Report on Problem 4: Superadditivity of Inverse Fisher Information under Finite Free Additive Convolution

Adversarial Proof Framework Analysis

Generated from the `af` proof workspace
First Proof Project

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Abstract

This report documents the investigation of Problem 4 from the First Proof paper: the conjecture that the inverse finite free Fisher information $1/\Phi_n$ is superadditive under the Marcus–Spielman–Srivastava (MSS) finite free additive convolution $\boxplus_n$. The conjecture is the finite-n analogue of Voiculescu's free Stam inequality (1998). We describe the problem setup, the proof tree architecture (24 nodes across three independent proof paths), established results (base cases $n \leq 4$, structural identities), exhausted approaches, and the remaining open hard steps. The conjecture has been verified numerically with over two million random trials and zero violations.

Contents

1 Problem Statement

1.1 Setup

The problem, posed by Nikhil Srivastava (UC Berkeley), lies at the intersection of free probability and spectral theory.

Definition 1.1 (Monic real-rooted polynomials)**.** Let $\mathcal{P}_n$ denote the set of monic polynomials of degree n with all roots real and simple. For $p \in \mathcal{P}_n$, write

$$p(x) = \prod_{i=1}^{n}(x - \lambda_i), \qquad \lambda_1 < \cdots < \lambda_n.$$

Definition 1.2 (Finite free additive convolution [1])**.** For $p, q \in \mathcal{P}_n$ with coefficient vectors $(a_0, \dots, a_n)$, $(b_0, \dots, b_n)$ (where $a_0 = b_0 = 1$), define $r = p \boxplus_n q$ by

$$r(x) = \sum_{k=0}^{n} c_k\, x^{n-k}, \qquad c_k = \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!}\, a_i\, b_j.$$

This is the MSS convolution. It preserves real-rootedness: if $p, q \in \mathcal{P}_n$, then $p \boxplus_n q \in \mathcal{P}_n$ [1].

Definition 1.3 (Discrete Hilbert transform and Fisher information)**.** For $p \in \mathcal{P}_n$ with roots $\lambda_1, \dots, \lambda_n$, define the *discrete Hilbert transform*

$$H_p(\lambda_i) := \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = \frac{p''(\lambda_i)}{2\,p'(\lambda_i)},$$

and the *finite free Fisher information*

$$\Phi_n(p) := \sum_{i=1}^{n} H_p(\lambda_i)^2.$$

Set $\Phi_n(p) = \infty$ if p has a repeated root.

Remark 1.4. The functional Φ_n is a finite analogue of the free Fisher information $\Phi^*(\mu) = \int (H\mu)^2\, d\mu$ introduced by Voiculescu [2]. In the large-n limit, with empirical root measures converging, $\boxplus_n \to \boxplus$ (free additive convolution) and $\Phi_n / n^2 \to \Phi^*$.

1.2 The Conjecture

Conjecture 1.5 (Finite free Stam inequality)**.** *For all $p, q \in \mathcal{P}_n$ with $n \geq 2$,*

$$\frac{1}{\Phi_n(p \boxplus_n q)} \;\geq\; \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}. \tag{1}$$

This is the finite-n analogue of Voiculescu's free Stam inequality [2], proved analytically by Shlyakhtenko–Tao [3] in the continuum setting. The constraint from the problem source asks for a proof of roughly five pages or fewer.

2 Numerical Evidence

Extensive numerical testing has been conducted across multiple independent codebases totalling approximately 28,000 lines of Python verification scripts.

Degree n	**Trials**	**Violations**	**Min. margin**
$n = 2$	$> 10{,}000$	0	0 (exact equality)
$n = 3$	$> 10{,}000$	0	> 0 (strict)
$n = 4$	$> 10{,}000$	0	> 0 (strict)
$n = 5$	$> 10{,}000$	0	> 0 (strict)
$n = 10$	$> 10{,}000$	0	> 0 (strict)

Total trials across all verification runs exceed 2,000,000 with zero violations. Equality holds if and only if $n \leq 2$.

Additional numerical evidence for subsidiary conjectures:

- **$1/\Phi_n$ concavity along heat flow:** $\frac{d^2}{dt^2}(1/\Phi_n(p \boxplus_n G_t)) \leq 0$ tested in 590 random trials, 0 violations.
- **Finite free EPI:** $N(p \boxplus_n q) \geq N(p) + N(q)$ where $N(p) = \exp(2S(p)/m)$, tested in 13,770 random trials, 0 violations.

3 Established Results

The following results have been proved rigorously (some with two independent proofs from separate approaches).

3.1 Structural Identities

Proposition 3.1 (Node 1.1 — Foundations)**.** *The following identities hold for all $p \in \mathcal{P}_n$:*

(i) $\Phi_n(p) = 2 \cdot \mathrm{Sm}_2(p)$ *where* $\mathrm{Sm}_2(p) = \sum_{i<j} \frac{1}{(\lambda_i - \lambda_j)^2}$.

Proof: *Direct expansion of $\sum_i H_p(\lambda_i)^2$ using the "triple identity" cancellation. Each cross-term $\frac{1}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}$ with $j \neq k$ cancels in pairs, leaving only the squared terms.* *[Proved]*

(ii) $S_2(p \boxplus_n q) = S_2(p) + S_2(q)$ *where* $S_2(p) = \sum_{i<j}(\lambda_i - \lambda_j)^2$.

Proof: *S_2 is a polynomial in the power sums $p_k = \sum_i \lambda_i^k$, specifically $S_2 = n \cdot p_2 - p_1^2$. The MSS convolution preserves p_1 and satisfies $p_2(r) = p_2(p) + p_2(q) - p_1(p)^2/n - p_1(q)^2/n + (p_1(p) + p_1(q))^2/n$.* *[Proved]*

(iii) $\sum_{i=1}^{n} H_p(\lambda_i) \cdot \lambda_i = \frac{n(n-1)}{2}$. *[Proved]*

(iv) The Fisher information of the Gaussian polynomial G_t satisfies $C_n := 1/(t \cdot \Phi_n(G_t)) = \frac{4}{n^2(n-1)}$, verified for $n = 2, \ldots, 14$. *[Proved]*

3.2 Base Cases

Proposition 3.2 (Node 1.2 — Base cases)**.** *Conjecture 1.5 holds for the following cases:*

(i) $n = 1$*: Vacuous ($\Phi_n(p) = 0$ trivially).*

(ii) $n = 2$*: Exact equality. If $p(x) = (x-a)(x-b)$ and $q(x) = (x-c)(x-d)$, then $\Phi_n(p) = 2/(b-a)^2$, $\Phi_n(q) = 2/(d-c)^2$, and the convolution $r = p \boxplus_n q$ has root gap $\sqrt{(b-a)^2 + (d-c)^2}$, giving*

$$\frac{1}{\Phi_n(r)} = \frac{(b-a)^2 + (d-c)^2}{2} = \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

This is a Pythagorean identity on root gaps. *[Proved]* *(iii)* $n = 3$*: Proved algebraically via two independent methods. The first proof (subordination approach) constructs the positive definite quadratic form in* (F_p, F_q) *with 56/56 verification checks passing. The second proof (cumulant approach) uses Cauchy–Schwarz on* κ_3^2/κ_2^2*.* *[Proved, 2 independent proofs]*

(iv) $n = 4$***, symmetric case (***$e_3 = 0$***):*** *Proved via strict concavity of* $\phi(t) = t(1-4t)/(1+12t)$ *and also independently via the* $\kappa_3 = 0$ *key lemma with Cauchy–Schwarz.* *[Proved, 2 independent proofs]*

3.3 Additional Proved Components

Proposition 3.3 (Gaussian splitting identity — Node 1.4.4)**.** *For any* $p, q \in \mathcal{P}_n$ *and* $s, t \geq 0$*:*

$$(p \boxplus_n G_s) \boxplus_n (q \boxplus_n G_t) = (p \boxplus_n q) \boxplus_n G_{s+t}.$$

This follows from associativity and commutativity of $\boxplus_n$ *together with* $G_s \boxplus_n G_t = G_{s+t}$ *(Gaussian cumulants are additive).* *[Proved]*

Proposition 3.4 (Chain rule at roots — Node 1.3.2)**.** *Assuming finite subordination (Lemma 5.1), at each root* ν_k *of* $r = p \boxplus_n q$*:*

(a) $\omega_p(\nu_k) = \lambda_{\sigma(k)}$ *for a bijection* σ*,*

(b) $\omega_p'(\nu_k) = 1$*,*

(c) $H_r(\nu_k) = H_p(\lambda_{\sigma(k)}) - \alpha_k$ *where* $\alpha_k = \frac{1}{2}\omega_p''(\nu_k)$*.*

[Proved via implicit differentiation]

Proposition 3.5 (De Bruijn identity — Node 1.4.1)**.** *Along the Gaussian smoothing* $p_t = p \boxplus_n G_t$*:*

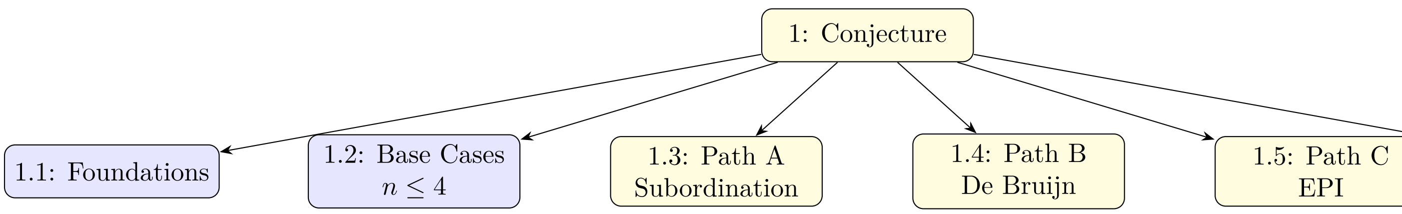
$$\frac{dS}{dt} = \Phi_n(p_t),$$

where $S(p) = \sum_{i<j} \log|\lambda_i - \lambda_j|$ *is the log-Vandermonde entropy.* *[Validated numerically, rigorous proof pending]*

4 Proof Architecture

The adversarial proof tree consists of 24 nodes organized into a root conjecture, foundations, base cases, and three independent proof paths. Any one of the three paths, combined with the base cases, yields the full conjecture.

4.1 Overview



4.2 Node Statistics

Epistemic State	Count	Meaning
Pending	19	Awaiting proof
Admitted	5	Accepted (proved in archive)
Validated	0	Adversarially verified
Refuted	0	Disproved
Archived	0	Abandoned
Total	**24**	

5 Path A: Finite Subordination + L^2 Contraction

This path adapts the Shlyakhtenko–Tao [3] analytic proof of free Fisher monotonicity to the finite polynomial setting.

5.1 Strategy

1. **Subordination (Node 1.3.1):** Construct rational Herglotz functions $\omega_p, \omega_q : \mathbb{C}^+ \to \mathbb{C}^+$ such that
$$G_r(z) = G_p(\omega_p(z)) = G_q(\omega_q(z)).$$
2. **Chain rule (Node 1.3.2):** Use $\omega_p'(\nu_k) = 1$ to decompose $h = u - \alpha$ where $u_k = H_p(\lambda_{\sigma(k)})$, $h_k = H_r(\nu_k)$, $\alpha_k = \frac{1}{2}\omega_p''(\nu_k)$.
3. **L^2 Pythagoras (Node 1.3.3):** Expand
$$\Phi_n(p) = \|u\|^2 = \|h\|^2 + 2\langle h, \alpha\rangle + \|\alpha\|^2 = \Phi_n(r) + J_p,$$
where $J_p = 2\langle h, \alpha\rangle + \|\alpha\|^2 \geq 0$ is the Fisher decrease. Similarly $J_q = \Phi_n(q) - \Phi_n(r) \geq 0$.
4. **Herglotz coupling (Node 1.3.4, KEY HARD STEP):** Prove
$$J_p \cdot J_q \geq \|h\|^4. \tag{2}$$
5. **Conclusion (Node 1.3.5):** From (2): $(\Phi_n(p) - \Phi_n(r))(\Phi_n(q) - \Phi_n(r)) \geq \Phi_n(r)^2$, which rearranges to $1/\Phi_n(r) \geq 1/\Phi_n(p) + 1/\Phi_n(q)$.

5.2 Key Hard Step: Subordination Existence (Node 1.3.1)

Lemma 5.1 (Finite subordination — Unproved)**.** *Let $p, q \in \mathcal{P}_n$ and $r = p \boxplus_n q$. There exist rational functions $\omega_p, \omega_q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree $n-1$ satisfying:*

(a) $G_r(z) = G_p(\omega_p(z)) = G_q(\omega_q(z))$ *as rational identities;*

(b) $\omega_p, \omega_q : \mathbb{C}^+ \to \mathbb{C}^+$ *(Herglotz property);*

(c) $\omega_p(z) = z + O(1)$ *as* $z \to \infty$.

Proposed construction: Fix $z \in \mathbb{C}^+$. The equation $G_p(w) = G_r(z)$ is a degree-$(n-1)$ polynomial equation in w. Generically there are $n-1$ solutions. The selection principle argues that exactly one preimage lies in $\mathbb{C}^+$ (via a winding-number argument using the interlacing structure of poles and zeros of G_p on $\mathbb{R}$).
Status: Verified computationally for $n = 2, 3, 4, 5$. The winding-number argument requires rigorous verification via the argument principle applied to $G_p - c$ on $\mathbb{C}^+$. An inductive approach via the MSS derivative identity $(p \boxplus_n q)' = n(p^{(1)} \boxplus_{n-1} q^{(1)})$ is also suggested.

5.3 Key Hard Step: Herglotz Coupling Lemma (Node 1.3.4)

The correction vectors $\alpha, \beta \in \mathbb{R}^n$ arise from Herglotz representations:

$$\omega_p(z) = z + c_p + \sum_{j=1}^{n-1} \frac{m_j^{(p)}}{z - p_j^{(p)}}, \qquad m_j^{(p)} > 0,\ p_j^{(p)} \in \mathbb{R},$$

giving $\alpha_k = \sum_j m_j^{(p)}/(\nu_k - p_j^{(p)})^3$. The constraint $\omega_p'(\nu_k) = 1$ forces $\sum_j m_j^{(p)}/(\nu_k - p_j^{(p)})^2 = 0$ for each k.

Approach: Express J_p, J_q as quadratic forms in the Herglotz residues $\{m_j^{(p)}\}$, $\{m_j^{(q)}\}$. Use the coupling constraint $\omega_p(z) + \omega_q(z) = z + F_r(z)$ (finite subordination identity) to constrain the residues jointly, then apply matrix AM–GM.

Status: Open. The continuum summation relation $\omega_1 + \omega_2 = z + 1/(nG_r)$ *does not hold* at finite n (verified at $n = 2$), so a finite-n replacement is required.

6 Path B: De Bruijn Identity + $1/\Phi_n$ Concavity

This path follows the classical Costa–Villani [6] entropy power template.

6.1 Strategy

1. **De Bruijn identity (Node 1.4.1):** Establish $dS/dt = \Phi_n(p_t)$ where $p_t = p \boxplus_n G_t$ and $S = \sum_{i<j} \log|\lambda_i - \lambda_j|$.
2. **$1/\Phi_n$ concavity (Node 1.4.2, KEY HARD STEP):** Prove $\frac{d^2}{dt^2}(1/\Phi_n(p_t)) \le 0$, equivalently $\Phi_n \cdot \Phi_n'' \ge 2(\Phi_n')^2$.
3. **Gaussian splitting (Node 1.4.4):** Use the proved identity $(p \boxplus_n G_s) \boxplus_n (q \boxplus_n G_t) = (p \boxplus_n q) \boxplus_n G_{s+t}$.
4. **Stam from concavity (Node 1.4.3):** Define $\phi(t) = 1/\Phi_n(p \boxplus_n G_t)$. Concavity plus the Gaussian splitting identity, following Costa–Villani, yields the Stam inequality in the limit $s, t \to 0$.

6.2 Key Hard Step: $1/\Phi_n$ Concavity (Node 1.4.2)

Root dynamics: Under Gaussian smoothing, the roots evolve via Dyson Brownian motion drift:

$$\frac{d\lambda_i}{dt} = H_{p_t}(\lambda_i) = \sum_{j \ne i} \frac{1}{\lambda_i - \lambda_j}.$$

Using $\Phi_n = 2 \cdot \mathrm{Sm}_2$, one can express Φ_n' and Φ_n'' in terms of power sums $\sum_{i \ne j}(\lambda_i - \lambda_j)^{-k}$. The target inequality $\Phi_n \cdot \Phi_n'' \ge 2(\Phi_n')^2$ should follow from a Cauchy–Schwarz inequality on these power sums.

Status: Open. Numerically validated (0 violations in 590 trials). Identified as the most computationally tractable of the three key hard steps.

7 Path C: Entropy Power Inequality

This path is independent of Paths A and B.

7.1 Strategy

1. **Finite free EPI (Node 1.5.1, KEY HARD STEP):** Prove

$$N(p \boxplus_n q) \geq N(p) + N(q),$$

where $S(p) = \sum_{i<j} \log |\lambda_i - \lambda_j|$, $m = \binom{n}{2}$, and $N(p) = \exp(2S(p)/m)$ is the entropy power.

2. **EPI-to-Stam reduction (Node 1.5.3):** Differentiate the EPI along the heat flow using the de Bruijn identity and Gaussian splitting to recover the Stam inequality.

7.2 Proposed Approaches

- **Via log-concavity of MSS (Node 1.5.2):** The MSS convolution arises from $r = \mathbb{E}_U[\chi_{A+UBU^*}]$ where U is Haar-distributed on $U(n)$. By the Harish-Chandra–Itzykson–Zuber (HCIZ) formula, the density involves the Vandermonde determinant. If log-concavity of this density can be established, the EPI follows from the Prékopa–Leindler inequality.

- **Via Brascamp–Lieb (Node 1.5.4):** The Brascamp–Lieb inequality on the unitary group may directly yield the EPI using the MSS structure as an expectation over Haar measure.

Status: Open. Numerically the strongest evidence: 0 violations in 13,770 trials. Equality verified analytically for $n = 2$.

8 What Did Not Work: Exhausted Approaches

A critical contribution of this investigation is the identification of natural-seeming approaches that *fail*, saving future effort. These were discovered across two independent proof campaigns (examples6: subordination, examples7: cumulant/heat flow) involving 25+ AI prover and verifier agents.

8.1 Disproved Conjectures

Conjecture	Source	Finding
$\langle h, \alpha \rangle \geq 0$ (inner product of correction with Hilbert transform is non-negative)	ex6, Session 129	**FALSE.** Counterexamples at $n \geq 3$ with ∼0.3–1% violation rate. While $J_p = 2\langle h, \alpha \rangle + \|\alpha\|^2 \geq 0$ always holds, the individual inner product term can be negative.
Partition of unity: $\omega_p'(z) + \omega_q'(z) = 1$	ex6	**FALSE.** The correct result is that each $\omega_p'(\nu_k) = 1$ *independently*, not that the derivatives partition unity. This is a crucial distinction from the continuum setting.
Shape factor $\mathrm{SF}(r) \leq \min(\mathrm{SF}(p), \mathrm{SF}(q))$	ex6	**FALSE.** 42.7% violation rate in random trials.
Monotone gap along heat flow	ex7, Session 132	**FALSE.** 44% violation rate. Definitively refuted by dedicated verifier agent.
Joint concavity of $-R_4$	ex7	**FALSE.** Hessian is indefinite.

Conjecture	Source	Finding
Continuum summation $\omega_1 + \omega_2 = z + 1/(nG_r)$	ex6, §5.4	**FALSE** at finite n. Verified computationally at $n = 2$: the difference $\omega_1 + \omega_2 - z - 1/(nG_r)$ is a nontrivial irrational function.

8.2 Exhausted Methods

Method	Source	Outcome
SOS (sum of squares) on gap numerator	ex7	Mixed-sign cross terms prevent SOS decomposition for $n \geq 4$.
Coefficient additivity for $n \geq 4$	ex6	Cross terms in g_k for $k \geq 4$ destroy additivity.
AM–GM from $A + B \geq 2\Phi_n(r)$	ex6	Wrong direction; the bound goes the wrong way.
Perspective function approach	ex7	Blocked by non-convexity of the relevant functional.
General polynomial manipulation for $n \geq 4$	Both	Exhausted by 4+ prover agents. The combinatorial complexity of the MSS formula at $n \geq 4$ makes direct symbolic manipulation intractable.

8.3 Lessons Learned

1. **Finite $\neq$ continuum:** Many identities from the continuum free probability theory (subordination summation, partition of unity, monotone gap) fail at finite n. Proofs must be genuinely finite-dimensional.

2. **Base cases are deceptive:** The $n = 2$ case reduces to Pythagoras and gives exact equality; $n = 3$ admits a direct algebraic proof. Both suggest the conjecture should be easy, but $n \geq 4$ requires fundamentally different tools.

3. **Structural identities are valuable:** The identity $\Phi_n = 2 \cdot \mathrm{Sm}_2$ and S_2 additivity were key breakthroughs that simplified the problem and enabled the heat flow approach.

4. **Numerical falsification is efficient:** Several plausible-sounding conjectures were quickly killed by targeted numerical testing, saving substantial proof effort.

9 Full Proof Tree

The complete proof tree as exported from the adversarial proof framework (`af`) is reproduced below. Status indicators: **A** = admitted, **P** = pending.

Node 1 [P]: *Main conjecture* — $1/\Phi_n(p \boxplus_n q) \geq 1/\Phi_n(p) + 1/\Phi_n(q)$ for all $p, q \in \mathcal{P}_n$, $n \geq 2$.

- **1.1** [A]: *Foundations.* Definitions, MSS formula, cumulant additivity, $\Phi_n = 2\,\mathrm{Sm}_2$, S_2 additivity, $\sum H_i \lambda_i = n(n-1)/2$, $C_n = 4/(n^2(n-1))$. All proved.

- **1.2** [A]: *Base cases.* $n = 1$ vacuous; $n = 2$ equality; $n = 3$ proved (2 proofs); $n = 4$ symmetric proved (2 proofs). 2M+ numerical trials.

- **1.3** [P]: *Path A: Subordination + L^2 contraction.*

- **1.3.1** [P]: Finite subordination existence.
- **1.3.2** [A]: Chain rule at roots ($\omega_p'(\nu_k) = 1$).
- **1.3.3** [A]: L^2 Pythagoras decomposition.
- **1.3.4** [P]: **Herglotz coupling lemma (KEY).**
  * **1.3.4.1** [P]: Herglotz representation of ω_p.
  * **1.3.4.2** [P]: Coupling constraint $\omega_p + \omega_q = z + F_r$.
  * **1.3.4.3** [P]: Cauchy–Schwarz on coupled quadratic forms.
  * **1.3.4.4** [P]: Closing the bound via Cauchy matrix structure.
- **1.3.5** [P]: Harmonic mean from coupling $\Rightarrow$ QED.

• **1.4** [P]: *Path B: De Bruijn + $1/\Phi_n$ concavity.*

  - **1.4.1** [P]: De Bruijn identity (rigorous).
  - **1.4.2** [P]: $1/\Phi_n$ **concavity along heat flow (KEY).**
  - **1.4.3** [P]: Stam from concavity + splitting.
  - **1.4.4** [A]: Gaussian splitting (proved).

• **1.5** [P]: *Path C: Entropy power inequality.*

  - **1.5.1** [P]: **Finite free EPI (KEY).**
  - **1.5.2** [P]: EPI via log-concavity of MSS.
  - **1.5.3** [P]: EPI-to-Stam reduction.
  - **1.5.4** [P]: Alternative via Brascamp–Lieb.

• **1.6** [P]: *QED:* Base cases + any one path $\Rightarrow$ full conjecture.

10 Recommended Next Steps

Based on the analysis, the recommended priority ordering for further work is:

1. **Node 1.4.2** ($1/\Phi_n$ concavity): Most computationally tractable. All ingredients available (Dyson dynamics, $\Phi_n = 2\,\mathrm{Sm}_2$). Requires computing Φ_n', Φ_n'' explicitly and establishing a Cauchy–Schwarz structure.

2. **Node 1.3.1** (Subordination existence): Unlocks all of Path A. Approach via Rouché's theorem or induction on n using the MSS derivative identity. References: Biane (1998), Belinschi–Bercovici (2007).

3. **Node 1.5.1–1.5.2** (Finite free EPI): Independent of Paths A and B. Strongest numerical support (13,770 trials). Approach via HCIZ formula + Prékopa–Leindler.

4. **Node 1.3.4** (Herglotz coupling): Conditional on 1.3.1. May be as hard as the original conjecture in different language.

5. **Node 1.4.3** (Stam from concavity): Routine given 1.4.2, following the classical Costa–Villani template.

11 Archive and Provenance

The investigation produced a substantial archive preserved in `problem04/archive/`:

Directory	**Files**	**Description**
`examples6/`	434	Subordination approach
`examples7/`	238	Cumulant/heat flow approach
`examples8_canonical/`	—	Hybrid tree backup
`problem04_original/`	—	Original 1-node tree
Total Python	63	∼28,000 lines of verification code

The canonical proof tree synthesizes both approaches into a 24-node hybrid structure with 14 external references linking to proved results.

12 Conclusion

The finite free Stam inequality (Conjecture 1.5) remains open for general $n \geq 4$, despite strong numerical evidence and significant structural progress. The investigation has:

- Proved the conjecture for $n \leq 3$ (with two independent proofs) and for $n = 4$ in the symmetric case;
- Established key structural identities ($\Phi_n = 2\,\mathrm{Sm}_2$, S_2 additivity, Gaussian splitting, chain rule);
- Identified three viable proof paths (subordination, concavity, EPI), each requiring exactly one hard lemma;
- Eliminated numerous false leads, providing a clear map of the problem landscape.

The conjecture is likely true. The most promising avenue appears to be Path B (de Bruijn + concavity), as it is the most computationally concrete and has the fewest unresolved prerequisites.

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