Introduction to general relativity: tangent space

19 April 2021 12:16

Directional derivatives as vectors

Let $(v^1, v^2, \dots, v^n) \in \mathbb{R}^n$. This defines a directional derivative at a point $p \in \mathbb{R}^n$ as follows. Suppose f is C^1 function from $\mathbb{R}^n \to \mathbb{R}$ and define

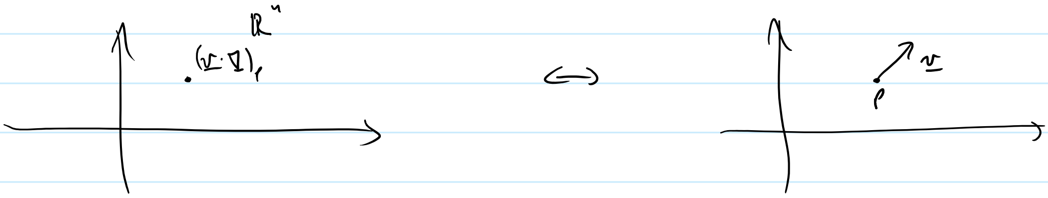
$$\left(\underline{v} \cdot \underline{\nabla} \equiv \sum_{\mu=1}^{n} v^\mu \frac{\partial}{\partial x^\mu} \right) \qquad \boxed{v(f) \equiv (\underline{v} \cdot \underline{\nabla}) f(x) \Big|_{\underline{x} = p}}$$

$$= \sum_{\mu=1}^{n} v^\mu \frac{\partial f}{\partial x^\mu}(p)$$

directional derivative @ p.

Conversely, given $(\underline{v} \cdot \underline{\nabla})_p$ we obtain a vector in $\mathbb{R}^n$

Further directional derivatives at p form a vector space



Note: $(\underline{v} \cdot \underline{\nabla})_p$ satisfies

(1) $v(af + bg) = a\, v(f) + b\, v(f)$ $\forall f, g$ functions & $\forall a, b \in \mathbb{R}$

(2) $v(fg) = f(p)\, v(g) + g(p)\, v(f)$ (Leibniz property)

Notation: $\mathcal{F}(M)$ set of all C^∞ functions from M to $\mathbb{R}$

Idea: define a vector space V_p associated to point $p \in M$ to be set of all maps $v : \mathcal{F}(M) \to \mathbb{R}$ which obey (1) & (2)

Ex: (a) Convince yourself if $h \in \mathcal{F}(M)$ is constant then $v(h) = 0$, using only (1) & (2)

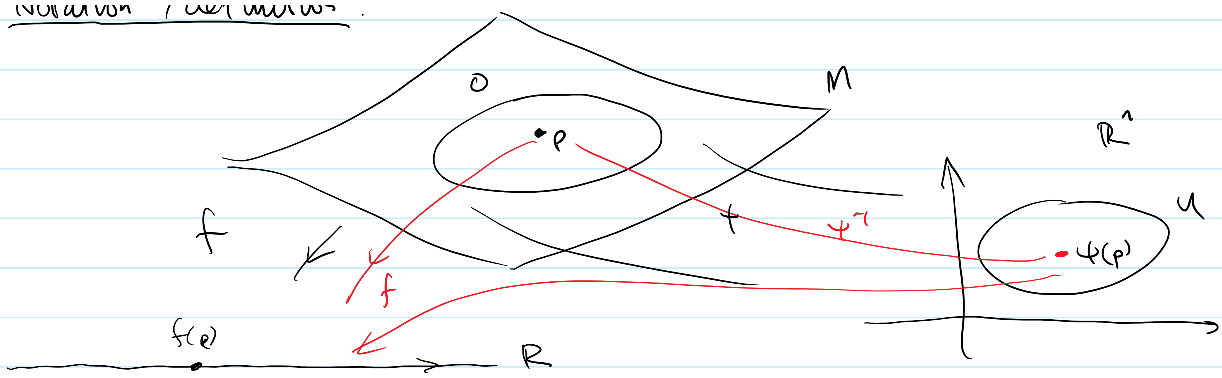
(b) Prove V_p is a vector space

Have we created a monster ?! (eg $\dim(V_p) = \infty$!)

Theorem Let M be an n-dimensional (real smooth) manifold. Let $p \in M$ and let V_p denote "tangent space at p" above. Then $\dim(V_p) = n$

Notation / definition:

Notation / definitions:



Let $\psi : O \to U \subset \mathbb{R}^n$ be a chart with $p \in O$

If $f \in \mathcal{F}(M)$ then $f \circ \psi^{-1} : U \subset \mathbb{R}^n \to \mathbb{R}$ is C^∞

Define for $\mu = 1, \dots, n$

X capital X

x lower case x

$$X_\mu(f) = \frac{\partial}{\partial x^\mu} \left(f \circ \psi^{-1} \right) \Big|_{\psi(p)}$$

where $(x^1, \dots, x^n)$ are the coordinates of $\mathbb{R}^n$

Ex X_μ so defined, are tangent vectors (ie elements of V_p)

Proof: Now suppose $F : \mathbb{R}^n \to \mathbb{R}$ is C^∞, then for all $\underline{a} = (a^1, \dots, a^n)$ $\exists$ C^∞ functions $H_\mu : \mathbb{R}^n \to \mathbb{R}$ s.t. $\forall \underline{x} \in \mathbb{R}^n$

$$(*) \quad F(\underline{x}) = F(\underline{a}) + \sum_{\mu=1}^{n} (x^\mu - a^\mu) H_\mu(\underline{x})$$

with $H_\mu(\underline{a}) = \frac{\partial F}{\partial x^\mu} \Big|_{\underline{x} = \underline{a}}$

(HW see bonus problem: prove this statement)

Let $F(\underline{x}) = (f \circ \psi^{-1})(\underline{x})$ and $\underline{a} = \psi(p)$. Then by $(*)$ we have for all $q \in O$

$$f(q) = f(p) + \sum_{\mu=1}^{n} \Big((x^\mu \circ \psi)(q) - (x^\mu \circ \psi)(p) \Big) H_\mu(\psi(q)) \quad (**)$$

Suppose $v \in V_p$. Apply v to f

$$v(f) \overset{(**)}{=} v\Big(f(p) + \sum_{\mu=1}^{n} (\cdots) H_\mu(\psi(q)) \Big)$$

linearity

$$\overset{\text{Leibniz}}{=} \underbrace{v(f(p))}_{0} + \sum_{\mu=1}^{n} \Big((x^\mu \circ \psi)(q) - \cancel{(x^\mu \circ \psi)(p)} \Big) \Big|_{\substack{x^\mu = a^\mu \\ q = p}} v\big(H_\mu(\psi(q)) \big)$$

$$+ (H_\mu \circ \psi)\big|_p \; v\big(x^\mu \circ \psi - x^\mu \cancel{\circ \psi(p)} \big)$$

$$= 0 + 0 + \sum_{\mu=1}^{n} (H_\mu \circ \psi)(p) \cdot v(x^\mu \circ \psi)$$

v^μ

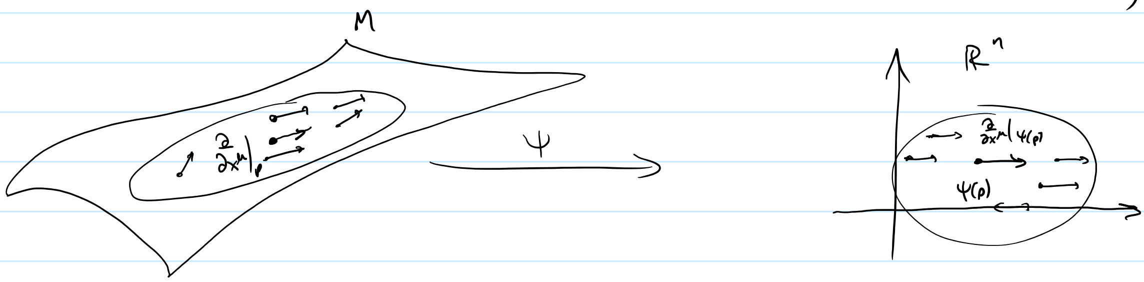
(Remember $H_\mu(a) = \left.\frac{\partial F}{\partial x^\mu}\right|_{x=a}$)

$$v(f) = \sum_{\mu=1}^{n} \underbrace{\left.\frac{\partial (f \circ \psi^{-1})}{\partial x^\mu}\right|_{\psi(p)}}_{X_\mu(f)} \; v^\mu$$

$$\Rightarrow \quad v(f) = \sum_{\mu=1}^{n} v^\mu X_\mu(f) . \qquad \square$$

The basis $\{X_\mu \mid \mu = 1, \dots, n\}$ of V_p is called the coordinate basis, often denoted

$$X_\mu = \left.\frac{\partial}{\partial x^\mu}\right|_p \; ; \left(\text{or } \partial_\mu|_p , \; \frac{\partial}{\partial x^\mu}, \dots \; e_\mu \right)$$



Suppose we had chosen a different chart ψ'. We would have coord. basis $\{X'_\nu\}$.
Chain rule (ex)

$$X_\mu = \sum_{\nu=1}^{n} \left.\frac{\partial x'^\nu}{\partial x^\mu}\right|_{\psi(p)} X'_\nu$$

where x'^ν denotes νth component of $\psi' \circ \psi^{-1}$

Given a tangent vector in basis X_μ:

$$v = \sum_{\mu=1}^{n} v^\mu X_\mu = \sum_{\mu,\nu} v^\mu \left.\frac{\partial x'^\nu}{\partial x^\mu}\right|_{\psi(p)} X'_\nu$$

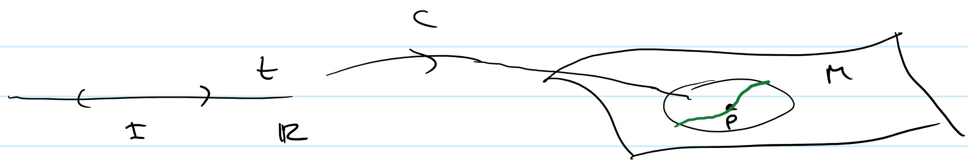
$$\Rightarrow \quad \boxed{v'^\nu = \sum_{\mu=1}^{n} v^\mu \left.\frac{\partial x'^\nu}{\partial x^\mu}\right|_{\psi(p)}}$$

Vector transformation law.

Vector transformation law.

Mechanics on manifolds

Definition: A smooth curve C on a manifold M is a C^∞ map $C : \mathbb{R} \to M$ (or an interval $I \subset \mathbb{R}$)
$C : I \to M$



To each $p \in M$ on C we associate a tangent vector $T \in V_p$ as follows: Set, for $f \in \mathcal{F}(M)$

$$T(f) \equiv \frac{d}{dt}(f \circ C)\Big|_p = \sum_\mu \frac{\partial}{\partial x^\mu}(f \circ \psi^{-1})\Big|_{\psi(p)} \frac{dx^\mu}{dt} = \sum_\mu \frac{dx^\mu}{dt} X_\mu(f)$$

where $\underline{x^\mu \circ (\psi \circ C) \equiv x^\mu(t)}$.

This expansion works for any coord. basis. Components of $T^\mu = \frac{dx^\mu}{dt}$.

We call V_p the tangent space at p.

$$\bigcup_p V_p \equiv TM \quad \text{tangent space for } M.$$

Warning: although $\dim(V_p) = \dim(V_q)$ $\forall p, q \in M$, and thus $V_p \cong V_q$, these isomorphisms are not natural. No standard way to choose this isomorphism. $\Rightarrow$ Isomorphisms could be wild. To get "good" choice need extra data!

Definition: a tangent field v on a manifold M is an assignment of a tangent vector $v|_p \in V_p$, $\forall p \in M$. We say v is smooth if $\forall f \in \mathcal{F}(M)$, $v(f)$ is a C^∞ function

Lemma: The coord. basis fields X_μ are smooth.
Proof. $X_\mu(f)(p) = \frac{\partial}{\partial x^\mu}(f \circ \psi^{-1})\Big|_{\psi(p)}$ is C^∞ function of p.

Since an arbitrary tangent vector v is a linear combination of X_μ: v is smooth $\Leftrightarrow$ its components $v^\mu \in \mathcal{F}(M)$

Velocity field v is tangent vector field

Solution to equations of motion is a smooth curve C with

$$T(f) = v(f).$$