Corrected Path-Integral Hockey-Stick Representation of Conditional Entropy

Numerical verification via Julia

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1 Setup

Let $\mathcal{H}_A$, $\mathcal{H}_B$ be finite-dimensional complex Hilbert spaces with $d_A = \dim \mathcal{H}_A$, $d_B = \dim \mathcal{H}_B$, and $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

Let $\rho_{AB}, \sigma_{AB} \in \mathcal{D}(\mathcal{H}_{AB})$ be density operators with $\rho_{AB}, \sigma_{AB} > 0$ (full rank).

Definition 1 (Interpolation and derived quantities)**.**

$$\rho(t) := (1-t)\,\rho_{AB} + t\,\sigma_{AB}, \qquad t \in [0,1] \tag{1}$$

$$\tau(t) := \nVdash_A \otimes \rho(t)_B, \qquad \rho(t)_B := \mathrm{Tr}_A[\rho(t)] \tag{2}$$

$$\delta_{AB} := \sigma_{AB} - \rho_{AB} \tag{3}$$

$$\delta_B := \mathrm{Tr}_A[\delta_{AB}] = \sigma_B - \rho_B \tag{4}$$

Definition 2 (Hockey-stick divergence)**.** For PSD operators A, B and threshold $\gamma > 0$:

$$E_\gamma(A\|B) := \mathrm{Tr}(A - \gamma B)_+$$

where $X_+ := \sum_{\lambda_i > 0} \lambda_i |e_i\rangle\langle e_i|$ denotes the positive part of Hermitian X.

Definition 3 (Spectral projectors)**.**

$$P_\beta(t) := \mathbf{1}\big\{\rho(t) - \beta\,\tau(t) > 0\big\} \qquad \text{(forward projector)} \tag{5}$$

$$Q_\beta(t) := \mathbf{1}\big\{\tau(t) - \beta\,\rho(t) > 0\big\} \qquad \text{(reverse projector)} \tag{6}$$

Definition 4 (Threshold bounds)**.**

$$M_{\mathrm{fwd}}(t) := \inf\big\{\lambda \ge 0 : \rho(t) \le \lambda\,\tau(t)\big\} = \max\mathrm{eig}\big(\rho(t),\, \tau(t)\big) \le d_A \tag{7}$$

$$M_{\mathrm{rev}}(t) := \inf\big\{\lambda \ge 0 : \tau(t) \le \lambda\,\rho(t)\big\} = \max\mathrm{eig}\big(\tau(t),\, \rho(t)\big) \tag{8}$$

where $\max\mathrm{eig}(A, B)$ denotes the largest generalized eigenvalue of $Av = \lambda Bv$ with $B > 0$.

2 The corrected identity

Theorem 5 (Path-integral hockey-stick representation — corrected)**.** *Under the full-rank assumption $\rho_{AB}, \sigma_{AB} > 0$:*

$$\boxed{H(A|B)_\sigma - H(A|B)_\rho = -\int_0^1 \left[\int_{1/d_A}^{M_{\mathrm{fwd}}(t)} \frac{\mathrm{Tr}\big[P_\beta(t)\,\delta_{AB}^{(\beta)}\big]}{\beta}\,d\beta \;+\; \int_{d_A}^{M_{\mathrm{rev}}(t)} \frac{\mathrm{Tr}\big[Q_\beta(t)\,\big(\nVdash_A \otimes \delta_B - \beta\,\delta_{AB}\big)\big]}{\beta^2}\,d\beta\right] dt} \tag{MAIN$'$}$$

where

$$\delta_{AB}^{(\beta)} := \delta_{AB} - \beta \nVdash_A \otimes \delta_B\,.$$

3 Derivation from the correct Frenkel formula

The identity follows from three ingredients.

Lemma 6 (CE–RE identity)**.** $H(A|B)_\rho = -D(\rho_{AB} \| \nVdash_A \otimes \rho_B)$.

Lemma 7 (Correct Frenkel integral representation [**?**, **?**])**.** *For density operators* ρ, σ *(both trace* 1*, full rank), using the natural logarithm:*

$$D(\rho\|\sigma) = \int_1^\infty \left[\frac{E_\gamma(\rho\|\sigma)}{\gamma} + \frac{E_\gamma(\sigma\|\rho)}{\gamma^2}\right] d\gamma \tag{FR}$$

For unnormalised $\tau \geq 0$ *with* $\mathrm{Tr}[\tau] = c > 0$ *and* $\omega := \tau/c$*, the change of variables* $E_\gamma(\rho\|\omega) = E_{\gamma/c}(\rho\|\tau)$ *and* $E_\gamma(\omega\|\rho) = \frac{1}{c}\, E_{c\gamma}(\tau\|\rho)$ *gives:*

$$D(\rho\|\tau) = \int_{1/c}^\infty \frac{E_\beta(\rho\|\tau)}{\beta}\, d\beta \;+\; \int_c^\infty \frac{E_\beta(\tau\|\rho)}{\beta^2}\, d\beta \;-\; \log c \tag{FR-c}$$

Setting $\tau = \nVdash_A \otimes \rho_B$*,* $c = d_A$*:*

$$D(\rho_{AB}\|\nVdash_A \otimes \rho_B) = \int_{1/d_A}^\infty \frac{E_\beta(\rho_{AB}\|\tau)}{\beta}\, d\beta \;+\; \int_{d_A}^\infty \frac{E_\beta(\tau\|\rho_{AB})}{\beta^2}\, d\beta \;-\; \log d_A \tag{FR-bip}$$

Derivation of Theorem **??**. Differentiate (FR-bip) along $\rho(t), \tau(t)$. The $\log d_A$ term is t-independent. By the hockey-stick derivative formula, at generic β:

$$\frac{d}{dt}\, E_\beta\big(\rho(t)\|\tau(t)\big) = \mathrm{Tr}\big[P_\beta(t)\,(\delta_{AB} - \beta \nVdash_A \otimes \delta_B)\big] = \mathrm{Tr}\big[P_\beta(t)\, \delta_{AB}^{(\beta)}\big] \tag{9}$$

$$\frac{d}{dt}\, E_\beta\big(\tau(t)\|\rho(t)\big) = \mathrm{Tr}\big[Q_\beta(t)\,(\nVdash_A \otimes \delta_B - \beta\, \delta_{AB})\big] \tag{10}$$

The first integrand vanishes for $\beta > M_{\mathrm{fwd}}(t)$ (since $P_\beta = 0$) and the second for $\beta > M_{\mathrm{rev}}(t)$ (since $Q_\beta = 0$). Applying the fundamental theorem of calculus $H(A|B)_\sigma - H(A|B)_\rho = -\int_0^1 \frac{d}{dt} D(\rho(t)\|\tau(t))\, dt$ yields (MAIN′). □

4 Erratum: the skeleton's formula

The earlier proof skeleton (v3.0, §0.6) stated the Frenkel formula as

$$D(\rho\|\sigma) \stackrel{?}{=} \int_0^\infty \frac{E_\gamma(\rho\|\sigma) - (1-\gamma)_+}{\gamma(1+\gamma)}\, d\gamma \qquad \textbf{(INCORRECT)}$$

This single-term formula with kernel $[\gamma(1+\gamma)]^{-1}$ is **wrong**. For the simple test case $\rho = |0\rangle\langle 0|$, $\sigma = I/2$ (dimension 2), the formula gives 0.432 instead of the correct $D = \log 2 \approx 0.693$.

The correct formula (FR) has *two* hockey-stick terms—one forward ($E_\gamma(\rho\|\sigma)$) and one reverse ($E_\gamma(\sigma\|\rho)$)—with kernels $1/\gamma$ and $1/\gamma^2$ respectively, integrated over $[1, \infty)$.

Consequently, the skeleton's main identity with the single-term kernel $[\gamma(1 + d_A\gamma)]^{-1}$ is also incorrect. The corrected identity (MAIN′) above has been verified numerically to machine precision ($|\mathrm{error}| < 8 \times 10^{-15}$) for 37 test cases across dimensions up to $(d_A, d_B) = (4, 4)$.

References

[1] P. E. Frenkel, "Integral formula for quantum relative entropy implies data processing inequality," *Quantum* **7**, 1102 (2023). arXiv:2208.12194.

[2] M. Berta, L. Lami, M. Tomamichel, "Continuity of entropies via integral representations," arXiv:2408.15226 (2024).