Introduction to general relativity: trajectories of null geodesics in Schwarzschild

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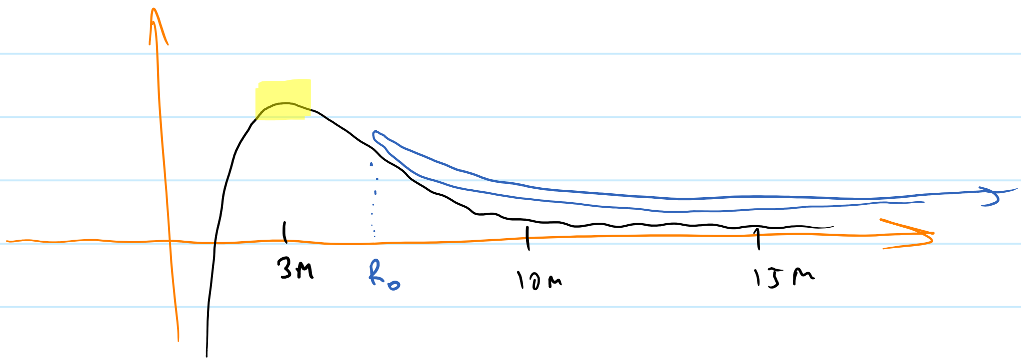
Recall : "Energy equation" for null geodesic

$$\frac{1}{2}E^2 = \frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1-\frac{2M}{r}\right)\left(\frac{L^2}{r^2}\right)$$

Effective radial potential

$$V_{eff} = \frac{L^2}{2r^3}(r-2M)$$

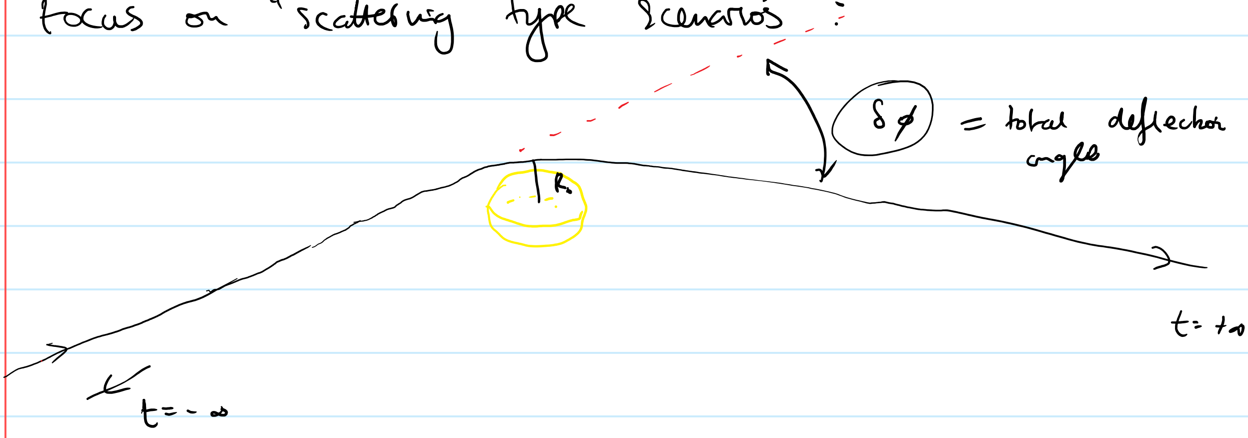


$$0 = \frac{dV}{dr} = -3\frac{L^2}{2r^4}(r-2M) + \frac{L^2}{2r^3}$$

$$0 = \frac{L^2}{2r^4}\left(-3(r-2M)+r\right)$$

Maxima is at $r = 3M$

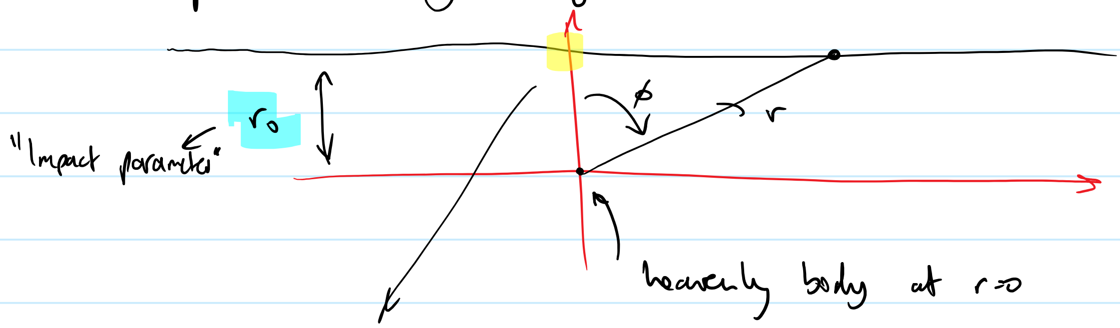
Focus on "scattering" type scenarios :



One can argue that $L/E = r^2 \frac{d\phi}{d\lambda} = r_0$

One can argue that $L/E \equiv r^2 \frac{d\phi}{dt} = r_0$

represents the distance of closest approach to centre $r=0$ of heavenly body: <u>Consider $\mu = 0$</u>

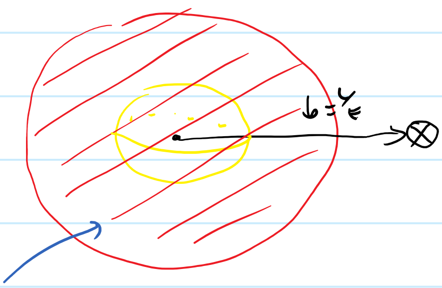


<u>Trajectory</u>: $y = r_0$ $x = t$

The Schwarzschild geometry is asymptotically flat: For a light ray very far from origin ($r \gg M$) the constant L/E as an <u>apparent impact parameter</u>

$$b = L/E$$

of the light ray.



If incoming light ray has $E^2 > V_{max} = 3M$ then it is captured.

→ Because some trajectories can reach $r=0$ we estimate the "capture radius" or, "capture cross section" of the object (ie area of target that captures geodesic) <u>red circle</u>

$$\sigma \equiv \pi b_c^2$$

To calculate b_c we need E to exceed highest potential, ie

$$\frac{1}{2}E^2 = V(r=3M) = \frac{L^2 M}{2(3M)^3}$$

Rearranging :

$$\frac{L^2}{E^2} = 27M^2 \Rightarrow$$

$$b_c = 3^{3/2} M$$

and thus

$$\sigma = 27\pi M^2$$

Trajectories with $b > b_c$ will be deflected

From lecture 22 :

$$\boxed{\frac{d\phi}{dr} = \frac{L}{r^2\sqrt{E^2 - \frac{L^2}{r^3}(r-2M)}}}$$

Goal is to calculate the total angular shift

$$\Delta\phi = \phi(\tau=+\infty) - \phi(\tau=-\infty)$$

For such trajectories turning point is at $r = R_0$

$$E^2 = \frac{L^2}{R_0^3}(R_0-2M) \Rightarrow \quad 1 = \frac{b^2}{R_0^3}(R_0-2M)$$

ie

(i) $\underline{R_0^3 - b^2(R_0-2M) = 0}$

The largest radius for which $V(R_0) = E^2/2$ is the largest zero :

$$R_0 = \frac{2b}{\sqrt{3}} \cos\left[\frac{1}{3}\cos^{-1}\left(-\frac{3^{3/2}m}{b}\right)\right]$$

Now:

$$\Delta\phi = 2\int_{R_0}^{\infty} \frac{dr}{\sqrt{r^4 b^{-2} - r(r-2m)}}$$

(symmetric about $r = R_0$)

Make substitution $u = 1/r$

$$\Delta\phi = 2\int_0^{1/R_0} \frac{du}{\sqrt{1/b^2 - u^2 + 2mu^3}}$$

$m = 0$ (flat spacetime) (ex):

$$\Delta\phi = \pi$$

What is $\Delta\phi(m)$: nontrivial because $\frac{1}{R_0}$ becomes singular at $m = 0$. Promote m & R_0 to independent variables: compare

$$\Delta\phi(m, R_0)$$

for light rays with same R_0. We eliminate b via (i)

$$\Delta\phi = 2\int_0^{1/R_0} \frac{du}{(R_0^{-2} - 2mR_0^{-3} - u^2 + 2mu^3)^{1/2}}$$

Expand

$$\Delta\phi = \underbrace{\Delta\phi(m=0)}_{\pi} + \underbrace{\frac{\partial}{\partial m}(\Delta\phi(m))\Big|_{m=0}}\cdot m + \dots$$

Now

$$\frac{\partial}{\partial m}(\Delta\phi)\Big|_{m=0} = 2\int_0^{1/R_0} \frac{(R_0^{-3} - u^3)}{(R_0^{-2} - 2mR_0^{-3} - u^2 + 2mu^3)^{3/2}}\Bigg|_{m=0}$$

$$\partial_m \Big|_{m=0} \int_0 \left(R_0^{-2} - 2mR_0^{-3} - u^2 + 2Mu^3\right)^{\frac{1}{2}} \Big|_{m=0}$$

$$= 2\int_0^{1/b} \frac{\left(b^{-3} - u^3\right)}{\left(b^{-2} - u^2\right)^{3/2}}\, du$$

$$= \frac{4}{b}$$

Here we have used fact that

$$\cos\left(\tfrac{1}{3}\cos^{-1}(x)\right) \approx \frac{\sqrt{3}}{2} + \frac{x}{6} - \frac{x^2}{12\sqrt{3}} + \cdots$$

So

$$R_0 = \frac{2b}{\sqrt{3}}\left(\frac{\sqrt{3}}{2} + \frac{1}{6}\left(\frac{-3^{3/2} m}{b}\right) - \cdots\right)$$

$$= b - 3m$$

ie $R_0 = b$ to $O(1)$ in m.

So to $O(m)$:

$$\Delta\phi \approx \pi + \frac{4m}{b}$$

Total change of angular variable throughout trajectories "lifetime"

Correction $\delta\phi$

For light ray touching edge of sun this correction is about 1.75 seconds of arc. (Confirmed in 1919)
Bending of radio waves of quasars by sun give better confirmation