Introduction to general relativity: the Schwarzschild solution cont.

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Interior solutions (static, spherically symmetric)

We look for solutions of Einstein's field equation which are static and spherically with

(*) $$\boxed{T_{ab} = \rho u_a u_b + p(g_{ab} + u_a u_b)}$$

To be compatible with the Schwarzschild solution we choose

$$u^a \equiv -\sqrt{f}(dt)^a$$

where f appears in the vacuum metric.

Apply Einstein's field equations ($G_{ab} = 8\pi T_{ab}$) gives three independent equations (ex):

$$\boxed{\begin{aligned}
8\pi T_{00} &= G_{00} = 8\pi\rho = (rh^2)^{-1}h' + r^{-2}(1-h^{-1}) && (1)\\
8\pi T_{11} &= G_{11} = 8\pi P = (rfh)^{-1}f' - r^{-2}(1-h^{-1}) && (2)\\
8\pi T_{22} &= G_{22} = 8\pi P = \tfrac{1}{2}(fh)^{-\frac{1}{2}}\frac{d}{dr}\left[(fh)^{-\frac{1}{2}}f'\right] && (3)\\
&\quad + \tfrac{1}{2}(rfh)^{-1}f' - \tfrac{1}{2}(rh^2)^{-1}h'
\end{aligned}}$$

where we've defined

$$\begin{aligned}
(e_0)_a &\equiv \sqrt{f}(dt)_a\\
(e_1)_a &\equiv \sqrt{h}(dr)_a\\
(e_2)_a &= r(d\theta)_a\\
(e_3)_a &= r\sin\theta\,(d\phi)_a
\end{aligned}$$

and

$$G_{\alpha\beta} \equiv G_{ab} (e_\alpha)^a (e_\beta)^b$$

$$T_{\alpha\beta} = T_{ab} (e_\alpha)^a (e_\beta)^b$$

(1) Involves h only, and may be written

$$\frac{1}{r^2} \frac{d}{dr} \left[r(1-h^{-1}) \right] = 8\pi \rho$$

So that

$$h(r) = \left[1 - \frac{2m(r)}{r} \right]^{-1} \qquad (**)$$

Where

$$m(r) = 4\pi \int_0^r \rho(r') r'^2 \, dr' + \overset{\text{constant}}{a}$$

Smoothness of g_{ab} as $r \to 0 \to h(r) \to 1$

$\Rightarrow a = 0$.

Because Σ must be spacelike, necessarily $h \geq 0$, ie

$$r \geq 2m(r)$$

If $\rho = 0$ for $r > R$ then solution for h is continued onto vacuum solution with

$$M = m(R) = 4\pi \int_0^R \rho(r) r^2 \, dr$$

Write now

$$f = e^{2\phi}$$

Equation (2) becomes

$$\boxed{\frac{d\phi}{dr} = \frac{m(r) + 4\pi r^3 P}{r(r - 2m(r))}} \qquad (***)$$

In Newtonian limit $r^3 P \ll m(r)$ and $m(r) \ll r$ this reduces to

$$\frac{d\phi}{dr} \simeq \frac{m(r)}{r^2}$$

$\to$ Poisson's equation for gravitational potential

Substituting $(**)$ & $(***)$ into (3) gives
(ex: lots of algebra)

$$\boxed{\frac{dP}{dr} = -(P+\rho)\frac{m(r) + 4\pi r^3 p}{r(r-2m(r))}}$$

Tolman – Oppenheimer – Volkoff equation of hydrostatic equilibrium

Thus the spacetime geometry inside a static spherically symmetric star is

$$\boxed{ds^2 = -e^{2\phi}dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1}dr^2 + r^2 d\Omega^2}$$

where $m(r) = 4\pi\int_0^r \rho(r')\, r'^2\, dr'$

<u>Geodesics of Schwarzschild Solution</u>

Now we derive the geodesics followed by test bodies and light rays in the <u>exterior region</u> $r > 2M$

<u>Proposition</u>. Let ξ^a be a Killing vector field and let γ be a geodesic with tangent u^a. Then

$\xi^a u_a$ is constant along γ.

Proof. We have

$$u^b \nabla_b (\xi_a u^a) = \underbrace{u^a u^b \nabla_b \xi_a}_{=0 \text{ Because } \nabla_a \xi_b + \nabla_b \xi_a = 0} + \underbrace{\xi_a u^b \nabla_b u^a}_{\text{geodesic eq}^n} = 0$$

Another symmetry: is parity reflection $\theta \mapsto \pi - \theta$:
If the initial position and tangent vector of a geodesic lie in the equatorial plane $\theta = \pi/2$, then the entire geodesic must lie in the plane.
Hence via a rotation every geodesic may be brought to an equatorial plane.