General relativity

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Equivalence principle implies it is impossible, in general, to set up a family of inertial observers to measure "gravitational force": we cannot "insulate" an observer from gravity!

Fundamental hypothesis of general relativity:

Spacetime is a manifold M on which is defined a Lorentz metric g_{ab}. (M may be much more general than $\mathbb{R}^{1,3}$)

Further, spacetime is not necessarily flat. The world lines of freely falling bodies are geodesics of (curved) metric g_{ab}

Note: for cases of time-translation symmetry, one can set up preferred background family of observers and "measure" a "gravitational force" w.r.t. family

While we cannot measure gravitational force in general situations we can measure relative acceleration of nearby geodesics $\Rightarrow$ we speak of tidal forces.

Two key principles to determine laws of physics in GR:

(1) General covariance: the metric g_{ab} (and quantities derivable from g_{ab}) are the only spacetime quantities that can appear in laws of physics (equations)

(2) Equations of physics must reduce to their SR versions for case g_{ab} is flat

According to (1) and (2) we continue to represent physical quantities via same tensorial quantities as in SR. Thus

(*) particle motion is represented by timelike curves C

(4) The 4-velocity of a particle is the unit tangent to its worldline: u^a (measured w.r.t. g_{ab}).

We need to amend equations of motion:

Free-particle equation of motion is the geodesic equation

$$\boxed{u^a \nabla_a u^b = 0}$$

where ∇_a is determined by g_{ab}. Acceleration is defined analogously:

$$a^b \equiv u^a \nabla_a u^b$$

When $a \neq 0$ we say a force

$$\boxed{f^b = m a^b}$$

where m is rest mass of particle.

The 4-momentum of the particle is defined as

$$\boxed{p^a = m u^a}$$

The energy, as determined by observer present at the particle's world line at which the energy measured is

$$\boxed{E = -p_a v^a}$$

where v^a is velocity of observer.

A given observer cannot define energy of distant particle because parallel transport is path-dependent.

"GR-friendly" equations of motion can be found (usually) by applying "minimal substitution" rules

SR: $\eta_{ab} \longmapsto$ GR: g_{ab}

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SR : $\partial_a \longmapsto$ GR : ∇_a

Examples : the Klein-Gordon field (in SR)

$$\mathcal{L} = \frac{1}{2}(\partial_a \phi)(\partial^a \phi) - \frac{1}{2} m^2 \phi^2$$

$\Rightarrow$ equation of motion

$$\partial_a \partial^a \phi - m^2 \phi = 0$$

In GR : $\partial_a \longmapsto \nabla_a$

$$\boxed{\nabla_a \nabla^a \phi - m^2 \phi = 0}$$

Not only generalisation possible!
e.g. could add $\alpha R \phi$ to equation of motion

Stress-energy tensor is then

$$T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} (\nabla_c \phi \nabla^c \phi + m^2 \phi)$$

$$\boxed{\nabla^a T_{ab} = 0}$$

Example (perfect fluid in SR)
pertains to a continuous distribution of matter with Stress-Energy tensor T_{ab} of form

$$T_{ab} = \rho u_a u_b + P(\eta_{ab} + u_a u_b)$$

where ρ is density, P is pressure field, u^a is a unit timelike vector field represents 4-velocity of fluid. Equation of motion of for perfect fluid :

$$\partial^a T_{ab} = 0$$

In GR :

$$\boxed{T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b)}$$

Impose / guess : equation of motion

(**). $$\boxed{\nabla^a T_{ab} = 0}$$

By projecting onto components parallel & perpendicular to u^a:
$\Rightarrow$

$$u^a \nabla_a \rho + (\rho + P) \nabla^a u_a = 0$$

$$(P+\rho) u^a \nabla_a u_b + (g_{ab} + u_a u_b) \nabla^a P = 0$$

<u>Comment on interpretation of T_{ab} in GR.</u>

In SR: An observer with 4-velocity v^a interprets

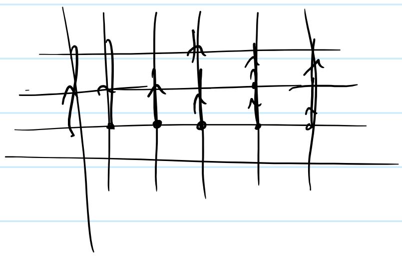
$$T_{ab} v^a v^b$$

as energy density, ie mass-energy density / unit volume for the observer. (ex.) Further, if x^a is orthogonal to v^a then

$$-T_{ab} v^a x^b$$

is interpreted as momentum density of matter in x^a direction.

So, in SR, $\partial^a T_{ab} = 0$ may then be interpreted as conservation law, eg. apply Gauss law in following situation. We assume we can set up a family of inertial observers with parallel 4-velocities v^a, so $\partial_b v^a = 0$

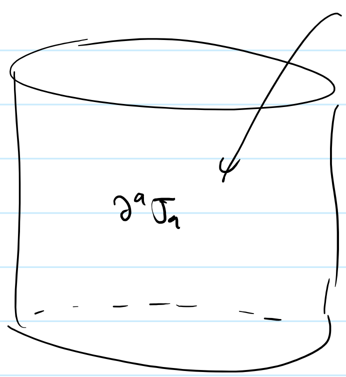


Define

$$J_a = -T_{ab} v^b$$

So

$$\partial^a T_{ab} = 0 \Rightarrow \partial^a J_a = 0$$



<u>In GR</u>: A family of observers is represented by v^a (unit & timelike). The condition that 4 velocities are parallel:

$$\nabla^a v_b = 0 \quad \text{or}$$

equivalently,

$$\nabla_{(a} v_{b)} = 0 \qquad \text{(Killing's equation)}$$

<u>However</u>: in curved spacetime it is generally impossible to find v^a such that

$$v^a v_a = -1$$

&

$$\nabla_{(a} v_{b)} = 0$$

<u>Counterexample</u>: de Sitter spacetime.

Therefore $\nabla^a T_{ab} = 0$ does not imply strict <u>global</u> energy conservation.

Physically, the gravitational field can do work on fluid via tidal forces.

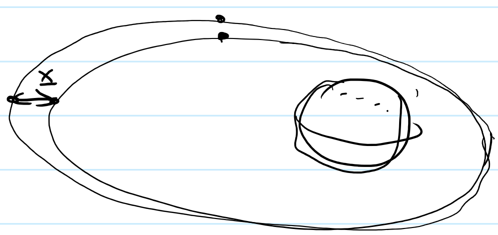
Can only regard (★) as a <u>local conservation</u> of material energy. valid in small regions of spacetime.

<u>Einstein's equation</u>: spacetime is dynamical in GR

We need <u>equation of motion for metric g_{ab}</u>.

<u>Mach's principle</u>: spacetime geometry is influenced by matter distribution

To make this quantitative we look at how tidal forces are calculated in Newtonian physics and GR.



Newton: gravitation field described by potential ϕ, tidal acceleration vector $\underline{a}$

$$\underline{a} = -(\underline{x}\cdot\underline{\nabla})\underline{\nabla}\phi$$

where $\underline{x}$ is relative separation vector.

In GR: tidal acceleration described by geodesic deviation

$$a^a = -R_{cbd}{}^a v^c x^b v^d$$

here v^a is 4-velocity of particles and x^a the deviation

Strongly suggests correspondence:

$$R_{cbd}{}^a v^c v^d \longleftrightarrow \partial_b \partial^a \phi$$

However ϕ is determined by ρ according to Poisson's equation

$$\nabla^2\phi = 4\pi\rho$$

Energy-density of matter is described by stress-energy tensor

$$T_{ab} v^a v^b \longleftrightarrow \rho$$

Thus: this suggests

$$R_{cad}{}^a v^c v^d = 4\pi T_{cd} v^c v^d$$

$\Rightarrow$

$$R_{cd} = 4\pi T_{cd} \qquad ??!$$

This was Einstein's original equation. However it has a flaw: namely

$$\nabla^a T_{ab} \neq 0$$

because of Bianchi identity:

$$\nabla^c\left(R_{cd} - \tfrac{1}{2}g_{cd}R\right) = 0$$

ie we would need

$$\nabla_a R = 0 \quad !$$

Resolution: consider instead, the equation

$$\boxed{G_{ab} = R_{ab} - \tfrac{1}{2} R g_{ab} = 8\pi T_{ab}}$$

This is Einstein's equation.