Introduction to general relativity: prerelativity gravitation

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Text: Wald, "General Relativity"

Other sources:
- Misner, Thorne, & Wheeler
- Weinberg

Structure: Mathematical methods $\Rightarrow$ Physics

$\Uparrow$

Differential geometry

$\Downarrow$ $\Downarrow$ $\Downarrow$

Quantum gravity & string theory.

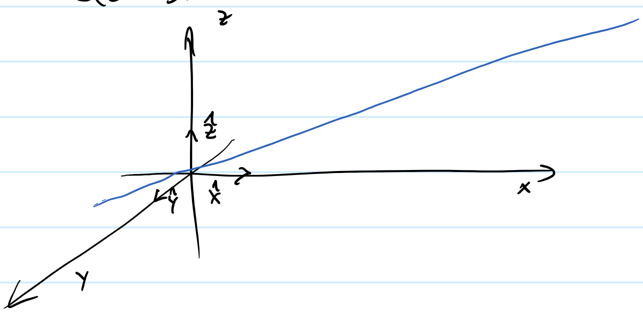
Videos: Lenny Susskind

Assume:
* Special relativity (Theoretische Physik A)
* Calculus
* Linear algebra
* Combinatorics

Lecture 1: prerelativity gravitation

Assumption

Newtonian physics: existence of a preferred reference system & clocks.



A free test mass (i.e. no force acts) moves along a straight trajectory at a constant rate uniformly in time.

$\Rightarrow$ Inertial referance frame

Newton's 2nd law is written w.r.t. an inertial reference system

$$\boxed{\underline{F}(\underline{x}(t), t) = m_I \ddot{\underline{x}}(t)}$$

Force   inertial mass   acceleration

W.r.t non-inertial reference frame ?

Define three unit vectors $\hat{\underline{x}}'(t)$, $\hat{\underline{y}}'(t)$, $\hat{\underline{z}}'(t)$

In reference frame, with respect to old basis

coordinates

$$\underline{w}(t) = w_x(t)\,\hat{\underline{x}} + w_y(t)\,\hat{\underline{y}} + w_z(t)\,\hat{\underline{z}}$$

trajectory

$$\overset{!}{=} w_{x'}\,\hat{\underline{x}}'(t) + w_{y'}(t)\,\hat{\underline{y}}'(t) + w_{z'}(t)\,\hat{\underline{z}}'(t)$$

Notation:

$\underline{a} \Leftarrow$ denotes vector

$\underline{\underline{M}} \Leftarrow$ matrix

$$\underline{\underline{M}}\,\underline{x} = \underline{y}$$

Tensors: won't use underline notation

Index notation:

$[\underline{a}]_x \equiv x$ component

or $\equiv a_x$

$[\underline{\underline{M}}]_{xy} = x\text{-}y$ component

$\equiv M_{xy}$

Physics in non-inertial frames (as captured by Newton's 2nd law) gives rise to additional force terms - inertial forces -

Eg. centrifugal, coriolis, euler, etc..

HW) Distinguish "true forces" from such

apparent forces". In particular in Newtonian physics Gravity is a true force (not be true in GR)
general relativity

Mass: (at least) 3 conceptually distinct quantities

- **Inertial mass**: constant of proportionality in Newton's 2nd law

$$\boxed{\underline{F} = m_I \, \underline{a}}$$

- **"active" gravitational mass**: this is the mass that produces the gravitational field that other masses respond to

$$\underline{g}(\underline{x}) \equiv -G \, m_g^{(a)} \, \frac{\underline{x} - \underline{x}'}{\|\underline{x} - \underline{x}'\|^3}$$

$\underline{x}'$ = location of generating mass and $G \to$ gravitational constant

$$\boxed{G = 6.67430\,(15) \times 10^{-11} \ m^3 kg^{-1} s^{-2}}$$

- **"passive" gravitational mass**: the mass that responds to an external gravitational field by accelerating.

$$\underline{F}_g = m_g^{(p)} \cdot \underline{g}$$

It is assumed that $m_I, m_g^{(a)}, m_g^{(p)} \geq 0$

Relationships between these three masses.

Exercise: using Newton's 3rd law argue that

$$\frac{m_g^{(a)}}{m_g^{(p)}} = \text{universal constant}$$

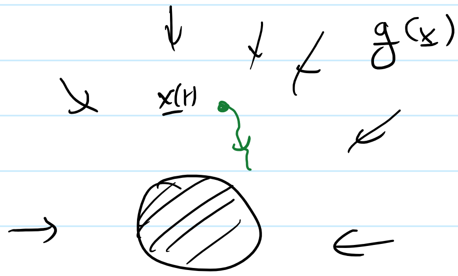
Choosing units: $m_g^{(a)} = m_g^{(p)} = m_g$

Relationship between m_g & m_I

Extensive experimental evidence suggests $m_g = m_I$

Universality of Free Fall (UFF): In a gravitational field g the orbit $\underline{x}(t)$ of a test mass only depends on initial conditions (IC). Same ICs $\Rightarrow$ same trajectories $\forall$ test masses.

Test mass ("a point")



$$m_g \, g(\underline{x}(t)) = m_I \, \ddot{\underline{x}}(t)$$

or

$$\ddot{\underline{x}}(t) = \frac{m_g}{m_I} \, g(\underline{x}(t))$$

A universal constant

Choose units so $m_g = m_I$.

Remark: (1) UFF is not a consequence of Newtonian theory! It is a falsifiable hypothesis (2) UFF relates to test masses (subtle concept)

$\boxed{\text{From Newtonian gravity to GR}}$

→ Newton gravity as a field theory

Electrodynamics and SR are compatible
Newtonian gravity <~> Electrostatics

Highlight similarities between Newton gravity & Electrostatics

Gravitational field:

$$g : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$$

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$$(t, \underline{x}) \to g(t, \underline{x})$$

Sources: (gravitational) mass density

$$\rho : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^+$$

with

$$\boxed{\underline{\nabla} \cdot g = -4\pi G \rho}$$

Just like Coulomb's law!

We have $\underline{\nabla} \times g = 0$, so g is conservative

Thus by Poincaré's lemma $\exists$

$$\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$$

s.t. convention!

$$\boxed{g = -\underline{\nabla} \phi}$$

Just as for electrostatics

$$\boxed{\Delta \phi = 4\pi G \rho}$$ Poisson's equation

Equation of motion for test masses

$$\boxed{\ddot{\underline{x}}(t) = -\nabla \phi (t, \underline{x}(t))}$$

NB Assumed here is that

closure

$$\text{supp}(\rho) = \overline{\{ \underline{x} \in \mathbb{R}^3 \mid \rho(x) \neq 0 \}}$$
$$= \text{compact} \quad (\text{Heine-Borel} = \text{closed \& Bounded})$$

With boundary conditions

$$\phi(t, \|\underline{x}\| \to \infty) \to 0$$

unique solution

$$\phi(t,\underline{x}) = -G \int_{\mathbb{R}^3} d^3\underline{x}' \frac{\rho(t,\underline{x})}{\|\underline{x}-\underline{x}'\|}.$$

Thus we have an analogue "gravitostatics"

How to add dynamics to grav. field?

(0) Assume instantaneous propagation?

(1) Assume scalar wave equation for ϕ?

(2) Complete $\phi \to$ to a 4-vector, (eg. electrodynamics)

(3) Complete $\phi \to$ a tensor (symmetric)

MTW chapter 7

(0) obvious causality problems ✗

(1) No bending of light

(2) Perihelion precession problems (negative energy waves)

(3) This route leads to linearised Einstein field equations $\to$ GR