Introduction to general relativity: curvature cont.

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Action of $(\nabla_a \nabla_b - \nabla_b \nabla_a)$ $\left(\equiv [\nabla_a, \nabla_b] \right)$ on vectors & tensors may be obtained as follows

Let t^a & ω_a are arbitrary vector & dual vector fields. Consider $t^a \omega_a$:

$$0 = (\nabla_a \nabla_b - \nabla_b \nabla_a)(t^c \omega_c) = \nabla_a (\omega_c \nabla_b t^c + t^c \nabla_b \omega_c) - \nabla_b (\omega_c \nabla_a t^c + t^c \nabla_a \omega_c)$$

$$= \omega_c (\nabla_a \nabla_b - \nabla_b \nabla_a) t^c + t^c (\nabla_a \nabla_b - \nabla_b \nabla_a) \omega_c$$

$$0 = \omega_c (\text{———}) t^c + t^c \omega_d R_{abc}{}^d$$

$\Rightarrow$

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) t^c = - R_{abd}{}^c t^d$$

Similarly, (ex 1): we find $T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$:

$$(*) \quad \boxed{(\nabla_a \nabla_b - \nabla_b \nabla_a) T^{c_1 \cdots c_k}{}_{d_1 \cdots d_l} = - \sum_{j=1}^{k} R_{abe}{}^{c_j} T^{c_1 \cdots e \cdots c_k}{}_{d_1 \cdots d_l} + \sum_{j=1}^{l} R_{abd_j}{}^{e} T^{c_1 \cdots c_k}{}_{d_1 \cdots e \cdots d_l}}$$

The Riemann tensor has four important properties:

1. $R_{abc}{}^d = - R_{bac}{}^d$
2. $R_{[abc]}{}^d = 0$
3. For ∇_a defined by metric $\nabla_a g_{bc} = 0$:
   $$R_{abcd} = - R_{abdc}$$
4. $\nabla_{[a} R_{bc]d}{}^e = 0$ (Bianchi identity)

Proof: 1. follows by definition

2. Note for arbitrary ω_a & ∇_a we have
$$\nabla_{[a} \nabla_b \omega_{c]} = 0$$

Check this as follows:

LHS:

Antisymmetrise over abc, leave d alone.

$$\text{LHS} = \nabla_{[a} \partial_b \omega_{c]} + \nabla_{[a} C_{bc]}{}^d \omega_d$$

$$= \partial_{[a} \partial_b \omega_{c]} + C_{[ab}{}^d \partial_{|d|} \omega_{c]} + C_{[ac}{}^d \partial_{b]} \omega_d$$

$$+ \partial_{[a} \cancel{\Gamma_{bc]}}{}^d \omega_d + C_{[ab}{}^e \cancel{\Gamma}_{|e|c]}{}^d \omega_d + C_{[a|c|}{}^e \cancel{\Gamma}_{b]e}{}^d \omega_d$$

$= 0$ Thus

$$0 = 2\nabla_{[a}\nabla_b \omega_{c]} = \nabla_{[a}\nabla_b\omega_{c]} - \nabla_{[b}\nabla_a\omega_{c]} = R_{[abc]}{}^d\omega_d$$

3. Consider:

$$0 = (\nabla_a\nabla_b - \nabla_b\nabla_a) g_{cd} \overset{!}{\underset{(*)}{=}} R_{abc}{}^e g_{ed} + R_{abd}{}^e g_{ce} = R_{abcd} + R_{abdc}$$

4. Consider

(i) $= (\nabla_a\nabla_b - \nabla_b\nabla_a)\nabla_c\omega_d \overset{(*)}{=} R_{abc}{}^e\nabla_e\omega_d + R_{abd}{}^f\nabla_c\omega_f$

vs.

(ii) $= \nabla_a(\nabla_b\nabla_c\omega_d - \nabla_c\nabla_b\omega_d) = \nabla_a(R_{bcd}{}^e\omega_e) = \omega_e\nabla_a R_{bcd}{}^e + R_{bcd}{}^e\nabla_a\omega_e$

Antisymmetrise (i) & (ii) over abc → both expressions become equal:

$$\overset{0}{\cancel{R_{[abc]}{}^e}}\nabla_e\omega_d + R_{[ab|d|}{}^f\nabla_{c]}\omega_f = \omega_e\nabla_{[a}R_{bc]d}{}^e + R_{[bc|d|}{}^e\nabla_{a]}\omega_e$$

Thus, for all ω_e: $\omega_e\nabla_{[a}R_{bc]d}{}^e = 0$ □

Riemann curvature tensor:

vector vector vector dual vector

$$R(\ ,\ ,\ ,\ )$$

This argument defines one coordinate of 2D subspace of M

defines 2nd coordinate axis

$$R(v, w, \cdot, \cdot) \cong M_a{}^b$$

$\forall \quad R(v, w, \cdot, \cdot) \cong M_a{}^b$

a matrix or linear transformation of V_p

Defn

Ricci tensor:

$$\boxed{R_{ac} \equiv R_{abc}{}^b}$$

Note.

$$R_{ab} = R_{ba} \qquad (\text{by } \underline{3}.)$$

Scalar curvature

$$\boxed{R \equiv R_a{}^a}$$

Contract Bianchi identity:

$$\nabla_a R_{bcd}{}^a + \nabla_b R_{cd} - \nabla_c R_{bd} = 0$$

contract with g^{bd}:

$$\nabla_a R_c{}^a + \nabla_b R_c{}^b - \nabla_c R = 0$$

Written

$$\nabla^a G_{ab} = 0 \qquad (\text{"Divergence" definition of})$$

where:

$$\boxed{G_{ab} = R_{ab} - \tfrac{1}{2} R g_{ab}}$$

Is the Einstein tensor.

Geodesic deviation equation

- Tells us how nearby geodesics accelerate away or toward each other.

Suppose we have a smooth one-parameter family $\gamma_s(t)$ of geodesics. That is, γ_s is a geodesic $\forall s \in [-\varepsilon, \varepsilon]$, and $(s, t) \longrightarrow \gamma_s(t)$ is smooth, one-to-one map with smooth inverse. Let Σ be 2-D submanifold described by $\gamma_s(t)$; (s,t) are a coord system of Σ

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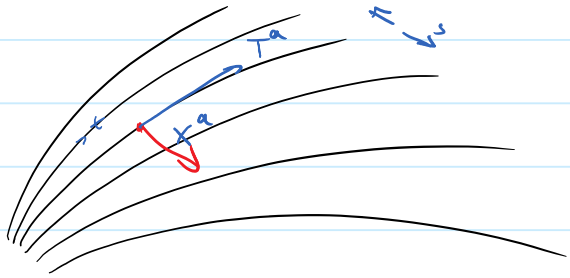

$T^a \equiv \left(\partial/\partial t\right)^a$ is tangent to $\gamma_s(t)$, we have

$$T^a \nabla_a T^b = 0$$

The vector field $X^a = \left(\partial/\partial s\right)^a$ is <u>displacement</u> to infinitesimally close geodesic $\gamma_{s+\delta}(t)$

Rescale t with a s-dependent factor so that $g_{ab} T^a T^b$ does not vary with s. Since X^a, T^a are coordinate fields so

$$T^b \nabla_b X^a = X^b \nabla_b T^a$$

$\Rightarrow$ $X^a T_a$ is constant. A reparametrisation by addition of (s-dependent) constant

$$X^a T_a = 0$$

The quantity $v^a \equiv T^b \nabla_b X^a$ measures rate of change of displacement X^a along geodesic ; "relative velocity"

"Relative acceleration"

$$\begin{aligned}
A^a \equiv T^c \nabla_c v^a &= \underline{T^c \nabla_c} (\underline{T^b \nabla_b} X^a) \\
&= T^c \nabla_c (X^b \nabla_b T^a) \qquad (X^a \text{ \& } T^a \text{ commute}) \\
&= (T^c \nabla_c X^b)(\nabla_b T^a) + X^b T^c \nabla_c \nabla_b T^a \\
&= (X^c \nabla_c T^b)(\nabla_b T^a) + X^b T^c \nabla_b \nabla_c T^a - R_{cbd}{}^a X^b T^c T^d \\
&= X^c \nabla_c (T^b \cancel{\nabla_b T^a}) - R_{cba}{}^a X^b T^c T^d \\
&= -R_{cbd}{}^a X^b T^c T^d
\end{aligned}$$

$\rightarrow$ <u>Geodesic deviation equation</u>

If.

$A^a = 0$ for all geodesic families $\gamma_s(t) \Leftrightarrow R_{abc}{}^d = 0$

If $\exists$ relatively accelerating geodesics (families)
$\Rightarrow \quad R_{abc}{}^{d} \neq 0$

Computing curvature

Given manifold M with metric g_{ab} then $R_{abc}{}^{d}$ is determined uniquely by ∇_a

In practice we choose coordinates (ψ, x^μ)

determined by ψ.

$$\nabla_b \omega_c = \partial_b \omega_c - \Gamma^d_{bc}\, \omega_d$$

So

$$\nabla_a \nabla_b \omega_c = \partial_a (\partial_b \omega_c - \Gamma^d_{bc} \omega_d) - \Gamma^e_{ab} (\partial_e \omega_c - \Gamma^d_{ec} \omega_d) - \Gamma^e_{ac} (\partial_b \omega_e - \Gamma^d_{be} \omega_d)$$

Thus:

$$R_{abc}{}^{d} \omega_d = 2 \nabla_{[a} \nabla_{b]} \omega_c \qquad \text{(ex.)}$$

$$R_{abc}{}^{d} \omega_d = \left(-2\, \partial_{[a} \Gamma^d_{b]c} + 2\, \Gamma^e_{c[a} \Gamma^d_{b]e} \right) \omega_d$$

This holds for all ω_d, so in coordinates ψ:

$$\boxed{R_{\mu\nu\rho}{}^{\sigma} = \frac{\partial}{\partial x^\nu} \Gamma^\sigma_{\mu\rho} - \frac{\partial}{\partial x^\mu} \Gamma^\sigma_{\nu\rho} + \sum_\alpha \left(\Gamma^\alpha_{\mu\rho} \Gamma^\sigma_{\alpha\nu} - \Gamma^\alpha_{\nu\rho} \Gamma^\sigma_{\alpha\mu} \right)}$$

Ricci tensor is simply $R_{\mu\rho} \equiv \sum_\nu R_{\mu\nu\rho}{}^{\nu}$

Some useful formulas

Define $\quad g = \det(g_{\mu\nu})$

Use formula for inverse of matrix: (ex)

$$\sum_{\nu,\alpha} g^{\nu\alpha} \frac{\partial g_{\nu\alpha}}{\partial x^\mu} = \frac{1}{g} \frac{\partial g}{\partial x^\mu}$$

and

$$= \partial_\mu \log(|g|)$$

∂x^μ

This equation appears in formula for Christoffel symbols

$$\Gamma^{a}_{a\mu} = \sum_\nu \Gamma^{\nu}_{\nu\mu} = \frac{1}{2} \sum_{\nu,\alpha} g^{\nu\alpha} \frac{\partial}{\partial x^\mu} g_{\nu\alpha}$$

$\Rightarrow$ by above $\Rightarrow$ $= \frac{1}{2} \frac{1}{g} \frac{\partial}{\partial x^\mu} g = \frac{\partial}{\partial x^\mu} \log\left(\sqrt{|g|}\right)$

Appears as well in <u>divergence</u> $\nabla_a T^a$ of vector field T^a

$$\boxed{\begin{aligned} \nabla_a T^a &= \partial_a T^a + \Gamma^{a}_{ab} T^b \\ &= \sum_\mu \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\mu} \left(\sqrt{|g|}\, T^\mu\right). \end{aligned}}$$