice is rolled until a sum of 5 or 7 appears. Find the probability that a 5 occurs first. *Hint:* Calculate the probability of E_n = the event that a 5 occurs on the n^{th} roll and no 5 or 7 occurs on the first $n-1$ rolls. Combine these to create a union that represents the set of all the possible specified outcomes, and calculate its probability.

6. Suppose you have an experiment with sample space S that you then perform n times. For any event E in S, let $n(E)$ be the number of times the event E occurred, and define $f(E) = n(E)/n$. Show that $f(\cdot)$ is a probability.

7. Tchebysheff's Inequality: Let $g(x)$ be a non-negative function of the random variable x, which has a probability density function $p(x)$ of variance σ^2. Show that, if the expectation value for g exists (i.e. $E(g)$ is finite), then the set of x with $g(x) \geq c$ satisfies

$$P(x \text{ with } g(x) \geq c) \leq \frac{1}{c} E(g(x)) \tag{1}$$

In particular, set $g(x) = (x - E(x))^2$, and $c = \lambda^2\sigma^2$, and so prove that the fraction of the distribution $p(x)$ that is more than $\lambda\sigma$ away from the mean satisfies

$$P(x \text{ with } |x - E(x)| \geq \lambda\sigma) \leq \frac{1}{\lambda^2}. \tag{2}$$

This is known as the *Tchebysheff Inequality.*

8. Write a program to simulate an experiment which has 100 independent Bernoulli trials each with a probability $p = 0.2$ of success. That is, this pseudo-experiment has some random number of successes, we hope with an expected mean of 20. Now, write another loop that runs this pseudo-experiment N times. Make a histogram of the pseudo-experiment results when $N = 100$. Overlay a histogram of the Binomial distribution, suitably normalized to the number of entries N. Repeat for $N = 1000$ and $N = 100000$.

9. Prove that the correlation coefficient between any two variables satisfies

$$-1 \leq \rho(x, y) \leq 1 \tag{3}$$

10. Suppose you measure the position of an object on a plane as (x, y), that the uncertainties ‘on the measurements are σ_x, σ_y and that these are uncorrelated. If you express the position in polar coordinates r, θ, with $x = r\cos\theta, y = r\sin\theta$, what is the covariance matrix $V(r, \theta)$? Under what conditions is the covariance (and so correlation coeff) between r, θ zero? When is it positive, and when negative? Draw sketches of the measurement uncertainty ellipses in the various cases to demonstrate the answer?

11. Given uncorrelated original probability densities for x, y (i.e. $V_{ij} = \mathrm{diag}(\sigma_x^2, \sigma_y^2)$), find $\sigma_z = \sqrt{V_z}$ for the following functional transformations:

(a) $z = x^2$

(b) $z = \sin(ax), a =$ const

(c) $z = e^{-x/\tau}, \tau =$ const

(d) $z = \log(x)$

(e) $z = x + y$

(f) $z = x - y$

(g) $z = x * y$

(h) $z = x/y$

12. Suppose you measure the voltage V across a resistor of resistance R, and you estimate the square root of the variances of the underlying probability distributions of the measurements to be σ_V and σ_R respectively. We then estimate the Power $P = V^2/R$ and Current $I = V/R$.

    (a) Calculate the covariance matrix $V(P, I)$, starting from the covariance matrix $V(V, R)$ (sorry for the double use of the symbol V, but it should be clear which one is meant from the context).

    (b) You can also express $P = VI$. So you could do the calculation in 2 steps: First calculate I, then calculate P from V and I, rather than V, R. Do this, (i.e. first transform from V, R to V, I, then transform those variables to P, I) and compare the final covariance matrix you get this way with the direct transformation.

13. The Geometric Distribution.
    Consider a Bernoulli process, with probability of success in each trial being p.

    (a) Show that the probability that the first success occurs on the n^{th} trial is given by the *Geometric Distribution*

    $$P_1(n; p) = p(1-p)^{n-1} \tag{4}$$

    Verify that this gives a properly normalized set of probabilities, i.e.

    $$\sum_{n=1}^{\infty} P_1(n; p) = 1 \tag{5}$$

    (b) Show that the mean and variance of this distribution are:

    $$E(r) = 1/p \tag{6}$$
    $$V(r) = (1-p)/p^2 \tag{7}$$

(c) Make a plot of this distribution for $p = 0.4$.

14. The Negative Binomial Distribution.
Suppose, instead of the usual binomial situation in which we fix the number of trials n and ask how many successes r we get, we reverse the problem – we keep performing trials *until* we get r successes and ask how many trials n we had to make.

  (a) Generalize the argument for the geometric distribution to show that the probability that the r^{th} success is on the n^{th} trial is given by the *negative binomial distribution*

$$P_r(n;p) = \binom{n-1}{r-1} p^r (1-p)^{n-r}, \quad n = r, r+1, ... \tag{8}$$

  (b) Show that the mean and variance of this distribution are:

$$E(n) = r/p \tag{9}$$
$$V(n) = r(1-p)/p^2 \tag{10}$$

  (c) Make a plot of this distribution for $p = 0.4, r = 10$ and $p = 0.6, r = 5$.