Sum of Divisors of 9! with Units Digit 3

Alethfeld Proof System

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Abstract

We prove that the sum of all positive divisors of 9! having units digit 3 equals 66. This corrects an erroneous problem statement claiming the sum equals 105. The proof proceeds by prime factorization and systematic case analysis on units digits modulo 10.

1 Main Result

Theorem 1 (Sum of Divisors with Units Digit 3). *The sum of all positive divisors of* 9! *that have units digit* 3 *equals* 66. *Symbolically:*

$$\sum_{\substack{d|9! \\ d\equiv 3 \pmod{10}}} d = 66.$$

Remark 2. The original problem statement claimed this sum equals 105, which is incorrect. Computational verification confirms the only such divisors are $\{3, 63\}$.

2 Proof

Definition 3 (Units Digit). For $n \in \mathbb{N}$, the *units digit* of n is defined as $n \bmod 10$.

Proof of Theorem 1. We proceed via Lamport-style structured proof.

$\langle 1 \rangle 1$. **Factorization.** $9! = 362880 = 2^7 \cdot 3^4 \cdot 5 \cdot 7$. [algebraic computation] ✓

Proof. Direct computation:

$$9! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9$$
$$= 362880$$
$$= 2^7 \cdot 3^4 \cdot 5 \cdot 7.$$

The exponents are obtained by Legendre's formula: for prime p,

$$\nu_p(9!) = \sum_{i=1}^{\infty} \left\lfloor \frac{9}{p^i} \right\rfloor.$$

For $p = 2$: $\lfloor 9/2 \rfloor + \lfloor 9/4 \rfloor + \lfloor 9/8 \rfloor = 4 + 2 + 1 = 7$. For $p = 3$: $\lfloor 9/3 \rfloor + \lfloor 9/9 \rfloor = 3 + 1 = 4$. For $p = 5$: $\lfloor 9/5 \rfloor = 1$. For $p = 7$: $\lfloor 9/7 \rfloor = 1$. □

$\langle 1 \rangle 2$. **Divisor Form.** Every divisor of 9! has the form

$$d = 2^a \cdot 3^b \cdot 5^c \cdot 7^e$$

where $0 \le a \le 7$, $0 \le b \le 4$, $0 \le c \le 1$, $0 \le e \le 1$. [from $\langle 1 \rangle 1$, definition of divisor] ✓

⟨1⟩3. **Divisibility by 5.** If $c = 1$ (i.e., $5 \mid d$), then $d \bmod 10 \in \{0, 5\}$. Therefore, $c = 0$ for any divisor with units digit 3. [from ⟨1⟩2, Definition 3] ✓

Proof. If $5 \mid d$, then $d = 5k$ for some $k \in \mathbb{N}$.

- If $2 \mid k$, then $10 \mid d$, so $d \equiv 0 \pmod{10}$.
- If $2 \nmid k$, then $d = 5(2m + 1) = 10m + 5$, so $d \equiv 5 \pmod{10}$.

In either case, $d \not\equiv 3 \pmod{10}$. □

⟨1⟩4. **Reduced Form.** We seek divisors $d = 2^a \cdot 3^b \cdot 7^e$ with $d \equiv 3 \pmod{10}$. [from ⟨1⟩2, ⟨1⟩3] ✓

⟨2⟩1. **Power Cycles mod 10.** The units digits of powers cycle as follows:

$$2^a \bmod 10: \quad 1, 2, 4, 8, 6, 2, 4, 8 \quad \text{for } a = 0, 1, \ldots, 7$$
$$3^b \bmod 10: \quad 1, 3, 9, 7, 1 \quad \text{for } b = 0, 1, 2, 3, 4$$
$$7^e \bmod 10: \quad 1, 7 \quad \text{for } e = 0, 1$$

[algebraic computation] ✓

⟨2⟩2. **Case $e = 0$.** We need $(2^a \cdot 3^b) \bmod 10 = 3$. [from ⟨1⟩4] ✓

Checking all $8 \times 5 = 40$ combinations of (a, b) using ⟨2⟩1:

$2^a \backslash 3^b$	$b = 0$	$b = 1$	$b = 2$	$b = 3$	$b = 4$
$a = 0$ (1)	1	**3**	9	7	1
$a = 1$ (2)	2	6	8	4	2
$a = 2$ (4)	4	2	6	8	4
$a = 3$ (8)	8	4	2	6	8
$a = 4$ (6)	6	8	4	2	6
$a = 5$ (2)	2	6	8	4	2
$a = 6$ (4)	4	2	6	8	4
$a = 7$ (8)	8	4	2	6	8

Only $(a, b) = (0, 1)$ gives units digit 3, yielding divisor $d = 2^0 \cdot 3^1 = 3$. □

⟨2⟩3. **Case $e = 1$.** We need $(2^a \cdot 3^b \cdot 7) \bmod 10 = 3$. [from ⟨1⟩4] ✓

Since $7 \cdot 3 \equiv 21 \equiv 1 \pmod{10}$, the multiplicative inverse of 7 modulo 10 is 3. Thus:

$$2^a \cdot 3^b \cdot 7 \equiv 3 \pmod{10} \iff 2^a \cdot 3^b \equiv 3 \cdot 3 \equiv 9 \pmod{10}.$$

From the table in ⟨2⟩2, units digit 9 occurs only at $(a, b) = (0, 2)$.
This yields divisor $d = 2^0 \cdot 3^2 \cdot 7 = 9 \cdot 7 = 63$.
Verification: $63 \bmod 10 = 3$. ✓ □

⟨1⟩5. **Completeness.** The set D_3 of all divisors of 9! with units digit 3 is exactly $\{3, 63\}$. [from ⟨2⟩2, ⟨2⟩3, ⟨1⟩3] ✓

Proof. By ⟨1⟩3, we only consider $c = 0$. Cases ⟨2⟩2 and ⟨2⟩3 exhaustively check all combinations of (a, b, e) with $e \in \{0, 1\}$. The only solutions are:

- $(a, b, e) = (0, 1, 0)$ giving $d = 3$.
- $(a, b, e) = (0, 2, 1)$ giving $d = 63$.

Both divide 9!: $362880/3 = 120960$ and $362880/63 = 5760$. □

⟨1⟩6. **Sum.** $\sum_{d \in D_3} d = 3 + 63 = 66$. [from ⟨1⟩5, arithmetic] ✓

⟨1⟩7. **QED.** The sum of positive divisors of 9! with units digit 3 is 66. [from ⟨1⟩6] ✓

□

3 Refutation of Original Claim

Proposition 4. *The sum of divisors of* 9! *with units digit* 3 *is* not *equal to* 105.

Proof. By Theorem 1, the sum equals $66 \neq 105$. $\qquad\qquad\qquad\qquad\qquad\qquad$ □

4 Formal Verification

This proof has been formalized in Lean 4 with decidable verification:

```
theorem sum_divisors_units_digit_3 :
    d  divisorsWithUnitsDigit3, d = 66 := by native_decide

theorem sum_not_105 :
    d  divisorsWithUnitsDigit3, d  105 := by native_decide
```

The Lean formalization is available at:

```
lean/AlethfeldLean/NumberTheory/DivisorSum9Factorial.lean
```

Proof Metadata

Graph ID:	`graph-div9fac-a1b2c3`
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