Comparative Analysis: Official Solution vs. Adversarial Proof Attempt for Problem 2 (Whittaker Functions for Rankin–Selberg Integrals)

First Proof Benchmark

Automated Analysis
First Proof Project

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Abstract

We critically compare the official solution to Problem 2 of the First Proof benchmark (posed by Paul D. Nelson) with an automated adversarial proof (`af`) attempt. The problem asks whether, for a generic irreducible admissible representation Π of $\mathrm{GL}_{n+1}(F)$ over a non-archimedean local field, there exists a single Whittaker function W such that for every generic π of $\mathrm{GL}_n(F)$, the local Rankin–Selberg integral is finite and nonzero for all $s \in \mathbb{C}$. Both approaches answer **YES**. The official solution uses the Godement–Jacquet functional equation to relate the unipotent-shifted Rankin–Selberg integral to a $K_1(\mathfrak{q})$-integral of the newvector, obtaining an explicit constant in two pages. The `af` attempt instead pursued a three-case strategy (unramified, supercuspidal, non-supercuspidal ramified) via Iwasawa unfolding and Kirillov model support arguments, accumulating 57 open challenges over 17 nodes without reaching a complete proof. We identify which of the five systematic issues found by `af` were genuine mathematical obstacles versus artifacts of the wrong proof strategy, and extract lessons for future automated theorem proving in representation theory.

Contents

1 Problem Statement

Let F be a non-archimedean local field with ring of integers $\mathfrak{o}$, maximal ideal $\mathfrak{p}$, and residue field cardinality q. Let $\psi : F \to \mathbb{C}^\times$ be a nontrivial additive character of conductor $\mathfrak{o}$. Set $G_r := \mathrm{GL}_r(F)$, $K_r := \mathrm{GL}_r(\mathfrak{o})$, and let $N_r < G_r$ denote the upper-triangular unipotent subgroup.

Let Π be a generic irreducible admissible representation of G_{n+1}, realized in its ψ^{-1}-Whittaker model $\mathcal{W}(\Pi, \psi^{-1})$.

Theorem 1.1 (Nelson). *There exists $W \in \mathcal{W}(\Pi, \psi^{-1})$ with the following property. For every generic irreducible admissible π of G_n with conductor ideal $\mathfrak{q}$ and generator $Q \in F^\times$ of $\mathfrak{q}^{-1}$, setting $u_Q := I_{n+1} + Q\,E_{n,n+1}$, there exists $V \in \mathcal{W}(\pi, \psi)$ such that the local Rankin–Selberg integral*

$$\ell_{\mathrm{RS}}(s, W, V) := \int_{N_n \backslash G_n} W\big(\mathrm{diag}(g, 1) \cdot u_Q\big)\, V(g)\, |\det g|^{s-1/2}\, dg$$

is finite and nonzero for all $s \in \mathbb{C}$.

The critical subtlety, emphasized in the authors' commentary, is that W must be *independent of π*. Without this universality requirement, the problem would be much easier and well-known.

2 The Official Solution

2.1 Overview

The official solution (Nelson, with Jana) is remarkably compact—roughly two pages of mathematics—and proceeds via a single unified argument that avoids any case splitting by the type of π. The key ideas are:

(i) **Choice of W:** Define $W_0 \in \mathcal{W}(\Pi, \psi^{-1})$ via the Kirillov model extension of the function

$$W_0(g) := \int_{N_n} \mathbf{1}_{K_n}(xg)\, \psi(x)\, dx$$

on G_n, extended to G_{n+1} by the theory of the Kirillov model (Bernstein). The actual test vector for Π is $u_Q W_0$ (a unipotent translate of W_0).

(ii) **Choice of V:** Take V to be the normalized newvector of π—the unique $K_1(\mathfrak{q})$-invariant vector with $V(1) = 1$.

(iii) **Godement–Jacquet bridge:** Apply the Godement–Jacquet local functional equation (Lemma 2 in the solution) with a carefully chosen Schwartz function $\phi \in \mathcal{S}(M_n(F))$ and a function $\beta \in \mathcal{S}^e(F^\times)$ (depending on π only through its conductor) to transform the Rankin–Selberg integral into a $K_1(\mathfrak{q})$-integral.

(iv) **Reduction to newvector theory:** The transformed integral evaluates to $|Q|^n \cdot \mathrm{vol}(K_1(\mathfrak{q}))$ by $K_1(\mathfrak{q})$-invariance of V and the normalization $V(1) = 1$.

2.2 Key Technical Ingredients

The Schwartz function ϕ and the weight β. The solution defines $\phi_0 = \mathbf{1}_{M_n(\mathfrak{o})}$ and

$$\phi(x) := \psi(-x_{nn})\, \phi_0(x\, d_Q^{-1}),$$

where $d_Q = \mathrm{diag}(1, \ldots, 1, Q)$. The function β is defined via Mellin inversion:

$$\widetilde{\beta}(s) := \frac{\varepsilon(\frac{1}{2} + s, \pi, \psi)}{L(\frac{1}{2} + s, \pi)},$$

and has the crucial properties that β is supported on $\{y : |Q| \leq |y| \leq |Q|q^n\}$ while $\beta^\sharp$ (the transform of β through the gamma factor) is supported on $\{y : 1 \leq |y| \leq q^n\}$ and takes the value 1 on $\mathfrak{o}^\times$.

The Godement–Jacquet identity (Lemma 4). For unitary generic π and $f = V$ (a Whittaker function treated as a generalized matrix coefficient), the identity

$$\int_{G_n} \phi(g)\, f(g)\, \beta(\det g)\, |\det g|^{n/2}\, dg = \int_{G_n} \phi^\wedge(g)\, f^\vee(g)\, \beta^\sharp(\det g)\, |\det g|^{n/2}\, dg$$

holds with absolute convergence on both sides. This is established by inserting the Mellin expansion of β, recognizing the Godement–Jacquet zeta integral, applying the local functional equation, and re-expanding.

The factorization identity (Lemma 5). Two identities link the Schwartz data (ϕ, β) to the Rankin–Selberg integral on the left side and to the congruence subgroup on the right:

$$\text{(Left):} \quad \int_{N_n} \beta(\det xg)\, \phi(xg)\, \psi(x)\, dx = \varepsilon(\tfrac{1}{2}, \pi, \psi)\, W_0(g\, t_Q), \tag{1}$$

$$\text{(Right):} \quad \beta^\sharp(\det g)\, \phi^\wedge(g) = |Q|^n\, \mathbf{1}_{K_1(\mathfrak{q})}(g). \tag{2}$$

The final computation. Combining these, the Rankin–Selberg integral evaluates to:

$$\ell_{\text{RS}}(s, u_Q W_0, d_Q V) = \varepsilon(\tfrac{1}{2}, \pi, \psi)^{-1}\, |Q|^{n-n/2}\, \text{vol}(K_1(\mathfrak{q})),$$

which is a *nonzero constant* (independent of s).

2.3 Why the Solution Is Elegant

Several features are worth highlighting:

- **No case splitting:** The argument works uniformly for all π—unramified, supercuspidal, and non-supercuspidal ramified—because the Godement–Jacquet functional equation is universal.

- **The s-independence is immediate:** Since W_0 is supported on $\det^{-1}(\mathfrak{o}^\times)$, the translate $t_Q W_0$ is supported on $\det^{-1}(Q\mathfrak{o}^\times)$, so the torus sum collapses to a single determinant value automatically. This makes $\ell_{\text{RS}}(s)$ a constant times $|Q|^s$, and the homogeneity property (equation (19) in the solution) absorbs this into a constant.

- **Newvector theory does all the work:** The right-hand side of the Godement–Jacquet identity (after applying (2)) reduces to an integral over $K_1(\mathfrak{q})$, where $K_1(\mathfrak{q})$-invariance and the normalization $V(1) = 1$ immediately give a computable nonzero answer.

- **The choice of W_0 is explicit and canonical:** It is the Whittaker function corresponding to $\mathbf{1}_{K_n}$ in the Kirillov model. This is *not* the newvector of Π (unless Π is unramified), contrary to what the `af` attempt assumed.

3 The `af` Automated Attempt

3.1 Answer and Strategy

The `af` attempt correctly identified the answer as **YES** and proposed that the essential Whittaker function (newvector) W° of Π works universally. The proof was organized as a 17-node tree with the following high-level structure:

1. **Algebraic reduction** (Nodes 1.1, 1.2): Establish the commutation identity $W(\mathrm{diag}(g,1)\,u_Q) = \psi^{-1}(Qg_{nn})\,W(\mathrm{diag}(g,1))$ and the monomial characterization (a rational function in q^{-s} with no zeros or poles on $\mathbb{C}^\times$ is a monomial).

2. **Three-case strategy:** The choice of test vector V depends on π:

   - *Unramified π:* Construct V_0 with compact Kirillov model support $\phi_0 = \mathbf{1}_{(\mathfrak{o}^\times)^{n-1}}$, aiming to collapse the torus sum to a single term.

   - *Supercuspidal ramified π:* Use the newvector V°; argue that compact Kirillov support of supercuspidal representations forces the torus sum to collapse.

   - *Non-supercuspidal ramified π:* Use V° and attempt to identify the integral with the Rankin–Selberg epsilon factor $\varepsilon(s, \Pi \times \pi, \psi)$.

3. **Iwasawa unfolding** (Node 1.4): For ramified π, unfold via $g = nak$ and exploit W°-factorization and conductor analysis to show $J_K(m) = 0$ unless $m_n = 0$.

4. **Nonvanishing** (Node 1.5): Argue that the surviving terms are nonzero.

3.2 Status at Termination

After three sessions, the proof had:

- 5 validated nodes (1.1, 1.2, 1.4.1, 1.4.2, plus the root by propagation),

- 12 pending nodes with 57 open challenges (18 critical, 19 major, 15 minor),

- 0 refutations.

The five systematic issues identified were:

1. Kirillov model evaluation for $n \geq 2$ (pointwise evaluation off the mirabolic),

2. Casselman–Shalika formula scope (unramified Π only),

3. K-projection dilemma (unramified case),

4. Test vector theory for general $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$,

5. $J_K(0)$ nonvanishing for $n \geq 2$.

4 Critical Comparison

4.1 The Fundamental Strategic Divergence

The most important difference between the two approaches is **structural**: the official solution avoids case splitting entirely, while the `af` attempt committed to a three-case strategy that created three independent proof obligations, each of which proved intractable.

The official solution achieves this by using the Godement–Jacquet functional equation as a *bridge* between the Rankin–Selberg world (where the integral lives) and the congruence subgroup world (where the newvector is well-understood). The key insight is that one can choose Schwartz data (ϕ, β) that simultaneously:

- match the Whittaker function $W_0(g\,t_Q)$ on the Rankin–Selberg side (via identity (1)), and

- reduce to the indicator function $\mathbf{1}_{K_1(\mathfrak{q})}$ on the dual side (via identity (2)).

The `af` attempt, by contrast, tried to analyze the Rankin–Selberg integral *directly* via the Iwasawa decomposition and Kirillov/Whittaker model support. This "direct analysis" approach works for $n = 1$ (where it reduces to Gauss sums) but becomes increasingly unwieldy for $n \geq 2$ because:

- The torus sum has n indices, and collapsing it to a single term requires controlling the support of both W° and V simultaneously.

- The Kirillov model, while powerful for P_n-orbits, does not give pointwise evaluation for elements outside the mirabolic subgroup.

- Different types of π (unramified, supercuspidal, non-supercuspidal ramified) have qualitatively different Kirillov/Whittaker support, necessitating case splits.

4.2 Choice of W: Newvector vs. Kirillov Extension of $\mathbf{1}_{K_n}$

A second major divergence concerns the choice of W itself:

	Official solution	af attempt
W	W_0 defined via the Kirillov model extension of $g \mapsto \int_{N_n} \mathbf{1}_{K_n}(xg)\,\psi(x)\,dx$	W°, the newvector (essential Whittaker function) of Π
Dependence on Π	Only through the Kirillov model embedding $\mathcal{W}(\pi_n, \psi) \hookrightarrow \mathcal{W}(\Pi, \psi^{-1})$	Through the conductor $c(\Pi)$ and the $K_1(\mathfrak{p}^{c(\Pi)})$-invariance
Key property	Supported on $\det^{-1}(\mathfrak{o}^\times)$ when restricted to G_n	$K_1(\mathfrak{p}^{c(\Pi)})$-invariant

The `af` choice of $W = W^\circ$ is not *wrong*—both choices produce valid Whittaker functions. However, W° lacks the determinantal support property that makes the official argument work: since W° is not necessarily supported on $\det^{-1}(\mathfrak{o}^\times)$ when restricted to G_n, the torus sum does not automatically collapse, and one is forced into the case-by-case analysis that generated all 57 challenges.

The official choice of W_0 is specifically engineered so that its translate $t_Q W_0$ is supported on $\det^{-1}(Q\mathfrak{o}^\times)$, which immediately pins the determinant and eliminates all but one term in the torus sum. This is the single most important technical idea that the `af` attempt missed.

4.3 Choice of V: Universal Newvector vs. Case-Dependent Construction

	Official solution	af attempt
V	Always $d_Q V^\circ$, where V° is the normalized newvector of π	V° for ramified π; a compact-Kirillov-support V_0 for unramified π
Universality	Uniform across all π (with the natural d_Q-scaling)	Two different constructions depending on $c(\pi)$

The official solution's choice of $V = d_Q V^\circ$ is arguably more natural: the d_Q-twist accounts for the conductor in the same way that the u_Q-shift accounts for it on the Π-side. The `af` attempt's construction of a special V_0 for the unramified case was motivated by the need to avoid L-function poles in the torus sum—a problem that simply does not arise in the official approach because the torus sum is already collapsed by the support of W_0.

4.4 Role of the Godement–Jacquet Functional Equation

The Godement–Jacquet theory plays no role in the `af` attempt. This is the single biggest missed opportunity. The official solution uses it as the central engine:

1. The functional equation relates $Z(\phi, f, s)$ to $Z(\phi^\wedge, f^\vee, 1 - s)$.

2. With the choice $f = V$ (a Whittaker function, not just a matrix coefficient), and the Schwartz data (ϕ, β) from Lemma 5, the left side becomes $\varepsilon(\frac{1}{2}, \pi, \psi) \cdot \ell_{\mathrm{RS}}(\frac{n+1}{2}, t_Q W_0, V)$ (after unfolding).

3. The right side becomes $|Q|^n \int_{K_1(\mathfrak{q})} V(g^{-1}) \, dg = |Q|^n \mathrm{vol}(K_1(\mathfrak{q}))$ (by (2) and newvector properties).

The `af` attempt's Node 1.5.2 *gestured* toward using the JPSS functional equation to establish the monomial property, but this was framed as an identification $I(s) = C \cdot \varepsilon(s, \Pi \times \pi, \psi)$ via test vector theory—a much harder claim that requires knowing the precise proportionality constant, which the `af` attempt could not establish. The official solution sidesteps this entirely by working with the Godement–Jacquet functional equation (for π alone, not for the pair $\Pi \times \pi$), which is a simpler and more classical tool.

5 Assessment of the Five Systematic Issues

We now evaluate each of the five systematic issues identified by the `af` attempt, distinguishing between genuine mathematical obstacles and artifacts of the proof strategy.

5.1 Issue 1: Kirillov Model Evaluation for $n \geq 2$

Verdict: Artifact of the wrong approach.
The `af` attempt needed pointwise evaluation of $V_0(ak)$ for $k \in K_n$ not in the mirabolic subgroup P_n, which the Kirillov model does not provide. The official solution avoids this entirely: V is always the newvector V°, which is defined globally on G_n (not just on P_n), and the Godement–Jacquet approach reduces everything to an integral over $K_1(\mathfrak{q})$ where V° is simply a constant.

However, the underlying mathematical issue—that the Kirillov model is a P_n-representation, not a G_n-representation—is a genuine fact that any proof using Kirillov-model support arguments for G_n-integrals must contend with. The `af` attempt correctly identified this as a problem; it simply should not have been using the Kirillov model in this way.

5.2 Issue 2: Casselman–Shalika Formula Scope

Verdict: Artifact of the wrong approach.
The `af` attempt invoked the Casselman–Shalika formula to control the torus support of W° (the newvector of Π), but this formula applies only when Π is unramified. The official solution uses W_0 (not W°), whose torus support is controlled not by Casselman–Shalika but by the elementary fact that $\mathbf{1}_{K_n}(xg)$ forces $g \in K_n$ (after integrating over N_n), giving $\det g \in \mathfrak{o}^\times$.

5.3 Issue 3: K-Projection Dilemma

Verdict: Genuine obstacle for the Iwasawa-unfolding approach, but irrelevant to the correct proof.
For unramified π, the K_n-average of any Whittaker function is either proportional to the spherical vector (reproducing L-function poles) or zero. This is a real phenomenon—it reflects

the fact that the spherical vector generates the unique K_n-fixed line in π. The `af` attempt correctly identified this as a dilemma for the unramified case.

The official solution resolves this "dilemma" implicitly: by using $V = d_Q V^\circ$ and working through the Godement–Jacquet functional equation rather than through a K_n-average, the issue simply does not arise. The integral over $K_1(\mathfrak{q})$ (which equals K_n when π is unramified, since $\mathfrak{q} = \mathfrak{o}$) yields $\mathrm{vol}(K_n) \cdot V^\circ(1) = 1$—there is no projection dilemma because V° *is K_n*-fixed.

5.4 Issue 4: Test Vector Theory for General $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$

Verdict: Genuine gap in the literature, but irrelevant to the correct proof.

The `af` attempt tried to identify the Rankin–Selberg integral with $C \cdot \varepsilon(s, \Pi \times \pi, \psi)$, invoking Humphries (2021, GL_2-specific) and Assing–Blomer (2024, does not cover the general case). This is indeed an open problem in the test vector literature: the precise proportionality constant for new vectors in $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$ Rankin–Selberg integrals is not fully established for general n.

However, the official solution does not need this. It computes the integral *directly* as $c \cdot |Q|^{-n/2}$ using the Godement–Jacquet bridge, without ever identifying it with an epsilon factor of the pair $\Pi \times \pi$. The epsilon factor of π alone (not of the pair) appears in the Schwartz function β, but this is elementary (Godement–Jacquet theory for a single representation).

5.5 Issue 5: $J_K(0)$ Nonvanishing for $n \geq 2$

Verdict: Genuine mathematical question, partially relevant.

The `af` attempt needed to show that $\sum_{x \in (\mathfrak{o}/\mathfrak{q})^\times} \psi^{-1}(Qx) \Phi(x) \neq 0$, where Φ is a fiber sum of the newvector over $\mathrm{GL}_n(\mathfrak{o}/\mathfrak{q})$. This is related to the nonvanishing of certain character sums on finite groups of Lie type, which is a non-trivial question.

The official solution encounters a related but much simpler nonvanishing: it reduces to showing that $\int_{K_1(\mathfrak{q})} V^\circ(g^{-1}) \, dg = \mathrm{vol}(K_1(\mathfrak{q})) \cdot V^\circ(1) = \mathrm{vol}(K_1(\mathfrak{q})) \neq 0$, which is immediate from the normalization $V^\circ(1) = 1$ and $K_1(\mathfrak{q})$-invariance of V°. The "hard" nonvanishing problem that the `af` attempt struggled with is thus an artifact of not having the Godement–Jacquet reduction.

5.6 Summary Table

#	Issue	Genuine?	Relevant to official proof?
1	Kirillov model evaluation	Genuine fact	No
2	Casselman–Shalika scope	Genuine fact	No
3	K-projection dilemma	Genuine for Iwasawa approach	No
4	Test vector theory for $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$	Open problem	No
5	$J_K(0)$ nonvanishing	Non-trivial question	No (trivially resolved)

All five issues are real mathematical phenomena, but *none* of them are relevant to the official proof. They are all artifacts of the `af` attempt's strategic choice to analyze the Rankin–Selberg integral directly via Iwasawa unfolding and Kirillov model support, rather than using the Godement–Jacquet functional equation as a bridge.

6 What the `af` Attempt Got Right

Despite the strategic divergence, the `af` attempt produced several correct and valuable contributions:

1. **Correct answer (YES):** The `af` attempt correctly identified that the answer is affirmative, which is non-trivial—the authors' commentary notes that some LLM attempts constructed W depending on π, solving a weaker problem. The `af` attempt understood the universality requirement from the start.

2. **Node 1.1 (Commutation identity):** The identity $W(\mathrm{diag}(g,1)\,u_Q) = \psi^{-1}(Qg_{nn})\,W(\mathrm{diag}(g,1))$ is correct, useful, and appears implicitly in the official solution (in the computation $W_0(gt_Q) = \psi(-g_{nn})W_0(gd_Q^{-1})$, equation (12)).

3. **Node 1.2 (Monomial characterization):** The fact that a rational function in q^{-s} with no zeros or poles on $\mathbb{C}^\times$ is a monomial is used in both approaches. The official solution obtains an explicit constant directly, but the underlying algebraic fact is the same.

4. **Node 1.4.1 (W° factorization):** The factorization of W° out of the K-integral via $K_1(\mathfrak{p}^{c(\Pi)})$-invariance is a valid and standard argument. While the official solution uses a different W (not W°), the factorization principle is the same.

5. **Node 1.4.2 (Vanishing for $m_n > c(\pi)$):** The argument that $\pi^{K_n} = 0$ for ramified π is correct. The official solution does not need this step (the torus collapse is automatic), but the underlying representation-theoretic fact is valid.

6. **Correct identification of difficulty:** The `af` attempt's systematic issues (Section 5) accurately reflect the genuine obstacles to a "direct analysis" proof. The self-diagnosis was remarkably accurate—the attempt knew where it was stuck and why.

7. **Correct use of references:** The references to JPSS, Matringe, Miyauchi, Casselman–Shalika, and Godement–Jacquet are all appropriate and correctly cited (even if some were applied outside their valid scope).

7 What Went Wrong: Root Cause Analysis

7.1 The Core Error: Missing the Godement–Jacquet Bridge

The `af` attempt's fundamental error was not a mathematical mistake but a *strategic* one: it tried to analyze the Rankin–Selberg integral directly, without using the Godement–Jacquet functional equation as an intermediary.

This is understandable: the Rankin–Selberg integral is defined directly in terms of Whittaker functions, and the most natural approach is to unfold it via the Iwasawa decomposition and analyze the resulting torus sum. This is, in fact, the standard approach in much of the literature on Rankin–Selberg integrals. The problem is that this approach requires controlling the torus support of both W and V simultaneously, which (as the `af` attempt discovered) is extremely difficult for general π.

The official solution's insight is that the Godement–Jacquet theory for the *smaller* group G_n provides a way to "dualize" the integral, replacing the Whittaker function V (which has complicated torus support) with the indicator function $\mathbf{1}_{K_1(\mathfrak{q})}$ (which has trivial support properties). This duality is achieved by the identity $\beta^\sharp(\det g)\,\phi^\wedge(g) = |Q|^n\,\mathbf{1}_{K_1(\mathfrak{q})}(g)$, which is the key computation in Lemma 5.

7.2 The Secondary Error: Wrong Choice of W

The `af` attempt chose $W = W^\circ$ (the newvector of Π), while the official solution uses W_0 (the Kirillov extension of $\mathbf{1}_{K_n}$). These are different functions (unless Π is unramified, in which case W_0 may coincide with W° up to normalization).

The newvector W° is a natural first guess—it is the most canonical element of $\mathcal{W}(\Pi, \psi^{-1})$—but it does not have the determinantal support property that makes the official argument work. The official choice of W_0 is engineered precisely to have compact determinantal support $(\det^{-1}(\mathfrak{o}^\times)$ on $G_n)$, which collapses the torus sum to a single term without any case analysis.

Interestingly, the authors' commentary notes that the best LLM attempt (ChatGPT 5.2 Pro) *did* identify a suitable W and reduced to the nonvanishing of $\int_{K_n} V(g)\, \psi(-Qg_{nn})\, dg$. This suggests that the choice of W is the easier part; the harder part is the nonvanishing argument, which is where the Godement–Jacquet bridge is essential.

7.3   The Cascade of Complications

The wrong strategic choices cascaded into increasingly complex proof obligations:

1. Wrong $W \Rightarrow$ torus sum does not collapse $\Rightarrow$ need to analyze multi-dimensional torus sums.

2. Multi-dimensional torus sums $\Rightarrow$ need to control support of both W° and $V \Rightarrow$ case splitting by type of π.

3. Case splitting $\Rightarrow$ three independent proof branches, each with its own difficulties.

4. Unramified branch $\Rightarrow$ need Kirillov model for non-mirabolic elements $\Rightarrow$ Issue 1.

5. Supercuspidal branch $\Rightarrow$ need $J_K(0) \neq 0$ for $n \geq 2 \Rightarrow$ Issue 5.

6. Non-supercuspidal branch $\Rightarrow$ need test vector theory for general $n \Rightarrow$ Issue 4.

Each issue, in turn, generated multiple challenges in the adversarial framework, leading to the 57 open challenges.

8   Lessons Learned

8.1   For Automated Theorem Proving

1. **Strategy selection matters more than local correctness.** The `af` attempt produced many locally correct arguments (5 validated nodes), but the global strategy was suboptimal. An automated prover that can search over proof strategies—not just refine a fixed strategy—would have been far more effective.

2. **Functional equations are not just for computing L-values.** The Godement–Jacquet functional equation is usually presented as a tool for establishing analytic properties of L-functions. The official solution uses it as a *duality transform*—a way to move from the Whittaker model (complicated) to the congruence subgroup (simple). Automated provers should be trained to recognize functional equations as potential "simplification bridges," not just as identities to be verified.

3. **Case splitting is a warning sign.** When a proof of a universal statement ($\forall \pi$) requires case splitting by the type of π, this often indicates that the proof is not using the right level of abstraction. The official solution's universality (no case splitting) is a hallmark of the "right" proof.

4. **The adversarial framework correctly identified failure modes.** The 5 systematic issues were accurate diagnoses of why the proof was stuck. The `af` framework's strength is in *verification* (catching errors) rather than *synthesis* (finding the right approach). This suggests a complementary architecture: use a creative "strategy search" module to propose proof outlines, and use the adversarial framework to verify them.

5. **Beware of "natural" choices.** The newvector W° is the most natural element of $\mathcal{W}(\Pi, \psi^{-1})$, and it was the `af` attempt's first (and only) choice. The official solution's W_0 is less obvious but more effective. Automated provers should be encouraged to explore multiple candidate constructions before committing to a proof strategy.

8.2 For the Representation Theory of p-Adic Groups

1. **The Godement–Jacquet theory for a single group is underappreciated as a tool for Rankin–Selberg problems.** The standard approach to Rankin–Selberg integrals uses the JPSS theory (zeta integrals for the *pair* $\Pi \times \pi$). The official solution instead uses the Godement–Jacquet theory for π alone, combined with a clever choice of Schwartz data. This is a genuinely novel technique.

2. **Determinantal support is a powerful tool.** The collapse of the torus sum in the official solution comes from the fact that W_0 (restricted to G_n) is supported on $\det^{-1}(\mathfrak{o}^\times)$. This is a much simpler condition than the detailed torus support analysis (Casselman–Shalika, Matringe formulas) that the `af` attempt attempted.

3. **The test vector problem for $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$ remains open for general n.** The `af` attempt's Issue 4 is a real gap in the literature. The official solution circumvents it, but does not solve it.

8.3 Comparison with the Authors' Commentary on LLM Attempts

The authors' commentary identifies three failure modes in LLM-generated solutions:

(a) Constructing W depending on π (solving a weaker problem).

(b) Choosing V so the integrand is constant on its support (unviable due to central characters).

(c) Asserting false support properties of V via a "standard Howe-vector existence result" (contradicts central character).

The `af` attempt avoided failure mode (a)—it correctly understood the universality requirement. It partially fell into failure mode (c) via the Kirillov model evaluation issue (treating V_0 as having simpler support than it actually does). It did not exhibit failure mode (b). Overall, the `af` attempt was more sophisticated than the typical LLM attempt described in the commentary, but still failed to find the correct proof strategy.

9 Conclusion

The comparison between the official solution and the `af` attempt illustrates a fundamental challenge in automated theorem proving for research mathematics: *finding the right proof strategy is often harder than executing it.*

The official solution is elegant precisely because it reduces a difficult representation-theoretic problem to a routine computation via a single well-chosen duality transform (the Godement–Jacquet functional equation). The `af` attempt, lacking this key insight, was forced into an increasingly complex case analysis that generated genuine mathematical difficulties—all of which dissolve when the right approach is used.

The `af` attempt's strengths—correct answer identification, accurate error diagnosis, rigorous local verification—are valuable but insufficient without the strategic insight that makes the proof work. Future automated provers would benefit from a "strategy search" capability that explores multiple high-level proof outlines before committing to detailed verification.

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