Introduction to general relativity: flows and tensors

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Definition: Let M be a manifold. A one-parameter group of diffeomorphisms ϕ_t is a C^∞ map from

$$\phi_t : \mathbb{R} \times M \to M$$

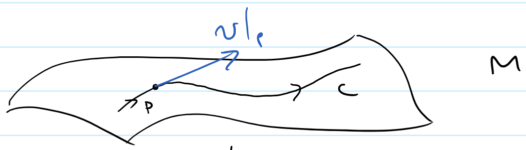
such that for fixed $t \in \mathbb{R}$, $\phi_t : M \to M$ is a diffeomorphism and for all $s, t \in \mathbb{R}$

$$\phi_s \circ \phi_t = \phi_{t+s}$$

Thus

$$\phi_0 = \text{identity.} \qquad (\text{choose } s = -t)$$

Given such $\phi_t \Rightarrow$ obtain tangent vector field v:



for fixed $p \in M$, $\phi_t(p) : \mathbb{R} \to M$ is a curve c the orbit of p under ϕ_t. Define $v|_p$ to be the tangent vector to c @ $t=0$

Converse: let v be a smooth vector field on $M \Rightarrow$ try to find integral curves of v: Family of curves in M, exactly one curve through each point $p \in M$, and tangent vector at p to this curve is $v|_p$

(i) $$T(f) = \frac{d}{dt}(f \circ c) = \sum_\mu \frac{dx^\mu}{dt} X_\mu(f)$$

c is the integral curve through p (we are solving for this!)

(ii) $$v(f) = \sum_\mu v^\mu \frac{\partial}{\partial x^\mu}\bigg|_p f = \sum_\mu v^\mu X_\mu(f)$$

$\Rightarrow$ To find c we set (i) = (ii) we must solve

$$\boxed{\frac{dx^\mu}{dt} = v^\mu(x^1, \dots, x^n)} \qquad \mu = 1, \dots, n$$

$\Rightarrow$ Coupled ODEs. There is a unique solution $\Rightarrow$ $\exists\, \varepsilon > 0$ & $\forall\, t \in (-\varepsilon, \varepsilon)$. (if M is compact we can say more)

New vector fields from old

Definition: Let v, w be smooth vector fields on manifold M
Define u via $\forall f \in \mathcal{F}(M)$

$$\boxed{u(f) = v(w(f)) - w(v(f))}$$

This is a vector field (ex.) called commutator of v & w. Notation: $u \equiv [v, w]$

New vector spaces from old

(i) Taking duals $V \mapsto V^*$
~~(ii) Taking direct sums $V, W \mapsto V \oplus W$~~
(iii) Take tensor products $V, W \mapsto V \otimes W$
(iv) Take subspaces $U \subset V$
~~(v) Take intersections $V, W \mapsto V \wedge W$~~
~~(vi) Take joins $V, W \mapsto V \vee W$~~
(vii) ...

(i) **Definition:** Let V be a real vector space. Denote by V^* the vector space of linear functions
$$f : V \to \mathbb{R}$$
(Ex.) Show V^* is a real vector space.
Elements of V^* are called dual vectors

Remark: If $v_1, \dots, v_n$ is a basis for V we obtain a basis for V^* via
$$(v^\mu)^* (v_\nu) = \delta^\mu{}_\nu \qquad \forall \mu, \nu$$
Ex. Show $(V^*)^* \cong V$
Ex. $\dim(V^*) = \dim(V)$.

Remark: $v_\mu \longmapsto (v^\mu)^*$ gives an isomorphism between V & V^*. BUT it is a non canonical

(Informal) Tensor Product of two vector spaces V & W ($\dim(V) < \infty$ $\dim(W) < \infty$) is denoted
$$V \otimes W$$
and consists of the elements
$$v \otimes w \qquad \forall v \in V \quad \forall w \in W$$
and their linear combinations.
The elements $v \otimes w$ obey the bilinearity condition
$$(*) \quad (a_1 v_1 + a_2 v_2) \otimes (b_1 w_1 + b_2 w_2) =$$
$$a_1 b_1 \, v_1 \otimes w_1 + a_1 b_2 \, v_1 \otimes w_2 + a_2 b_1 \, v_2 \otimes w_1 + a_2 b_2 \, v_2 \otimes w_2$$

$$\forall a_1, a_2 \in \mathbb{R} \quad \forall b_1, b_2 \in \mathbb{R} \quad \forall v_1, v_2 \in V$$
$$\forall w_1, w_2 \in W$$

Ex. $V = \mathbb{R}^2$, $W = \mathbb{R}^2$. $V \otimes W = \mathbb{R}^2 \otimes \mathbb{R}^2$
Look at $\binom{a_1}{a_2} \otimes \binom{b_1}{b_2}$ and write this as linear

combination of $\binom{1}{0}\otimes\binom{1}{0}$ etc. Convince yourself
$\mathbb{R}^2\otimes\mathbb{R}^2 \cong \mathbb{R}^4$.

Δ — track More precisely $V\otimes W \equiv F(V\times W)/\sim$
where F denotes free vector space generated by $V\times W$
where equivalence relation $\sim$ is defined by $\forall v\in V$
$\forall w \in W$

(identity) (1) $(v,w) \sim (v,w)$
(symmetry) (2) $(v,w) \sim (v',w') \Rightarrow (v',w') \sim (v,w)$
(transitivity) (3) $(v,w) \sim (v',w')$ & $(v',w') \sim (v'',w'') \Rightarrow (v,w)\sim(v'',w'')$
(distributivity) (4) $(v,w)+(v',w) \sim (v+v',w)$ and
$(v,w)+(v,w') \sim (v, w+w')$
(scalar multiples) (5) $c(v,w) \sim (cv,w) \sim (v,cw)$ $\forall c\in\mathbb{R}$

END of Δ

Elements of $V\otimes W$ are linear combinations of simple tensors
$v_j\otimes w_k$: given arbitrary $u\in V\otimes W$

$$u = \sum_{\substack{j=1\\k=1}}^{M,N} v_j \otimes w_k \qquad (\text{here } M=\dim V,\ N=\dim W)$$

Warning! given $u\in V\otimes W$ it is not always true
that

$$u = v\otimes w$$

Generically: $u \neq v\otimes w$

Lemma (ex): $\dim(V\otimes W) \overset{!}{=} \dim(V)\dim(W)$
Also: $U\otimes(V\otimes W) \cong (U\otimes V)\otimes W \Rightarrow$
use notation $U\otimes V\otimes W$

Approach 2. (Dual version of tensor product)
Let V and W be real vector spaces. A tensor is
a multilinear map

$$T: V\times W \to \mathbb{R}$$

where:
$T(a_1v_1+a_2v_2,\ b_1w_1+b_2w_2) = a_1b_1T(v_1,w_1) + a_1b_2T(v_1,w_2) +$
$a_2b_1T(v_1,w_2) + a_2b_2T(v_2,w_2)$
$\forall a_1,a_2,b_1,b_2\in\mathbb{R},\ v_1,v_2\in V,\ w_1,w_2\in W$

$\Rightarrow$ "Classical approach to $\otimes$": This is just dual
of (in)formal definition:

$$(V\otimes W)^* = \{T: V\times W\to\mathbb{R} \mid T \text{ multilinear}\}$$

Ex: $(V\otimes W)^* \cong V^*\otimes W^*$

Apply these constructions to V_p:

Apply these constructions to V_p:

$V_p \longmapsto V_p^*$; $\quad \underbrace{V_p \otimes V_p \otimes \cdots V_p}_{k} \otimes \underbrace{V_p^* \otimes V_p^* \otimes \cdots \otimes V_p^*}_{\ell} \cong V_p^{\otimes k} \otimes (V_p^*)^{\otimes \ell}$

Also build, eg. $V_p \otimes V_p^* \otimes V_p$, but these don't play a major role in GR

Elements of $V_p^{\otimes k} \otimes (V_p^*)^{\otimes \ell}$ can also be interpreted as multilinear maps

$$T: (V_p^*)^{\otimes k} \otimes V_p^{\otimes \ell} \to \mathbb{R}$$

why?, $(V_p^*)^* = V_p$ so $t \in V_p$ can be thought of as a function on (V_p^*) as well

$T \in V_p^{\otimes k} \otimes (V_p^*)^{\otimes \ell}$ can be thought of as an element of $\left(\underbrace{(V_p^*)^{\otimes k} \otimes V_p^{\otimes \ell}}\right)^*$ or as a multilinear function of

Such a map T is tensor of type (k, ℓ)
The vector space of all tensors of type (k, ℓ) is denoted $\mathcal{T}(k, \ell)$

We have $\dim(\mathcal{T}(k, \ell)) = n^{k+\ell}$, where $\dim(V_p) = n$

Example: (i) a tensor of type $(0,1)$ is an element of

$$V_p^{\otimes 0} \otimes (V_p^*)^{\otimes 1} \cong \mathbb{R} \otimes V_p^* \equiv V_p^*$$
$\Rightarrow$ dual vector

Define $V^{\otimes 0} \cong \mathbb{R}$

(ii) a tensor of type $(1,0)$ is an element of

$$V_p^{\otimes 1} \otimes (V_p^*)^{\otimes 0} \cong V_p \otimes \mathbb{R} \cong V_p$$
$\Rightarrow$ a vector

(iii) An element of type $(1,1)$ is an element of $V_p \otimes V_p^*$
(or, equivalently, a multilinear map from $V_p^* \otimes V_p \to \mathbb{R}$)

We can interpret T of type $(1,1)$ in three ways

(i) as a vector in in vectorspace $V_p \otimes V_p^*$
(ii) as a multilinear map from $V_p^* \otimes V_p \to \mathbb{R}$
(or dual vector in $V_p^* \otimes V_p$)
(iii) as a linear transformation (matrix) $T: V_p \to V_p$
$\to$

Choose a basis $\underline{v}_\mu$ of $V_p \Rightarrow (v^\nu)^*$ for V_p^*

$$T = \sum_{\mu,\nu} \underbrace{t^\mu{}_\nu}_{\in \mathbb{R}} \underbrace{v_\mu \otimes (v^\nu)^*}_{\text{simple tensor basis of } V_p \otimes V_p^*} \in V_p \otimes V_p^*$$

Call $[\underline{\underline{T}}]^\mu{}_\nu = t^\mu{}_\nu \Rightarrow$ matrix elements of T

The matrix $\underline{\underline{T}}$ acts on elements of V_p as follows

let $\underline{w} \in V_p$

$$\underline{\underline{T}}\,\underline{w} \equiv \left(\sum_{\mu,\nu} t^\mu{}_\nu\, v_\mu \otimes (v^\nu)^*\right) \cdot (\underline{w})$$

$$= \sum_{\mu,\nu} t^\mu{}_\nu\, v_\mu \otimes \left((v^\nu)^*(\underline{w})\right)$$

$$= \sum_{\mu,\nu,\lambda} t^\mu{}_\nu\, v_\mu \otimes (v^\nu)^* (w^\lambda v_\lambda)$$

$$= \sum_{\mu,\nu,\lambda} t^\mu{}_\nu\, w^\lambda\, \delta^\nu{}_\lambda\, v_\mu$$

$$= \sum_{\mu,\nu} t^\mu{}_\nu\, w^\nu\, v_\mu$$

So

$$[\underline{\underline{T}}\,\underline{w}]_\mu = \sum_\nu t^\mu{}_\nu\, w^\nu \quad \in V_p$$

This is an example of tensor contraction

Contraction: this is a map $C_{j,j'}$ from the vector space $\mathcal{T}(k,l)$ to $\mathcal{T}(k-1,l-1)$ defined for $T \in \mathcal{T}(k,l)$

$$C_{j,j'} T \equiv \sum_{\sigma=1}^{n} T(\ldots, \underbrace{(v^\sigma)^*}_{j\text{th}}, \ldots; \ldots, \underbrace{v_\sigma}_{j'\text{th}}, \ldots)$$

where $\{v_\sigma\}$ is a basis for V_p, $(v^\sigma)^*$ is dual basis and we interpret T as a multilinear map from $(V_p^*)^{\otimes k} \otimes V_p^{\otimes l} \to \mathbb{R}$

Write $T \in \mathcal{T}(k,l)$ in terms of basis:

$$T = \sum_{\substack{\mu_1 \cdots \mu_k = 1 \\ \nu_1 \cdots \nu_l = 1}}^{n} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}\, v_{\mu_1} \otimes \cdots \otimes v_{\mu_k} \otimes (v^{\nu_1})^* \otimes \cdots \otimes (v^{\nu_l})^*$$

(Ex) $$\left(C_{j,j'} T\right)^{\mu_1 \cdots \mu_{k-1}}{}_{\nu_1 \cdots \nu_{l-1}} = \sum_{\sigma=1}^{n} T^{\mu_1 \cdots \sigma_j \cdots \mu_{k-1}}{}_{\nu_1 \cdots \sigma_{j'} \cdots \nu_{l-1}}$$

Outer product Let $S \in \mathcal{T}(k,l)$, $T \in \mathcal{T}(k',l')$ their outer product $S \otimes T$ is simply the tensor

$$U \in \mathcal{T}(k+k', l+l')$$

with components

$$U^{\mu_1 \cdots \mu_k \cdots \mu_{k+k'}}{}_{\nu_1 \cdots \nu_l \cdots \nu_{l+l'}} \equiv S^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}\, T^{\mu_{k+1} \cdots \mu_{k+k'}}{}_{\nu_{l+1} \cdots \nu_{l+l'}}$$

ie $$U \equiv S \otimes T \in (V_p \otimes \cdots \otimes V_p) \otimes (V_p^* \otimes \cdots \otimes V_p^*)$$

ii $\quad U \cong S \otimes T \in \underbrace{(V_p \otimes \cdots \otimes V_p)}_{k+k'} \otimes \underbrace{(V_p^* \otimes \cdots \otimes V_p^*)}_{l+l'}$

(I.e. reorder V_p & V_p^* tensor factors so all V_p factors lie on left)