

A note on ∂_a as a covariant derivative

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Let $T_{b_1 \dots b_k}^{a_1 \dots a_k}$ be tensor of type $T(k, n)$
let ψ be chart:

$$T = \sum_{\substack{\mu_1 \dots \mu_n \\ \nu_1 \dots \nu_k}} T_{\nu_1 \dots \nu_k}^{\mu_1 \dots \mu_n} \frac{\partial}{\partial x^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_n}} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_k}$$

Define $\partial_a T_{b_1 \dots b_k}^{a_1 \dots a_k}$ to be the tensor T'
whose representation in this basis is

$$T' = \sum_{\substack{\mu_1 \dots \mu_n \\ \nu_1 \dots \nu_k}} \frac{\partial T_{\nu_1 \dots \nu_k}^{\mu_1 \dots \mu_n}}{\partial x^\sigma} \frac{\partial}{\partial x^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_n}} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_k}$$

When we change coords

$$\begin{aligned} \frac{\partial}{\partial x^\mu} &\mapsto \frac{\partial x'^{m'}}{\partial x^\mu} \frac{\partial}{\partial x^{m'}} \\ dx^\nu &\mapsto \frac{\partial x^\nu}{\partial x'^{\nu'}} dx'^{\nu'} \end{aligned}$$

Thus in new coord basis

$$T' = \sum_{\mu_1 \dots \mu_n} \sum_{\nu'_1 \dots \nu'_k} \sum_{\sigma'} \frac{\partial T_{\nu'_1 \dots \nu'_k}^{\mu_1 \dots \mu_n}}{\partial x^{\sigma'}} \frac{\partial x'^{m'}}{\partial x^\sigma} \frac{\partial x'^{m'}}{\partial x^{\mu_1}} \dots - \frac{\partial x'^{m'}}{\partial x'^{\nu'}} \frac{\partial x'^{m'}}{\partial x^{\mu_n}}$$

$$\frac{\partial x'^{m'}}{\partial x^{\mu_1}} \otimes \dots \otimes \frac{\partial x'^{m'}}{\partial x^{\mu_n}} \otimes dx'^{\nu_1} \otimes \dots \otimes dx'^{\nu_k}$$

Thus, when you change coordinates the jacobian-type factors come from the

$$\frac{\partial}{\partial x^\mu} \quad \text{and} \quad dx'^{\nu}$$

The components of the tensor itself do not transform, rather, the basis elements of $\sqrt{\rho}$ & $\sqrt{\rho}^*$ transform

Thus, the tensor T' does, as defined above, transform like a tensor. The reason why this isn't our favourite covariant derivative is that had we defined $\partial_a T$ with respect to a different chart we would get different components when we transform it to our original chart. This derivative operator is basis dependent.