Introduction to general relativity: gravitational radiation

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In GR the gravitational field is dynamic $\Rightarrow$ possibility of <u>gravitational radiation</u>

We use the source-free linearised Einstein's field equations:

(i) $\partial^a \bar{\gamma}_{ab} = 0 \implies$ gauge choice

(ii) $\partial^c \partial_c \bar{\gamma}_{ab} = 0$

(i) doesn't completely fix gauge: we are still free to make transformation

$$\gamma_{ab} \longmapsto \gamma_{ab} + \partial_a \xi_b + \partial_b \xi_a$$

as long as

(iii) $\partial^b \partial_b \xi^a = 0$

<u>Radiation gauge</u>:

(iv) $\gamma = 0$ and $\gamma_{0j} = 0 \qquad j = 1,2,3$
$(\gamma = \gamma^a{}_a)$

If, further, $T_{ab} = 0$ everywhere, then we can further achieve

$$\gamma_{00} = 0$$

We can transform to radiation gauge via ξ_a solving the following equations on a fixed initial surface $t = t_0$

$$2\left(-\frac{\partial \xi_0}{\partial t} + \underline{\nabla} \cdot \underline{\xi}\right) = -\gamma$$

$$2\left(-\nabla^2 \xi_0 + \underline{\nabla} \cdot \partial \underline{\xi}\right) = -\partial\gamma \qquad \text{(take time derivative)}$$

($\frac{\partial}{\partial t}$, $\frac{\partial}{\partial t}$ of previous eq^n)

$$\frac{\partial \xi_\mu}{\partial t} + \frac{\partial \xi_0}{\partial x^\mu} = -\gamma_{0\mu} \qquad (\mu = 1,2,3)$$

$$\nabla^2 \xi_\mu + \frac{\partial}{\partial x^\mu}\left(\frac{\partial \xi_0}{\partial t}\right) = -\frac{\partial \gamma_{0\mu}}{\partial t} \qquad (\mu=1,2,3)$$

Initial value problem: given ξ_μ & $\frac{\partial \xi_\mu}{\partial t}$ on $t = t_0$ we solve for ξ_μ. One then has (ex.) that $\gamma = 0$ & $\gamma_{0j} = 0$ in the source-free region

If $T_{ab} = 0$ everywhere we achieve $\gamma_{00} = 0$ as follows:

$$\gamma = 0 \Rightarrow \qquad \bar{\gamma}_{0b} = \gamma_{0b} . \qquad (i) \Rightarrow$$

$$\frac{\partial \gamma_{00}}{\partial t} = 0$$

Solve linearised Einstein's field equations

$$\nabla^2 \gamma_{00} = -16\pi T_{00} = 0$$

$\Rightarrow$ γ_{00} = constant. Then shift γ_{00} by constant

To find wavelike solutions assume

$$\boxed{\gamma_{ab} = H_{ab}\, e^{i \sum_{\mu=0}^{3} k_\mu x^\mu}}$$

where $H_{ab} \equiv$ constant.

Substituting into (ii) :

$$\sum_{\mu=0}^{3} k^\mu k_\mu = 0 .$$

Equations (i) & (iv) :

(a) $$\sum_{\mu=0}^{3} k^\mu H_{\mu\nu} = 0 .$$

(a) $$\sum_{\mu=0} k^{\mu} H_{\mu\nu} = 0 .$$

(b) $$H_{0\nu} = 0 .$$

(c) $$\sum_{\mu=0}^{3} H^{\mu}{}_{\mu} = 0 .$$

Only 8 equations are independent as (a) & (b) $\Rightarrow$

$$\sum_{\nu} H_{0\nu} k^{\nu} = 0 .$$

A symmetric tensor of type $(0,2)$ has 10 independent components

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$\Rightarrow$ only two linearly independent solutions for h_{ab}

<u>Lower dimensions?</u> Eg. $(2+1)d$?

Here gauge conditions $\Rightarrow$ 6 independent equations. However A type $(0,2)$ tensor has 6 components. $\Rightarrow$ no degrees of freedom!

We could have guessed this already!!

In $(2+1)d$:

$$\boxed{R_{abcd} = g_{ac} R_{bd} + g_{bd} R_{ac} - g_{bc} R_{ad} - g_{ad} R_{bc} - \tfrac{1}{2}(g_{ac} g_{bd} - g_{ad} g_{bc}) R}$$

This is because the Riemann tensor may be decomposed in terms of R_{ab}, R, and a completely trace-free tensor C_{abcd} called Weyl tensor. In $(2+1)d$ C_{abcd} vanishes

Look at vacuum EFE:

$$R_{ab} - \tfrac{1}{2} g_{ab} R = 0$$

$$\Rightarrow \quad R = 0 \quad \& \quad R_{ab} = 0$$

A final note: Gravitational waves were detected Sep. 14, 2015 via laser interferometry.

Homogeneous isotropic cosmology

To solve

$$G_{ab} = 8\pi T_{ab}$$

we assume sufficiently homogeneous & isotropic distribution of matter.

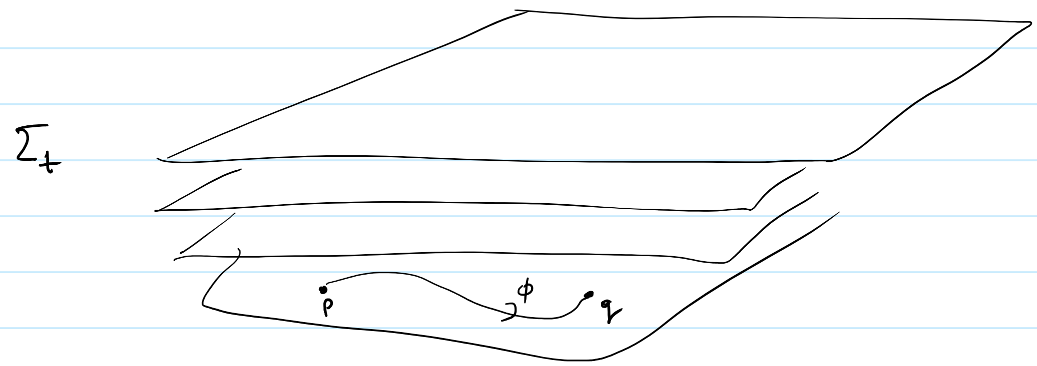
Homogeneity and isotropy

Assumption 1 (homogeneity): We don't occupy a privileged position in universe $\Rightarrow$ universe should look same no matter where we are

Assumption 2 (isotropy): no preferred direction in space

A most compelling supporting observation: isotropy of cosmic background radiation

Definition: A spacetime manifold M is homogeneous if there exists a one-parameter foliation of M via hypersurfaces Σ_t such that for each t and any points p & $q \in \Sigma_t$ there exists an isometry of g_{ab} which takes $p \to q$.



Remark (isotropy): Only one observer at each point can perceive universe as isotropic (boosted observers see background distorted)

<u>Definition (isotropy)</u>: A spacetime M is <u>(spatially) isotropic</u> at each point if there exists a congruence of timelike curves with tangents u^a filling M with property that $\forall p \in M$ and any two unit <u>spatial tangent</u> vectors s_1^a & $s_2^a \in V_p$, there exists isometry of g_{ab} leaving p & u^a fixed but rotates $s_1^a \to s_2^a$. (Here $u^a s_{1a} = u^a s_{2a} = 0$)

<u>Remark</u>: for a homogeneous and isotropic spacetime the surfaces Σ_t of homogeneity must be orthogonal to u^a.