# Introduction to general relativity: derivative operators and parallel transport

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<u>Covariant derivative operator</u> $\nabla$ } notation
$\equiv$ <u>affine connections</u>

<u>Notation</u>: Let $M$ be a manifold, denote by $\mathfrak{X}(M)$ the space of smooth vector fields on $M$.

<u>Definition</u> an affine connection $\nabla$ is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

I immediately extend domain of definition of $\nabla$ to

$$\nabla : \mathfrak{X}(M) \times \mathcal{T}(k,l) \longrightarrow \mathcal{T}(k,l)$$

(0) Let $X, Y \in \mathfrak{X}(M)$ $Z \in \mathcal{T}(k,l)$ :

$$\boxed{\nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z}$$

(0') $\forall$ $f \in \mathcal{F}(M)$, $X \in \mathfrak{X}(M)$, $Y \in \mathcal{T}(k,l)$

$$\boxed{\nabla_{fX} Y = f \nabla_X Y}$$

(1) <u>Linearity</u> : $A, B \in \mathcal{T}(k,l)$

$$\boxed{\nabla_X (A+B) = \nabla_X A + \nabla_X B}$$

(In <u>AIN</u>: note

$$\nabla_X \equiv X^a \nabla_a$$
)

$$\nabla_c \left( A^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} + B^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} \right) = \nabla_c A^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} + \nabla_c B^{a_1 \ldots a_k}{}_{b_1 \ldots b_l}$$

2. <u>Leibniz rule</u>: For all $A \in \mathcal{T}(k,l)$, $B \in \mathcal{T}(k',l')$

$$\boxed{\nabla_X (A \otimes B) = (\nabla_X A) \otimes B + A \otimes (\nabla_X B)}$$

<u>In AIN</u>:

$$\nabla_e \left( A^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} B^{c_1 \ldots c_{k'}}{}_{d_1 \ldots d_{l'}} \right) = \left( \nabla_e A^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} \right) B^{c_1 \ldots c_{k'}}{}_{d_1 \ldots d_{l'}} + A^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} \left( \nabla_e B^{c_1 \ldots c_{k'}}{}_{d_1 \ldots d_{l'}} \right)$$

In AIN:

$$\nabla_e \left( A^{a_1 \dots a_k}{}_{b_1 \dots b_l} B^{c_1 \dots c_{k'}}{}_{d_1 \dots d_{l'}} \right) = \left( \nabla_e A^{a_1 \dots a_k}{}_{b_1 \dots b_l} \right) B^{c_1 \dots c_{k'}}{}_{d_1 \dots d_{l'}} + A^{a_1 \dots a_k}{}_{b_1 \dots b_l} \left( \nabla_e B^{c_1 \dots c_{k'}}{}_{d_1 \dots d_{l'}} \right)$$

(3) Commutativity with contraction: For all $A \in \mathcal{T}(k,l)$

$$\boxed{\nabla_X (\mathcal{C}_{j,j'} A) = \mathcal{C}_{j+1, j'+1} (\nabla_X A)}$$

In AIN:

$$\nabla_d \left( A^{a_1 \dots c \dots a_k}{}_{b_1 \dots c \dots b_l} \right) = (\nabla_d A)^{a_1 \dots c \dots a_k}{}_{b_1 \dots c \dots b_l}$$

(4) Reduction to vector fields on scalar functions

Let $f \in \mathcal{F}(M)$ then

$$\boxed{\nabla_X (f) = X(f)}$$

AIN:

$$X(f) = X^a \nabla_a f$$

(5) Torsion free:

$$\boxed{\nabla_X Y - \nabla_Y X - [X,Y] = 0}$$

$\Rightarrow$

In AIN (ex). $\forall f \in \mathcal{F}(M)$

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f$$

Simple consequences: $\forall X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(M)$, $f \in \mathcal{F}(M)$

(*) $\nabla_X (fY) = X(f) Y + f \nabla_X Y$

(**)
$$\begin{aligned}
[v,w](f) &= v(w(f)) - w(v(f)) \\
&= v^a \nabla_a (w^a \nabla_a (f)) - w^a \nabla_a (v^b \nabla_b (f)) \\
&= v^a (\nabla_a w^b) \nabla_b f - w^a (\nabla_a v^b) \nabla_b f \\
&\quad + v^a w^b \nabla_a \nabla_b f - v^a w^b \nabla_b \nabla_a f
\end{aligned}$$

So vector field $[v,w]$ is given by

$$[v,w]^b = v^a \nabla_a w^b - w^a \nabla_a v^b$$

There are many derivative operators obeying (0), ..., (5)

The simplest example is the ordinary derivative operator

$\partial_a$ defined as follows. Let $\psi$ be a chart & $T \in \mathcal{T}(k,l)$ with components $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$ in coord. basis

$\partial_a$ acts via

$$\partial_c T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} \xrightarrow[\text{coord. basis } \psi]{\text{in}} \frac{\partial}{\partial x^\sigma} \left( T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \right)$$

$\rightarrow$ obviously defined only in chart $\psi$. (ex check $\partial_a$ obeys (1) $\rightarrow$ (5))

Suppose $\nabla_a$ & $\tilde{\nabla}_a$ are two derivative operators obeying (1) $-$ (5). As a map on $f \in \mathcal{F}(M)$

$$(\tilde{\nabla}_a - \nabla_a)(f) = \tilde{\nabla}_a f - \nabla_a f \quad = 0$$

Now consider

$$(\tilde{\nabla}_a - \nabla_a) \text{ on } \mathcal{T}(0,1):$$

$$(\tilde{\nabla}_a - \nabla_a) : \mathcal{T}(0,1) \longrightarrow \mathcal{T}(0,2)$$

Let $f \in \mathcal{F}(M)$ & $\omega \in \mathcal{T}(0,1)$. Then $f\omega_a \in \mathcal{T}(0,1)$

We find

$$(\tilde{\nabla}_a - \nabla_a)(f\omega_b) \overset{ex}{=} f(\tilde{\nabla}_a - \nabla_a)(\omega_b) \qquad (*)$$

( via Leibniz & (4) )

<u>Claim</u>: $(*)$ implies that $\tilde{\nabla}_a \omega_b - \nabla_a \omega_b$ only depends on $\omega_b$ at $p$. Strategy:

Let $\omega_b' \in \mathcal{T}(0,1)$ have property

$$\underline{\omega_b'|_p = \omega_b|_p}$$

we will argue:

$$\tilde{\nabla}_a \omega_b - \nabla_a \omega_b = \tilde{\nabla}_a \omega_b' - \nabla_a \omega_b'$$

One can find (ex) smooth functions $f_\alpha$ which vanish at $p$ and smooth $\mu_b(\alpha) \in \mathcal{T}(0,1)$ such that

$$\omega_b' - \omega_b = \sum_{\alpha=1}^{n} f_\alpha \, \mu_b(\alpha) \qquad f_\alpha|_p = 0$$

Apply $(\tilde{\nabla}_a - \nabla_a)$:

$$(\tilde{\nabla}_a - \nabla_a)(\omega_b' - \omega_b) = \sum_{\alpha=1}^{n} f_\alpha \, (\tilde{\nabla}_a - \nabla_a) \mu_b(\alpha)$$

$= 0$ at $p$ because $f_\alpha|_p = 0$

Hence $(\underbrace{\tilde{\nabla}_a - \nabla_a}) \omega_b' = (\tilde{\nabla}_a - \nabla_a)\omega_b$ at $p$

doesn't depend on $\omega_b$ away from $p$

$\underbrace{(\tilde{\nabla}_a - \nabla_a)}_{C}$ : as a map of tensors of type $(0,1)$ at $p$ to $\mathcal{T}(0,2)$

$$C : V_p^* \to V_p^* \otimes V_p^*$$

By duality $V_p^{**} \cong V_p$, $C$ can be interpreted as a map from $V_p^* \otimes V_p \otimes V_p \to \mathbb{R}$ or as an element of $V_p \otimes V_p^* \otimes V_p^*$. We write $C_{ab}^{\ \ c}$ for the AIN of $C$

$$\boxed{\nabla_a \omega_b = \tilde{\nabla}_a \omega_b - C_{ab}^{\ \ c} \omega_c}$$

WARNING! BUT $C_{ab}^{\ \ c}$ does not transform as a tensor field of type $\mathcal{T}(1,2)$ if we choose $\tilde{\nabla}_a = \partial_a$

Consider $\omega_b = \nabla_b f \overset{(5)}{=} \tilde{\nabla}_b f \qquad f \in \mathcal{F}(M)$

$$\underbrace{\nabla_a \nabla_b f}_{\substack{\text{symmetric} \\ (5)}} = \underbrace{\tilde{\nabla}_a \tilde{\nabla}_b}_{\substack{\text{symmetric} \\ (5)}} f - C_{ab}^{\ \ c} \nabla_c f$$

$$\Downarrow$$

$$C_{ab}^{\ \ c} = C_{ba}^{\ \ c}$$

The Leibniz property & (4) now determines $(\tilde{\nabla}_a - \nabla_a)$ on all tensor fields : let $\omega \in \mathcal{T}(0,1)$ and $t \in \mathcal{T}(1,0) \cong \mathfrak{X}(M)$.

$$\omega_a t^a \in \mathcal{F}(M)$$

(4) :

$$(\tilde{\nabla}_a - \nabla_a)(\omega_b t^b) = 0$$

Leibniz & (3) :

$$t^b (\underbrace{\tilde{\nabla}_a - \nabla_a}_{C}) \omega_b + \omega_b (\tilde{\nabla}_a - \nabla_a) t^b = 0$$

$$\Rightarrow \quad t^b C_{ab}^{\ \ c} \omega_c + \omega_b \underbrace{(\tilde{\nabla}_a - \nabla_a) t^b}_{} = 0$$

$$\Rightarrow \quad \omega_b \left( (\tilde{\nabla}_a - \nabla_a) t^b + t^c C_{ac}^{\ \ b} \right) = 0$$

or

$$\boxed{\nabla_a t^b = \tilde{\nabla}_a t^b + C_{ac}^{\ \ b} t^c}$$

$$\boxed{\nabla_a t^b = \tilde{\nabla}_a t^b + C^b{}_{ac} t^c}$$

ex: for $T \in \mathcal{T}(k,l)$

$$\boxed{\nabla_a T^{b_1 \ldots b_k}{}_{c_1 \ldots c_l} = \tilde{\nabla}_a T^{b_1 \ldots b_k}{}_{c_1 \ldots c_l} + \sum_j C^{b_j}{}_{ad} T^{b_1 \ldots d \ldots b_k}{}_{c_1 \ldots c_l} - \sum_j C^d{}_{ac_j} T^{b_1 \ldots b_k}{}_{c_1 \ldots d \ldots c_l} \quad (\ast\ast)}$$

In other words action $\tilde{\nabla}_a - \nabla_a$ on arbitrary $\mathcal{T}(k,l)$ is determined by $C^c{}_{ab} \in \boxed{V_p \otimes V_p^* \otimes V_p^*}$

Converse: $C^c{}_{ab} \in V_p \otimes V_p^* \otimes V_p^*$ and a derivative operator $\tilde{\nabla}_a$ then $\nabla_a$ defined by $(\ast\ast)$ is also a derivative operator

$C^c{}_{ab}$ doesn't always transform as a tensor field of type $\mathcal{T}(1,2)$ under change of coord's (eg. when $\tilde{\nabla}_a \equiv \partial_a$)

Most important example: let $\tilde{\nabla}_a = \partial_a$. In this case $C^c{}_{ab} \longmapsto \Gamma^c{}_{ab}$, called Christoffel symbols:

$$\nabla_a t^b = \partial_a t^b + \Gamma^b{}_{ac} t^c$$

in coords.

$$\boxed{\nabla_\mu t^\nu = \partial_\mu t^\nu + \Gamma_{\mu\nu}{}^{\lambda} t^\lambda}$$