Introduction to general relativity: FLRW cont.

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$$\boxed{ds^2 = -d\tau^2 + a^2(\tau)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]}$$

FLRW metric; $k=-1$ (hyperboloid)

$k=0$ (flat spatial) $k=+1$ (sphere)

Case $k=0$:

$$ds^2 = -d\tau^2 + a^2(\tau)\left(dx^2 + dy^2 + dz^2\right)$$

Nonvanishing Christoffel symbols are (ex):

$$\Gamma^\tau_{xx} = \Gamma^\tau_{yy} = \Gamma^\tau_{zz} = \dot{a}a \qquad \left[\dot{a} = \frac{d}{d\tau}a\right.$$

$$\Gamma^x_{x\tau} = \Gamma^x_{\tau x} = \Gamma^y_{y\tau} = \Gamma^y_{\tau y} = \Gamma^z_{z\tau} = \Gamma^z_{\tau z} = \dot{a}/a$$

$k \neq 0$ (ex) show that

$$\Gamma^\tau_{rr} = \frac{a\dot{a}}{1-kr^2}, \quad \Gamma^\tau_{\theta\theta} = a\dot{a}r^2, \quad \Gamma^\tau_{\phi\phi} = a\dot{a}r^2\sin^2\theta$$

$$\Gamma^r_{r\tau} = \Gamma^r_{\tau r} = \dot{a}/a, \quad \Gamma^r_{rr} = \frac{kr}{1-kr^2}, \quad \Gamma^r_{\theta\theta} = -r(1-kr^2)$$

$$\Gamma^r_{\phi\phi} = -r(1-kr^2)\sin^2\theta, \quad \Gamma^\theta_{\theta\tau} = \Gamma^\theta_{\tau\theta} = \dot{a}/a$$

$$\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = 1/r, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi_{\phi\tau} = \Gamma^\phi_{\tau\phi} = \dot{a}/a \qquad \Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = 1/r$$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta.$$

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Ricci tensor becomes (ex)

$$R_{\tau\tau} = -3\ddot{a}/a$$

$$R_{rr} = (\ddot{a}a + 2\dot{a}^2 + 2k)/(1-kr^2)$$

$$R_{\theta\theta} = (\ddot{a}a + 2\dot{a}^2 + 2k)\, r^2$$

$$R_{\phi\phi} = (\ddot{a}a + 2\dot{a}^2 + 2k)\, r^2 \sin^2\theta$$

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)$$

Thus: (ex).

$$G_{\tau\tau} = 3\frac{\dot{a}^2}{a^2} + \frac{3k}{a^2}$$

$$G_{rr} = \frac{-2\ddot{a}a}{(1-kr^2)} - \frac{(\dot{a}^2+k)}{1-kr^2}$$

$$G_{\theta\theta} = r^2(-2\ddot{a}a - \dot{a}^2 - k)$$

$$G_{\phi\phi} = r^2\sin^2\theta\,(-2\ddot{a}a - \dot{a}^2 - k)$$

Einstein's field equations

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$$

τ-τ component

$$\frac{3\dot{a}^2}{a^2} + \frac{3k}{a^2} - \Lambda = 8\pi\rho \qquad (i)'$$

a^2 a^2

r-r component:

$$\frac{-2\ddot{a}a - \dot{a}^2 - k}{(1-kr^2)} + \frac{\Lambda a^2}{(1-kr^2)} = \frac{8\pi p a^2}{(1-kr^2)}$$

$\Rightarrow$

$$-\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} + \Lambda = 8\pi p \qquad \text{(ii)}'$$

(ex) (i)' and (ii)' are equivalent to

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p) + \frac{\Lambda}{3}$$

Make substitution:

$$\rho \mapsto \rho + \frac{\Lambda}{8\pi} \qquad p \mapsto p - \frac{\Lambda}{8\pi}$$

$\Rightarrow$ This eliminates Λ.

In this way Λ represents a form of energy with a negative pressure equal in magnitude to its energy density:

$$p = -\rho$$

This is known as dark energy.

Thus we look at equations

$$\boxed{\begin{aligned} \frac{3\dot{a}^2}{a^2} &= 8\pi\rho - \frac{3k}{a^2} & \text{(i)} \\ \frac{3\ddot{a}}{a} &= -4\pi(\rho + 3p) & \text{(ii)} \end{aligned}}$$

Remark: in case $\rho > 0$, $P \geq 0$ we can deduce immediately that universe cannot be static. In this case $\ddot{a} < 0$. That is, the universe must always be contracting or expanding

$\dot{a} > 0$ or $\dot{a} < 0$

Write R for the distance between two isotropic observers (at some constant τ)

The change in R is

$$v \equiv \frac{dR}{d\tau} = \frac{R}{a}\frac{da}{d\tau} = HR.$$

where

$$H(\tau) = \frac{\dot{a}}{a} \quad \text{is Hubble's constant}$$

(isn't constant!)

If R is large, v can exceed speed of light

→ No violation of relativity: only locally measured relative velocities at a single event do not exceed speed of light. Global relative velocities such as v are not so constrained.

Given $\dot{a} > 0$, we know $\ddot{a} < 0$:
Thus universe must have been accelerating at a faster and faster rate as τ goes in negative direction.

Even at constant expansion, then at time $T = \dot{a}/a = H^{-1}$ we learn $a = 0 \Rightarrow$ singular state where all distances between points of space was zero and density of matter was infinite. $\Rightarrow$ "Big bang"

This is <u>not</u> an <u>explosion</u> of preexisting matter (there was no "before" T) because it is impossible to extend spacetime manifold to "before" big bang.