Wilde's Path-Integral Hockey-Stick Representation

Adversarial Proof Tree — Faithful Node-by-Node Rendering

Generated from `af` (Adversarial Proof Framework)

36 nodes · 36 validated · 36 clean · 30 challenges

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Contents

1 Definitions

The proof workspace defines the following 8 terms.

hilbert_spaces. $\mathcal{H}_A$, $\mathcal{H}_B$: finite-dimensional complex Hilbert spaces. $d_A := \dim(\mathcal{H}_A)$, $d_B := \dim(\mathcal{H}_B)$. $\mathcal{H}_{AB} := \mathcal{H}_A \otimes \mathcal{H}_B$, dimension $d_{AB} = d_A d_B$. $\mathbf{1}_X$: identity on $\mathcal{H}_X$.

states. Density operator: $\rho \geq 0$, $\mathrm{Tr}[\rho] = 1$. Set $\mathcal{D}(\mathcal{H})$. Partial trace: $\rho_B := \mathrm{Tr}_A[\rho_{AB}]$. Trace distance: $T(\rho, \sigma) := \frac{1}{2}\|\rho - \sigma\|_1$.

operators. $X \geq 0$: positive semidefinite (PSD). $X > 0$: strictly positive definite. Positive part: $X_+ := \sum_{\lambda_i > 0} \lambda_i |e_i\rangle\langle e_i|$. Spectral projector: $\mathbf{1}\{X > 0\} := \sum_{\lambda_i > 0} |e_i\rangle\langle e_i|$. Trace norm: $\|X\|_1 := \mathrm{Tr}(\sqrt{X^\dagger X})$.

entropy. Von Neumann entropy: $S(\rho) := -\mathrm{Tr}[\rho \log \rho]$ (natural log, $0 \log 0 := 0$). Conditional entropy: $H(A|B)_\rho := S(\rho_{AB}) - S(\rho_B)$. Relative entropy (Umegaki): $D(\rho\|\sigma) := \mathrm{Tr}[\rho(\log \rho - \log \sigma)]$ when $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$, else $+\infty$. Binary entropy: $h_2(\varepsilon) := -\varepsilon \log \varepsilon - (1 - \varepsilon) \log(1 - \varepsilon)$. Key identity CE-RE: $H(A|B)_\rho = -D(\rho_{AB}\|\mathbf{1}_A \otimes \rho_B)$.

hockey_stick. For PSD operators ρ, σ and threshold $\gamma > 0$: $E_\gamma(\rho\|\sigma) := \mathrm{Tr}(\rho - \gamma\sigma)_+$. SDP form: $E_\gamma(\rho\|\sigma) = \max_{0 \leq M \leq \mathbf{1}} \mathrm{Tr}[M(\rho - \gamma\sigma)]$. The maximiser is $P_\gamma = \mathbf{1}\{\rho - \gamma\sigma > 0\}$.

max_relative_entropy. $M(\rho, \sigma) := \inf\{\lambda \geq 0 : \rho \leq \lambda\sigma\}$. For bipartite states: $M(\rho_{AB}, \mathbf{1}_A \otimes \rho_B) \leq d_A$.

frenkel_integral. Frenkel's integral representation (FR-1): For density operators ρ, σ (trace 1) with $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$:

$$D(\rho\|\sigma) = \int_0^\infty \frac{E_\gamma(\rho\|\sigma) - (1 - \gamma)_+}{\gamma(1 + \gamma)}\, d\gamma.$$

For unnormalised reference τ with $\mathrm{Tr}[\tau] = c > 0$ (FR-c):

$$D(\rho\|\tau) = \int_0^\infty \frac{E_\beta(\rho\|\tau) - (1 - c\beta)_+}{\beta(1 + c\beta)}\, d\beta - \log c.$$

interpolating_path. $\rho(t) := (1 - t)\rho_{AB} + t\,\sigma_{AB}$, $t \in [0, 1]$. $\delta_{AB} := \sigma_{AB} - \rho_{AB}$ (traceless Hermitian). $\delta_B := \sigma_B - \rho_B = \mathrm{Tr}_A[\delta_{AB}]$. $\rho(t)_B := \mathrm{Tr}_A[\rho(t)] = (1 - t)\rho_B + t\,\sigma_B$. $\tau(t) := \mathbf{1}_A \otimes \rho(t)_B$ (conditional reference, $\mathrm{Tr}[\tau(t)] = d_A$). $\omega(t) := \tau(t)/d_A$ (normalised reference, $\mathrm{Tr}[\omega(t)] = 1$).

2 Proof Tree

The complete proof tree consists of 36 nodes. Every node is shown with its ID, type (claim, qed, or local_assume), epistemic status, and taint status. Child nodes are nested inside their parent.

Node 1 [claim] validated / clean

Path-Integral Hockey-Stick Representation: Under full-rank assumption ($\rho_{AB}, \sigma_{AB} > 0$),

$$H(A|B)_\sigma - H(A|B)_\rho = -\int_0^1 \int_0^{M(t)} \frac{\mathrm{Tr}\big[P_\gamma(t)\,\delta_{AB}^{(\gamma)}\big]}{\gamma(1+d_A\gamma)}\,d\gamma\,dt,$$

where $\rho(t) = (1-t)\rho_{AB} + t\,\sigma_{AB}$, $\tau(t) = \mathbf{1}_A \otimes \rho(t)_B$, $P_\gamma(t) = \mathbf{1}\{\rho(t) - \gamma\,\tau(t) > 0\}$, $\delta_{AB}^{(\gamma)} = \delta_{AB} - \gamma\,\mathbf{1}_A \otimes \delta_B$, $M(t) = M(\rho(t), \tau(t)) \le d_A$.

Node 1.1 [local_assume] validated / clean

(Full Rank Assumption) Assume $\rho_{AB}, \sigma_{AB} > 0$. This ensures $\rho(t) > 0$ and $\rho(t)_B > 0$ for all $t \in [0,1]$.

Node 1.2 [claim] validated / clean

(CE-RE Identity) $H(A|B)_{\rho(t)} = -D(\rho(t)_{AB} \| \mathbf{1}_A \otimes \rho(t)_B)$ for all $t \in [0,1]$. This follows from the definition of conditional entropy and relative entropy.

Node 1.2.1 [claim] validated / clean

By definition, $D(\rho_{AB}\|\mathbf{1}_A \otimes \rho_B) = \mathrm{Tr}[\rho_{AB}\log\rho_{AB}] - \mathrm{Tr}[\rho_{AB}\log(\mathbf{1}_A \otimes \rho_B)]$.

Node 1.2.2 [claim] validated / clean

$\log(\mathbf{1}_A \otimes \rho_B) = \mathbf{1}_A \otimes \log(\rho_B)$, by the spectral mapping theorem for tensor products.

Node 1.2.3 [claim] validated / clean

$\mathrm{Tr}[\rho_{AB}(\mathbf{1}_A \otimes \log\rho_B)] = \mathrm{Tr}[\rho_B \log\rho_B] = -S(\rho_B)$, by the partial trace property $\mathrm{Tr}[\rho_{AB}(\mathbf{1}_A \otimes X_B)] = \mathrm{Tr}[\rho_B X_B]$.

Node 1.2.4 [qed] validated / clean

Therefore $D(\rho_{AB}\|\mathbf{1}_A \otimes \rho_B) = -S(\rho_{AB}) + S(\rho_B) = -(S(\rho_{AB}) - S(\rho_B)) = -H(A|B)_\rho$, giving $H(A|B)_\rho = -D(\rho_{AB}\|\mathbf{1}_A \otimes \rho_B)$.

Node 1.3 [claim] validated / clean

(FTC) $H(A|B)_\sigma - H(A|B)_\rho = \int_0^1 (d/dt)\,H(A|B)_{\rho(t)}\,dt$. This holds by the fundamental theorem of calculus since $H(A|B)_{\rho(t)}$ is C^1 under the full-rank assumption.

Node 1.3.1 [claim] validated / clean

Under the full-rank assumption (node 1.1), $\rho(t)$ and $\rho(t)_B$ are strictly positive for all $t \in [0,1]$. Hence $S(\rho(t))$ and $S(\rho(t)_B)$ are smooth (C^∞) functions of t, since the eigenvalues of $\rho(t)$ are smooth and positive.

Node 1.3.2 [claim] validated / clean

Since $H(A|B)_{\rho(t)} = S(\rho(t)_{AB}) - S(\rho(t)_B)$ and both terms are C^1, $H(A|B)_{\rho(t)}$ is C^1 on $[0,1]$. The fundamental theorem of calculus applies: $H(A|B)_{\rho(1)} - H(A|B)_{\rho(0)} = \int_0^1 \frac{d}{dt} H(A|B)_{\rho(t)}\, dt$.

Node 1.3.3 [qed] validated / clean

Since $\rho(0) = \rho_{AB}$ and $\rho(1) = \sigma_{AB}$, we obtain $H(A|B)_\sigma - H(A|B)_\rho = \int_0^1 \frac{d}{dt} H(A|B)_{\rho(t)}\, dt$.

Node 1.4 [claim] validated / clean

(DER) $(d/dt)\, H(A|B)_{\rho(t)} = \mathrm{Tr}[\delta_{AB}(\mathbf{1}_A \otimes \log \rho(t)_B - \log \rho(t)_{AB})]$. Derivative of conditional entropy along the interpolating path.

Node 1.4.1 [claim] validated / clean

By CE-RE (node 1.2), $\frac{d}{dt} H(A|B)_{\rho(t)} = -\frac{d}{dt} D(\rho(t)\|\tau(t))$. Write $D(\rho(t)\|\tau(t)) = \mathrm{Tr}[\rho(t)\log\rho(t)] - \mathrm{Tr}[\rho(t)\log\tau(t)]$.

Node 1.4.2 [claim] validated / clean

For the first term: $\frac{d}{dt}\,\mathrm{Tr}[\rho(t)\log\rho(t)] = \mathrm{Tr}[\delta_{AB}\log\rho(t)] + \mathrm{Tr}[\rho(t)\, D_{\log}(\rho(t))[\delta_{AB}]]$. By the Fréchet derivative identity $\mathrm{Tr}[A\, D_{\log}(B)[C]] = \mathrm{Tr}[B^{-1}AC]$ in the full-rank case, the second part gives $\mathrm{Tr}[\delta_{AB}] = 0$ (since δ_{AB} is traceless). So $\frac{d}{dt}\,\mathrm{Tr}[\rho(t)\log\rho(t)] = \mathrm{Tr}[\delta_{AB}\log\rho(t)]$.

Node 1.4.2.1 [qed] validated / clean

For the first term: $\frac{d}{dt}\,\mathrm{Tr}[\rho(t)\log\rho(t)] = \mathrm{Tr}[\delta_{AB}\log\rho(t)] + \mathrm{Tr}[\rho(t)\, D_{\log}(\rho(t))[\delta_{AB}]]$. By the identity $\mathrm{Tr}[B\, D_{\log}(B)[C]] = \mathrm{Tr}[C]$ for $B > 0$ (which follows from the integral representation $D_{\log}(B)[C] = \int_0^\infty (B+sI)^{-1}\, C\, (B+sI)^{-1}\, ds$ and cyclicity of trace), the second part with $B = \rho(t) > 0$ (by node 1.1) and $C = \delta_{AB}$ gives $\mathrm{Tr}[\delta_{AB}] = 0$ (since $\delta_{AB} = \sigma_{AB} - \rho_{AB}$ is traceless, both being partial traces of density matrices). So $\frac{d}{dt}\,\mathrm{Tr}[\rho(t)\log\rho(t)] = \mathrm{Tr}[\delta_{AB}\log\rho(t)]$.

Depends on: 1.1

Node 1.4.3 [claim] validated / clean

For the second term: $\frac{d}{dt}\,\mathrm{Tr}[\rho(t)\log\tau(t)] = \mathrm{Tr}[\delta_{AB}\log\tau(t)] + \mathrm{Tr}[\rho(t)\, D_{\log}(\tau(t))[\dot\tau(t)]]$. Since $\tau(t) = \mathbf{1}_A \otimes \rho(t)_B$, $\log\tau(t) = \mathbf{1}_A \otimes \log\rho(t)_B$ and $\dot\tau(t) = \mathbf{1}_A \otimes \delta_B$. The Fréchet derivative term gives $\mathrm{Tr}[\rho(t)_B\, \delta_B\, \rho(t)_B^{-1}] = \mathrm{Tr}[\delta_B] = 0$.

Node 1.4.4 [qed] validated / clean

Combining: $\frac{d}{dt} D(\rho(t)\|\tau(t)) = \mathrm{Tr}[\delta_{AB} \log\rho(t)] - \mathrm{Tr}[\delta_{AB} \log\tau(t)] = \mathrm{Tr}[\delta_{AB}(\log\rho(t) - \mathbf{1}_A \otimes \log\rho(t)_B)]$. Negating: $\frac{d}{dt} H(A|B)_{\rho(t)} = \mathrm{Tr}[\delta_{AB}(\mathbf{1}_A \otimes \log\rho(t)_B - \log\rho(t))]$.

Node 1.5 [claim] validated / clean

(HS-DER) $\frac{d}{dt} E_\gamma(\rho(t)\|\tau(t)) = \mathrm{Tr}[P_\gamma(t)\,(\delta_{AB} - \gamma\,\mathbf{1}_A \otimes \delta_B)]$ at generic γ (no zero eigenvalue of $\rho(t) - \gamma\,\tau(t)$).

Node 1.5.1 [claim] validated / clean

Write $E_\gamma(\rho(t)\|\tau(t)) = \mathrm{Tr}(\rho(t) - \gamma\,\tau(t))_+$. Let $A(t) = \rho(t) - \gamma\,\tau(t)$. By spectral decomposition $A(t) = \sum_i \lambda_i(t)\,|e_i(t)\rangle\langle e_i(t)|$, and $A(t)_+ = \sum_{\lambda_i>0} \lambda_i(t)\,|e_i(t)\rangle\langle e_i(t)| = P_\gamma(t)\,A(t)\,P_\gamma(t)$ where $P_\gamma(t) = \mathbf{1}\{A(t) > 0\}$.

Node 1.5.2 [claim] validated / clean

At generic γ (where no eigenvalue of $A(t)$ equals zero), the rank of $A(t)_+$ is locally constant in t. The eigenvalues and eigenprojections are smooth in t. Therefore $\mathrm{Tr}(A(t)_+)$ is differentiable.

Node 1.5.3 [claim] validated / clean

Differentiating via the product rule: $\frac{d}{dt}\,\mathrm{Tr}(A(t)_+) = \frac{d}{dt}\,\mathrm{Tr}[P_\gamma(t)\,A(t)] = \mathrm{Tr}[\frac{dP_\gamma}{dt} \cdot A(t)] + \mathrm{Tr}[P_\gamma(t) \cdot \dot{A}(t)]$. For the first term: since $P_\gamma(t)$ is a spectral projector of $A(t)$, we have $[P_\gamma(t), A(t)] = 0$. Differentiating $P^2 = P$ gives $P'P + PP' = P'$, so $PP'P = 0$. Since $PA = AP$ (commutativity), $\mathrm{Tr}[P'A] = \mathrm{Tr}[P'PA] = \mathrm{Tr}[(PP'P)A] = 0$ (using $PP'P = 0$ and cyclicity of trace). Therefore $\frac{d}{dt}\,\mathrm{Tr}(A(t)_+) = \mathrm{Tr}[P_\gamma(t)\,\dot{A}(t)]$. Here $\dot{A}(t) = \delta_{AB} - \gamma(\mathbf{1}_A \otimes \delta_B) = \delta^{(\gamma)}_{AB}$.

Node 1.5.3.1 [claim] validated / clean

Differentiating via the product rule: $\frac{d}{dt}\,\mathrm{Tr}(A(t)_+) = \frac{d}{dt}\,\mathrm{Tr}[P_\gamma(t)\,A(t)] = \mathrm{Tr}[\frac{dP_\gamma}{dt} \cdot A(t)] + \mathrm{Tr}[P_\gamma(t) \cdot \dot{A}(t)]$. For the first term: since $P = P_\gamma(t)$ is a spectral projector of $A = A(t)$, we have $[P, A] = 0$ (both block-diagonal in the P-decomposition). Differentiating $P^2 = P$ gives $P'P + PP' = P'$, hence $PP'P = 0$ and $(I - P)P'(I - P) = 0$, so $P' = PP'(I-P) + (I-P)P'P$ is off-block-diagonal. Since A is block-diagonal (commutes with P), the product $P'A$ is off-block-diagonal. Off-block-diagonal operators have zero trace:

$$\begin{aligned}
\mathrm{Tr}[P'A] &= \mathrm{Tr}[PP'(I-P)A] + \mathrm{Tr}[(I-P)P'PA] \\
&= \mathrm{Tr}[PP'A(I-P)] + \mathrm{Tr}[(I-P)P'AP] \\
&= \mathrm{Tr}[(I-P)PP'A] + \mathrm{Tr}[P(I-P)P'A] = 0 + 0 = 0
\end{aligned}$$

(using $P(I - P) = 0$ and cyclicity). Therefore $\frac{d}{dt}\,\mathrm{Tr}(A(t)_+) = \mathrm{Tr}[P_\gamma(t)\,\dot{A}(t)]$. Here $\dot{A}(t) = \delta_{AB} - \gamma(\mathbf{1}_A \otimes \delta_B) = \delta^{(\gamma)}_{AB}$.

Depends on: 1.5.1, 1.5.2

Node 1.5.4 [qed] validated / clean

Therefore $\frac{d}{dt}\, E_\gamma(\rho(t)\|\tau(t)) = \mathrm{Tr}[P_\gamma(t)\,(\delta_{AB} - \gamma\, \mathbf{1}_A \otimes \delta_B)] = \mathrm{Tr}[P_\gamma(t)\, \delta^{(\gamma)}_{AB}]$.

Node 1.6 [claim] validated / clean

(DER-HS) $\frac{d}{dt}\, D(\rho(t)\|\tau(t)) = \int_0^{M(t)} \frac{\mathrm{Tr}[P_\gamma(t)\,\delta^{(\gamma)}_{AB}]}{\gamma(1+d_A\gamma)}\, d\gamma$. The derivative of relative entropy expressed via hockey-stick divergence using the corrected Frenkel formula FR-bip.

Node 1.6.1 [claim] validated / clean

By the Frenkel bipartite formula FR-bip (with $c = d_A$): $D(\rho(t)\|\tau(t)) = \int_0^\infty \frac{E_\gamma(\rho(t)\|\tau(t)) - (1-d_A\gamma)_+}{\gamma(1+d_A\gamma)}\, d\gamma - \log(d_A)$. The term $(1 - d_A\gamma)_+$ and $\log(d_A)$ are independent of t.

Node 1.6.2 [claim] validated / clean

Differentiate under the integral sign: $\frac{d}{dt}\, D(\rho(t)\|\tau(t)) = \int_0^\infty \frac{\frac{d}{dt}\, E_\gamma(\rho(t)\|\tau(t))}{\gamma(1+d_A\gamma)}\, d\gamma$. The exchange of $\frac{d}{dt}$ and $\int$ is justified as follows: Under the full-rank assumption (node 1.1), $\rho(t) > 0$ for all $t \in [0, 1]$. By compactness of $[0, 1]$ and continuity of the minimum eigenvalue $\lambda_{\min}(\rho(t))$, there exists $\gamma_0 > 0$ (uniform in t) such that $\rho(t) - \gamma\,\tau(t) > 0$ for all $\gamma < \gamma_0$ and all t. For such γ, $E_\gamma(\rho(t)\|\tau(t)) = \mathrm{Tr}[\rho(t) - \gamma\,\tau(t)] = 1 - \gamma d_A$, which is t-independent, so $\frac{d}{dt}\, E_\gamma = 0$. Therefore the integrand vanishes on $(0, \gamma_0)$. On $[\gamma_0, \infty)$, the bound $|\frac{d}{dt}\, E_\gamma| \le \|\delta_{AB}\|_1 + \gamma\, \|\mathbf{1}_A \otimes \delta_B\|_1$ is finite, and $[\gamma(1+d_A\gamma)]^{-1} \le [\gamma_0(1+d_A\gamma_0)]^{-1}$ is bounded. Dominated convergence applies on $[\gamma_0, \infty)$.

Node 1.6.3 [claim] validated / clean

By node 1.5 (HS-DER), $\frac{d}{dt}\, E_\gamma(\rho(t)\|\tau(t)) = \mathrm{Tr}[P_\gamma(t)\, \delta^{(\gamma)}_{AB}]$ at generic γ. Since the set of non-generic γ has measure zero, the integral is unaffected.

Node 1.6.4 [claim] validated / clean

For $\gamma > M(t)$, we have $\rho(t) \le \gamma\,\tau(t)$, so $E_\gamma(\rho(t)\|\tau(t)) = 0$ identically in t, hence its t-derivative is also 0. The upper limit effectively becomes $M(t) \le d_A$.

Node 1.6.5 [qed] validated / clean

Combining: $\frac{d}{dt}\, D(\rho(t)\|\tau(t)) = \int_0^{M(t)} \frac{\mathrm{Tr}[P_\gamma(t)\,\delta^{(\gamma)}_{AB}]}{\gamma(1+d_A\gamma)}\, d\gamma$.

Node 1.7 [qed] validated / clean

(MAIN) Combining FTC (node 1.3), $H(A|B) = -D$ (node 1.2), and DER-HS (node 1.6):

$$H(A|B)_\sigma - H(A|B)_\rho = -\int_0^1 \int_0^{M(t)} \frac{\mathrm{Tr}[P_\gamma(t)\,\delta^{(\gamma)}_{AB}]}{\gamma(1+d_A\gamma)}\, d\gamma\, dt.$$

Fubini applies because, under the full-rank assumption (node 1.1), the integrand vanishes for γ below a uniform threshold $c > 0$ (since $\mathrm{Tr}[\delta_{AB}] = 0$ and $\mathrm{Tr}[\delta_B] = 0$ when $P_\gamma = I$), so the effective domain $[0, 1] \times [c, d_A]$ is compact with bounded integrand (see node 1.7.5).

Node 1.7.1 [claim] validated / clean

By FTC (node 1.3): $H(A|B)_\sigma - H(A|B)_\rho = \int_0^1 \frac{d}{dt} H(A|B)_{\rho(t)}\, dt$.

Node 1.7.2 [claim] validated / clean

By CE-RE (node 1.2): $\frac{d}{dt} H(A|B)_{\rho(t)} = -\frac{d}{dt} D(\rho(t)\|\tau(t))$.

Node 1.7.3 [claim] validated / clean

By DER-HS (node 1.6): $\frac{d}{dt} D(\rho(t)\|\tau(t)) = \int_0^{M(t)} \frac{\mathrm{Tr}[P_\gamma(t)\, \delta_{AB}^{(\gamma)}]}{\gamma(1+d_A\gamma)}\, d\gamma$.

Node 1.7.4 [claim] validated / clean

Substituting into the FTC integral:

$$H(A|B)_\sigma - H(A|B)_\rho = -\int_0^1 \int_0^{M(t)} \frac{\mathrm{Tr}[P_\gamma(t)\, \delta_{AB}^{(\gamma)}]}{\gamma(1+d_A\gamma)}\, d\gamma\, dt.$$

Node 1.7.5 [claim] validated / clean

Fubini–Tonelli applies to justify the iterated integral. Under the full-rank assumption (node 1.1), $\rho(t) > 0$ for all $t \in [0,1]$. By compactness of $[0,1]$ and continuity of the minimum eigenvalue of $\rho(t)$, there exists $c > 0$ (uniform in t) such that for all $\gamma \in (0,c)$ and all $t \in [0,1]$, $\rho(t) - \gamma\, \tau(t) > 0$, hence $P_\gamma(t) = I$. Then $\mathrm{Tr}[P_\gamma(t)\, \delta_{AB}^{(\gamma)}] = \mathrm{Tr}[\delta_{AB}] - \gamma\ \mathrm{Tr}[\mathbf{1}_A \otimes \delta_B] = 0 - \gamma\, d_A\ \mathrm{Tr}[\delta_B] = 0$, since $\mathrm{Tr}[\delta_{AB}] = 0$ (both states have unit trace) and $\mathrm{Tr}[\delta_B] = 0$ (partial traces preserve trace). Therefore the integrand vanishes on $[0,1] \times (0,c)$. On the remaining domain $[0,1] \times [c, d_A]$, the factor $[\gamma(1+d_A\gamma)]^{-1}$ is bounded by $[c(1+d_A c)]^{-1}$, and the numerator is bounded by $\|\delta_{AB}\|_1 + d_A^2\, \|\delta_B\|_1$. Hence the integrand is bounded on this compact domain, and Fubini–Tonelli applies.

Node 1.7.6 [qed] validated / clean

Therefore $H(A|B)_\sigma - H(A|B)_\rho = -\int_0^1 \int_0^{M(t)} \frac{\mathrm{Tr}[P_\gamma(t)\, \delta_{AB}^{(\gamma)}]}{\gamma(1+d_A\gamma)}\, d\gamma\, dt$. QED.

3 Challenge Log

A total of 30 challenges were raised by adversarial verifier agents during the proof construction. Of these, 29 are resolved and 1 remains open.

Challenge ID	Node	Severity	Status	Reason (abbreviated)
`ch-1383541abc5`	1.2.2	note	resolved	Nomenclature: "spectral mapping theorem" should be "functional calculus for tensor products"
`ch-a5b1fe0a200`	1.2.3	note	resolved	Partial trace property not declared as external reference; ambiguous use of Tr
`ch-be1a55d9a13`	1.2.4	minor	resolved	No declared dependencies on 1.2.1–1.2.3
`ch-df131e734d5`	1.3.1	minor	resolved	"Eigenvalues are smooth" imprecise; smoothness of $S(\rho(t))$ follows from analyticity of $A \mapsto -\mathrm{Tr}[A \log A]$ on PD cone
`ch-94b49f661ce`	1.4.1	minor	resolved	References node 1.2 but no declared dependency
`ch-36dc7252b6f`	1.4.2	minor	resolved	Fréchet derivative identity misstated for general A; special case $A = B$ used is correct
`ch-c2e0c60129d`	1.4.2	minor	resolved	No declared dependencies on node 1.1 or interpolating_path
`ch-64c786eed61`	1.4.2.1	note	resolved	Minor wording: ρ_{AB}, σ_{AB} are density matrices, not "partial traces of density matrices"
`ch-2631666bed3`	1.4.3	minor	resolved	Unstated intermediate steps: tensor-product compatibility of $D_{\log}$ and partial trace identity
`ch-470c2904317`	1.4.4	note	resolved	No declared dependencies on 1.4.1–1.4.3
`ch-57e44965f0d`	1.5.2	minor	resolved	"Eigenprojections smooth" imprecise; aggregate spectral projection $P_\gamma(t)$ is smooth
`ch-2da126bd81f`	1.5.3	minor	resolved	No declared dependencies on 1.5.1, 1.5.2
`ch-3b6d07bdd06`	1.5.3	major	resolved	"$P_\gamma(t)$ locally constant" is false; $P(t)$ varies smoothly as eigenvectors rotate
`ch-9644ff271ba`	1.5.3	major	resolved	"Eigenvalues crossing zero contribute measure zero" invalid for pointwise derivative
`ch-d76c2ce1134`	1.5.3	minor	resolved	Dependencies still not added after prior challenges
`ch-d89bf26d847`	1.5.3	critical	resolved	Algebraic errors in $\mathrm{Tr}[P'A] = \mathrm{Tr}[P'PA] = \mathrm{Tr}[(PP'P)A]$; correct proof uses off-block-diagonal structure
`ch-ed64acfec8b`	1.5.3	major	resolved	Third time dependency issue raised; dependencies field still shows none
`ch-dbbdb582ef8`	1.6.1	note	resolved	No declared dependencies on frenkel_integral definition, interpolating_path, or full-rank assumption
`ch-cc812a5c870`	1.6.2	minor	**open**	DCT argument on $[\gamma_0, \infty)$ incomplete: dominating function $O(1/\gamma)$ not integrable; should restrict to $[\gamma_0, d_A]$

Challenge ID	Node	Severity	Status	Reason (abbreviated)
`ch-ccf446a99bb`	1.6.2	major	resolved	Dominating function not integrable near $\gamma = 0$; fixed by noting integrand vanishes for $\gamma < \gamma_0$
`ch-ea3d8723544`	1.6.4	note	resolved	"Identically in t" imprecise; $M(t)$ varies with t
`ch-e789d1c6095`	1.6.5	minor	resolved	No declared dependencies on 1.6.2–1.6.4
`ch-2d84a890d86`	1.7	note	resolved	Structural assessment: logical chain sound; Fubini text needs correction
`ch-6e886672b55`	1.7	minor	resolved	No declared dependencies on 1.2, 1.3, 1.6
`ch-de8e84eddfd`	1.7	major	resolved	"Bounded integrand" claim incorrect due to $1/\gamma$ singularity; integrand vanishes near $\gamma = 0$
`ch-a27fd88c37f`	1.7.1	minor	resolved	Cites node 1.3 but no declared dependency
`ch-6d9506bb668`	1.7.2	minor	resolved	Cites node 1.2 but no declared dependency
`ch-48442a6e7ac`	1.7.4	note	resolved	No declared dependencies on 1.7.1–1.7.3
`ch-40e6ca8a4a7`	1.7.5	critical	resolved	Original "bounded integrand" argument invalid; corrected via full-rank vanishing near $\gamma = 0$
`ch-bdc7601d90b`	1.7.6	note	resolved	No declared dependencies on 1.7.4, 1.7.5

4 Statistics

Metric	**Value**
Total nodes	36
of type claim	26
of type qed	9
of type local_assume	1
Epistemic: validated	36
Epistemic: pending	0
Epistemic: admitted	0
Epistemic: refuted	0
Taint: clean	36
Taint: self_admitted	0
Taint: tainted	0
Taint: unresolved	0
Total challenges	30
resolved	29
open	1
Challenge severity breakdown	
critical	2
major	6
minor	14
note	8
Tree depth	4 (root $\rightarrow$ 1.5.3.1, 1.4.2.1)

Node inventory. The 36 nodes are: `1`, `1.1`, `1.2`, `1.2.1`, `1.2.2`, `1.2.3`, `1.2.4`, `1.3`, `1.3.1`, `1.3.2`, `1.3.3`, `1.4`, `1.4.1`, `1.4.2`, `1.4.2.1`, `1.4.3`, `1.4.4`, `1.5`, `1.5.1`, `1.5.2`, `1.5.3`, `1.5.3.1`, `1.5.4`, `1.6`, `1.6.1`, `1.6.2`, `1.6.3`, `1.6.4`, `1.6.5`, `1.7`, `1.7.1`, `1.7.2`, `1.7.3`, `1.7.4`, `1.7.5`, `1.7.6`.