Introduction to general relativity: Geodesics cont.; curvature

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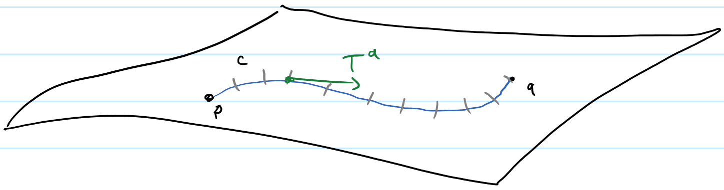


M

∇_a

g_{ab}

Length of C (w.r.t. g_{ab}, ∇_a):

(*) $$\boxed{l \equiv \int (g_{ab} T^a T^b)^{1/2} \, dt}$$

where T^a is the tangent to C, t is curve parameter

⌈ Assume that our metric g_{ab} has signature $(+,+,\dots,+)$ ⌋

For metric with signature $(-,+,+,\dots,+)$, a curve C is said to be timelike if

$$g_{ab} T^a T^b < 0$$

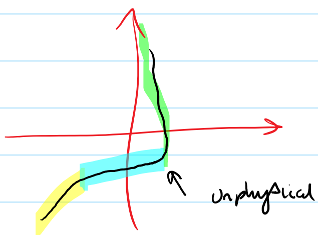
everywhere along curve; null if

$$g_{ab} T^a T^b = 0$$

and spacelike

$$g_{ab} T^a T^b > 0$$

Everywhere along C.



For timelike curve C define the proper time τ via

$$\boxed{\tau = \int \sqrt{-g_{ab} T^a T^b} \, dt}$$

⌈ If a curve C changes from eg, timelike → spacelike, ⇒ length is not defined ⌋

Since, for a geodesic the tangent vector T^a is parallel transported along curve itself, its norm

$$(T^a, T^a) = g_{ab} T^a T^b$$

cannot change ⇒ a geodesic cannot change from null to timelike etc.

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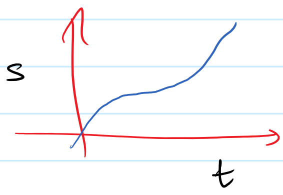
Change in norm: $\quad (T^c \nabla_c)\, \text{norm} = (T^c \nabla_c)(g_{ab} T^a T^b)$

$$= \cancel{(T^c \nabla_c g_{ab})}(T^a T^b) + g_{ab}\,(T^c \cancel{\nabla_c T^a})\,T^b + g_{ab} T^a (\cancel{T^c \nabla_c T^b})$$

$$= 0$$

l (k τ) are reparameterisation invariant:
suppose we chose $s(t)$, instead of t, to parametrise C: New tangent vector (ex.):

$$\boxed{S^a = \frac{dt}{ds} T^a}$$



The length l defined w.r.t. s is

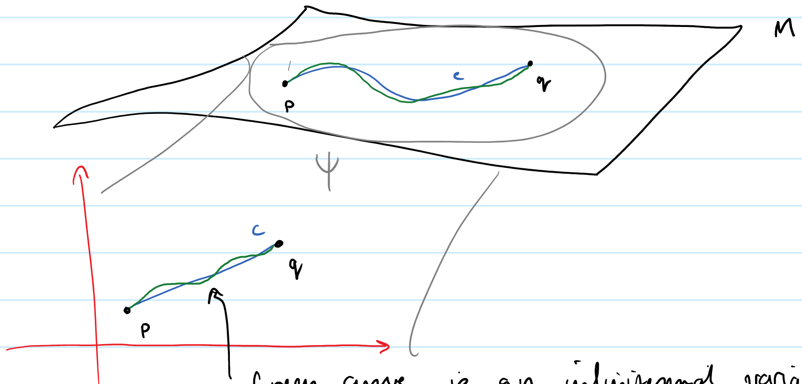
$$l' = \int \sqrt{g_{ab} S^a S^b}\, ds = \int \sqrt{g_{ab} T^a T^b} \cdot \frac{dt}{ds}\, ds = l$$

Claim: geodesics extremise length of curve joining two points p & q.

Suppose p & q are in a common chart $\phi: M \to \mathbb{R}^n$.
Assume C is spacelike.

$$(*) \quad l = \int_a^b \sqrt{\sum_{\mu,\nu} g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}}\, dt$$

where $C(a) = p$; $C(b) = q$ are end points.



Green curve is an infinitesimal variation of the C

$$x^\mu(t) \longmapsto x^\mu(t) + \delta x^\mu(t)$$

where $\delta x^\mu(a) = \delta x^\mu(b) = 0$ $\qquad \searrow l(x+\delta x) = l(x) + \delta l(x) + \cdots$

How does the length of the $C' = x^\mu + \delta x^\mu$ change?
(Ex.)

$$\delta l = \int_a^b \left(\sum_{\mu,\nu} g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \right)^{-1/2} \sum_{\alpha,\beta} \left\{ g_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{d(\delta x^\beta)}{dt} + \frac{1}{2} \sum_\sigma \frac{\partial g_{\alpha\beta}}{\partial x^\sigma} \delta x^\sigma \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \right\} dt$$

Assume we have chosen a parametrisation which so that

$g_{ab} T^a T^b = 1$ $\qquad$ curve $\to \int_0 \to$ number

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$$g_{ab}T^aT^b = 1$$

curve $\rightarrow \boxed{l} \rightarrow$ number

<u>Extremality</u> : $\delta l = 0$
$\forall\ \delta x^\alpha$

[Remember l is a functional of functions, i.e. curves]

$$0 = -\sum_{\alpha} g_{\alpha\beta}\frac{d^2x^\alpha}{dt^2} - \sum_{\alpha,\lambda}\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}\frac{dx^\lambda}{dt}\frac{dx^\alpha}{dt} + \frac{1}{2}\sum_{\alpha,\lambda}\frac{\partial g_{\alpha\lambda}}{\partial x^\beta}\frac{dx^\alpha}{dt}\frac{dx^\lambda}{dt}$$

$\Rightarrow$ This <u>is</u> the <u>geodesic equation</u>.

A curve extremises the length $\Leftrightarrow$ it <u>is</u> a geodesic.

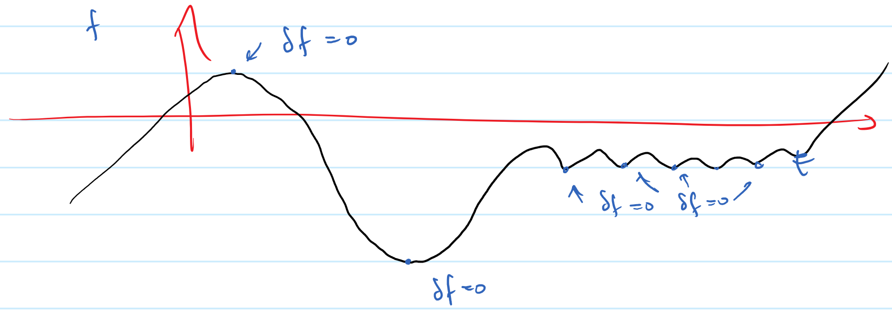
A similar derivation shows that geodesic equation can be obtained by varying / extremising <u>Lagrangian</u>

$$\boxed{L = \sum_{\mu,\nu} g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}$$

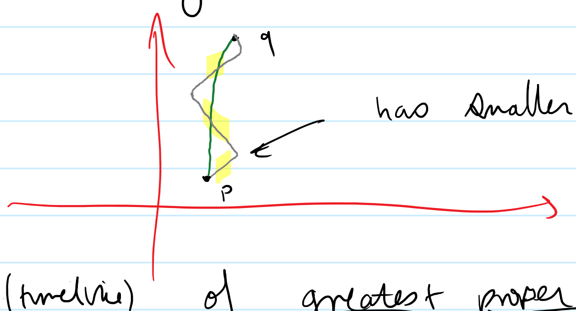
$\Rightarrow$ This can give a very efficient way to calculate Christoffel symbols.

$$\left[\ \frac{\partial L}{\partial x^\mu} - \frac{d}{dt}\left(\frac{\partial L}{\partial\left(\frac{dx^\mu}{dt}\right)}\right) = 0\ \right]$$

<u>Extremise</u> vs <u>minimise</u>.



For Lorentz signature manifolds M a curve (timelike) joining two points may have arbitrarily small proper time



If a curve (timelike) of <u>greatest proper time</u> exists, it must be a timelike geodesic. Note a given geodesic for $p \to q$ need not maximise proper time.

Curvature

Path-dependence of parallel transport of a vector from V_p to V_q gives an intrinsic proxy for curvature of M.

Riemann curvature tensor $\equiv$ measure of failure of successive parallel transport operations to commute.

Start by studying action $\nabla_a \nabla_b$ on an arbitrary dual vector field ω_c. Suppose $f \in \mathcal{T}(M)$ & consider

$$\nabla_a \nabla_b (f\omega_c) = \nabla_a (\omega_c \nabla_b f + f \nabla_b \omega_c)$$
$$= (\nabla_a \nabla_b f)\omega_c + (\nabla_b f)(\nabla_a \omega_c) + (\nabla_a f)(\nabla_b \omega_c) + f \nabla_a \nabla_b \omega_c$$

Consider

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)(f\omega_c) \overset{ex}{=} f(\nabla_a \nabla_b - \nabla_b \nabla_a)\omega_c \quad (\text{true } \forall f \in \mathcal{T}(M))$$

Thus $(\nabla_a \nabla_b - \nabla_b \nabla_a)\omega_c$ only depends on value of ω_c at p.
So

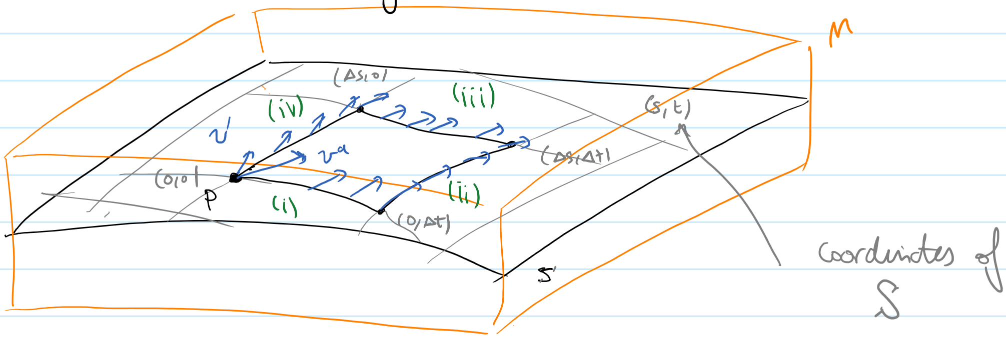
$$(\nabla_a \nabla_b - \nabla_b \nabla_a) : \mathcal{T}(0,1) \to \mathcal{T}(0,3),$$

ie there exists $R_{abc}{}^d \in \mathcal{T}(1,3)$ such that $\forall \omega_c \in \mathcal{T}(0,1)$

$$\boxed{(\nabla_a \nabla_b - \nabla_b \nabla_a)\omega_c = R_{abc}{}^d \omega_d}$$

$R_{abc}{}^d \equiv$ Riemann curvature tensor.

Relate $R_{abc}{}^d$ to failure of successive parallel transports:



Consider a loop starting at $p \in M$ defined by a 2D surface S through p. Let coordinates of S be (s,t), with $(s,t) = (0,0) = \psi(p)$. The loop is then defined by

$$(0,0) \xrightarrow{(i)} (0,\Delta t) \xrightarrow{(ii)} (\Delta s, \Delta t) \xrightarrow{(iii)} (\Delta s, 0) \xrightarrow{(iv)} (0,0).$$

Let v^a be an arbitrary vector at p and parallel transport it around loop. Suppose $\omega_a \in \mathcal{T}(0,1)$ is arbitrary.

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Consider $v^a\omega_a \in \mathcal{T}(M)$

The change δ_1 in $v^a\omega_a$ along (i) is

$$\delta_1 = \Delta t \; \frac{\partial}{\partial t}(v^a\omega_a)\Big|_{(\frac{\Delta t}{2},0)}$$

evaluating at midpoint ensures expression is correct to 2nd order in Δt.

$\Rightarrow$

$$\delta_1 = \Delta t \; T^b\nabla_b(v^a\omega_a)\Big|_{(\frac{\Delta t}{2},0)}$$

$$= \Delta t \; v^a \, T^b\nabla_b\omega_a\Big|_{(\frac{\Delta t}{2},0)} \qquad (v^a \text{ is parallel transported})$$

Here T^a is the tangent vector to curve with constant s.

Similarly: we obtain

$$\delta_3 = -\Delta t \; v^a \, T^b \nabla_b \, \omega_a\Big|_{(\frac{\Delta t}{2},\Delta s)}$$

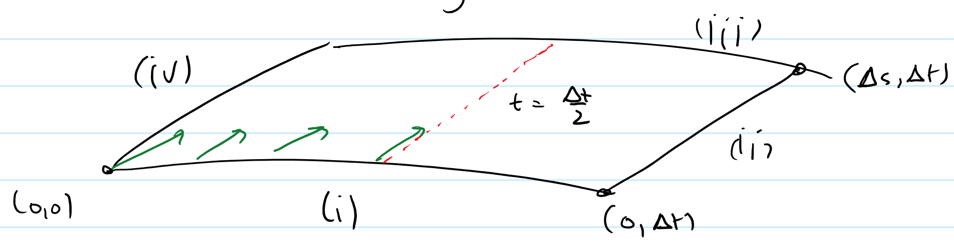
Combine $\delta_1 + \delta_3$:

$$\delta_1+\delta_3 = \Delta t\left(v^a T^b \nabla_b\omega_a\Big|_{(\frac{\Delta t}{2},0)} - v^a T^b \nabla_b\,\omega_a\Big|_{(\frac{\Delta t}{2},\Delta s)}\right)$$

Note: $\delta_1+\delta_3 \to 0$ as $\Delta s \to 0$. Also $\delta_2+\delta_4 \to 0$ as $\Delta t \to 0$

So $\delta_1+\delta_2+\delta_3+\delta_4 = 0$ to first order in $(\Delta s, \Delta t)$!
ie parallel transport around loop is path-independent to first order.

To get 2nd order dependence consider parallel transport of v^a and $T^b\nabla_b\omega_a$ along $t = \Delta t/2$:



To first order v^a at $(\Delta s, \Delta t/2)$ is equal to v^a at $(0, \Delta t/2)$

The quantity $T^b\nabla_b\omega_a$ at $(\Delta s, \Delta t/2)$ differs from $T^b\nabla_b\omega_a$ at $(0, \Delta t/2)$ parallel transported to $(\Delta s, \Delta t/2)$

via (ex)

(**) $\Delta s\, S^c \nabla_c (T^b \nabla_b \omega_a)$

where S^c is tangent to curve of constant t. Substituting (**) into $\delta_1 + \delta_3$:

$$\delta_1 + \delta_3 = -\Delta t\, \Delta s\; v^a S^c \nabla_c (T^b \nabla_b \omega_a) \qquad (\text{similar for } \delta_2 + \delta_4)$$

Add together (ex)

$$\begin{aligned}
\delta(v^a \omega_a) &= \Delta s\, \Delta t\; v^a (T^c \nabla_c (S^b \nabla_b \omega_a) - S^c \nabla_c (T^b \nabla_b \omega_c))) \\
&= \Delta s\, \Delta t\; v^a\, T^c S^b (\nabla_c \nabla_b - \nabla_b \nabla_c) \omega_a \\
&= \Delta s\, \Delta t\; v^a\, T^c S^b\, R_{cba}{}^d\, \omega_d
\end{aligned}$$

① ②

Here we used the fact that

$$[v, w]^b = v^a \nabla_a w^b - w^a \nabla_a v^b = 0$$

for coordinate vector fields $v^a = \frac{\partial}{\partial s}$ $\quad w^b = \frac{\partial}{\partial t}$

This variation $\delta(v^a \omega_a)$ can hold for all ω_a only if (to 2nd order in Δs & Δt)

$$\boxed{\delta v^a = \Delta s\, \Delta t\; v^d T^c S^b\, R_{cbd}{}^a}$$