Finite sample properties

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For the fractional unobserved components model, this section examines the finite sample properties of the proposed estimation methods relative to popular competitors for the data-generating mechanism

$$y_t = d_t + x_t + c_t + u_t, \qquad \Delta_+^d x_t = \eta_t, \qquad c_t - b_1 c_{t-1} - b_2 c_{t-2} = \epsilon_t,$$

where d_t controls for deterministic terms, u_t is an additional measurement error, $b_{1,0} = 1.6$, $b_{2,0} = -0.8$ reflect strong cyclical patterns, and

$$\mathrm{Var}\begin{pmatrix}\eta_t \\ \epsilon_t\end{pmatrix} = Q.$$

The subsection below vary over Q, d_t, u_t. They consider sample sizes $n \in \{100, 200, 300\}$, integration order $d_0 \in \{0.75, 1.00, 1.75\}$, and variance ratios

$$\nu_0 \in \left\{1, \frac{n^{-1}\sum_{t=1}^n \sum_{j=0}^{t-1} \pi_j^2(-d_0)}{\sum_{j=0}^\infty a_j^2(\varphi_0)} r^{-1}\right\}, \qquad r \in \{0.1, 1, 10, 30\}, \tag{1}$$

with $\varphi_0 = (b_{1,0}, b_{2,0})$, and $a(L, \varphi_0) = \sum_{j=0}^\infty a_j(\varphi_0) L^j = (1 - b_{1,0}L - b_{2,0}L^2)^{-1}$. While the choices for n and d_0 cover empirically relevant sample sizes and integration orders in macroeconomics and finance and allow for a comparison with the $I(1)$ UC model when $d_0 = 1$, the choice for ν_0 is justified as follows: Trivially, setting $\nu_0 = \sigma_{\epsilon,0}^2/\sigma_{\eta,0}^2 = 1$ assigns equal variation to long- and short-run innovations. By its non-stationary nature, the trend then dominates the overall variance of y_t, i.e. $\mathrm{Var}(x_t)/\mathrm{Var}(c_t) = O(t^{2d_0-1})$, which constitutes a favorable scenario for estimating d_0. At contrast, letting ν_0 depend on d_0 and φ_0 controls for the diverging variance ratio: The numerator is the mean variance of the trend component (under $\sigma_{\eta,0}^2 = 1$ as in the simulations), hence $\mathrm{Var}(c_t) = \mathrm{Var}(a(L,\varphi_0)\epsilon_t) = \nu \sum_{j=0}^\infty a_j^2(\varphi_0) = r^{-1} n^{-1} \sum_{t=1}^n \sum_{j=0}^{t-1} \pi_j^2(-d_0)$ is proportional to the mean variance of x_t, and $n^{-1}\sum_{t=1}^n \mathrm{Var}(x_t)/\mathrm{Var}(c_t) = r$. This fixes the variance ratio of x_t and c_t (instead of $\sigma_{\epsilon,0}^2/\sigma_{\eta,0}^2$), and the lower r, the weaker the relative contribution of the trend to the overall variation, and the less favorable the scenario for estimating d_0 and x_t.

Each Monte Carlo simulation consists of 1000 replications. For the QML estimator, the trend is initialized with variance zero (as implied by the type II definition of long memory), whereas the cycle is initialized with its long-run variance. Once the prediction error variance satisfies $\left|\frac{\mathrm{Var}_\theta(v_{t+1}(\theta)|y_1,\dots,y_t) - \mathrm{Var}_\theta(v_t(\theta)|y_1,\dots,y_{t-1})}{\mathrm{Var}_\theta(v_t(\theta)|y_1,\dots,y_{t-1})}\right| < 0.01$, the optimization switches to the steady-state Kalman filter, which assumes the prediction error variance to be constant from that point on. Both CSS and QML estimator are initialized by first evaluating their objective function at a large grid for the model parameters, and the grid point referring to the lowest value of the CSS objective function or the lowest negative likelihood is chosen as the starting point for numerical optimization. As a benchmark, the exact local Whittle estimator of Shimotsu and Phillips (2005) is introduced, using $m = \lfloor n^j \rfloor$ Fourier frequencies, $j \in \{.50, .60, .70\}$. Moreover, CSS and QML estimation results from the $I(1)$

UC model (setting $d = 1$) are reported, as well as QML estimates from the approximate fractional UC model of Hartl and Jucknewitz (2022). The latter approximates the fractional differencing operator by an ARMA(3, 3) polynomial, which yields a low-dimensional state space model so that Kalman filter and smoother remain computationally feasible. Parameter estimates are compared by the root mean squared error (RMSE), as well as by the bias. To judge trend and cycle estimates, the coefficients of determination R_x^2 and R_c^2 from regressing x_t and c_t on their estimates from the Kalman smoother are reported.

1 Uncorrelated innovations

The first simulation considers the prototypical fractional UC model with no measurement error $u_t = 0$, no deterministic terms $d_t = 0$, and diagonal covariance matrix Q. The resulting fractional UC model is thus

$$y_t = x_t + c_t, \qquad \Delta_+^d x_t = \eta_t, \qquad c_t - b_1 c_{t-1} - b_2 c_{t-2} = \epsilon_t, \qquad Q = \begin{pmatrix} 1 & 0 \\ 0 & \nu \end{pmatrix}.$$

Table 1 shows the RMSE and the bias for the estimated integration orders. As can be expected, bias and RMSE decrease both in n and r, and are significantly smaller as compared to the nonparametric Whittle estimators. The difference is particularly striking when the signal of the trend is drowned by the cycle (i.e. r_0 small / ν_0 large), which biases the Whittle estimates towards zero, whereas the estimates for the fractional UC model are hardly affected. Noticeably, for the fractional UC model QML is slightly superior to CSS, and using ARMA approximations as suggested by [**?**] yields estimates that are close to the exact fractional UC models in terms of RMSE and bias.

Table 1: Simulation with uncorrelated innovations: root mean squared errors (RMSE) and bias for the integration order estimates.

Tables 2 and 3 detail RMSE and bias for ν_0 and the autoregressive parameters. In addition to the fractional UC model, the table also displays the estimation results for the $I(1)$ UC benchmark. For $b_{1,0}$ and $b_{2,0}$, CSS and QML estimates for the fractional UC model behave equally well, while those from the approximate fractional UC model and the integer-integrated benchmarks come with a slightly higher RMSE, particularly for $d_0 = 0.75$. Interestingly, the estimates for ν_0 show a clear dominance of QML over CSS, as the latter comes with a much higher RMSE and strong, positive bias. QML - at contrast - does not appear biased. Misspecifying $d = 1$ in integer-integrated models has a rather small effect on the estimates for the cyclical autoregressive coefficients, but a comparably strong effect on the estimate for ν_0, thus shifting variation from trend to cycle and vice versa.

Table 2: Simulation with uncorrelated innovations: root mean squared errors (RMSE) for the other parameter estimates.

Table 3: Simulation with uncorrelated innovations: bias for the other parameter estimates.

Table 4 compares the estimates for x_t and c_t for the different models by regressing the respective Kalman smoother-based estimates on the true trend and cycle and reporting the coefficient of determination. As can be seen, differences between the coefficients of determination are almost negligible for the CSS and the QML estimator of the fractional UC model, with the latter exhibiting slightly larger coefficients of determination. Strikingly, for $d_0 = 1$ the fractional UC model shows no loss in efficiency compared to the $I(1)$ UC model. For non-integer d_0, the fractional model shows a slightly higher R^2 than the integer-integrated

Table 1:

n	r	d_0	ν	RMSE d CSS	d QML	d ARMA	d EW .5	d EW .6	d EW .7	bias d CSS	d QML	d ARMA	d EW .5	d EW .6	d EW .7
100		0.75	1.00	0.26	0.13	0.52	0.64	0.41	0.57	-0.01	-0.03	0.18	-0.62	-0.36	0.52
		1.00	1.00	0.33	0.13	0.13	0.68	0.46	0.40	0.03	-0.03	-0.02	-0.65	-0.42	0.33
		1.75	1.00	0.20	0.12	0.18	0.55	0.43	0.16	0.02	-0.03	0.00	-0.50	-0.38	-0.06
	1	0.75	0.65	0.21	0.12	0.49	0.60	0.37	0.50	-0.02	-0.02	0.16	-0.58	-0.32	0.45
		1.00	3.82	0.49	0.18	0.31	0.85	0.61	0.52	0.13	-0.06	0.02	-0.83	-0.58	0.45
		1.75	1053.88	0.65	0.42	0.32	1.58	1.36	0.36	-0.16	-0.18	-0.10	-1.57	-1.35	-0.23
	10	0.75	0.07	0.13	0.12	0.34	0.34	0.21	0.18	0.03	0.05	0.07	-0.24	-0.10	0.10
		1.00	0.38	0.24	0.12	0.11	0.54	0.34	0.28	0.00	-0.02	-0.02	-0.49	-0.28	0.21
		1.75	105.39	0.49	0.24	0.22	1.31	1.13	0.34	-0.04	-0.07	-0.09	-1.29	-1.11	-0.22
	30	0.75	0.02	0.08	0.07	0.35	0.27	0.19	0.14	0.02	0.02	0.06	-0.12	-0.04	0.04
		1.00	0.13	0.20	0.13	0.10	0.38	0.24	0.18	0.00	0.03	-0.01	-0.30	-0.15	0.10
		1.75	35.13	0.41	0.19	0.20	1.14	0.98	0.32	-0.03	-0.05	-0.04	-1.12	-0.96	-0.21
200		0.75	1.00	0.13	0.09	0.23	0.62	0.57	0.14	-0.03	-0.02	0.03	-0.60	-0.56	0.03
		1.00	1.00	0.17	0.08	0.08	0.60	0.56	0.15	0.00	-0.02	-0.01	-0.57	-0.55	-0.10
		1.75	1.00	0.15	0.07	0.13	0.39	0.44	0.24	0.03	-0.01	0.04	-0.34	-0.41	-0.22
	1	0.75	0.93	0.13	0.08	0.22	0.61	0.56	0.14	-0.02	-0.02	0.02	-0.59	-0.55	0.03
		1.00	7.59	0.43	0.15	0.64	0.87	0.83	0.23	0.13	-0.04	0.39	-0.86	-0.82	-0.18
		1.75	5871.84	0.60	0.38	0.29	1.60	1.58	0.96	-0.22	-0.16	-0.16	-1.60	-1.58	-0.95
	10	0.75	0.09	0.11	0.13	0.13	0.32	0.26	0.10	0.02	0.06	0.01	-0.26	-0.22	0.01
		1.00	0.76	0.14	0.08	0.07	0.56	0.52	0.14	0.00	-0.02	-0.01	-0.53	-0.50	-0.08
		1.75	587.18	0.47	0.17	0.19	1.35	1.35	0.89	-0.06	-0.05	-0.03	-1.34	-1.34	-0.88
	30	0.75	0.03	0.06	0.05	0.13	0.23	0.17	0.10	0.01	0.01	0.01	-0.13	-0.10	0.01
		1.00	0.25	0.11	0.09	0.06	0.39	0.35	0.11	-0.01	0.00	-0.01	-0.34	-0.33	-0.04
		1.75	195.73	0.41	0.13	0.16	1.20	1.21	0.83	-0.03	-0.03	-0.08	-1.19	-1.20	-0.82
300		0.75	1.00	0.10	0.07	0.14	0.51	0.60	0.22	-0.02	-0.01	0.01	-0.49	-0.59	-0.19
		1.00	1.00	0.10	0.07	0.06	0.45	0.58	0.27	-0.01	-0.01	-0.01	-0.42	-0.57	-0.26
		1.75	1.00	0.13	0.06	0.11	0.24	0.43	0.28	0.03	-0.01	0.05	-0.18	-0.41	-0.27
	1	0.75	1.14	0.11	0.07	0.16	0.53	0.62	0.22	-0.02	-0.01	0.01	-0.51	-0.61	-0.20
		1.00	11.37	0.33	0.13	0.90	0.79	0.88	0.49	0.10	-0.03	0.74	-0.77	-0.88	-0.48
		1.75	16098.90	0.56	0.32	0.44	1.55	1.63	1.27	-0.25	-0.12	-0.29	-1.54	-1.63	-1.27
	10	0.75	0.11	0.11	0.14	0.08	0.25	0.31	0.10	0.02	0.06	0.00	-0.19	-0.29	-0.05
		1.00	1.14	0.10	0.07	0.06	0.47	0.60	0.29						

Table 2:

n	r	d~0	ν	$\nu$$\hat\nu$~CSS$ $\hat\nu$~QML$ $\hat\nu$~ARMA(1)$ $H_{(1)}$~CSS $H_{(1)}$~QML $H_{(1)}$~ARMA(1) $H_{(2)}$~CSS $H_{(2)}$~QML $H_{(2)}$~ARMA(1) $H_{(C)}$~QML~														
100		0.75	1.00	7044803.31	[illegible]	0.33	7740506.21	[illegible]	0.12	0.11	0.11	0.19	0.14	0.12	0.10	0.10	0.18	0.13
		1.00	1.00	700720.630	[illegible]	0.30	2846802.21	[illegible]	0.14	0.09	0.09	0.12	0.09	0.12	0.08	0.09	0.12	0.09
		1.75	1.00	153.59	0.28	0.32	24425273.86	[illegible]	0.11	0.11	0.25	0.11	0.19	0.10	0.09	0.23	0.09	0.08
	1	0.75	0.65	4740202.27	[illegible]	0.27	4943604.35	[illegible]	0.13	0.11	0.12	0.21	0.17	0.12	0.11	0.12	0.21	0.16
		1.00	3.82	1764270.74	[illegible]	0.76	5493501.67	[illegible]	0.09	0.08	0.08	0.09	0.08	0.09	0.07	0.07	0.09	0.08
		1.75	1053.88	2945015.8513	[illegible]	148.51	2136641127.86	[illegible]	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.09
	10	0.75	0.07	1545.80.09	[illegible]	0.10	121890.812	[illegible]	0.14	0.12	0.35	0.18	0.20	0.11	0.08	0.21	0.15	0.14
		1.00	0.38	3762809.21	[illegible]	0.19	1375400.21	[illegible]	0.14	0.12	0.13	0.15	0.12	0.15	0.11	0.13	0.16	0.12
		1.75	105.39	10347918.522	[illegible]	15.09	7398714966.93	[illegible]	0.07	0.07	0.07	0.08	0.11	0.07	0.07	0.07	0.08	0.12
	30	0.75	0.02	0.31	0.03	0.06	606.27	0.04	0.15	0.15	0.63	0.18	0.18	0.10	0.10	0.26	0.15	0.14
		1.00	0.13	2838.680.15	[illegible]	0.12	147380.645	[illegible]	0.17	0.13	0.25	0.18	0.13	0.15	0.09	0.17	0.18	0.10
		1.75	35.13	433556.887	[illegible]	6.70	502282680.66	[illegible]	0.07	0.07	0.08	0.09	0.12	0.07	0.07	0.07	0.10	0.11
200		0.75	1.00	5201.750.25	[illegible]	0.23	74011022.32	[illegible]	0.07	0.06	0.06	0.14	0.09	0.08	0.06	0.06	0.14	0.09
		1.00	1.00	2952309.85	[illegible]	0.21	1897205.62	[illegible]	0.07	0.06	0.06	0.07	0.06	0.08	0.06	0.06	0.07	0.06
		1.75	1.00	21.10	0.19	0.23	5222.94372.94	[illegible]	0.07	0.07	0.10	0.11	0.24	0.08	0.06	0.12	0.10	0.11
	1	0.75	0.93	3162009.24	[illegible]	0.22	6178300.41	[illegible]	0.07	0.06	0.06	0.14	0.09	0.08	0.06	0.06	0.14	0.09
		1.00	7.59	209249.01	[illegible]	1.59	7036626.18	[illegible]	0.05	0.05	0.08	0.05	0.05	0.05	0.05	0.07	0.05	0.05
		1.75	5871.84	446416561.75	[illegible]	[illegible]	1162941370920.07	[illegible]	[illegible]	0.06	0.04	0.04	0.06	0.05	0.06	0.04	0.04	0.08
	10	0.75	0.09	1634.310.10	[illegible]	0.09	379.83	0.13	0.10	0.08	0.16	0.15	0.17	0.09	0.06	0.14	0.13	0.13
		1.00	0.76	2607600.23	[illegible]	0.18	2200.70.19	[illegible]	0.08	0.06	0.06	0.08	0.06	0.08	0.06	0.06	0.08	0.06
		1.75	587.18	2092447.21	[illegible]	60.16	30899241.55	[illegible]	0.05	0.04	0.04	0.05	0.08	0.05	0.04	0.05	0.06	0.10
	30	0.75	0.03	454.66	0.02	0.05	0.02	0.02	0.11	0.10	0.38	0.17	0.15	0.06	0.06	0.19	0.14	0.11
		1.00	0.25	1985503.07	[illegible]	0.11	1438500.94	[illegible]	0.11	0.09	0.10	0.12	0.09	0.12	0.09	0.11	0.13	0.09
		1.75	195.73	94274247.63	[illegible]	26.51	7810.55302.33	[illegible]	0.05	0.04	0.04	0.06	0.12	0.05	0.05	0.05	0.07	0.13
300		0.75	1.00	1656.91.21	[illegible]	0.18	4370506.28	[illegible]	0.06	0.05	0.05	0.11	0.08	0.06	0.05	0.05	0.11	0.07
		1.00	1.00	725.28	0.22	0.17	5238.818	[illegible]	0.06	0.05	0.05	0.06	0.05	0.06	0.05	0.05	0.06	0.05
		1.75	1.00	10.34	0.15	0.21	4387.24446.30	[illegible]	0.06	0.05	0.06	0.11	0.24	0.06	0.04	0.08	0.11	0.12
	1	0.75	1.14	2187.560.24	[illegible]	0.19	6213800.39	[illegible]	0.06	0.05	0.05	0.11	0.07	0.06	0.05	0.05	0.11	0.07
		1.00	11.37	160943.80	[illegible]	1.77	1065371.87	[illegible]	0.04	0.04	0.10	0.04	0.04	0.04	0.04	0.07	0.04	0.04
		1.75	16098.4698228381617788.48316327679205.04	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.08	0.04	0.04	0.08	0.04	0.05	0.04	0.04	0.06
	10	0.75	0.11	143.68	0.11	0.07	161290.45	[illegible]	0.09	0.06	0.11	0.19	0.17	0.10	0.06	0.11	0.18	0.15
		1.00	1.14	653.32	0.23	0.19	1083.50.19	[illegible]	0.06	0.05	0.05	0.06	0.05	0.06	0.05	0.04	0.06	0.05
		1.75	1609.83	30662186.5309	[illegible]	133.76	1617.94703.84	[illegible]	0.04	0.04	0.04	0.04	0.08	0.04	0.04	0.04	0.05	0.09
	30	0.75	0.04	0.17	0.03	0.05	0.03	0.03	0.08	0.07	0.24	0.15	0.13	0.04	0.03	0.17	0.12	0.08
		1.00	0.38	2.04	0.16	0.11	1.02	0.11	0.07	0.07	0.06	0.07	0.06	0.08	0.07	0.06	0.08	0.06
		1.75	536.63	1568745.1027	[illegible]	46.85	532.65	253.72	0.04	0.04	0.04	0.05	0.10	0.04	0.04	0.04	0.06	0.10

Table 3:

n	r	d	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
100		0.75	1.00	6477.95	0.01	0.01	15815.48	0.19	-0.03	-0.02	-0.03	0.08	0.08	0.02	0.01	0.02	-0.06	-0.06
		1.00	1.00	7646.94	0.03	-0.05	3048.25	0.02	-0.03	-0.02	-0.02	0.02	0.03	0.02	0.01	0.01	-0.02	-0.02
		1.75	1.00	11.04	-0.01	0.04	1665.89	150.50	-0.01	-0.02	-0.07	0.02	-0.17	-0.02	0.01	0.08	-0.04	-0.03
	1	0.75	0.65	2921.70	0.02	0.01	8468.21	0.16	-0.02	-0.02	-0.04	0.07	0.09	0.01	0.01	0.02	-0.05	-0.08
		1.00	3.82	38902.53	-0.15	-0.15	8828.59	0.05	-0.02	-0.01	-0.02	0.02	0.03	0.02	0.01	0.01	-0.02	-0.02
		1.75	1053.81	17926.52	2.23	-21.76	54801.20	204.18	-0.01	-0.02	-0.02	-0.01	-0.02	0.00	0.01	0.01	0.01	-0.05
	10	0.75	0.07	53.70	-0.04	0.00	661.92	-0.03	-0.07	-0.06	-0.08	0.11	0.11	0.03	0.02	-0.05	-0.06	-0.04
		1.00	0.38	2630.61	0.02	-0.03	1163.93	0.00	-0.03	-0.03	-0.02	0.03	0.03	0.02	0.01	0.00	-0.02	-0.02
		1.75	105.39	20987.73	3.65	-2.21	8068.91	111.29	-0.01	-0.02	-0.02	-0.04	-0.06	0.00	0.01	0.01	0.05	-0.08
	30	0.75	0.02	0.01	-0.01	0.01	19.30	-0.01	-0.10	-0.10	-0.27	0.16	0.15	0.05	0.04	-0.03	-0.11	-0.08
		1.00	0.13	202.76	-0.05	-0.03	1150.23	-0.05	-0.06	-0.06	-0.02	0.07	0.05	0.03	0.01	-0.06	-0.04	-0.01
		1.75	35.13	5771.29	2.47	-0.15	5954.43	136.20	0.00	-0.02	-0.02	-0.06	-0.09	-0.01	0.01	0.02	0.06	-0.08
200		0.75	1.00	185.26	0.03	0.00	8846.97	0.18	-0.01	-0.01	-0.01	0.05	0.06	0.01	0.01	0.01	-0.05	-0.05
		1.00	1.00	1575.31	0.10	-0.02	649.94	0.01	-0.01	-0.01	-0.01	0.01	0.01	0.01	0.01	0.01	-0.01	-0.01
		1.75	1.00	3.84	0.02	0.11	269.09	243.88	0.00	-0.01	-0.03	0.05	-0.21	-0.01	0.01	0.06	-0.05	0.01
	1	0.75	0.93	1018.32	0.03	0.00	7116.17	0.18	-0.01	-0.01	-0.01	0.05	0.06	0.01	0.01	0.01	-0.04	-0.05
		1.00	7.59	49187.05	0.20	0.03	9136.07	0.16	-0.01	-0.01	-0.03	0.01	0.01	0.01	0.01	0.02	-0.01	-0.01
		1.75	5871.84	215947.38	3.46	-58.19	15728.17	764.32	0.00	-0.01	-0.01	-0.01	-0.03	0.00	0.01	0.01	0.01	-0.04
	10	0.75	0.09	51.68	-0.06	0.01	12.94	-0.05	-0.03	-0.02	-0.03	0.06	0.07	0.01	0.01	0.00	-0.03	-0.03
		1.00	0.76	1081.62	0.10	-0.01	139.18	0.01	-0.01	-0.02	-0.01	0.01	0.01	0.01	0.02	0.01	-0.01	-0.01
		1.75	587.18	62952.33	4.63	-5.96	995.95	208.52	0.00	-0.01	-0.01	-0.03	-0.07	0.00	0.01	0.01	0.04	-0.08
	30	0.75	0.03	14.36	-0.02	0.00	-0.02	-0.02	-0.06	-0.05	-0.11	0.14	0.12	0.02	0.02	-0.03	-0.09	-0.06
		1.00	0.25	648.64	0.03	-0.01	736.57	-0.01	-0.02	-0.03	-0.01	0.02	0.02	0.01	0.03	0.01	-0.02	-0.01
		1.75	195.73	19792.12	27.04	-1.74	141.41	226.25	0.00	-0.01	-0.01	-0.05	-0.08	0.00	0.01	0.01	0.06	-0.09
300		0.75	1.00	60.94	0.03	0.00	3487.79	0.17	-0.01	-0.01	-0.01	0.04	0.05	0.01	0.01	0.00	-0.03	-0.04
		1.00	1.00	24.60	0.12	-0.01	166.30	0.01	-0.01	-0.01	0.00	0.01	0.01	0.01	0.01	0.00	0.00	-0.01
		1.75	1.00	1.93	0.01	0.12	147.03	322.11	0.00	-0.01	-0.02	0.06	-0.21	-0.01	0.00	0.05	-0.06	0.01
	1	0.75	1.14	80.27	0.04	0.00	5875.34	0.18	-0.01	-0.01	-0.01	0.04	0.05	0.01	0.01	0.00	-0.04	-0.04
		1.00	11.37	29892.04	4.37	-0.72	15376.11	0.12	0.00	0.00	-0.06	0.00	0.01	0.00	0.00	0.04	0.00	-0.01
		1.75	16098.23	[illegible]	[illegible]	213.83	-4223.49	5284.75	0.00	-0.01	0.00	0.00	-0.03	0.00	0.00	0.01	0.01	-0.04
	10	0.75	0.11	5.62	-0.05	0.01	623.24	-0.05	-0.01	-0.02	-0.02	0.07	0.06	0.01	0.01	0.01	-0.05	-0.04
		1.00	1.14	24.91	0.11	-0.02	44.23	0.01	-0.01	-0.01	0.00	0.01	0.01	0.01	0.01	0.00	0.00	-0.01

benchmarks, particularly when $d_0 = 1.75$. However, as will become clear, integer-integrated UC models often provide a good approximation to fractionally integrated trends as long as correlation between trend and cycle innovations is ruled out.

Table 4: Simulation with uncorrelated innovations: Coefficient of determination from regressing true trend and cycle on their respective estimates from the Kalman smoother.

2 Correlated innovations

The second simulation generalizes the prototypical fractional UC model to correlated innovations, allowing for a non-diagonal covariance matrix Q. The resulting fractional UC model is thus

$$y_t = x_t + c_t, \qquad \Delta_+^d x_t = \eta_t, \qquad c_t - b_1 c_{t-1} - b_2 c_{t-2} = \epsilon_t, \qquad Q = \begin{pmatrix} 1 & \rho\sqrt{\nu} \\ \rho\sqrt{\nu} & \nu \end{pmatrix},$$

where $\rho = -0.8$ is set to mimic strong but not perfect correlation.

Table 5 shows the RMSE and the bias for the estimated integration orders. As before, bias and RMSE decrease both in n and r, and are significantly smaller as compared to the nonparametric Whittle estimators. The performance of QML and CSS is now somewhat more similar as compared to the uncorrelated case, which may be due to the additional parameter that QML now has to estimate. Nonetheless, QML appears to be less biased.

Table 5: Simulation with correlated innovations: root mean squared errors (RMSE) and bias for the integration order estimates.

Tables 6 and 7 detail RMSE and bias for the covariance and autoregressive parameters, both for the fractional UC model and the integer-integrated benchmark.

Table 6: Simulation with correlated innovations: root mean squared errors (RMSE) for the other parameter estimates.

Table 7: Simulation with correlated innovations: bias for the other parameter estimates.

Table 8 compares the estimates for x_t and c_t for the different models by regressing the respective Kalman smoother-based estimates on the true trend and cycle and reporting the coefficient of determination.

Table 8: Simulation with correlated innovations: Coefficient of determination from regressing true trend and cycle on their respective estimates from the Kalman smoother.

3 References

Hartl, Tobias, and Roland Jucknewitz. 2022. "Approximate State Space Modelling of Unobserved Fractional Components." *Econometric Reviews* 41 (1): 75–98.

Table 4:

n	r	d_0	ν_0	Trend R^2~CSS	Trend R^2~QML	Trend R^2~ARMA	Trend [illegible]	Trend [illegible]	Cycle [illegible]	Cycle [illegible]	Cycle [illegible]	Cycle [illegible]	Cycle [illegible]
100		0.75	1.00	0.57	0.59	0.59	0.52	0.58	0.79	0.80	0.80	0.75	0.80
		1.00	1.00	0.61	0.64	0.65	0.62	0.66	0.88	0.90	0.90	0.89	0.90
		1.75	1.00	0.78	0.78	0.80	0.78	0.74	0.96	0.96	0.96	0.95	0.78
	1	0.75	0.65	0.50	0.53	0.53	0.46	0.52	0.84	0.85	0.85	0.81	0.84
		1.00	3.82	0.75	0.78	0.78	0.75	0.79	0.77	0.79	0.79	0.77	0.79
		1.75	1053.88	0.99	0.99	0.99	0.94	0.80	0.56	0.60	0.58	0.49	0.04
	10	0.75	0.07	0.93	0.93	0.91	0.92	0.93	0.23	0.23	0.15	0.22	0.22
		1.00	0.38	0.88	0.90	0.91	0.88	0.90	0.42	0.43	0.40	0.42	0.43
		1.75	105.39	0.95	0.95	0.95	0.92	0.70	0.87	0.89	0.89	0.80	0.24
	30	0.75	0.02	0.83	0.83	0.84	0.81	0.82	0.40	0.40	0.34	0.38	0.37
		1.00	0.13	0.82	0.85	0.85	0.82	0.85	0.63	0.65	0.65	0.63	0.65
		1.75	35.13	0.91	0.91	0.92	0.91	0.70	0.91	0.92	0.92	0.89	0.40
200		0.75	1.00	0.63	0.65	0.65	0.58	0.64	0.84	0.85	0.85	0.82	0.84
		1.00	1.00	0.69	0.71	0.70	0.69	0.72	0.92	0.93	0.92	0.93	0.93
		1.75	1.00	0.82	0.82	0.83	0.82	0.75	0.98	0.98	0.98	0.97	0.78
	1	0.75	0.93	0.62	0.64	0.64	0.57	0.63	0.85	0.86	0.86	0.82	0.85
		1.00	7.59	0.87	0.88	0.88	0.87	0.88	0.79	0.80	0.80	0.80	0.81
		1.75	5871.84	1.00	1.00	1.00	0.99	0.81	0.60	0.63	0.61	0.55	0.01
	10	0.75	0.09	0.93	0.93	0.92	0.93	0.93	0.29	0.29	0.22	0.28	0.26
		1.00	0.76	0.92	0.93	0.94	0.92	0.93	0.59	0.60	0.61	0.59	0.61
		1.75	587.18	0.97	0.97	0.97	0.97	0.74	0.93	0.94	0.94	0.92	0.25
	30	0.75	0.03	0.84	0.84	0.87	0.82	0.83	0.49	0.49	0.48	0.45	0.45
		1.00	0.25	0.88	0.89	0.89	0.88	0.89	0.76	0.77	0.77	0.77	0.78
		1.75	195.73	0.94	0.94	0.94	0.94	0.77	0.95	0.96	0.96	0.94	0.41
300		0.75	1.00	0.67	0.68	0.68	0.62	0.67	0.87	0.87	0.87	0.84	0.86
		1.00	1.00	0.74	0.75	0.74	0.74	0.76	0.94	0.94	0.93	0.94	0.95
		1.75	1.00	0.85	0.86	0.83	0.85	0.74	0.98	0.98	0.96	0.98	0.79
	1	0.75	1.14	0.69	0.70	0.70	0.65	0.69	0.86	0.86	0.86	0.83	0.85
		1.00	11.37	0.91	0.91	0.91	0.91	0.92	0.80	0.81	0.81	0.80	0.81
		1.75	16098.90	1.00	1.00	1.00	1.00	0.83	0.62	0.64	0.62	0.58	0.01
	10	0.75	0.11	0.93	0.93	0.93	0.93	0.93	0.34	0.35	0.29	0.32	0.30
		1.00	1.14	0.94	0.95	0.95	0.94	0.95	0.68	0.69	0.69	0.68	0.69
		1.75	1609.89	0.98	0.98	0.98	0.97	0.77	0.95	0.96	0.95	0.94	0.25
	30	0.75	0.04	0.86	0.85	0.88	0.82	0.83	0.54	0.54	0.55	0.50	0.50
		1.00	0.38	0.90	0.91	0.91	0.90	0.91	0.81	0.82	0.82	0.82	0.82
		1.75	536.63	0.95	0.95	0.96	0.95	0.76	0.97	0.97	0.97	0.96	0.42

Table 5:

n	r	d_0	ν	RMSE						bias					
				d CSS	d QML	d ARMA	d EW$^{.5}$	d EW$^{.6}$	d EW$^{.7}$	d CSS	d QML	d ARMA	d EW$^{.5}$	d EW$^{.6}$	d EW$^{.7}$
100		0.75	1.00	0.30	0.13	0.14	0.64	0.42	0.68	0.08	-0.03	-0.02	-0.62	-0.37	0.64
		1.00	1.00	0.27	0.17	0.15	0.62	0.36	0.62	0.08	-0.01	-0.02	-0.58	-0.29	0.57
		1.75	1.00	0.24	0.27	0.21	0.46	0.22	0.24	-0.06	-0.04	0.06	-0.40	-0.09	0.24
	1	0.75	0.65	0.19	0.12	0.13	0.59	0.37	0.57	0.01	-0.03	-0.02	-0.57	-0.31	0.53
		1.00	3.82	0.49	0.22	0.19	0.82	0.56	0.69	0.30	-0.03	-0.03	-0.81	-0.52	0.64
		1.75	1053.88	0.56	0.57	0.42	1.55	1.30	0.33	-0.38	-0.35	-0.30	-1.54	-1.29	-0.20
	10	0.75	0.07	0.08	0.09	0.10	0.24	0.19	0.14	0.02	-0.02	-0.02	-0.01	0.00	0.03
		1.00	0.38	0.18	0.14	0.14	0.41	0.21	0.40	0.01	-0.03	-0.04	-0.34	-0.08	0.36
		1.75	105.39	0.52	0.39	0.26	1.27	1.05	0.28	-0.36	-0.12	-0.15	-1.25	-1.04	-0.14
	30	0.75	0.02	0.04	0.09	0.10	0.29	0.19	0.14	0.01	-0.03	-0.02	0.13	0.03	0.00
		1.00	0.13	0.15	0.11	0.13	0.25	0.21	0.21	0.05	-0.03	-0.04	-0.02	0.10	0.15
		1.75	35.13	0.48	0.29	0.23	1.11	0.90	0.24	-0.34	-0.07	-0.11	-1.08	-0.88	-0.08
200		0.75	1.00	0.22	0.10	0.12	0.61	0.57	0.16	0.05	-0.02	0.00	-0.59	-0.56	0.05
		1.00	1.00	0.20	0.13	0.12	0.55	0.50	0.15	0.05	0.00	-0.01	-0.53	-0.48	0.07
		1.75	1.00	0.18	0.19	0.18	0.42	0.33	0.19	-0.03	-0.03	0.13	-0.38	-0.30	0.17
	1	0.75	0.93	0.19	0.10	0.11	0.60	0.56	0.16	0.03	-0.02	-0.01	-0.58	-0.55	0.05
		1.00	7.59	0.45	0.17	0.15	0.85	0.81	0.21	0.28	0.00	-0.01	-0.84	-0.80	-0.11
		1.75	5871.84	0.54	0.48	0.33	1.58	1.56	0.94	-0.42	-0.29	-0.24	-1.57	-1.55	-0.92
	10	0.75	0.09	0.10	0.06	0.08	0.20	0.15	0.10	0.03	-0.02	-0.02	-0.07	-0.07	0.02
		1.00	0.76	0.16	0.11	0.10	0.50	0.44	0.16	0.03	-0.01	-0.01	-0.47	-0.42	0.09
		1.75	587.18	0.41	0.35	0.22	1.33	1.32	0.84	-0.24	-0.11	-0.14	-1.32	-1.31	-0.83
	30	0.75	0.03	0.03	0.07	0.07	0.21	0.15	0.10	0.00	-0.02	-0.01	0.10	0.07	0.02
		1.00	0.25	0.18	0.09	0.10	0.30	0.20	0.17	0.06	-0.01	-0.02	-0.24	-0.15	0.14
		1.75	195.73	0.41	0.31	0.19	1.18	1.18	0.76	-0.25	-0.07	-0.11	-1.17	-1.17	-0.75
300		0.75	1.00	0.19	0.08	0.10	0.49	0.60	0.22	0.04	-0.01	0.00	-0.47	-0.59	-0.20
		1.00	1.00	0.15	0.10	0.10	0.41	0.53	0.17	0.03	0.00	0.00	-0.38	-0.52	-0.14
		1.75	1.00	0.15	0.15	0.17	0.31	0.38	0.09	-0.02	-0.02	0.15	-0.26	-0.36	0.03
	1	0.75	1.14	0.24	0.09	0.11	0.51	0.62	0.24	0.07	-0.01	0.00	-0.49	-0.61	-0.21
		1.00	11.37	0.42	0.15	0.14	0.78	0.88	0.47	0.26	0.00	-0.01	-0.77	-0.87	-0.45
		1.75	16098.90	0.57	0.44	0.30	1.52	1.62	1.25	-0.48	-0.28	-0.21	-1.51	-1.61	-1.24
	10	0.75	0.11	0.13	0.05	0.07	0.18	0.18	0.08	0.06	-0.01	-0.01	-0.06	-0.14	-0.01
		1.00	1.14	0.19	0.12	0.11	0.43	0.55	0.19	0.05	0.01	0.01			

Table 6:

n	r	d~0	ν~0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
100		0.75	1.00	364.29	373.53	0.74	0.37	0.52	1.38	0	0.57	0.66	0	0.63	0.90	0.35
		1.00	1.00	213.86	111.15	0.70	0.45	0.51	0.50	0	0.80	0.71	0	0.79	0.74	0.35
		1.75	1.00	3.16	0.72	0.31	0.33	0.22	0.20	0	28.79	1.08	0	27.32	0.98	0.35
	1	0.75	0.65	185.01	328.32	0.80	0.36	0.55	1.55	0	0.75	0.98	0	0.89	1.14	0.00
		1.00	3.82	540.81	106.32	0.49	0.25	0.28	0.29	0	1.51	1.14	0	1.77	1.29	3.16
		1.75	1053.88	734.50	1000.81	0.66	0.65	0.63	0.58	0	109.23	24.26	0	356.24	217.11	1053.23
	10	0.75	0.07	0.11	0.31	0.15	0.22	0.26	0.51	0	1.52	1.14	0	1.05	1.05	0.59
		1.00	0.38	70.92	55.05	0.86	0.46	0.52	0.78	0	2.75	2.67	0	2.93	2.78	0.27
		1.75	105.39	295.19	116.80	0.67	0.63	0.67	0.26	0	19.10	5.00	0	54.86	26.37	104.73
	30	0.75	0.02	0.01	0.03	0.10	0.21	0.23	0.21	0	1.63	1.48	0	1.09	1.11	0.63
		1.00	0.13	31.62	31.63	0.35	0.24	0.32	0.63	0	2.28	3.89	0	2.04	3.81	0.53
		1.75	35.13	144.60	55.01	0.65	0.56	0.68	0.19	0	8.38	3.04	0	21.54	10.52	34.48
200		0.75	1.00	279.87	283.25	0.67	0.38	0.56	1.45	0	0.41	0.44	0	0.36	0.37	0.35
		1.00	1.00	119.73	7.85	0.55	0.37	0.39	0.28	0	0.49	0.51	0	0.42	0.47	0.35
		1.75	1.00	1.44	0.59	0.19	0.19	0.17	0.20	0	6.63	0.76	0	6.27	0.65	0.35
	1	0.75	0.93	233.03	280.91	0.70	0.37	0.56	1.49	0	0.44	0.44	0	0.42	0.40	0.28
		1.00	7.59	607.91	99.99	0.29	0.17	0.21	0.17	0	1.36	1.45	0	1.56	1.55	6.94
		1.75	5871.84	4107.13	3772.30	0.68	0.67	0.60	0.58	0	259.78	97.84	0	1101.16	729.24	5871.19
	10	0.75	0.09	0.18	0.35	0.19	0.26	0.35	0.92	0	0.73	0.97	0	0.52	1.07	0.56
		1.00	0.76	55.25	6.03	0.64	0.36	0.42	0.33	0	0.46	0.46	0	0.41	0.46	0.11
		1.75	587.18	585.32	580.56	0.68	0.59	0.57	0.29	0	50.77	10.82	0	160.09	78.28	586.53
	30	0.75	0.03	0.02	0.02	0.09	0.18	0.20	0.17	0	1.01	0.95	0	0.58	0.79	0.62
		1.00	0.25	0.50	1.23	0.76	0.40	0.48	0.76	0	1.46	1.31	0	1.48	1.36	0.40
		1.75	195.73	475.19	193.53	0.57	0.57	0.55	0.17	0	19.14	5.49	0	69.66	28.19	195.07
300		0.75	1.00	217.22	187.49	0.64	0.32	0.54	1.49	0	0.35	0.41	0	0.27	0.28	0.35
		1.00	1.00	78.71	2.39	0.43	0.29	0.34	0.19	0	0.41	0.46	0	0.30	0.37	0.35
		1.75	1.00	1.06	0.50	0.15	0.14	0.15	0.20	0	4.95	0.61	0	4.81	0.48	0.35
	1	0.75	1.14	316.57	212.40	0.65	0.34	0.53	1.38	0	0.39	0.44	0	0.29	0.28	0.49
		1.00	11.37	653.67	55.76	0.26	0.17	0.21	0.17	0	1.83	2.09	0	2.00	1.78	10.72
		1.75	16098.95	[illegible]	12978.76	0.71	0.64	0.61	0.66	0	566.79	46.17	0	2310.22	2511.45	16098.25
	10	0.75	0.11	0.41	1.56	0.42	0.31	0.44	1.24	0	0.53	0.87	0	0.45	0.98	0.54
		1.00	1.14	131.89	2.64	0.39	0.29	0.33	0.17	0	0.44	0.49	0	0.32	0.38	0.48
		1.75	1609.89	1152.61	1599.49	0.63	0.54	0.50	0.37	0	146.74	9.88	0	324.29	164.54	1609.24
	30	0.75	0.04	0.03	0.03	0.09	0.18	0.21	0.17	0	1.00	0.88	0	0.62	0.81	0.62
		1.00	0.38	0.22	0.57	0.67	0.43	0.45	0.40	0	0.50	1.01	0	0.44	0.95	0.28
		1.75	536.63	586.26	533.28	0.58	0.51	0.47	0.21	0	38.41	5.12	0	128.31	56.55	535.98

Table 7:

n	r	d~0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
100		0.75	1.00	141.74	165.94	0.27	0.03	0.07	1.15	0	0.14	0.07	0	0.10	0.11	-0.35
		1.00	1.00	53.43	16.60	0.23	0.07	0.08	0.17	0	0.20	0.19	0	0.15	0.08	-0.35
		1.75	1.00	0.41	0.09	0.01	0.02	0.06	-0.20	0	3.01	-0.33	0	2.71	-0.26	-0.35
	1	0.75	0.65	36.96	116.68	0.34	0.04	0.10	1.39	0	0.15	0.07	0	0.16	0.15	0.00
		1.00	3.82	319.15	24.89	0.04	-0.06	-0.08	-0.02	0	0.54	0.40	0	0.15	0.07	-3.16
		1.75	1053.88	-516.24	-974.58	0.14	0.09	0.07	0.24	0	30.64	8.18	0	138.60	45.22	-1053.23
	10	0.75	0.07	-0.04	-0.04	0.03	-0.13	-0.15	0.01	0	0.44	0.30	0	0.21	0.24	0.59
		1.00	0.38	5.41	3.65	0.38	0.07	0.07	0.35	0	0.48	0.35	0	0.47	0.37	0.27
		1.75	105.39	39.32	-96.24	0.18	0.11	0.16	-0.03	0	7.48	1.68	0	26.33	2.79	-104.73
	30	0.75	0.02	-0.01	-0.01	-0.05	-0.17	-0.17	-0.15	0	0.52	0.38	0	0.26	0.24	0.63
		1.00	0.13	0.94	1.09	0.06	-0.10	-0.12	0.08	0	0.48	0.72	0	0.30	0.67	0.53
		1.75	35.13	14.43	-31.43	0.19	0.12	0.23	-0.14	0	3.04	0.80	0	9.85	-1.07	-34.48
200		0.75	1.00	83.20	104.31	0.22	0.05	0.11	1.31	0	0.08	0.00	0	0.05	0.05	-0.35
		1.00	1.00	18.10	1.18	0.17	0.06	0.05	0.09	0	0.07	0.10	0	0.05	0.04	-0.35
		1.75	1.00	0.30	0.07	-0.01	0.00	0.09	-0.20	0	0.77	-0.50	0	0.64	-0.38	-0.35
	1	0.75	0.93	58.45	99.41	0.25	0.04	0.11	1.36	0	0.08	0.00	0	0.06	0.06	-0.28
		1.00	7.59	393.43	22.38	-0.07	-0.08	-0.11	-0.06	0	0.28	0.36	0	0.04	0.06	-6.94
		1.75	5871.84	-5022.96	-5764.63	0.16	0.11	0.07	0.28	0	59.53	13.84	0	387.86	93.36	-5871.19
	10	0.75	0.09	-0.06	-0.05	0.10	-0.04	-0.09	0.41	0	0.23	0.22	0	0.11	0.23	0.56
		1.00	0.76	3.82	0.68	0.24	0.04	0.07	0.12	0	0.10	0.08	0	0.07	0.04	-0.11
		1.75	587.18	-93.74	-580.27	0.19	0.09	0.08	0.15	0	11.79	2.64	0	81.61	12.31	-586.53
	30	0.75	0.03	-0.02	-0.02	-0.02	-0.15	-0.17	-0.16	0	0.36	0.19	0	0.14	0.12	0.62
		1.00	0.25	-0.01	0.18	0.32	0.05	0.05	0.33	0	0.22	0.17	0	0.17	0.16	0.40
		1.75	195.73	54.56	-193.39	0.14	0.10	0.09	-0.02	0	5.76	1.13	0	37.33	2.54	-195.07
300		0.75	1.00	50.22	57.61	0.22	0.03	0.11	1.40	0	0.07	-0.03	0	0.04	0.04	-0.35
		1.00	1.00	8.47	0.37	0.11	0.04	0.06	0.06	0	0.03	0.04	0	0.02	0.00	-0.35
		1.75	1.00	0.22	0.04	-0.01	-0.01	0.10	-0.20	0	0.43	-0.59	0	0.35	-0.45	-0.35
	1	0.75	1.14	104.83	77.62	0.21	0.03	0.10	1.26	0	0.08	-0.02	0	0.03	0.03	-0.49
		1.00	11.37	449.56	17.27	-0.07	-0.06	-0.10	-0.04	0	0.51	0.65	0	0.20	0.06	-10.72
		1.75	16098.90	-14975.03	-15972.50	0.21	0.09	0.09	0.32	0	95.05	13.90	0	768.99	185.42	-16098.25
	10	0.75	0.11	-0.05	0.03	0.17	0.01	0.00	0.86	0	0.16	0.17	0	0.08	0.21	0.54
		1.00	1.14	21.93	0.41	0.08	0.05	0.05	0.05	0	0.03	0.05	0	0.01	0.00	-0.48
		1.75	1609.89	-882.58	-1598.85	0.17	0.06	0.04	0.26	0	28.10	2.69	0	169.51	22.78	-1609.24

Table 8:

n	r	d_0	ν_0	Trend					Cycle				
				R^2 CSS	R^2 QML	R^2 ARMA	${R^2}^{(1)}$ CSS	${R^2}^{(1)}$ QML	R^2 CSS	R^2 QML	R^2 ARMA	${R^2}^{(1)}$ CSS	${R^2}^{(1)}$ QML
100		0.75	1.00	0.54	0.61	0.55	0.27	0.12	0.79	0.83	0.77	0.64	0.00
		1.00	1.00	0.62	0.65	0.58	0.65	0.18	0.86	0.88	0.80	0.87	0.02
		1.75	1.00	0.65	0.65	0.61	0.63	0.26	0.77	0.79	0.68	0.76	0.02
	1	0.75	0.65	0.49	0.56	0.50	0.28	0.08	0.82	0.86	0.78	0.73	0.01
		1.00	3.82	0.75	0.78	0.75	0.77	0.41	0.79	0.82	0.78	0.82	0.02
		1.75	1053.88	0.81	0.80	0.81	0.76	0.80	0.54	0.55	0.57	0.22	0.06
	10	0.75	0.07	0.92	0.78	0.61	0.91	0.87	0.21	0.16	0.13	0.15	0.01
		1.00	0.38	0.82	0.84	0.72	0.81	0.83	0.09	0.38	0.28	0.20	0.02
		1.75	105.39	0.93	0.96	0.94	0.91	0.81	0.84	0.91	0.87	0.75	0.03
	30	0.75	0.02	0.78	0.73	0.55	0.77	0.73	0.10	0.27	0.20	0.05	0.01
		1.00	0.13	0.73	0.79	0.76	0.76	0.66	0.51	0.64	0.60	0.59	0.03
		1.75	35.13	0.87	0.91	0.84	0.87	0.65	0.87	0.93	0.83	0.84	0.02
200		0.75	1.00	0.64	0.71	0.66	0.34	0.13	0.85	0.89	0.86	0.71	0.00
		1.00	1.00	0.69	0.71	0.66	0.70	0.18	0.88	0.90	0.86	0.89	0.02
		1.75	1.00	0.67	0.68	0.65	0.66	0.23	0.79	0.79	0.71	0.78	0.01
	1	0.75	0.93	0.63	0.70	0.65	0.34	0.12	0.86	0.89	0.86	0.73	0.00
		1.00	7.59	0.84	0.85	0.85	0.86	0.55	0.81	0.83	0.82	0.85	0.02
		1.75	5871.84	0.79	0.79	0.79	0.78	0.79	0.58	0.59	0.60	0.17	0.04
	10	0.75	0.09	0.91	0.85	0.81	0.90	0.87	0.15	0.20	0.18	0.07	0.01
		1.00	0.76	0.83	0.89	0.87	0.84	0.82	0.36	0.65	0.58	0.54	0.02
		1.75	587.18	0.96	0.98	0.96	0.96	0.79	0.92	0.95	0.93	0.85	0.02
	30	0.75	0.03	0.76	0.80	0.75	0.76	0.74	0.05	0.43	0.36	0.03	0.01
		1.00	0.25	0.86	0.87	0.86	0.88	0.66	0.77	0.82	0.80	0.83	0.03
		1.75	195.73	0.93	0.95	0.90	0.92	0.63	0.94	0.96	0.90	0.91	0.02
300		0.75	1.00	0.69	0.76	0.72	0.41	0.14	0.88	0.90	0.89	0.76	0.00
		1.00	1.00	0.71	0.73	0.70	0.72	0.18	0.88	0.89	0.87	0.89	0.02
		1.75	1.00	0.68	0.70	0.66	0.66	0.21	0.79	0.80	0.72	0.78	0.01
	1	0.75	1.14	0.71	0.77	0.73	0.39	0.16	0.87	0.90	0.88	0.72	0.00
		1.00	11.37	0.87	0.87	0.87	0.87	0.62	0.82	0.84	0.83	0.85	0.02
		1.75	16098.90	0.79	0.79	0.79	0.78	0.79	0.60	0.60	0.61	0.15	0.03
	10	0.75	0.11	0.90	0.87	0.85	0.89	0.87	0.12	0.22	0.22	0.04	0.00
		1.00	1.14	0.91	0.93	0.92	0.93	0.82	0.70	0.76	0.74	0.77	0.03
		1.75	1609.89	0.98	0.98	0.98	0.97	0.78	0.95	0.97	0.96	0.90	0.02
	30	0.75	0.04	0.76	0.84	0.79	0.76	0.74	0.06	0.54	0.45	0.05	0.00
		1.00	0.38	0.90	0.90	0.90	0.91	0.66	0.85	0.86	0.85	0.87	0.03
		1.75	536.63	0.95	0.96	0.94	0.95	0.62	0.96	0.97	0.94	0.94	0.01

Shimotsu, Katsumi, and Peter C. B. Phillips. 2005. "Exact Local Whittle Estimation of Fractional Integration." *The Annals of Statistics* 33 (4): 1890–1933.