Introduction to general relativity: manifolds

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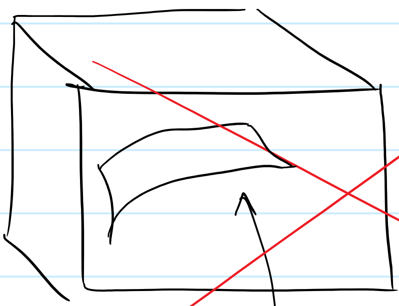


$\mathbb{R}^m$ $\quad m > 4$

↑ Ambient space

$M \equiv$ our spacetime $\qquad M \subset \mathbb{R}^m$

We discuss curved continua intrinsically

$\Rightarrow$ Manifolds

Notation: $\quad \mathbb{R}^n \equiv \{ (x^1, \dots, x^n) \mid x^j \in \mathbb{R} \}$

$$\underline{x} = (x^1, \dots, x^n)$$

$$|\underline{x} - \underline{y}| \equiv \left(\sum_{\mu=1}^{n} (x^\mu - y^\mu)^2 \right)^{1/2}$$

$$B_r(\underline{y}) = \{ \underline{x} \in \mathbb{R}^n \mid |\underline{x} - \underline{y}| < r \}$$

$\rightarrow$ open ball around $\underline{y}$

Open set U: $\quad \forall \underline{x} \in U \quad \exists \varepsilon > 0 \quad$ s.t. $B_\varepsilon(\underline{x}) \subset U$

$C^n \equiv$ set of n-times differentiable functions (on $\mathbb{R}^n$ to $\mathbb{R}$)

$C^0 \equiv$ continuous

$C^\infty \equiv$ infinitely differentiable functions

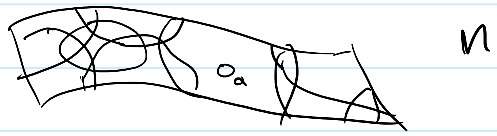
smooth

$\boxed{\text{We do things in } n \text{ dimensions}}$

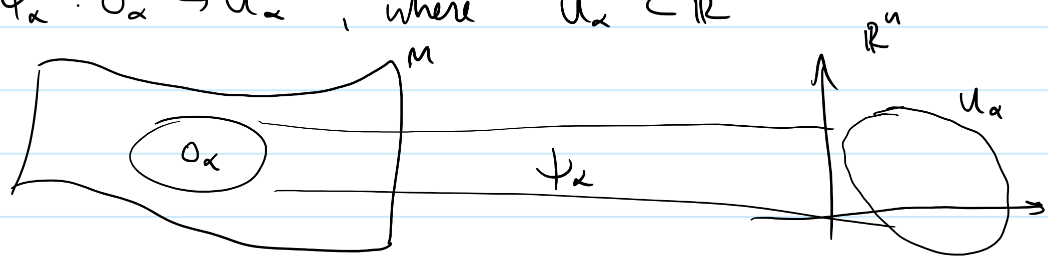
Definition: An n-dimensional, C^∞, real manifold M is a set together with a collection $\{O_\alpha\}$ of subsets satisfying

(0) M is a topological space, Hausdorff & paracompact
ψ_α are homeomorphisms. optional

(1) The set $\{O_\alpha\}$ covers M, ie $\bigcup_\alpha O_\alpha = M$. Or $\forall\, p \in M$ $\exists$ at least one α s.t. $p \in O_\alpha$



(2) For each α there is a 1-1 and onto map $\psi_\alpha : O_\alpha \to U_\alpha$, where $U_\alpha \subset \mathbb{R}^n$

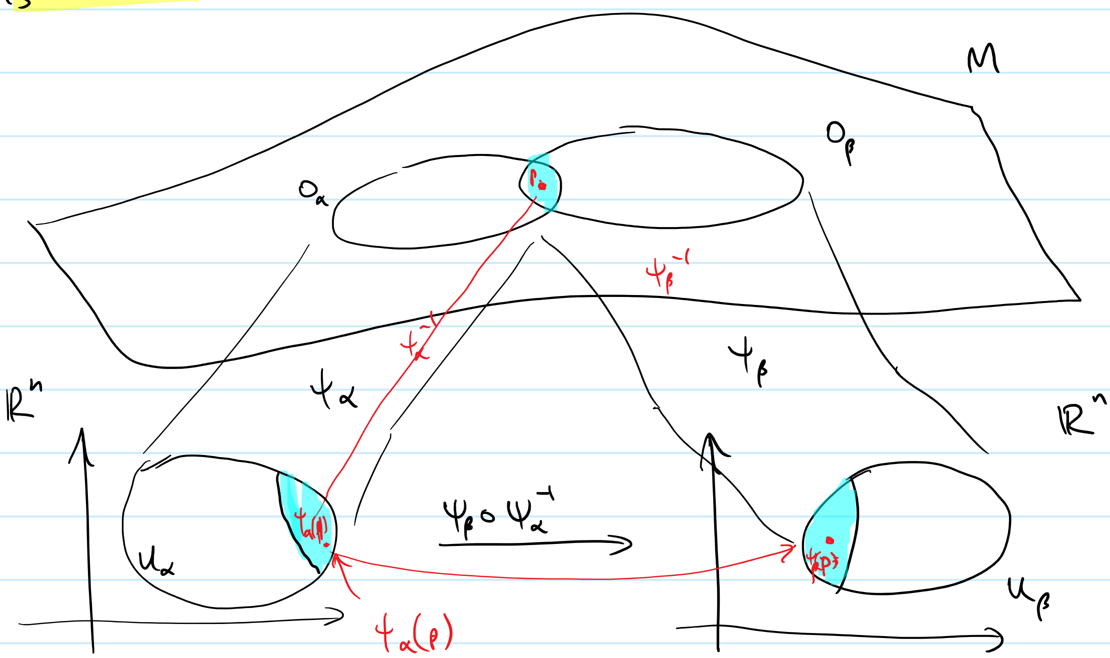


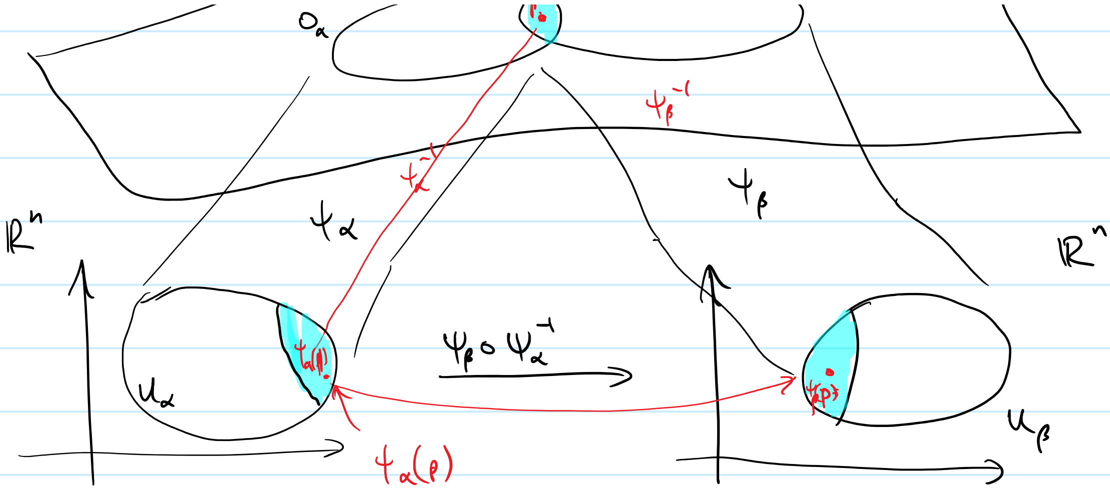
(3) If any O_α and O_β overlap, ie $O_\alpha \cap O_\beta \neq \emptyset$ then the map $\psi_\beta \circ \psi_\alpha^{-1}$ which acts via

$$\psi_\beta \circ \psi_\alpha^{-1} : \underset{\substack{\cap \\ U_\alpha}}{\psi_\alpha[O_\alpha \cap O_\beta]} \to \underset{\substack{\cap \\ U_\beta}}{\psi_\beta[O_\alpha \cap O_\beta]}$$

subset of

is C^∞

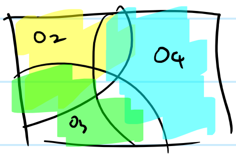




The maps ψ_α are called charts (mathematics) or coordinate systems (physics)

The definition so far depends on $\{O_\alpha\}, \{\psi_\alpha\}$: if we add a new set P & new chart ψ' $\Rightarrow$ new manifold even if P & ψ' contain no new info! To eliminate this arbitrariness, we require, that the cover $\{O_\alpha\}$ & chart family $\{\psi_\alpha\}$ is maximal. That is, all coordinate systems compatible with (2) & (3) are required included

Examples: (1) $\mathbb{R}^n$ is a trivial example, can be covered by a single chart $O_\alpha = \mathbb{R}^n$, ψ = identity ($\mathbb{R}^n$ as a manifold has uncountably many covers)



$O_1 = \mathbb{R}^n$ $\quad \psi_1$ = identity

O_2 $\quad \psi_2$ = identity

O_3

O_4

$\vdots$

(2) Spacetime $\mathbb{R}^1 \times \mathbb{R}^3 \cong \mathbb{R}^{1,3} \cong \mathbb{R}^4$

(Lorentzian structure comes later)

(3) The sphere S^n (embedded in $n+1$ dimensions)

$$S^n = \{ (x^1, \dots, x^{n+1}) \mid \sum^{n+1} (x^{\mu+1})^2 = 1 \} \subseteq \mathbb{R}^{n+1}$$

$$S^n \equiv \{ (x^1, \dots, x^{n+1}) \mid \sum_{\mu=1}^{n+1} (x^{\mu+1})^2 = 1 \} \subseteq \mathbb{R}^{n+1}$$

Let $O_\alpha^+ \equiv \{ (x^1, x^2, \dots, x^{n+1}) \in S^n \mid x^\alpha > 0 \}$

$O_\alpha^- = \{ (x^1, x^2, \dots, x^{n+1}) \in S^n \mid x^\alpha < 0 \}$

There $2(n+1)$ such sets $\Rightarrow$ cover S^n

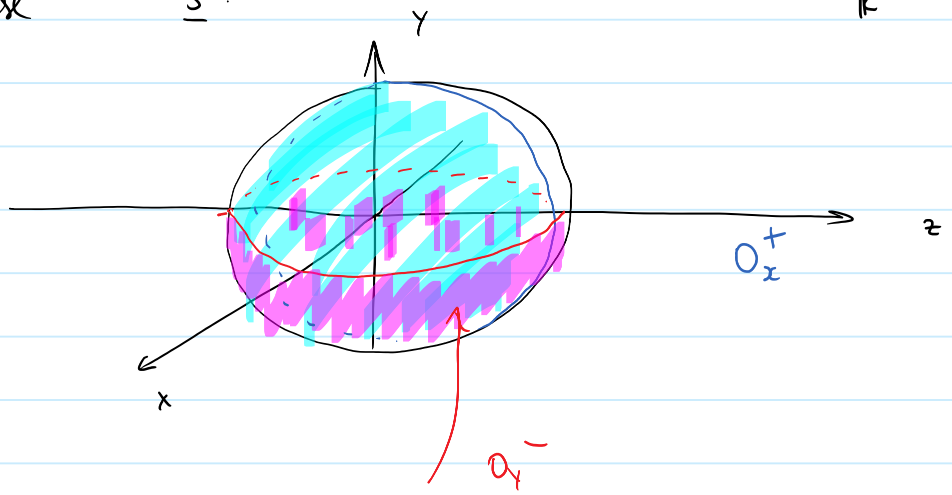
Need coordinate systems

$$\psi_\alpha^+ : O_\alpha^+ \to \mathbb{R}^n \quad \& \quad \psi_\alpha^- : O_\alpha^- \to \mathbb{R}^n$$

via

$$\psi_\alpha^\pm (x^1, \dots, x^{n+1}) = (x^1, \dots, x^{\alpha-1}, x^{\alpha+1}, \dots, x^{n+1}) \in \mathbb{R}^n$$

Special case $\underline{S^2}$.



$$\left(\psi_y^- \circ (\psi_x^+)^{-1} \right) (y, z) \overset{\text{ex}}{=} \left(\sqrt{1-y^2-z^2}, z \right)$$

$\Downarrow$ transition function goes from $U_\alpha \to U_\beta$

Ex. find the rest of these transition functions & show they are all C^∞.

Comment: do we need all this manifold stuff anyways?

YES: Cosmology, Black holes

New manifolds from old

Suppose M and M' are manifolds of dimension n & n' (respectively). We can define <u>product space</u> $M \times M'$

Suppose $\psi_\alpha : O_\alpha \to U_\alpha$ & $\psi'_\beta : O'_\beta \to U'_\beta$ are charts for M & M' (respectively). We define for $M \times M'$ via

$$\psi_{\alpha\beta} : O_{\alpha\beta} \to U_{\alpha\beta} \subset \mathbb{R}^{n+n'}$$

with

$$O_{\alpha\beta} \equiv O_\alpha \times O'_\beta \; ; \quad U_{\alpha\beta} = U_\alpha \times U'_\beta \quad \&$$

$$\psi_{\alpha\beta}(p, p') \equiv (\psi_\alpha(p), \psi'_\beta(p')) \in \mathbb{R}^{n+n'} \quad \forall (p,p') \in M \times M'$$

<u>Ex</u> prove this is a manifold

<u>Ex</u>, realise $\mathbb{R}^n = \mathbb{R}^1 \times \mathbb{R}^1 \times \cdots \times \mathbb{R}^1$ in this way

With just $\mathbb{R}$ & S^n and their products, one can build many relevant manifolds for GR

<u>Torus</u>: $S^1 \times S^1$ etc etc

Endow class of manifolds with <u>category structure</u> by describing <u>morphisms</u> between them

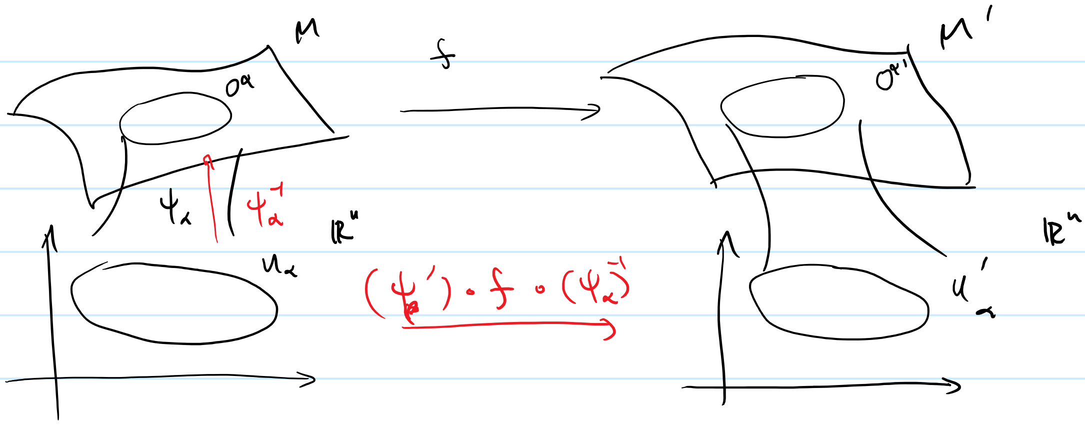
Let M and M' be manifolds and let $\{\psi_\alpha\}$ & $\{\psi'_\beta\}$ be their charts. A map

$$f : M \to M'$$

is said to C^∞ (smooth) if $\forall \alpha, \beta$ the map

$$\psi'_\beta \circ f \circ (\psi_\alpha^{-1}) : U_\alpha \to U'_\alpha$$

is C^∞ (ie smooth)



When are two manifolds the <u>same</u>?

If $f: M \to M'$ is C^∞, one-to-one, and onto, <u>and</u> has C^∞ <u>inverse</u>, then it is called a <u>diffeomorphism</u>.

The manifolds M and M' are said to be <u>diffeomorphic</u>
The manifolds M and M' have identical manifold structure.

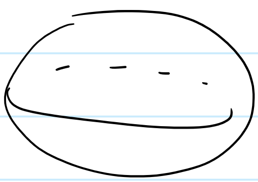
Vectors

<u>Euclidean</u> (& Minkowski) space has a natural vector <u>space structure</u>: ie

$$V = \mathbb{R}^n$$

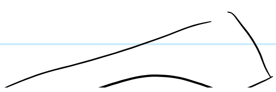
is <u>both</u> a <u>manifold</u> & <u>vector space</u> (globally). For general manifolds one <u>loses</u> the <u>vector space</u> <u>structure</u> (no natural additive structure).

<u>Example</u> S^2



is not a vector space

It is, however, still possible to associate vector <u>spaces</u> to (embedded) manifolds.



$S^2 \subset \mathbb{R}^3$

$S^2 \subset \mathbb{R}^3$

Tangent space at p

Problem: how to define things intrinsically? ! ?

Answer: first identify vector space of tangent vectors with vector space of directional derivatives, then we directional derivatives intrinsically.

Definition: A locally Euclidean space M of dimension d is a Hausdorff topological space M for which each point has a neighbourhood homeomorphic to open subset of $\mathbb{R}^d$

Definition: A differentiable structure $\mathcal{F}$ of class C^k $(1 \leq k \leq \infty)$ on a locally Euclidean space M is a collection coordinate systems $\{(O_\alpha, \psi_\alpha) \mid \alpha \in A\}$ s.t.

(a) $\bigcup_{\alpha \in A} O_\alpha = M$

(b) $\psi_\alpha \circ \psi_\beta^{-1}$ is C^k for all $\alpha, \beta \in A$

(c) the collection $\mathcal{F}$ is maximal w.r.t (b)

A d-dimensional differentiable manifold of class C^k (C^k can generalised to C^∞, complex, etc) is a pair $(M, \mathcal{F})$ of a d-dimensional, second countable, locally Euclidean, space M together with a differentiable structure $\mathcal{F}$ of class C^k.