

A note on ∂_a as a covariant derivative

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Let $T^{a_1 \dots a_k}_{b_1 \dots b_k}$ be tensor of type $T(k,k)$
 Let ϕ be chart:

$$T = \sum_{\substack{\mu_1 \dots \mu_k \\ \nu_1 \dots \nu_k}} T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_k} \frac{\partial}{\partial x^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_k}} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_k}$$

Define $\partial_a T^{a_1 \dots a_k}_{b_1 \dots b_k}$ to be the tensor T'
 whose representation in this basis is

$$T' = \sum_{\substack{\mu_1 \dots \mu_k \\ \nu_1 \dots \nu_k}} \frac{\partial T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_k}}{\partial x^\sigma} \frac{\partial}{\partial x^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_k}} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_k}$$

when we change coords

$$\frac{\partial}{\partial x^\sigma} \mapsto \frac{\partial x'^{\mu'}}{\partial x^\sigma} \frac{\partial}{\partial x'^{\mu'}}$$

$$dx^\nu \mapsto \frac{\partial x^\nu}{\partial x'^{\mu'}} dx'^{\mu'}$$

Thus in new coord basis

$$T' = \sum_{\mu_1 \dots \mu_k} \sum_{\nu_1 \dots \nu_k} \sum_{\sigma'} \frac{\partial T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_k}}{\partial x^{\sigma'}} \frac{\partial x'^{\mu'}}{\partial x^{\sigma'}} \frac{\partial x'^{\nu'}}{\partial x^{\mu_1}} \dots \frac{\partial x'^{\nu_k}}{\partial x^{\mu_k}}$$

$$\frac{\partial}{\partial x^{\sigma'}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_k}} \otimes dx'^{\nu_1} \otimes \dots \otimes dx'^{\nu_k}$$

Thus, when you change coordinates the jacobian-type factors come from the $\frac{\partial}{\partial x^{\mu'}}$ and $dx^{\nu'}$ terms.

The components of the tensor itself do not transform, rather, the basis elements of V_p & V_p^* transform

Thus, the tensor T' does, as defined above, transform like a tensor. The reason why this isn't our favourite covariant derivative is that had we defined $\partial_a T$ with respect to a different chart we would get different components when we transform it to our original chart. This derivative operator is basis dependent.