Report on the Quantum Group Generalization of the BLM Melonic Model

Adversarial Proof Framework Analysis

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Abstract

This report documents the adversarial proof investigation of a conjectured quantum group generalization of the Biggs–Lin–Maldacena (BLM) melonic quantum mechanical model. The BLM model is an $\mathcal{N} = 2$ supersymmetric quantum mechanics of $N = 2j + 1$ complex fermions with deterministic interactions governed by SU(2) Wigner 3j symbols. The conjecture proposes that replacing SU(2) 3j symbols with $U_q(\mathfrak{su}(2))$ quantum 3j symbols yields a one-parameter family $H_q = \{Q_q, Q_q^\dagger\}$ exhibiting three distinct large-j geometric regimes: (I) $q = 1$: Euclidean 3D gravity (Ponzano–Regge) with melonic dominance; (II) fixed $q > 0$, $q \neq 1$: hyperbolic regime with Volume Conjecture physics; (III) $q = e^{2\pi i/r}$: topological 3D gravity (Turaev–Viro / Chern–Simons). Over two versions of the proof tree, we have constructed a 17-node v2 proof tree with 11 nodes validated, 5 refined (awaiting re-verification), and 1 archived. A critical error in the original treatment of SUSY at roots of unity was discovered and corrected. The established results (Part 0 and Part I) rest on firm mathematical ground, while Parts II and III remain largely conjectural with clearly identified open problems.

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1 Problem Statement

1.1 The BLM Model

The Biggs–Lin–Maldacena (BLM) model [1] is an $N = 2$ supersymmetric quantum mechanical model of $N = 2j + 1$ complex fermions ψ_m ($m = -j, \ldots, j$) with deterministic three-body interactions governed by the SU(2) Wigner 3j symbol.

Definition 1.1 (BLM supercharge and Hamiltonian)**.** Let j be a positive odd integer and $N = 2j + 1$. Let $C^j_{m_1 m_2 m_3}$ denote the Wigner 3j symbol $\begin{pmatrix} j & j & j \\ m_1 & m_2 & m_3 \end{pmatrix}$. The supercharge is

$$Q = \frac{1}{3!}\sqrt{2JN} \sum_{m_1,m_2,m_3} C^j_{m_1 m_2 m_3}\, \psi_{m_1}\, \psi_{m_2}\, \psi_{m_3},$$

and the Hamiltonian is $H = \{Q, Q^\dagger\}$, where $J > 0$ is a coupling constant.

Remark 1.2. The parameter j must be *odd*. The fermion pair angular momentum $\ell = 1, 3, \ldots, 2j - 1$ must be odd for antisymmetry, so j must be odd. Even j values give unphysical negative energies.

The model possesses SU(2) symmetry, a $U(1)_R$ R-symmetry ($R = N_\psi/3$), and charge conjugation invariance. The normal-ordered form of the Hamiltonian is

$$H = \frac{J}{3}\Big[(2j+1) - 3(N_\psi + j + \tfrac{1}{2}) + 3\sum_m O^\dagger_{j,m} O_{j,m}\Big],$$

where $O_{\ell,m}$ projects fermion pairs onto angular momentum ℓ. Only the $\ell = j$ Haldane pseudopotential channel contributes — a connection to quantum Hall physics on the fuzzy sphere.

Key established properties of the BLM model include:

- $N = 2$ SUSY algebra: $Q^2 = 0$, $(Q^\dagger)^2 = 0$, $H \geq 0$.
- BPS state degeneracy: $D^{\mathrm{BPS}}(j, R = \pm 1/6) = 3^j$ (numerically verified for $j = 1, 3, 5, 7, 9, 11$).
- Melonic dominance at large j: bubble diagrams dominate, with non-melonic corrections suppressed as $(\log j)/j$.
- Emergent 2D CFT at large SU(2) charge, with BPS partition function $Z^{\mathrm{BPS}}_q(j) = 2\prod_{m=1}^{j}(1 + q^m_{\mathrm{BPS}} + q^{2m}_{\mathrm{BPS}})$.

1.2 The Quantum Group Generalization

Conjecture 1.3 (q-deformed BLM model)**.** *Replacing* SU(2) *Wigner 3j symbols* $C^j_{m_1 m_2 m_3}$ *with* $U_q(\mathfrak{su}(2))$ *quantum 3j symbols* $C^j_{m_1 m_2 m_3}(q)$ *in the supercharge yields a one-parameter family of* $N = 2$ *SUSY quantum mechanical models* $H_q = \{Q_q, Q^\dagger_q\}$ *(*$q > 0$ *real) exhibiting three distinct large-*j *geometric regimes:*

(I) $q = 1$ ***(Euclidean):*** *Recovery of the original BLM model. Ponzano–Regge 3D Euclidean gravity. Melonic dominance with SYK-like Schwinger–Dyson equations.*

(II) ***Fixed*** $q > 0$***,*** $q \neq 1$ ***(Hyperbolic):*** *Breakdown of melonic dominance due to exponential growth of quantum 6j symbols governed by hyperbolic tetrahedron volumes (Volume Conjecture regime).*

(III) $q = e^{2\pi i/r}$ ***root of unity (Topological):*** *Topological 3D gravity described by the Turaev–Viro state sum / Chern–Simons theory at level* $k = r - 2$*, with spin truncation* $j \leq (r-2)/2$*.*

1.3 Why This Is Hard

Several features make this conjecture non-trivial:

1. The quantum 3j symbols satisfy different symmetry properties from their classical counterparts; total antisymmetry for j odd survives, but the behavior under column permutations involves phase factors dependent on q.

2. The classical melonic dominance relies on 3j orthogonality (bubble identity) and 6j asymptotic suppression. At $q \neq 1$, the 6j asymptotics change qualitatively: they grow exponentially rather than oscillating, governed by the volume of the associated hyperbolic tetrahedron.

3. At roots of unity, the representation theory truncates ($j \leq (r-2)/2$), and fundamental questions about SUSY preservation arise. The original treatment contained a critical error confusing conjugated coefficients $\bar{Q}$ with the Fock adjoint $Q^\dagger$.

4. The $q \leftrightarrow q^{-1}$ symmetry of quantum 6j symbols constrains which asymptotic formulas are valid: the Costantino–Murakami formula (log-polynomial growth) holds only at roots of unity, while the fixed-real-q asymptotics remain conjectural.

2 Proof Strategy

The v2 proof tree is organized into four parts, corresponding to progressively less established claims.

2.1 Part 0: Well-Definedness and SUSY (Nodes 1.1, 1.1.1, 1.1.2)

Node 1.1 — Established Foundation. For all $q > 0$ real: $Q_q^2 = 0$ (by total antisymmetry of q-3j symbols), $Q_q^\dagger$ is the true Hermitian adjoint (q-3j symbols are real for $q > 0$), and $H_q \geq 0$. The q-bubble identity $\sum_{m_1,m_2} C^j(q)C^j(q) = \delta/[2j+1]_q$ holds. The q-9j symbol with all spins j (odd) vanishes.

Status: VALIDATED. Established from v1 proof tree (nodes 1.1, 1.1.2, 1.1.4, 1.2.1, 1.2.4).

Node 1.1.1 — Braided Fermion / U_q Covariance. The q-deformed supercharge transforms covariantly under $U_q(\mathfrak{su}(2))$. This requires understanding the braided tensor product structure of the fermion Fock space.

Status: VALIDATED. An open problem for the full braided framework, but the basic covariance is established.

Node 1.1.2 — q-Melonic Self-Energy. The q-melonic self-energy is $\Sigma_{\text{melonic}}(q) = J\delta_{m,m'}/[N]_q$, with the quantum dimension $[N]_q$ replacing N.

Status: VALIDATED.

2.2 Part I: Euclidean Regime $q = 1$ (Node 1.2)

Node 1.2 — Original BLM Paper. At $q = 1$, the model reduces to the original BLM construction. Melonic dominance, SYK-like physics, Ponzano–Regge state sum interpretation, and BPS state counting are all established in [1].

Status: VALIDATED.

2.3 Part II: Hyperbolic Regime (Nodes 1.3, 1.3.1–1.3.5)

Node 1.3 — Phase Structure at $q \neq 1$. For fixed $q > 0$, $q \neq 1$, the quantum 6j asymptotics change qualitatively: the Ponzano–Regge oscillatory regime is replaced by exponential growth governed by hyperbolic tetrahedron volumes. This signals a qualitative change in the model's large-j behavior.

Status: VALIDATED.

Node 1.3.1 — 6j Asymptotics. The quantum 6j symbol asymptotics split by regime:

- Root of unity: Costantino–Murakami formula (established).
- Fixed real $q > 0$: $q \leftrightarrow q^{-1}$ symmetry constrains the asymptotics; the exponential growth conjecture remains open.

Status: VALIDATED.

Node 1.3.2 — Non-Melonic Scaling. Classical melonic dominance established; q-deformed non-melonic scaling is conjectural.

Status: VALIDATED.

Nodes 1.3.3, 1.3.4 — Qualitative Change and Volume Conjecture. Node 1.3.3: The qualitative change at $q = 1$ is conjectural, conditional on 1.3.1/1.3.2. Node 1.3.4: The Volume Conjecture provides a motivating *analogy*, not mathematical equivalence.

Status: VALIDATED (both).

Node 1.3.5 — BPS Survival at $q \neq 1$. Whether BPS states survive q-deformation remains an open question.

Status: VALIDATED.

2.4 Part III: Root-of-Unity Regime (Nodes 1.4, 1.4.1–1.4.4)

Node 1.4 — Root-of-Unity Framework. At $q = e^{2\pi i/r}$, the representation theory truncates to $j \leq (r-2)/2$. The regime connects to Turaev–Viro / Chern–Simons topology. Restructured with epistemic labels after critical SUSY correction.

Status: REFINED (awaiting re-verification).

Node 1.4.1 — Turaev–Viro State Sum. The Turaev–Viro state sum at level r is established mathematics: it computes a topological invariant of 3-manifolds equivalent to $|Z_{\mathrm{CS}}|^2$. The TV normalization $D_r^{-2|V|}$ and the admissibility constraint $j \leq (r-2)/2$ are standard.

Status: REFINED (TV normalization, $\Lambda = 4\pi^2/r^2$, conventions updated).

Node 1.4.2 — Boulatov GFT Analogy. The Boulatov group field theory provides a structural analogy with the BLM model (both use 3j-symbol vertices), but this is *not* a mathematical equivalence.

Status: VALIDATED.

Node 1.4.3 — SUSY at Root of Unity. Critical Discovery: The original claim that SUSY breaks at roots of unity was **wrong**. The error confused $\bar{Q}$ (conjugated coefficients) with $Q^\dagger$ (Fock adjoint). In fact, $\{Q, Q^\dagger\} \geq 0$ holds *tautologically* for any operator Q:

$$\langle v|\{Q, Q^\dagger\}|v\rangle = \|Q^\dagger v\|^2 + \|Qv\|^2 \geq 0.$$

The correct obstruction is *representation-theoretic*: at roots of unity, the admissibility constraint $j \leq (r-2)/2$ restricts the model, and the interplay between SUSY and truncation requires careful analysis.

Status: REFINED (rewritten with correct analysis).

Node 1.4.4 — Open Problems and $r \to \infty$ Limit. The $r \to \infty$ limit should recover the $q = 1$ model. The precise rate of convergence and the behavior of BPS states in this limit are open questions.

Status: REFINED (awaiting re-verification).

3 Numerical Results

Exact diagonalization of the BLM Hamiltonian ($q = 1$) has been performed using a Julia implementation with sector-resolved construction and Krylov (Lanczos) eigensolvers.

3.1 Computational Scaling

j	Sites N	Full dim	Largest sector	Build H	Eigensolve
7	15	32,768	289	0.2s	full diag
9	19	524,288	2,934	0.5s	$\sim$30s (Lanczos)
11	23	8,388,608	32,540	6s	$\sim$80s (Lanczos)

Phase 4 implementation uses parallel sector diagonalization via Julia threads; at $j = 11$ with 4 threads, all 2048 sectors are processed (1042 by full diagonalization, 1006 by Lanczos).

3.2 BPS State Verification

j	Observed BPS	Predicted 2×3^j	Match
1	6	6	✓
3	54	54	✓
5	486	486	✓
7	4,374	4,374	✓
9	39,366	39,366	✓
11	354,294	354,294	✓

All BPS states lie at $R = \pm 1/6$ (fermion numbers $n = j$ and $n = j + 1$).

3.3 Hamiltonian Validation

Hermiticity, positive semi-definiteness, vacuum energy $E_{\text{vac}} = J(2j + 1)/3$, and commutators $[H, J_3] = [H, N_\psi] = 0$ have been verified numerically. MPO vs. ED matrix comparison at $j = 1$ gives Frobenius norm error $\sim 10^{-15}$.

4 Current Status

4.1 Proof Tree Evolution

The proof tree has undergone two major revisions:

Version	**Nodes**	**Validated**	**Outcome**
v1	23	5	18 challenged $\to$ archived
v2	17	11	5 refined, 1 archived

The v1 tree was overly optimistic, claiming established results for regimes where the physics is genuinely conjectural. The v2 tree introduces epistemic labels distinguishing three levels:

- **Part 0/I (Established):** Results proven in the literature or by direct computation.
- **Part II (Conjectural):** Plausible claims supported by analogy but lacking rigorous proof.
- **Part III (Mixed):** Established mathematics (Turaev–Viro) combined with open problems (SUSY boundary, $r \to \infty$ limit).

4.2 Node Statistics (v2)

Epistemic State	Count	Meaning
Validated	11	Passed adversarial verification
Refined	5	Challenges resolved, awaiting re-verification
Archived	1	Duplicate (1.3.1.1)
Total	**17**	

4.3 Validated Nodes

Node	Content	Status
1.1	Well-definedness and SUSY (Part 0)	VALIDATED
1.1.1	Braided fermion / U_q covariance	VALIDATED
1.1.2	q-melonic self-energy	VALIDATED
1.2	Euclidean regime $q = 1$ (Part I)	VALIDATED
1.3	Phase structure at $q \neq 1$	VALIDATED
1.3.1	6j asymptotics ($q \leftrightarrow q^{-1}$ constraint)	VALIDATED
1.3.2	Non-melonic scaling	VALIDATED
1.3.3	Qualitative change at $q = 1$	VALIDATED
1.3.4	Volume Conjecture analogy	VALIDATED
1.3.5	BPS survival at $q \neq 1$	VALIDATED
1.4.2	Boulatov GFT analogy	VALIDATED

4.4 Refined Nodes (Awaiting Re-Verification)

Node	Content	Refinement Summary
1	Root conjecture	Epistemic labels added (I=established, II=conjectural, III=mixed)
1.4	Root-of-unity framework	Part III no longer claims TV/CS equivalence
1.4.1	Turaev–Viro state sum	TV normalization $D_r^{-2\|V\|}$, $\Lambda = 4\pi^2/r^2$, admissibility
1.4.3	SUSY at root of unity	Critical rewrite: SUSY preserved, representation-theoretic constraints
1.4.4	Open problems / $r \to \infty$	Updated with corrected SUSY analysis

5 Critical Discovery: SUSY at Roots of Unity

The most significant outcome of the adversarial verification process was the discovery and correction of a fundamental error in the treatment of SUSY at roots of unity.

5.1 The Error

The original v1 proof tree (Node 1.4.3) claimed that SUSY *breaks* at $q = e^{2\pi i/r}$ because the quantum 3j symbols become complex, so $Q_q^\dagger \neq \bar{Q}_q$. This was used to argue that $H_q = \{Q_q, Q_q^\dagger\}$ could fail to be positive semi-definite.

5.2 The Correction

This argument confused two different operations:

- $\bar{Q}$: obtained by conjugating the *coefficients* $C^j_{m_1 m_2 m_3}(q)$.
- $Q^\dagger$: the Fock-space adjoint, defined by $\langle Q^\dagger v, w\rangle = \langle v, Qw\rangle$.

The Fock adjoint $Q^\dagger$ satisfies $\{Q, Q^\dagger\} \geq 0$ *tautologically* for any linear operator Q:

$$\langle v, \{Q, Q^\dagger\} v\rangle = \|Q^\dagger v\|^2 + \|Qv\|^2 \geq 0.$$

Thus SUSY ($H \geq 0$) is *preserved* at roots of unity. The genuine obstruction is representation-theoretic: the spin truncation $j \leq (r-2)/2$ at roots of unity restricts the Hilbert space, and the interplay between the supercharge structure and this truncation requires careful analysis that has not been completed.

5.3 Impact

This correction propagated through the entire Part III of the proof tree, requiring rewrites of nodes 1, 1.4, 1.4.1, 1.4.3, and 1.4.4. All five have been refined and await re-verification.

6 Session History

6.1 v1 Proof Tree Construction

- Initial 23-node proof tree created covering all four parts of the conjecture.
- Organized into: Part 1.1 (well-definedness, 4 children), Part 1.2 (melonic dominance, 5 children), Part 1.3 (q-BPS states, 4 children), Part 1.4 (Turaev–Viro, 5 children).
- 5 nodes validated: basic SUSY properties, 3j antisymmetry, bubble identity, 9j vanishing, positive semi-definiteness.
- 18 nodes challenged: primarily for overclaiming established status on conjectural results.
- Key finding: the $q \leftrightarrow q^{-1}$ symmetry of quantum 6j symbols constrains the Taylor–Woodward asymptotic formula to roots of unity only.

6.2 v2 Proof Tree: First Verification Wave

- Restructured into 17 nodes with explicit epistemic labels.
- Part II (hyperbolic regime) relabeled as conjectural throughout.
- Part III restructured into 4 children (TV state sum, Boulatov analogy, SUSY obstruction, open problems).
- 11 nodes validated on first pass.

6.3 v2 Proof Tree: Prover Fixes

- 5 nodes challenged by adversarial verifiers; all challenges resolved by provers.
- **Critical fix:** SUSY at roots of unity (see Section 5).
- Node 1.4.1: TV normalization, cosmological constant, admissibility conditions corrected.
- Root node: epistemic labels added, Part III no longer claims TV/CS equivalence.
- All 5 refined nodes await re-verification.

7 Open Problems and Gaps

The remaining open problems cluster into four categories.

7.1 Gap 1: Fixed-Real-q Asymptotics of Quantum 6j Symbols

Affected nodes: 1.3.1, 1.3.2, 1.3.3.

The problem: The quantum 6j symbol asymptotics at fixed real $q > 0$ are not rigorously established. The Costantino–Murakami formula [7] gives log-polynomial growth $\sim \exp(c \cdot r)$ at roots of unity $q = e^{2\pi i/r}$, but this formula does not apply at fixed real q. The expected exponential growth governed by hyperbolic tetrahedron volumes (the "Volume Conjecture regime") lacks a rigorous asymptotic formula.

Status: This is correctly labeled as CONJECTURAL in the v2 proof tree. No repair is needed; the gap is acknowledged.

What would resolve it: A rigorous asymptotic formula for quantum 6j symbols at fixed real q, extending the Ponzano–Regge ($q = 1$) and Costantino–Murakami (root of unity) results.

7.2 Gap 2: Representation-Theoretic SUSY Constraints at Roots of Unity

Affected nodes: 1.4.3, 1.4.4.

The problem: While $H_q \geq 0$ is tautological, the spin truncation $j \leq (r-2)/2$ at roots of unity restricts which representations appear. The questions are:

1. Does $Q_q^2 = 0$ still hold in the truncated Hilbert space?
2. How do BPS states behave under truncation?
3. What is the $r \to \infty$ limit of the truncated spectrum?

Status: Node 1.4.3 has been rewritten to pose these as open questions rather than claiming false answers. The correct formulation is in place; the mathematics remains to be done.

7.3 Gap 3: q-Deformed Melonic Dominance

Affected nodes: 1.3.2.

The problem: At $q = 1$, melonic dominance follows from 3j orthogonality (bubble identity) and the $1/\sqrt{j}$ suppression of the tetrahedron (6j) diagram. At $q \neq 1$, the bubble identity generalizes to q-bubble with $[2j+1]_q$ replacing $2j+1$, but the 6j suppression fails: the quantum 6j symbols grow exponentially rather than being suppressed.

Status: Correctly labeled conjectural. The classical ($q = 1$) melonic dominance is established; the q-deformed version is a genuine open question about whether an entirely different large-j regime emerges.

7.4 Gap 4: BPS State Survival Under q-Deformation

Affected nodes: 1.3.5.

The problem: The $\mathbb{Z}_3$-graded Witten index $W_r = \omega^{-(2j+1)/2}(1-\omega^r)^{2j+1}$ is q-independent (it depends only on SUSY algebra, not on the specific Hamiltonian). However, the *detailed* BPS spectrum (which states are BPS, at which R-charges) could change under q-deformation.

Status: This is an open question requiring q-deformed numerics (implementing quantum 3j symbols in the Julia ED code).

8 Assessment of Correctness

8.1 What Is Secure

Part	Content	Confidence
Part 0	Well-definedness, SUSY, q-bubble identity, q-9j vanishing	**High**
Part I	Euclidean regime ($q = 1$) = original BLM	**High**
Node 1.4.2	Boulatov GFT is a structural analogy	**High**

These results rest on established quantum group theory and the published BLM paper. They have survived adversarial verification without challenge.

8.2 What Is Plausible but Conjectural

Part	Content	Confidence
Part II	Hyperbolic regime, Volume Conjecture physics	**Medium**
1.3.1	6j asymptotics at fixed q	**Medium**
1.3.2–1.3.4	Non-melonic scaling, phase change, VC analogy	**Medium–Low**
1.3.5	BPS survival at $q \neq 1$	**Open**

Part II is correctly labeled conjectural in the v2 tree. The physical intuition is compelling (the quantum 6j symbols *do* encode hyperbolic geometry), but rigorous asymptotic results are lacking for fixed real q.

8.3 What Faces Fundamental Obstacles

Part	Content	Confidence
Part III	Root-of-unity regime as a whole	**Mixed**
1.4.1	TV state sum mathematics	**High**
1.4.3	SUSY boundary at truncation	**Open**
1.4.4	$r \to \infty$ limit convergence	**Open**

The Turaev–Viro mathematics is established, but connecting it to the BLM model's physics (SUSY, BPS states, melonic structure) at roots of unity involves genuinely open mathematical questions.

8.4 Overall Assessment

The v2 proof tree is *honest*: it clearly distinguishes established results from conjectures and open problems. The adversarial verification process successfully identified and corrected a critical error (SUSY at roots of unity) and forced the removal of overclaimed equivalences (TV/CS not equivalent to BLM, Boulatov is analogy not identity). The 11 validated nodes form a solid foundation; the 5 refined nodes need only re-verification of the corrected statements, not new mathematical content.

9 Prospects and Recommended Next Steps

9.1 Most Promising Directions

9.1.1 Direction A: q-Deformed Numerics

Modify the Julia ED code to use quantum 3j symbols $C^j_{m_1 m_2 m_3}(q)$. Validate: $q \to 1$ limit recovers original spectrum. Then:

1. Test BPS count stability under q-deformation (Gap 4).
2. Compute non-melonic diagram ratios at $q \neq 1$ to test exponential growth (Gap 3).
3. Measure spectral statistics at various q to detect phase transitions.

Difficulty: Low–Medium. Requires implementing q-deformed Clebsch–Gordan coefficients, which are available in closed form.

9.1.2 Direction B: Root-of-Unity Investigation

Implement the truncated model at $q = e^{2\pi i/r}$ with $j \leq (r-2)/2$:

1. Verify $Q_q^2 = 0$ numerically in the truncated Hilbert space (Gap 2).
2. Count BPS states and compare with $q = 1$ prediction 2×3^j.
3. Study the $r \to \infty$ limit empirically.

Difficulty: Medium. Complex coefficients require careful numerics; the truncated Hilbert space is smaller, easing computation.

9.1.3 Direction C: Rigorous 6j Asymptotics

The missing ingredient for Part II is a rigorous asymptotic formula for quantum 6j symbols at fixed real q. This is a problem in asymptotic analysis / special functions, independent of the BLM model. **Difficulty:** Hard. This is an open problem in quantum topology.

9.1.4 Direction D: Re-Verify Refined Nodes

Run adversarial verifiers on the 5 refined nodes. Since the challenges have been resolved and the mathematical content corrected, re-verification should proceed smoothly. **Difficulty:** Low. Automated process.

9.2 Recommended Priority Order

1. **Priority 1:** Re-verify the 5 refined nodes (Direction D). Cost: minimal. Clears the backlog.
2. **Priority 2:** Implement q-deformed numerics (Direction A). This is the fastest path to new physics results and tests the core conjecture directly.
3. **Priority 3:** Root-of-unity investigation (Direction B). Addresses the most novel part of the conjecture (topological gravity connection).
4. **Priority 4:** Rigorous 6j asymptotics (Direction C). Long-term mathematical goal; not on critical path for numerical validation.

10 Key References

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A Full Proof Tree (v2)

The complete v2 proof tree as maintained in the adversarial proof framework. Status key: **V** = validated, **R** = refined (awaiting re-verification), **A** = archived.

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1 [R] Root conjecture: The q-deformed BLM model H_q = {Q_q, Q_q†}
  |  exhibits three distinct large-j geometric regimes:
  |  (I) q=1 Euclidean, (II) fixed q>0 hyperbolic, (III) root of unity topological.
  |  Epistemic labels: I=established, II=conjectural, III=mixed.
  |
  +-- PART 0: WELL-DEFINEDNESS AND SUSY (ESTABLISHED)
  |
  +-- 1.1 [V] For all q > 0: Q_q^2 = 0, Q_q† is Hermitian adjoint,
  |   |  H_q >= 0. q-bubble identity, q-9j vanishing hold.
  |   |
  |   +-- 1.1.1 [V] Braided fermion / U_q covariance
  |   |
  |   +-- 1.1.2 [V] q-melonic self-energy: Sigma = J delta / [N]_q
  |
  +-- PART I: EUCLIDEAN REGIME q=1 (ESTABLISHED)
  |
  +-- 1.2 [V] q=1 recovers original BLM. Ponzano-Regge, melonic
  |      dominance, SYK-like SD equations all established.
  |
  +-- PART II: HYPERBOLIC REGIME q>0, q1 (CONJECTURAL)
  |
  +-- 1.3 [V] Phase structure at fixed q != 1.
  |   |  6j asymptotics change qualitatively. q<->q^{-1} constraint.
  |   |
  |   +-- 1.3.1 [V] 6j asymptotics: root-of-unity proven (CM),
  |   |      fixed-real-q conjectural.
  |   |
  |   +-- 1.3.2 [V] Non-melonic scaling: classical established,
  |   |      q-deformed conjectural.
  |   |
  |   +-- 1.3.3 [V] Qualitative change at q=1: conjectural,
  |   |      conditional on 1.3.1/1.3.2.
  |   |
  |   +-- 1.3.4 [V] Volume Conjecture: motivating analogy,
  |   |      NOT mathematical equivalence.
  |   |
  |   +-- 1.3.5 [V] BPS survival at q != 1: OPEN.
  |
  +-- PART III: ROOT-OF-UNITY REGIME (MIXED)
  |
  +-- 1.4 [R] Root-of-unity framework. q = exp(2i/r),
      |  spin truncation j <= (r-2)/2.
      |
      +-- 1.4.1 [R] Turaev-Viro state sum (established math).
      |      D_r^{-2|V|} normalization,  = 4²/r².
      |
      +-- 1.4.2 [V] Boulatov GFT: structural analogy, NOT equivalence.
      |
      +-- 1.4.3 [R] SUSY preserved at root of unity.
      |      {Q,Q†} >= 0 tautological. Representation-theoretic
      |      constraints from spin truncation are the real issue.
      |
      +-- 1.4.4 [R] Open problems: r -> infinity limit, BPS under
```

truncation, convergence rate.

1.3.1.1 [A] Archived (duplicate of 1.3.1).

B Key Definitions

Quantum 3j Symbol

The $U_q(\mathfrak{su}(2))$ quantum 3j symbol $C^j_{m_1 m_2 m_3}(q)$ is the q-analog of the Wigner 3j symbol $\begin{pmatrix} j & j & j \\ m_1 & m_2 & m_3 \end{pmatrix}$, defined via the quantum Clebsch–Gordan decomposition of $V_j^{\otimes 3}$. For $q > 0$ real, $C^j(q)$ is real-valued; for j odd, it is totally antisymmetric under column permutation.

Quantum Dimension

$[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$. At $q = 1$: $[n]_1 = n$. At $q = e^{2\pi i/r}$: $[n]_q$ vanishes when $n \equiv 0 \pmod r$, which drives the spin truncation.

Turaev–Viro State Sum

For a triangulation $\mathcal{T}$ of a closed 3-manifold M with $|V|$ vertices and $|E|$ edges:

$$Z_{\mathrm{TV}}(M; r) = D_r^{-2|V|} \sum_{\text{colorings}} \prod_{\text{edges}} [2j_e + 1]_q \prod_{\text{tetrahedra}} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}_q ,$$

where $D_r^2 = \sum_{j=0}^{(r-2)/2} [2j+1]_q^2$ and the sum is over admissible colorings ($j_e \leq (r-2)/2$).

BPS States

A state $|v\rangle$ is BPS if $H|v\rangle = 0$, equivalently $Q|v\rangle = Q^\dagger|v\rangle = 0$. The BPS degeneracy is counted by the $\mathbb{Z}_3$-graded Witten index:

$$W_r = \mathrm{Tr}\big[(-1)^F \omega^{rR}\big] = \omega^{-(2j+1)/2}(1 - \omega^r)^{2j+1}, \qquad \omega = e^{2\pi i/3}.$$

C Physics Summary Table

Feature	$q = 1$ **(Euclidean)**	$q > 0$, $q \neq 1$ **(Hyperbolic)**	$q = e^{2\pi i/r}$ **(Root of unity)**
3D gravity	Ponzano–Regge	Hyperbolic volume	Turaev–Viro / CS
Melonic dom.	Yes ($1/\sqrt{j}$)	Breaks (exp. growth)	Truncated ($j \leq (r-2)/2$)
SUSY	$H \geq 0$, $Q^2 = 0$	$H \geq 0$, $Q^2 = 0$	$H \geq 0$ tautological; $Q^2 = 0$ open
BPS count	2×3^j	Open	Open
6j asymptotics	PR oscillatory	Conjectural exp. growth	CM log-polynomial
$q \leftrightarrow q^{-1}$	Trivial	Constraining	N/A
Status	Established	Conjectural	Mixed