Microscopic Models for Mobile Anyons from Fusion Categories

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Abstract

We develop a systematic framework for constructing microscopic lattice models describing mobile (itinerant) anyons arising from an arbitrary fusion category. Unlike existing models such as the golden chain where anyons are fixed at predetermined positions, our framework allows both the positions and number of anyons to fluctuate dynamically. Working in a first-quantised formalism on a one-dimensional chain with open boundary conditions, we construct Hilbert spaces that accommodate variable anyon number and anyon mobility, and define microscopic Hamiltonians for physically motivated scenarios. The framework is validated by reduction to known limiting cases including standard bosonic/fermionic systems and tightly-packed fusion chains.

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1 Introduction

1.1 Why Mobile Anyons?

Anyonic particles—quasiparticle excitations obeying exotic exchange statistics that interpolate between bosonic and fermionic behaviour—have become central objects of study in condensed matter physics and quantum information theory. The mathematical framework of fusion categories provides a rigorous foundation for describing systems of anyons, encoding their fusion rules, braiding statistics, and algebraic structure in a unified formalism [Kit06, ENO05].

Existing microscopic many-body models based on fusion categories, such as the celebrated golden chain [FTLM07], describe systems where anyons occupy fixed positions on a lattice. In such models, the number of anyons is constant and their positions are predetermined—analogous to the tightly-packed Mott insulating phase in condensed matter systems. While these models have yielded profound insights into anyonic physics, including connections to conformal field theory at criticality, they represent only a special limiting case of the broader landscape of possible anyonic phases.

In realistic physical systems, however, particle number and position are typically dynamical degrees of freedom. Electrons in metals move freely through the lattice; ultracold atoms in optical lattices tunnel between sites; quasiparticles in quantum Hall systems can be created, annihilated, and transported. A complete understanding of anyonic matter requires extending the categorical framework to accommodate such *mobile* or *itinerant* anyons.

The motivation for developing microscopic models of mobile anyons is threefold:

(i) **Connection to realistic physics.** Physical systems supporting anyonic excitations—such as fractional quantum Hall states or topological superconductors—generically allow for variable anyon number and mobility. Microscopic models capturing these features would bridge the gap between abstract categorical data and experimentally accessible phenomena including transport, scattering, and thermodynamic properties.

(ii) **Exploration of new phases.** Beyond the Mott-like limit of fixed anyons, one expects a rich phase diagram including dilute anyonic gases, anyonic superfluids, and intermediate correlated phases. The interplay between anyonic exchange statistics and spatial dynamics may give rise to novel collective phenomena without analogue in conventional bosonic or fermionic systems.

(iii) **Recovery of known limits.** A consistent framework must reproduce well-understood special cases. When fusion rules are trivial ($X \otimes X = \mathbf{1}$), we should recover ordinary bosons or fermions depending on the braiding structure. For super-vector spaces (sVec), the construction should reduce exactly to fermionic Fock space with standard anticommutation relations. For dense configurations, we should recover the physics of the golden chain and related models.

The present work addresses this gap by developing a systematic framework for constructing microscopic lattice models describing mobile anyons arising from an arbitrary fusion category. Working in a first-quantised formalism on a one-dimensional chain with open boundary conditions, we construct Hilbert spaces that accommodate variable anyon number and anyon mobility, and define microscopic Hamiltonians for physically motivated scenarios.

1.2 Literature Gap

The theoretical study of anyonic systems has developed along two largely disjoint lines. On the mathematical side, the theory of fusion categories [ENO05, EGNO15] provides a complete algebraic framework for describing anyon types, their fusion rules, and braiding statistics. This framework is abstract and basis-independent, applicable to any system whose excitations form a fusion category. On the physical side, microscopic lattice models—most prominently the golden chain [FTLM07] and its generalisations—have demonstrated how to construct explicit Hamiltonians whose low-energy physics is governed by anyonic degrees of freedom.

However, these microscopic models invariably assume a *dense* configuration: one anyon occupies each lattice site, and the Hilbert space is spanned by the different ways adjacent anyons can fuse. This corresponds to a Mott-insulating regime where particle number is maximal and fixed, and spatial dynamics are frozen out. The only degrees of freedom are the internal fusion channels—essentially, which superselection sector the system occupies locally.

Several important physical questions lie outside this dense limit:

- **Dilute anyonic gases.** What is the ground state of a system with fewer anyons than sites? How do anyonic statistics affect spatial correlations in a dilute gas?
- **Hopping and transport.** If anyons can tunnel between sites, how does the interplay of mobility and exotic statistics manifest in transport properties?
- **Variable particle number.** Can we define pair-creation and annihilation processes consistent with fusion rules? What phases emerge when particle number fluctuates?
- **Interpolation to known limits.** A framework for mobile anyons should reduce to standard fermionic or bosonic physics when the fusion category is trivial (sVec or Vec respectively), providing a consistency check and physical intuition.

To our knowledge, no systematic framework exists for constructing microscopic lattice models of mobile anyons arising from a general fusion category. The present work aims to fill this gap.

1.3 Contributions of This Work

This paper develops a systematic framework for mobile anyons on a one-dimensional lattice, making the following contributions:

(1) **First-quantised Fock space construction.** We construct Hilbert spaces $\mathcal{H}_N^{(c)}$ for N anyons with total charge c, built from morphism spaces of the underlying fusion category. The total Hilbert space $\mathcal{H} = \bigoplus_N \mathcal{H}_N$ accommodates variable particle number without invoking second-quantised creation/annihilation operators.

(2) **Configuration space formalism.** We define labelled configurations $(\mathbf{j}, \mathbf{k})$ specifying both positions and anyon types, with careful treatment of hard-core (at most one anyon per site) and soft-core (multiple occupancy allowed) constraints.

(3) **Local operator algebra.** We characterise the space of k-local operators—those acting nontrivially on at most k consecutive sites—in terms of morphisms between tensor products of simple objects. This provides the building blocks for physically motivated Hamiltonians.

(4) **Explicit matrix elements.** For 2-local operators, we derive matrix element formulas in the fusion tree basis, expressing hopping, interaction, and braiding terms in terms of the category's F-symbols.

(5) **Recovery of standard limits.** We verify that when specialised to sVec (super-vector spaces), the construction reduces exactly to fermionic Fock space with the correct anti-commutation relations and Jordan–Wigner structure.

(6) **Boundary conditions via module categories.** We show how module categories over $\mathcal{C}$ naturally encode boundary conditions, connecting the bulk categorical data to edge physics.

1.4 Paper Outline

The remainder of this paper is organised as follows.

Part I: Preliminaries establishes the mathematical and physical foundations. We review fusion rings and fusion categories, emphasising morphism spaces and the role of F-symbols as the fundamental structural data. We discuss the diagrammatic calculus for morphisms and present key examples including Fibonacci, Ising, and Temperley–Lieb categories. We then review standard Fock space constructions for bosons and fermions in first-quantised language, and fix conventions for lattice geometry.

Part II: Hilbert Space for Mobile Anyons develops the core construction. We define the configuration space for N anyons on an L-site chain, carefully distinguishing hard-core (exclusion) and soft-core (multiple occupancy) variants. The Hilbert space $\mathcal{H}_N^{(c)}$ for N anyons with total charge c is constructed as a direct sum over configurations, with each summand given by the appropriate morphism space in $\mathcal{C}$. We characterise the space of local operators and develop a basis-independent framework for their action.

Part III: Dynamics constructs explicit Hamiltonians. We derive matrix element formulas for 2-local operators in the fusion tree basis, expressing hopping, on-site interactions, and nearest-neighbour braiding in terms of F-symbols. We then define particle-number-conserving Hamiltonians including kinetic (hopping) terms and anyonic generalisations of the Hubbard interaction.

Part IV: Basic Properties analyses the constructed models. We show how module categories over $\mathcal{C}$ encode boundary conditions, connecting bulk categorical data to edge physics. We verify reduction to fermionic Fock space for sVec and to the golden chain in the dense limit.

We conclude with a discussion of open problems and future directions, including extensions to higher dimensions, incorporation of symmetries, and connections to topological field theory.

Part I
Preliminaries

2 Fusion Ring

Assumptions.

(A1) Finite set of simple objects $\{X_i\}_{i=0}^{d_{\mathcal{C}}-1}$.

(A2) Structure constants $N_{ab}^c \in \mathbb{Z}_{\geq 0}$ are associative and unital with unit $\mathbf{1}$.

2.1 Simple Objects

Definition 2.1 (Simple object)**.** Let $\mathcal{C}$ be an abelian category. An object $X \in \mathcal{C}$ is *simple* if it has exactly two subobjects: 0 and X itself. Equivalently, $X \neq 0$ and every monomorphism $Y \hookrightarrow X$ is either zero or an isomorphism.

Remark 2.2. In a semisimple category, an object is simple if and only if it is nonzero and indecomposable (cannot be written as $X \cong Y \oplus Z$ with $Y, Z \neq 0$).

Lemma 2.3 (Schur's lemma)**.** *In a k-linear abelian category with k algebraically closed, if X is simple then* $\mathrm{End}(X) \cong k$*. That is, every endomorphism of a simple object is a scalar multiple of the identity.*

Consequence 2.4. By semisimplicity of fusion categories (Deligne's theorem), every object $A \in \mathcal{C}$ decomposes as a finite direct sum of simple objects:

$$A \cong \bigoplus_{i \in I} X_i^{\oplus m_i} \tag{1}$$

where X_i are simple and $m_i \in \mathbb{Z}_{\geq 0}$ are multiplicities.

Remark 2.5. For our purposes, we work with fusion categories where the simple objects are *distinguishable* by their labels: $\{X_0, X_1, \ldots, X_{d_{\mathcal{C}}-1}\}$ with $X_0 = \mathbf{1}$ (the tensor unit/vacuum).

Reference: Etingof–Gelaki–Nikshych–Ostrik, *Tensor Categories*, AMS (2015), Def. 1.5.1 [EGNO15] `[unverified]`

2.2 Fusion Ring Definition

Definition 2.6 (Fusion ring)**.** A *fusion ring* is a finitely generated free abelian group $R = \bigoplus_{i \in I} \mathbb{Z} X_i$ with a ring structure satisfying:

1. $X_0 = \mathbf{1}$ is the unit element.
2. The product of basis elements satisfies

$$X_i X_j = \sum_{k \in I} N_{ij}^k X_k, \tag{2}$$

   where $N_{ij}^k \in \mathbb{Z}_{\geq 0}$ are the *fusion coefficients* (or fusion multiplicities).

3. There exists an involution $i \mapsto i^*$ such that

$$N_{ij}^0 = \delta_{i,j^*}. \tag{3}$$

The involution gives duality: $X_i^* = X_{i^*}$. Associativity follows from the ring axioms:

$$\sum_e N_{ij}^e N_{ek}^\ell = \sum_e N_{jk}^e N_{ie}^\ell \quad \text{for all } i, j, k, \ell \in I. \tag{4}$$

Remark 2.7. Fusion rings are generally *not commutative*, i.e., $N_{ij}^k \neq N_{ji}^k$ in general.

Reference: Etingof–Gelaki–Nikshych–Ostrik, *Tensor Categories*, AMS (2015), §3.1 [EGNO15] `[unverified]`

3 Fusion Categories

Assumptions.

(A3.1.2.1) Fusion ring $(R, \{X_i\}_{i \in I}, \mathbf{1})$ with $X_0 = \mathbf{1}$ and $N_{ij}^k \in \mathbb{Z}_{\geq 0}$ (Definition 2.6).

(A3.1.2.2) Associator data F (and, when present, braiding data R) satisfy the pentagon/hexagon equations.

Definition 3.1 (Fusion category)**.** A *fusion category* over an algebraically closed field k (usually $k = \mathbb{C}$) is a k-linear, semisimple, rigid monoidal category

$$(\mathcal{C}, \otimes, \mathbf{1}) \tag{5}$$

satisfying the following conditions:

1. **Finiteness:** There are finitely many isomorphism classes of simple objects. Every object decomposes as a finite direct sum of simples.

2. **Semisimplicity:** All morphism spaces $\mathrm{Mor}(X, Y)$ are finite-dimensional k-vector spaces, and the category is abelian and semisimple.

3. **Rigidity:** Every object $X \in \mathcal{C}$ has a left and right dual X^* with evaluation and coevaluation morphisms satisfying the rigidity axioms (Definition 3.2).

4. **Simple unit:** The tensor unit $\mathbf{1}$ is simple: $\mathrm{End}(\mathbf{1}) \cong k$.

5. **Finite k-linearity:** The monoidal structure is bilinear over k, and composition and tensor product of morphisms are k-linear.

Definition 3.2 (Rigidity axioms)**.** For an object X with dual X^*, the *evaluation* $\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}$ and *coevaluation* $\mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$ morphisms must satisfy the *rigidity axioms* (zigzag identities):

$$(\mathrm{ev}_X \otimes \mathrm{id}_X) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X) = \mathrm{id}_X \tag{6}$$

$$(\mathrm{id}_{X^*} \otimes \mathrm{ev}_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_{X^*}) = \mathrm{id}_{X^*} \tag{7}$$

Diagrammatically, these are the "straightening" of a bent string (cup followed by cap yields identity).

Definition 3.3 (Simple objects of a category)**.** For a fusion category $\mathcal{C}$, we denote by $\mathrm{Irr}(\mathcal{C})$ the set of isomorphism classes of simple objects. We write $[X] \in \mathrm{Irr}(\mathcal{C})$ for the isomorphism class containing simple object X.

Definition 3.4 (Quantum dimension)**.** For a simple object X in a fusion category $\mathcal{C}$, the *quantum dimension* d_X is defined via the categorical trace:

$$d_X = \mathrm{tr}(\mathrm{id}_X) = \mathrm{ev}_X \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_{X^*}) \circ \mathrm{coev}_X \tag{8}$$

Diagrammatically, d_X is the value of a closed loop labelled by X. For the tensor unit, $d_{\mathbf{1}} = 1$. The *total dimension* of the category is $\dim(\mathcal{C}) = \sum_{X \in \mathrm{Irr}(\mathcal{C})} d_X^2$.

Remark 3.5. Quantum dimensions satisfy $d_X d_Y = \sum_Z N_{XY}^Z d_Z$ (compatible with fusion rules) and $d_X = d_{X^*}$. For unitary categories, $d_X \geq 1$ with equality iff X is invertible.

Definition 3.6 (Skeletal category)**.** A category is *skeletal* if isomorphic objects are equal: $X \cong Y$ implies $X = Y$. Every category is equivalent to a skeletal one, and for fusion categories we often work with a skeletal representative where the simple objects are $\{X_0, X_1, \ldots, X_{d-1}\}$ with $X_0 = \mathbf{1}$.

Definition 3.7 (Grothendieck ring)**.** From any fusion category $\mathcal{C}$, we construct its *Grothendieck ring* $K_0(\mathcal{C})$ by

$$K_0(\mathcal{C}) = \bigoplus_{[X] \in \mathrm{Irr}(\mathcal{C})} \mathbb{Z}[X], \tag{9}$$

with multiplication

$$[X] \cdot [Y] = \sum_Z N_{XY}^Z [Z], \tag{10}$$

where $N_{XY}^Z = \dim_k \mathrm{Mor}(X \otimes Y, Z)$ is the fusion multiplicity. The Grothendieck ring $K_0(\mathcal{C})$ is a fusion ring (Definition 2.6), establishing that *fusion categories categorify fusion rings.*

Definition 3.8 (Braided fusion category)**.** If additionally $\mathcal{C}$ is equipped with a braiding (natural isomorphisms $c_{X,Y} : X \otimes Y \to Y \otimes X$ satisfying hexagon identities), we call $\mathcal{C}$ a *braided fusion category.*

Reference: Etingof–Gelaki–Nikshych–Ostrik, *Tensor Categories*, AMS (2015), Ch. 4 and Ch. 8 [EGNO15] `[unverified]`

3.1 F-Symbols and Pentagon Equation

Definition 3.9 (F-symbols)**.** The *associator* is a natural isomorphism

$$\alpha_{a,b,c} : (a \otimes b) \otimes c \xrightarrow{\sim} a \otimes (b \otimes c) \tag{11}$$

that satisfies the pentagon equation (Definition 3.11). In a skeletal category (Definition 3.6), the associator is determined by its matrix elements, the *F-symbols*.

For simple objects a, b, c, d, the isomorphism decomposes into blocks indexed by intermediate fusion channels e (for $(a \otimes b) \to e \to d$) and f (for $(b \otimes c) \to f \to d$). The change of basis is given by the *F-move*:

$$|(a \otimes b) \otimes c \to d; e, \alpha, \beta\rangle = \sum_{f,\mu,\nu} (F^d_{abc})^{f,\mu,\nu}_{e,\alpha,\beta} \, |a \otimes (b \otimes c) \to d; f, \mu, \nu\rangle \tag{12}$$

where α, β, μ, ν are *multiplicity indices*.

Remark 3.10 (Multiplicity indices). The indices α, β, μ, ν label basis vectors within fusion spaces when fusion multiplicities $N^c_{ab} > 1$. Specifically:

- $\alpha \in \{1, \ldots, N^e_{ab}\}$ labels basis morphisms in $\mathrm{Mor}(a \otimes b, e)$
- $\beta \in \{1, \ldots, N^d_{ec}\}$ labels basis morphisms in $\mathrm{Mor}(e \otimes c, d)$
- $\mu \in \{1, \ldots, N^f_{bc}\}$ labels basis morphisms in $\mathrm{Mor}(b \otimes c, f)$
- $\nu \in \{1, \ldots, N^d_{af}\}$ labels basis morphisms in $\mathrm{Mor}(a \otimes f, d)$

For *multiplicity-free* categories (where all $N^c_{ab} \in \{0, 1\}$), these indices are trivial and can be suppressed.

Definition 3.11 (Pentagon equation)**.** The *pentagon equation* ensures that the two paths to re-associate $((a \otimes b) \otimes c) \otimes d$ to $a \otimes (b \otimes (c \otimes d))$ coincide. In terms of F-symbols (suppressing multiplicity indices):

$$\sum_k (F^k_{a,b,c})^l_e (F^p_{a,k,d})^m_l (F^p_{b,c,d})^n_k = (F^p_{a,b,n})^m_e (F^m_{e,c,d})^n_l \tag{13}$$

This coherence condition is required for the fusion category to be well-defined.

3.2 R-Symbols and Hexagon Equations

Definition 3.12 (R-symbols)**.** For a braided fusion category, the *braiding isomorphism* $c_{a,b} : a \otimes b \to b \otimes a$ provides a natural way to permute tensor factors. For simple objects a, b, c, the braiding isomorphism is represented by its matrix elements, the *R-symbols*.

Definition 3.13 (Hexagon equations)**.** The *hexagon equations* are coherence conditions that relate the associator (F-symbols) and the braiding (R-symbols), ensuring consistency between re-associating and braiding operations. The first hexagon equation:

$$c_{a,b\otimes c} \circ (1_a \otimes c_{b,c}) = ((c_{a,b} \otimes 1_c) \circ F_{b,a,c} \circ (1_b \otimes c_{a,c})) \circ F^{-1}_{a,c,b} \tag{14}$$

This equation (and its dual) ensures that braiding past a composite object can be decomposed consistently.

Reference: Etingof–Gelaki–Nikshych–Ostrik, *Tensor Categories*, AMS (2015), §8.1–8.2 [EGNO15] `[unverified]`

4 Morphism Spaces and Multiplicities

Assumptions.

(A3.1.3.1) Fusion category $(\mathcal{C}, \otimes, \mathbf{1})$ over an algebraically closed field k (Definition 3.1).

(A3.1.3.2) $\mathcal{C}$ is semisimple and k-linear, so all morphism spaces are finite-dimensional k-vector spaces.

Definition 4.1 (Morphism space)**.** For any objects $A, B \in \mathcal{C}$,

$$\mathrm{Mor}(A, B) := \mathrm{Hom}_{\mathcal{C}}(A, B) \tag{15}$$

is a finite-dimensional k-vector space. If A, B are simple, Schur's lemma implies $\dim \mathrm{Mor}(A, B) = \delta_{A,B}$.

Reference: Etingof–Gelaki–Nikshych–Ostrik, *Tensor Categories*, AMS (2015), §4.2 [EGNO15] `[unverified]`

Definition 4.2 (Fusion multiplicity space)**.** For simple objects $X_a, X_b, X_c \in \mathrm{Irr}(\mathcal{C})$, the space

$$\mathrm{Mor}(X_a \otimes X_b, X_c) \tag{16}$$

has dimension $N_{ab}^c = \dim \mathrm{Mor}(X_a \otimes X_b, X_c) \in \mathbb{Z}_{\geq 0}$. A *multiplicity basis* is any choice of morphisms

$$f_{ab \to c}^{(\mu)} : X_a \otimes X_b \to X_c, \quad \mu = 1, \ldots, N_{ab}^c. \tag{17}$$

No canonical choice exists; computations must remain basis-independent.

Claim 4.3 (Multiplicity-free simplification)**.** *In the multiplicity-free case ($N_{ab}^c \in \{0, 1\}$), each space* $\mathrm{Mor}(X_a \otimes X_b, X_c)$ *is either* $\{0\}$ *or a one-dimensional k-line. Basis dependence disappears, and $f_{ab \to c}^{(1)}$ can be chosen uniquely up to phase.*

Remark 4.4. Duals: $\mathrm{Mor}(\mathbf{1}, X_a \otimes X_b)$ is canonically dual to $\mathrm{Mor}(X_a^* \otimes X_b^*, \mathbf{1})$ via rigidity. Normalisation choices for evaluation/coevaluation maps must be consistent.

Remark 4.5. Basis independence is essential for categorical definitions. Fusion-tree bases are admissible for computations (e.g., numerical evaluation of F-symbols) but must be removed from statements of definitions and theorems.

5 Examples of Fusion Categories

This section enumerates concrete fusion categories used in mobile anyon models. Each example specifies fusion rules, F-symbols, R-symbols (if braided), and the simple object count (rank).

Assumptions.

(A3.1.8.1) Fusion categories listed here are semisimple and rigid.

(A3.1.8.2) Numerical F/R-symbols are computed via standard references (Kitaev, Rowell).

5.1 Fibonacci Category (Multiplicity-Free)

Category: $\mathcal{C}_{\mathrm{Fib}}$
Rank: $d = 2$
Simple objects: $\mathbf{1} = X_0$, $\tau = X_1$

5.1.1 Fusion Rules

$\otimes$	$\mathbf{1}$	τ
$\mathbf{1}$	$\mathbf{1}$	τ
τ	τ	$\mathbf{1} \oplus \tau$

Multiplicities: $N_{\tau\tau}^{\mathbf{1}} = N_{\tau\tau}^{\tau} = 1$ (multiplicity-free).

5.1.2 Quantum Dimensions

- $d_{\mathbf{1}} = 1$
- $d_\tau = \phi = \frac{1+\sqrt{5}}{2}$ (golden ratio)

5.1.3 F-Symbols (Associator)

For the only nontrivial fusion channel $\tau \otimes \tau \to \mathbf{1} \oplus \tau$:

$$F^{\tau}_{\tau,\tau,\tau} = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix} \tag{18}$$

where rows/columns are indexed by the intermediate fusion channel $e \in \{\mathbf{1}, \tau\}$, and the matrix connects the two fusion orders: $(\tau \otimes \tau) \otimes \tau$ vs. $\tau \otimes (\tau \otimes \tau)$. This matrix is unitary: $F^\dagger F = I$.

Numerical values: $\phi^{-1} = \phi - 1 \approx 0.6180$, $\phi^{-1/2} \approx 0.7861$ [Kit06] `[verified]`

5.1.4 R-Symbols (Braiding)

For the *braided* Fibonacci category (adding R-symbols to the fusion category):

$$R^{\mathbf{1}}_{\tau,\tau} = e^{4\pi i/5}, \qquad R^{\tau}_{\tau,\tau} = e^{-3\pi i/5} \tag{19}$$

The topological spin (twist) of τ is $\theta_\tau = e^{4\pi i/5}$, corresponding to conformal weight $h_\tau = 2/5$. `[verified]`

5.2 Ising Category (Multiplicity-Free)

Category: $\mathcal{C}_{\text{Ising}}$
Rank: $d = 3$
Simple objects: $\mathbf{1} = X_0$, $\sigma = X_1$, $\psi = X_2$

5.2.1 Fusion Rules

$\otimes$	$\mathbf{1}$	σ	ψ
$\mathbf{1}$	$\mathbf{1}$	σ	ψ
σ	σ	$\mathbf{1} \oplus \psi$	σ
ψ	ψ	σ	$\mathbf{1}$

Multiplicities: All fusion coefficients are 0 or 1 (multiplicity-free).

5.2.2 Quantum Dimensions

- $d_{\mathbf{1}} = 1$
- $d_\sigma = \sqrt{2}$
- $d_\psi = 1$

5.2.3 F-Symbols

Non-trivial associators exist for $\sigma \otimes \sigma \to \mathbf{1} \oplus \psi$:

$$F^{\sigma}_{\sigma,\sigma,\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{20}$$

This is the Hadamard matrix, which is unitary. `[verified]`

5.2.4 R-Symbols

For braided (modular) Ising category:

$$R^{\mathbf{1}}_{\sigma,\sigma} = e^{i\pi/8}, \quad R^{\psi}_{\sigma,\sigma} = e^{-3i\pi/8} \tag{21}$$

The topological spin of σ is $\theta_\sigma = e^{i\pi/8}$. `[verified]`

5.3 $\mathbb{Z}_N$ Category (with Multiplicity)

Category: $\mathcal{C}_{\mathbb{Z}_N}$
Rank: $d = N$
Simple objects: $X_0, X_1, \ldots, X_{N-1}$ (cyclic group)

5.3.1 Fusion Rules

$$X_a \otimes X_b = X_{(a+b) \bmod N} \tag{22}$$

Multiplicities: All $N^c_{ab} = 0$ or 1 (multiplicity-free for standard abelian fusion).

5.3.2 Quantum Dimensions

$d_{X_a} = 1$ for all a. (For abelian/pointed categories, all objects are invertible and hence have quantum dimension 1; see EGNO §8.4 [EGNO15].)

5.3.3 F-Symbols

For abelian (group-like) fusion, all nontrivial associators are trivial:

$$F^{e}_{a,b,c} = 1 \tag{23}$$

5.3.4 R-Symbols

Braiding given by a 2-cocycle $\sigma(a,b) \in U(1)$:

$$R^{a+b}_{a,b} = \sigma(a,b) \tag{24}$$

Example 5.1 ($\mathbb{Z}_2$ with fermionic statistics)**.**

$$R^{\mathbf{1}}_{\mathrm{f},\mathrm{f}} = -1 \tag{25}$$

5.4 sVec Category (Fermionic)

Category: $\mathcal{C}_{\mathrm{sVec}}$
Rank: $d = 2$
Simple objects: $\mathbf{1} = X_0$ (boson), $\psi = X_1$ (fermion)

5.4.1 Fusion Rules

$\otimes$	$\mathbf{1}$	ψ
$\mathbf{1}$	$\mathbf{1}$	ψ
ψ	ψ	$\mathbf{1}$

Multiplicities: Multiplicity-free ($N_{\psi\psi}^{\mathbf{1}} = 1$).

5.4.2 Quantum Dimensions

- $d_{\mathbf{1}} = 1$
- $d_\psi = 1$ (fermionic dimension)

5.4.3 F-Symbols

For fermionic fusion (super-case), the associator has an extra sign:

$$F_{\psi,\psi,\psi}^{\mathbf{1}} = 1 \quad \text{(standard)} \tag{26}$$

But crossing rules differ due to Fermi statistics.

5.4.4 R-Symbols

$$R_{\psi,\psi}^{\mathbf{1}} = e^{i\pi} = -1 \tag{27}$$

(fermionic exchange is anticommuting; eigenvalue is -1).

5.5 Summary Table

Category	Rank	Mult-Free?	Braided?	Modular?	Status
Fibonacci	2	✓	×	—	Implemented
Ising	3	✓	✓	✓	Implemented
$\mathbb{Z}_N$	N	✓	✓	×	Template
sVec	2	✓	✓	—	Fermionic

Table 1: Summary of fusion category examples.

Remark 5.2. **Unverified:** All F/R-symbol values marked `[unverified]` pending HITL review against literature.

Remark 5.3. **Not exhaustive:** Other fusion categories (e.g., Haagerup, $SU(2)_k$, Tambara–Yamagami) can be added as needed. Note: Potts models and Virasoro algebras are related CFT structures but are not themselves fusion categories.

6 Temperley–Lieb Categories

Assumptions.

(A1) Loop fugacity $n = q + q^{-1}$ for quantum parameter q.

(A2) At roots of unity $q = e^{i\pi/p}$, the category truncates to finitely many simples.

(A3) Standard normalisation: loop evaluates to $\delta = -[2]_q$.

6.1 Overview

The *Temperley–Lieb (TL) category* is a fundamental example connecting fusion categories to statistical mechanics, loop models, and conformal field theory. TL categories provide:

1. A continuous interpolation between solvable models via the loop fugacity parameter.
2. Concrete realisations of Fibonacci ($q = e^{i\pi/5}$) and Ising ($q = e^{i\pi/4}$) anyons.
3. Direct connection to critical phenomena and CFT via the central charge formula.

6.2 Loop Fugacity Parametrisation

The TL category is parametrised by the *loop fugacity* n, related to the quantum parameter q by:

$$n = q + q^{-1} = 2\cos\theta, \qquad q = e^{i\theta}. \tag{28}$$

Example 6.1 (Key values of loop fugacity)**.**

p (for $q = e^{i\pi/p}$)	$n = 2\cos(\pi/p)$	Physical model
$p = 3$	$n = 1$	Percolation
$p = 4$	$n = \sqrt{2}$	Ising (σ-anyon)
$p = 5$	$n = \phi = \frac{1+\sqrt{5}}{2}$	Fibonacci (τ-anyon)
$p = 6$	$n = \sqrt{3}$	3-state Potts
$p \to \infty$	$n \to 2$	Free fermion

At roots of unity, the representation theory truncates, yielding *modular tensor categories* with finitely many simple objects.

6.3 Quantum Numbers

Definition 6.2 (Quantum integer)**.** For $q = e^{i\pi/p}$, the *quantum integer* is:

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{\sin(n\pi/p)}{\sin(\pi/p)}. \tag{29}$$

Definition 6.3 (Quantum factorial and binomial)**.** The *quantum factorial* and *quantum binomial* are:

$$[n]!_q = [1]_q[2]_q \cdots [n]_q, \qquad \binom{n}{k}_q = \frac{[n]!_q}{[k]!_q[n-k]!_q}. \tag{30}$$

Remark 6.4. At roots of unity, $[p-1]_q = 0$, causing the truncation of the representation theory.

6.4 Simple Objects and Fusion Rules

At generic q, the TL category has infinitely many simple objects labelled by half-integers $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$, corresponding to spins in the representation theory of $U_q(\mathfrak{sl}_2)$.

At roots of unity $q = e^{i\pi/p}$, the category *truncates* to:

$$j \in \left\{0, \frac{1}{2}, 1, \ldots, \frac{p-2}{2}\right\}. \tag{31}$$

Definition 6.5 (TL fusion rules)**.** The fusion rules are the $SU(2)_q$ rules with truncation:

$$j_1 \otimes j_2 = \bigoplus_{j=|j_1-j_2|}^{\min(j_1+j_2, p-2-j_1-j_2)} j. \tag{32}$$

6.5 Connection to Key Examples

Example 6.6 (Fibonacci anyons, $p = 5$)**.** Two simple objects $\{\mathbf{1}, \tau\}$ with $\tau \otimes \tau = \mathbf{1} \oplus \tau$.

Example 6.7 (Ising anyons, $p = 4$)**.** Three simple objects $\{\mathbf{1}, \sigma, \psi\}$ with:

$$\sigma \otimes \sigma = \mathbf{1} \oplus \psi, \tag{33}$$

$$\psi \otimes \psi = \mathbf{1}, \tag{34}$$

$$\sigma \otimes \psi = \sigma. \tag{35}$$

6.6 Critical Behaviour

Theorem 6.8 (Central charge)**.** *At $q = e^{i\pi/p}$, the TL algebra describes critical points with central charge:*

$$c = 1 - \frac{6(p-1)^2}{p}. \tag{36}$$

This connects TL categories to minimal model CFTs $\mathcal{M}(p, p-1)$.

Reference: Kauffman–Lins, *Temperley–Lieb Recoupling Theory* (1994) `[unverified]`; Jones, *Inventiones Math.* **72** (1983), 1–25 `[unverified]`

7 Fock Space Perspective (First Quantisation)

Assumptions.

(A3.2.1) First-quantised formalism only (no $a^\dagger, a$).

(A3.2.2) Fock space is a direct sum of fixed-number sectors.

7.1 Standard Fock Space

Definition 7.1 (Fock space)**.** The *Fock space* $\mathcal{F}$ is the direct sum of N-particle Hilbert spaces $\mathcal{H}_N$ for all possible particle numbers $N \geq 0$:

$$\mathcal{F} = \bigoplus_{N=0}^{\infty} \mathcal{H}_N = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \tag{37}$$

where $\mathcal{H}_0 \cong \mathbb{C}$ is the vacuum sector spanned by the vacuum state $|\Omega\rangle$.

Remark 7.2. In this first-quantised approach, a state $|\Psi\rangle \in \mathcal{F}$ is a sequence of wavefunctions (or categorical states) $|\Psi\rangle = (\psi_0, \psi_1, \psi_2, \ldots)$, where $\psi_N \in \mathcal{H}_N$ is the projection of the state onto the N-particle sector.

Constraint 7.3. We strictly avoid the use of second-quantised creation/annihilation operators $(a_i^\dagger, a_i)$ as fundamental building blocks. While convenient for bosons/fermions, they obscure the categorical data (braiding, fusion) essential for anyons.

7.2 Direct Sum as "OR Quantifier"

The direct sum ($\oplus$) operation represents a logical "OR" or superposition of different particle number sectors.

- A state in $\mathcal{H}_1 \oplus \mathcal{H}_2$ describes a system that is in a superposition of having 1 particle **OR** 2 particles.
- This contrasts with the tensor product (see §7.3).

This perspective highlights that the total Hilbert space allows for quantum fluctuations in particle number, even if dynamics (Hamiltonian) conserve it.

7.3 Tensor Product as "AND Quantifier"

The tensor product ($\otimes$) operation represents a logical "AND" or composition of subsystems.

- Within a fixed N-particle sector $\mathcal{H}_N$, the structure involves tensor products of single-particle spaces (or local site spaces):

$$\mathcal{H}_N \sim \mathcal{H}_{\text{loc}} \otimes \cdots \otimes \mathcal{H}_{\text{loc}} \quad \text{(schematically)} \tag{38}$$

- A state $|\phi\rangle \otimes |\chi\rangle$ describes a system where part A is in state ϕ **AND** part B is in state χ.

Remark 7.4 (Anyonic nuance). For anyons, $\mathcal{H}_N$ is not a simple tensor product of single-particle spaces due to fusion constraints (fusion spaces are not product spaces). However, the *ambient* space in which $\mathcal{H}_N$ is embedded (before fusion constraints) often has a tensor product structure (e.g., sites on a lattice).

7.4 First-Quantised Operators

Definition 7.5 (Operator on Fock space)**.** An operator $\hat{O} : \mathcal{F} \to \mathcal{F}$ is defined by its action on each sector $\mathcal{H}_N$ and maps between sectors. It can be represented as a matrix of operators $\hat{O}_{MN} : \mathcal{H}_N \to \mathcal{H}_M$.

Definition 7.6 (Number-conserving operator)**.** An operator $\hat{H}$ is *number-conserving* if it maps each sector $\mathcal{H}_N$ to itself ($\hat{H}_{MN} = 0$ for $M \neq N$). It decomposes as a direct sum of operators acting on fixed-number sectors:

$$\hat{H} = \bigoplus_{N=0}^{\infty} \hat{H}_N \tag{39}$$

where $\hat{H}_N : \mathcal{H}_N \to \mathcal{H}_N$.

Example 7.7. The Hamiltonian for mobile anyons is typically number-conserving (unless studying source terms), so we construct it by defining a sequence of Hamiltonians H_N for each N-anyon configuration space.

7.5 Summary

Structure	**Symbol**	**Interpretation**	**Logical Equiv.**
Direct Sum	$\oplus$	Superposition of sectors	**OR**
Tensor Product	$\otimes$	Composition of parts	**AND**
Fock Space	$\mathcal{F}$	$\bigoplus_N \mathcal{H}_N$	Variable particle number
Operator	$\hat{O}$	$\bigoplus_N \hat{O}_N$ (if conserved)	Collection of N-particle ops

8 Quantum Mechanics for Lattice Models

Assumptions.

(A3.3.1) All Hilbert spaces are finite-dimensional.

(A3.3.2) Hamiltonians are Hermitian.

8.1 Hilbert Spaces, States, Observables

We consider a finite-dimensional *Hilbert space* $\mathcal{H} \cong \mathbb{C}^d$ with inner product $\langle \cdot | \cdot \rangle$. A *state* is a unit vector $|\psi\rangle \in \mathcal{H}$ (defined up to phase), and an *observable* is a Hermitian operator $A \in \mathrm{End}(\mathcal{H})$ such that $A = A^\dagger$. Measurements yield eigenvalues of A with probabilities given by the Born rule.

8.2 Hamiltonians for Lattice Systems

Definition 8.1 (Hamiltonian)**.** A *Hamiltonian* is a Hermitian operator $H : \mathcal{H} \to \mathcal{H}$ representing the total energy of the system (generator of time evolution).

Definition 8.2 (Local Hamiltonian)**.** A Hamiltonian H is *local* if it decomposes as a sum of terms $H = \sum_j h_j$, where each term h_j has finite *support* localised near site j. The *support* of an operator A on a tensor product space $\mathcal{H} = \bigotimes_i \mathcal{H}_i$ is the minimal set of sites S such that A acts as the identity on all sites $k \notin S$ (i.e., A acts trivially outside S, $A = A_S \otimes \mathbb{I}_{S^c}$).

Definition 8.3 (Nearest-neighbour Hamiltonian)**.** A Hamiltonian is *nearest-neighbour* if each term h_j acts nontrivially only on sites j and $j+1$.

8.3 Ground States, Spectrum, Partition Function

Definition 8.4 (Spectrum)**.** The *spectrum* of H is the set of eigenvalues $\{E_0, E_1, \ldots\}$ ordered so that $E_0 \leq E_1 \leq \cdots$.

Definition 8.5 (Ground state)**.** A *ground state* is an eigenvector of H with eigenvalue E_0 (the minimum eigenvalue).

Definition 8.6 (Spectral gap)**.** The *spectral gap* is $\Delta = E_1 - E_0$.

Definition 8.7 (Partition function)**.** The *partition function* at inverse temperature β is

$$Z(\beta) = \mathrm{Tr}\Big(e^{-\beta H}\Big) = \sum_n e^{-\beta E_n} \tag{40}$$

8.4 Tensor Product Structure

Definition 8.8 (Tensor product of Hilbert spaces)**.** Given Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, the *tensor product* $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a Hilbert space with:

- $\dim(\mathcal{H}_1 \otimes \mathcal{H}_2) = \dim(\mathcal{H}_1) \cdot \dim(\mathcal{H}_2)$
- Basis: $\{|i\rangle \otimes |j\rangle\}$ for bases $\{|i\rangle\}$ of $\mathcal{H}_1$ and $\{|j\rangle\}$ of $\mathcal{H}_2$

Definition 8.9 (Multi-site Hilbert space)**.** For n sites with local Hilbert space $\mathcal{H}_{\mathrm{loc}}$ of dimension d, the total Hilbert space is

$$\mathcal{H}_{\mathrm{total}} = \bigotimes_{j=1}^{n} \mathcal{H}_{\mathrm{loc}} \cong \mathbb{C}^{d^n} \tag{41}$$

Remark 8.10. The tensor product represents the "AND quantifier": a state in $\mathcal{H}_1 \otimes \mathcal{H}_2$ describes a system where both subsystem 1 AND subsystem 2 exist simultaneously. Cf. §7.

8.5 Summary

Concept	Symbol	Definition
Hilbert space	$\mathcal{H}$	Finite-dim complex inner product space
State	$\lvert\psi\rangle$	Nonzero vector in $\mathcal{H}$
Observable	A	Hermitian operator
Hamiltonian	H	Hermitian operator (energy)
Ground state	$\lvert\psi_0\rangle$	Eigenvector with minimal eigenvalue
Partition function	$Z(\beta)$	$\mathrm{Tr}\left(e^{-\beta H}\right)$
Tensor product	$\otimes$	"AND quantifier" for composite systems

9 Lattice Geometry

Assumptions.

(A3.4.1) Spatial dimension is 1.

(A3.4.2) Open boundary conditions (no PBCs).

(A3.4.3) Finite number of sites.

9.1 One-Dimensional Chain

Definition 9.1 (Lattice)**.** A *lattice* Λ is a finite set of sites. In this work:

$$\Lambda = \{0, 1, 2, \ldots, n-1\} \tag{42}$$

where $n = |\Lambda|$ is the number of sites.

Remark 9.2. We use 0-based indexing for sites to align with physical position $x_j = \epsilon \cdot j$.

Definition 9.3 (Neighbour relation)**.** Sites i and j are *neighbours* if $|i - j| = 1$. We write $i \sim j$.

Definition 9.4 (Boundary sites)**.** The *boundary* of Λ consists of sites 0 and $n-1$. The *bulk* consists of sites $\{1, \ldots, n-2\}$.

9.2 Lattice Spacing and Physical Position

Definition 9.5 (Physical length)**.** The *physical length* of the system is $L > 0$.

Definition 9.6 (Lattice spacing)**.** The *lattice spacing* is

$$\epsilon = \frac{L}{n} \tag{43}$$

Definition 9.7 (Physical position)**.** The *physical position* of site j is

$$x_j = \epsilon \cdot j = \frac{L \cdot j}{n} \tag{44}$$

so that $x_0 = 0$ and $x_{n-1} = L(1 - 1/n) < L$.

Convention 9.8. Where convenient, we set $\epsilon = 1$ (equivalently, $L = n$), so that site index equals physical position.

9.3 Open Boundary Conditions

Definition 9.9 (Open boundary conditions)**.** A system has *open boundary conditions* (OBC) if:

1. The lattice is a finite chain with distinct endpoints
2. Site 0 has only one neighbour (site 1)
3. Site $n-1$ has only one neighbour (site $n-2$)

Remark 9.10. This contrasts with *periodic boundary conditions* (PBC) where site $n-1$ is also a neighbour of site 0. We exclude PBCs in this work (Assumption A3.4.2).

Remark 9.11. OBC implies:

- No topological ground state degeneracy from nontrivial cycles
- Edge effects may be present
- Total charge is well-defined without ambiguity from winding

9.4 Local Hilbert Spaces

Definition 9.12 (Local Hilbert space)**.** For an anyonic system described by a fusion category $\mathcal{C}$, the *local Hilbert space* $\mathcal{H}_j$ at site j is identified with the vector space spanned by the simple objects of $\mathcal{C}$ (viewed as possible particle types at that site):

$$\mathcal{H}_j \cong \bigoplus_{a \in \mathrm{Irr}(\mathcal{C})} \mathbb{C}|a\rangle_j \tag{45}$$

where $\mathrm{Irr}(\mathcal{C}) = \{\mathbf{1} = X_0, X_1, \ldots, X_{d-1}\}$ is the set of isomorphism classes of simple objects. The dimension is $d_{\mathrm{loc}} = |\,\mathrm{Irr}(\mathcal{C})|$, the rank of the category.

Remark 9.13. This definition naturally accommodates the hard-core constraint where at most one anyon occupies a site. In this context, the "vacuum" state corresponds to the unit object $|X_0\rangle_j = |\mathbf{1}\rangle_j$ (no topological charge). For soft-core models allowing multiple anyons per site, the local space would be larger, involving fusion spaces of multiple anyons (see §13).

9.5 Summary

Concept	**Symbol**	**Value/Definition**
Number of sites	n	$\lvert\Lambda\rvert$
Site indices	j	$0, 1, \ldots, n-1$
Physical length	L	System size
Lattice spacing	ϵ	L/n
Physical position	x_j	$\epsilon \cdot j$
Local Hilbert space	$\mathcal{H}_j$	Space at site j
Local dimension	d_{loc}	$\dim(\mathcal{H}_{\mathrm{loc}})$
Total dimension	—	d^n

9.6 Notation Conventions

Throughout this project:

- Sites are **0-indexed**: $j \in \{0, 1, \ldots, n-1\}$

- Boundary conditions are **open** (OBC)
- Default: $\epsilon = 1$ unless stated otherwise
- Tensor products are ordered left-to-right: $\mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{n-1}$

Part II
Hilbert Space for Mobile Anyons

10 Classical Configuration Space

Assumptions.

(A4.1.1) Fusion category $\mathcal{C}$ with simples $\{X_0, X_1, \ldots, X_{d-1}\}$ where $X_0 = \mathbf{1}$.

(A4.1.2) Lattice $\Lambda = \{0, \ldots, n-1\}$ with OBC.

10.1 Anyon Positions

Definition 10.1 (Configuration)**.** A *configuration* of N anyons is a tuple

$$\mathbf{j} = (j_1, j_2, \ldots, j_N) \in \Lambda^N \tag{46}$$

satisfying $0 \leq j_1 \leq j_2 \leq \cdots \leq j_N \leq n-1$.

Remark 10.2. The ordering $j_1 \leq \cdots \leq j_N$ reflects indistinguishability up to exchange. Distinct orderings related by permutation represent the same physical configuration.

10.2 Anyon Labels

Definition 10.3 (Labelled configuration)**.** A *labelled configuration* is a pair $(\mathbf{j}, \mathbf{k})$ where:

- $\mathbf{j} = (j_1, \ldots, j_N)$ is a configuration
- $\mathbf{k} = (k_1, \ldots, k_N)$ with $k_m \in \{1, \ldots, d-1\}$ labels the anyon type at site j_m

Convention 10.4. Label 0 denotes vacuum (no anyon). Labels $1, \ldots, d-1$ denote the nontrivial simple objects $X_1, \ldots, X_{d-1}$.

10.3 Configuration Space

Definition 10.5 (N-anyon configuration space)**.**

$$\mathrm{Conf}_N = \{(\mathbf{j}, \mathbf{k}) : \mathbf{j} \in \Lambda^N \text{ ordered, } k_m \in \{1, \ldots, d-1\}\} \tag{47}$$

Definition 10.6 (Hard-core configuration space)**.** With hard-core constraint:

$$\mathrm{Conf}_N^{\mathrm{HC}} = \{(\mathbf{j}, \mathbf{k}) \in \mathrm{Conf}_N : j_1 < j_2 < \cdots < j_N\} \tag{48}$$

Claim 10.7. $|\mathrm{Conf}_N^{\mathrm{HC}}| = \binom{n}{N}(d-1)^N$.

10.4 Mixed Character

Remark 10.8. Configurations have mixed identical/distinguishable character:

- Anyons with *same label* $k_i = k_j$ are identical
- Anyons with *different labels* $k_i \neq k_j$ are distinguishable

This is handled by the morphism space structure in §11.

11 Hilbert Space for Mobile Anyons

11.1 Notation: Disambiguation of Symbol N

Convention 11.1 (Disambiguation of N)**.** The symbol N is overloaded in this section with three distinct meanings. Context distinguishes them:

Context	**Meaning**	**Example**
Subscript: $\mathcal{H}_N$, $\mathcal{H}_N^{(c)}$	Particle number sector	$\mathcal{H}_3$ = sector with 3 anyons
Superscript: N_{ab}^c	Fusion multiplicity	$N_{12}^0 \in \{0, 1\}$ in Fibonacci
Variable: $N \in \{1, \ldots, n\}$	Particle number (variable)	$\bigoplus_{N=0}^n$

All three use the same letter but are syntactically distinct. Avoid using plain N alone; always include subscript/superscript or context.

Assumptions.

(A4.2.1) Fusion category $\mathcal{C}$ with unit $\mathbf{1}$ and simples $X_0 = \mathbf{1}, X_1, \ldots, X_{d-1}$.

(A4.2.2) n lattice sites, OBC.

11.2 Morphism Spaces

Definition 11.2 (Morphism space)**.** For objects $A, B \in \mathcal{C}$:

$$\mathrm{Mor}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B) \tag{49}$$

This is a finite-dimensional complex vector space with $\dim \mathrm{Mor}(A, B) = \delta_{A,B}$ for simple A, B (in the multiplicity-free case).

Remark 11.3. With multiplicity: $\dim \mathrm{Mor}(X_a \otimes X_b, X_c) = N_{ab}^c$.

11.3 N-Anyon Sector with Definite Charge

Definition 11.4 (Site occupation)**.** For a labelled configuration $(\mathbf{j}, \mathbf{k})$ on n sites, define the *site label* $s_m \in \{0, 1, \ldots, d-1\}$ for each site $m \in \{0, \ldots, n-1\}$:

$$s_m = \begin{cases} k_i & \text{if } j_i = m \text{ for some } i \in \{1, \ldots, N\} \\ 0 & \text{otherwise (site unoccupied, i.e., vacuum } \mathbf{1}) \end{cases} \tag{50}$$

Definition 11.5 (Object for configuration)**.** The *tensor product object* for a labelled configuration $(\mathbf{j}, \mathbf{k})$ is:

$$\mathcal{O}(\mathbf{j}, \mathbf{k}) = X_{s_0} \otimes X_{s_1} \otimes \cdots \otimes X_{s_{n-1}} \tag{51}$$

This is a tensor product over all n sites, with $X_0 = \mathbf{1}$ (vacuum) at unoccupied sites.

Remark 11.6 (Tensor product ordering convention). The tensor product is ordered by *site index* $m = 0, 1, \ldots, n-1$ (left-to-right on the lattice). This ordering is a convention: any fixed ordering is equivalent up to associators. The fusion tree formalism (§11.6) provides an explicit way to handle the associativity ambiguity by specifying a parenthesisation. Different parenthesisations (different fusion trees) give orthogonal basis vectors related by F-moves.

Definition 11.7 (N-anyon space, definite charge)**.** The space of N anyons with total charge c:

$$\mathcal{H}_N^{(c)} = \bigoplus_{(\mathbf{j},\mathbf{k})\in \mathrm{Conf}_N} \mathrm{Mor}(X_c, \mathcal{O}(\mathbf{j},\mathbf{k})) \tag{52}$$

Remark 11.8. A state $|\psi\rangle \in \mathcal{H}_N^{(c)}$ is a superposition over configurations, with amplitudes in the appropriate morphism spaces. The morphism $\mathrm{Mor}(X_c, \mathcal{O}(\mathbf{j},\mathbf{k}))$ describes how anyons at the specified positions fuse to total charge X_c.

11.4 N-Anyon Sector with Indefinite Charge

Definition 11.9 (N-anyon space, indefinite charge)**.**

$$\mathcal{H}_N = \bigoplus_{c=0}^{d-1} \mathcal{H}_N^{(c)} \tag{53}$$

Remark 11.10. Direct sum = "OR quantifier": the system has charge X_0 OR charge X_1 OR ...

11.5 Total Hilbert Space

Definition 11.11 (Total Hilbert space)**.**

$$\mathcal{H} = \bigoplus_{N=0}^{n} \mathcal{H}_N \tag{54}$$

where $\mathcal{H}_0 = \mathbb{C}$ (vacuum sector).

Remark 11.12. Direct sum over N = "OR quantifier" for particle number. This is the first-quantised Fock space structure (cf. §7).

11.6 Fusion Trees

Fusion trees provide an equivalent and explicit way to handle the tensor product ordering ambiguity. Rather than treating $X_{s_0} \otimes X_{s_1} \otimes \cdots \otimes X_{s_{n-1}}$ as an abstract n-fold tensor product, a fusion tree specifies a concrete parenthesisation (binary bracketing) that determines which pairs fuse first.

Definition 11.13 (Fusion tree)**.** A *fusion tree* τ for a configuration $(\mathbf{j},\mathbf{k})$ with N anyons is a labelling of internal nodes in a **binary tree** that encodes how the anyons fuse step-by-step to produce a final total charge. Formally:

1. **Leaves:** The N anyons occupy the leaves, labelled with their types $X_{k_1}, \ldots, X_{k_N}$.
2. **Internal nodes:** Each internal node is labelled with an intermediate fusion result X_e (some simple object of $\mathcal{C}$).
3. **Binary structure:** Each internal node has exactly two children (left and right), and the label at that node represents the fusion result: left child $\otimes$ right child.

4. **Root:** The root node is labelled with the final total charge X_c.

Remark 11.14. The fusion tree is not unique (not canonical) due to associativity of tensor products: different parenthesisations yield different tree structures, all yielding the same tensor product result. For multiplicity-free categories ($N_{ab}^c \in \{0, 1\}$), all choices span the same morphism space $\mathrm{Mor}(X_c, \mathcal{O}(\mathbf{j}, \mathbf{k}))$, but individual basis vectors depend on the choice.

Claim 11.15. *Different fusion trees $\tau \neq \tau'$ for the same configuration and charge are orthogonal basis vectors. The transition amplitudes between different tree bases are determined by the F-symbols of the category.*

11.7 Explicit Construction

For hard-core anyons, a basis of $\mathcal{H}_N^{(c)}$ is indexed by:

1. Configuration $(\mathbf{j}, \mathbf{k}) \in \mathrm{Conf}_N^{\mathrm{HC}}$
2. Fusion tree τ from $\mathcal{O}(\mathbf{j}, \mathbf{k})$ to X_c
3. Multiplicity indices (if $N_{ab}^c > 1$)

Claim 11.16. *For multiplicity-free categories:*

$$\dim \mathcal{H}_N^{(c)} = \sum_{(\mathbf{j},\mathbf{k})} \dim \mathrm{Mor}(X_c, \mathcal{O}(\mathbf{j}, \mathbf{k})) \tag{55}$$

12 Operators as Morphisms

Planning ref: §4.3
Dependency note: This section is logically prior to §11. We define operators as abstract categorical morphisms first, then show in §11 how they act on the concrete Hilbert space.

Assumptions.

(A4.3.1) We work with finite tensor products of simple objects.

(A4.3.2) The fusion category $\mathcal{C}$ has finitely many simple objects.

12.1 Abstract Operator Definition (Category-Level)

Definition 12.1 (Morphism space operator)**.** An *operator* is an element of a morphism space:

$$\mathcal{O} \in \mathrm{Mor}(A, B) \tag{56}$$

where A, B are tensor products of simple objects from $\mathcal{C}$.

Remark 12.2. This is a purely categorical definition, independent of any representation or Hilbert space. Morphisms are abstract; they follow categorical axioms (composition, associativity, identities).

Definition 12.3 (Operator as morphism sum)**.** The space of all operators in this categorical sense is:

$$\mathfrak{Op}_{\mathrm{cat}} = \bigoplus_{n_A, n_B \in \mathbb{Z}_{\geq 0}} \bigoplus_{\substack{(a_1, \ldots, a_{n_A}) \\ a_i \in \{1, \ldots, d-1\}}} \bigoplus_{\substack{(b_1, \ldots, b_{n_B}) \\ b_j \in \{1, \ldots, d-1\}}} \mathrm{Mor}(X_{a_1} \otimes \cdots \otimes X_{a_{n_A}}, X_{b_1} \otimes \cdots \otimes X_{b_{n_B}}) \tag{57}$$

where the nested direct sums run over:

- $n_A, n_B \in \mathbb{Z}_{\geq 0}$ — number of non-vacuum factors
- Labels $a_i, b_j \in \{1, \ldots, d-1\}$ — indices of simple objects (excluding vacuum $X_0 = \mathbf{1}$)

By semisimplicity, $\mathrm{Mor}(A, B)$ is a finite-dimensional vector space for each choice of A, B.

Remark 12.4. Objects with $n_A = 0$ are identified with $\mathbf{1}$ (tensor unit), so $X_\emptyset = \mathbf{1}$. This definition is independent of the Hilbert space $\mathcal{H}$ and does not require it to be defined yet.

12.2 Action on Hilbert Space (Concrete Representation)

Once the Hilbert space $\mathcal{H}$ is defined (Definition 11.11), morphism operators are promoted to linear maps via a representation:

Definition 12.5 (Representation of operators on $\mathcal{H}$)**.** A *representation* of the morphism space $\mathrm{Mor}(A, B)$ on the Hilbert space $\mathcal{H}$ is a linear embedding:

$$\rho : \mathrm{Mor}(A, B) \to \mathrm{Lin}(\mathcal{H}_A \to \mathcal{H}_B) \tag{58}$$

where $\mathcal{H}_A$ denotes the sector of $\mathcal{H}$ corresponding to object A, and Lin denotes linear maps.

The full operator algebra on $\mathcal{H}$ is:

$$\mathrm{End}(\mathcal{H}) = \bigoplus_{A,B} \mathrm{Lin}(\mathcal{H}_A \to \mathcal{H}_B) \tag{59}$$

12.3 Particle-Number Conservation (at Category Level)

Definition 12.6 (Particle-number of object)**.** For an object $A = X_{a_1} \otimes \cdots \otimes X_{a_n}$, define:

$$\mathrm{N}(A) = n \tag{60}$$

(the number of nontrivial factors, excluding vacuum $\mathbf{1}$).

Definition 12.7 (Particle-conserving morphism)**.** A morphism $\phi \in \mathrm{Mor}(A, B)$ is *particle-conserving* if:

$$\mathrm{N}(A) = \mathrm{N}(B) \tag{61}$$

Definition 12.8 (Particle-changing morphism)**.** A morphism $\phi \in \mathrm{Mor}(A, B)$ is *particle-changing* if:

$$\mathrm{N}(A) \neq \mathrm{N}(B) \tag{62}$$

Example 12.9.

- $\mathrm{Mor}(X_a \otimes X_b, X_c)$ is particle-annihilating: $\mathrm{N}(A) = 2$, $\mathrm{N}(B) = 1$.
- $\mathrm{Mor}(X_a, X_b \otimes X_c)$ is particle-creating: $\mathrm{N}(A) = 1$, $\mathrm{N}(B) = 2$.

12.4 Locality

Definition 12.10 (Support of object)**.** For an object $A = X_{a_1} \otimes \cdots \otimes X_{a_n}$, define:

$$\mathrm{supp}(A) = \{j : a_j \neq 0\} \tag{63}$$

(the set of factor positions that are nontrivial).

Definition 12.11 (k-local morphism)**.** A morphism $\phi \in \mathrm{Mor}(A, B)$ is *k-local* if the number of sites (factor positions) involved in the transition is at most k. Formally, this is satisfied if:

$$|\mathrm{supp}(A) \cup \mathrm{supp}(B)| \leq k \tag{64}$$

where A and B are viewed as objects on a local subset of sites.

Remark 12.12 (Well-definedness and embedding). This definition is well-defined for *primitive* morphisms acting on a small number of anyons. When such a morphism is embedded into a larger lattice Λ with existing anyons at sites $j \notin \mathrm{supp}(A) \cup \mathrm{supp}(B)$, it is understood to act as the identity on those "spectator" anyons. Thus, a 2-local hopping morphism remains 2-local even when acting on a many-particle state, as its non-trivial action is restricted to $k = 2$ sites.

Example 12.13.

- $\mathrm{Mor}(X_a \otimes X_b, X_c \otimes X_d)$ is 2-local: only factors 0 and 1 are involved.
- $\mathrm{Mor}(X_a, X_a)$ is 1-local: only factor 0 is involved.
- $\mathrm{Mor}(X_a \otimes \mathbf{1}, \mathbf{1} \otimes X_a)$ is 2-local: acts on positions 0 and 1.

Remark 12.14. In the context of lattice systems, locality also specifies which lattice sites the morphism acts on. This is covered when embedding into the lattice in §15.

13 Hard-Core vs Soft-Core Anyons

13.1 Hard-Core Constraint

Definition 13.1 (Hard-core anyons)**.** Anyons are *hard-core* if at most one anyon occupies each site:

$$x_1 < x_2 < \cdots < x_N \tag{65}$$

Consequence 13.2. Maximum particle number is $N_{\max} = n$ (number of sites).

Remark 13.3. Hard-core is natural for:

- Impenetrable particles (infinite on-site repulsion)
- Lattice models where sites represent localised orbitals

13.2 Soft-Core: Multiple Occupancy

Definition 13.4 (Soft-core anyons)**.** Anyons are *soft-core* if multiple anyons may occupy the same site:

$$x_1 \leq x_2 \leq \cdots \leq x_N \tag{66}$$

Definition 13.5 (On-site fusion space)**.** When anyons X_a, X_b occupy the same site, the local state space is:

$$\bigoplus_c N_{ab}^c \cdot \mathrm{Mor}(X_a \otimes X_b, X_c) \tag{67}$$

representing the possible fusion outcomes.

Remark 13.6. Soft-core requires tracking on-site fusion structure.

13.3 Hilbert Space Modifications

Regime	**Config space**	**Local structure**	$N_{\max}$
Hard-core	$x_i < x_j$ for $i < j$	One anyon/site	n
Soft-core	$x_i \leq x_j$ for $i < j$	Fusion at each site	∞ (needs cutoff)

Definition 13.7 (Hard-core Hilbert space)**.**

$$\mathcal{H}^{\mathrm{HC}} = \bigoplus_{N=0}^{n} \mathcal{H}_N^{\mathrm{HC}} \tag{68}$$

where $\mathcal{H}_N^{\mathrm{HC}}$ uses $\mathrm{Conf}_N^{\mathrm{HC}}$.

13.4 Physical Motivation

Hard-core regime:

- Models impenetrable anyons
- Connection to Girardeau mapping (§**??**)
- Simpler Hilbert space structure

Soft-core regime:

- Models "bosonic" anyons that can bunch
- Richer on-site physics
- Connection to Levin–Wen models when particles can annihilate

Convention 13.8. Unless stated otherwise, we work in the **hard-core** regime.

Part III
Dynamics for Mobile Anyons

14 Matrix Elements of 2-Local Operators

Planning ref: §4.5
Status: Draft

Assumptions.

(A4.5.1) Fusion category $\mathcal{C}$ with unit $\mathbf{1}$ and simples $X_0 = \mathbf{1}, X_1, \ldots, X_{d-1}$.

(A4.5.2) Hard-core constraint: at most one anyon per site.

(A4.5.3) Fusion tree basis fixed for $\mathcal{H}_N^{(c)}$ (per Definition 11.13).

(A4.5.4) Multiplicity-free fusion rules (i.e., $N_{ab}^c \in \{0, 1\}$).

14.1 Fusion Trees as Basis Data

Definition 14.1 (Fusion tree basis states)**.** A fusion tree τ (defined in Definition 11.13) specifies one of several basis choices for the morphism space $\mathrm{Mor}(X_c, \mathcal{O}(\mathbf{j}, \mathbf{k}))$. We denote basis states as:

$$|(\mathbf{j}, \mathbf{k}), \tau, c\rangle \in \mathcal{H}_N^{(c)} \tag{69}$$

where:

- $(\mathbf{j}, \mathbf{k})$ is the configuration (site indices and anyon types)
- τ encodes the fusion tree structure
- c is the total charge

Remark 14.2. The fusion tree is not canonical; different choices of parenthesisation yield different bases. For multiplicity-free categories, all choices span the same space, but individual basis vectors depend on the choice.

Remark 14.3. When two basis states have different fusion trees $\tau \neq \tau'$ (even for the same configuration and charge), they are orthogonal basis vectors. The Gram matrix between them is determined by the F-symbols of the category.

14.2 Matrix Element Definitions

Definition 14.4 (Matrix element for 2-local morphism)**.** Given:

- Basis states $|\psi\rangle = |(\mathbf{j}, \mathbf{k}), \tau, c\rangle$ and $|\phi\rangle = |(\mathbf{j}', \mathbf{k}'), \tau', c\rangle$ in $\mathcal{H}_N^{(c)}$
- A morphism $f \in \mathrm{Mor}(X_a \otimes X_b, X_c \otimes X_d)$ acting on sites $m, m+1$

The matrix element is the complex number:

$$\langle \phi | f_{m,m+1} | \psi \rangle \in \mathbb{C} \tag{70}$$

Interpretation: $f_{m,m+1}$ acts as a morphism between the anyon types at sites $m, m+1$ in the source and target configurations.

14.3 Type 1: Two-Particle Interaction

Definition 14.5 (Two-particle interaction morphism)**.** A morphism $f_{ab\to cd} \in \mathrm{Mor}(X_a \otimes X_b, X_c \otimes X_d)$ acts on two neighbouring sites with anyons of type X_a, X_b and outputs X_c, X_d.

Configuration support: Non-zero matrix elements occur only when:

- Source configuration: $s_m = a, s_{m+1} = b$
- Target configuration: $s'_m = c, s'_{m+1} = d$
- All other sites identical: $s_i = s'_i$ for $i \notin \{m, m+1\}$
- Total charge preserved

In this case, the matrix element is:

$$\langle (\mathbf{j}', \mathbf{k}'), \tau' | f_{ab\to cd} | (\mathbf{j}, \mathbf{k}), \tau \rangle = \alpha_{ab\to cd}(\tau, \tau') \tag{71}$$

where $\alpha_{ab\to cd}(\tau, \tau')$ is determined by the morphism and the fusion tree compatibility.

Remark 14.6 (Multiplicity-free case). $N_{ab}^c \in \{0, 1\}$ for all triples, so the morphism space $\mathrm{Mor}(X_a \otimes X_b, X_c \otimes X_d)$ is either $\{0\}$ or one-dimensional. If non-zero, the coefficient $\alpha_{ab\to cd}$ is a phase/scalar set by normalisation.

Remark 14.7 (Forbidden fusion). If $N_{ab}^c = 0$ (fusion forbidden), then $\mathrm{Mor}(X_a \otimes X_b, X_c) = \{0\}$, and the operator component is identically zero.

Claim 14.8 (Hermiticity)**.** *If $f_{ab\to cd}$ is a unitary morphism (or paired with its adjoint), the physical operator $\mathcal{O} = f_{m,m+1} + f_{m,m+1}^\dagger$ is Hermitian and preserves the sector $\mathcal{H}_N^{(c)}$.*

14.4 Type 2: Right Hopping

Definition 14.9 (Right-hopping morphism)**.** A morphism $h_R \in \mathrm{Mor}(X_a \otimes \mathbf{1}, \mathbf{1} \otimes X_a)$ describes translating an anyon of type X_a from site m to site $m+1$.

Hard-core configuration support:

- Source: anyon X_a at site m, vacuum ($\mathbf{1}$) at site $m+1$
- Target: vacuum at site m, anyon X_a at site $m+1$

Matrix element:

$$\langle (\mathbf{j}', \mathbf{k}'), \tau' | h_{R,m,m+1} | (\mathbf{j}, \mathbf{k}), \tau \rangle = \begin{cases} \beta_a(\tau, \tau') & \text{if } s_m = a, s_{m+1} = 0 \text{ and } s'_m = 0, s'_{m+1} = a \\ 0 & \text{otherwise} \end{cases} \tag{72}$$

where $\beta_a(\tau, \tau')$ is determined by the fusion tree evolution and normalisation convention.

Claim 14.10 (Locality of hopping)**.** *Right hopping is a 2-local operator: it couples only configurations differing by a single anyon displacement at neighbouring sites.*

14.5 Type 3: Left Hopping

Definition 14.11 (Left-hopping morphism)**.** A morphism $h_L \in \mathrm{Mor}(\mathbf{1} \otimes X_a, X_a \otimes \mathbf{1})$ describes translating an anyon of type X_a from site $m+1$ to site m.

Hard-core configuration support:

- Source: vacuum at site m, anyon X_a at site $m+1$
- Target: anyon X_a at site m, vacuum at site $m+1$

Remark 14.12 (Relation to right hopping). By rigidity of the fusion category, h_L and h_R are related via duality. In particular, if both use consistent normalisation for $X_a \otimes \mathbf{1} \leftrightarrow \mathbf{1} \otimes X_a$, then $h_L = h_R^\dagger$ (up to phase).

14.6 Operator Matrix Representation

Definition 14.13 (Full operator matrix in $\mathcal{H}_N^{(c)}$)**.** Let $\{|\psi_i\rangle = |(\mathbf{j}_i, \mathbf{k}_i), \tau_i, c\rangle\}_{i=1}^{d_N^{(c)}}$ be an orthonormal basis of $\mathcal{H}_N^{(c)}$ (in a chosen fusion tree basis). The matrix representation of a 2-local operator $\mathcal{O}$ is:

$$[\mathcal{O}]_{ij} := \langle \psi_i | \mathcal{O} | \psi_j \rangle \in \mathbb{C} \tag{73}$$

Remark 14.14. This is basis-dependent; different fusion tree bases yield different matrix coordinates. However, the eigenvalues and trace (and all basis-independent properties) are invariant.

Claim 14.15 (Sparsity of 2-local operators)**.** *For 2-local operators in the hard-core sector:*

- *Type 1 (two-particle interaction) couples configurations with anyons at the same pair of sites $(m, m+1)$; expect $O(n)$ nonzero entries per row.*
- *Type 2, 3 (hopping) couple configurations differing by a single displacement; expect $O(1)$ nonzero entries per row.*

Total matrix sparsity: $O(n \cdot d_N^{(c)})$ nonzero entries.

14.7 Example: Two Fibonacci Anyons

Example 14.16 (Fibonacci τ anyons, $N = 2$ on 3 sites)**.** Fusion category: Fibonacci, $d = 2$ (objects $\mathbf{1}, \tau$), $\tau \otimes \tau = \mathbf{1} \oplus \tau$ (multiplicity 1 each).

Hard-core configurations with 2 anyons on 3 sites: $\mathrm{Conf}_2^{\mathrm{HC}}(3) = \{(0,1), (0,2), (1,2)\}$ (pairs of occupied sites).

Basis states in $\mathcal{H}_2^{(\mathbf{1})}$ (two τ anyons fusing to $\mathbf{1}$):

- $|\psi_1\rangle = |(0,1), \tau, \mathbf{1}\rangle$ = anyons at sites 0, 1 fusing to $\mathbf{1}$
- $|\psi_2\rangle = |(0,2), \tau, \mathbf{1}\rangle$ = anyons at sites 0, 2 fusing to $\mathbf{1}$
- $|\psi_3\rangle = |(1,2), \tau, \mathbf{1}\rangle$ = anyons at sites 1, 2 fusing to $\mathbf{1}$

Right-hopping operator $h_{R,0,1}$ on sites 0, 1 (moves anyon from site 0 to site 1):

- Source: anyon at site 0, vacuum at site 1 $\to$ only $|\psi_2\rangle = |(0,2)\rangle$ qualifies
- Target: vacuum at site 0, anyon at site 1 $\to$ only $|\psi_3\rangle = |(1,2)\rangle$ qualifies
- Non-zero element: $\langle \psi_3 | h_{R,0,1} | \psi_2 \rangle$

Matrix of $h_{R,0,1}$ in basis $\{\psi_1, \psi_2, \psi_3\}$:

$$h_{R,0,1} = \begin{pmatrix} 0 & \beta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{74}$$

where β is the amplitude (normalisation-dependent).

14.8 Critical Observations and Open Questions

Observation 1 (Fusion tree dependence). The matrix elements explicitly depend on the choice of fusion tree basis via τ, τ'. This choice is not canonical for multiplicity $N_{ab}^c > 1$. All published definitions must remain basis-independent; fusion trees are a computational tool only.

Observation 2 (Missing fusion tree reduction). Computing actual matrix elements requires:

1. Extracting the local fusion tree structure at sites $j, j+1$ from τ
2. Composing with the morphism f or h
3. Tracking how τ evolves to τ'
4. Applying R-matrices if the operator causes anyon braiding

This requires implementation of fusion tree reduction coefficients and F-symbols.

Question 4.5.1 (Hard-core + arbitrary charge). For N anyons with definite charge c in hard-core sector, is the dimension of $\mathcal{H}_N^{(c)}$ always $O\left(\binom{n}{N}\right)$? What are the constraints on c?

Question 4.5.2 (Multiplicity handling). How do matrix elements change if $N_{ab}^c > 1$? Each morphism space becomes multi-dimensional, introducing index-dependent amplitudes $\alpha_{ab\to cd}^{(\mu)}$.

Question 4.5.3 (Normalisation convention). What is the standard normalisation for h_R and h_L? Should they be unitary? Hermitian? How does this interact with R-matrices and the fusion category structure?

15 Particle-Conserving Local Hamiltonians

Planning ref: §5.1.1
Status: Draft

Assumptions.

A5.1.1. Fusion category $\mathcal{C}$ with unit $\mathbf{1}$ and simples $X_0 = \mathbf{1}, X_1, \ldots, X_{d-1}$
A5.1.2. n lattice sites labelled $0, \ldots, n-1$, OBC
A5.1.3. Hard-core regime: at most one anyon per site
A5.1.4. Hilbert space $\mathcal{H} = \bigoplus_{N=0}^{n} \mathcal{H}_N$ as in §4.2

15.1 Number-Conserving Hamiltonians (§5.1.1.1)

Definition 15.1 (Number operator). The *number operator* $\hat{N} : \mathcal{H} \to \mathcal{H}$ acts as

$$\hat{N}\,|\psi\rangle = N\,|\psi\rangle \quad \text{for } |\psi\rangle \in \mathcal{H}_N \tag{75}$$

Definition 15.2 (Number-conserving Hamiltonian). A Hamiltonian H is *number-conserving* (or *particle-conserving*) if it commutes with the number operator:

$$[\hat{N}, H] = 0 \tag{76}$$

Equivalently, H preserves each N-particle sector:

$$H : \mathcal{H}_N \to \mathcal{H}_N \quad \text{for all } N \in \{0, 1, \ldots, n\} \tag{77}$$

Remark 15.3. Number-conserving Hamiltonians do not create or annihilate anyons. They may move existing anyons between sites or introduce interactions between them.

15.2 Local Hamiltonians (§5.1.1.2)

Definition 15.4 (Local Hamiltonian)**.** A Hamiltonian H is *local* if it decomposes as

$$H = \sum_{j=0}^{n-2} h_j \tag{78}$$

where each h_j acts nontrivially only on a bounded neighbourhood of site j.

Definition 15.5 (Nearest-neighbour Hamiltonian)**.** A Hamiltonian H is *nearest-neighbour* if each local term h_j acts nontrivially only on sites j and $j+1$:

$$h_j : \mathcal{H} \to \mathcal{H}, \quad h_j = \nVdash_{<j} \otimes \tilde{h}_j \otimes \nVdash_{>j+1} \tag{79}$$

where $\tilde{h}_j$ acts on the two-site space.

15.3 Morphism Decomposition (§5.1.1.3)

For mobile anyons, local terms decompose into morphism spaces. A number-conserving, nearest-neighbour term h_j has components:

Definition 15.6 (Morphism components of local term)**.** A nearest-neighbour term h_j acting on sites $j, j+1$ decomposes as:

$$h_j \in \bigoplus_{A,B} \mathrm{Mor}(A, B) \tag{80}$$

where A, B are objects of the form $X_a \otimes X_b$ with $a, b \in \{0, 1, \ldots, d-1\}$, and the sum is over pairs (A, B) with equal numbers of nontrivial (non-vacuum) factors.

Proposition 15.7 (Number conservation criterion)**.** *A local term h_j is number-conserving if and only if for every nonzero component in* $\mathrm{Mor}(X_a \otimes X_b, X_c \otimes X_d)$*:*

$$|\{a, b\} \cap \{1, \ldots, d-1\}| = |\{c, d\} \cap \{1, \ldots, d-1\}| \tag{81}$$

where $|S|$ denotes the count of indices in S that are nonzero (nontrivial).

15.4 Classification of Two-Site Processes (§5.1.1.4)

For hard-core anyons, the possible two-site configurations and number-conserving transitions are:

Source $X_a \otimes X_b$	**Target** $X_c \otimes X_d$	**Process**	**Particle count**
$\mathbf{1} \otimes \mathbf{1}$	$\mathbf{1} \otimes \mathbf{1}$	Vacuum identity	0
$X_a \otimes \mathbf{1}$	$\mathbf{1} \otimes X_a$	Hop right	1
$\mathbf{1} \otimes X_a$	$X_a \otimes \mathbf{1}$	Hop left	1
$X_a \otimes \mathbf{1}$	$X_a \otimes \mathbf{1}$	Stay left	1
$\mathbf{1} \otimes X_a$	$\mathbf{1} \otimes X_a$	Stay right	1
$X_a \otimes X_b$	$X_a \otimes X_b$	Two-anyon identity	2
$X_a \otimes X_b$	$X_c \otimes X_d$	Two-anyon scattering	2

Remark 15.8. Note that hopping does NOT involve braiding: the anyon moves to an empty site without passing through another anyon. Braiding processes are treated in §5.1.3.

15.5 Hermiticity (§5.1.1.5)

Definition 15.9 (Hermitian local term)**.** A local term h_j is *Hermitian* if $h_j = h_j^\dagger$.

For morphism components, this means:

$$\langle B|h_j|A\rangle = \overline{\langle A|h_j|B\rangle} \tag{82}$$

where the bar denotes complex conjugation.

Proposition 15.10 (Hermiticity in terms of morphisms)**.** *A local term with component $\alpha \in$ Mor(A,B) is Hermitian if and only if it also has component $\alpha^\dagger \in$ Mor(B,A) with matching coefficient.*

15.6 Julia Implementation

```
# file: src/julia/MobileAnyons/hamiltonian_v0.jl
using LinearAlgebra

"""
    NumberConservingTerm

A nearest-neighbour term that conserves particle number.
Acts on sites (site, site+1).
"""
struct NumberConservingTerm
    site::Int # left site index (0-based)
    components::Dict{Tuple{Tuple{Int,Int}, Tuple{Int,Int}}, ComplexF64}
    # (source, target) => coefficient
    # source/target are (label_left, label_right), 0 = vacuum
end

"""
Check that all components conserve particle number.
"""
function is_number_conserving(term::NumberConservingTerm)
    for ((a, b), (c, d)) in keys(term.components)
        n_source = (a != 0) + (b != 0)
        n_target = (c != 0) + (d != 0)
        if n_source != n_target
            return false
        end
    end
    return true
end

"""
Check that the term is Hermitian.
"""
function is_hermitian(term::NumberConservingTerm)
    for (k, v) in term.components
        ((a,b), (c,d)) = k
        conj_key = ((c,d), (a,b))
        if !haskey(term.components, conj_key)
            return false
        end
        if term.components[conj_key] != conj(v)
            return false
        end
```

```
    end
    return true
end

"""
    LocalHamiltonian

A number-conserving nearest-neighbour Hamiltonian.
"""
struct LocalHamiltonian
    n_sites::Int
    terms::Vector{NumberConservingTerm}
end

"""
Build a uniform nearest-neighbour Hamiltonian from a single local term.
"""
function uniform_nn_hamiltonian(n_sites::Int, local_components::Dict)
    terms = [NumberConservingTerm(j, local_components) for j in 0:(n_sites-2)]
    return LocalHamiltonian(n_sites, terms)
end
```

15.7 Summary

Concept	Symbol	Definition
Number operator	$\hat{N}$	Counts anyons: $\hat{N}\lvert\psi\rangle = N\lvert\psi\rangle$ for $\lvert\psi\rangle \in \mathcal{H}_N$
Number-conserving	$[\hat{N}, H] = 0$	Preserves particle number
Nearest-neighbour	h_j	Acts on sites $j, j+1$ only
Morphism component	$\mathrm{Mor}(A, B)$	Transitions between local configurations

15.8 Next Steps

- §5.1.1.2: Laplacian-type models (free hopping)
- §5.1.1.3: Hard-core blocking behaviour
- §5.1.2: Interactions without braiding
- §5.1.3: Free anyons with braiding

Part IV
Basic Properties of the Models

16 Module Categories

Assumptions.

(A1) Fusion category $\mathcal{C}$ over $\mathbb{C}$ (from §**??**).

(A2) $\mathcal{C}$ is rigid (has duals).

(A3) Module categories are semisimple and finite.

16.1 Overview

Module categories provide the mathematical framework for classifying boundary conditions in topological phases. A (left) $\mathcal{C}$-module category is a category $\mathcal{M}$ equipped with an action of the fusion category $\mathcal{C}$, analogous to how a module is a set with an action of a ring.

In the context of anyonic chains, module categories classify:

- Boundary conditions for open chains
- Edge modes and boundary excitations
- Domain walls between different phases

16.2 Definition of Module Categories

Definition 16.1 (Left module category)**.** A *left $\mathcal{C}$-module category* is a category $\mathcal{M}$ equipped with:

1. **Action functor:** $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, written $(X, M) \mapsto X \triangleright M$.
2. **Module associator:** Natural isomorphism
$$m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M). \tag{83}$$
3. **Unit constraint:** Natural isomorphism $\ell_M : \mathbf{1} \triangleright M \xrightarrow{\sim} M$.

These satisfy coherence conditions (pentagon and triangle diagrams for modules).

Definition 16.2 (Right module category)**.** A *right $\mathcal{C}$-module category* is defined analogously with action $\triangleleft : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$.

Definition 16.3 (Bimodule category)**.** A $(\mathcal{C}, \mathcal{D})$*-bimodule category* is a category $\mathcal{M}$ that is simultaneously a left $\mathcal{C}$-module and right $\mathcal{D}$-module, with compatible associators.

Reference: Etingof–Nikshych–Ostrik, *Adv. Math.* **226** (2011) `[unverified]`

16.3 Simple Module Objects

Definition 16.4 (Simple module object)**.** An object $M \in \mathcal{M}$ is *simple* if it has no proper subobjects. The simple objects of $\mathcal{M}$ form a finite set $\mathrm{Irr}(\mathcal{M})$.

Remark 16.5. Simple module objects correspond to *boundary excitations* or *edge modes*—the elementary degrees of freedom localised at the boundary.

16.4 Internal Hom

Definition 16.6 (Internal Hom)**.** For $M, N \in \mathcal{M}$, the *internal Hom* $\underline{\mathrm{Hom}}(M, N) \in \mathcal{C}$ is defined by:
$$\mathrm{Hom}_{\mathcal{M}}(X \triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M, N)). \tag{84}$$

This captures how bulk anyons ($X \in \mathcal{C}$) can transform one boundary excitation into another.

16.5 The Regular Module

Example 16.7 (Regular module)**.** Every fusion category $\mathcal{C}$ is a module over itself via the tensor product:
$$X \triangleright Y := X \otimes Y. \tag{85}$$
This is called the *regular module* $\mathcal{C}_{\mathcal{C}}$.

The regular module corresponds to the "trivial" or "smooth" boundary condition.

16.6 Module Functors

Definition 16.8 (Module functor)**.** A $\mathcal{C}$*-module functor* between $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{N}$ is a functor $F : \mathcal{M} \to \mathcal{N}$ with natural isomorphisms:

$$s_{X,M} : F(X \triangleright M) \xrightarrow{\sim} X \triangleright F(M) \tag{86}$$

satisfying coherence conditions.

Module functors describe *boundary-changing operators* or *defects* between different boundary conditions.

16.7 Morita Equivalence

Definition 16.9 (Morita equivalence)**.** Two fusion categories $\mathcal{C}$ and $\mathcal{D}$ are *Morita equivalent* if there exists an invertible $(\mathcal{C}, \mathcal{D})$-bimodule category.

Theorem 16.10 (Boundary-bulk correspondence)**.** *The bulk topological order determines, and is determined by, the set of all possible boundary conditions (module categories) up to Morita equivalence.*

Reference: Kitaev–Kong, *Commun. Math. Phys.* **313** (2012), 351–373 `[unverified]`

17 Boundary Conditions for Anyonic Chains

Assumptions.

(A1) Bulk fusion category $\mathcal{C}$ (from §**??**).

(A2) 1D chain with open boundary conditions (from §9).

(A3) Boundary conditions classified by $\mathcal{C}$-module categories.

17.1 Overview

For anyonic chains with open boundary conditions, the choice of *boundary conditions* at each end significantly affects the Hilbert space structure and dynamics. Following the Kitaev–Kong framework, boundary conditions are classified by *module categories* over the bulk fusion category $\mathcal{C}$.

Different boundary conditions lead to different edge mode structures, affecting ground state degeneracy, edge excitations, and partition functions.

17.2 Boundary Hilbert Space

Definition 17.1 (Boundary Hilbert space)**.** For a chain with:

- Bulk category $\mathcal{C}$
- Left boundary condition $\mathcal{M}_L$ (left $\mathcal{C}$-module category)
- Right boundary condition $\mathcal{M}_R$ (right $\mathcal{C}$-module category)

The boundary-modified Hilbert space involves the *relative tensor product* $\mathcal{M}_L \boxtimes_{\mathcal{C}} \mathcal{M}_R$.

17.3 Trivial Boundary Conditions

Definition 17.2 (Trivial/smooth boundary)**.** The *trivial boundary condition* corresponds to the regular module $\mathcal{C}_\mathcal{C}$, where $\mathcal{C}$ acts on itself by tensor product.

Anyons can freely approach the boundary without restriction. No edge modes beyond the bulk structure.

17.4 Gapped Boundaries via Lagrangian Algebras

Definition 17.3 (Lagrangian algebra)**.** A *Lagrangian algebra* $A \in \mathcal{C}$ is a commutative algebra object satisfying:

$$\dim(A)^2 = \dim(\mathcal{C}), \tag{87}$$

where $\dim(\mathcal{C}) = \sum_i d_i^2$ is the total quantum dimension.

Theorem 17.4 (Classification of gapped boundaries)**.** *Gapped boundary conditions for a topological phase with bulk $\mathcal{C}$ are in bijection with Lagrangian algebras in $\mathcal{C}$ (for modular $\mathcal{C}$).*

Reference: Kong–Wen, *JHEP* (2014) `[unverified]`

A Lagrangian algebra specifies which bulk anyons can "condense" at the boundary.

17.5 Examples

Example 17.5 (Fibonacci anyons)**.** For the Fibonacci category $\mathcal{C} = \mathrm{Fib}$ with simples $\{\mathbf{1}, \tau\}$:

Module category	Simple objects	Physical meaning
$\mathrm{Fib}_{\mathrm{Fib}}$ (regular)	$\{\mathbf{1}, \tau\}$	Smooth boundary
Vec (condensed)	$\{\mathbf{1}\}$	τ condensed at boundary

Example 17.6 (Ising anyons)**.** For the Ising category $\mathcal{C} = \mathrm{Ising}$ with simples $\{\mathbf{1}, \sigma, \psi\}$:

Module category	Simple objects	Physical meaning
$\mathrm{Ising}_{\mathrm{Ising}}$	$\{\mathbf{1}, \sigma, \psi\}$	Smooth boundary
$\mathrm{Vec}(\mathbb{Z}_2)$	$\{\mathbf{1}, \psi\}$	σ condensed

17.6 Application to Mobile Anyons

For mobile anyons on an open chain (our setting from §11):

1. **Standard construction** (§11): Uses trivial boundary conditions implicitly.
2. **With general boundaries:** The Hilbert space becomes:

$$\mathcal{H} = \bigoplus_{M_L \in \mathcal{M}_L} \bigoplus_{M_R \in \mathcal{M}_R} \mathcal{H}(M_L, M_R), \tag{88}$$

   where $\mathcal{H}(M_L, M_R)$ is the space of bulk configurations interpolating between boundary states.

3. **Boundary Hamiltonians:** Additional terms can describe:
   - Boundary potentials (energy cost for edge modes)
   - Boundary-bulk coupling (anyons interacting with edge)
   - Boundary-changing operators (transitions between boundary conditions)

17.7 Connection to Golden Chain

Example 17.7 (Golden chain boundaries)**.** The golden chain (Fibonacci anyons at unit filling) has been studied with various boundary conditions:

- **Free boundaries:** Regular module, leading to CFT edge modes.
- **Fixed boundaries:** Specific module object pinned at edge, breaking some symmetry.

Reference: Aasen–Fendley–Mong, *J. Phys. A* (2020) `[unverified]`

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