# Introduction to general relativity: Einstein's field equations, linearised solutions

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$$(*) \qquad R_{ab} - \tfrac{1}{2} R \, g_{ab} = 8 \pi T_{ab}$$

<u>Basic properties</u>: Take trace of $(*)$ we obtain

$$R = -8 \pi T$$

Here $T$ is trace of $T_{ab}$. Substitute back into $(*)$ so that

$$R_{ab} = 8\pi \left( T_{ab} - \tfrac{1}{2} g_{ab} T \right)$$

Geometry ←    Energy
               Matter Stress etc.

Sometimes this equation can yield solutions to $(*)$.

Do we actually recover, eg., SR, and Newtonian physics as solutions to $(*)$ in some limits?

<u>Goal for next couple of lectures</u>: argue that GR reduces to SR / Newtonian in correct limits.

In case where energy of matter as measured by an observer "at rest" with respect to masses is greater than stresses (in units where $c = 1$) then

$$T \simeq -\rho = - T_{ab} v^a v^b$$

So that $(*)$

$$R_{ab} v^a v^b \simeq 4\pi T_{ab} v^a v^b$$

The physical content of GR is thus

Spacetime $M$ is a manifold with Lorentzian metric $g_{ab}$. The curvature of $g_{ab}$ is related to the matter distribution in spacetime via (*)

(1) Equation (*) is highly nonlinear system of coupled PDEs involving $g_{\mu\nu}$ (in some coord. system). It is 2nd order in derivatives

(2) Unlike, eg., Maxwell's equations, we cannot regard $T_{ab}$ as a "source" and solve for $g_{ab}$! This is because $T_{ab}$ is defined in terms of $g_{ab}$ for eg. Perfect fluids. → we can't even interpret $T_{ab}$ before solving (*).

(3) Einstein's field equations imply $\nabla^a T_{ab} = 0$. It may argued that $\nabla^a T_{ab} = 0$, for a "dust" of grains exerting no force on each other, implies that grains move on geodesics. So, effectively (*) Contain geodesic hypothesis.

## Linearised gravity

We now attempt solve (*) in case where gravity is "weak": ie spacetime metric $g_{ab}$ is nearly flat, which we assume means

$$g_{ab} = \eta_{ab} + \gamma_{ab}$$

with $\gamma_{ab}$ "small"

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in case of linearised gravity we say $g_{ab}$ is __small__ if there exists a __global__ __inertial coordinate system__ such that $\gamma_{\mu\nu}$ are __small__ w.r.t. 1

__Linearised gravity__ is what results from substituting $g_{ab} = \eta_{ab} + \gamma_{ab}$ into (†) (and collecting terms to first order in $\gamma_{ab}$)

We raise & lower indices with $\eta_{ab}$: exception $g^{ab}$ denotes true inverse to $g_{ab}$ ($\underline{NOT}$ $\eta^{aa'}\eta^{bb'}g_{ab}$)

Calculate $g^{ab} = \eta^{ab} - \gamma^{ab}$

Check:

$$g^{ab}g_{bc} = (\eta^{ab} - \gamma^{ab})(\eta_{bc} + \gamma_{bc})$$
$$= \delta^a_c - \gamma^a_c + \gamma^a_c - \gamma^{ab}\gamma_{bc}$$
$$= \delta^a_b$$

In __global coordinate system__, to linear in $\gamma_{ab}$, we find

$$\Gamma^c_{ab} = \tfrac{1}{2}(\eta^{cd} - \gamma^{cd})(\partial_a \gamma_{bd} + \partial_b \gamma_{ad} - \partial_d \gamma_{ab})$$

Thus, to linear order in $\gamma_{ab}$, Ricci tensor is (ex:)

$$\overset{(1)}{R_{ab}} = \partial_c \Gamma^c_{ab} - \partial_a \Gamma^c_{cb}$$

$$= \partial^c \partial_{(b}\gamma_{a)c} - \tfrac{1}{2}\partial^c\partial_c \gamma_{ab} - \tfrac{1}{2}\partial_a\partial_b\gamma$$

where $\gamma \equiv \gamma^c_c$.

$$\overset{(1)}{G_{ab}} = \overset{(1)}{R_{ab}} - \tfrac{1}{2}\eta_{ab}R^{(1)}:$$

$$\overset{(1)}{G_{ab}} = \partial^c\partial_{(b}\gamma_{a)c} - \tfrac{1}{2}\partial^c\partial_c\gamma_{ab} - \tfrac{1}{2}\partial_a\partial_b\gamma - \tfrac{1}{2}\eta_{ab}(\partial^c\partial^d\gamma_{cd} - \partial^c\partial_c\gamma)$$

If we define $\bar{\gamma}_{ab} = \gamma_{ab} - \tfrac{1}{2}\eta_{ab}\gamma$, then (†) to linear order in $\gamma_{ab}$, become (†):

linear order in $\gamma_{ab}$, become $(*)$:
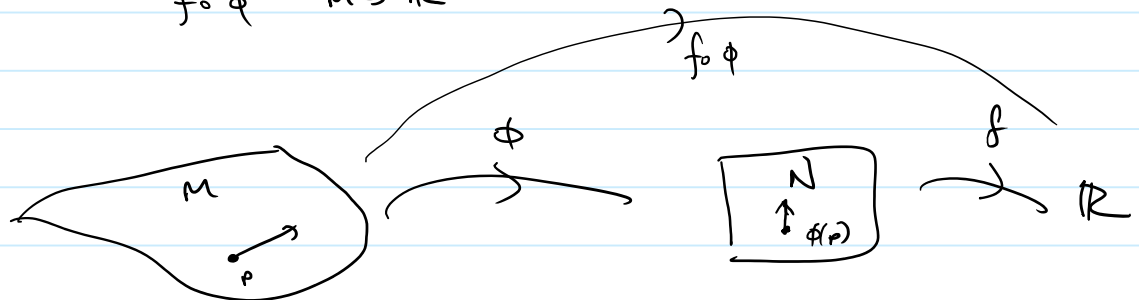
$$-\frac{1}{2}\partial^c\partial_c \bar{\gamma}_{ab} + \partial^c\partial_{(b}\bar{\gamma}_{a)c} - \frac{1}{2}\eta_{ab}\partial^c\partial^d \bar{\gamma}_{cd} = 8\pi T_{ab}$$

→ Linear in $\gamma_{ab}$. This equation may be simplified using the __gauge freedom__ of general relativity: if $\phi: M \to M$ is a diffeomorphism then $g_{ab}$ and its image $g'_{ab}$ represent the __same__ spacetime geometry. This will be explained below.

__Maps of manifolds__

Let $M$ and $N$ be manifolds. Let $\phi: M \to N$ be a $C^\infty$ map. Associated to any function $f: N \to \mathbb{R}$ there is a "pullback" function

$$f \circ \phi : M \to \mathbb{R}$$



To any tangent vector at $p \in M$ there is a "push forward" vector defined via

$$\phi^* : V_p \to V_{\phi(p)}$$

where $\phi^*$ is obtained as follows. For $v \in V_p$ define $\phi^* v \in V_{\phi(p)}$ by  (ex: check it is a tangent vector)

$$(\phi^* v)(f) \equiv v(f \circ \phi)$$

for all $f \in \mathcal{F}(M)$.

The map $\phi^*$ is given by Jacobian of $\phi$: Let $x^\mu$ denote coordinates of $M$  (resp. $y^\nu$ coords. of $N$)

Then matrix of $\phi^*$ is then (ex.)

$$[\phi^*]^\mu{}_\nu \equiv \partial y^\mu / \partial x^\nu .$$

Similarly, we can "pullback" dual vectors at $\phi(p)$ by defining $\phi_*: V_{\phi(p)}^* \to V_p^*$ so that for all $v \in V_p$
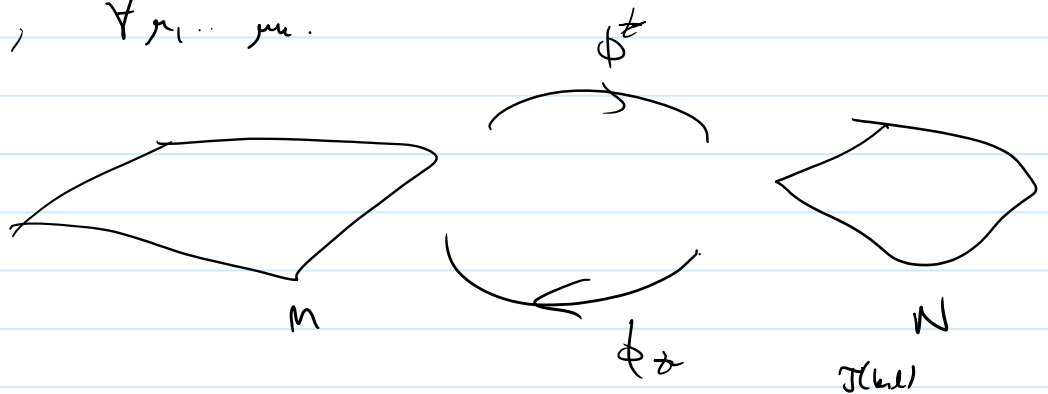
$$(\phi_* \mu)_a v^a = \mu_a (\phi^* v)^a$$

Extend to tensors in $J(0,\ell)$ via:

$$(\phi_* T)_{a_1 \cdots a_\ell} v_1^{a_1} \cdots v_\ell^{a_\ell} = T_{a_1 \cdots a_\ell} (\phi^* v_1)^{a_1} \cdots (\phi^* v_\ell)^{a_\ell}$$

and to $J(k,0)$ via

$$(\phi^* T)^{b_1 \cdots b_k} (\mu_1)_{b_1} \cdots (\mu_k)_{b_k} = T^{b_1 \cdots b_k} (\phi_* \mu_1)_{b_1} \cdots (\phi_* \mu_k)_{b_k}$$

$$\forall \ v_1 \cdots v_\ell \ , \quad \forall \mu_1 \cdots \mu_k .$$



We <u>do</u> <u>not</u> get an action on mixed tensors, because arrows go wrong way. However, if $\phi$ is diffeomorphism (Thus $M=N$) we can exploit $\phi^{-1}$ to build/extend $\phi^*$ to mixed tensors ( Because $(\phi^{-1})^*: V_{\phi(p)} \to V_p$ )
<u>We define</u>:

$$(\phi^* T)^{b_1 \cdots b_k}{}_{a_1 \cdots a_\ell} (\mu_1)_{b_1} (\mu_2)_{b_2} \cdots (\mu_k)_{b_k} (v_1)^{a_1} \cdots (v_\ell)^{a_\ell}$$

$$\equiv T^{b_1 \cdots b_k}{}_{a_1 \cdots a_\ell} (\phi_* \mu_1)_{b_1} \cdots ([\phi^{-1}]^* v_\ell)^{a_\ell}$$

Note (ex.) $\phi_* = (\phi^{-1})^*$

**Definition** : a diffeomorphism $\phi: M \to M$ such that $(\phi^* g_{ab}) = g_{ab}$ is called an _isometry_.