# Microscopic Models for Mobile Anyons from Fusion Categories

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**Abstract**

We develop a systematic framework for constructing microscopic lattice models describing mobile (itinerant) anyons arising from an arbitrary fusion category. Unlike existing models such as the golden chain where anyons are fixed at predetermined positions, our framework allows both the positions and number of anyons to fluctuate dynamically. Working in a first-quantised formalism on a one-dimensional chain with open boundary conditions, we construct Hilbert spaces that accommodate variable anyon number and anyon mobility, and define microscopic Hamiltonians for physically motivated scenarios. The framework is validated by reduction to known limiting cases including standard bosonic/fermionic systems and tightly-packed fusion chains.

## Contents

# Part I
# Preliminaries

## 1 Fusion Ring

**Assumptions.**

(A1) Finite set of simple objects $\{X_i\}_{i=0}^{d_{\mathcal{C}}-1}$.

(A2) Structure constants $N_{ab}^c \in \mathbb{Z}_{\geq 0}$ are associative and unital with unit $\mathbf{1}$.

**Definition 1.1** (Fusion ring)**.** A *fusion ring* is a finitely generated free abelian group $R = \bigoplus_{i\in I} \mathbb{Z}X_i$ with a ring structure satisfying:

1. $X_0 = \mathbf{1}$ is the unit element.
2. The product of basis elements satisfies
$$X_i X_j = \sum_{k\in I} N_{ij}^k X_k, \tag{1}$$
where $N_{ij}^k \in \mathbb{Z}_{\geq 0}$ are the *fusion coefficients* (or fusion multiplicities).
3. There exists an involution $i \mapsto i^*$ such that
$$N_{ij}^0 = \delta_{i,j^*}. \tag{2}$$

The involution gives duality: $X_i^* = X_{i^*}$. Associativity follows from the ring axioms:
$$\sum_e N_{ij}^e N_{ek}^\ell = \sum_e N_{jk}^e N_{ie}^\ell \quad \text{for all } i,j,k,\ell \in I. \tag{3}$$

*Remark* 1.2. Fusion rings are generally *not commutative*, i.e., $N_{ij}^k \neq N_{ji}^k$ in general.

*Reference:* Etingof–Nikshych–Ostrik, *Ann. Math.* **162** (2005), 581–642, Def. 3.1 [ENO05] `[unverified]`

## 2 Fusion Categories

**Assumptions.**

(A3.1.2.1) Fusion ring $(R, \{X_i\}_{i\in I}, \mathbf{1})$ with $X_0 = \mathbf{1}$ and $N_{ij}^k \in \mathbb{Z}_{\geq 0}$ (Definition 1.1).

(A3.1.2.2) Associator data $F$ (and, when present, braiding data $R$) satisfy the pentagon/hexagon equations.

**Definition 2.1** (Fusion category)**.** A *fusion category* over an algebraically closed field $k$ (usually $k = \mathbb{C}$) is a $k$-linear, semisimple, rigid monoidal category
$$(\mathcal{C}, \otimes, \mathbf{1}) \tag{4}$$
satisfying the following conditions:

1. **Finiteness:** There are finitely many isomorphism classes of simple objects. Every object decomposes as a finite direct sum of simples.
2. **Semisimplicity:** All morphism spaces $\mathrm{Mor}(X, Y)$ are finite-dimensional $k$-vector spaces, and the category is abelian and semisimple.
3. **Rigidity:** Every object $X \in \mathcal{C}$ has a left and right dual $X^*$ with evaluation and coevaluation morphisms satisfying the rigidity axioms.
4. **Simple unit:** The tensor unit $\mathbf{1}$ is simple: $\mathrm{End}(\mathbf{1}) \cong k$.
5. **Finite $k$-linearity:** The monoidal structure is bilinear over $k$, and composition and tensor product of morphisms are $k$-linear.

**Definition 2.2** (Grothendieck ring)**.** From any fusion category $\mathcal{C}$, we construct its *Grothendieck ring* $K_0(\mathcal{C})$ by

$$K_0(\mathcal{C}) = \bigoplus_{[X]\in \mathrm{Irr}(\mathcal{C})} \mathbb{Z}[X], \tag{5}$$

with multiplication

$$[X] \cdot [Y] = \sum_Z N_{XY}^Z [Z], \tag{6}$$

where $N_{XY}^Z = \dim_k \mathrm{Mor}(X \otimes Y, Z)$ is the fusion multiplicity. The Grothendieck ring $K_0(\mathcal{C})$ is a fusion ring (Definition 1.1), establishing that *fusion categories categorify fusion rings.*

**Definition 2.3** (Braided fusion category)**.** If additionally $\mathcal{C}$ is equipped with a braiding (natural isomorphisms $c_{X,Y} : X \otimes Y \to Y \otimes X$ satisfying hexagon identities), we call $\mathcal{C}$ a *braided fusion category.*

*Reference:* Etingof–Nikshych–Ostrik, *Ann. Math.* **162** (2005), 581–642, Def. 2.1 [ENO05] `[unverified]`

## 2.1 F-Symbols and Pentagon Equation

**Definition 2.4** (F-symbols)**.** The *associator* is a natural isomorphism

$$\alpha_{a,b,c} : (a \otimes b) \otimes c \xrightarrow{\sim} a \otimes (b \otimes c) \tag{7}$$

that satisfies the pentagon equation. In a skeletal category (where objects are direct sums of simple objects $X_i$), the associator is determined by its matrix elements, the *F-symbols.*

For simple objects $a, b, c, d$, the isomorphism decomposes into blocks indexed by intermediate fusion channels $e$ (for $(a \otimes b) \to e \to d$) and $f$ (for $(b \otimes c) \to f \to d$). The change of basis is given by the *F-move*:

$$|(a \otimes b) \otimes c \to d; e, \alpha, \beta\rangle = \sum_{f,\mu,\nu} (F_{abc}^d)_{e,\alpha,\beta}^{f,\mu,\nu} \, |a \otimes (b \otimes c) \to d; f, \mu, \nu\rangle \tag{8}$$

where $\alpha, \beta, \mu, \nu$ are multiplicity indices.

**Definition 2.5** (Pentagon equation)**.** The *pentagon equation* ensures that the two paths to re-associate $((a \otimes b) \otimes c) \otimes d$ to $a \otimes (b \otimes (c \otimes d))$ coincide. In terms of F-symbols (suppressing multiplicity indices):

$$\sum_k (F_{a,b,c}^k)_e^l (F_{a,k,d}^p)_l^m (F_{b,c,d}^p)_k^n = (F_{a,b,n}^p)_e^m (F_{e,c,d}^m)_l^n \tag{9}$$

This coherence condition is required for the fusion category to be well-defined.

## 2.2 R-Symbols and Hexagon Equations

**Definition 2.6** (R-symbols)**.** For a braided fusion category, the *braiding isomorphism* $c_{a,b} : a \otimes b \to b \otimes a$ provides a natural way to permute tensor factors. For simple objects $a, b, c$, the braiding isomorphism is represented by its matrix elements, the *R-symbols.*

**Definition 2.7** (Hexagon equations)**.** The *hexagon equations* are coherence conditions that relate the associator (F-symbols) and the braiding (R-symbols), ensuring consistency between re-associating and braiding operations. The first hexagon equation:

$$c_{a,b\otimes c} \circ (1_a \otimes c_{b,c}) = ((c_{a,b} \otimes 1_c) \circ F_{b,a,c} \circ (1_b \otimes c_{a,c})) \circ F^{-1}_{a,c,b} \tag{10}$$

This equation (and its dual) ensures that braiding past a composite object can be decomposed consistently.

*Reference:* Etingof–Nikshych–Ostrik, *Ann. Math.* **162** (2005), §2.1 [ENO05] `[unverified]`

# 3 Morphism Spaces and Multiplicities

**Assumptions.**

(A3.1.3.1) Fusion category $(\mathcal{C}, \otimes, \mathbf{1})$ over an algebraically closed field $k$ (Definition 2.1).

(A3.1.3.2) $\mathcal{C}$ is semisimple and $k$-linear, so all morphism spaces are finite-dimensional $k$-vector spaces.

**Definition 3.1** (Morphism space)**.** For any objects $A, B \in \mathcal{C}$,

$$\mathrm{Mor}(A, B) := \mathrm{Hom}_{\mathcal{C}}(A, B) \tag{11}$$

is a finite-dimensional $k$-vector space. If $A, B$ are simple, Schur's lemma implies $\dim \mathrm{Mor}(A, B) = \delta_{A,B}$.

*Reference:* Etingof–Nikshych–Ostrik, §2 [ENO05] `[unverified]`

**Definition 3.2** (Fusion multiplicity space)**.** For simple objects $X_a, X_b, X_c \in \mathrm{Irr}(\mathcal{C})$, the space

$$\mathrm{Mor}(X_a \otimes X_b, X_c) \tag{12}$$

has dimension $N^c_{ab} = \dim \mathrm{Mor}(X_a \otimes X_b, X_c) \in \mathbb{Z}_{\geq 0}$. A *multiplicity basis* is any choice of morphisms

$$f^{(\mu)}_{ab\to c} : X_a \otimes X_b \to X_c, \quad \mu = 1, \ldots, N^c_{ab}. \tag{13}$$

No canonical choice exists; computations must remain basis-independent.

**Claim 3.3** (Multiplicity-free simplification)**.** *In the multiplicity-free case ($N^c_{ab} \in \{0, 1\}$), each space* $\mathrm{Mor}(X_a \otimes X_b, X_c)$ *is either* $\{0\}$ *or a one-dimensional $k$-line. Basis dependence disappears, and $f^{(1)}_{ab\to c}$ can be chosen uniquely up to phase.*

*Remark* 3.4. Duals: $\mathrm{Mor}(\mathbf{1}, X_a \otimes X_b)$ is canonically dual to $\mathrm{Mor}(X^*_a \otimes X^*_b, \mathbf{1})$ via rigidity. Normalisation choices for evaluation/coevaluation maps must be consistent.

*Remark* 3.5. Basis independence is essential for categorical definitions. Fusion-tree bases are admissible for computations (e.g., numerical evaluation of $F$-symbols) but must be removed from statements of definitions and theorems.

# 4 Fock Space Perspective (First Quantisation)

**Assumptions.**

(A3.2.1) First-quantised formalism only (no $a^\dagger, a$).

(A3.2.2) Fock space is a direct sum of fixed-number sectors.

## 4.1 Standard Fock Space

**Definition 4.1** (Fock space)**.** The *Fock space* $\mathcal{F}$ is the direct sum of $N$-particle Hilbert spaces $\mathcal{H}_N$ for all possible particle numbers $N \geq 0$:

$$\mathcal{F} = \bigoplus_{N=0}^{\infty} \mathcal{H}_N = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \tag{14}$$

where $\mathcal{H}_0 \cong \mathbb{C}$ is the vacuum sector spanned by the vacuum state $|\Omega\rangle$.

*Remark* 4.2*.* In this first-quantised approach, a state $|\Psi\rangle \in \mathcal{F}$ is a sequence of wavefunctions (or categorical states) $|\Psi\rangle = (\psi_0, \psi_1, \psi_2, \ldots)$, where $\psi_N \in \mathcal{H}_N$ is the projection of the state onto the $N$-particle sector.

**Constraint 4.3.** We strictly avoid the use of second-quantised creation/annihilation operators $(a_i^\dagger, a_i)$ as fundamental building blocks. While convenient for bosons/fermions, they obscure the categorical data (braiding, fusion) essential for anyons.

## 4.2 Direct Sum as "OR Quantifier"

The direct sum ($\oplus$) operation represents a logical "OR" or superposition of different particle number sectors.

- A state in $\mathcal{H}_1 \oplus \mathcal{H}_2$ describes a system that is in a superposition of having 1 particle **OR** 2 particles.
- This contrasts with the tensor product (see §4.3).

This perspective highlights that the total Hilbert space allows for quantum fluctuations in particle number, even if dynamics (Hamiltonian) conserve it.

## 4.3 Tensor Product as "AND Quantifier"

The tensor product ($\otimes$) operation represents a logical "AND" or composition of subsystems.

- Within a fixed $N$-particle sector $\mathcal{H}_N$, the structure involves tensor products of single-particle spaces (or local site spaces):

$$\mathcal{H}_N \sim \mathcal{H}_{\text{loc}} \otimes \cdots \otimes \mathcal{H}_{\text{loc}} \quad \text{(schematically)} \tag{15}$$

- A state $|\phi\rangle \otimes |\chi\rangle$ describes a system where part A is in state $\phi$ **AND** part B is in state $\chi$.

*Remark* 4.4 (Anyonic nuance)*.* For anyons, $\mathcal{H}_N$ is not a simple tensor product of single-particle spaces due to fusion constraints (fusion spaces are not product spaces). However, the *ambient* space in which $\mathcal{H}_N$ is embedded (before fusion constraints) often has a tensor product structure (e.g., sites on a lattice).

### 4.4 First-Quantised Operators

**Definition 4.5** (Operator on Fock space)**.** An operator $\hat{O} : \mathcal{F} \to \mathcal{F}$ is defined by its action on each sector $\mathcal{H}_N$ and maps between sectors. It can be represented as a matrix of operators $\hat{O}_{MN} : \mathcal{H}_N \to \mathcal{H}_M$.

**Definition 4.6** (Number-conserving operator)**.** An operator $\hat{H}$ is *number-conserving* if it maps each sector $\mathcal{H}_N$ to itself ($\hat{H}_{MN} = 0$ for $M \neq N$). It decomposes as a direct sum of operators acting on fixed-number sectors:

$$\hat{H} = \bigoplus_{N=0}^{\infty} \hat{H}_N \tag{16}$$

where $\hat{H}_N : \mathcal{H}_N \to \mathcal{H}_N$.

**Example 4.7.** The Hamiltonian for mobile anyons is typically number-conserving (unless studying source terms), so we construct it by defining a sequence of Hamiltonians $H_N$ for each $N$-anyon configuration space.

### 4.5 Summary

| **Structure** | **Symbol** | **Interpretation** | **Logical Equiv.** |
|---|---|---|---|
| Direct Sum | $\oplus$ | Superposition of sectors | **OR** |
| Tensor Product | $\otimes$ | Composition of parts | **AND** |
| Fock Space | $\mathcal{F}$ | $\bigoplus_N \mathcal{H}_N$ | Variable particle number |
| Operator | $\hat{O}$ | $\bigoplus_N \hat{O}_N$ (if conserved) | Collection of $N$-particle ops |

## 5 Quantum Mechanics for Lattice Models

**Assumptions.**

(A3.3.1) All Hilbert spaces are finite-dimensional.

(A3.3.2) Hamiltonians are Hermitian.

### 5.1 Hilbert Spaces, States, Observables

**Definition 5.1** (Hilbert space)**.** A *Hilbert space* $\mathcal{H}$ is a complex vector space equipped with an inner product $\langle\cdot|\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ that is complete with respect to the induced norm. In this work, all Hilbert spaces are finite-dimensional.

**Definition 5.2** (State)**.** A *state* is a nonzero vector $|\psi\rangle \in \mathcal{H}$. Two states are physically equivalent if they differ by a nonzero scalar. Normalised states satisfy $\langle\psi|\psi\rangle = 1$.

**Definition 5.3** (Observable)**.** An *observable* is a Hermitian operator $A : \mathcal{H} \to \mathcal{H}$, i.e., $A = A^\dagger$ where $\langle\phi|A\psi\rangle = \langle A\phi|\psi\rangle$ for all $|\phi\rangle, |\psi\rangle$.

### 5.2 Hamiltonians for Lattice Systems

**Definition 5.4** (Hamiltonian)**.** A *Hamiltonian* is a Hermitian operator $H : \mathcal{H} \to \mathcal{H}$ representing the total energy of the system.

**Definition 5.5** (Local Hamiltonian)**.** A Hamiltonian $H$ is *local* if it decomposes as a sum of terms, each acting nontrivially on a bounded number of sites:

$$H = \sum_j h_j \tag{17}$$

where each $h_j$ acts on sites in a neighbourhood of $j$.

**Definition 5.6** (Nearest-neighbour Hamiltonian)**.** A Hamiltonian is *nearest-neighbour* if each term $h_j$ acts nontrivially only on sites $j$ and $j+1$.

### 5.3 Ground States, Spectrum, Partition Function

**Definition 5.7** (Spectrum)**.** The *spectrum* of $H$ is the set of eigenvalues $\{E_0, E_1, \ldots\}$ ordered so that $E_0 \leq E_1 \leq \cdots$.

**Definition 5.8** (Ground state)**.** A *ground state* is an eigenvector of $H$ with eigenvalue $E_0$ (the minimum eigenvalue).

**Definition 5.9** (Spectral gap)**.** The *spectral gap* is $\Delta = E_1 - E_0$.

**Definition 5.10** (Partition function)**.** The *partition function* at inverse temperature $\beta$ is

$$Z(\beta) = \mathrm{Tr}\left(e^{-\beta H}\right) = \sum_n e^{-\beta E_n} \tag{18}$$

### 5.4 Tensor Product Structure

**Definition 5.11** (Tensor product of Hilbert spaces)**.** Given Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, the *tensor product* $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a Hilbert space with:

- $\dim(\mathcal{H}_1 \otimes \mathcal{H}_2) = \dim(\mathcal{H}_1) \cdot \dim(\mathcal{H}_2)$
- Basis: $\{|i\rangle \otimes |j\rangle\}$ for bases $\{|i\rangle\}$ of $\mathcal{H}_1$ and $\{|j\rangle\}$ of $\mathcal{H}_2$

**Definition 5.12** (Multi-site Hilbert space)**.** For $n$ sites with local Hilbert space $\mathcal{H}_{\mathrm{loc}}$ of dimension $d$, the total Hilbert space is

$$\mathcal{H}_{\mathrm{total}} = \bigotimes_{j=1}^{n} \mathcal{H}_{\mathrm{loc}} \cong \mathbb{C}^{d^n} \tag{19}$$

*Remark* 5.13. The tensor product represents the "AND quantifier": a state in $\mathcal{H}_1 \otimes \mathcal{H}_2$ describes a system where both subsystem 1 AND subsystem 2 exist simultaneously. Cf. §4.

### 5.5 Summary

| **Concept** | **Symbol** | **Definition** |
|---|---|---|
| Hilbert space | $\mathcal{H}$ | Finite-dim complex inner product space |
| State | $\lvert\psi\rangle$ | Nonzero vector in $\mathcal{H}$ |
| Observable | $A$ | Hermitian operator |
| Hamiltonian | $H$ | Hermitian operator (energy) |
| Ground state | $\lvert\psi_0\rangle$ | Eigenvector with minimal eigenvalue |
| Partition function | $Z(\beta)$ | $\mathrm{Tr}\left(e^{-\beta H}\right)$ |
| Tensor product | $\otimes$ | "AND quantifier" for composite systems |

# 6 Lattice Geometry

**Assumptions.**

(A3.4.1) Spatial dimension is 1.

(A3.4.2) Open boundary conditions (no PBCs).

(A3.4.3) Finite number of sites.

## 6.1 One-Dimensional Chain

**Definition 6.1** (Lattice)**.** A *lattice* $\Lambda$ is a finite set of sites. In this work:

$$\Lambda = \{0, 1, 2, \ldots, n-1\} \tag{20}$$

where $n = |\Lambda|$ is the number of sites.

*Remark* 6.2. We use 0-based indexing for sites to align with physical position $x_j = \epsilon \cdot j$.

**Definition 6.3** (Neighbour relation)**.** Sites $i$ and $j$ are *neighbours* if $|i - j| = 1$. We write $i \sim j$.

**Definition 6.4** (Boundary sites)**.** The *boundary* of $\Lambda$ consists of sites 0 and $n-1$. The *bulk* consists of sites $\{1, \ldots, n-2\}$.

## 6.2 Lattice Spacing and Physical Position

**Definition 6.5** (Physical length)**.** The *physical length* of the system is $L > 0$.

**Definition 6.6** (Lattice spacing)**.** The *lattice spacing* is

$$\epsilon = \frac{L}{n} \tag{21}$$

**Definition 6.7** (Physical position)**.** The *physical position* of site $j$ is

$$x_j = \epsilon \cdot j = \frac{L \cdot j}{n} \tag{22}$$

so that $x_0 = 0$ and $x_{n-1} = L(1 - 1/n) < L$.

**Convention 6.8.** Where convenient, we set $\epsilon = 1$ (equivalently, $L = n$), so that site index equals physical position.

## 6.3 Open Boundary Conditions

**Definition 6.9** (Open boundary conditions)**.** A system has *open boundary conditions* (OBC) if:

1. The lattice is a finite chain with distinct endpoints
2. Site 0 has only one neighbour (site 1)
3. Site $n-1$ has only one neighbour (site $n-2$)

*Remark* 6.10. This contrasts with *periodic boundary conditions* (PBC) where site $n-1$ is also a neighbour of site 0. We exclude PBCs in this work (Assumption A3.4.2).

*Remark* 6.11. OBC implies:

- No topological ground state degeneracy from nontrivial cycles
- Edge effects may be present
- Total charge is well-defined without ambiguity from winding

### 6.4 Local Hilbert Spaces

**Definition 6.12** (Local Hilbert space)**.** Each site $j \in \Lambda$ carries a *local Hilbert space* $\mathcal{H}_j$. For identical sites:

$$\mathcal{H}_j \cong \mathcal{H}_{\text{loc}} \quad \forall j \tag{23}$$

**Definition 6.13** (Local dimension)**.** The *local dimension* is $d_{\text{loc}} = \dim(\mathcal{H}_{\text{loc}})$.

**Definition 6.14** (Total Hilbert space)**.** The *total Hilbert space* for the lattice is

$$\mathcal{H}_{\text{total}} = \bigotimes_{j=0}^{n-1} \mathcal{H}_j \cong \mathbb{C}^{d^n} \tag{24}$$

*Remark* 6.15. For mobile anyons, the local Hilbert space structure is more subtle—see §8. The "local dimension" depends on the occupation at that site.

### 6.5 Summary

| Concept | Symbol | Value/Definition |
|---|---|---|
| Number of sites | $n$ | $\lvert\Lambda\rvert$ |
| Site indices | $j$ | $0, 1, \ldots, n-1$ |
| Physical length | $L$ | System size |
| Lattice spacing | $\epsilon$ | $L/n$ |
| Physical position | $x_j$ | $\epsilon \cdot j$ |
| Local Hilbert space | $\mathcal{H}_j$ | Space at site $j$ |
| Local dimension | $d_{\text{loc}}$ | $\dim(\mathcal{H}_{\text{loc}})$ |
| Total dimension | — | $d^n$ |

### 6.6 Notation Conventions

Throughout this project:

- Sites are **0-indexed**: $j \in \{0, 1, \ldots, n-1\}$
- Boundary conditions are **open** (OBC)
- Default: $\epsilon = 1$ unless stated otherwise
- Tensor products are ordered left-to-right: $\mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{n-1}$

# Part II
# Hilbert Space for Mobile Anyons

## 7 Classical Configuration Space

**Assumptions.**

(A4.1.1) Fusion category $\mathcal{C}$ with simples $\{X_0, X_1, \ldots, X_{d-1}\}$ where $X_0 = \mathbf{1}$.

(A4.1.2) Lattice $\Lambda = \{0, \ldots, n-1\}$ with OBC.

### 7.1 Anyon Positions

**Definition 7.1** (Configuration)**.** A *configuration* of $N$ anyons is a tuple

$$\mathbf{x} = (x_1, x_2, \dots, x_N) \in \Lambda^N \tag{25}$$

satisfying $0 \le x_1 \le x_2 \le \cdots \le x_N \le n-1$.

*Remark* 7.2. The ordering $x_1 \le \cdots \le x_N$ reflects indistinguishability up to exchange. Distinct orderings related by permutation represent the same physical configuration.

### 7.2 Anyon Labels

**Definition 7.3** (Labelled configuration)**.** A *labelled configuration* is a pair $(\mathbf{x}, \mathbf{k})$ where:

- $\mathbf{x} = (x_1, \dots, x_N)$ is a configuration
- $\mathbf{k} = (k_1, \dots, k_N)$ with $k_j \in \{1, \dots, d-1\}$ labels the anyon type at position $x_j$

**Convention 7.4.** Label 0 denotes vacuum (no anyon). Labels $1, \dots, d-1$ denote the nontrivial simple objects $X_1, \dots, X_{d-1}$.

### 7.3 Configuration Space

**Definition 7.5** ($N$-anyon configuration space)**.**

$$\mathrm{Conf}_N = \{(\mathbf{x}, \mathbf{k}) : \mathbf{x} \in \Lambda^N \text{ ordered}, \, k_j \in \{1, \dots, d-1\}\} \tag{26}$$

**Definition 7.6** (Hard-core configuration space)**.** With hard-core constraint:

$$\mathrm{Conf}_N^{\mathrm{HC}} = \{(\mathbf{x}, \mathbf{k}) \in \mathrm{Conf}_N : x_1 < x_2 < \cdots < x_N\} \tag{27}$$

**Claim 7.7.** $|\mathrm{Conf}_N^{\mathrm{HC}}| = \binom{n}{N}(d-1)^N$.

### 7.4 Mixed Character

*Remark* 7.8. Configurations have mixed identical/distinguishable character:

- Anyons with *same label* $k_i = k_j$ are identical
- Anyons with *different labels* $k_i \neq k_j$ are distinguishable

This is handled by the morphism space structure in §8.

## 8 Hilbert Space for Mobile Anyons

**Assumptions.**

(A4.2.1) Fusion category $\mathcal{C}$ with unit $\mathbf{1}$ and simples $X_0 = \mathbf{1}, X_1, \dots, X_{d-1}$.

(A4.2.2) $n$ lattice sites, OBC.

## 8.1 Morphism Spaces

**Definition 8.1** (Morphism space)**.** For objects $A, B \in \mathcal{C}$:

$$\mathrm{Mor}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B) \tag{28}$$

This is a finite-dimensional complex vector space with $\dim \mathrm{Mor}(A, B) = \delta_{A,B}$ for simple $A, B$ (in the multiplicity-free case).

*Remark* 8.2. With multiplicity: $\dim \mathrm{Mor}(X_a \otimes X_b, X_c) = N_{ab}^c$.

## 8.2 $N$-Anyon Sector with Definite Charge

**Definition 8.3** (Site occupation)**.** For a labelled configuration $(\mathbf{x}, \mathbf{k})$ on $n$ sites, define the *site label* $s_j \in \{0, 1, \ldots, d-1\}$ for each site $j \in \{0, \ldots, n-1\}$:

$$s_j = \begin{cases} k_i & \text{if } x_i = j \text{ for some } i \in \{1, \ldots, N\} \\ 0 & \text{otherwise (site unoccupied, i.e., vacuum } \mathbf{1}) \end{cases} \tag{29}$$

**Definition 8.4** (Object for configuration)**.** The *tensor product object* for a labelled configuration $(\mathbf{x}, \mathbf{k})$ is:

$$\mathcal{O}(\mathbf{x}, \mathbf{k}) = X_{s_0} \otimes X_{s_1} \otimes \cdots \otimes X_{s_{n-1}} \tag{30}$$

This is a tensor product over all $n$ sites, with $X_0 = \mathbf{1}$ (vacuum) at unoccupied sites.

**Definition 8.5** ($N$-anyon space, definite charge)**.** The space of $N$ anyons with total charge $c$:

$$\mathcal{H}_N^{(c)} = \bigoplus_{(\mathbf{x}, \mathbf{k}) \in \mathrm{Conf}_N} \mathrm{Mor}(X_c, \mathcal{O}(\mathbf{x}, \mathbf{k})) \tag{31}$$

*Remark* 8.6. A state $|\psi\rangle \in \mathcal{H}_N^{(c)}$ is a superposition over configurations, with amplitudes in the appropriate morphism spaces. The morphism $\mathrm{Mor}(X_c, \mathcal{O}(\mathbf{x}, \mathbf{k}))$ describes how anyons at the specified positions fuse to total charge $X_c$.

## 8.3 $N$-Anyon Sector with Indefinite Charge

**Definition 8.7** ($N$-anyon space, indefinite charge)**.**

$$\mathcal{H}_N = \bigoplus_{c=0}^{d-1} \mathcal{H}_N^{(c)} \tag{32}$$

*Remark* 8.8. Direct sum = "OR quantifier": the system has charge $X_0$ OR charge $X_1$ OR ...

## 8.4 Total Hilbert Space

**Definition 8.9** (Total Hilbert space)**.**

$$\mathcal{H} = \bigoplus_{N=0}^{n} \mathcal{H}_N \tag{33}$$

where $\mathcal{H}_0 = \mathbb{C}$ (vacuum sector).

*Remark* 8.10. Direct sum over $N$ = "OR quantifier" for particle number. This is the first-quantised Fock space structure (cf. §4).

### 8.5 Explicit Construction

For hard-core anyons, a basis of $\mathcal{H}_N^{(c)}$ is indexed by:

1. Configuration $(\mathbf{x}, \mathbf{k}) \in \mathrm{Conf}_N^{\mathrm{HC}}$
2. Fusion tree $\tau$ from $\mathcal{O}(\mathbf{x}, \mathbf{k})$ to $X_c$
3. Multiplicity indices (if $N_{ab}^c > 1$)

**Claim 8.11.** *For multiplicity-free categories:*

$$\dim \mathcal{H}_N^{(c)} = \sum_{(\mathbf{x},\mathbf{k})} \dim \mathrm{Mor}(X_c, \mathcal{O}(\mathbf{x}, \mathbf{k})) \tag{34}$$

## 9 Operators as Morphisms

### 9.1 Operator Space

**Definition 9.1** (Operator)**.** An operator on $\mathcal{H}$ is a linear map $\mathcal{H} \to \mathcal{H}$.

**Definition 9.2** (Operator as morphism sum)**.** Operators decompose as:

$$\mathcal{A} \in \bigoplus_{A,B} \mathrm{Mor}(A, B) \tag{35}$$

where $A, B$ range over tensor products of $\mathbf{1}, X_1, \ldots, X_{d-1}$.

### 9.2 Particle-Number Conservation

**Definition 9.3** (Particle-conserving operator)**.** An operator $\mathcal{A}$ is *particle-conserving* if it maps $\mathcal{H}_N \to \mathcal{H}_N$ for all $N$.

**Definition 9.4** (Particle-changing operator)**.** An operator with components in $\mathrm{Mor}(A, B)$ where $A$ and $B$ have different numbers of nontrivial factors changes particle number.

**Example 9.5.** $\mathrm{Mor}(X_a \otimes X_b, X_c \otimes \mathbf{1})$ annihilates a particle (if $X_c$ is nontrivial) or two particles (if $X_c = \mathbf{1}$).

### 9.3 Locality

**Definition 9.6** (Local operator)**.** An operator is *local* if it acts nontrivially only on a bounded number of adjacent sites.

**Definition 9.7** ($k$-local operator)**.** A $k$*-local* operator acts on at most $k$ consecutive sites.

For mobile anyons, a 2-local operator has components:

- $\mathrm{Mor}(X_a \otimes X_b, X_c \otimes X_d)$ — acts on two neighbouring anyons
- $\mathrm{Mor}(X_a \otimes \mathbf{1}, \mathbf{1} \otimes X_a)$ — hops anyon right
- $\mathrm{Mor}(\mathbf{1} \otimes X_a, X_a \otimes \mathbf{1})$ — hops anyon left

# 10 Hard-Core vs Soft-Core Anyons

## 10.1 Hard-Core Constraint

**Definition 10.1** (Hard-core anyons)**.** Anyons are *hard-core* if at most one anyon occupies each site:

$$x_1 < x_2 < \cdots < x_N \tag{36}$$

*Consequence* 10.2. Maximum particle number is $N_{\max} = n$ (number of sites).

*Remark* 10.3. Hard-core is natural for:

- Impenetrable particles (infinite on-site repulsion)
- Lattice models where sites represent localised orbitals

## 10.2 Soft-Core: Multiple Occupancy

**Definition 10.4** (Soft-core anyons)**.** Anyons are *soft-core* if multiple anyons may occupy the same site:

$$x_1 \leq x_2 \leq \cdots \leq x_N \tag{37}$$

**Definition 10.5** (On-site fusion space)**.** When anyons $X_a, X_b$ occupy the same site, the local state space is:

$$\bigoplus_c N_{ab}^c \cdot \mathrm{Mor}(X_a \otimes X_b, X_c) \tag{38}$$

representing the possible fusion outcomes.

*Remark* 10.6. Soft-core requires tracking on-site fusion structure.

## 10.3 Hilbert Space Modifications

| **Regime** | **Config space** | **Local structure** | $N_{\max}$ |
|---|---|---|---|
| Hard-core | $x_i < x_j$ for $i < j$ | One anyon/site | $n$ |
| Soft-core | $x_i \leq x_j$ for $i < j$ | Fusion at each site | $\infty$ (needs cutoff) |

**Definition 10.7** (Hard-core Hilbert space)**.**

$$\mathcal{H}^{\mathrm{HC}} = \bigoplus_{N=0}^{n} \mathcal{H}_N^{\mathrm{HC}} \tag{39}$$

where $\mathcal{H}_N^{\mathrm{HC}}$ uses $\mathrm{Conf}_N^{\mathrm{HC}}$.

## 10.4 Physical Motivation

**Hard-core regime:**

- Models impenetrable anyons
- Connection to Girardeau mapping (§**??**)
- Simpler Hilbert space structure

**Soft-core regime:**

- Models “bosonic” anyons that can bunch
- Richer on-site physics
- Connection to Levin–Wen models when particles can annihilate

**Convention 10.8.** Unless stated otherwise, we work in the **hard-core** regime.

## References

[ENO05] P. Etingof, D. Nikshych, and V. Ostrik. On fusion categories. *Annals of Mathematics*, 162(2):581–642, 2005.

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