WEEK 1: SINGLE QUBIT STATES

The questions

1. What is a qubit?

2. Describe the qubit's Hilbert space (the structure of the state vector, i.e. the "wavefunction"). How many (physically relevant) degrees of freedom does the qubit's state have?

3. Consider the so-called Pauli matrices σ_x, σ_y, σ_z. For a spin-1/2 system, find the quantum states which are the eigenstates of these Pauli matrices. What is the physical meaning of such quantum states (google it)?

4. To describe the physical behaviour of the qubit, one needs to give the Hamiltonian for it. What would be the general form of such a Hamiltonian? How many degrees of freedom (d.o.f.) does it have? How to express it with the Pauli matrices?

5. Consider the hermitian operator $\vec{n} \cdot \vec{\sigma} \equiv n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$, for $n_x^2 + n_y^2 + n_z^2 = 1$. What are the eigenvalues of such a combination? What are the eigenstates? How to use these eigenstates to recover the eigenstates of the Hamiltonian parametrized as in the previous exercize?

6. Consider a unitary transformation of a qubit generated by the σ_z, σ_y Pauli matrices, i.e. $\exp(i\phi\sigma_z)$ and $\exp(i\theta\sigma_y)$. What do they do to an arbitrary qubit state?

7. To properly represent the d.o.f. that the qubit has, one needs to come up with a smart parametrization of the states. Google the Bloch sphere parametrization: how does it work? How to understand it as a sequence of unitary rotations from the previous question, applied to a σ_z-polarized state?

8. Interprete the unit vector $\vec{n}$ from p.5, as a vector on a sphere with coordinates θ, ϕ. How to understand the Bloch sphere states in terms of the eigenstates of the matrix $\vec{n} \cdot \vec{\sigma}$?

The hints

1. A two-level quantum system which evolves under external control. (What does this mean? Possible examples?)

2. (a) Hilbert space is the vector space where the states of the system live. The only thing you need is to determine its dimensionality; whether it's a complex or a real vector space; and what's the physical interpretation of its basis vectors (as quantum states).

   (b) For the degrees of freedom, take into account the normalization condition and the fact that the overall phase is physically irrelevant.

3. To find the eigensystem of a Pauli matrix, solve the characteristic equation. Is there a faster way to do it? (think of the following properties: $\sigma_\alpha^2 = 1$, tr $\sigma_\alpha = 0$)

4. (a) The Hamiltonian for a quantum system with d-dimensional Hilbert space is a d x d Hermitian matrix. What would be the general form of such a matrix in this case?

   (b) Express the general Hamiltonian as a linear combination of the three Pauli matrices, and the fourth matrix, namely the identity.

5. To find the eigenvalues of $\vec{n} \cdot \vec{\sigma}$, consider the identity $(\vec{n} \cdot \vec{\sigma})^2 = 1$, tr $\vec{n} \cdot \vec{\sigma} = 0$. For the eigenstates, solve the characteristic equation. Try to simplify your answers as much as possible, so that they take the neat form.

6. To give a nice expression for the exponential, consider the series for the exponential function, and the property $\sigma_\alpha^2 = 1$. You can use the outcome to evaluate the action on the qubit state.

7. Try $\exp(i\theta\sigma_y)\exp(i\phi\sigma_z)$ applied to a σ_z-polarized state. How to relate the answer to the expression that you found in the literature?

8. For this, try to massage the answer for the eigenstates from p.5, in terms of the spherical coordinates of $\vec{n}$, θ and ϕ.

WEEK 2: QUANTUM CORRELATIONS AND ENTANGLEMENT

Some preliminaries

The Hilbert space of N-qubit quantum states is a tensor product of N single-qubit Hilbert spaces. This can be represented as a Kronecker product, which is a space of 2^N-dimensional vectors with a specific meaning associated to its components. For 2 qubits, the space is 4-dimensional and the vector components are $(c_{00}, c_{01}, c_{10}, c_{11})$. The state will then be:

$$|\Psi\rangle = c_{00}\,|00\rangle + c_{01}\,|01\rangle + c_{10}\,|10\rangle + c_{11}\,|11\rangle\,. \tag{1}$$

The following notation will be used interchangeably: $|00\rangle$ ($|01\rangle$) and $|0\rangle \otimes |0\rangle$ ($|0\rangle \otimes |1\rangle$). The state $|\Psi\rangle$ is called a tensor product of $|\psi\rangle^A$ and $|\psi\rangle^B$, if its components are related to the components of $|\psi\rangle^{A,B}$ in the following way:

$$|\Psi\rangle \equiv |\psi\rangle^A \otimes |\psi\rangle^B = c_0^A c_0^B\,|00\rangle + c_0^A c_1^B\,|01\rangle + c_1^A c_0^B\,|10\rangle + c_1^A c_1^B\,|11\rangle\,, \text{ for: } |\psi\rangle^{A,B} = c_0^{A,B}\,|0\rangle + c_1^{A,B}\,|1\rangle \tag{2}$$

The operator O is called to be a tensor product of O^A and O^B, if its action on the tensor product states gives a tensor product of actions on the tensor factors:

$$O(|\psi\rangle^A \otimes |\psi\rangle^B) = (O^A \otimes O_B)\,|\psi\rangle^A \otimes |\psi\rangle^B \equiv (O^A\,|\psi\rangle^A) \otimes (O^B\,|\psi\rangle^B) \tag{3}$$

The questions

1. The Kronecker product form of the tensor product $O^A \otimes O^B$ for qubits A, B is the following 4x4 matrix:

$$O^A \otimes O^B = \begin{pmatrix} (O^A)^0_0 \cdot O^B & (O^A)^0_1 \cdot O^B \\ (O^A)^1_0 \cdot O^B & (O^A)^1_1 \cdot O^B \end{pmatrix} \tag{4}$$

   Check that this is equivalent to the definition of the tensor product above, by checking the specific example of $\sigma_z \otimes \sigma_y$. Note that it is sufficient to check how it acts on the states $|00\rangle\,, |01\rangle\,, |10\rangle\,, |11\rangle$.

2. Show that $(O^A \otimes O^B)(\tilde{O}^A \otimes \tilde{O}^B) = O^A\tilde{O}^A \otimes O^B\tilde{O}^B$ for general operators O^A, O^B, $\tilde{O}^A$, $\tilde{O}^B$. Then, show that $(\sigma_\alpha \otimes \sigma_\beta)^2 = I^A \otimes I^B$.

3. Evaluate the matrix exponent $U(\theta) = e^{i\theta\sigma_y^A\sigma_x^B}$. Evaluate a θ-dependent state: $|\Psi(\theta)\rangle = U(\theta)\,|00\rangle$. Calculate the following expectation values in the state $|\Psi(\theta)\rangle$: $S_A = \langle\frac{\sigma_x^A + \sigma_z^A}{\sqrt{2}}\rangle$, $S_B = \langle\frac{\sigma_x^B + \sigma_z^B}{\sqrt{2}}\rangle$, $S_{AB} = \langle\frac{\sigma_x^A + \sigma_z^A}{\sqrt{2}} \cdot \frac{\sigma_x^B + \sigma_z^B}{\sqrt{2}}\rangle$. For which states $|\Psi(\theta)\rangle$ does the following hold: $S_{AB} = S_A S_B$?

4. What is a density matrix? How to understand, if the density matrix corresponds to a pure state (check the property $\rho^2 = \rho$)? Use Sakurai's book to check this and other basic facts about the density matrices.

   Check if the following one-qubit density matrices correspond to pure states: $\rho = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $\rho = \frac{1}{2}\begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}$.

5. What is a partial trace of a density matrix? What is a reduced density matrix of a state? Find a reduced density matrix of a qubit A that comes from the state $|\Psi(\theta)\rangle$ (from the previous exercise). I.e., evaluate $\rho_A(\theta) = \mathrm{tr}_B\left(|\Psi(\theta)\rangle \langle\Psi(\theta)|\right)$. For which θ does this reduced density matrix correspond to a pure state?

6. Consider a state $|\Psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$. Calculate the expectation values S_A, S_B, S_{AB} in this state. Does the following hold: $S_{AB} = S_A S_B$? Evaluate a reduced density matrix in this state, $\rho_A = \mathrm{tr}_B\left(|\Psi\rangle \langle\Psi|\right)$. Does this density matrix correspond to a pure state?

7. Given the information you obtained about the state $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$, would you conclude that this state is a product or a non-product (entangled) state? If it's a product state, what are the tensor factors of it?