Introduction to general relativity: parallel transport continued

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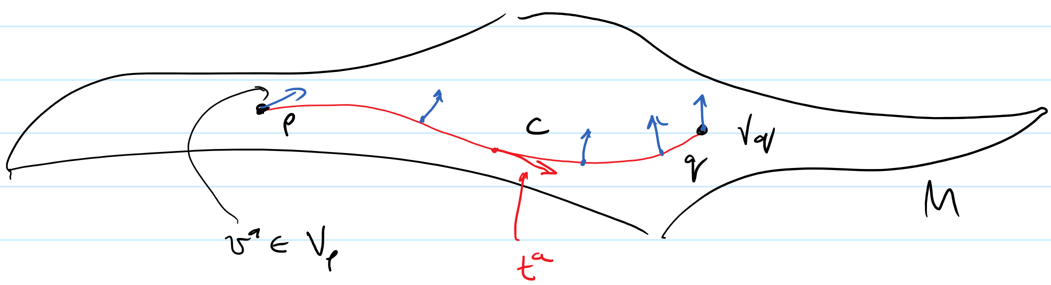
Let M be a manifold & ∇_a is a derivative operator



$\mathcal{U}_C : V_p \to V_q$

Definition: Let C be a curve in M with tangent vector t^a. A vector v^a given at each point in C is said to be parallelly transported along C if

$$\boxed{t^a \nabla_a v^b = 0}$$

$\forall p \in C$. Similarly, if $T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$ is a tensor along C we demand

$$\boxed{t^a \nabla_a T^{a_1 \cdots a_k}{}_{c_1 \cdots c_l} = 0}$$

In the case $\nabla_a \equiv \partial_a$, we have in the coordinate chart defining ∂_a:

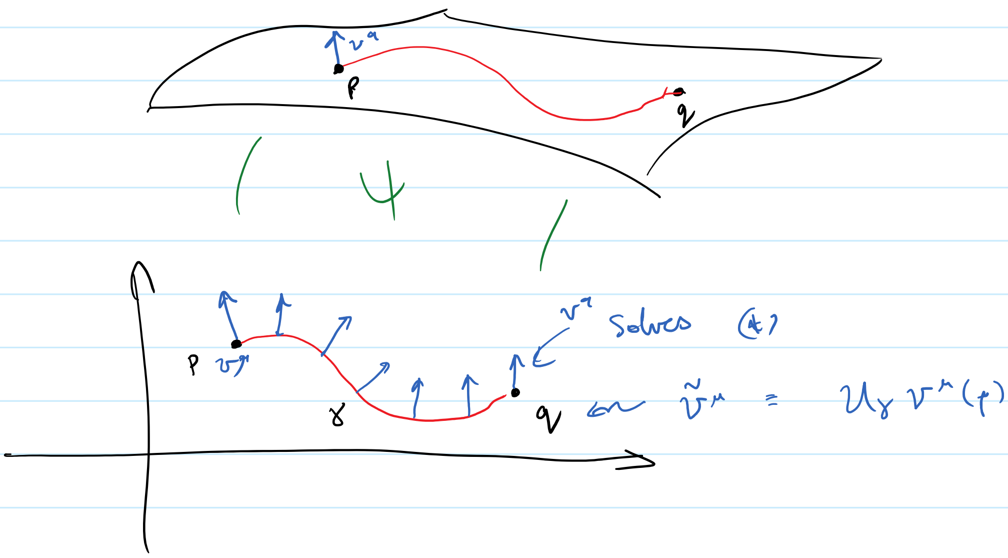
$$\underbrace{\frac{dx^\mu}{dt}}_{t^a}\underbrace{\frac{\partial v^\nu}{\partial x^\mu}}_{\partial_a} + \frac{dx^\mu}{dt}\Gamma^\nu_{\mu\lambda} v^\lambda = 0$$

(4)
$$\boxed{\frac{dv^\nu}{dt} + \frac{dx^\mu}{dt}\Gamma^\nu_{\mu\lambda} v^\lambda = 0}$$

Parallel transport only depends on v^a on the curve C:

we don't need that v^a is a vector field

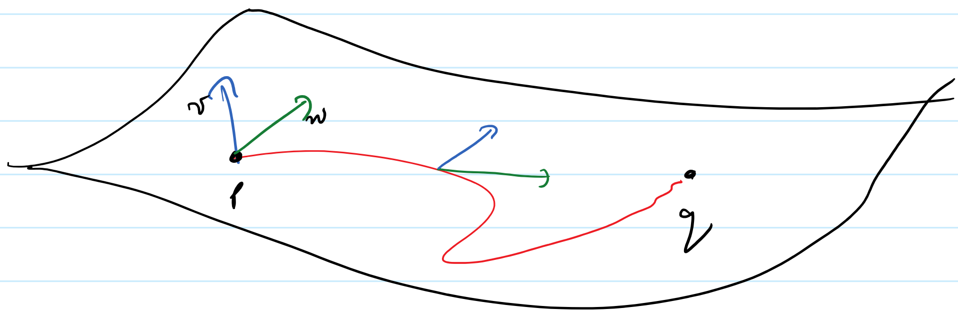
(*) is a set of nonlinear ODEs. Existence & uniqueness results for ODEs guarantee solution of (*) for a given initial v^a. Given p & $q \in M$ & curve γ connecting them we build a "parallel transporter" $\mathcal{U}_\gamma$, (also known as a connection)



If we have data of a metric g_{ab}, we can make a canonical choice of ∇_a

metric $\longrightarrow$ length $\Leftarrow$

metric $\longrightarrow$ inner product $(v,w)_p$



Requiring $(v,w)_p \equiv g_{ab}v^a w^b$ is preserved during parallel transport determines canonical choice of ∇_a: $v^a, w^a \in V_p$; $t^a \nabla_a v^b = t^a \nabla_a w^b = 0$

$$(t^a \nabla_a)(\underbrace{g_{bc} v^b w^c}_{f(p)}) = 0$$

⇒ Use Leibniz:

$$0 = (t^a \nabla_a g_{bc})(v^b w^c) + g_{bc}\,\cancel{(t^a \nabla_a v^b)}\, w^c + g_{bc} v^b \cancel{(t^a \nabla_a w^c)}$$

⇒ we demand $v^b w^c t^a \nabla_a g_{bc} = 0$

for all v^b, w^c, $\forall$ curves, $\forall\, t^a$

$$\Rightarrow \quad (**) \quad \boxed{\nabla_a g_{bc} = 0}$$

there is a family of ∇_as, which are constrained by choice g_{bc}

Theorem: Let g_{ab} be metric. Then there exists a unique derivative operator ∇_a satisfying $\nabla_a g_{bc} = 0$

Proof: Suppose $\tilde{\nabla}_a$ is a derivative operator, eg. ∂_a

impose $(**)$:

$$0 = \nabla_a g_{bc} \overset{\text{def}^n \text{ of } \nabla_a \text{ on tensors}}{=} \tilde{\nabla}_a g_{bc} - C^d{}_{ab}\, g_{dc} - C^d{}_{ac}\, g_{bd}$$

So that:

$$\begin{aligned} \tilde{\nabla}_a g_{bc} &= C_{cab} + C_{bac} \\ + \quad \tilde{\nabla}_b g_{ac} &= C_{cba} + C_{abc} \qquad a \to b \quad b \to a \\ - \quad \tilde{\nabla}_c g_{ab} &= C_{bca} + C_{acb} \qquad c \to b,\ b \to a,\ a \to c \end{aligned}$$

$$\tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} - \tilde{\nabla}_c g_{ab} = C_{cab} + \cancel{C_{bac}} + C_{cba} + \cancel{C_{abc}} - \cancel{C_{bca}} - \cancel{C_{acb}}$$

$$= 2\, C_{cab}$$

⇒ Contract both sides with g^{cd}:

$$g^{cd} C_{dab} = \frac{1}{2} g^{cd} \left(\tilde{\nabla}_a g_{bd} + \tilde{\nabla}_b g_{ad} - \tilde{\nabla}_d g_{ab} \right)$$

$$C^c{}_{ab} = \frac{1}{2} g^{cd} \left(\tilde{\nabla}_a g_{bd} + \tilde{\nabla}_b g_{ad} - \tilde{\nabla}_d g_{ab} \right)$$

$\Rightarrow$ Solution is uniquely determined by $\hat{\nabla}_a$ & g_{ab}.

$\square$

Thus g_{ab} determines ∇_a. From now on, when we have a metric g_{ab} we choose ∇_a to be this operator

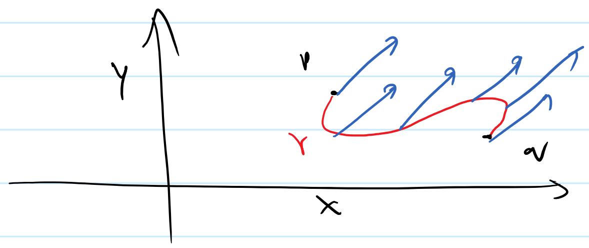
In the specific case where $\tilde{\nabla}_a = \partial_a$, the <u>Christoffel symbols</u> are

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} \{ \partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \}$$

which, in coordinate chart ψ determining ∂_a:

$$\boxed{\Gamma^{\rho}_{\nu\mu} = \frac{1}{2} \sum_{\sigma} g^{\rho\sigma} \left\{ \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right\}} \quad (\star\star\star)$$

<u>Example</u>: Let $M = \mathbb{R}^2$. Choose a chart ψ furnishing the usual Cartesian coordinate system (x, y).



Let v be a vector in this coordinate system:

$$v = \sum_{\mu = x, y} v^{\mu} \left. \frac{\partial}{\partial x^{\mu}} \right|_p$$

Let $g_{\mu\nu} = dx^{\mu} \otimes dx^{\mu} = dx \otimes dx + dy \otimes dy$
(ex: $\Gamma^{\lambda}_{\mu\nu} = 0$)

Choose $\tilde{\nabla}_a \equiv \partial_a$ w.r.t Cartesian coordinates. Hence $\tilde{\nabla}_a v$ has components at $(x, y) = \psi(p)$

$$\frac{\partial v^{\mu}}{\partial x^{\nu}} (x, y)$$

<u>Demanding</u> $t^a \tilde{\nabla}_a v = 0 \Rightarrow$

$$\boxed{t^{\nu} \frac{\partial}{\partial x^{\nu}} v^{\mu} = 0}$$

$\Rightarrow$ if v_p is a tangent vector at p, the parallel transported vector at q: $\mathcal{U}_\gamma v_p = \tilde{v}_q$ has components at $(x', y') = \phi(q)$

$$\tilde{v}^\mu(x', y') = v^\mu(x, y)$$

Usual notion of parallel transport

Let's now go to another chart, in this case polar coordinates.

$$(x, y) = (r\cos\phi,\ r\sin\phi)$$

<u>Metric transforms as</u> $\quad dx = \left(\frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \phi} d\phi\right);\ dy = \left(\frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \phi} d\phi\right)$

$$\underset{(ex)}{g} = dx \otimes dx + dy \otimes dy =$$

$$= dr \otimes dr + r^2 d\phi \otimes d\phi = g'_{\mu'\nu'} dx'^{\mu'} \otimes dx'^{\nu'}$$

$$\Rightarrow \quad g'_{rr} = 1 \qquad g'_{\phi r} = g'_{r\phi} = 0 \qquad g'_{\phi\phi} = r^2$$

$$\underset{\sim}{g}' = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}; \qquad (\underset{\sim}{g}')^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$$

(Ex)

$$\Gamma^r_{rr} = 0, \quad \Gamma^r_{r\phi} = 0, \quad \Gamma^\phi_{rr} = 0, \quad \Gamma^\phi_{r\phi} = ?$$

Apply ~~[illegible]~~:

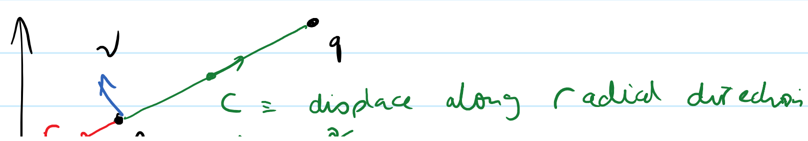
$$\Gamma^\phi_{r\phi} = \frac{1}{2} g'^{\phi\phi} \left\{ \frac{\partial}{\partial r} \underbrace{g'_{\phi\phi}}_{r^2} + \cancel{\frac{\partial}{\partial \phi} g'_{r\phi}} - \cancel{\frac{\partial g'_{r\phi}}{\partial \phi}} \right\}$$

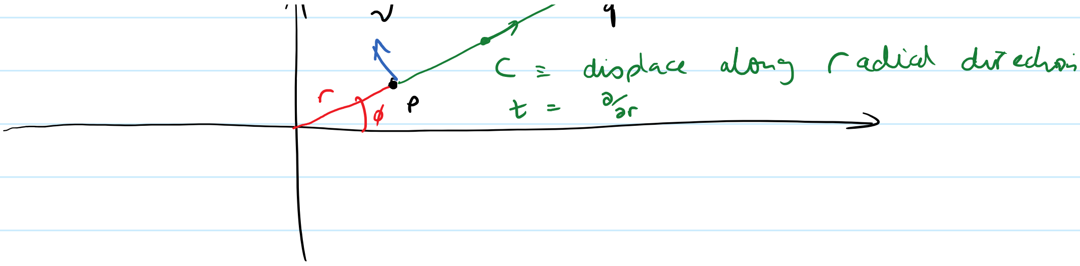
$$= \frac{1}{2} \frac{1}{r^2} \left\{ \frac{\partial}{\partial r} r^2 \right\} = \frac{1}{r}$$

Similarly (ex):

$$\Gamma^r_{\phi r} = 0, \quad \Gamma^r_{\phi\phi} = -r, \quad \Gamma^\phi_{\phi r} = \frac{1}{r}, \quad \Gamma^\phi_{\phi\phi} = 0$$

<u>Parallel transport condition</u>:





$$t = \frac{\partial}{\partial r} = t^{\mu'} \frac{\partial}{\partial x'^{\mu'}} \rightarrow t^{\phi} = 0 \quad t^{r} = 1$$

Parallel transport condition

$$t^a \nabla_a v^b = 0$$

$\Downarrow$ (ex)

(i) $$\frac{dv^r}{dr} + \cancel{\Gamma^r_{rr}} v^r + \cancel{\Gamma^r_{r\phi}} v^\phi = 0 \Rightarrow \frac{dv^r}{dr} = 0$$

(ii) $$\frac{dv^\phi}{dr} + \frac{1}{r} v^\phi = 0 \qquad \frac{dv^\phi}{dr} = -\frac{1}{r} v^\phi$$

$$\int \frac{1}{v^\phi} dv^\phi = -\int \frac{1}{r} dr$$

$$\log v^\phi = -\log(r) + \text{Const.}$$

$$v^\phi = \frac{A}{r}$$

$A \equiv$ const. determined by ICs

Suppose ∇_X is an affine connection. Let ω be a tensor of type $(0,1)$ (a "1"-form)

The object

$$\nabla_X \omega \in \mathcal{T}(0,1)$$

Let ψ be a chart for a coordinate system. Choose

$$X = \frac{\partial}{\partial x^\mu}\Big|_p \qquad \omega = dx^\nu$$

$$\nabla_{\frac{\partial}{\partial x^\mu}} (dx^\nu) = -\Gamma_{\mu\lambda}^{\nu} dx^\lambda$$

For each choice of $\mu, \nu \Rightarrow$ we get a 1-form (or tensor of type $(0,1)$). $\rightarrow$ connection 1-forms

Transformation properties of connection coefficients

Let $\tilde{\nabla}_a = \partial_a$ for a chart (U, ψ). Let (V, ψ') be another chart such that $U \cap V \neq \emptyset$ with coords $y = \psi'(p)$.

$$\nabla_a v^b = \partial_a v^b + \Gamma_{ac}^{b} v^c$$

Note:

$$\Gamma_{\alpha\beta}^{\gamma} \frac{\partial}{\partial x^\gamma} = \nabla_{\frac{\partial}{\partial x^\alpha}} \left(\frac{\partial}{\partial x^\beta} \right)$$

Write this out in coord. system ψ'

Define:

$$f_a = \frac{\partial}{\partial y^\alpha} = \left(\frac{\partial x^\mu}{\partial y^\alpha} \underbrace{\frac{\partial}{\partial x^\mu}}_{\text{old basis}} \right)$$

Define/Note:

$$\nabla_{\frac{\partial}{\partial y^\alpha}} \left(\frac{\partial}{\partial y^\beta} \right) = \tilde{\Gamma}_{\alpha\beta}^{\gamma} \frac{\partial}{\partial y^\beta}$$

$$= \frac{\partial^2 x^\mu}{\partial y^\alpha \partial y^\beta} \frac{\partial}{\partial x^\mu} + \frac{\partial x^\lambda}{\partial y^\alpha} \frac{\partial x^\mu}{\partial y^\beta} \nabla_{\frac{\partial}{\partial x^\lambda}} \left(\frac{\partial}{\partial x^\mu} \right)$$

$$= \left(\frac{\partial^2 x^\mu}{\partial y^\alpha \partial y^\beta} + \frac{\partial x^\lambda}{\partial y^\alpha} \frac{\partial x^\mu}{\partial y^\beta} \Gamma_{\lambda\mu}^{\nu} \right) \frac{\partial}{\partial x^\nu}$$

Thus:

$$\boxed{\tilde{\Gamma}_{\alpha\beta}^{\gamma} = \frac{\partial x^\lambda}{\partial y^\alpha} \frac{\partial x^\mu}{\partial y^\beta} \frac{\partial y^\gamma}{\partial x^\nu} \Gamma_{\lambda\mu}^{\nu} + \frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta} \frac{\partial y^\gamma}{\partial x^\nu}}$$

When Γ transforms in this way we ensure

$$\nabla_X Y \text{ is a vector}$$

where

$$X = X^\mu \frac{\partial}{\partial x^\mu} \qquad Y = Y^\mu \frac{\partial}{\partial x^\mu}$$

$$X = \tilde{X}^\alpha \frac{\partial}{\partial y^\alpha} \qquad Y = Y^\alpha \frac{\partial}{\partial y^\alpha}$$

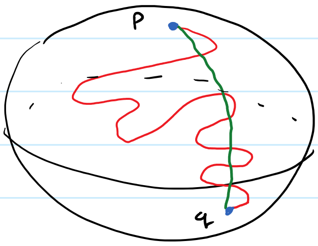
Ex: Show

$$X^a \nabla_{\frac{\partial}{\partial x^\alpha}} \left(Y^\beta \frac{\partial}{\partial x^\beta} \right) = \nabla_X Y \overset{!}{=} \tilde{X}^\alpha \nabla_{\frac{\partial}{\partial y^\alpha}} \left(\tilde{Y}^\beta \frac{\partial}{\partial y^\beta} \right)$$

Geodesics: Let (M, g_{ab}) be a manifold with metric.

Intuitively:

A geodesic is the "straightest possible" curve one can trace through M



Definition: A geodesic is a curve whose tangent vector T^a is a parallel-transported along itself, ie it is a curve satisfying

$$\boxed{T^a \nabla_a T^b = 0}$$

Choose coordinate system ψ: the geodesic is a curve

$$x^\mu(t) \text{ in } \mathbb{R}^n$$

According to (*):

$$\frac{dT^\mu}{dt} + \sum_{\sigma,\nu} \Gamma^\mu_{\sigma\nu} T^\sigma T^\nu = 0$$

Since $T^\mu \equiv \frac{dx^\mu}{dt}$ we have geodesic equation

$$\boxed{\frac{d^2 x^\mu}{dt^2} + \sum_{\sigma,\nu} \Gamma^\mu_{\sigma\nu} \frac{dx^\sigma}{dt} \frac{dx^\nu}{dt} = 0}$$

$\Rightarrow$ Coupled system of n ODEs.