Introduction to general relativity: the Schwarzschild solution

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$$\boxed{ds^2 = -f(r)\,dt^2 + h(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)}$$

we need to solve for these

The nonzero components of metric

$$g_{tt} = -f(r)\,; \quad g_{rr} = h(r)\,; \quad g_{\theta\theta} = r^2\,; \quad g_{\phi\phi} = r^2\sin^2\theta$$

Inverse

$$g^{tt} = -f^{-1}(r)\,; \quad g^{rr} = h^{-1}(r)\,; \quad g^{\theta\theta} = r^{-2}\,; \quad g^{\phi\phi} = r^{-2}\sin^{-2}\theta$$

Christoffel symbols

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^{\nu}} + \frac{\partial g_{\rho\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}}\right)$$

Show (ex).

$$\Gamma^{r}_{rr} = \frac{1}{2h(r)}\frac{dh(r)}{dr}\,, \quad \Gamma^{r}_{\theta\theta} = -\frac{r}{h(r)}\,, \quad \Gamma^{r}_{\phi\phi} = \frac{-r\sin^2\theta}{h(r)}$$

$$\Gamma^{r}_{tt} = \frac{1}{2h(r)}\frac{df}{dr}\,, \quad \Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r}\,, \quad \Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^{\phi}_{\phi r} = \Gamma^{\phi}_{r\phi} = \frac{1}{r}\,, \quad \Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \cot\theta\,, \quad \Gamma^{t}_{tr} = \Gamma^{t}_{rt} = \frac{1}{2f(r)}\frac{df(r)}{dr}$$

The Ricci tensor is found from:

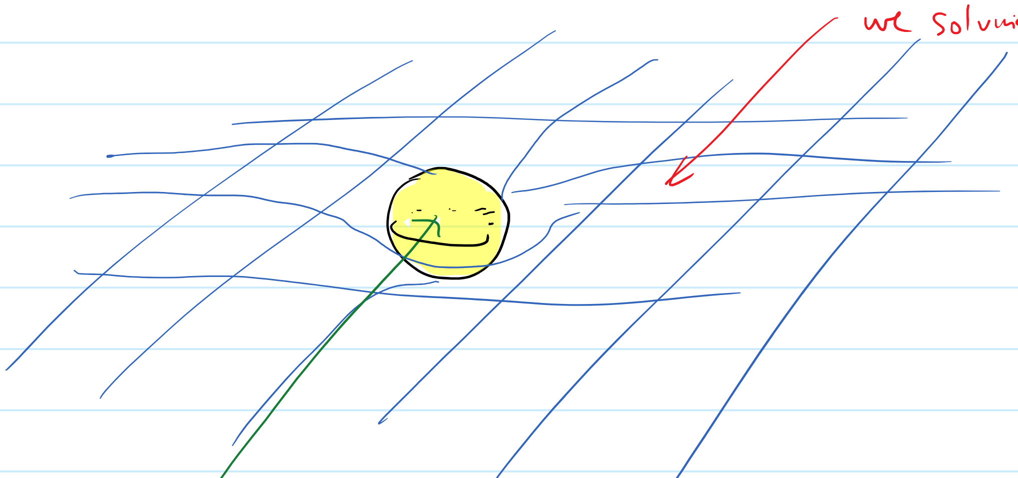
$$R_{\mu\kappa} = \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\lambda}} - \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\lambda\eta}$$

$$\partial x^k \qquad \partial x^\lambda$$

<u>Show (ex.)</u> $\quad R_{\mu\nu} = 0 \quad$ for $\quad \mu \neq \nu$

$$\boxed{\begin{aligned}
R_{rr} &= -\left(\frac{f''(r)}{2f(r)} - \frac{1}{4}\left(\frac{f'(r)}{f(r)}\right)\left(\frac{h'(r)}{h(r)} + \frac{f'(r)}{f(r)}\right) - \frac{1}{r}\left(\frac{h'(r)}{h(r)}\right)\right) \\
R_{\theta\theta} &= -\left(-1 + \frac{r}{2h(r)}\left(-\frac{h'(r)}{h(r)} + \frac{f'(r)}{f(r)}\right) + \frac{1}{h(r)} \right) \\
R_{\phi\phi} &= \sin^2\theta \, R_{\theta\theta} \\
R_{tt} &= -\left(-\frac{f''(r)}{2h(r)} + \frac{1}{4}\left(\frac{f'(r)}{f(r)}\right)\left(\frac{h'(r)}{h(r)} + \frac{f'(r)}{f(r)}\right) - \frac{1}{r}\left(\frac{f'(r)}{h(r)}\right)\right)
\end{aligned}}$$

we solving for g's outside the celestial body



inside / interior solution do in next lecture

Thus for exterior solution we use <u>vacuum Einstein's field equations</u>.

Recall from lecture 13 the "trace-reversed" form

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \tfrac{1}{2} T g_{\mu\nu}\right)$$

Which, in vacuum $T_{\mu\nu} = 0$ & $T = 0$:

$$R_{\mu\nu} = 0$$

It suffices to set just $R_{tt}, R_{rr}, R_{\theta\theta} = 0$

Note that:

$$\frac{R_{tt}}{f(r)} + \frac{R_{rr}}{h(r)} = -\frac{1}{rh(r)}\left(\frac{h'(r)}{h(r)} + \frac{f'(r)}{f(r)}\right) = 0$$

ie

$$\frac{f'}{f} + \frac{h'}{h} = 0$$

or

$$f = K h^{-1}$$

Where K is a constant. By rescaling time $t \mapsto \sqrt{K}\, t$ we may set $K = 1$

Equation $R_{\theta\theta} = 0$ then implies (ex):

$$-f' + \frac{1-f}{r} = 0$$

So that

$$\frac{d}{dr}(rf) = 1$$

That is

$$f = 1 + \frac{c}{r}$$

where c is a constant.

We have thus solved for the metric:

$$(*) \quad \boxed{ds^2 = -\left(1 + \frac{c}{r}\right) dt^2 + \left(1 + \frac{c}{r}\right)^{-1} dr^2 + r^2 d\Omega^2}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

This is the Schwarzschild metric (1916)

As $r \to \infty$ the metric approaches the Minkowski metric. To interpret constant we consider the behaviour of test body and compare with that of Newtonian theory (in weak-field limit).

Exercise: (*) suggests

$$C = -2M$$

Thus Schwarzschild metric becomes

$$\boxed{ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2}$$

There are two singularities, one at

$r = 0$ and one at $r = 2M$

The singularity at $r_s = 2M$ is a coordinate singularity (coordinate system breaks down; we won't argue this here.)

For "usual" objects (e.g. not Black holes) $r = 2M$ lies deep within the object. E.g. for the Sun

$$r_s \approx 2.95 \text{ km}$$

where vacuum Einstein's equations do not apply