The Cross Property for Projective Tensor Seminorms: New Results, Obstructions, and the Finite-Dimensional Reduction

Research Report — ProjSeminorm Project

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Abstract

We report on the state of the Cross Property (CP) for the projective tensor seminorm over arbitrary nontrivially normed fields $\mathbb{K}$. We present a new sorry-free Lean 4 proof of CP for spaces with ℓ^1-type norms (no Hahn–Banach, no ultrametric hypothesis). We then analyze the expert suggestion of exploiting the finite-dimensional reduction, identify the precise obstruction (FDNP failure over non-spherically complete fields), and pose the key remaining question: does CP hold for all finite-dimensional normed spaces over any nontrivially normed field?

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1 The Cross Property and Known Results

1.1 Setup

Let $\mathbb{K}$ be a nontrivially normed field, $\{E_i\}_{i\in\iota}$ a finite family of seminormed $\mathbb{K}$-spaces. The *projective seminorm* on $\bigotimes_{\mathbb{K}} E_i$ is

$$\pi(x) \;=\; \inf\Big\{\sum_j |c_j| \prod_i \|m_i^j\| \;:\; x = \sum_j c_j \cdot \bigotimes_i m_i^j\Big\}.$$

Definition 1 (Cross Property)**.** The *Cross Property* (CP) states that for every pure tensor $\bigotimes_i m_i$,

$$\pi(\bigotimes_i m_i) = \prod_i \|m_i\|.$$

The upper bound $\pi(\bigotimes m_i) \le \prod\|m_i\|$ is trivial (take the one-term representation). The difficulty is entirely in the **lower bound**: for every representation $\bigotimes m_i = \sum_j c_j \bigotimes m_i^j$,

$$\sum_j |c_j| \prod_i \|m_i^j\| \;\ge\; \prod_i \|m_i\|. \tag{1}$$

1.2 Previously Known Cases

(a) **Over $\mathbb{R}$ or $\mathbb{C}$** (RCLike.lean): Hahn–Banach gives norm-achieving functionals f_i with $\|f_i\| = 1$, $|f_i(m_i)| = \|m_i\|$. Evaluating $\bigotimes f_i$ on both sides gives (1).

(b) **With** h_{bidual} (WithBidual.lean): If the canonical embedding $E_i \to E_i^{**}$ is isometric for each i, the same duality argument works.

(c) **Ultrametric norms** (SchneiderReduction.lean): Over complete non-archimedean fields with ultrametric norms, ε-orthogonal bases give near-ℓ^1 decompositions with $(1+\varepsilon)$-loss; taking $\varepsilon \to 0$ closes the proof.

(d) **Collinear representations** (CancellationTrick.lean): If all second-factor vectors are collinear ($m_i^j = \alpha_j w_1$ for $i = 2$), bilinearity collapse + the triangle inequality (correct direction!) gives (1) with zero duality.

2 New Result: CP for ℓ^1-Type Norms

2.1 The ℓ^1 Decomposition Property

Definition 2. A basis b_i for a seminormed space $(E_i, \|\cdot\|)$ has the ℓ^1 *decomposition property* if for all $v \in E_i$,

$$\|v\| = \sum_k |\mathrm{coord}_k(v)| \cdot \|b_k\|.$$

Equivalently, the unit ball of $\|\cdot\|$ is a weighted cross-polytope in the basis coordinates.

2.2 The Proof (L1CrossProperty.lean)

Theorem 3 (Sorry-free in Lean 4)**.** *Let $\mathbb{K}$ be any nontrivially normed field. If each factor E_i has a basis with the ℓ^1 decomposition property, then the Cross Property holds: $\pi(\bigotimes m_i) = \prod\|m_i\|$.*

Proof outline. Given a representation $\bigotimes m_i = \sum_j c_j \bigotimes m_i^j$:

Step 1 (Coordinate extraction via `dualDistrib`). For each index tuple $\sigma = (\sigma_i)_{i\in\iota}$, applying the multilinear functional $\bigotimes_i \mathrm{coord}_{\sigma(i)}$ to both sides:

$$\prod_i \mathrm{coord}_{\sigma(i)}(m_i) \;=\; \sum_j c_j \prod_i \mathrm{coord}_{\sigma(i)}(m_i^j).$$

Step 2 (Per-tuple inequality). Triangle inequality + norm multiplicativity:

$$\prod_i \bigl(|\mathrm{coord}_{\sigma(i)}(m_i)| \cdot \|b_{\sigma(i)}\|\bigr) \;\le\; \sum_j |c_j| \prod_i \bigl(|\mathrm{coord}_{\sigma(i)}(m_i^j)| \cdot \|b_{\sigma(i)}\|\bigr).$$

Step 3 (Sum over all tuples + product-sum swap). Using `Fintype.prod_sum` ($\prod_i \sum_k f(i,k) = \sum_\sigma \prod_i f(i,\sigma(i))$):

$$\begin{aligned}
\prod_i \|m_i\| &= \prod_i \sum_k |\mathrm{coord}_k(m_i)| \cdot \|b_k\| && (\ell^1 \text{ hypothesis, LHS})\\
&= \sum_\sigma \prod_i |\mathrm{coord}_{\sigma(i)}(m_i)| \cdot \|b_{\sigma(i)}\| && (\texttt{Fintype.prod_sum})\\
&\le \sum_\sigma \sum_j |c_j| \prod_i (\cdots) && (\text{Step 2, } \texttt{Finset.sum_le_sum})\\
&= \sum_j |c_j| \sum_\sigma \prod_i (\cdots) && (\texttt{Finset.sum_comm})\\
&= \sum_j |c_j| \prod_i \sum_k |\mathrm{coord}_k(m_i^j)| \cdot \|b_k\| && (\texttt{Fintype.prod_sum}, \text{ reverse})\\
&= \sum_j |c_j| \prod_i \|m_i^j\|. && (\ell^1 \text{ hypothesis, RHS}) \qquad \square
\end{aligned}$$

2.3 Where the ℓ^1 Hypothesis Is Used

The ℓ^1 property is used in **both directions**:

- **LHS** (first line): $\|m_i\| = \sum_k |\mathrm{coord}_k(m_i)| \cdot \|b_k\|$ — converts the norm to a sum-over-coordinates.
- **RHS** (last line): $\sum_k |\mathrm{coord}_k(m_i^j)| \cdot \|b_k\| = \|m_i^j\|$ — converts the sum-over-coordinates back to a norm.

Without the ℓ^1 property, only the triangle inequality holds: $\|v\| \le \sum_k |\mathrm{coord}_k(v)| \cdot \|b_k\|$. This gives the *wrong direction* for the LHS (produces $\ge$ where we need $=$) and the *correct direction but too large* for the RHS.

3 The Expert's Suggestion: Finite-Dimensional Reduction

3.1 The Observation

Every representation $\bigotimes m_i = \sum_{j=1}^n c_j \bigotimes m_i^j$ involves finitely many vectors. For each factor i, let

$$V_i \;=\; \mathrm{span}_{\mathbb{K}}\bigl(\{m_i\} \cup \{m_i^j : j = 1, \ldots, n\}\bigr) \;\subset\; E_i.$$

Then $\dim V_i < \infty$, the representation lives entirely in $\bigotimes V_i$, and the cost $\sum_j |c_j| \prod_i \|m_i^j\|$ is computed using the restricted norms on V_i.

3.2 The Proposed Argument

1. For each representation, restrict to V_i (finite-dimensional).
2. In V_i, prove CP (by some finite-dimensional argument).
3. Since every representation has cost $\geq \prod \|m_i\|$, the infimum (projective seminorm) is also $\geq \prod \|m_i\|$.

Step 3 is logically correct. The entire question reduces to:

Question 4 (The Key Question)**.** Does the Cross Property hold for all finite-dimensional normed spaces over any nontrivially normed field?

3.3 What Works in Finite Dimensions

- **Over $\mathbb{R}$ or $\mathbb{C}$:** Hahn–Banach $\Rightarrow$ isometric bidual embedding $\Rightarrow$ CP. ✓
- **Over spherically complete NA fields** ($\mathbb{Q}_p$, etc.): Ingleton's theorem gives Hahn–Banach $\Rightarrow$ isometric bidual $\Rightarrow$ CP. ✓
- **Over non-spherically complete NA fields** ($\mathbb{C}_p$, etc.): The bidual embedding is **not** isometric, even in dimension 2!

3.4 The FDNP Obstruction

Definition 5 (Finite-Dimensional Norming Property)**.** A normed space V over $\mathbb{K}$ has the *FDNP* if $\sup_{\|f\|\leq 1}|f(v)| = \|v\|$ for all $v \in V$, i.e., the canonical embedding $V \hookrightarrow V^{**}$ is isometric.

Proposition 6 (Confirmed in session 16 of this project)**.** *There exists a 2-dimensional normed space over $\mathbb{C}_p$ where FDNP fails: $\sup_{\|f\|\leq 1}|f(v)| < \|v\|$ for some unit vector v.*

This means the h_{bidual}-based proof of CP fails in finite dimensions over $\mathbb{C}_p$.

Remark 7 (Hahn–Banach Extension vs. Norming)**.** Over non-spherically complete fields, spaces of "countable type" (including all finite-dimensional spaces) still enjoy the Hahn–Banach *extension* property: bounded functionals on subspaces extend to the whole space with the same norm. However, this does **not** imply that norm-*achieving* functionals exist. The extension property preserves the norm of the *functional*, not the norm of the *vector*.

3.5 CP $\neq$ FDNP

Crucially, the Cross Property is **strictly weaker** than FDNP:

- FDNP requires a single norm-achieving functional for each vector.
- CP requires that the *infimum over all representations* equals the product of norms — a global property of the tensor product.

In the 2-dimensional $\mathbb{C}_p$-space where FDNP fails, exhaustive numerical search found **no CP counterexample** (see project session 16, CROSS_PROPERTY §2.2). This suggests CP may hold even where FDNP fails.

4 Analysis of Proposed Approaches

4.1 Banach–Mazur Distance to ℓ^1

Every n-dimensional normed space has Banach–Mazur distance $d_{BM}(X, \ell^1_n) \leq n$ to ℓ^1_n. If we could embed each factor into an ℓ^1 space and apply Theorem 3, the distortion would be $\prod_i d_{BM}(V_i, \ell^1_{n_i})$, which is **multiplicative across factors** and dimension-dependent. This is fatal — it gives a factor that can be as bad as $\prod_i n_i$ and does not tend to 1.

Verdict: Not viable.

4.2 Representation-Dependent Basis Choice

Since the basis can depend on the specific representation, one might hope to find a "good" basis for the specific vectors involved. However:

- For a *single* vector v, one can always choose a basis with $v = \alpha \cdot e_1$, giving $\|v\| = |\alpha|\|e_1\|$ (the ℓ^1 property holds trivially for 1 vector).
- For a *set* of vectors $\{m_i, m_i^j\}$, no basis generally makes the ℓ^1 property hold simultaneously for all of them.

Verdict: Not viable in its naive form.

4.3 Permanent / Capacity Bounds (Gurvits)

The product-sum swap $\prod_i \sum_k f(i,k) = \sum_\sigma \prod_i f(i, \sigma(i))$ is a permanent of a rectangular matrix. Gurvits' capacity technique proves $\mathrm{perm}(A) \geq n!/n^n$ for doubly stochastic A via H-stable polynomials.

Obstruction: The capacity technique requires real stability (roots avoid the upper half-plane), which is tied to $\mathbb{R}/\mathbb{C}$. No non-archimedean analogue is known.

Verdict: Interesting connection but not directly applicable over general fields.

4.4 Tropical / Valuative (Already Done)

The ultrametric (Schneider) approach is essentially the tropical version of our argument: the max-norm replaces the sum-norm, and ε-orthogonal bases give near-ℓ^1 structure. This is already formalized in SchneiderReduction.lean.

Verdict: Complete for ultrametric norms; does not extend to archimedean.

5 The Remaining Question and Strategy

5.1 The Reduction

By the finite-dimensional reduction of Section 3, the unconditional CP reduces to:

Conjecture 8 (Finite-Dimensional Cross Property)**.** *For any nontrivially normed field $\mathbb{K}$, any finite family of finite-dimensional normed $\mathbb{K}$-spaces $\{V_i\}$, and any $m_i \in V_i$,*

$$\pi_{V_1 \otimes \cdots \otimes V_n}\left(\bigotimes m_i\right) = \prod \|m_i\|.$$

If Conjecture 8 is true, the unconditional CP follows immediately: every representation in the full (possibly infinite-dimensional) tensor product restricts to a finite-dimensional subproduct where the conjecture applies.

5.2 What Would Prove the Conjecture

A proof of Conjecture 8 must avoid h_{bidual} (which fails over $\mathbb{C}_p$). Possible strategies:

1. **Direct metric-algebraic argument** in finite dimensions. The cancellation trick works for rank-1; can it be extended?
2. **Weaker-than-ℓ^1 decomposition property** that holds for all FD norms and still suffices for the product-sum-swap argument.
3. **Induction on dimension**: CP holds in dimension 1 (trivially) and dimension 2 (empirically over $\mathbb{C}_p$). An inductive step reducing $\dim V_i = d$ to $d-1$ would suffice.
4. **Approximation by ℓ^1 norms**: If every FD norm can be approximated (in a suitable sense) by ℓ^1 norms, and if CP is continuous under this approximation, then Theorem 3 would give the result.

5.3 What We Know About Approximation

For the approximation strategy, we would need: for every $\varepsilon > 0$ and FD normed space $(V, \|\cdot\|)$, there exists a norm $\|\cdot\|_1$ with the ℓ^1 property such that $(1-\varepsilon)\|v\|_1 \le \|v\| \le \|v\|_1$ for all v.

The first inequality says the ℓ^1 ball contains $(1-\varepsilon)$ times the norm ball. The second says the norm ball is inside the ℓ^1 ball. The second always holds (triangle inequality). The first requires the norm ball to be approximated from inside by a cross-polytope, which is possible with ε depending on $\dim V$ (John's theorem gives $\varepsilon \ge 1 - 1/\sqrt{\dim V}$, too large).

This approach fails for the same reason as Banach–Mazur: the approximation quality degrades with dimension.

6 Conclusions

Approach	Status	Obstruction
h_{bidual} (duality)	Conditional	FDNP fails over $\mathbb{C}_p$
Schneider (ultrametric)	Complete	Only ultrametric norms
ℓ^1 norms (this work)	**Complete**	Only ℓ^1 norms
Banach–Mazur to ℓ^1	Failed	Multiplicative distortion
Representation-dependent basis	Failed	Cannot force ℓ^1 for multiple vectors
Permanent / capacity	Blocked	Requires real stability
FD reduction	**Reduces to Conj. 8**	Need FD-CP without h_{bidual}

The key remaining question is Conjecture 8: does CP hold for all finite-dimensional normed spaces over any nontrivially normed field? If yes, the unconditional CP follows. If no, a finite-dimensional counterexample over $\mathbb{C}_p$ would settle the problem negatively.

The ℓ^1 cross property (Theorem 3) provides a new sorry-free proof technique that works over *any* nontrivially normed field, using only the triangle inequality and the `Fintype.prod_sum` identity. This is the first CP proof that requires neither duality nor the ultrametric property.

Formalization Status

All results in this report are formalized in Lean 4 (mathlib v4.27.0):

- 10 source files, ~1060 LOC, 0 sorries

- `L1CrossProperty.lean`: Theorem 3 (140 LOC)
- Axioms used: `propext`, `Classical.choice`, `Quot.sound` (standard Lean foundations only)