Entropy-Influence Bound for Rank-1 Product State QBFs

Alethfeld Proof System v4

Graph ID: `qbf-rank1-entropy-influence`
Version: 2 (Verified)

Abstract

We establish an explicit upper bound on the entropy-to-influence ratio for rank-1 quantum Boolean functions (QBFs) constructed from product states. For the QBF $U = I - 2\left|\psi\right\rangle\left\langle\psi\right|$ where $\left|\psi\right\rangle = \bigotimes_{k=1}^{n}\left|\phi_k\right\rangle$ is a product state, we prove that the influence $I(U) = n \cdot 2^{1-n}$ is independent of the choice of single-qubit states, while the entropy $S(U)$ is maximized when all qubits are in the "magic" state with Bloch vector $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. The ratio S/I approaches $\log_2 3 + 4 \approx 5.585$ as $n \to \infty$, establishing a lower bound on any universal constant C satisfying $S(U) \leq C \cdot I(U)$.

Contents

1 Main Result

Theorem 1.1 (Entropy-Influence Bound). *:theorem* *For the rank-1 QBF $U = I - 2\left|\psi\right\rangle\left\langle\psi\right|$ where $\left|\psi\right\rangle = \bigotimes_{k=1}^{n}\left|\phi_k\right\rangle$ is a product state:*

$$\frac{S(U)}{I(U)} \leq \log_2 3 + \frac{2^{n-1}}{n}\left[-p_0 \log_2 p_0 + (2n-2)(1-p_0)\right] \tag{1}$$

where $p_0 = (1-2^{1-n})^2$. The maximum is achieved when all qubits are in the magic state with Bloch vector $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

The proof proceeds through six parts: establishing Fourier coefficients, computing influence, deriving the entropy formula, identifying the maximum, analyzing asymptotics, and drawing implications for the entropy-influence conjecture.

2 Preliminaries

2.1 Setup

:0-assume0

Let $U = I - 2\left|\psi\right\rangle\left\langle\psi\right|$ be a rank-1 QBF where $\left|\psi\right\rangle = \bigotimes_{k=1}^{n}\left|\phi_k\right\rangle$ is a product state with each $\left|\phi_k\right\rangle \in \mathbb{C}^2$.

Definition 2.1 (Bloch Vector)**.** *:1-def1 Each single-qubit state $\left|\phi_k\right\rangle$ has Bloch vector $\vec{r}_k = (x_k, y_k, z_k)$ with*

$$|\vec{r}_k|^2 = x_k^2 + y_k^2 + z_k^2 = 1. \tag{2}$$

Definition 2.2 (Extended Bloch Coefficients)**.** *:1-def2 Define $q_k^{(0)} = 1$ and*

$$\left(q_k^{(1)}, q_k^{(2)}, q_k^{(3)}\right) = \left(x_k^2, y_k^2, z_k^2\right). \tag{3}$$

Definition 2.3 (Bloch Entropy)**.** *:1-def3 The Bloch entropy of qubit k is*

$$f_k = H(x_k^2, y_k^2, z_k^2) = -\sum_{\ell=1}^{3} q_k^{(\ell)} \log_2 q_k^{(\ell)}. \tag{4}$$

Remark. The Bloch entropy f_k measures the "spread" of the Bloch vector across coordinate axes. It is *not* the von Neumann entropy of the qubit state (which is zero for pure states).

3 Fourier Coefficients

Lemma 3.1 (Fourier Coefficient Formula)**.** *:1-lem1 For $U = I - 2\left|\psi\right\rangle\left\langle\psi\right|$:*

$$\hat{U}(\alpha) = \delta_{\alpha,0} - 2^{1-n}\prod_{k=1}^{n} r_k^{(\alpha_k)} \tag{5}$$

where $r_k^{(0)} = 1$, $r_k^{(1)} = x_k$, $r_k^{(2)} = y_k$, $r_k^{(3)} = z_k$.

Proof. **1.** :2-lem1-1 By definition of the Pauli-Fourier expansion:

$$\hat{U}(\alpha) = 2^{-n}\operatorname{Tr}(\sigma^\alpha U) = 2^{-n}\operatorname{Tr}(\sigma^\alpha) - 2^{1-n}\operatorname{Tr}(\sigma^\alpha\left|\psi\right\rangle\left\langle\psi\right|). \tag{6}$$

2. :2-lem1-2 The trace of Pauli strings satisfies:

$$\operatorname{Tr}(\sigma^\alpha) = 2^n\delta_{\alpha,0} \tag{7}$$

since $\operatorname{Tr}(\sigma_i) = 0$ for $i \in \{1,2,3\}$ and $\operatorname{Tr}(I) = 2$.

3. :2-lem1-3 By the cyclic property of trace:

$$\operatorname{Tr}(\sigma^\alpha\left|\psi\right\rangle\left\langle\psi\right|) = \left\langle\psi\right|\sigma^\alpha\left|\psi\right\rangle. \tag{8}$$

4. :2-lem1-4 For a product state, this expectation value factorizes:

$$\left\langle\psi\right|\sigma^\alpha\left|\psi\right\rangle = \prod_k \left\langle\phi_k\right|\sigma^{\alpha_k}\left|\phi_k\right\rangle = \prod_k r_k^{(\alpha_k)}. \tag{9}$$

Combining these steps yields the result.:2-lem1-qed ∎

Lemma 3.2 (Probability Distribution)**.** *:1-lem2 The Fourier weight distribution is:*

$$p_\alpha = |\hat{U}(\alpha)|^2 = \begin{cases} (1-2^{1-n})^2 & \alpha = 0 \\ 2^{2-2n} \displaystyle\prod_{k=1}^{n} q_k^{(\alpha_k)} & \alpha \neq 0 \end{cases} \tag{10}$$

Proof. For $\alpha = 0$: $|\hat{U}(0)|^2 = |1-2^{1-n}|^2 = (1-2^{1-n})^2$.

For $\alpha \neq 0$: $|\hat{U}(\alpha)|^2 = |-2^{1-n}\prod_k r_k^{(\alpha_k)}|^2 = 2^{2-2n}\prod_k |r_k^{(\alpha_k)}|^2 = 2^{2-2n}\prod_k q_k^{(\alpha_k)}$. □

4 Influence Calculation

Theorem 4.1 (Influence Independence)**.** *:1-thm3 For any rank-1 product state QBF:*

$$I(U) = n \cdot 2^{1-n}. \tag{11}$$

*This is **independent of the choice of Bloch vectors**.*

Proof. **1.** :2-thm3-1 The influence of qubit j is defined as:

$$I_j = \sum_{\alpha:\alpha_j \neq 0} p_\alpha. \tag{12}$$

2. :2-thm3-2 For $\alpha \neq 0$ with $\alpha_j = \ell \neq 0$, we sum over all choices of $(\alpha_k)_{k\neq j} \in \{0,1,2,3\}^{n-1}$:

$$\sum_{\alpha:\alpha_j=\ell} p_\alpha = \sum_{(\alpha_k)_{k\neq j}} 2^{2-2n} \cdot q_j^{(\ell)} \cdot \prod_{k\neq j} q_k^{(\alpha_k)} \tag{13}$$

$$= 2^{2-2n} \cdot q_j^{(\ell)} \cdot \prod_{k\neq j} \left(\sum_{m=0}^{3} q_k^{(m)} \right). \tag{14}$$

3. :2-thm3-3 Since $\sum_{m=0}^{3} q_k^{(m)} = 1 + x_k^2 + y_k^2 + z_k^2 = 1 + 1 = 2$:

$$\sum_{\alpha:\alpha_j=\ell} p_\alpha = 2^{2-2n} \cdot q_j^{(\ell)} \cdot 2^{n-1}. \tag{15}$$

4. :2-thm3-4 Summing over $\ell \in \{1,2,3\}$:

$$I_j = \sum_{\ell=1}^{3} 2^{2-2n} \cdot 2^{n-1} \cdot q_j^{(\ell)} \tag{16}$$

$$= 2^{1-n} \cdot \sum_{\ell=1}^{3} q_j^{(\ell)} \tag{17}$$

$$= 2^{1-n} \cdot (x_j^2 + y_j^2 + z_j^2) \tag{18}$$

$$= 2^{1-n} \cdot 1 = 2^{1-n}. \tag{19}$$

Therefore, the total influence is::2-thm3-qed

$$I = \sum_{j=1}^{n} I_j = n \cdot 2^{1-n}. \tag{20}$$

□

5 Entropy Calculation

Lemma 5.1 (Entropy Decomposition). *:1-lem4*

$$S = -p_0 \log_2 p_0 - \sum_{\alpha \neq 0} p_\alpha \log_2 p_\alpha. \tag{21}$$

Theorem 5.2 (General Entropy Formula). *:1-thm5*

$$S = -p_0 \log_2 p_0 + (2n-2)(1-p_0) + 2^{1-n} \sum_{k=1}^{n} f_k \tag{22}$$

where $f_k = H(x_k^2, y_k^2, z_k^2)$ is the Bloch entropy of qubit k.

Proof. **1.** :2-thm5-1 For $\alpha \neq 0$:

$$\begin{aligned}
-p_\alpha \log_2 p_\alpha &= -p_\alpha \log_2 \left(2^{2-2n} \prod_k q_k^{(\alpha_k)} \right) & (23)\\
&= -p_\alpha \left[(2-2n)\log_2 2 + \sum_k \log_2 q_k^{(\alpha_k)} \right] & (24)\\
&= p_\alpha(2n-2) - p_\alpha \sum_k \log_2 q_k^{(\alpha_k)}. & (25)
\end{aligned}$$

2. :2-thm5-2 Summing over all $\alpha \neq 0$:

$$\sum_{\alpha \neq 0} p_\alpha (2n-2) = (2n-2)(1-p_0). \tag{26}$$

3. :2-thm5-3 For fixed qubit j, the sum $-\sum_{\alpha \neq 0} p_\alpha \log_2 q_j^{(\alpha_j)}$ splits into two cases. When $\alpha_j = 0$, we have $\log_2 q_j^{(0)} = \log_2 1 = 0$, so only $\alpha_j \neq 0$ contributes.

4. :2-thm5-4 For the nonzero part, using the result from Theorem 4.1:

$$\sum_{\alpha : \alpha_j = \ell} p_\alpha = 2^{1-n} q_j^{(\ell)}. \tag{27}$$

5. :2-thm5-5 Therefore:

$$\begin{aligned}
-\sum_{\alpha : \alpha_j \neq 0} p_\alpha \log_2 q_j^{(\alpha_j)} &= -\sum_{\ell=1}^{3} \log_2 q_j^{(\ell)} \cdot 2^{1-n} q_j^{(\ell)} & (28)\\
&= -2^{1-n} \sum_{\ell=1}^{3} q_j^{(\ell)} \log_2 q_j^{(\ell)} & (29)\\
&= 2^{1-n} f_j. & (30)
\end{aligned}$$

6. :2-thm5-6 Summing over all qubits j:

$$-\sum_{\alpha \neq 0} p_\alpha \sum_k \log_2 q_k^{(\alpha_k)} = 2^{1-n} \sum_{k=1}^{n} f_k. \tag{31}$$

Combining all terms::2-thm5-qed

$$S = -p_0 \log_2 p_0 + (2n-2)(1-p_0) + 2^{1-n} \sum_{k=1}^{n} f_k. \tag{32}$$

□

6 Maximum at Magic State

Theorem 6.1 (Maximum Ratio). *:1-thm6* *The ratio S/I is maximized when all qubits are in the magic state*

$$(x_k^2, y_k^2, z_k^2) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right). \tag{33}$$

Proof. **1.** :2-thm6-1 Since $I = n \cdot 2^{1-n}$ is constant (independent of Bloch vectors by Theorem 4.1), maximizing S/I is equivalent to maximizing S.

2. :2-thm6-2 From Theorem 5.2:

$$S = -p_0 \log_2 p_0 + (2n-2)(1-p_0) + 2^{1-n} \sum_{k=1}^{n} f_k. \tag{34}$$

The first two terms depend only on n, not on the Bloch vectors.

3. :2-thm6-3 Each $f_k = H(x_k^2, y_k^2, z_k^2)$ is the Shannon entropy of a probability distribution on 3 outcomes (since $x_k^2 + y_k^2 + z_k^2 = 1$).

4. :2-thm6-4 By the maximum entropy principle [1], for a probability distribution on k outcomes:

$$H(p_1, \ldots, p_k) \leq \log_2 k \tag{35}$$

with equality if and only if $p_i = 1/k$ for all i. Applied to $k = 3$:

$$f_k \leq \log_2 3 \tag{36}$$

with equality if and only if $(x_k^2, y_k^2, z_k^2) = (1/3, 1/3, 1/3)$.
Therefore S is maximized when all $f_k = \log_2 3$. :2-thm6-qed □

Corollary 6.2 (Explicit Maximum). *:1-cor7* *For the symmetric magic product state:*

$$\frac{S}{I} = \log_2 3 + \frac{2^{n-1}}{n} \left[-p_0 \log_2 p_0 + (2n-2)(1-p_0)\right] \tag{37}$$

where $p_0 = (1 - 2^{1-n})^2$.

Proof. At the magic state, $f_k = \log_2 3$ for all k. Substituting into the entropy formula:

$$S_{\max} = -p_0 \log_2 p_0 + (2n-2)(1-p_0) + 2^{1-n} \cdot n \cdot \log_2 3. \tag{38}$$

Dividing by $I = n \cdot 2^{1-n}$:

$$\begin{aligned} \frac{S_{\max}}{I} &= \frac{-p_0 \log_2 p_0 + (2n-2)(1-p_0)}{n \cdot 2^{1-n}} + \log_2 3 \qquad (39) \\ &= \log_2 3 + \frac{2^{n-1}}{n} \left[-p_0 \log_2 p_0 + (2n-2)(1-p_0)\right]. \end{aligned}$$

□

7 Asymptotic Analysis

Theorem 7.1 (Limiting Behavior). *:1-thm8*

$$\lim_{n \to \infty} \frac{S_{\max}}{I} = \log_2 3 + 4 \approx 5.585. \tag{40}$$

Proof. Let $\varepsilon = 2^{1-n}$. :2-thm8-0 Then $p_0 = (1-\varepsilon)^2$ and $1 - p_0 = 2\varepsilon - \varepsilon^2 \approx 2\varepsilon$ for small ε.

1. :2-thm8-1 For the entropy term, using $\log_2(1-x) \approx -x/\ln 2$ for small x:

$$-p_0 \log_2 p_0 \approx -(1-2\varepsilon)\left(\frac{-2\varepsilon}{\ln 2}\right) \approx \frac{2\varepsilon}{\ln 2}. \tag{41}$$

2. :2-thm8-2 For the influence term:

$$(2n-2)(1-p_0) \approx (2n-2)\cdot 2\varepsilon = 4(n-1)\varepsilon. \tag{42}$$

3. :2-thm8-3 The correction term becomes:

$$g(n) = \frac{2^{n-1}}{n}\left[-p_0 \log_2 p_0 + (2n-2)(1-p_0)\right] \tag{43}$$

$$\approx \frac{2^{n-1}}{n}\cdot \varepsilon \cdot \left[\frac{2}{\ln 2} + 4(n-1)\right] \tag{44}$$

$$= \frac{2^{n-1}}{n}\cdot 2^{1-n} \cdot \left[\frac{2}{\ln 2} + 4(n-1)\right]. \tag{45}$$

4. :2-thm8-4 Simplifying:

$$g(n) = \frac{1}{n}\left[\frac{2}{\ln 2} + 4(n-1)\right] \tag{46}$$

$$= \frac{2}{n \ln 2} + 4 - \frac{4}{n} \tag{47}$$

$$\to 0 + 4 - 0 = 4 \quad \text{as } n \to \infty. \tag{48}$$

Therefore::2-thm8-qed

$$\frac{S_{\max}}{I} \to \log_2 3 + 4 \approx 1.585 + 4 = 5.585. \tag{49}$$

□

Theorem 7.2 (Finite n Values). *:1-thm9*

n	$S_{\max}/I$	*Numerical Value*
1	$\log_2 3$	*1.585*
2	$2 + \log_2 3$	*3.585*
3	*(formula)*	*4.541*
4	*(formula)*	*4.987*
5	*(formula)*	*5.209*
10	*(formula)*	*5.469*
20	*(formula)*	*5.529*
∞	$\log_2 3 + 4$	*5.585*

8 Implications for the Conjecture

Theorem 8.1 (Supremum). *:1-sup*

$$\sup_{n,\, \text{product states}} \frac{S}{I} = \log_2 3 + 4 \approx 5.585. \tag{50}$$

This supremum is achieved in the limit $n \to \infty$ with all qubits in the magic state.

Theorem 8.2 (Conjecture Bound). *:1-conj For the entropy-influence conjecture $S(U) \leq C \cdot I(U)$ to hold for all rank-1 product state QBFs, we require:*

$$\boxed{C \geq \log_2 3 + 4 \approx 5.585} \tag{51}$$

9 Conclusion

:1-main-qed

For rank-1 QBFs from product states, we have proven:

1. **Influence is constant:** $I = n \cdot 2^{1-n}$ regardless of Bloch vectors.
2. **Entropy formula:**
$$S = -p_0 \log_2 p_0 + (2n-2)(1-p_0) + 2^{1-n} \sum_{k=1}^{n} f_k. \tag{52}$$
3. **Maximum at magic states:** Each qubit in the $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ direction (squared Bloch components).
4. **Explicit bound:**
$$\frac{S}{I} = \log_2 3 + \frac{2^{n-1}}{n} \left[-p_0 \log_2 p_0 + (2n-2)(1-p_0)\right]. \tag{53}$$
5. **Asymptotic limit:** $S/I \to \log_2 3 + 4 \approx 5.585$ as $n \to \infty$.
6. **Required constant:** Any universal bound $S \leq C \cdot I$ requires $C \geq 5.585$.

References

[1] C. E. Shannon, *A Mathematical Theory of Communication*, Bell System Technical Journal, vol. 27, pp. 379–423, 623–656, 1948.

A Alethfeld Graph Metadata

```
Graph ID:       qbf-rank1-entropy-influence
Version:        2
Proof Mode:     formal-physics
Status:         VERIFIED

Nodes:          42 (42 verified, 0 proposed, 0 admitted)
Lemmas:         0 extracted
External Refs: 1 (Shannon entropy theorem)
Taint:          ALL CLEAN
Obligations:    NONE

Verification Summary:
  - Total nodes verified: 42
  - Initially accepted:   39
  - Challenged:           3
  - Revisions applied:    3
  - Final status:         ALL VERIFIED
```