

Collective states of interacting Fibonacci anyons

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We show that chains of interacting Fibonacci anyons can support a wide variety of collective ground states ranging from extended critical, gapless phases to gapped phases with ground-state degeneracy and quasiparticle excitations. In particular, we generalize the Majumdar-Ghosh Hamiltonian to anyonic degrees of freedom by extending recently studied pairwise anyonic interactions to three-anyon exchanges. The energetic competition between two- and three-anyon interactions leads to a rich phase diagram that harbors multiple critical and gapped phases. For the critical phases and their higher symmetry endpoints we numerically establish descriptions in terms of two-dimensional conformal field theories. A topological symmetry protects the critical phases and determines the nature of gapped phases.

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Two-dimensional topological quantum liquids such as the fractional quantum Hall (FQH) states harbor exotic quasiparticle excitations which due to their unusual exchange statistics are referred to as anyons. Interchanging two anyons may result in not only a fractional exchange phase, but may also give rise to a unitary rotation of the original wave function in a degenerate ground-state manifold. This latter case of non-Abelian statistics is proposed to be exploited in the context of topological quantum computation [1, 2]. Intense experimental efforts [3] are currently under way to demonstrate the non-Abelian character of quasiparticle excitations in certain FQH states, as proposed theoretically [4, 5].

Given a set of several non-Abelian anyons we can ask what kind of collective states are formed if these anyons are *interacting* with each other. A first step in this direction has recently been taken by studying chains of “Fibonacci anyons” with nearest-neighbor interactions [6]. Fibonacci anyons represent the non-Abelian part of the quasiparticle statistics in the $k = 3$ Z_k -parafermion ‘Read-Rezayi’ state [4], an effective theory for FQH liquids at filling fraction $\nu = 12/5$ [7]. A single Fibonacci anyon carries a topological charge τ . Two such anyons may combine (“fuse”) so the pair has charge τ or has no charge, which is denoted 1. This is analogous to two $SU(2)$ spin-1/2’s combining to either spin-1 or spin-zero total spin. A two-anyon interaction assigns different energy to the two possible charges of the pair, just as a Heisenberg exchange interaction does for the two possible total values of spin of a pair of $SU(2)$ spin-1/2’s. For a chain of Fibonacci anyons with a uniform pairwise nearest-neighbor interaction of either sign it has been explicitly shown [6] that the Hamiltonian has a topological symmetry, which was predicted to stabilize one of the gapless phases. In this manuscript, we give a broader perspective on possible collective phases of interacting Fibonacci anyons and phase transitions between them.

The specific model we will focus on has, in addition to the

two-anyon term, an additional three-anyon interaction both of which may arise from tunneling [8]. We find a rich ground state phase diagram that harbors multiple critical, gapless and gapped phases. We also mention in passing other one-dimensional models that we have investigated and will report on in a forthcoming longer paper [9]. The topological symmetry, introduced in Ref. 6, that measures the topological flux through a ring of Fibonacci anyons plays an essential role in determining the nature of the observed phases and phase transitions. In particular, we find that this topological symmetry *protects* all the critical phases against spatially uniform local perturbations. These extended critical phases can be described in terms of 2D conformal field theories (CFT) with central charges $c = 7/10$ and $c = 4/5$ and can be respectively mapped exactly onto the tricritical Ising and 3-state Potts critical points of the generalized hard hexagon model [6, 10]. At the phase transitions out of the tricritical Ising phase into adjacent gapped phases the system exhibits even higher symmetries which we identify as *tetracritical* Ising and 3-state Potts critical points. Remarkably, this demonstrates that these 2D classical models share an identical non-local symmetry which is the classical analog of the topological symmetry in the 1D quantum chains. At the transition into the gapped phases this topological symmetry is spontaneously broken which results in a non-trivial ground-state degeneracy in the gapped phases.



FIG. 1: (color online) Illustration of a) the alternating chain and b) the generalized Majumdar-Ghosh chain with a three-anyon interaction term. Shaded enclosures indicate the fusion products that are energetically biased by the Hamiltonian.

Two-anyon interactions.— For a uniform chain of Fibonacci anyons the Hamiltonian introduced in Ref. 6 energetically favors one or the other of the possible fusion products of two neighboring τ -particles which, by the fusion rule $\tau \times \tau = 1 + \tau$, can be either a 1 or a τ . The energy of the former is lower for a coupling that is termed ‘antiferromagnetic’ (AFM), in analogy to the familiar $SU(2)$ spins-chains, while that of the latter is lower with a coupling termed ‘ferromagnetic’ (FM). The underlying Hilbert space is spanned by an orthonormal basis of states, each state corresponding to one possible labeling of the chain [6] of repeated fusions with τ . Each site along this chain of fusions has either a 1 or a τ , with a constraint forbidding two adjacent 1’s.

By performing a sequence of local basis transformations and projection onto one of the two fusion channels for each pair of neighboring anyons, the resulting two-anyon interaction Hamiltonian can be written as a sum of local 3-site operators $H = J_2 \sum_i H_2^i$ which take the explicit form (‘ i ’ denotes the first in a triple of adjacent sites)

$$H_2^i = -\mathcal{P}_{1\tau 1} - \phi^{-2} \mathcal{P}_{\tau 1 \tau} - \phi^{-1} \mathcal{P}_{\tau \tau \tau} - \phi^{-3/2} (|\tau 1 \tau\rangle \langle \tau \tau \tau| + \text{h.c.}) , \quad (1)$$

where \mathcal{P}_a projects onto the state $|a\rangle$, e.g. $\mathcal{P}_{1\tau 1} = |1\tau 1\rangle \langle 1\tau 1|$ and $\phi = (1 + \sqrt{5})/2$ is the golden ratio [6].

Here we want to explore a larger space of models than that given by this uniform chain with only nearest-neighbor two-anyon interactions. One way is to let the strength J_2 of the interaction alternate along the chain, as illustrated in Fig. 1a). Two chains can be coupled to form a two-leg ladder. Another way is to add a spatially-uniform three-anyon interaction, as indicated in Fig. 1b), which because of its rich phase diagram we discuss in the following.

Majumdar-Ghosh (MG) chain.— Three $SU(2)$ spin-1/2’s can combine to a total spin 3/2 or 1/2. For a uniform $SU(2)$ spin-1/2 chain, Majumdar and Ghosh showed that an AFM coupling (favoring total spin 1/2) for each set of three neighboring spins gives rise to a gapped phase with the two possible dimer coverings being the exact ground states [11]. In the same spirit we have asked what possible phases can be stabilized by a spatially uniform three-particle interaction term in our ‘anyonic generalization of the $SU(2)$ Heisenberg model’, i.e. a term that energetically favors each set of three adjacent Fibonacci anyons to fuse together [8] into either a 1 or a τ , as illustrated in Fig. 1b). Like the pairwise interaction term such a three-particle interaction term respects both the translational and topological symmetries. We find that the energetic competition between such two- and three-anyon interactions gives rise to the rich ground-state phase diagram shown in Fig. 2, which we discuss in some detail in the following. Similar to the derivation for the pairwise interaction term (2) we can obtain a local form $H = J_3 \sum_i H_3^i$ of the three-anyon interaction term by a sequence of basis transformations and projections which then takes the explicit form of a 4-site interaction

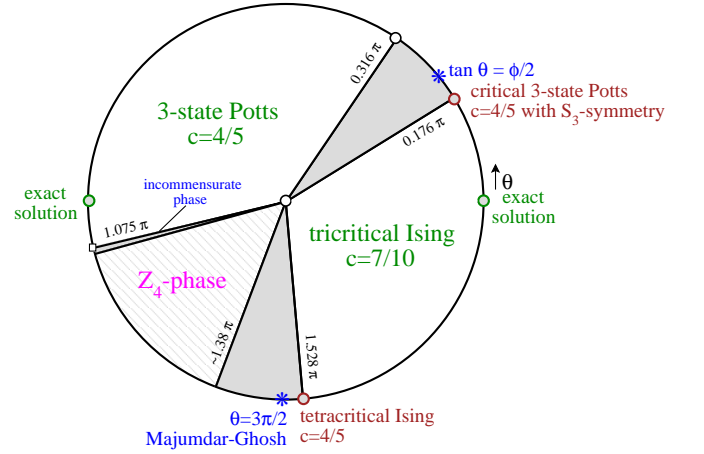


FIG. 2: (color online) The phase diagram of our anyonic Majumdar-Ghosh chain on the circle parameterized by θ , with pairwise fusion term $J_2 = \cos \theta$ and three-particle fusion term $J_3 = \sin \theta$. Besides extended critical phases around the exactly solvable points ($\theta = 0, \pi$) that can be mapped to the tricritical Ising model and the 3-state Potts model, there are two gapped phases (grey filled). The phase transitions (red circles) out of the tricritical Ising phase exhibit higher symmetries and are both described by CFTs with central charge 4/5. In the gapped phases exact ground states are known at the positions marked by the stars. In the lower left quadrant a small sliver of an incommensurate phase occurs and a phase which has Z_4 -symmetry. These latter two phases also appear to be critical.

between consecutive labels along the chain of fusions

$$H_3^i = \mathcal{P}_{\tau 1 \tau 1} + \mathcal{P}_{1 \tau 1 \tau} + \mathcal{P}_{\tau \tau \tau 1} + \mathcal{P}_{1 \tau \tau \tau} + 2\phi^{-2} \mathcal{P}_{\tau \tau \tau \tau} + \phi^{-1} (\mathcal{P}_{\tau 1 \tau \tau} + \mathcal{P}_{\tau \tau 1 \tau}) - \phi^{-2} (|\tau \tau 1 \tau\rangle \langle \tau 1 \tau \tau| + \text{h.c.}) + \phi^{-5/2} (|\tau 1 \tau \tau\rangle \langle \tau \tau \tau \tau| + |\tau \tau 1 \tau\rangle \langle \tau \tau \tau \tau| + \text{h.c.}) , \quad (2)$$

where the site i denotes the first position in each ‘quad’ of sites. The full Hamiltonian with competing fusion terms then becomes $H_{J_2, J_3} = \sum_i (J_2 H_2^i + J_3 H_3^i)$, where we parameterize the couplings by the angle θ as $J_2 = \cos \theta$ and $J_3 = \sin \theta$. We study periodic chains of L anyons.

The phase diagram of this model, shown in Fig. 2, exhibits two critical phases that contain the two exactly solvable points ($\theta = 0, \pi$). These extended critical phases can be described by 2D conformal field theories and are thereby related to 2D classical critical points to which an exact mapping was established at the two solvable points [6]. For AFM pair interaction ($J_2 > 0$) this is the tricritical Ising model ($c = 7/10$), while for FM pair interaction ($J_2 < 0$) it is the critical point of the 3-state Potts model ($c = 4/5$). In particular, we note that the critical phases found at the exactly solvable points are *stable* upon introducing a small three-anyon fusion term. While the J_3 -term respects both translational and topological symmetries, all translational invariant operators with scaling dimension < 2 at the exactly solvable points are found to break the topological symmetry [6]. This shows that the topological symmetry *protects* the gaplessness in the vicinity of these points, somewhat analogously to the much-discussed notion

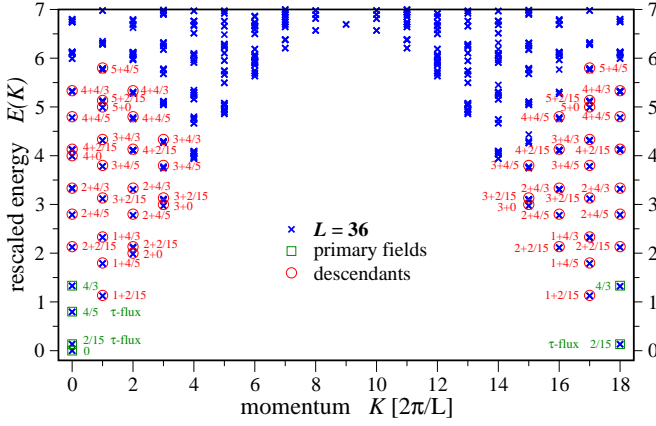


FIG. 3: (color online) Energy spectrum at the S_3 -symmetric point ($\theta \cong 0.176\pi$) of the Majumdar-Ghosh chain. The energies have been shifted and rescaled so that the two lowest eigenvalues match the CFT scaling dimensions. The open boxes indicate the positions of the primary fields of the parafermionic subset of the $c = 4/5$ CFT; fields with topological flux τ are marked. The open circles give the positions of multiple descendant fields as indicated.

that a topological symmetry protects a ground-state degeneracy in a gapped topological phase in 2D space.

For large three-particle interactions these critical phases eventually give way to other phases, such as the two distinct gapped phases indicated by the grey shaded arcs in the phase diagram. Remarkably, the transition to the gapped phase from the tricritical Ising phase when both interaction terms are AFM ($J_2 > 0$ and $J_3 > 0$) apparently has an “emergent” S_3 (3-state Potts) symmetry. Our numerical analysis shows that this transition occurs at $\theta \cong 0.176\pi$ and is described by the parafermion CFT with central charge $c = 4/5$, indicative of an additional S_3 -symmetry at this point. Fig. 3 shows the rescaled energy spectrum at this critical point whose (universal) low-energy part is in spectacular agreement with the CFT predictions. Note the relevant operator with zero momentum, zero flux and scaling dimension $4/3$, which breaks the S_3 -symmetry. It is the leading operator present in the Hamiltonian away from this special point, and drives the system into either the gapped or the tricritical Ising phase. In the gapped phase, the topological symmetry is spontaneously broken and the resulting ground state, which has zero total momentum, is two-fold degenerate in the thermodynamic limit. In the tricritical Ising phase, the Z_2 sublattice-symmetry breaking order parameter corresponds to a more relevant continuum operator than the topological order parameter, while it is the state corresponding to the Z_2 order parameter which acquires a higher energy in the gapped phase, where only the topological symmetry is broken. At the transition, both order parameters are degenerate, see Fig. 3, and together they form the order parameter of a critical 3-state Potts model with S_3 -symmetry.

In the case of FM three-particle interaction $J_3 < 0$ the transition at $\theta \cong -0.472\pi$ between tricritical Ising and gapped phases is described in terms of CFT by the full $c = 4/5$ minimal model representing [12] the *tetracritical* Ising model.

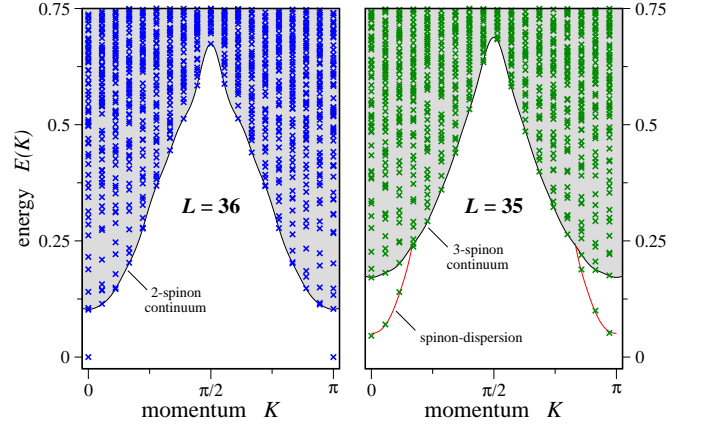


FIG. 4: (color online) Energy spectra at the generalized Majumdar-Ghosh point with ferromagnetic three-anyon interaction term only ($\theta = 3\pi/2$). With both translational and topological symmetry spontaneously broken there are four ground states for even length chains (left panel) with two states each at momenta $K = 0$ and $K = \pi$. The gapped low-energy spectrum is a continuum of two-spinon scattering states (grey shaded). For odd system size (right panel) a distinct single-spinon dispersion emerges below a gapped continuum of 3-spinon scattering states.

Again we have unambiguously identified the CFT description of this critical endpoint by assigning the low-energy states in the energy spectrum similar to Fig. 3 (not depicted here). In particular, the topological symmetry forces the system onto the integrable renormalization group trajectory [13] into the gapped phase or tricritical Ising fixed point, driven again by the relevant operator with dimension $4/3$. At this tetracritical Ising transition into the gapped phase the system spontaneously breaks *both* the translational and topological symmetries. As a consequence, we observe a four-fold ground-state degeneracy throughout this gapped phase for chains with even length. The nature of this gapped phase is best characterized at the point $\theta = 3\pi/2$ ($J_3 = -1$, $J_2 = 0$) that constitutes the anyonic analog of the Majumdar-Ghosh point of the spin-1/2 Heisenberg chain. At this point the four ground states for even L take the exact form

$$|\psi_{\text{no-flux}}\rangle = |\tau_x \tau_x \tau_x \tau_x \dots\rangle + \phi^{-1} |\tau_1 \tau_1 \tau_1 \dots\rangle \pm |\tau_x \tau_x \tau_x \tau_x \dots\rangle + \phi^{-1} |1 \tau_1 \tau_1 \dots\rangle \quad (3)$$

$$|\psi_{\tau\text{-flux}}\rangle = \phi^{-1} |\tau_x \tau_x \tau_x \tau_x \dots\rangle - |\tau_1 \tau_1 \tau_1 \dots\rangle \pm \phi^{-1} |\tau_x \tau_x \tau_x \tau_x \dots\rangle - |1 \tau_1 \tau_1 \dots\rangle, \quad (4)$$

where $\tau_x = \phi^{-1} |1\rangle + \phi^{-1/2} |\tau\rangle$ denotes a normalized superposition of the states $|1\rangle$ and $|\tau\rangle$ on a single site. Note these ground states have total momenta $K = 0$ or $K = \pi$, indicating the two-sublattice ordering. There are two states at each momentum, one with a τ -flux and the other without. Of course, we can instead make the simpler linear combinations of these ground states that explicitly break both the topological and sublattice symmetries: these four states are $|\tau_x \tau_x \tau_x \tau_x \dots\rangle$, $|\tau_1 \tau_1 \tau_1 \dots\rangle$ and the equivalent states under translation by one site. Note that the density of 1's [which for

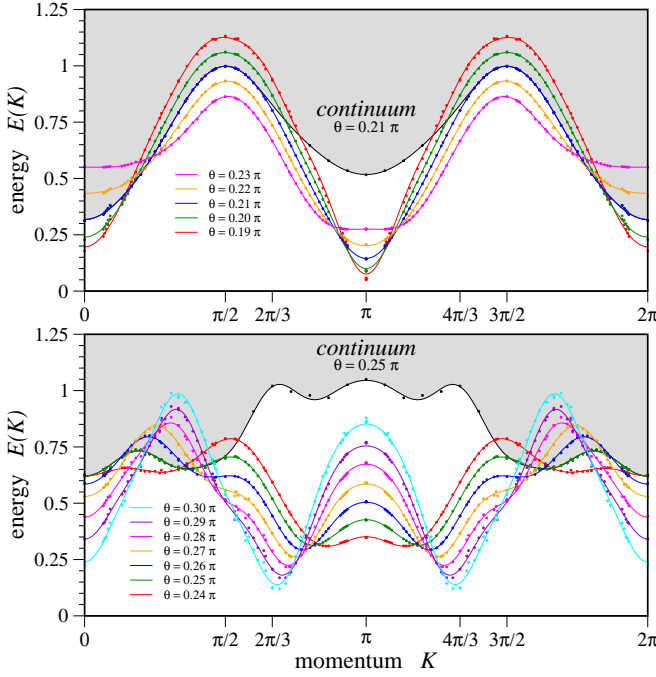


FIG. 5: (color online) Schematic energy spectra in the gapped phase for $0.19 \leq \theta/\pi \leq 0.30$. Below a continuum of scattering states (grey shaded) a distinct quasiparticle band forms. The plot combines data with $24 \leq L \leq 36$. The solid lines are a guide to the eye.

these states is $1/(2\phi^2)$ and $1/2$, respectively] is a simple order parameter that reflects the topological symmetry breaking. A pair of these latter states also becomes the exact ground states if the Hamiltonian explicitly breaks the sublattice symmetry by alternating the two-anyon interaction along the chain as shown in Fig. 1a). For a uniform chain in the tricritical Ising phase this alternation corresponds to a relevant operator that immediately drives the system into a gapped phase. On the other hand, the uniform chain in the Potts critical phase has Z_3 sublattice symmetry and no relevant operators with spatial period two, so this chain remains critical when a small alternation is added [9].

The low-energy excitations in the gapped phase around the MG point are domain walls between the two sublattice-ordered ground states with a low density of 1's – similar to spinon states in the spin-1/2 MG chain. For chains of odd L the periodic boundary conditions force the presence of an odd number of such spinons/walls, and indeed we observe a distinct single-spinon dispersion relation over part of the Brillouin zone, as shown in the right panel of Fig. 4. Explicitly evaluating the topological symmetry operator [6] we find that these one-spinon states all have a τ -flux. For even length rings we observe a continuum of two-spinon scattering states in the spectrum shown in the left panel of Fig. 4.

At the S_3 -symmetric transition out of the tricritical Ising phase ($\theta \cong 0.176\pi$) the system spontaneously breaks only the topological symmetry resulting in a two-fold degenerate ground state at total momentum $K = 0$ in the gapped phase.

With increasing coupling J_3 , away from the transition, an energy gap opens in the spectrum, with a distinct quasiparticle dispersion forming below a continuum of scattering states as shown in Fig. 5. Explicitly evaluating the topological symmetry operator [6] we find that these quasiparticle states all have a τ -flux. For small J_3 , the lowest energy quasiparticle remains at momentum $K = \pi$, with both the gap and the mass increasing with increasing J_3 . Near $\theta \cong 0.24\pi$ the gap reaches a maximum value and the mass diverges. The minimum of the quasiparticle dispersion bifurcates and continuously moves away from $K = \pi$, see the lower panel in Fig. 5. Eventually, the two minima approach the commensurate momenta $K = 2\pi/3, 4\pi/3$ and as these modes soften the system enters the 3-state Potts critical phase at $\theta \cong 0.316\pi$. Similar to the MG point we can identify a point in this gapped phase at which we can determine the exact form of the two ground states. For $J_3 = \frac{\phi}{2}J_2$, we note that the off-diagonal term in Hamiltonian (2) is exactly cancelled by its counterpart in the J_3 -Hamiltonian (2), and at this angle, $\tan \theta = \phi/2$, the ground states take the explicit form

$$|\psi_{\text{no-flux}}\rangle = |\tilde{\tau}_x \tilde{\tau}_x \tilde{\tau}_x \dots\rangle + (-1)^L \phi^{-1} |\tau \tau \tau \dots\rangle \quad (5)$$

$$|\psi_{\tau\text{-flux}}\rangle = \phi^{-1} |\tilde{\tau}_x \tilde{\tau}_x \tilde{\tau}_x \dots\rangle - (-1)^L |\tau \tau \tau \dots\rangle \quad (6)$$

where $\tilde{\tau}_x = \phi^{1/2} |1\rangle + \phi^{-1} |\tau\rangle$ is a single site superposition. (A projector onto the states in the Hilbert space is implicitly assumed.) Again note that the density of 1's is a simple order parameter for topological symmetry breaking.

Finally, we note that when both interaction terms are FM the critical 3-state Potts phase gives way to a small sliver of an incommensurate phase and then a phase with Z_4 sublattice symmetry. All of these phases appear to be critical or nearly critical. In the incommensurate phase, correlations in the local density of 1's oscillate with a spatial period varying between 3 and 4 lattice spacings.

Breaking the topological symmetry.— To further elucidate the role of the topological symmetry in determining the nature of the observed phases, we can *explicitly* break this symmetry in a modified Hamiltonian. Since the number of 1's, N_1 , is a simple order parameter for this symmetry, we add a term hN_1 to the Hamiltonian, where h is the field that breaks the symmetry. Within the gapped phases the ground-state degeneracy is immediately lifted by $h \neq 0$, with the state(s) with lower (higher) density of 1's favored by positive (negative) h . At the point $\tan \theta = \phi/2$ these states are precisely the “vacuum” state of 1's $|\tau \tau \tau \dots\rangle$ ($N_1 = 0$) for $h > 0$ and the state $|\tilde{\tau}_x \tilde{\tau}_x \tilde{\tau}_x \dots\rangle$ ($N_1 > 0$) for $h < 0$. The whole gapped phase for AFM J_2 and J_3 can thus be identified as a first-order transition with a ‘liquid-gas’ coexistence of 1's at $h = 0$. A similar picture, but with broken sublattice symmetry, holds for the gapped phase around the MG point. Varying h in the *critical* phases of our phase diagram, there is a weaker feature in the ground-state energy $E(h)$: the ‘topological susceptibility’ $\chi_t(h) \propto d^2 E/dh^2$ diverges with system size at $h = 0$. This indication of a continuous phase transition on varying h confirms that the topological symmetry is a full participant in the critical behavior at $h = 0$.

In conclusion, the anyonic quantum chains possess a topological symmetry that forces their corresponding 2D classical models onto a highly fine-tuned submanifold of their respective phase diagrams. The competition of anyonic exchange interactions allows one to move within this manifold containing a plethora of both, (multi)critical phases such as tricritical Ising, 3-state Potts, tetracritical Ising, and various gapped phases with spontaneously broken topological symmetry including an analog of the Majumdar-Ghosh point.

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