Path-Integral Hockey-Stick Representation of Conditional Entropy Differences

Adversarial Proof Framework (af)

Verified by independent prover and verifier agents

36 nodes, all validated and clean

February 2026

Abstract

We present a complete proof of the path-integral hockey-stick representation for the difference of conditional entropies of two bipartite quantum states. Under a full-rank assumption, the conditional entropy difference $H(A|B)_\sigma - H(A|B)_\rho$ is expressed as a double integral involving the spectral projector of an interpolating operator and the hockey-stick divergence kernel. The proof was constructed and adversarially verified within the `af` (Adversarial Proof Framework), in which independent prover agents propose proof steps and independent verifier agents challenge them until all nodes reach a validated state.

Contents

1 Notation and Definitions

Hilbert spaces. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite-dimensional complex Hilbert spaces with $d_A := \dim \mathcal{H}_A$ and $d_B := \dim \mathcal{H}_B$. Write $\mathcal{H}_{AB} := \mathcal{H}_A \otimes \mathcal{H}_B$ with $\dim \mathcal{H}_{AB} = d_A d_B$, and let $\mathbb{I}_X$ denote the identity operator on $\mathcal{H}_X$.

States and operators. A *density operator* is a positive semidefinite operator $\rho \geq 0$ with $\mathrm{Tr}[\rho] = 1$. We write $X > 0$ for strictly positive definite operators. The *partial trace* is $\rho_B := \mathrm{Tr}_A[\rho_{AB}]$. The *trace norm* is $\|X\|_1 := \mathrm{Tr}\sqrt{X^\dagger X}$.

Entropy and divergence. The *von Neumann entropy* is $S(\rho) := -\mathrm{Tr}[\rho \log \rho]$ (natural logarithm, with $0 \log 0 := 0$). The *conditional entropy* is

$$H(A|B)_\rho \;:=\; S(\rho_{AB}) - S(\rho_B).$$

The *(Umegaki) relative entropy* is

$$D(\rho\|\sigma) \;:=\; \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$$

when $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$, and $+\infty$ otherwise. A key identity (**CE-RE**) is

$$H(A|B)_\rho \;=\; -D(\rho_{AB}\|\mathbb{I}_A \otimes \rho_B). \tag{1}$$

Hockey-stick divergence. For positive semidefinite operators ρ, σ and $\gamma > 0$,

$$E_\gamma(\rho\|\sigma) \;:=\; \mathrm{Tr}(\rho - \gamma\sigma)_+ \;=\; \max_{0\le M\le \mathbb{I}} \mathrm{Tr}[M(\rho - \gamma\sigma)].$$

The maximiser is $P_\gamma = \mathbf{1}\{\rho - \gamma\sigma > 0\}$, the spectral projector onto the strictly positive eigenspace.

Max-relative entropy. $M(\rho, \sigma) := \inf\{\lambda \ge 0 : \rho \le \lambda\sigma\}$. For bipartite states, $M(\rho_{AB}, \mathbb{I}_A \otimes \rho_B) \le d_A$.

Frenkel integral representation. For a density operator ρ and an unnormalised reference τ with $\mathrm{Tr}[\tau] = c > 0$ and $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\tau)$ (**FR-bip**):

$$D(\rho\|\tau) \;=\; \int_0^\infty \frac{E_\gamma(\rho\|\tau) - (1 - c\gamma)_+}{\gamma(1 + c\gamma)}\, d\gamma \;-\; \log c. \tag{2}$$

Interpolating path. Given density operators ρ_{AB}, σ_{AB} on $\mathcal{H}_{AB}$, define

$$\begin{aligned}
\rho(t) &:= (1-t)\,\rho_{AB} + t\,\sigma_{AB}, \qquad t \in [0,1],\\
\delta_{AB} &:= \sigma_{AB} - \rho_{AB}, \qquad \delta_B := \sigma_B - \rho_B = \mathrm{Tr}_A[\delta_{AB}],\\
\rho(t)_B &:= \mathrm{Tr}_A[\rho(t)] = (1-t)\,\rho_B + t\,\sigma_B,\\
\tau(t) &:= \mathbb{I}_A \otimes \rho(t)_B, \qquad \mathrm{Tr}[\tau(t)] = d_A.
\end{aligned}$$

We also define the shifted difference operator $\delta_{AB}^{(\gamma)} := \delta_{AB} - \gamma\,\mathbb{I}_A \otimes \delta_B$, the spectral projector $P_\gamma(t) := \mathbf{1}\{\rho(t) - \gamma\,\tau(t) > 0\}$, and the t-dependent upper cutoff $M(t) := M(\rho(t), \tau(t)) \le d_A$.

2 Statement of the Main Theorem

Theorem 1 (Path-Integral Hockey-Stick Representation). *Let $\rho_{AB}, \sigma_{AB} > 0$ be strictly positive density operators on $\mathcal{H}_A \otimes \mathcal{H}_B$. Then*

$$H(A|B)_\sigma - H(A|B)_\rho \;=\; -\int_0^1 \int_0^{M(t)} \frac{\mathrm{Tr}\big[P_\gamma(t)\,\delta_{AB}^{(\gamma)}\big]}{\gamma\,(1 + d_A\gamma)}\, d\gamma\, dt. \tag{3}$$

The proof assembles five propositions, which we now develop in turn.

3 Proof

Assumption 2 (Full Rank)**.** Assume $\rho_{AB}, \sigma_{AB} > 0$. Since the set of positive-definite operators is convex, $\rho(t) = (1-t)\rho_{AB} + t\,\sigma_{AB} > 0$ for all $t \in [0,1]$, and consequently $\rho(t)_B = \mathrm{Tr}_A[\rho(t)] > 0$ as well.

Proposition 3 (CE-RE Identity)**.** *For all* $t \in [0,1]$,

$$H(A|B)_{\rho(t)} \;=\; -D\big(\rho(t)_{AB} \,\big\|\, \mathbb{I}_A \otimes \rho(t)_B\big).$$

Proof. By definition of relative entropy,

$$D(\rho_{AB} \| \mathbb{I}_A \otimes \rho_B) \;=\; \mathrm{Tr}[\rho_{AB} \log \rho_{AB}] \;-\; \mathrm{Tr}[\rho_{AB} \log(\mathbb{I}_A \otimes \rho_B)].$$

By the spectral mapping theorem for tensor products, $\log(\mathbb{I}_A \otimes \rho_B) = \mathbb{I}_A \otimes \log \rho_B$. Using the partial-trace identity $\mathrm{Tr}[\rho_{AB}(\mathbb{I}_A \otimes X_B)] = \mathrm{Tr}[\rho_B X_B]$,

$$\mathrm{Tr}[\rho_{AB}(\mathbb{I}_A \otimes \log \rho_B)] \;=\; \mathrm{Tr}[\rho_B \log \rho_B] \;=\; -S(\rho_B).$$

Therefore

$$D(\rho_{AB} \| \mathbb{I}_A \otimes \rho_B) \;=\; -S(\rho_{AB}) + S(\rho_B) \;=\; -\big(S(\rho_{AB}) - S(\rho_B)\big) \;=\; -H(A|B)_\rho,$$

giving $H(A|B)_\rho = -D(\rho_{AB} \| \mathbb{I}_A \otimes \rho_B)$. The same argument applies at every point $\rho(t)$ along the interpolating path. □

Proposition 4 (Fundamental Theorem of Calculus)**.**

$$H(A|B)_\sigma - H(A|B)_\rho \;=\; \int_0^1 \frac{d}{dt} H(A|B)_{\rho(t)}\, dt. \tag{4}$$

Proof. Under the full-rank assumption (Assumption 2), $\rho(t)$ and $\rho(t)_B$ are strictly positive for all $t \in [0,1]$. The eigenvalues of $\rho(t)$, being eigenvalues of a matrix depending affinely on t, are smooth and strictly positive; hence $S(\rho(t))$ and $S(\rho(t)_B)$ are C^∞ functions of t. Since $H(A|B)_{\rho(t)} = S(\rho(t)_{AB}) - S(\rho(t)_B)$, the map $t \mapsto H(A|B)_{\rho(t)}$ is C^1 on $[0,1]$. The fundamental theorem of calculus gives

$$H(A|B)_{\rho(1)} - H(A|B)_{\rho(0)} \;=\; \int_0^1 \frac{d}{dt} H(A|B)_{\rho(t)}\, dt.$$

Since $\rho(0) = \rho_{AB}$ and $\rho(1) = \sigma_{AB}$, the result follows. □

Proposition 5 (Derivative of Conditional Entropy)**.**

$$\frac{d}{dt} H(A|B)_{\rho(t)} \;=\; \mathrm{Tr}\big[\delta_{AB}\big(\mathbb{I}_A \otimes \log \rho(t)_B - \log \rho(t)\big)\big]. \tag{5}$$

Proof. By Proposition 3, $\frac{d}{dt} H(A|B)_{\rho(t)} = -\frac{d}{dt} D(\rho(t) \| \tau(t))$. Write

$$D(\rho(t) \| \tau(t)) \;=\; \mathrm{Tr}[\rho(t) \log \rho(t)] - \mathrm{Tr}[\rho(t) \log \tau(t)].$$

First term. Using $\dot\rho(t) = \delta_{AB}$,

$$\frac{d}{dt}\, \mathrm{Tr}[\rho(t) \log \rho(t)] \;=\; \mathrm{Tr}[\delta_{AB} \log \rho(t)] + \mathrm{Tr}\big[\rho(t)\, D_{\log}(\rho(t))[\delta_{AB}]\big].$$

Here $D_{\log}(B)[C] = \int_0^\infty (B + s\mathbb{I})^{-1} C (B + s\mathbb{I})^{-1} ds$ denotes the Fréchet derivative of the matrix logarithm. The identity $\mathrm{Tr}[B D_{\log}(B)[C]] = \mathrm{Tr}[C]$ for $B > 0$ (which follows from the integral representation and cyclicity of trace) gives, with $B = \rho(t) > 0$ and $C = \delta_{AB}$,

$$\mathrm{Tr}\big[\rho(t) D_{\log}(\rho(t))[\delta_{AB}]\big] = \mathrm{Tr}[\delta_{AB}] = 0,$$

since $\delta_{AB} = \sigma_{AB} - \rho_{AB}$ is traceless (both states have unit trace). Therefore $\frac{d}{dt} \mathrm{Tr}[\rho(t) \log \rho(t)] = \mathrm{Tr}[\delta_{AB} \log \rho(t)]$.

Second term. Since $\tau(t) = \mathbb{I}_A \otimes \rho(t)_B$, we have $\log \tau(t) = \mathbb{I}_A \otimes \log \rho(t)_B$ and $\dot{\tau}(t) = \mathbb{I}_A \otimes \delta_B$. Therefore

$$\frac{d}{dt} \mathrm{Tr}[\rho(t) \log \tau(t)] = \mathrm{Tr}[\delta_{AB} \log \tau(t)] + \mathrm{Tr}\big[\rho(t) D_{\log}(\tau(t))[\dot{\tau}(t)]\big].$$

The Fréchet derivative term reduces (via the partial-trace identity and $\rho(t)_B > 0$) to

$$\mathrm{Tr}\big[\rho(t)_B \, \delta_B \, \rho(t)_B^{-1}\big] = \mathrm{Tr}[\delta_B] = 0.$$

Combining.

$$\frac{d}{dt} D(\rho(t)\|\tau(t)) = \mathrm{Tr}[\delta_{AB} \log \rho(t)] - \mathrm{Tr}[\delta_{AB}(\mathbb{I}_A \otimes \log \rho(t)_B)] = \mathrm{Tr}\big[\delta_{AB}\big(\log \rho(t) - \mathbb{I}_A \otimes \log \rho(t)_B\big)\big].$$

Negating yields (5). □

Proposition 6 (Hockey-Stick Derivative)**.** *At generic γ (i.e., when no eigenvalue of $\rho(t) - \gamma\,\tau(t)$ equals zero),*

$$\frac{d}{dt} E_\gamma(\rho(t)\|\tau(t)) = \mathrm{Tr}\big[P_\gamma(t)\, \delta_{AB}^{(\gamma)}\big]. \tag{6}$$

Proof. Let $A(t) := \rho(t) - \gamma\,\tau(t)$. By spectral decomposition,

$$A(t) = \sum_i \lambda_i(t) \, |e_i(t)\rangle\langle e_i(t)|, \qquad A(t)_+ = \sum_{\lambda_i > 0} \lambda_i(t) \, |e_i(t)\rangle\langle e_i(t)| = P_\gamma(t)\, A(t)\, P_\gamma(t).$$

At generic γ, no eigenvalue of $A(t)$ is zero, so the rank of $A(t)_+$ is locally constant in t, and the eigenvalues and eigenprojections are smooth. Hence $\mathrm{Tr}\, A(t)_+$ is differentiable.

Since $A(t)_+ = P_\gamma(t)\, A(t)$ (using $[P_\gamma(t), A(t)] = 0$),

$$\frac{d}{dt} \mathrm{Tr}\, A(t)_+ = \mathrm{Tr}\Big[\frac{dP_\gamma}{dt} A(t)\Big] + \mathrm{Tr}\big[P_\gamma(t)\, \dot{A}(t)\big].$$

We claim $\mathrm{Tr}[P'A] = 0$ where $P = P_\gamma(t)$ and $P' = dP_\gamma/dt$. Since P is a spectral projector of A, we have $[P, A] = 0$; both operators are block-diagonal in the decomposition $\mathcal{H} = P\mathcal{H} \oplus (\mathbb{I} - P)\mathcal{H}$. Differentiating $P^2 = P$ gives $P'P + PP' = P'$, from which $PP'P = 0$ and $(\mathbb{I} - P)P'(\mathbb{I} - P) = 0$. Thus P' is *off-block-diagonal*:

$$P' = PP'(\mathbb{I} - P) + (\mathbb{I} - P)P'P.$$

Since A is block-diagonal and P' is off-block-diagonal, their product $P'A$ is off-block-diagonal. Any off-block-diagonal operator has zero trace:

$$\begin{aligned}
\mathrm{Tr}[P'A] &= \mathrm{Tr}\big[PP'(\mathbb{I} - P)A\big] + \mathrm{Tr}\big[(\mathbb{I} - P)P'PA\big] \\
&= \mathrm{Tr}\big[(\mathbb{I} - P)A\, PP'\big] + \mathrm{Tr}\big[PA\,(\mathbb{I} - P)P'\big] \\
&= \mathrm{Tr}\big[(\mathbb{I} - P)P \cdot AP'\big] + \mathrm{Tr}\big[P(\mathbb{I} - P) \cdot AP'\big] = 0,
\end{aligned}$$

using $P(\mathbb{I} - P) = 0$, $[A, P] = 0$, and cyclicity of trace. Therefore

$$\frac{d}{dt} E_\gamma(\rho(t)\|\tau(t)) = \mathrm{Tr}\big[P_\gamma(t)\, \dot{A}(t)\big] = \mathrm{Tr}\big[P_\gamma(t)\, (\delta_{AB} - \gamma\, \mathbb{I}_A \otimes \delta_B)\big] = \mathrm{Tr}\big[P_\gamma(t)\, \delta_{AB}^{(\gamma)}\big]. \qquad \square$$

Proposition 7 (Derivative via Hockey-Stick)**.**

$$\frac{d}{dt}\,D(\rho(t)\|\tau(t)) \;=\; \int_0^{M(t)} \frac{\mathrm{Tr}\big[P_\gamma(t)\,\delta_{AB}^{(\gamma)}\big]}{\gamma\,(1+d_A\gamma)}\,d\gamma. \tag{7}$$

Proof. By the Frenkel bipartite formula (2) with $c = d_A$,

$$D(\rho(t)\|\tau(t)) = \int_0^\infty \frac{E_\gamma(\rho(t)\|\tau(t)) - (1-d_A\gamma)_+}{\gamma(1+d_A\gamma)}\,d\gamma \;-\; \log d_A.$$

The term $(1-d_A\gamma)_+$ and the constant $\log d_A$ are independent of t.

Differentiation under the integral sign. We justify the exchange of $\frac{d}{dt}$ and $\int$ via the dominated convergence theorem. Under the full-rank assumption, $\rho(t) > 0$ for all $t \in [0,1]$. By compactness of $[0,1]$ and continuity of the minimum eigenvalue $\lambda_{\min}(\rho(t))$, there exists $\gamma_0 > 0$ (uniform in t) such that for all $\gamma < \gamma_0$ and all $t \in [0,1]$, $\rho(t) - \gamma\,\tau(t) > 0$. For such γ, $E_\gamma(\rho(t)\|\tau(t)) = \mathrm{Tr}[\rho(t) - \gamma\tau(t)] = 1 - \gamma d_A$, which is independent of t, so $\frac{d}{dt}E_\gamma = 0$. Thus the integrand vanishes on $(0,\gamma_0)$. On $[\gamma_0,\infty)$, the bound $\left|\frac{d}{dt}E_\gamma\right| \le \|\delta_{AB}\|_1 + \gamma\,\|\mathbb{I}_A \otimes \delta_B\|_1$ is finite, and $[\gamma(1+d_A\gamma)]^{-1} \le [\gamma_0(1+d_A\gamma_0)]^{-1}$ provides a uniform bound. Dominated convergence applies.

Substituting the hockey-stick derivative. By Proposition 6, $\frac{d}{dt}E_\gamma(\rho(t)\|\tau(t)) = \mathrm{Tr}[P_\gamma(t)\,\delta_{AB}^{(\gamma)}]$ at generic γ. The set of non-generic γ has measure zero and does not affect the integral.

Truncation of the upper limit. For $\gamma > M(t)$, we have $\rho(t) \le \gamma\,\tau(t)$, so $E_\gamma(\rho(t)\|\tau(t)) = 0$ identically in t and its t-derivative vanishes. The effective upper limit is $M(t) \le d_A$.

Combining yields (7). □

Proof of Theorem 1. We assemble the five propositions.

Step 1 (FTC). By Proposition 4,

$$H(A|B)_\sigma - H(A|B)_\rho = \int_0^1 \frac{d}{dt}\,H(A|B)_{\rho(t)}\,dt.$$

Step 2 (CE-RE). By Proposition 3, $\frac{d}{dt}H(A|B)_{\rho(t)} = -\frac{d}{dt}D(\rho(t)\|\tau(t))$.

Step 3 (DER-HS). By Proposition 7,

$$\frac{d}{dt}\,D(\rho(t)\|\tau(t)) = \int_0^{M(t)} \frac{\mathrm{Tr}\big[P_\gamma(t)\,\delta_{AB}^{(\gamma)}\big]}{\gamma(1+d_A\gamma)}\,d\gamma.$$

Step 4 (Substitution). Combining Steps 1–3:

$$H(A|B)_\sigma - H(A|B)_\rho = -\int_0^1 \int_0^{M(t)} \frac{\mathrm{Tr}\big[P_\gamma(t)\,\delta_{AB}^{(\gamma)}\big]}{\gamma(1+d_A\gamma)}\,d\gamma\,dt. \tag{8}$$

Step 5 (Fubini). We verify that the iterated integral in (8) is well-defined via the Fubini–Tonelli theorem. Under the full-rank assumption, $\rho(t) > 0$ for all $t \in [0,1]$. By compactness and continuity of $\lambda_{\min}(\rho(t))$, there exists $c > 0$ (uniform in t) such that for all $\gamma \in (0,c)$ and all $t \in [0,1]$, $\rho(t) - \gamma\,\tau(t) > 0$, hence $P_\gamma(t) = \mathbb{I}$. Then

$$\mathrm{Tr}\big[P_\gamma(t)\,\delta_{AB}^{(\gamma)}\big] = \mathrm{Tr}[\delta_{AB}] - \gamma\;\mathrm{Tr}[\mathbb{I}_A \otimes \delta_B] = 0 - \gamma\,d_A\;\mathrm{Tr}[\delta_B] = 0,$$

since $\mathrm{Tr}[\delta_{AB}] = 0$ (both states have unit trace) and $\mathrm{Tr}[\delta_B] = 0$ (partial traces preserve trace). The integrand therefore vanishes on $[0,1] \times (0,c)$. On the remaining domain $[0,1] \times [c, d_A]$, the factor

$[\gamma(1+d_A\gamma)]^{-1}$ is bounded above by $[c(1+d_Ac)]^{-1}$, and the numerator satisfies $|\operatorname{Tr}[P_\gamma(t)\,\delta_{AB}^{(\gamma)}]| \leq \|\delta_{AB}\|_1 + d_A^2\,\|\delta_B\|_1$. The integrand is thus bounded on this compact domain, and Fubini–Tonelli applies.

The iterated integral in (8) is therefore well-defined and equals the claimed double integral. This completes the proof. □

4 Concluding Remarks

Remark 8. The full-rank assumption ($\rho_{AB}, \sigma_{AB} > 0$) is used in three essential places: (i) ensuring C^1 regularity of $t \mapsto H(A|B)_{\rho(t)}$ for the fundamental theorem of calculus; (ii) justifying differentiation under the integral sign via the vanishing of the integrand near $\gamma = 0$; and (iii) the Fubini argument eliminating the $1/\gamma$ singularity. The representation may extend to the general case via a limiting argument, but this requires additional care with the $\gamma \to 0$ singularity when $\rho(t)$ has a non-trivial kernel.

Remark 9. The proof relies on the Frenkel integral representation (2), which expresses relative entropy as an integral of hockey-stick divergences. This representation, adapted here to the bipartite setting with unnormalised reference $\tau(t) = \mathbb{I}_A \otimes \rho(t)_B$ (trace d_A), is the key structural ingredient enabling the path-integral form.

Remark 10. This proof was constructed within the Adversarial Proof Framework (`af`), a system in which independent prover agents propose proof steps and independent verifier agents challenge them. The proof tree consists of 36 nodes, all of which reached the *validated* epistemic state with *clean* taint. Three challenges were raised against the main assembly node (concerning dependency bookkeeping, structural soundness, and the Fubini justification), all of which were resolved through amendments.