Introduction to general relativity: FLRW cont.

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$$\boxed{\begin{aligned} \frac{3\dot{a}^2}{a^2} &= 8\pi\rho - \frac{3k}{a^2} \qquad (i) \\ \frac{3\ddot{a}}{a} &= -4\pi(\rho + 3p) \qquad (ii) \end{aligned}}$$

Using equation

$$\dot{\rho} + 3(\rho + p)\dot{a}/a = 0$$

We find for "dust" distribution of matter $p = 0$

$$\dot{\rho} + 3\rho(\dot{a}/a) = 0$$

$\Rightarrow$

$$\frac{d}{dt}(\rho a^3) = 0$$

$\Rightarrow$

$$\rho a^3 = \text{const.}$$

This is a conservation of rest mass.

For radiation $(\rho = p/3)$

$\Rightarrow$

$$\boxed{\rho a^4 = \text{const.}}$$

Here one interprets the result as follows

photon number density scales as a^{-3} and each photon is redshifted a^{-1}.

Hence, radiation dominated in past $(a \to 0)$

If $k = 0$ or $k = -1$ in (i) and (ii) (and $\rho, p > 0$)

then $\dot{a} > 0$. Expansion implies eternal expansion.

Further ρ must decrease as a increases : thus

$$\rho a^2 \xrightarrow[a\to\infty]{} 0$$

For $k=0$ "the expansion velocity" $\dot a \xrightarrow[\tau\to\infty]{} 0$

For $k=-1$ $\dot a \xrightarrow[\tau\to\infty]{} 1$

For $k=+1$ the universe ends in a "big crunch"

Now turn to exact solutions. For dust $(p=0)$

$$\dot a^2 - c/a + k = 0$$

where $c = 8\pi \rho a^3/3$ is constant.

For radiation

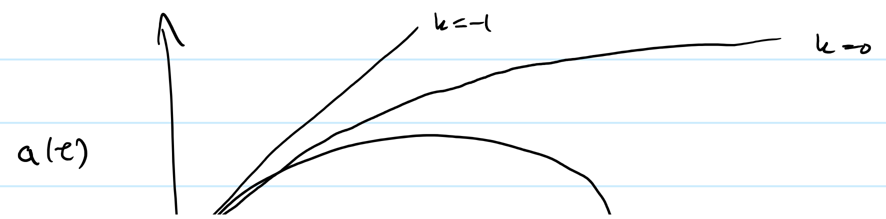
$$\dot a^2 - c'/a^2 + k = 0$$

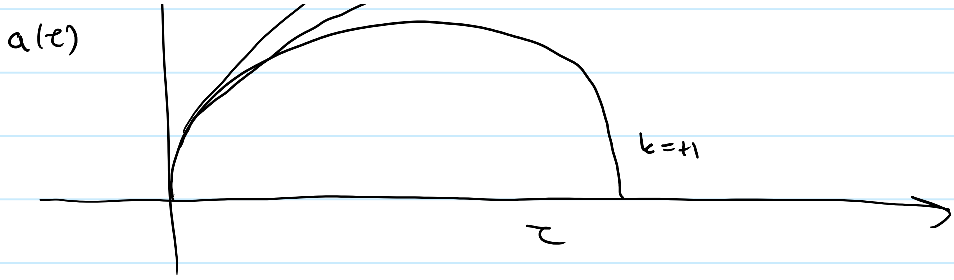
where $c' = 8\pi \rho a^4/3$.

Solve via standard techniques (ex.)

Geometry	Dust	Radiation
Sphere $k=+1$	$a = \frac{1}{2}c(1-\cos\eta)$ $\tau = \frac{1}{2}c(\eta - \sin\eta)$	$a = \sqrt{c'}\,(1-(1-\tau/\sqrt{c'})^2)^{1/2}$
$k=0$	$a = \left(\frac{9c}{4}\right)^{1/3}\tau^{2/3}$	$a = (4c')^{1/4}\tau^{1/2}$
$k=-1$	$a = \frac{1}{2}c(\cosh\eta - 1)$ $\tau = \frac{1}{2}c(\sinh\eta - \eta)$	$a = \sqrt{c'}\,((1+\tau/\sqrt{c'})^2 - 1)^{1/2}$

The spherical solution (or, indeed, sometimes, all) are called the Friedman (LRW) cosmologies





The Schwarzschild solution

Definition: a spacetime M is *stationary* if there exists a one-parameter group ϕ_t of isometries, whose orbits are timelike curves ("time translations")

Remark: Equivalently, there exists a timelike killing vector field ξ^a

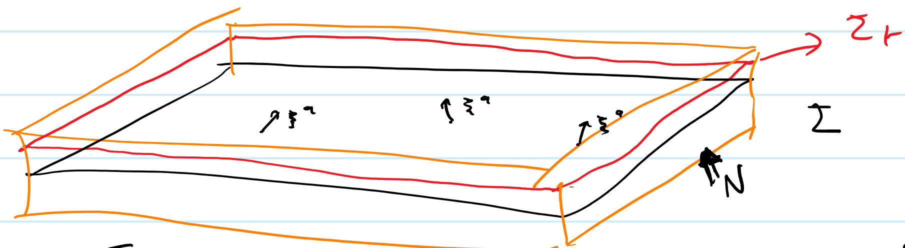
Definition: a spacetime M is *static* if, it is stationary, and there exists a spacelike hypersurface Σ which is orthogonal to the orbits of ϕ_t

Remark: this is equivalent, via Frobenius theorem, to requiring that ξ^a additionally satisfies

$$\xi_{[a} \nabla_b \xi_{c]} = 0$$

(Won't prove this here: it is motivation)

In *coordinates* we can describe these considerations



If $\xi^a \neq 0$ on Σ then, in a neighbourhood N of Σ every point of Σ will lie on a unique orbit of ξ^a. Choose coordinates $\{x^\mu\}$ on Σ and label each point

$p \in N$ by parameter t of orbit reaching p and coordinates x^μ of the point in Σ connected to p

Since this coordinate system uses Killing parameter t the metric components in this system are independent of t.
As Σ_t, the set of points reached by ϕ_t from Σ, Σ_t is also orthogonal to orbits of ϕ_t. Thus in coordinates the metric takes the form

$$(*) \quad \boxed{ds^2 = -V^2(x^1, x^2, x^3)\, dt^2 + \sum_{\mu,\nu=1}^{3} h_{\mu\nu}(x^1, x^2, x^3)\, dx^\mu dx^\nu}$$

and $V^2 = -\xi^a \xi_a$

Definition: A spacetime M is spherically symmetric if its isometry group contains a subgroup isomorphic to $SO(3)$, and the orbits of this subgroup are S^2.

The spacetime metric g_{ab} induces a metric h_{ab}, on each orbit (S^2). Because $SO(3)$ isometry, h_{ab} is a multiple of the metric of the 2-sphere, which is characterised by area A of the orbit sphere

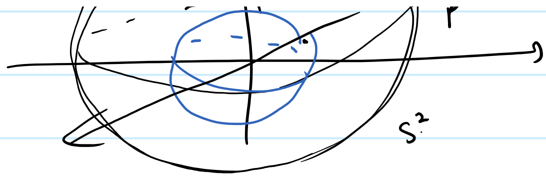
Introduce notation

$$r \equiv (A/4\pi)^{1/2}$$

Thus in spherical coordinates (θ, ϕ) the metric on each orbit sphere is

$$ds^2 = r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$$

$t = $ const.



$t = \text{const.}$

If spacetime is static as well, and assuming ξ^a is unique, the ξ^a is orthogonal to orbit spheres (Project ξ^a onto S^2 → invariant → vanishes). Thus orbit spheres lie in Σ_t.

<u>Coordinates</u>: Choose a fiducial sphere in $\Sigma = \Sigma_0$ and choose (θ, ϕ) (spherical coordinates on it). Use geodesics orthogonal to sphere carry coordinates to other spheres → gives (r, θ, ϕ) on Σ_t. Choose, using prescription for (t), coordinates on spacetime as

(t, r, θ, ϕ).

In these coordinates, our metric becomes

$$\boxed{ds^2 = -f(r)\,dt^2 + h(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)}$$

solve for these functions.