Lemma L1: Fourier Coefficient Formula

Alethfeld Verified Proof

Alethfeld Proof Orchestrator v4

Verification Status: **VERIFIED** | Taint: **CLEAN**

Dependencies

Assumption 1 (A1: Product State QBF)**.** Let $U = I - 2\left|\psi\right\rangle\left\langle\psi\right|$ be a rank-1 quantum Boolean function where $\left|\psi\right\rangle = \bigotimes_{k=1}^{n}\left|\phi_k\right\rangle$ is a product state.

Definition 1 (D1: Bloch Vector)**.** For each qubit k, the Bloch vector $\vec{r}_k = (x_k, y_k, z_k)$ satisfies $|\vec{r}_k|^2 = 1$. Define extended Bloch components:

$$r_k^{(0)} = 1, \quad r_k^{(1)} = x_k, \quad r_k^{(2)} = y_k, \quad r_k^{(3)} = z_k.$$

Statement

Lemma 1 (Fourier Coefficient Formula)**.** *Under Assumption 1 and Definition 1, for any multi-index $\alpha \in \{0, 1, 2, 3\}^n$:*

$$\hat{U}(\alpha) = \delta_{\alpha,0} - 2^{1-n}\prod_{k=1}^{n} r_k^{(\alpha_k)}$$

where $\delta_{\alpha,0} = 1$ if $\alpha = (0, \ldots, 0)$ and 0 otherwise.

Proof

Step 1: Definition Expansion

Claim 1 (L1-step1)**.** $\hat{U}(\alpha) = 2^{-n}\operatorname{Tr}(\sigma^\alpha U) = 2^{-n}\operatorname{Tr}(\sigma^\alpha) - 2^{1-n}\operatorname{Tr}(\sigma^\alpha\left|\psi\right\rangle\left\langle\psi\right|)$

Proof. By definition of the Pauli-Fourier coefficient, $\hat{U}(\alpha) = 2^{-n}\operatorname{Tr}(\sigma^\alpha U)$. Substituting $U = I - 2\left|\psi\right\rangle\left\langle\psi\right|$ from Assumption 1:

$$\hat{U}(\alpha) = 2^{-n}\operatorname{Tr}(\sigma^\alpha(I - 2\left|\psi\right\rangle\left\langle\psi\right|)) = 2^{-n}\operatorname{Tr}(\sigma^\alpha) - 2^{1-n}\operatorname{Tr}(\sigma^\alpha\left|\psi\right\rangle\left\langle\psi\right|).$$ □

Step 2: Trace of Pauli String

Claim 2 (L1-step2)**.** $\operatorname{Tr}(\sigma^\alpha) = 2^n\delta_{\alpha,0}$

Proof. We proceed in substeps.

(2a) By definition, $\sigma^\alpha = \sigma^{\alpha_1}\otimes\sigma^{\alpha_2}\otimes\cdots\otimes\sigma^{\alpha_n}$.

(2b) For tensor products, $\operatorname{Tr}(A\otimes B) = \operatorname{Tr}(A)\cdot\operatorname{Tr}(B)$.

(2b.1) By definition, $(A \otimes B)_{(i,j),(k,l)} = A_{ik} B_{jl}$.

(2b.2) Thus $\mathrm{Tr}(A \otimes B) = \sum_{i,j} (A \otimes B)_{(i,j),(i,j)} = \sum_{i,j} A_{ii} B_{jj} = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$.

(2c) Applying (2a) and (2b) inductively: $\mathrm{Tr}(\sigma^\alpha) = \prod_{k=1}^n \mathrm{Tr}(\sigma^{\alpha_k})$.
(2d) For single-qubit Pauli matrices:

(2d.1) $\sigma^0 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so $\mathrm{Tr}(\sigma^0) = 2$.

(2d.2) $\sigma^1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so $\mathrm{Tr}(\sigma^1) = 0$.

(2d.3) $\sigma^2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, so $\mathrm{Tr}(\sigma^2) = 0$.

(2d.4) $\sigma^3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so $\mathrm{Tr}(\sigma^3) = 0$.

(2e) Case analysis on the product:

(2e.1) If $\exists k : \alpha_k \neq 0$, then $\mathrm{Tr}(\sigma^{\alpha_k}) = 0$, so the product vanishes.

(2e.2) If $\forall k : \alpha_k = 0$, then each factor equals 2, giving 2^n.

Therefore $\mathrm{Tr}(\sigma^\alpha) = 2^n \delta_{\alpha,0}$. □

Step 3: Trace Cyclic Property

Claim 3 (L1-step3)**.** $\mathrm{Tr}(\sigma^\alpha \left|\psi\right\rangle \left\langle\psi\right|) = \left\langle\psi\middle|\sigma^\alpha\middle|\psi\right\rangle$

Proof. **(3a)** The operator $\left|\psi\right\rangle \left\langle\psi\right|$ is a rank-1 projector.
(3c) For any operator A and normalized state $\left|\psi\right\rangle$:

(3c.1) Let $\{\left|e_j\right\rangle\}$ be an orthonormal basis. Then

$$\mathrm{Tr}(A \left|\psi\right\rangle \left\langle\psi\right|) = \sum_j \left\langle e_j\middle|A\middle|\psi\right\rangle \left\langle\psi\middle|e_j\right\rangle .$$

(3c.2) Rearranging scalars: $= \sum_j \left\langle\psi\middle|e_j\right\rangle \left\langle e_j\middle|A\middle|\psi\right\rangle$.

(3c.3) By completeness ($\sum_j \left|e_j\right\rangle \left\langle e_j\right| = I$):

$$= \left\langle\psi\right| \left(\sum_j \left|e_j\right\rangle \left\langle e_j\right| \right) A \left|\psi\right\rangle = \left\langle\psi\middle|A\middle|\psi\right\rangle .$$

Setting $A = \sigma^\alpha$ yields the result. □

Step 4: Product State Factorization

Claim 4 (L1-step4)**.** $\langle\psi|\sigma^\alpha|\psi\rangle = \prod_{k=1}^n \langle\phi_k|\sigma^{\alpha_k}|\phi_k\rangle = \prod_{k=1}^n r_k^{(\alpha_k)}$

Proof. **(4a)** From Assumption 1: $|\psi\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle$ and $\sigma^\alpha = \sigma^{\alpha_1} \otimes \cdots \otimes \sigma^{\alpha_n}$.

(4c) Tensor product operator action: $(A \otimes B)(|a\rangle \otimes |b\rangle) = (A|a\rangle) \otimes (B|b\rangle)$.

(4d) Tensor product inner product: $(\langle a| \otimes \langle b|)(|c\rangle \otimes |d\rangle) = \langle a|c\rangle \langle b|d\rangle$.

(4e) Combining (4c) and (4d) inductively:

$$\langle\psi|\sigma^\alpha|\psi\rangle = \prod_{k=1}^n \langle\phi_k|\sigma^{\alpha_k}|\phi_k\rangle .$$

(4f) For a single qubit $|\phi\rangle$ with Bloch vector (x, y, z), parametrized as $|\phi\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle$ where $x = \sin\theta\cos\varphi$, $y = \sin\theta\sin\varphi$, $z = \cos\theta$:

(4f.2) $\langle\phi|I|\phi\rangle = 1 = r^{(0)}$. ✓

(4f.3) $\langle\phi|\sigma_x|\phi\rangle$: Since $\sigma_x|\phi\rangle = \cos(\theta/2)|1\rangle + e^{i\varphi}\sin(\theta/2)|0\rangle$,

$$\begin{aligned}\langle\phi|\sigma_x|\phi\rangle &= e^{-i\varphi}\sin(\theta/2)\cos(\theta/2) + e^{i\varphi}\cos(\theta/2)\sin(\theta/2) \\ &= 2\cos(\theta/2)\sin(\theta/2)\cos\varphi = \sin\theta\cos\varphi = x = r^{(1)}.\checkmark\end{aligned}$$

(4f.4) $\langle\phi|\sigma_y|\phi\rangle = \sin\theta\sin\varphi = y = r^{(2)}$. (Analogous calculation.)

(4f.5) $\langle\phi|\sigma_z|\phi\rangle$: Since $\sigma_z|0\rangle = |0\rangle$ and $\sigma_z|1\rangle = -|1\rangle$,

$$\langle\phi|\sigma_z|\phi\rangle = \cos^2(\theta/2) - \sin^2(\theta/2) = \cos\theta = z = r^{(3)}.\checkmark$$

Therefore $\langle\phi_k|\sigma^{\alpha_k}|\phi_k\rangle = r_k^{(\alpha_k)}$ for each k. □

Conclusion

Proof of Lemma 1. Combining Steps 1–4:

$$\begin{aligned}\hat{U}(\alpha) &= 2^{-n}\,\mathrm{Tr}(\sigma^\alpha) - 2^{1-n}\,\mathrm{Tr}(\sigma^\alpha|\psi\rangle\langle\psi|) && \text{(Step 1)} \\ &= 2^{-n} \cdot 2^n \delta_{\alpha,0} - 2^{1-n}\langle\psi|\sigma^\alpha|\psi\rangle && \text{(Steps 2, 3)} \\ &= \delta_{\alpha,0} - 2^{1-n}\prod_{k=1}^n r_k^{(\alpha_k)} && \text{(Step 4)}\end{aligned}$$

as claimed. □ □

Alethfeld Verification Report

Graph ID:	L1-fourier-verify-2024
Total Nodes:	46 (6 original + 40 expanded)
Max Depth:	5
Admitted Steps:	0
Taint Status:	CLEAN