MFE Bootcamp

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Abstract

These notes are written for part 2 of the UCLA Anderson Master of Financial Engineering Bootcamp. I have pulled together many concepts that will be useful for the MFE program and tried to make example code where I believe it might be useful. The material is gathered from lectures that I have taken myself, and some are my very own notes. If you find any typos (there are likely a lot of typos) or have any feedback, then please let me know.

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1 Linear Algebra

Motivation: As quantitative economists, we often rely on linear algebra as a tool for solving simple and complex problems. For example, linear algebra allows us to find solutions to linear systems:

$$\begin{aligned} y_1 &= \phi_1 x_{11} + \phi_2 x_{12} + ... + \phi_k x_{1k}, \\ &\vdots \\ y_n &= \phi_1 x_{n1} + \phi_2 x_{n2} + ... + \phi_k x_{nk}, \end{aligned}$$

where we are interested in the unknowns ϕ_i given series of observables x_i and y_i. An example of a system of interest is the famous Capital Asset Pricing Model:

$$r_{i,t} - r^f = \alpha_i + \beta_i (r_{m,t} - r^f),$$

where $r_{i,t}$ is the return of asset i on time t, $r_{m,t}$ is the market return on time t, and r^f is the risk-free rate of interest. In this model we try to explain the excess return of all assets by each assets sensitivity to the market (more on this later). Important things to consider when we work with these problems:

- How many solutions (if any) exists?
- If there are no closed-form solution to this system, can we still approximate it?
- How do we compute the solution?

Linear algebra can also simplify notation. Consider the simple case of computing the variance of a portfolio of two assets: We know that the total variance of two variables, in this case stock 1 and stock 2, s_1 and s_2 can be computed as

$$\text{Var}(s_1 + s_2) = Var(s_1) + Var(s_2) + 2cov(s_1, s_2).$$

Now, let ω_1 and ω_2 be the portfolio weights of stock 1 and stock 2. The portfolio variance is now:

$$\text{Var}(\omega_1 s_1 + \omega_2 s_2) = \omega_1^2 Var(s_1) + \omega_2^2 Var(s_2) + 2\omega_1\omega_2 cov(s_1, s_2).$$

For an n-asset portfolio this becomes

$$\text{Var}(\omega_1 s_1 + ... + \omega_n s_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_i \omega_j Cov(s_i, s_j).$$

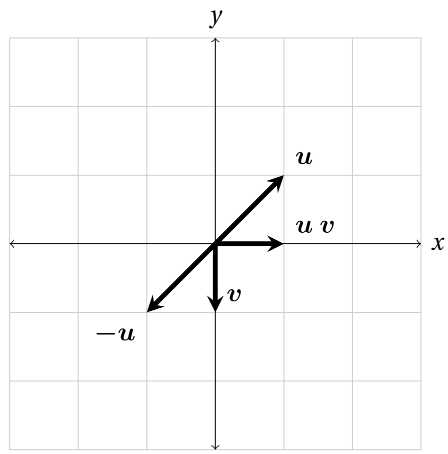
By using matrix notation we get

$$\text{Var}(\omega_1 s_1 + ... + \omega_n s_n) = \boldsymbol{\omega}' \boldsymbol{\Sigma} \boldsymbol{\omega},$$

where $\boldsymbol{\Sigma}$ is and $(n \times n)$ covariance matrix and $\boldsymbol{\omega}$ is a vector of weights.

Vectors

A vector in R^n is an n-tuple of numbers and represent a direction and magnitude.

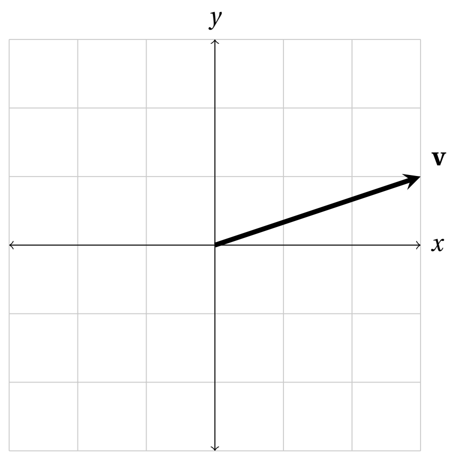


We can express the same vector in two ways:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{x}^t = [x_1, x_2, ..., x_3].$$

Example: In R^2 :

$$\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$



The ordering matters:

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

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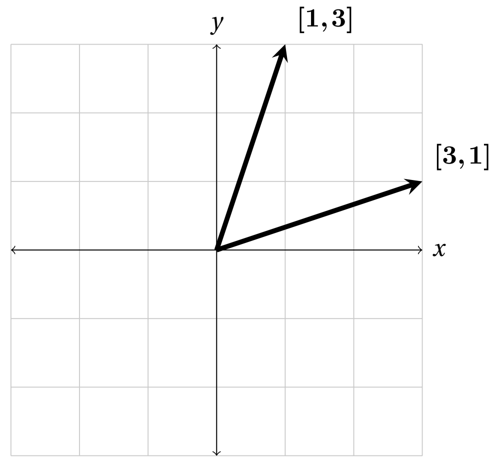


What can we do with vectors?

An n-vector, $\mathbf{u}$, can be multiplied by a number c. This is called scalar multiplication and yields a new vector n-vector, $c\mathbf{u}$.

Example: (n=3) Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $c = 3$. Then,

$$3\mathbf{u} = 3\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \times 2 \\ 3 \times (-1) \\ 3 \times 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 0 \end{bmatrix}.$$

We can also add and subtract to vectors $\mathbf{u}$, $\mathbf{v}$:

$$\mathbf{u}' + \mathbf{v}' = (u_1 + v_1, u_2 + v_2, +\ldots+, u_n + v_n),$$
$$\mathbf{u}' - \mathbf{v}' = (u_1 - v_1, u_2 - v_2, -\ldots-, u_n - v_n).$$

Definition 1.1: Vector Space

Formally, a vector space over $\mathbb{R}$ is a tuple $(V, +, \cdot)$ where V is a set, +: $V \times V \to V$ and $\cdot$: $\mathbb{R} \times V \to V$ such that

1. $x + y = y + x$ for all $x, y \in V$,
2. $(x + y) + z = x + (y + z)$ for all $x, y \in V$,
3. there exist $0 \in V$ such that $x + 0$=x for all $x \in V$,
4. for every $x \in V$, there exist $-x \in V$ such that $x + (-x) = 0$,
5. $1 \cdot x = x$ for all $x \in V$,
6. $(ab) \cdot x = a \cdot (b \cdot x)$ for all $a, b \in \mathbb{R}$ and $x \in V$,
7. $a \cdot (x + y) = a \cdot x + a \cdot y$ for all $a \in \mathbb{R}$ and $x, y \in V$.
8. $(a + b) \cdot x = a \cdot x + b \cdot x$ for all $a, b \in \mathbb{R}$ and $x \in V$

Example: Let $n \in \mathbb{N}$. Let

$$V = \mathbb{R}^n = \{(x_1, ..., x_n) | x_1, ..., x_n \in \mathbb{R}\}.$$

Given $(x_1, ..., x_n), (y_1, ..., y_n) \in V$ and $a \in \mathbb{R}$, define $(x_1, ..., x_n) + (y_1, ..., y_n) \in V$ and $a \cdot (x_1, ..., x_n) \in V$ by

$$(x_1, ..., x_n) + (y_1, ..., y_n) = (x_1 + y_1, ..., x_n + y_n)$$

and

$$a \cdot (x_1, ..., x_n) = (ax_1, ..., ax_n).$$

Then $(V, +, \cdot)$ is a vector space.

1.1 Exercise

1. Let S be a nonempty set. Let $V = \{f : S \to \mathbb{R}\}$. Given $f, g \in V$ and $a \in \mathbb{R}$, define $f + g \in V$ and $a \times f \in V$ by

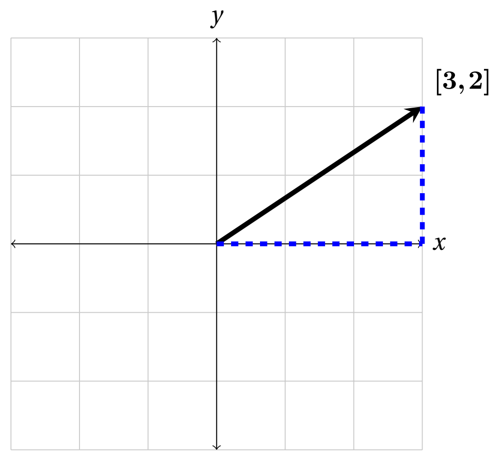
$$(f + g)(s) = f(s) \quad \text{for } s \in S,$$

and

$$(a \cdot f)(s) = af(s) \quad \text{for } s \in S.$$

Show that $(V, +, \cdot)$ is a vector space.

1.2 Length of a vector, inner product, and the Cauchy-Schwarz inequality



Lenght of a vector: This one is easy, we have a vector of length 2 and we remember the Pythagorean theorem:

$$c^2 = b^2 + a^2.$$

For a vector **x** of length n:

$$\|\mathbf{x}\| = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

This is called the **Euclidean norm**. We can verify it for the example above where $\mathbf{x}' = (3, 2) : \|\mathbf{x}\| = \sqrt{3^2 + 2^2}$, which is equal to what we learned using the Pythagorean theorem. Also, for any two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\|\mathbf{x} - \mathbf{y}\|$ measures the distance between **x** and **y**.

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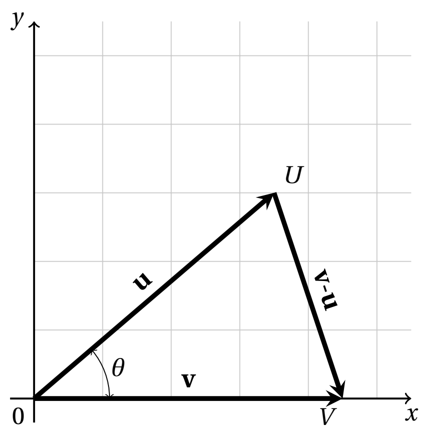
Inner product of two vectors $\mathbf{v}, \mathbf{u} \in \mathbb{R}^n$ is defined by $\mathbf{v} \cdot \mathbf{u} = \sum_{i=1}^{n} x_i y_i$. For example, let ω be a vector of portfolio weights and $\mathbf{p}$ be a price vector for every asset in the portfolio. The value of the portfolio is then given by $\omega \cdot \mathbf{p} = \sum_{i=1}^{n} \omega_i p_i$.

Cauchy-Schwarz inequality:

$$\mathbf{u} \cdot \mathbf{v} \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$$

In fact,

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cdot cos(\theta)$$



According to Pythagora's theorem, the angle θ between two vectors $\mathbf{u}$ and $\mathbf{v}$ is a right angle (= 90°) if and only if $(OU)^2 + (OB)^2 = (VU)^2$, or $\|\mathbf{u}\|^2 + \|\mathbf{u}\|^2 = \|\mathbf{v} - \mathbf{u}\|^2$. This implies that $\theta = 90°$ if and only if

$$\mathbf{v} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{u} = (\mathbf{v} - \mathbf{u})(\mathbf{v} - \mathbf{u}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v}.$$

Because of symmetry, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$, so we need $2\mathbf{u} \cdot \mathbf{v} = 0$ for the equality to hold, and so $\mathbf{u} \cdot \mathbf{v} = 0$. When the angle between two vectors $\mathbf{u}$ and $\mathbf{v}$ is $90°$, we say that they are **orthogonal.** If two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal we write $\mathbf{u} \perp \mathbf{v}$. Here we proved that two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are orthogonal if and only if their inner product is 0. For vectors in $\mathbb{R}^n$, we define orthogonality between $\mathbf{u}$ and $\mathbf{v}$ as

$$\mathbf{u} \perp \mathbf{v} \iff \mathbf{u} \cdot \mathbf{v} = 0.$$

Example: Suppose we have n observations of a commodity's price and quantity demanded $(p_1, d_1), (p_2, d_2)$, ..., (p_n, d_n). Define the means as

$$\overline{\mathbf{p}} = \frac{1}{n} \sum_{i=1}^{n} p_i, \quad \overline{\mathbf{d}} = \frac{1}{n} \sum_{i=1}^{n} d_i,$$

and

$$\mathbf{v} = (p_1 - \overline{\mathbf{p}}, p_2 - \overline{\mathbf{p}}, ..., p_n - \overline{\mathbf{p}}), \quad \mathbf{u} = (d_1 - \overline{\mathbf{d}}, d_2 - \overline{\mathbf{d}}, ..., d_n - \overline{\mathbf{d}}).$$

Then the **correlation coefficient** ρ can be computed as

$$\rho = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\| \cdot \|\mathbf{u}\|} = cos(\theta).$$

Projection

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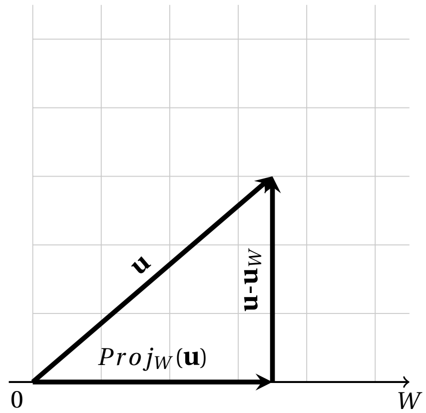


$\mathbf{u}_W$ is such that

1. $\mathbf{u}_W$ is parallell with W: There exist a scalar c such that $\mathbf{u}_W = cW$,
2. $\mathbf{u}_w \perp (\mathbf{u} - \mathbf{u}_W)$

Now we can find c, i.e. find $\mathbf{u}_w$. 2. means that

$$\begin{aligned} \mathbf{u}_W \cdot (\mathbf{u} - \mathbf{u}_W) &= 0, \\ \mathbf{u}_W \cdot \mathbf{u} - \mathbf{u}_W \cdot \mathbf{u}_W &= 0, \qquad \text{(Distributive law (VS 7))} \\ \mathbf{u}_W \cdot \mathbf{u} &= \|\mathbf{u}_W\|^2. \qquad (*) \end{aligned}$$

From 1. we have $\mathbf{u}_W = cW$. Insert $(*)$ into 1. to get $cW \cdot \mathbf{u} = \|cW\|^2 = c^2\|W\|^2$. Solve for c:

$$c = \frac{W \cdot \mathbf{u}}{\|W\|^2}.$$

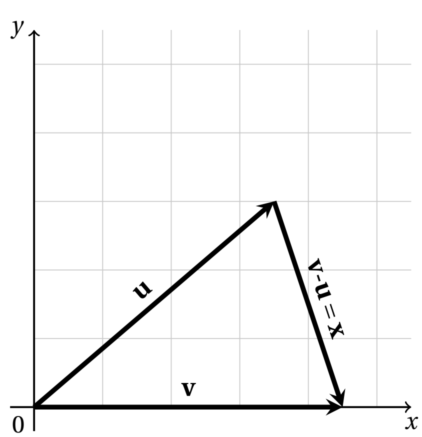
Conclusion:

$$Proj_W(\mathbf{u}) = \mathbf{u}_W = \frac{W \cdot \mathbf{u}}{\|W\|^2} \cdot W.$$

Example: Linear regression as a projection (later).

Linear independence of vectors, basis and linear subspace.

We have already seen that we can create a new vector $\mathbf{x}$ by two other vectors:



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A natural question then becomes: how many other vectors can we create by linear combinations of **u** and **v**?

Definition 1.2: Span

Assume $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are n-vectors. Then, span($\mathbf{v}_1, \ldots, \mathbf{v}_m$) are all the vectors that are linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ $(c_1\mathbf{v}_1, c_2\mathbf{v}_2, \ldots, c_m\mathbf{v}_m)$.

Example: (m=1) span($\mathbf{v}_1$) are all vectors $c_1\mathbf{v}_1$ where $c_1 \in \mathbb{R}$.

Exercise: (m=3) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$. What is span($\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$)?

Definition 1.3: Linear dependence

The vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ are **linearly dependent** if one or more of the vectors can be written as a linear combination of the others.

Example: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ are linear dependent since $\mathbf{v}_2 = \frac{1}{3}\mathbf{v}_3 - \frac{2}{3}\mathbf{v}_1$.

Definition 1.4: Basis

If the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ are not linearly dependent we say that they are **linearly independent**. Then the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ form a **basis** for span($\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$).

Example: Consider the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$. $\{\mathbf{v}_1, \mathbf{v}_2\}$ are linearly independent (why?) therefore forms a basis for the xy-plane. Note: $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^2$ but is not a basis since they are linearly dependent.

Definition 1.5: Dimension

The number of vectors is in the basis is called the dimension of the linear subspace.

Example: Consider the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$. The dimension of span($\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$) is 2.

Linear systems

$$\begin{aligned} 2x_1 - 4x_2 + 2x_3 &= 12, \\ 3x_1 - 5x_2 - 2x_3 &= 5, \\ -4x_1 + 7x_2 + 4x_3 &= -2. \end{aligned}$$

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Write $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -4 \\ -5 \\ 7 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 12 \\ 5 \\ -2 \end{bmatrix}$. The linear system now becomes

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 = \mathbf{v}.$$

Question: Does the above system have a solution? (i.e. is $\mathbf{v}$ in span$(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$?) → Gaussian Elimination.

Definition 1.6: Gaussian elimination

- a method for solving all linear systems.

1. Write the linear system as a matrix
2. From this matrix, create a new matrix in echelon form using elementary row operations
3. Translate the new matrix to a simpler linear system.
4. Solve this system using backwards substitution.

Definition 1.7: Echelon form

1. 0-rows are in the bottom rows of the matrix
2. Every pivot (leading coefficient) is longer to the right than the pivots in the rows above it (a pivot is the first number in a row that is non-zero.)

Definition 1.8: Free variables

The variables that corresponds to columns without pivots (but not the last column).

Example:

$$\begin{bmatrix} 6 & 4 & 0 & 1 \\ 0 & 4 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above matrix is in echelon form, where the blue numbers are **pivots** and the variable corresponding to the third column is free.

Exercise: Write the linear system

$$\begin{aligned} 2x_1 - 4x_2 + 2x_3 &= 12, \\ 3x_1 - 5x_2 - 2x_3 &= 5, \\ -4x_1 + 7x_2 + 4x_3 &= -2, \end{aligned}$$

in matrix form and solve it using Gaussian elimination.

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1.3 The solutions

Every linear system has either

1. No solution $\iff$ pivot in the last column.
2. A unique solution $\iff$ pivot in all columns except the last.
3. Infinitely many solutions $\iff$ no pivot in last column and at least one free variable.

1.4 Rank of a matrix

The rank of a matrix is the number of pivots in the echelon form.

Example:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -1 & 3 \\ -4 & -3 & 9 & 5 \\ 6 & 12 & -3 & 9 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 3 \\ 0 & 5 & 7 & 11 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here we have to pivots, so rk(A)=2.

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Here we have three pivots, so rk(B)=3.

```
from numpy.linalg import matrix_rank
A = np.matrix('2 4 -1 3;-4 -3 9 5;6 12 -3 9')

A
Out[10]:
matrix([[ 2,  4, -1,  3],
        [-4, -3,  9,  5],
        [ 6, 12, -3,  9]])

matrix_rank(A)
Out[11]: 2

B = np.eye(3)

B
Out[16]:
array([[1., 0., 0.],
       [0., 1., 0.],
       [0., 0., 1.]])

matrix_rank(B)
Out[17]: 3
```

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1.5 Applications

Let $\{\mathbf{u}_1, \mathbf{u}_2, ...\mathbf{u}_m\}$ be n-vectors.

1. The vectors are linearly independent $\iff x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + ... + x_m\mathbf{u}_m = 0$ only has the trivial solution $(x_1 = x_2 = ... = x_m = 0)$. [a system on the form $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + ... + x_m\mathbf{u}_m = 0$ is called a homogeneous system]
2. The vectors corresponding to pivot positions in the echelon form gives a basis for span$(\mathbf{u}_1, \mathbf{u}_2, ...\mathbf{u}_m)$

Matrices

Definition 1.9: Transpose

The transpose of an (m×n)-matrix A gives an (n×m) matrix A' where the first column in A becomes the first row of A', etc.

Example: $A = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 7 & 5, \end{bmatrix}$, $A' = \begin{bmatrix} 1 & 2 \\ 0 & 7 \\ -3 & 5 \end{bmatrix}$.

```
import numpy as np

A = np.matrix('1 0 -3; 2 7 5')


Out[4]:
matrix([[ 1,  0, -3],
        [ 2,  7,  5]])

A.T
Out[5]:
matrix([[ 1,  2],
        [ 0,  7],
        [-3,  5]])
```

What can we do with matrices? Almost the same operations as with scalars, however **the order is important. Example:**

$$\begin{aligned} A(B+C) &= AB + AC, \\ (A+B)^2 &= A^2 + AB + BA + B^2, \\ (A')' &= A, \\ c(A+B) &= cA + cB, \quad etc. \end{aligned}$$

Important: AB is not necessarily equal to BA:

```
import numpy as np
A = np.matrix(np.random.randint(0,10, size=(3, 3)))
B = np.matrix(np.random.randint(0,10, size=(3, 3)))

A
Out[28]:
matrix([[0, 0, 1],
        [3, 0, 4],
        [5, 7, 9]])

B
Out[29]:
matrix([[8, 0, 8],
        [4, 5, 9],
        [3, 0, 4]])

C = np.matmul(A,B)
D = np.matmul(B,A)
C
Out[32]:
matrix([[  3,   0,   4],
        [ 36,   0,  40],
        [ 95,  35, 139]])

D
Out[33]:
matrix([[ 40,  56,  80],
        [ 60,  63, 105],
        [ 20,  28,  39]])
```

Linear systems using matrix notation (m equations, n unknowns)

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m. \end{aligned}$$

Write

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{11} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

We can now write the linear system above as

$$A\mathbf{x} = \boldsymbol{b}.$$

Definition 1.10: The identity matrix

Let $A = (a_{ij})$ be a **quadratic** $(n \times n)$ matrix where $a_{ij} = 1$ for $i = j$ and 0 otherwise:

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is called the **identity matrix**. We then have that

$$AI = A = IA$$

Example:

```
import numpy as np
A = np.matrix(np.random.randint(0,10, size=(3, 3)))
I = np.eye(3)
A
Out[36]:
matrix([[5, 8, 7],
        [7, 1, 2],
        [5, 2, 1]])

I
Out[37]:
array([[1., 0., 0.],
       [0., 1., 0.],
       [0., 0., 1.]])

np.matmul(A, I)
Out[38]:
matrix([[5., 8., 7.],
        [7., 1., 2.],
        [5., 2., 1.]])

np.matmul(I,A)
Out[39]:
matrix([[5., 8., 7.],
        [7., 1., 2.],
        [5., 2., 1.]])
```

Definition 1.11: Invertability of a matrix

An $(n \times n)$ matrix A is invertible if there exist an $(n \times n)$ matrix B such that

$$AB = I = BA$$

If A is invertible, B is unique such that $AB = I = BA$.

```
import numpy as np
A = np.matrix(np.random.randint(0,10, size=(3, 3)))
B = np.linalg.inv(A)

A
Out[41]:
matrix([[1, 6, 1],
        [8, 7, 6],
        [4, 1, 7]])

B
Out[42]:
matrix([[-0.25443787,  0.24260355, -0.17159763],
        [ 0.18934911, -0.01775148, -0.01183432],
        [ 0.1183432 , -0.13609467,  0.24260355]])

np.matmul(A,B)
Out[43]:
matrix([[ 1.00000000e+00, -2.77555756e-17,  0.00000000e+00],
        [ 2.22044605e-16,  1.00000000e+00,  5.55111512e-17],
        [ 5.55111512e-17, -2.77555756e-17,  1.00000000e+00]])

np.matmul(B,A)
Out[44]:
matrix([[ 1.00000000e+00,  1.38777878e-16,  0.00000000e+00],
        [-6.93889390e-18,  1.00000000e+00, -5.20417043e-18],
        [-1.11022302e-16, -5.55111512e-17,  1.00000000e+00]])
```

The result above is very useful in economics since it allows us to solve linear systems very easily. Remember that we can write a linear system on the form

$$A\mathbf{x} = \boldsymbol{b}.$$

We might ask ourselves, what $\mathbf{x}$ will solve the system above? That is, we want to isolate $\mathbf{x}$ on the left side of the equation. Let B be the inverse of A (we often write A^{-1}), then:

$$\begin{aligned} A\mathbf{x} &= \boldsymbol{b}, \\ BA\mathbf{x} &= B\boldsymbol{b}, \\ I\mathbf{x} &= B\boldsymbol{b}. \end{aligned}$$

⇒ a very easy way to solve a system!

Definition 1.12: Determinants

For every (n×n) matrix A there exist a number det(A) (often $|A|$) so that the following is true:

1. det(AB) = det(A)· det(B)
2. det(A') = det(A)

Theorem: A invertible $\iff$ det(A)$\neq 0$.

If the determinant of A is not zero, then we say that is **nonsingular**.

Useful results: for an (n×n) matrix A where det(A) = 0:

1. The columns of A are dependent vectors in $\mathbb{R}^n$
2. The rows of A are dependent vectors in $\mathbb{R}^n$
3. A is not invertible

Definition 1.13: Eigenvalues and eigenvectors

The (n×n) matrix A has the eigenvalue λ (a number) if the equation

$$A\mathbf{x} = \lambda\mathbf{x}$$

has a non-trivial solution (i.e. $\mathbf{x} = \boldsymbol{b} \neq \mathbf{0}$). Then the vector $\boldsymbol{b}$ is an eigenvector to A with eigenvalue λ.

Example:

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad \text{then} \quad A\mathbf{x} = 3\mathbf{x},$$

so $\lambda = 3$ is an eigenvalue to A and all 2-vectors are eigenvectors to A with eigenvalue 3.

Example:

$$A = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}.$$

For what number λ does the system

$$A\mathbf{x} = \lambda\mathbf{x}$$

have non-trivial solutions?

$$\text{i.e.} \quad \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix}$$

$$\text{i.e.} \quad \begin{cases} 4x_1 + 2x_2 = \lambda x_1 \\ -3x_1 - x_2 = \lambda x_2 \end{cases}$$

$$\text{i.e.} \quad \begin{cases} (4-\lambda)x_1 + 2x_2 = 0 \\ -3x_1 - (1+\lambda)x_2 = 0 \end{cases}$$

$$\text{i.e.} \quad \underbrace{\begin{bmatrix} 4-\lambda & 2 \\ -3 & -(1+\lambda) \end{bmatrix}}_{B} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \text{a homogeneous system!}$$

Generally: A homogeneous system $B\mathbf{x} = \mathbf{0}$ has non-trivial solutions $\iff$ det(B) = 0. **Question:** Why? Therefore,

$$|B| = 0 \iff (4-\lambda)(-(1+\lambda)) - 2\cdot(-3) = \lambda^2 - 3\lambda + 2 = 0.$$

We solve this second-order equation to get

$$\lambda = \frac{-(-3) \pm \sqrt{(-3)^2 - 4\cdot 2}}{2} = \begin{cases} 1 \\ 2 \end{cases}$$

So $\lambda = 1$ and $\lambda = 2$ are the eigenvalues of A.

Exercise: Find the eigenvectors of A.

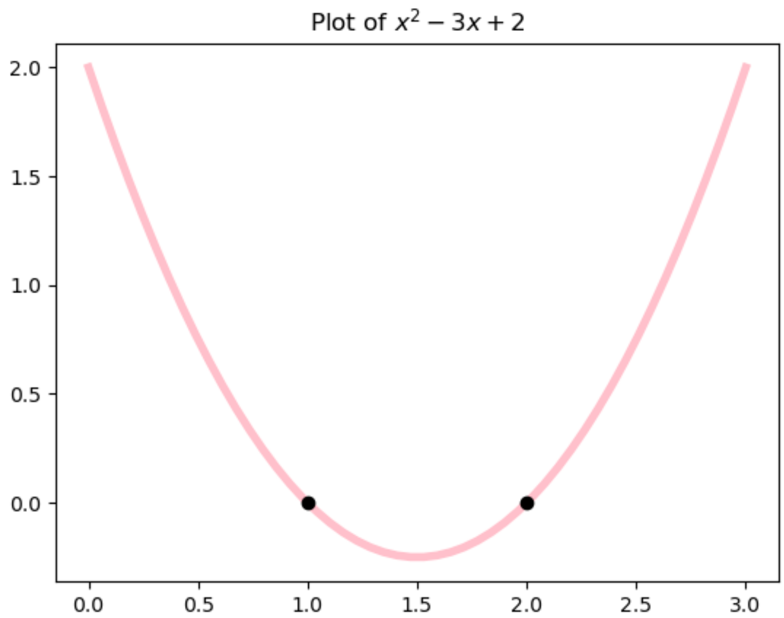

Figure 1: The roots are given by the black dots

1.5.1 Sidenote: Root-finding algorithms

In the previous example we solved a second-order equation of the form

$$x^2 - 3x + 2 = 0.$$

Although these can be straight-forward computed by hand, it's easy to make mistakes (and who solves problems by hand these days anyways?). Moreover, this becomes **very** complicated as the degree of the polynomial increase. A useful tool is therefore a root-finder. A root-finding algorithm is an algorithm for finding zeroes ("roots") of continuous functions.

```
import numpy as np

# Find the roots of f(x) = x^2 -3x +2
coeff = [1,-3,2]
np.roots(coeff)

Out[7]: array([2., 1.])
# The roots are 2 and 1.

# The figure
import matplotlib.pyplot as plt
def f(x):
    return (x**2 -3*x +2 )

x = np.linspace(0,3,50)
y = f(x)

plt.figure()
```

```
plt.plot(x,y,c='pink',linewidth=4, markevery=[1,2])
plt.plot(1,0,"-o",c='black')
plt.plot(2,0,"-o",c='black')
plt.title('Plot of $x^2-3x +2$')
```

1.6 Quadratic forms

Quadratic forms are polynomials where all terms are of degree two. Examples:

1. $Q_1(\mathbf{x}) = 3x_1^2 + 8x_1x_2 + 5x^2$
2. $Q_2(\mathbf{x}) = x_1^2 - 3x_2^2 + 5x_3^2 + 7x_2x_3 + 9x_1x_3 - 11x_1x_2$

For every quadratic form $Q(\mathbf{x})$ in n variables there exist a symmetrix $(n \times n)$ matrix A such that

$$Q(\mathbf{x}) = \mathbf{x}'A\mathbf{x}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{so } \mathbf{x}' = [x_1, x_2, \ldots, x_n].$$

Example:

$$A = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}.$$

$$\mathbf{x}'A\mathbf{x} = [x_1, x_2] \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1^2 + 8x_1x_2 + 5x_2^2 = Q(\mathbf{x}).$$

1.7 Definiteness of quadratic forms

Let $Q(\mathbf{x})$ be a quadratic form with symmetric matrix A.

Definition 1.14: Definiteness

$Q(\mathbf{x})$ is positive semidefinite definite if $Q(\mathbf{x}) \geq 0$ for all values of $\mathbf{x}$ (and positive definite if $Q(\mathbf{x}) > 0$ for all $x \neq 0$.)

$Q(\mathbf{x})$ is negative semidefinite definite if $Q(\mathbf{x}) \leq 0$ for all values of $\mathbf{x}$ (and negative definite if $Q(\mathbf{x}) < 0$ for all $x \neq 0$.)

If $Q(\mathbf{x})$ is neither positive or negative semidefinite, then it is indefinite.

Example:

$$Q(\mathbf{x}) = 3x_1^2 + 5x_2^2 > \forall \mathbf{x} \neq \mathbf{0},$$

i.e. $Q(\mathbf{x})$ is positive definite (and positive semidefinite).

What about $Q(\mathbf{x}) = 3x_1^2 + 8x_1x_2 + 5x_2^2$? $8x_1x_2$ is negative if x_1 and x_2 have opposite signs - it's not clear when we have crossterms in $Q(\mathbf{x})$.

Theorem Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of A. Then,

$Q(\mathbf{x})$

$$\begin{aligned} \text{positive semidefinite} &\iff \text{all } \lambda_i \geq 0 \\ \text{positive definite} &\iff \text{all } \lambda_i > 0 \\ \text{negative semdefinite} &\iff \text{all } \lambda_i \leq 0 \\ \text{negative definite} &\iff \text{all } \lambda_i < 0 \\ \text{indefinite} &\iff \lambda_i < 0 \text{ and } \lambda_j > 0 \text{ for some } i, j. \end{aligned}$$

2 Optimization

We are often faced with optimization problems in economics. A classic example is consumption and investment decisions, where an agent can choose to either consume some (or even all) of her wealth, and invest the remaining in assets which she can use to consume next period. This gives rise to yet another problem; what portfolio should she invest in? Should it be equal-weighted in risky stocks? Should she perhaps allocate some of her wealth to risk-free T-bills? These questions can be answered using optimization.

Definition 2.1: Extreme points

If $f(x)$ has domain D then

$$c \in D \text{ is a } \textbf{maximum point} \text{ for } f \iff f(x) \leq f(c) \text{ for all } x \in D,$$
$$d \in D \text{ is a } \textbf{minimum point} \text{ for } f \iff f(x) \geq f(d) \text{ for all } x \in D.$$

If the value of f at c is strictly larger than at any other point in D, then c is a **strict maximum point**. Similarly,d is a **strict minimum point** if $f(x) > f(d)$ for all $x \in D$, $x \neq 0$.

Note: If f is any function with domain D: $f(x) \leq f(c)$ for all x in D iff $-f(x) \geq -f(c)$ for all x in D. Implication: c maximizes f in D iff c minimizes $-f$ in D. This is sometimes useful in numerical optimization if the optimizer only minimize.

Example: find the extreme points for $g(x) = 3-(x-2)^2$.

Solution:

Since $(x-2)^2 \geq 0$ for all x, it follows that $f(x) \leq 3$ for all x. But $f(x) = 3$ when $(x-2)^2 = 0$ at $x = 2$. Therefore, $x = 2$ is a **maximum point** for f. Because $f(x) \to -\infty$ as $x \to \pm\infty$, f has no minimum.

Example: find the extreme points for $f(x) = \sqrt{x-5} - 100, \quad x \geq 5$.

Solution:

Because $\sqrt{x-5}$ is ≥ 0 for all $x \geq 5$, it follows that $f(x) \geq -100$ for all $x \geq 5$. Since $f(5) = -100$, we conclude that $x = 5$ is a **minimum point**. Since $f(x) \to \infty$ as $x \to \infty$, f has no maximum.

Definition 2.2: Derivative of a function

The derivative of a function f at point a, denoted by $f'(a)$, is given by the formula

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

$f'(a)$ = slope of the tangent to the curve $y = f(x)$ at the point $(a, f(a))$.

Definition 2.3: Necessary first-order condition

Suppose that a function f is differentiable in an interval I and that c is an interior point for I. For $x = c$ to be a maximum or minimum point for f in I, a necessary condition is that it is a

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stationary point for f, i.e. that $x = c$ satisfies the equation

$$f'(x) = 0.$$

We call this the **first-order condition.**

Definition 2.4: First-derivative test for max/min

If $f'(x) \geq 0$ for $x \leq c$ and $f'(x) \leq 0$ for $x \geq c$, then c is a maximum point for f. If $f'(x) \leq 0$ for $x \leq c$ and $f'(x) \geq 0$ for $x \geq c$, then c is a minimum point for f.

Exercise

Consider the function f defined for all x by

$$f(x) = e^{2x} - 5e^{x} + 4$$

(a) Find the zeros of $f(x)$ and compute its derivative $f'(x)$.
(b) Find the intervals where f increases and decreases, and determine possible extreme points and values.
(c) Examine the limit of $f(x)$ as $x \to -\infty$. Sketch the graph of f.
(d) Repeat (a)-(c) on the computer using Python. Try to avoid pre-programmed routines but rather write your own functions.

Definition 2.5: Second-order condition

Suppose that f is a $\mathscr{C}^2$ function (zeroth, first, and second derivative of f are continuous) and x be an interior point of I. Then:

(a) $f'(x) = 0$ and $f''(x) < 0 \Rightarrow$ c = x is a **strict** local maximum point
(b) $f'(x) = 0$ and $f''(x) > 0 \Rightarrow$ c = x is a **strict** local minimum point
(c) $f'(x) = 0$ and $f''(x) = 0 \Rightarrow$?

Exercise

(a) Classify the stationary points of

$$f(x) = {}^{1}/_{9}x^3 - {}^{1}/_{6}x^2 - {}^{2}/_{3}x + 1$$

by using the second-derivative test.
(b) Find a two examples that satisfies (c) while being different local extreme points.

Definition 2.6: Extreme Value Theorem

Suppose that f is a continuous function over a closed and bounded interval $[a, b]$. Then there exist a point d in $[a, b]$ where f has a minimum, and a point c in $[a, b]$ where f has a maximum, so that

$$f(d) \leq f(x) \leq f(c) \quad \text{for all } x \text{ in } [a, b].$$

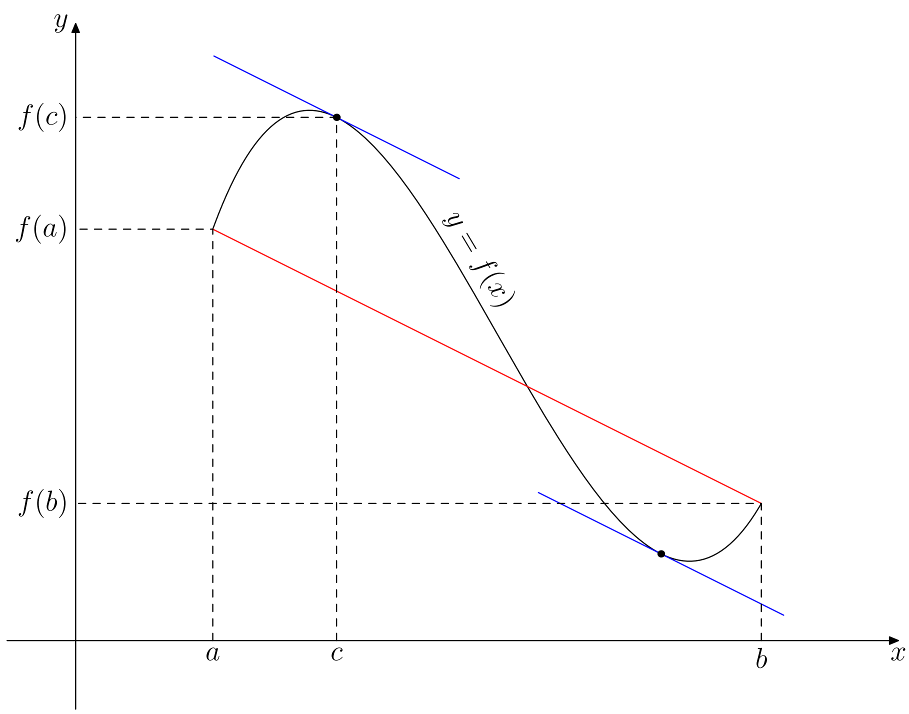
This result is useful in **constrained optimization problems** where we consider optimization problems over closed and bounded intervals.

Definition 2.7: The Mean Value Theorem

If f is continuous in the closed bounded interval $[a, b]$, and differentiable in the open interval (a, b), then there exists at least one interior point y in (a, b) such that

$$f'(y) = \frac{f(b) - f(a)}{b - a}.$$

"Proof":



2.1 Stationary points of quadratic functions

A second-order function:

$$\begin{aligned} f(\mathbf{x}) = & \underbrace{Q(\mathbf{x})}_{\text{quadr. form}} + \underbrace{L(\mathbf{x})}_{\text{linear form}} + \underbrace{c}_{\text{constant}} \\ = & \mathbf{x}' A\mathbf{x} + B\mathbf{x} + c. \end{aligned}$$

Then,

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} \partial f / \partial x_1 \\ \vdots \\ \partial f / \partial x_n \end{bmatrix} = 2A\mathbf{x} + B'.$$

Example:

$$f(x_1, x_2) = x_1^2 - 4x_1x_2 + 3x_2^2 + 7x_1 - 8x_2 + 5$$

which can be written as

$$[x_1, x_2] \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [7, -8] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 5.$$

$$\begin{aligned} \frac{\partial f}{\partial x_1} = & 2x_1 - 4x_2 + 7, \\ \frac{\partial f}{\partial x_2} = & 6x_2 - 4x_1 - 8, \\ & \text{so} \\ \frac{\partial f}{\partial \mathbf{x}} = & \begin{bmatrix} 2x_1 - 4x_2 + 7 \\ 6x_2 - 4x_1 - 8 \end{bmatrix}. \end{aligned}$$

$$2A\mathbf{x} = \begin{bmatrix} 2x_1 - 4x_2 \\ -4x_1 + 6x_2 \end{bmatrix}, \quad B' = \begin{bmatrix} 7 \\ -8 \end{bmatrix},$$

so

$$\frac{\partial f}{\partial \mathbf{x}} = 2A\mathbf{x} + B'.$$

2.2 Classification of stationary points

$$\frac{\partial f}{\mathbf{x}} = 0 \quad \text{i.e.} \quad 2A\mathbf{x} + B' = 0$$

which is the linear system

$$A\mathbf{x} = -\frac{1}{2}B'.$$

$$\begin{aligned} f \text{ convex} \quad & \Longleftrightarrow \quad A \text{ positive semidefinite} \\ f \text{ concave} \quad & \Longleftrightarrow \quad A \text{ negative semidefinite} \end{aligned}$$

If f convex:	All stationary points are global minimum.
If f concave:	All stationary points are global maximum.
If f indefinite:	All stationary points are saddle points.

3 Linear Regression

We are often interested in modeling relationships. For example, what is the relationship between returns and risk (what is risk?), what is the relationship between prices and quantities demanded, or does past returns inform us about future return dynamics? One of the workhorses in modeling relationships is linear regression.

3.1 Regression vs correlations

Why do we need regressions when we can easily compute correlations? Correlation between two variables x and y is the degree of linear association between them. This means that we treat the two variables in a completely symmetric way and we do not imply that the are any causal effect (i.e. that changes in x *causes* changes in y, or vice-versa.), but we rather say that there is evidence of a linear relationship, i.e. that they are related to an extent given by the correlation coefficient. When thinking in terms of regressions, the dependent variable (usally denoted by the vector $\boldsymbol{y}$) is treated as random and the independent variable(s) (usally denoted by the matrix X) is treated as fixed. This allows us to model relationships in a more meaningful way.

3.2 The Linear Regression Model

$$y_i = x_i'\beta + \varepsilon_i \quad i = 1, ..., N$$

x_i is a vector of observations on k independent variables. Usually, we include an intercept, so

$$x_i^t = [1, x_{1,i}, ..., x_{k,i}].$$

The conditional mean function, $r(x)$, then maps $\mathbb{R}^k \to \mathbb{R}$. Take the conditional expectation of both sides of the model:

$$\mathbb{E}[y|x] = \mathbb{E}[x'\beta + \varepsilon|x] = x'\beta + \mathbb{E}[\varepsilon|x].$$

If the conditional expectation of the error term, $\mathbb{E}[\varepsilon|x]$, is zero then we can interpret the linear term as the regression function, i.e. $r(x) = x'\beta$. If we include a constant term in the regressor matrix then this exogeneity condition will always be met.

We usually include another assumption, the homoskedasticity assumption:

$$\mathbb{V}[\varepsilon|x] = \sigma^2 < \infty.$$

Homoskedasticity refers to a constant variance (is this a reasonable assumption in financial time series?).

3.3 The Gauss-Markov Theorem

If the following assumptions about the residuals are true:

1. $\mathbb{E}[\varepsilon_i] = 0$,
2. $\mathbb{V}[\varepsilon_i] = \sigma^2 < \infty$,
3. $\mathrm{Cov}(\varepsilon_i, \varepsilon_j) = 0, \forall i \neq j$,

then the OLS estimator is the Best Linear Unbiased Estimator (BLUE).

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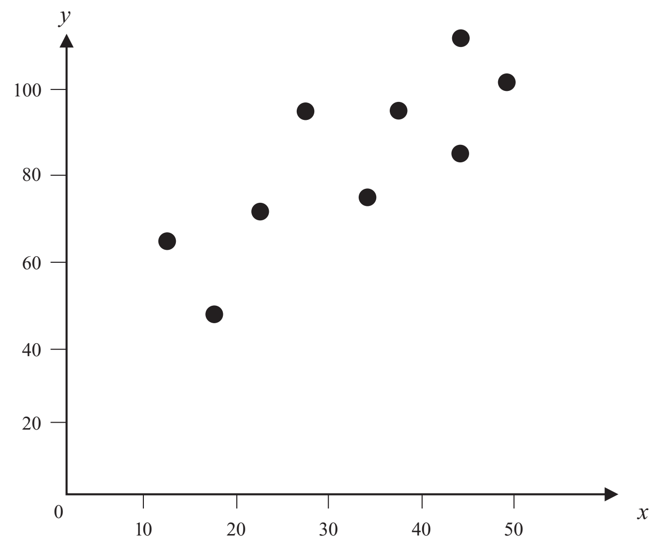
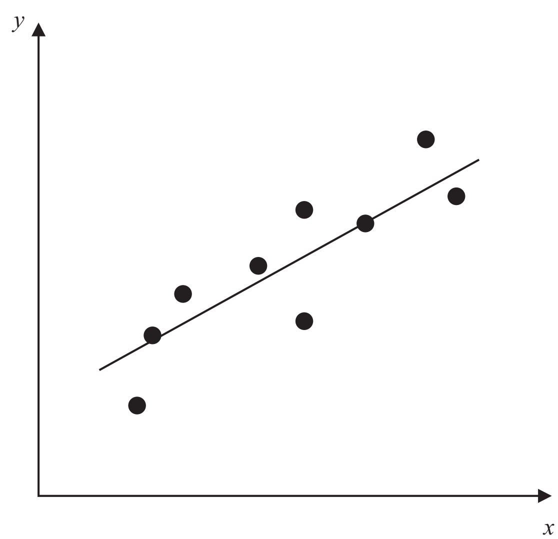
3.4 Estimation of β: Ordinary Least Squares - Univariate case



Let's say we are interested in estimating the following model:

$$y_i = \alpha + x_i\beta + \varepsilon_i.$$

What this means is that we are looking for the estimates of α and β that best captures the linear relationship between the two variables:



The method of Ordinary Least Squares (OLS) entails taking each vertical distance from the point to the line, squaring it and then minimizing the sum of the squares:

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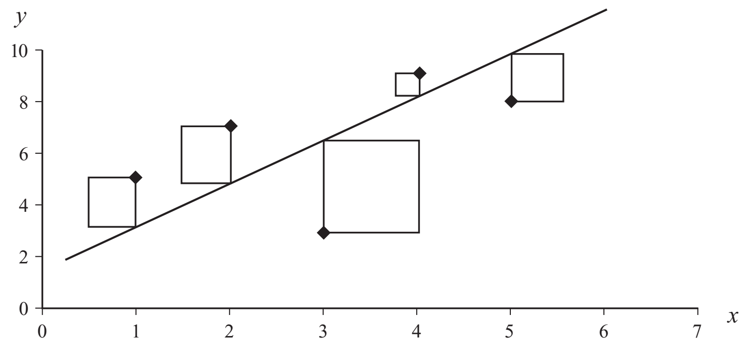

Write the fitted values of y_i as:

$$\hat{y}_i = \hat{\alpha} + \hat{\beta} x_i.$$

The residual, ε_i, is then defined as

$$\varepsilon_i = y_i - \hat{y}_i = y_i - \hat{\alpha} - \hat{\beta} x_i.$$

The problem of finding the $\hat{\alpha}$ and $\hat{\beta}$ that minimizes the sum of the squares can be written as

$$\underset{\alpha,\beta}{\arg\min} \sum_{i=i}^{N} \varepsilon_i^2 = (y_i - \hat{\alpha} - \hat{\beta} x_i)^2.$$

Let L denote the objective function, $\sum_{i=i}^{N} \varepsilon_i^2$. FOC:

$$\frac{\partial L}{d\alpha} = \sum_{i=1}^{N} -2(y_i - \hat{\alpha} - \hat{\beta} x_i) = 0 \tag{1}$$

$$\frac{\partial L}{d\beta} = \sum_{i=1}^{N} -2x_i(y_i - \hat{\alpha} - \hat{\beta} x_i) = 0 \tag{2}$$

From (1):

$$\begin{aligned}
\sum_{i=1}^{N} -2(y_i - \hat{\alpha} - \hat{\beta} x_i) &= 0 \\
\sum_{i=1}^{N} (y_i - \hat{\alpha} - \hat{\beta} x_i) &= 0 \\
N\overline{y} - N\hat{\alpha} - N\hat{\beta}\overline{x} &= 0 \\
\hat{\alpha} &= \overline{y} - \beta\overline{x}.
\end{aligned}$$

Now, insert this into (2):

$$
\begin{aligned}
\sum_{i=1}^{N} -2x_i(y_i - \hat{\alpha} - \hat{\beta}x_i) &= 0 \\
\sum_{i=1}^{N} -2x_i(y_i - (\overline{y} - \hat{\beta}\overline{x}) - \hat{\beta}x_i) &= 0 \\
\sum_{i=1}^{N} -2x_i(y_i - \overline{y} + \hat{\beta}\overline{x} - \hat{\beta}x_i) &= 0 \\
\sum_{i=1}^{N} x_i(y_i - \overline{y} + \hat{\beta}\overline{x} - \hat{\beta}x_i) &= 0 \\
\sum_{i=1}^{N} x_i y_i - \overline{y}x_i + \hat{\beta}\overline{x}x_i - \hat{\beta}x_i^2 &= 0 \\
\sum_{i=1}^{N} x_i y_i - \overline{y}\sum_{i=1}^{N} x_i + \hat{\beta}\overline{x}\sum_{i=1}^{N} x_i - \hat{\beta}\sum_{i=1}^{N} x_i^2 &= 0 \\
\hat{\beta} &= \frac{\sum_{i=1}^{N} x_i y_i - N\overline{xy}}{\sum_{i=1}^{N} x_i^2 - N\overline{x}^2} \\
\hat{\beta} &= \frac{\sum_{i=1}^{N}(x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{N}(x_i - \overline{x})^2}.
\end{aligned}
$$

We see that that the slope coefficient is the equivalent to the sample covariance between x and y divided by the sample variance of x. We also see from the equation for $\hat{\alpha}$ that the regression line will go through the mean of the observations, i.e. that the point $(\overline{y}, \overline{x})$ lies on the regression line.

3.5 Estimation of β: Ordinary Least Squares - General case

We can write the model as

$$\mathbf{y} = \boldsymbol{X}\boldsymbol{\beta} + \varepsilon,$$

where $\mathbf{y}$ is a vector of the dependent variable, X is an (N×k) matrix of the independent variables and a constant column, and ε is a vector of the residuals. $\boldsymbol{\beta}$ is the vector of estimates that we are interested in estimating. The objective is still to minimize the sum of the squared residuals:

$$\arg\min_{\boldsymbol{\beta}} \varepsilon'\varepsilon = (\mathbf{y} - \boldsymbol{X}\boldsymbol{\beta})'(\mathbf{y} - \boldsymbol{X}\boldsymbol{\beta}) = \mathbf{y}'\mathbf{y} - 2\mathbf{y}'\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta}.$$

FOC:

$$
\begin{aligned}
\frac{\partial \varepsilon'\varepsilon}{\partial \boldsymbol{\beta}} &= -2\boldsymbol{X}'\mathbf{y} + 2\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta} = 0 \\
\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta} &= \boldsymbol{X}'\mathbf{y}.
\end{aligned}
$$

Now we need to isolate $\boldsymbol{\beta}$. We do this by using the inverse trick:

$$A^{-1}A = I.$$

$$
\begin{aligned}
\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta} &= \boldsymbol{X}'\mathbf{y} \\
I\boldsymbol{\beta} &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\mathbf{y} \\
\boldsymbol{\beta} &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\mathbf{y}.
\end{aligned}
$$

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3.6 Sampling Properties of OLS

Let $\boldsymbol{b}$ denote the least squares estimator. Then,

$$\begin{aligned}
\boldsymbol{b} &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\mathbf{y},\\
\boldsymbol{b} &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\left(\boldsymbol{X}\beta+\varepsilon\right),\\
\boldsymbol{b} &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{X}\beta+(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\varepsilon,\\
\boldsymbol{b} &= \beta+\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\varepsilon
\end{aligned}$$

Taking the conditional expectation gives us

$$\begin{aligned}
\mathbb{E}[\boldsymbol{b}|\boldsymbol{X}] &= \boldsymbol{\beta}+\mathbb{E}\left[\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\varepsilon|\boldsymbol{X}\right],\\
&= \boldsymbol{\beta}+\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\mathbb{E}\left[\boldsymbol{X}'\varepsilon|\boldsymbol{X}\right],\\
&= \boldsymbol{\beta}+\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\mathbb{E}\left[\varepsilon|\boldsymbol{X}\right],\\
&= \boldsymbol{\beta},
\end{aligned}$$

due to the exogeneity condition. Thus, the Least Squares estimator is unbiased. This also holds unconditionally since $\mathbb{E}[b]=\mathbb{E}_X[\mathbb{E}[b|X]]=\mathbb{E}_X[\beta]=\beta$ by the Law of Iterated Expectations. What is the uncertainty associated with the estimates?

$$\begin{aligned}
\mathbb{V}(\boldsymbol{b}) &= \mathbb{E}\left[(\boldsymbol{b}-\boldsymbol{\beta})^2|\boldsymbol{X}\right],\\
&= \mathbb{E}\left[(\boldsymbol{\beta}+\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\varepsilon-\boldsymbol{\beta})^2|\boldsymbol{X}\right],\\
&= \mathbb{E}\left[\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\varepsilon\varepsilon'\boldsymbol{X}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}|\boldsymbol{X}\right],\\
&= \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\mathbb{E}\left[\varepsilon\varepsilon'|\boldsymbol{X}\right]\boldsymbol{X}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1},\\
&= \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\mathbb{V}(\varepsilon)\boldsymbol{X}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1},\\
&= \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\Lambda\boldsymbol{X}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1},
\end{aligned}$$

where Λ is the covariance matrix of the residuals. If we assume that the errors are homoskedastic and not autocorrelated, then Λ is simply $I\sigma^2$, thus:

$$\mathbb{V}(\boldsymbol{b}) = \sigma^2\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}.$$

We are seeking a sample estimator $\boldsymbol{b}$ of the true population coefficient β. Thus, if we are approaching the population, $\boldsymbol{b}$ should approach β:

$$\lim_{n\to\infty}\mathbb{P}(|\boldsymbol{b}-\beta|\geq\delta)=0 \quad \text{for every } \delta>0.$$

Equivalently:

$$\lim_{n\to\infty}\mathbb{V}(\boldsymbol{b}) = \sigma^2\lim_{n\to\infty}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}=0.$$

This property is called consistency. We can formally show this:

$$\begin{aligned}
b &= \beta+\left(X'X\right)^{-1}X'\varepsilon,\\
&= \beta+\left(\frac{X'X}{n}\right)^{-1}\frac{X'\varepsilon}{n}.
\end{aligned}$$

Taking the probability limit on both sides gives us

$$\begin{aligned}\operatorname{plim}(b) &= \beta + \operatorname{plim}\left(\left(\frac{X'X}{n}\right)^{-1}\frac{X'\varepsilon}{n}\right),\\ &= \beta + \operatorname{plim}\left(\left(\frac{X'X}{n}\right)^{-1}\right)\operatorname{plim}\left(\frac{X'\varepsilon}{n}\right).\end{aligned}$$

First, we know that

$$\operatorname{plim}\left(\left(\frac{X'X}{n}\right)^{-1}\right) = \Omega,$$

which is a positive definite matrix. Next,

$$\operatorname{plim}\left(\frac{X'\varepsilon}{n}\right) = 0,$$

which follows from the fact that $\mathbb{E}[\varepsilon|X] = 0$ which implies $\mathbb{E}[\varepsilon x] = 0$. This is because $\frac{X'\varepsilon}{n} = n^{-1}\sum_{i=1}^{n} x_i\varepsilon_i$. This leads us to the desired result:

$$\begin{aligned}\operatorname{plim}(b) &= \beta + \operatorname{plim}\left(\left(\frac{X'X}{n}\right)^{-1}\right)\operatorname{plim}\left(\frac{X'\varepsilon}{n}\right),\\ &= \beta + \Omega \times 0 = \beta.\end{aligned}$$

$\Rightarrow$ ***b*** is consistent!

3.7 Estimator uncertainty