Introduction to general relativity: homogeneity and isotropy cont.

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Suppose our manifold M (modelling spacetime) is homogeneous & isotropic. Consequences for metric g_{ab}?

First note g_{ab} induces a metric $h_{ab}(t)$ on Σ_t via restriction

Homogeneity: $\exists$ isometry of $h_{ab}(t)$ which takes any $p \in \Sigma_t$ to $q \in \Sigma_t$

Isotropy: implies it is impossible to construct geometrically preferred vector in Σ_t

This second constraint (isotropy) implies following: Build Riemann tensor for Σ_t from $h_{ab}(t)$:

$$ {}^{(3)}R_{abc}{}^{d} $$

Raise index c with $h_{ab}(t)$:

$$ {}^{(3)}R_{ab}{}^{cd} \equiv {}^{(3)}R_{abc'}{}^{d}\, h^{c'c} $$

$\rightarrow$ Now interpreted as a linear map from $\mathcal{J}(0,2) \rightarrow \mathcal{J}(0,2)$ (at point $p \in M$). (Ex: argue that ${}^{(3)}R_{ab}{}^{cd}$ annihilates symmetric tensors in $\mathcal{J}(0,2)$). We induce a linear map (from ${}^{(3)}R_{ab}{}^{cd}$)

$$ L : W \rightarrow W $$

where $W \equiv$ space of 2-forms (at p). The symmetric property $R_{abcd} = R_{cdab}$ implies that L is a symmetric matrix, on inner product space (inner product given by h_{ab}). Therefore L has real eigenvalues and eigenvectors.

$\dim W = \binom{3}{2} = 3$

Claim: all eigenvalues of L are equal. (otherwise we could construct preferred vector at p)

$$L \propto \mathbb{I}$$

the constant of proportionality

$\Rightarrow$ (ex)

$$^{(3)}R_{ab}{}^{cd} = \boxed{K}\, \delta^c{}_{[a} \delta^d{}_{b]}$$

K is constant due to homogeneity. Argue this independently: lower cd:

$$(*) \qquad ^{(3)}R_{abcd} = K\, h_{c[a} h_{b]d}$$

Bianchi identity:

$$0 = D_{[e} {}^{(3)}R_{ab]cd} = (D_{[e} K)\, h_{[c|a} h_{b]d}$$

(covariant derivative associated to h_{ab})

If $\dim(M) \geq 3$ then RHS vanishes $\Leftrightarrow D_e K = 0$

Definition: a spacetime manifold where $^{(3)}R_{abcd} = K\, h_{c[a} h_{b]d}$ (with K constant) is called a space of constant curvature

Now describe representative spacetimes for $K>0$, $K=0$, $K<0$.

$K>0$: is achieved by Σ_t = 3-spheres (described as embedded manifolds in $\mathbb{R}^4$)

$$x^2 + y^2 + z^2 + w^2 = R^2$$

The corresponding metric (in spherical coordinates):

$$ds^2 = d\psi^2 + \sin^2\psi\,(d\theta^2 + \sin^2\theta\,d\phi^2)$$

→ called a "closed" universe

<u>$K=0$</u>: flat 3D space;

$$ds^2 = dx^2 + dy^2 + dz^2$$

→ called an "open" universe

<u>$K<0$</u>: is obtained from 3D hyperboloids (defined in $\mathbb{R}^4$ by

$$t^2 - x^2 - y^2 - z^2 = R^2 \quad)$$

metric (in hyperbolic coordinates):

$$ds^2 = d\psi^2 + \sinh^2\psi\,(d\theta^2 + \sin^2\theta\,d\phi^2)$$

→ also an "open" universe

As isotropic observers move on worldlines orthogonal to Σ_t, we can express full 4D metric g_{ab} as

$$\boxed{g_{ab} = -u_a u_b + h_{ab}(t)}$$

Here $u_a \equiv$ tangent vector for isotropic observer(s) & $h_{ab}(t)$ is one of the metrics above. All isotropic observers attach same proper time τ to Σ_t. In these coordinates we have

$$\boxed{ds^2 = -d\tau^2 + a^2(\tau)\begin{cases} d\psi^2 + \sin^2\psi\,(d\theta^2 + \sin^2\theta\,d\phi^2) \\ dx^2 + dy^2 + dz^2 \\ d\psi^2 + \sinh^2\psi\,(d\theta^2 + \sin^2\theta\,d\phi^2) \end{cases}}$$

<u>Friedman–Lemaître–Robertson–Walker (FLRW) metric</u>

$\longrightarrow$ Just need to solve for $a(\tau)$.