Introduction to general relativity: abstract index notation; curvature

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$T \in \mathcal{T}(k,l)$: (at least) two ways to think about

(i) $T \in \underbrace{V_p \otimes V_p \otimes \cdots \otimes V_p}_{k} \otimes \underbrace{V_p^* \otimes \cdots \otimes V_p^*}_{l}$ (ie a vector in a vector space)

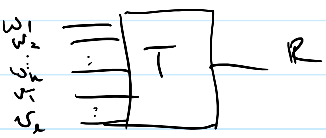
(ii) $T : \underbrace{V_p^* \otimes \cdots \otimes V_p^*}_{k} \otimes \underbrace{V_p \otimes \cdots \otimes V_p}_{l} \to \mathbb{R}$ (is a linear map from a vector space to $\mathbb{R}$)

In way (ii) we think T as a function with k entries/arguments from/of V_p^* and l entries/arguments from/of V_p:

$$T(\underbrace{\cdot,\cdot,\dots,\cdot}_{k}\,;\,\underbrace{\cdot,\cdot,\dots,\cdot}_{l}) \in \mathbb{R}$$

evaluation

$$T(\underbrace{\omega_1,\omega_2,\dots,\omega_k}_{\in V_p^*}\,;\,\underbrace{v_1,v_2,\dots,v_l}_{\in V_p})$$



AIN: a way to specify the type of a tensor by naming these entries.

$$T(\underset{a}{\cdot},\underset{b}{\cdot},\underset{c}{\cdot}\,;\,\underset{d}{\cdot},\underset{e}{\cdot},\underset{f}{\cdot}) \longrightarrow \mathbb{R}$$

specify the type as follows

$T^{abc}{}_{def}$ → not components, they label entries

$\Downarrow$

$\Downarrow$

<u>This says</u>: a tensor which takes 3 arguments from V_p^* & 3 arguments from V_p

The actual components of T with respect to a coordinate system $\{x^\mu\}$ are

$$\sum_{\mu,\nu} T^{\mu_1 \dots \mu_3}{}_{\nu_1 \nu_2 \nu_3} \left.\frac{\partial}{\partial x^{\mu_1}}\right|_p \otimes \left.\frac{\partial}{\partial x^{\mu_1}}\right|_p \otimes \left.\frac{\partial}{\partial x^{\mu_3}}\right|_p \otimes dx^{\nu_1} \otimes dx^{\nu_2} \otimes dx^{\nu_3}$$

$\Downarrow$

for each choice of $\mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3$ this is a function of M.

Certain tensor operations can be expressed compactly with <u>AIN</u>:

<u>Contraction</u>: if $T^{a_1 \dots a_k}{}_{b_1 \dots b_l} \in \mathcal{T}(k,l)$

then

$$\boxed{\mathcal{C}_{j,j'} T = T^{a_1 \dots c \dots a_k}{}_{b_1 \dots c \dots b_l}}$$

repeated letter

first c is in the jth entry

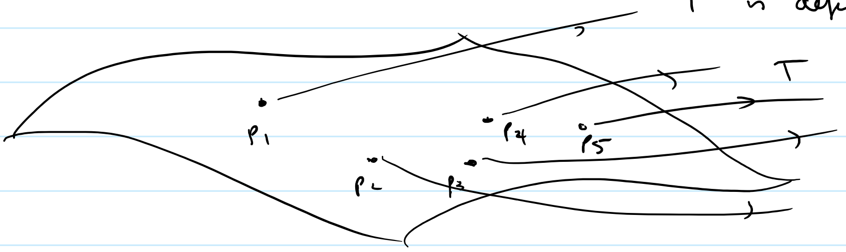
second c is in the j'th entry

What type of tensor do we have now? $\mathcal{C}_{j,j'} T \in \mathcal{T}(k-1, l-1)$

<u>AIN</u>: $T^{a_1 \dots a_k}{}_{b_1 \dots b_l} \to$ demand implicitly that T transforms as a tensor of type $\mathcal{T}(k,l)$ under change of coordinate basis.

$T \in \mathcal{T}(k,l)$ is really, in GR, an infinite <u>list</u> of tensors, one for each point $p \in M$.

T is defined



Outer products: if $T^{a_1 \dots a_k}{}_{b_1 \dots b_l} \in \mathcal{T}(k,l)$ & $S^{a_1' \dots a_{k'}'}{}_{\dots} \in \mathcal{T}(k', l')$

Outer products: if $T^{a_1\cdots a_k}{}_{b_1\cdots b_l} \in \mathcal{T}(k,l)$ & $S^{a_1'\cdots a_{k'}'}{}_{b_1'\cdots b_{l'}'} \in \mathcal{T}(k',l')$ then AIN for their outer product is <u>defined</u> to be

$$T^{a_1\cdots a_k}{}_{b_1\cdots b_l} S^{a_1'\cdots a_{k'}'}{}_{b_1'\cdots b_{l'}'} \in \mathcal{T}(k+k', l+l')$$

Metric: if $g \in \mathcal{T}(0,2)$ then AIN:

$g_{ab} \rightarrow g$ means metric

The inverse of g: think of g as a <u>matrix</u>
Form its matrix inverse $g^{-1} \in \mathcal{T}(2,0)$ (ex)

<u>AIN</u>: $(g^{-1})^{ab} \overset{\text{means}}{\Leftarrow} g^{ab} \in \mathcal{T}(2,0)$

Raising/lowering: "Apply g to vector v":

$$g \xrightarrow{\text{AIN}} g_{ab} \qquad v \xrightarrow{\text{AIN}} v^a$$

(1) <u>outer product</u>: $g \otimes v$:

(2) <u>Contraction</u>: $g \otimes v \overset{g_{ab}v^c}{\hookrightarrow} \mathcal{C}_{2,3}\, g \otimes v \in \mathcal{T}(0,1)$

<u>AIN</u>: $v^a \mapsto g_{ac}v^c \equiv v_a \in \mathcal{T}(0,1)$

$v^a \longmapsto v_a$ (really stands for $g\ \&\ v \mapsto g\otimes v \mapsto \mathcal{C}_{2,3}\, g\otimes v$)

Lemma: $g^{ab} g_{bc} \equiv \delta^a{}_c$

Proof: $\equiv \mathcal{C}_{2,3}(g^{-1} \otimes g) \Rightarrow \underline{g}^{-1} \cdot \underline{g} = \mathbb{I}$

$\mathbb{I}$ in AIN: $\delta^a{}_c$

Generalise raising/lowering: let $T^{abc}{}_{def} \in \mathcal{T}(3,3)$
Form: $\mathcal{C}_{2,3}\, g \otimes T$:

$$g_{aa'} T^{a'bc}{}_{def} \equiv T^{\ bc}{}_{a\ def} \in \mathcal{T}(2,4)$$

$$g_{aa'} T^{a'bc}{}_{def} \equiv T_a{}^{bc}{}_{def} \in \mathcal{T}(2,4)$$

Notation is consistent with repeated applications of g & g^{-1}

eg.

$$T^{abc}{}_{def} \mapsto g^{aa'} g_{a'a''} T^{a''bc}{}_{def} \mapsto \delta^a{}_{a''} T^{a''bc}{}_{def}$$
$$\equiv T^{abc}{}_{def}$$

Subspaces of symmetric / antisymmetric tensors

$\forall \quad T, T' \in \mathcal{T}(k,l)$

$$T + T' \in \mathcal{T}(k,l)$$

AIN

$$\equiv T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} + T'^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$$

Define AIN:

$$T_{(ab)} \equiv \frac{1}{2}(T_{ab} + T_{ba})$$

$$T_{[ab]} \equiv \frac{1}{2}(T_{ab} - T_{ba})$$

Ex:

$$T^{abc}{}_{def} \equiv T(\cdot_a, \cdot_b, \cdot_c; \cdot_d, \cdot_e, \cdot_f)$$

then define

$$T^{bac}{}_{def} \equiv T(\cdot_b, \cdot_a, \cdot_c; \cdot_d, \cdot_e, \cdot_f)$$

Define, for $T_{a_1 \cdots a_l} \in \mathcal{T}(0,l)$:

$$T_{(a_1 \cdots a_l)} \equiv \frac{1}{l!} \sum_{\pi \in S_l} T_{a_{\pi(1)} \cdots a_{\pi(l)}}$$

symmetric group of permutations of l entries / objects

$$T_{[a_1 \cdots a_l]} \equiv \frac{1}{l!} \sum_{\pi \in S_l} \varepsilon(\pi) T_{a_{\pi(1)} \cdots a_{\pi(l)}}$$

$$\varepsilon(\pi) = \begin{cases} +1 & \text{if } \pi \text{ is a product of even \# of transpositions} \\ -1 & \text{otherwise} \end{cases}$$

Can mix notation: eg. $T \in \mathcal{T}(3,2)$

$$T^{(ab)c}{}_{[de]} \equiv \frac{1}{4}\left[T^{abc}{}_{de} + T^{bac}{}_{de} - T^{abc}{}_{ed} - T^{bac}{}_{ed}\right]$$

A totally antisymmetric tensor of type $T(0,l)$ ie

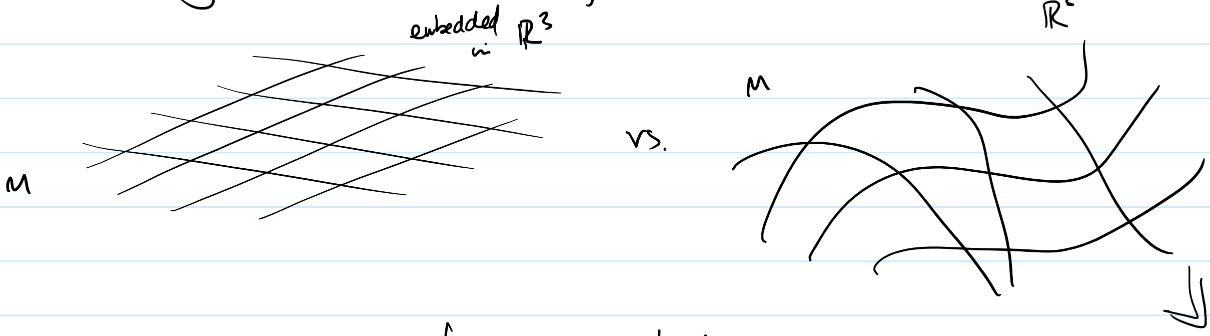
$$T_{a_1 \dots a_l} = T_{[a_1 \dots a_l]}$$

is called <u>differential l-form</u>

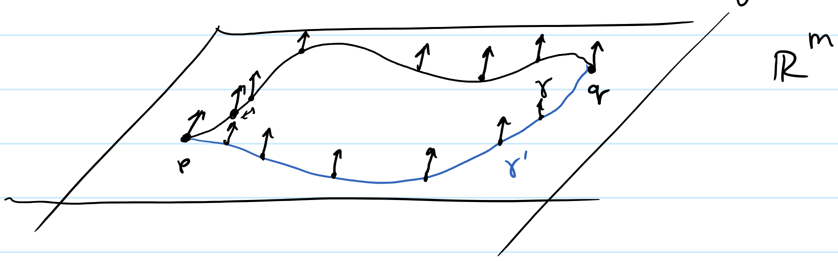
<u>Curvature</u>: <u>Spacetime is not embedded?</u>

We want an <u>intrinsic notion</u> of curvature

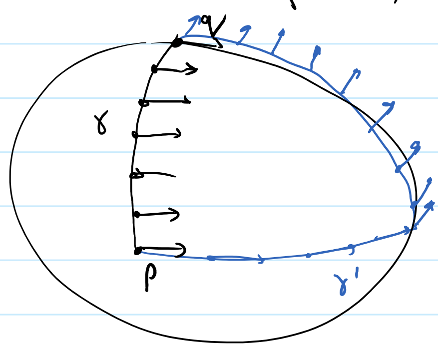
Γ if M were embedded in $\mathbb{R}^k$
it is "easy" to see if M is curved



We need a <u>proxy</u> for curvature: capture curvature by noticing that in flat manifolds we can move vectors around "in a parallel fashion" independent of path chosen.



On curved embedded manifold, eg surface of sphere

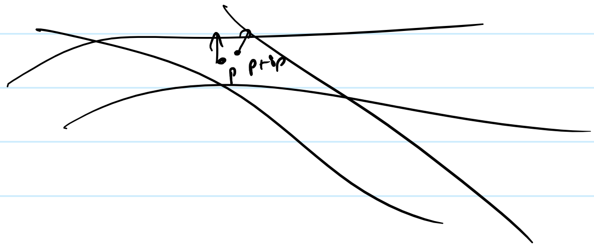


Curvature $\equiv$ dependence on path chosen of <u>parallel transport</u>

<u>Parallel transport</u>: Let M be a manifold with no additional structure. $\Rightarrow$ it turns out to be impossible

to define a natural notion of parallel transport

Problem: we want to move a vector from V_p at p to V_q at q. in as "parallel a way as possible".
$\Rightarrow$ no natural way to compare elements of V_p & V_q.
(or even of V_p and $V_{p+\delta p}$)



In euclidean space: $\mathbb{R}^n$ we take a vector at p and shift (using additive structure) to q $\Rightarrow$ this allows us to define derivatives of vectors $v^\mu \frac{\partial}{\partial x^\mu}\Big|_p \equiv v_p$

$$\frac{\partial v^\mu}{\partial x^\nu} = \lim_{\Delta x^\nu \to 0} \frac{v^\mu(\ldots, x^\nu + \Delta x^\nu, \ldots) - v^\mu(\ldots, x^\nu, \ldots)}{\Delta x^\nu}$$

defined at $V_{q = p+\delta p}$

We assume parallel transported version of $v \xrightarrow{\text{to}} q$ has same components as v at p.

Heuristic approach:

We need additional data, a parallel transporter

$$\boxed{U_\gamma : V_p \to V_q}$$

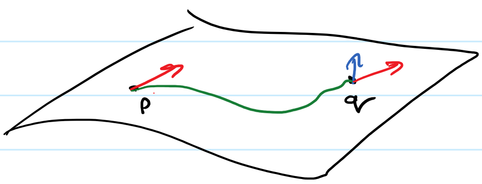
where γ is a smooth path connecting p and q.

If you had the data of a parallel transporter U_γ you can compare tangent vectors at p and q.

let $v \in V_p$, $w \in V_q$ then

define "v at q" to be

$$U_\gamma v \in V_q$$
$$w \in V_q$$



Demand $\mathcal{U}_\gamma$ is a linear transformation of vector spaces. What do we need to specify $\mathcal{U}_\gamma$? [$\Gamma\, \mathcal{U} : (\text{paths in } M) \times V_p \to V_q$]

Let's work infinitesimally: Let $v \in V_p$; Suppose p has components x^μ in a chart ψ

Consider q near p (infinitesimally close).
In chart ψ:

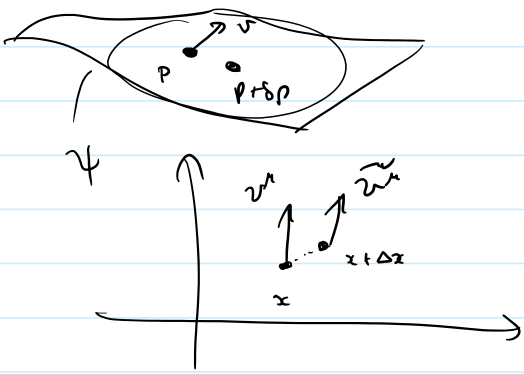
$$\psi(q) = x + \Delta x$$

Let γ be a smooth path connecting p & q (in chart ψ: $\psi \circ \gamma$). Demand for

$\tilde{v} = \mathcal{U}_\gamma v$ that components satisfy

(i) $\tilde{v}^\mu - v^\mu \propto \|\Delta x\|$

(ii) $\widetilde{(v^\mu + w^\mu)} = \tilde{v}^\mu + \tilde{w}^\mu$



We can satisfy both (i) & (ii) if we take (ex).

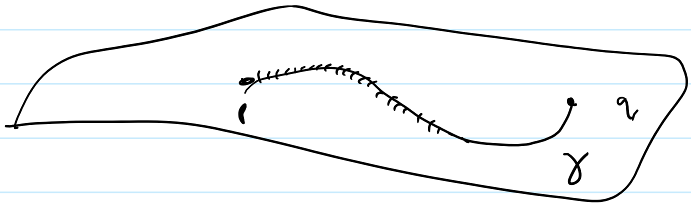
$$\boxed{\tilde{v}^\mu = v^\mu - v^\lambda\, \Gamma^\mu_{\nu\lambda}(x)\, \Delta x^\nu}$$

lives at $\psi \circ q = x + \Delta x$

original components

connection coefficients $\underbrace{n \cdot n \cdot n}$

we need these for each point x in chart



⇒ all this looks coordinate dependent: look for an intrinsic way to define this..

Note: to every infinitesimal notion of parallel transport U_γ → get derivative-type operator for vectors

vector at q; vector at $p \xrightarrow{U_\gamma} q$

$$\nabla_\nu \left(v^\mu \frac{\partial}{\partial x^\mu}\Big|_p \right) \equiv \lim_{\Delta x^\nu \to 0} \frac{v^\mu(x+\Delta x) - \tilde{v}^\mu(x+\Delta x)}{\Delta x^\nu} \frac{\partial}{\partial x^\mu}\Big|_q$$

$$= \left(\frac{\partial v^\mu}{\partial x^\nu} + v^\lambda \Gamma^\mu_{\nu\lambda} \right) \frac{\partial}{\partial x^\mu}\Big|_q$$