Introduction to general relativity: lie derivatives & Newtonian limit

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For diffeomorphism $\phi: M \to N$, then M and N have identical manifold structure. Since GR describes nature in terms of a manifold M and some tensor fields $T^{(j)}$ then $(M, T^{(j)})$ and $(N, \phi^* T^{(j)})$ have physically identical properties. This is because operational quantities are built from $T^{(j)}$ in a coordinate-free manner: Jacobian factors cancel under contraction / tensor transformations

In this way: diffeomorphisms comprise the gauge freedom of GR

Infinitesimal diffeomorphisms $\to$

Lie derivatives

Let M be a manifold and ϕ_t a one-parameter group of diffeomorphisms. Such a flow ϕ_t is generated by a vector field v. We can use ϕ_t^* on tensors to define a derivative-type operator — the Lie derivative — on tensors with respect to v:

$$\mathcal{L}_v T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} = \lim_{t \to 0} \left\{ \frac{\phi_{-t}^* T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} - T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}}{t} \right\}$$

evaluated at $\phi_t(p)$ is evaluated at p.

Ex: Check $\mathcal{L}_v T$ is a (tensor field); $\mathcal{L}_v$ obeys Leibniz property, and is linear

Remark: for $f \in \mathcal{F}(M)$:

$$\mathcal{L}_v f = v(f) \qquad (*)$$

Homework: Argue that

$$\mathcal{L}_v w^a = [v, w]^a \qquad (**)$$

Now note that $\forall \mu \in V_p^*$ and $w \in V_p$

true for all ∇_a !

$$\mathcal{L}_v(\mu_a w^a) \overset{(*)}{=} v(\mu_a w^a) = v^b \nabla_b(\mu_a w^a)$$

But Leibniz and

$$\mathcal{L}_v(\mu_a w^a) = w^a \mathcal{L}_v(\mu_a) + \mu_a [v,w]^a \quad (**)\ 2$$

So we find, noting:

$$v(\mu_a w^a) = v^b w^a \nabla_b \mu_a + v^b \mu_a \nabla_b w^a \quad \text{and} \quad [v,w]^a = v^b \nabla_b w^a - w^b \nabla_b v^a:$$

that

$$\mathcal{L}_v \mu_a = v^b \nabla_b \mu_a + \mu_b \nabla_a v^b$$

So for general T (ex):

$$\boxed{\mathcal{L}_v T^{a_1 \dots a_k}{}_{b_1 \dots b_l} = v^c \nabla_c T^{a_1 \dots a_k}{}_{b_1 \dots b_l} - \sum_{j=1}^{k} T^{a_1 \dots c \dots a_k}{}_{b_1 \dots b_l} \nabla_c v^{a_j} + \sum_{j=1}^{l} T^{a_1 \dots a_k}{}_{b_1 \dots c \dots b_l} \nabla_{b_j} v^c} \quad (***)$$

If $\phi_\lambda : M \to M$ is a one-parameter group of diffeomorphisms, then (M, g_{ab}) and $(M, \phi_\lambda^* g_{ab})$ represent same spacetime.
If, further, we allow g_{ab} to depend (smoothly) on parameter λ: we can ask what perturbations around $g_{ab}(0)$ ($\equiv g_{ab}$)

$$\gamma_{ab} = \frac{d}{d\lambda} g_{ab}(\lambda) \Big|_{\lambda=0}$$

are actually just infinitesimal diffeomorphisms, ie perturbations of form

$$^{*} g_{ab}))\Big|_{\lambda=0}$$

Ex: (writing out flow ϕ_λ^*) we see that

$$\gamma'_{ab} = \gamma_{ab} - \mathcal{L}_v g_{ab}$$

where v is the vector field generating flow ϕ_λ. By applying (***)

$$\mathcal{L}_v g_{ab} = v^c \overset{0}{\cancel{\nabla_c g_{ab}}} + g_{cb} \nabla_a v^c + g_{ac} \nabla_b v^c$$

$$= \nabla_a v_b + \nabla_b v_a$$

A <u>gauge transformation</u> of (linearised) GR is then

$$\gamma_{ab} \longrightarrow \gamma'_{ab} = \gamma_{ab} - \nabla_a v_b - \nabla_b v_a = \gamma_{ab} - \nabla_{(a} v_{b)}.$$

If $\nabla_{(a} v_{b)} = 0$ then v generates an infinitesimal isometry. In this case v^a is called a <u>Killing vector field</u>.

Killing's equation

$$\nabla_{(a} v_{b)} = 0$$

Return to linearised Einstein's equation:

$$\text{(i)} \qquad G^{(1)}_{ab} = -\tfrac{1}{2} \partial^c \partial_c \bar{\gamma}_{ab} + \partial^c \partial_{(b} \bar{\gamma}_{a)c} - \tfrac{1}{2} \eta_{ab} \partial^c \partial^d \bar{\gamma}_{cd} = 8\pi T_{ab}$$

We aim to find an infinitesimal diffeomorphism which simplifies (i). That is, we look for vector ξ^a s.t.

$$\gamma_{ab} \to \gamma_{ab} + \partial_a \xi_b + \partial_b \xi_a \qquad \text{(ii)}$$

simplifies (i). We achieve this by solving

$$\partial^b \partial_b \xi_a = -\partial^b \bar{\gamma}_{ab}$$

for ξ^a. Eq (ii) then ensures, that (ex.)

$$\partial^b \bar{\gamma}_{ab} = 0$$

In this gauge, linearised Einstein's equation become

$$\boxed{\partial^c \partial_c \bar{\gamma}_{ab} = -16\pi T_{ab}}$$

The vacuum case $T_{ab} = 0$ this exactly the field equations for spin-2 field (massless)

Newtonian limit

This is the limit where

(a) gravity weak

(b) relative motion is slower than c

(c) material stresses smaller than mass-energy density

The assumptions allow us infer existence of global inertial coordinate system of η_{ab} such that

(ii) $$T_{ab} \simeq \rho t_a t_b$$

where

$$t^a \equiv \left(\partial/\partial x^0\right)^a$$

is unit tangent vector pointing in time direction

<u>Approximation</u> (ii) means that momentum densities are small and stresses are small, Since sources are slowly moving we look for solutions where time derivatives of $\bar{\gamma}_{ab}$ are neglected.

Thus components of $\partial^c \partial_c \bar{\gamma}_{ab} = -16\pi T_{ab}$ in this coord. system become:

This is usual Laplace operator

$$\nabla^2 \bar{\gamma}_{\mu\nu} = 0$$

for all μ, ν except $\mu = \nu = 0$, where

$$\nabla^2 \bar{\gamma}_{00} = -16\pi \rho$$

Solution of $\nabla^2 \bar{\gamma}_{\mu\nu} = 0$, assumption $\bar{\gamma}_{\mu\nu} \to 0$ at infinity, is

$$\bar{\gamma}_{\mu\nu} = 0$$

Then we obtain the following solution.

$$\gamma_{ab} = \bar{\gamma}_{ab} - \tfrac{1}{2}\eta_{ab}\bar{\gamma} = -(4t_a t_b + 2\eta_{ab})\phi$$

where

$$\phi = -\tfrac{1}{4}\bar{\gamma}_{00} \quad \text{satisfies}$$

$$\nabla^2 \phi = 4\pi\rho$$

To understand motion of test particles we must solve geodesic equation:

$$\frac{d^2 x^\mu}{d\tau^2} + \sum_{\rho,\sigma} \Gamma^\mu_{\rho\sigma} \left(\frac{dx^\rho}{d\tau}\right)\left(\frac{dx^\sigma}{d\tau}\right) = 0$$

with respect to global inertial coordinates

When a test body moves slowly w.r.t. c, so

$$\frac{dx^\mu}{dt} \approx (1, 0, 0, 0)$$

and $\tau \simeq t$ (coord. time $\simeq$ proper time)

Geodesic equation becomes:

$$\frac{d^2 x^\mu}{dt^2} = -\Gamma^\mu_{00}$$

Ex: for γ_{ab} above:

$$\Gamma^\mu_{00} = -\frac{1}{2} \frac{\partial}{\partial x^\mu} \gamma_{00} = \frac{\partial \phi}{\partial x^\mu} \qquad \mu = 1, 2, 3$$

(time derivatives neglected)

Motion of test particle is then

$$\underline{a} = -\underline{\nabla} \phi$$

where

$\underline{a} = \frac{d^2 \underline{x}}{dt^2}$ are w.r.t. global inertial coords