Introduction to general relativity: The equivalence principle and Mach's principle

12 April 2021 13:06

Einstein's Equivalence Principle (EEP)

Consists of 3 parts

1 Universality of free fall (UFF)
(also known as weak equivalence principle (WEP))

UFF/WEP ~~verified~~ not rejected, eg., in Torsion balance experiments

2. Local Lorentz Invariance (LLI)
In freely falling reference frame, laws of physics are those of SR (no "preferred" reference frame)
ie. laws of physics don't depend on orientation or velocity.

Michelson - Morley: best bounds, which are
$$\frac{\Delta c}{c} < 5 \cdot 10^{-17}$$

3 Local position Invariance (LPI)
SR holds independent of spacetime position
(no "preferred" location)

Γ 3. is equivalent (ex) to universality of gravitational redshift (UGR). $\Rightarrow$ clock rates are universally affected by gravitational field.

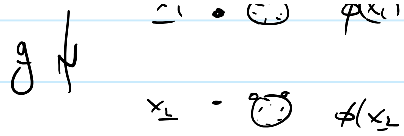
Exercise: show/argue that for small velocities and weak gravitational fields

x_1 • $\phi(x_1)$

$$\frac{\Delta \nu}{\cdot} = \frac{\Delta \phi}{\cdot} \qquad g \downarrow$$

Case gravitational fields

$$\frac{\Delta \nu}{\nu} = \frac{\Delta \phi}{c^2}$$



where $\nu(\underline{x}_1)$, $\nu(\underline{x}_2)$ = clock rates of two clocks 1, 2, @ $\underline{x}_1$ or $\underline{x}_2$ & $\Delta\phi = \phi(\underline{x}_1) - \phi(\underline{x}_2)$

Estimate: $\Delta\phi / c^2$ for $\| \underline{x}_1 - \underline{x}_2 \| \sim 30\text{cm}$

Observation: Using EEP it is possible to "derive" that gravity should be described geometrically (K. Thorne, D. Lee, A. Lightman, Phys. Rev. D, 7, 3563-3577 (1973))

All matter couples to same same geometry, namely that of spacetime

$\Rightarrow$ a single metric suffices to describe all of this couplings simultaneously.

Calculations with EEP

Statement: no external static homogeneous gravitational field can be detected in a freely falling elevator

Suppose we have N (non relativistic) particles moving in a pair force field $\underline{F}(\underline{x}_j - \underline{x}_k)$ and an external gravitational field $\underline{g}(x) = \underline{g}$

Equations of motion: $\underline{F}(\underline{0}) = \underline{0}$

$$m_j \frac{d^2 \underline{x}_j}{dt^2} = m_j \underline{g} + \sum_{k=1}^{N} \underline{F}(\underline{x}_j - \underline{x}_k)$$

If we make non-Galilean change to accelerating frame via

$$\underline{x}' = \underline{x} - \tfrac{1}{2} \underline{g} t^2 \qquad t' = t$$

$$x' = x - \tfrac{1}{2} g t^2 \qquad t' = t$$

Then equations of motion become (ex)

$$m_j \frac{d^2 \underline{x}_j'}{dt'^2} = \sum_k \underline{F}(\underline{x}_j' - \underline{x}_k')$$

Gravitational Forces

According to EEP $\exists$ for any particle moving purely under influence of gravitational field, a freely falling coordinate system $(\xi^0, \xi^1, \xi^2, \xi^3)$ s.t.

$$\frac{d^2 \xi^\alpha}{d\tau^2} = 0 \qquad (\equiv m f^\alpha)$$

where τ is proper time defined by

$$d\tau^2 = -\eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta$$

or

$$\boxed{-1 = \eta_{\alpha\beta} \frac{d\xi^\alpha}{d\tau} \frac{d\xi^\beta}{d\tau}}$$

Here $\underline{\eta} = \begin{bmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{bmatrix}$

and

$$[\underline{\eta}]_{\alpha\beta} \equiv \eta_{\alpha\beta} = \text{entry of row } \alpha \text{ \& col } \beta$$

Einstein's summation notation:

$$u_\alpha \xi^\alpha \equiv \sum_{\alpha=0,1,2,3} u_\alpha \xi^\alpha$$

'upstairs'

downstairs

Suppose x^μ denotes some other coordinates, e.g. lab, rotating, ...

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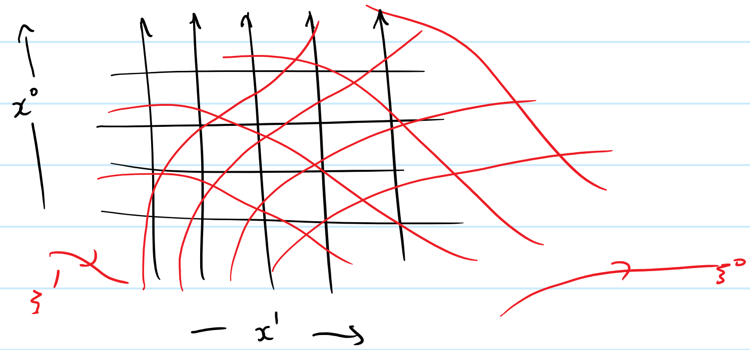

$$\xi^\alpha = \xi^\alpha (x^0, x^1, x^2, x^3)$$

The force-free equation of motion

$$0 = \frac{d^2 \xi^\alpha}{d\tau^2}$$

$$= \frac{d}{d\tau}\left(\frac{d\xi^\alpha}{d\tau} \right) = \frac{d}{d\tau}\left(\frac{\partial \xi^\alpha}{\partial x^\mu} \frac{dx^\mu}{d\tau} \right)$$

$$0 = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{d^2 x^\mu}{d\tau^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

Exploit

$$\boxed{\frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial x^\lambda}{\partial \xi^\alpha} = \delta^\lambda_\mu}$$

Multiply both sides by $\frac{\partial x^\lambda}{\partial \xi^\alpha}$ & sum:

$$\boxed{0 = \frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}$$

Here the <u>affine connection</u> is given by (exo)

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}$$

The <u>proper time</u> is given by

$$d\tau^2 = -\eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta = -\eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} dx^\mu \frac{\partial \xi^\beta}{\partial x^\nu} dx^\nu$$

$$\equiv -g_{\mu\nu}\, dx^\mu dx^\nu$$

where

$$g_{\mu\nu} \equiv \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta}$$ is

the <u>metric</u>.

<u>Mach's principle</u>: the second key set of ideas underlying GR; less precise than EEP.

<u>Summary</u>: All matter in the universe should contribute to the <u>local</u> definition of "non accelerating" and "non rotating"; in a universe devoid of matter there should be no meaning to these concepts!

<u>General Relativity</u> is the following

The <u>observer-independent</u> properties of spacetime are described by a <u>spacetime metric</u> which need not have the flat form of SR. <u>Curvature</u> accounts for the physical effects of gravitation. Further, curvature is determined by the distribution of stress-energy and momentum (& vice versa)

$$\begin{tikzcd} \text{Stress energy} \arrow[r, bend left, "\text{tells spacetime how to move}"] & \text{Curvature} \arrow[l, bend left, "\text{matter}"] \end{tikzcd}$$