# Introduction to general relativity: geodesics in Schwarzschild

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For a curve $\gamma$ parametrized by $\tau$; $u^a$ is tangent vector to $\gamma$. Coordinate basis components are

$$u^\mu = \frac{dx^\mu}{d\tau} \equiv \dot{x}^\mu$$

For timelike geodesics $\tau$ is chosen to be the proper time (for null geodesics, we choose $\tau$ so that $\underline{u^a \nabla_a u^b} = 0$ )

Remember that we focus on (without loss of generality) $\theta = \pi/2$. Therefore

$$(*) \quad \boxed{g_{ab} u^a u^b = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = \begin{cases} 1 & \text{timelike} \\ 0 & \text{null} \end{cases}}$$

[ what are we doing at the moment? We aim to solve for $t, r, \phi$ (also $\theta = \pi/2$). We need a complete set of differential equations for $t, r, \phi$. Could use geodesic equation. However, there is a much more economical way by writing down <u>Constants of motion</u> ]

According to Proposition in previous lecture we know that

$$\xi_a u^a = g_{ab}\, \xi^a u^b$$

is constant along curve for Killing vector field $\xi^a$

<u>Choose</u>: $\xi^a = \left(\frac{\partial}{\partial t}\right)^a$ : corresponding to $\xi^a$

(i) $E = \left(1 - \frac{2m}{r}\right)\dot{t}$ "the energy"

is constant → for timelike particles far from centre $E$ becomes SR energy per unit mass of particle measured by static observer.

In Null case $\hbar E$ represents total energy of photon.

The Killing field generator of rotations

$$\psi^a = \left(\partial/\partial\phi\right)^a$$

gives constant of motion

(ii) $L = g_{ab}\,\psi^a u^b = r^2\dot{\phi}$

This may be interpreted as angular momentum per unit rest mass of a particle / or as $\hbar L$ ⇒ angular momentum of photon.

Constancy of $L$, in Newtonian physics, Kepler's 2nd law.

Substituting (i) and (ii) into (*) :

$$\boxed{\underbrace{\tfrac{1}{2}\dot{r}^2}_{KE} + \underbrace{\tfrac{1}{2}\left(1-\frac{2m}{r}\right)\left(\frac{L^2}{r^2}+k\right)}_{PE} = \underbrace{\tfrac{1}{2}E^2}_{\text{Total energy}}}$$

(where $k = 1$ for timelike ; $k = 0$ for null)

This equation is that of a unit-mass particle moving in a potential

$$\boxed{V(r) = \tfrac{1}{2}K - K\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}}$$

with total energy $\tfrac{1}{2}E^2$.

New ingredient here is $-\frac{ML^2}{r^3}$ which dominates over centrifugal barrier for small $r$.

Solve for $\phi$ by eliminating $\tau$ from (ii) & (iv):

$$\phi = \pm \int \frac{L}{r^2\sqrt{E^2 - \left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2} + K\right)}}\, dr$$

$\Gamma_{(ii)} = L = r^2 \left(\frac{d\phi}{d\tau}\right)^2 \Rightarrow \frac{L^2}{r^2} = \left(\frac{d\phi}{d\tau}\right)^2$

Want $\frac{d\phi}{dr}$: $\quad \frac{d\phi}{dr} = \frac{d\phi}{d\tau}\frac{d\tau}{dr} \leftarrow$ get this from (iv)

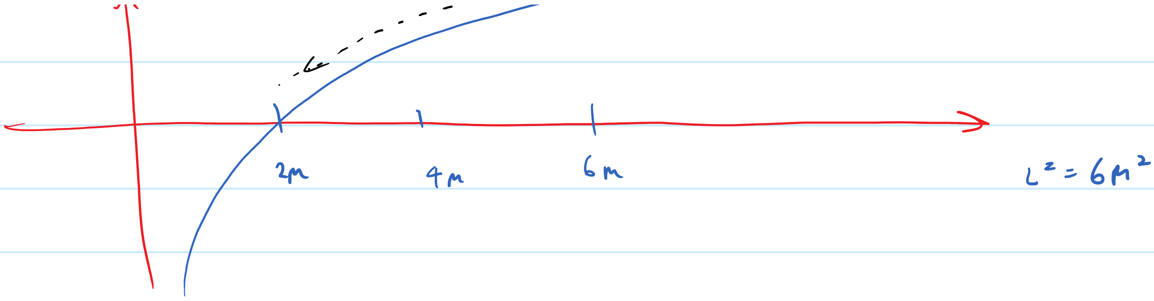
We consider stability of orbits. Suppose $K = 1$. The stationary points / extrema of effective potential are at

$$0 = \frac{dV}{dr} = \frac{1}{r^4}\left(Mr^2 - L^2 r + 3ML^2\right)$$

The zeros of this equation are at

$$r = R_\pm = \frac{L^2 \pm \left(L^4 - 12L^2M^2\right)^{1/2}}{2M}$$

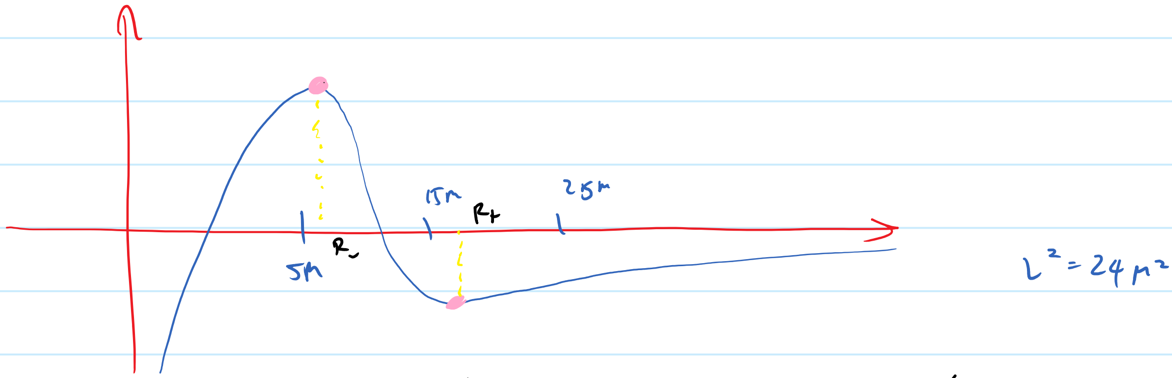
Hence, if $L^2 < 12M^2$, there are no extrema




For angular momentum

$$L^2 > 12M^2$$

(ex.) the extrema $R_+$ is a minima & $R_-$ is a maxima




Hence, there exist stable circular orbits (ie $\dot{r} = 0$) at $r = R_+$ and unstable circular orbits at $r = R_-$

In the limit $L \gg M$

$$R_+ \simeq \frac{L^2}{M}$$

( Same as Newtonian formula )

Because $R_+ > 6M$ there no stable circular orbits at radii smaller than $6M$.

The unstable circular orbits lie between

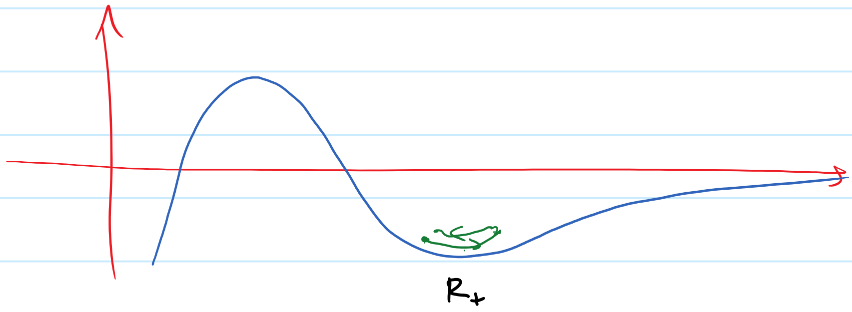
$$3M < R_- < 6M$$

The energy of a particle in a circular orbit is (ex.)

$$E(R) = \frac{R - 2M}{\sqrt{R}\sqrt{R - 3M}}$$

Since for $R \leq 4M$, $E \geq 1$, particles in unstable circular orbits between $3M$ & $4M$ escape to infinity if perturbed outwards.

Suppose a particle is in a stable circular orbit.



and is perturbed around $R_+$. For infinitesimal (sufficiently small) displacements the particle will execute simple harmonic motion with frequency (ex).

$$\omega_r^2 = k_{eff} = \left.\frac{d^2V}{dr^2}\right|_{R_+} = \frac{M(R_+ - 6M)}{R_+^3 (R_+ - 3M)}$$

Note that "time" here is proper time $\tau$ for the particle (not Schwarzschild coordinate time $t$).

The angular frequency of the circular orbit is found from

$$\omega_\phi \equiv \dot{\phi} \quad ; \quad L = r^2 \dot{\phi}$$

Using

$$0 = Mr^2 + 3ML^2 - L^2 r \Rightarrow L^2 = \frac{Mr^2}{r - 3M}$$

to eliminate $L$:

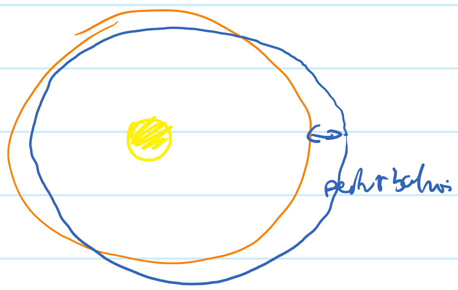
$$\omega_\phi = \frac{\sqrt{M}}{R_+\sqrt{R_+ - 3M}}$$

If $R_+ \gg M$ we find (Newtonian limit)

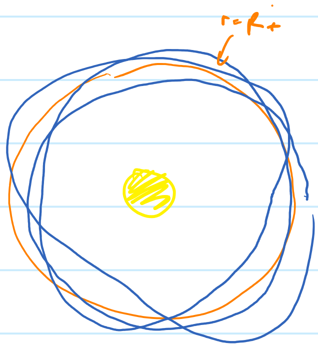
$$\omega_r^2 \approx \frac{M}{R_+^3}$$

$$\omega_\phi^2 \approx \frac{M}{R_+^3}$$

Coincide and oscillations are commensurate, orbit closes.




In general, $\omega_r \neq \omega_\phi$, oscillations are incommensurate, leading to precession of perihelion and aphelia:




The precession rate of perihelion is determined by

$$\omega_p = \omega_\phi - \omega_r = -\left(\sqrt{1 - \frac{6M}{R_+}} - 1\right)\omega_\phi$$

for $R_+ \gg M$

$$\omega_p \simeq \frac{3M^{3/2}}{R_+^{5/2}}$$

For Mercury $\Rightarrow$ 43 seconds of arc / century