Introduction to general relativity: tensors continued

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Let $p \in M$ be a point in a manifold M. V_p: tangent space at p.
Study behavior of $v \in V_p$, $v^* \in V_p^*$ etc. under changes of coordinates of M.

(i) <u>Dual of V_p</u>: $V_p^* \equiv$ <u>Cotangent space</u>; elements of V_p^* are called <u>covariant vectors</u>. Given basis

$$e_\mu \equiv \partial/\partial x^\mu \big|_p \qquad \text{formally define dual basis}$$

$$e^\mu \equiv dx^\mu \qquad \text{(Just a symbol, so far!)}$$

via

$$e^\mu(e_\nu) = \delta^\mu{}_\nu$$

$$\Rightarrow \quad \boxed{dx^\mu\left(\frac{\partial}{\partial x^\nu}\Big|_p\right) = \delta^\mu{}_\nu} \qquad (*)$$

dx^μ : linear function of tangent vectors <u>defined</u> by $(*)$

<u>Change of coordinate system</u>:

$$\boxed{v'^\mu = \sum_{\mu=1}^{n} v^\mu \frac{\partial x'^{\mu'}}{\partial x^\mu}} \qquad \text{(vector transformation law)}$$

Let $\omega \in V_p^*$: $\omega = \sum_{\mu=1}^{n} \omega_\mu \, dx^\mu \overset{?!?}{=} \sum_{\mu'} \omega'_{\mu'} \, dx'^{\mu'}$?

Apply ω to v:

$$\omega(v) = \omega\left(\sum_\mu v^\mu \frac{\partial}{\partial x^\mu}\Big|_p\right) = \sum_\mu v^\mu \, \omega\left(\frac{\partial}{\partial x^\mu}\Big|_p\right)$$

$$= \sum \omega_\mu v^\mu$$

$$\overset{!}{=} \omega\left(\sum_\mu v'^{\mu'} \frac{\partial}{\partial x'^{\mu'}}\Big|_p\right)$$

$$= \omega\left(\sum_{\mu',\mu} v^\mu \frac{\partial x'^{\mu'}}{\partial x^\mu} \frac{\partial}{\partial x'^{\mu'}}\Big|_p\right)$$

$$\sum_{\mu=1}^{n} \omega_\mu v^\mu = \sum_{\mu,\mu'=1}^{n} \omega'_{\mu'} v^{\mu} \frac{\partial x'^{\mu'}}{\partial x^{\mu}}$$

i.e.

$$\omega_\mu = \omega'_{\mu'} \frac{\partial x'^{\mu'}}{\partial x^\mu} \quad \text{or}$$

$$\boxed{\omega'_{\mu'} = \frac{\partial x^\mu}{\partial x'^{\mu'}} \omega_\mu} \qquad \text{covariant vector transformation law}$$

In general, for a tensor $T \in \mathcal{T}(k,l)$

$$T = \sum_{\underline{\mu},\underline{\nu}} T^{\underline{\mu}}{}_{\underline{\nu}} \; v_{\underline{\mu}} \otimes (v^{\underline{\nu}})^*$$

$\underline{\mu} = (\mu_1 \dots \mu_k)$ $\quad$ $\underline{\nu} = (\nu_1, \dots, \nu_l)$

$$v_{\underline{\mu}} \equiv \left.\frac{\partial}{\partial x^{\mu_1}}\right|_p \otimes \left.\frac{\partial}{\partial x^{\mu_2}}\right|_p \otimes \dots \otimes \left.\frac{\partial}{\partial x^{\mu_k}}\right|_p$$

$$(v^{\underline{\nu}})^* \equiv dx^{\nu_1} \otimes dx^{\nu_2} \otimes \dots \otimes dx^{\nu_l}$$

Components of T in new coord. system: $\left.\frac{\partial}{\partial x'^{\mu'}}\right|_p$

$$\boxed{T'^{\mu_1' \dots \mu_k'}{}_{\nu_1' \dots \nu_l'} = \sum_{\substack{\mu_1 \dots \mu_k \\ \nu_1 \dots \nu_l}} T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \frac{\partial x'^{\mu_1'}}{\partial x^{\mu_1}} \dots \frac{\partial x'^{\mu_k'}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x'^{\nu_1'}} \dots \frac{\partial x^{\nu_l}}{\partial x'^{\nu_l'}}}$$

$\Rightarrow$ Tensor transformation law $\quad (*)$

A collection of numbers $T^{\underline{\mu}}{}_{\underline{\nu}}(p)$ in M transforming like $(*)$ is classically called a tensor (field).

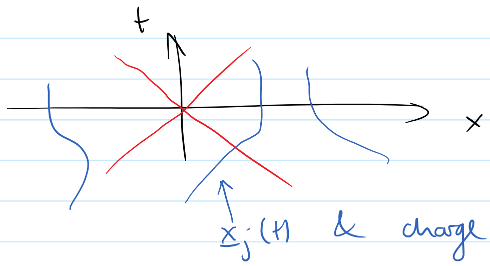
A smooth tensor field T of type (k,l) is one for which

$T(\omega^1, \dots, \omega^k; v_1, \dots, v_l)$ is smooth for all

smooth $\omega^1, \dots, \omega^k, v_1, \dots, v_l$. ($\omega \in V_p^*$ is smooth if $\forall$ smooth v, $\omega(v)$ is C^∞)

Examples: (i) Currents and densities in SR.

Let $M = \mathbb{R}^{1,3}$.



N particles: $j = 1, \dots, N$

Density (of charge) $\quad \varepsilon(\underline{x}, t) = \sum_{j=1}^{N} q_j \, \delta^{(3)}(\underline{x} - \underline{x}_j(t))$ (Dirac Delta function)

Current $\quad \underline{J}(\underline{x}, t) = \sum_{j=1}^{N} q_j \, \delta^{(3)}(\underline{x} - \underline{x}_j(t)) \frac{d\underline{x}_j}{dt}(t)$

Define a four vector J^μ by setting

$$J = \begin{pmatrix} \varepsilon \\ \underline{J} \end{pmatrix}$$

Ex. argue that J^μ is a vector (in V_p) field under change of coordinates via Lorentz transformations

$x'^\nu = \Lambda^\nu{}_\mu x^\mu$

of coordinates via Lorentz transformations

$$\boxed{x'^{\nu} = \Lambda^{\nu}{}_{\mu} x^{\mu}}$$

(ii) Energy-momentum tensor in SR. Let $M = \mathbb{R}^{1,3}$. Consider a collection N particles with energy-momentum four vectors p_j^{μ} ; $j = 1, \dots, N$

The density of μth component $p^{\mu}(t)$ is defined to be

$$\sim \quad T^{\mu 0}(\underline{x}, t) \equiv \sum_{j=1}^{N} p_j^{\mu}(t) \, \delta^{(3)}(\underline{x} - \underline{x}_j(t))$$

Corresponding current

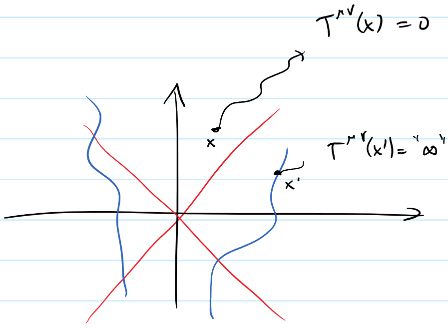
$$J^{k} \sim T^{\mu k}(\underline{x}, t) = \sum_{j=1}^{N} p_j^{\mu}(t) \frac{dx_j^{k}}{dt}(t) \, \delta^{(3)}(\underline{x} - \underline{x}_j(t))$$

Combine to a single formula $\quad (x = (\underline{x}, t))$

$$T^{\mu\nu}(x) = \sum_{j=1}^{N} p_j^{\mu} \frac{dx^{\nu}(t)}{dt} \, \delta^{(3)}(\underline{x} - \underline{x}_j(t))$$

(Here $x^{0}(t) = t$). Since

$$p_j^{\nu} = E_j \frac{dx_j^{\nu}}{dt}$$



we have

$$\boxed{T^{\mu\nu}(x) = \sum_{j} \frac{p_j^{\mu} p_j^{\nu}}{E_j} \, \delta^{(3)}(\underline{x} - \underline{x}_j(t))}$$

$\Rightarrow$

T is symmetric ; ie $T^{\mu\nu} = T^{\nu\mu}$

Writing

$$T^{\mu\nu}(x) = \sum_{j} \int d\tau \; p_j^{\mu} \frac{dx_j^{\nu}}{d\tau} \, \delta^{(4)}(x - \underline{x}_j(\tau))$$

argue (ex). under Lorentz transformations $x'^{\nu} = \Lambda^{\nu}{}_{\mu} x^{\mu}$

$$T'^{\mu'\nu'} = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} T^{\mu\nu}$$

$$\left(\Lambda^{\mu'}{}_{\mu} = \frac{\partial x'^{\mu'}}{\partial x^{\mu}} \right)$$

T is a tensor of type $(2,0)$

(iii) The metric tensor

A metric tensor g is a tensor field of type $(0,2)$, which is symmetric and non-degenerate, ie, $\forall v_1, v_2 \in V_p$

$$g(v_1, v_2) = g(v_2, v_1)$$

and

$$g(v, v_1) = 0 \quad \forall v \in V_p$$
$$\Rightarrow \quad v_1 = 0$$

A metric is the extra data we need to supply us with a notion of <u>infinitesimal length</u>:

<u>infinitesimal displacement</u> $\cong$ tangent vector

" infinitesimal squared distance " $\cong$ quadratic function of tangent vector

Choose coordinate basis $\frac{\partial}{\partial x^\mu}\Big|_p$: expand g:

$$g = \sum_{\mu\nu} g_{\mu\nu}\, dx^\mu \otimes dx^\nu \qquad (\equiv ds^2)$$

[We often omit "$\otimes$" sign:

$$ds^2 = g = \sum_{\mu\nu} g_{\mu\nu}\, dx^\mu dx^\nu$$]

A metric actually supplies us with extra data of an <u>inner product</u> on V_p, $\forall p \in M$:

$$(v, w)_p \equiv \sum_{\mu\nu} g_{\mu\nu} (dx^\mu \otimes dx^\nu)(v, w)$$
$$= \sum_{\mu\nu} g_{\mu\nu} (dx^\mu(v)\ dx^\nu(w))$$
$$= \sum_{\mu\nu} g_{\mu\nu}\, v^\mu w^\nu$$

[$v = \sum v^\mu \frac{\partial}{\partial x^\mu}\Big|_p$]

<u>Gram-Schmidt</u> procedure: orthonormal basis v_σ for V_p s.t.

$$(v_\mu, v_\nu) = g(v_\mu, v_\nu) = s_\mu\, \delta_{\mu\nu}$$

where $s_\mu \in \{\pm 1\}$

[<u>ex</u>: prove this]

The number of ± 1 is independent of orth. basis
$\Rightarrow$ <u>signature</u> of g

A metric g with $s_\mu = +1$, $\forall \mu$, is Riemannian (g is positive definite). The metric of spacetime has signature $(-1, +1, +1, +1)$

A metric g is simultaneously interpreted as a $(0,2)$ tensor and also as a multilinear map from
$$g: V_p \times V_p \to \mathbb{R}$$
also as a <u>linear map</u> from V_p to V_p^* induced via

$$v \longmapsto g(\cdot, v) = \ldots$$

also as a linear map from V_p to V_p^* induced via

$$\boxed{v \longmapsto g(\cdot, v) = \omega}$$

what is this thing?

$$g(\cdot, v) : V_p \longrightarrow \mathbb{R}$$

i.e.

$$g(\cdot, v) \in V_p^*$$

This map is 1 to 1 & onto and gives us a canonical basis-independent correspondence between vectors & dual vectors

Abstract index notation

Suppose $T \in \mathcal{T}(k, l)$

Think T: a multilinear map from $(V_p^*)^{\otimes k} \otimes (V_p)^{\otimes l} \to \mathbb{R}$

Can specify T via its components in a basis $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$

Often it is enough just to know which arguments of T take vectors / or dual vectors. Capture this by labelling each argument with a lower case latin letter

Superscript indices label contravariant entries & subscript indices —— covariant entries eg.

$$T^{ab}{}_{cd}$$

denotes a $(2,2)$ tensor. Here lower-case latin letters label arguments and their type (not components with respect to a basis)