P H Y S I C A L   R E V I E W   L E T T E R S

Anyon-Fermion Mapping and Applications to Ultracold Gases in Tight Waveguides

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The Fermi-Bose mapping method for one-dimensional Bose and Fermi gases with zero-range interactions is generalized to an anyon-fermion mapping and applied to exact solution of several models of ultracold gases with anyonic exchange symmetry in tight waveguides: anyonic Calogero-Sutherland model, anyons with point hard-core interaction (anyonic Tonks-Girardeau gas), and spin-aligned anyon gas with infinite zero-range odd-wave attractions (attractive anyonic Tonks-Girardeau, or AATG, gas). It is proved that for even $N \geq 4$ there are states of the AATG gas on a ring, with anyonic phase slips which are odd integral multiples of $\pi/(N-1)$, of energy lower than that of the corresponding fermionic ground state. A generalization to a spinor Fermi gas state with anyonic symmetry under purely spatial exchange enables energy lowering by the same mechanism.

The Fermi-Bose (FB) mapping method was introduced in 1960 [1] and used to obtain the exact N-particle ground and excited states of a 1D gas of impenetrable point bosons [now known as the Tonks-Girardeau (TG) gas], which has recently become the subject of extensive theoretical and experimental investigations because of the novelty and experimental realizability [2,3] of ultracold gases in tight atom waveguides with strong correlations induced by Feshbach resonance tuning [4] of the effective 1D interactions to very large values via confined-induced resonances [5–7]. It is now known [7–10] that the FB mapping is of much greater generality; when supplemented by an inversion and sign change of the coupling constant, it provides a mapping between the N-body energy eigenstates of a 1D Bose gas with delta-function interactions of any strength [Lieb-Liniger (LL) gas [11]] and those of a spin-aligned Fermi gas. For a recent review, see [12].

In the three-dimensional world, experimental evidence supports the *symmetrization postulate* (SP), according to which wave functions of identical particles are either completely symmetric (Bose) or antisymmetric (Fermi) under permutations of the particle coordinates [13]. The spin-statistics theorem entertains no possibilities other than bosons or fermions, excluding more complicated permutation symmetries from the start *by hypothesis* [14]. There is a trivial "proof" [15] of SP in some textbooks, but it is incorrect; by generalizing the original FB mapping [1], I identified the logical error in this proof and pointed out that for a 1D system of identical particles with hard-core interactions, wave functions which are neither completely symmetric (Bose) nor completely antisymmetric (Fermi) are physically allowed [16]. Leinaas and Myrheim [17] generalized and rigorized this approach, proved SP in three dimensions, and showed that permutation symmetries interpolating continuously between bosons and fermions are not excluded in one dimension and two dimensions. In recent years low-dimensional systems with anyonic symmetry have found application both in relativistic particle physics and condensed matter (quantized fractional Hall effect [18–20], anyonic superconductivity [21,22], rotating ultracold gases [23–25], Aharonov-Bohm and Aharonov-Casher effects and persistent current in 1D mesoscopic rings [26], and quantum-knot computation [27]).

Most of these applications involve systems which are essentially two dimensional, but anyonic symmetry can also occur in one dimension [26,28], where anyonic exchange symmetry leads to strong short-range correlations. An anyonic generalization of the FB mapping will be used herein to obtain exact solutions for several models of ultracold gases with 1D anyonic exchange symmetry in tight waveguides: the anyonic Calogero-Sutherland (CS) model, anyonic TG gas, and attractive anyonic TG (AATG) gas. The TG gas has already been physically realized via Feshbach resonance tuning of the effective 1D interactions in Bose gases in tight waveguides [2,3], the fermionic TG (FTG) gas may be realizable by the same mechanism, and it has recently been pointed out [29] that the CS model of bosons with inverse square repulsive potential should also be realizable.

1D anyon fields, anyonic symmetry, and zero-range interactions.—In one dimension, particles can physically exchange positions only by passing through each other, so exchange symmetry is inseparable from short-range interactions. This is the origin of the FB mapping from impenetrable point bosons (TG gas) to noninteracting spin-aligned fermions [1], and more generally of the Fermi-Bose duality in one dimension [30]. Several similar but nonequivalent definitions of 1D anyons appear in the literature. Kundu [28] defines anyon field operators $\hat{\psi}_A(x)$ in terms of Bose operators $\hat{\psi}_B(x)$ by $\hat{\psi}_A(x) = e^{-i\theta \int_{-\infty}^{x} dx' \hat{\rho}(x')} \hat{\psi}_B(x)$, where $\hat{\rho}(x) = \hat{\psi}_B^{\dagger}(x)\hat{\psi}_B(x) = \hat{\psi}_A^{\dagger}(x)\hat{\psi}_A(x)$ is the number density operator. These satisfy exchange relations $\hat{\psi}_A(x)\hat{\psi}_A^{\dagger}(x') = \delta(x-x') + e^{-i\theta\epsilon(x-x')}\hat{\psi}_A^{\dagger}(x')\hat{\psi}_A(x)$ and $\hat{\psi}_A(x)\hat{\psi}_A(x') = e^{i\theta\epsilon(x-x')}\hat{\psi}_A(x')\hat{\psi}_A(x)$, where $\epsilon(x) = +1(-1)$ for $x > 0(x < 0)$, and $\epsilon(0) = 0$. In this Letter x and x' are 1D

coordinates, bearing in mind the effectively 1D dynamics of ultracold gases in wave guides with transverse trapping so tight that the transverse excitation energy quantum exceeds the available longitudinal zero-point energy [5]. Kundu carried out a formal Bethe ansatz solution for the N-body energy eigenstates of such a system of anyons starting from a contact condition of LL form [11]. However, an attempt to generate these contact conditions as a zero-range limit of boundary conditions at the edges of a finite-range interaction potential leads to contradictions, and application of the kinetic energy operator to a wave function with contact discontinuities generates singularities which can be canceled only by highly singular and ill-defined interactions [28].

A different definition closely related to fermions eliminates these difficulties. Define the anyon field annihilation operator $\hat{\psi}_A(x)$ in terms of the Fermi field operator $\hat{\psi}_F(x)$ by $\hat{\psi}_A(x) = e^{-i\theta \int_{-\infty}^{x} dx' \hat{\rho}(x')} \hat{\psi}_F(x)$, where $\hat{\rho}(x) = \hat{\psi}_F^\dagger(x)\hat{\psi}_F(x) = \hat{\psi}_A^\dagger(x)\hat{\psi}_A(x)$ is the number density operator. Then

$$\hat{\psi}_A(x)\hat{\psi}_A^\dagger(x') + e^{-i\theta\epsilon(x-x')}\hat{\psi}_A^\dagger(x')\hat{\psi}_A(x) = \delta(x - x'),$$

$$\hat{\psi}_A(x)\hat{\psi}_A(x') + e^{i\theta\epsilon(x-x')}\hat{\psi}_A(x')\hat{\psi}_A(x) = 0. \tag{1}$$

Then the exclusion principle $\hat{\psi}^2(x) = [\hat{\psi}^\dagger(x)]^2 = 0$ follows from $\epsilon(x) = 0$, ensuring that anyonic phase discontinuities occur only at collisional nodes of the wave functions. At such zeros the phase is undefined, allowing phase slips consistent with those required by anyonic exchange symmetry.

N-particle anyon wave functions Ψ_A are the amplitudes in an N-anyon Fock state $|\Psi_A\rangle = (N!)^{-1/2} \times \int dx_1 \cdots dx_N \Psi_A(x_1, \ldots, x_N)\hat{\psi}_A^\dagger(x_1) \cdots \hat{\psi}_A^\dagger(x_N)|0\rangle$. Using Eq. (1) to exchange $\hat{\psi}_A^\dagger(x_j)$ and $\hat{\psi}_A^\dagger(x_{j+1})$ and interchanging the names of these integration variables, one proves that $\Psi_A(\ldots, x_j, x_{j+1}, \ldots) = -e^{-i\theta\epsilon(x_{j+1}-x_j)}\Psi_A(\ldots, x_{j+1}, x_j, \ldots)$; i.e., there is a sign change plus a phase slip $e^{i\theta}$ ($e^{-i\theta}$) when any particle passes its neighbor to the right (left). Iterating this one proves

$$\Psi_A(x_1, \ldots, x_j, \ldots, x_k, \ldots, x_N) = -e^{i\theta[\Sigma_{\ell=j+1}^{k}\epsilon(x_\ell-x_j)+\Sigma_{\ell=j+1}^{k-1}\epsilon(x_k-x_\ell)]}\Psi_A(x_1, \ldots, x_k, \ldots, x_j, \ldots, x_N), \tag{2}$$

which is similar to Kundu's Eq. (11) [28], but has the very important difference that these wave functions satisfy the exclusion principle; i.e., Ψ_A vanishes when $x_j = x_k$ for all $j \neq k$. Anyonic symmetry of this type may have applications to electrons in mesoscopic rings [26] as well as to ultracold gases in the 1D regime. Ψ_A cannot satisfy periodic boundary conditions with periodicity length L unless θ is a multiple of $2\pi/(N-1)$; instead, it satisfies *twisted boundary conditions* $\Psi_A(x_1, \ldots, x_j \pm L, \ldots, x_N) = (-1)^{N-1}e^{\mp i(N-1)\theta}\Psi_A(x_1, \ldots, x_j, \ldots, x_N)$. If one places the particles on a ring of circumference L and requires that wave functions be single valued, then the only values of θ allowed if N is odd are integral multiples of $2\pi/(N-1)$, and if N is even the only allowed values are odd integral multiples of $\pi/(N-1)$. As $N \to \infty$ this set of allowed θ values becomes dense.

Anyon-fermion mapping.—Define an anyon mapping function A_θ by $A_\theta(x_1, \ldots, x_N) = \prod_{1 \leq j < k \leq N} e^{(1/2)i\theta\epsilon(x_{jk})}$ where $x_{jk} = x_j - x_k$. This generalizes the original FB mapping [1,16], to which it reduces, apart from an irrelevant constant factor, when $\theta = \pi$. Define Ψ_F by $\Psi_F(x_1, \ldots, x_N) = A_\theta\Psi_A(x_1, \ldots, x_N)$ where Ψ_A is an N-anyon wave function satisfying Eq. (2). Then Ψ_F is totally antisymmetric (fermionic). Conversely, if one defines Ψ_A by $\Psi_A(x_1, \ldots, x_N) = A_{-\theta}\Psi_F(x_1, \ldots, x_N)$ where Ψ_F is fermionic, then Ψ_A satisfies (2). Finally, if one defines Ψ_B by $\Psi_B = A_{\theta+\pi}\Psi_A$ where Ψ_A satisfies (2), then Ψ_B will be completely symmetric (bosonic).

Anyonic CS gas.—Consider bosons or fermions on a ring of circumference L with Hamiltonian [31] $-\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + g\sum_{1 \leq j < k \leq N} d^{-2}(x_{jk})$ in units with $\hbar = 2m =$

1, where $d(x_{jk}) = (L/\pi)\sin(\pi|x_j - x_k|/L)$ is the chordal distance between x_j and x_k. It may be realizable via the strong dipolar interactions in ^{52}Cr [29,32]. The same Hamiltonian can be applied to wave functions with anyonic symmetry, and the anyonic ground state obtained by mapping from the fermionic one [31]: $\Psi_{A0}(x_1, \ldots, x_N) = \prod_{1 \leq j < k \leq N}\epsilon(x_{jk})e^{-(1/2)i\theta\epsilon(x_{jk})}|\sin(\pi x_{jk}/L)|^\lambda$ with $\lambda = \frac{1}{2} \times (1 + \sqrt{1 + 2g})$ and energy $E_0 = \frac{1}{3}\pi^2\lambda^2 N(N^2 - 1)/L^2$, reducing for $g = 0$ to the corresponding results for the TG gas [1].

Anyonic TG gas.—The TG gas is a 1D gas of impenetrable point bosons ($a \to 0+$ limit of hard cores of diameter a) with no particle-particle interactions except for the zero-diameter hard cores, which are equivalent to a constraint that all wave functions vanish at particle collision points $x_j = x_k$. It has been solved exactly by FB mapping to the ideal Fermi gas for the cases of periodic boundary conditions [1], harmonic trapping [33], and box enclosure [34]. Consider now an anyon gas. In the absence of interparticle interactions such a system is sometimes called an ideal anyon gas, but since (2) implies that its wave functions Ψ_A automatically satisfy the impenetrable point constraint of vanishing at contacts $x_j = x_k$, it is more properly viewed as an anyonic generalization of the TG gas, hence the name "anyonic TG gas." It maps to the 1D ideal Fermi gas via $\Psi_A(x_1, \ldots, x_N) = A_{-\theta}\Psi_F(x_1, \ldots, x_N)$. The Schrödinger equation is to be applied only when all interparticle separations $|x_j - x_k|$ are nonzero, being replaced at particle contact $x_j = x_k$ by the condition of vanishing wave function. The Hamiltonian consists only of the ki-

netic energy operator, and it follows just as for the TG gas [1,16] that its complete energy spectrum is identical with that of the corresponding ideal Fermi gas, as are all properties (both time independent and time dependent [12]) depending only on absolute squares $|\Psi_A|^2$ of its wave functions. For twisted boundary conditions its ground state wave function is $\Psi_{A0} = \prod_{1 \le j < k \le N} \sin(\pi x_{jk}/L) \times e^{-(1/2)i\theta\epsilon(x_{jk})}$, where $x_{jk} = x_j - x_k$, its energy in the thermodynamic limit is the same as that of the TG gas, $E_0/N = (\pi\hbar n)^2/6m$, and its low excitation spectrum is of phonon form with sound speed $c = \pi\hbar n/m$ [1]; here $n = N/L$, the particle number density. These thermodynamic limit results also apply to the anyonic TG gas on a ring of circumference L, since, as previously pointed out, the values of θ allowed by the requirement of single-valued wave functions become dense in the thermodynamic limit.

AATG gas.—The FTG gas is a spin-aligned 1D Fermi gas with infinitely strongly attractive zero-range odd-wave interaction induced by a p-wave Feshbach resonance. It is the infinite 1D scattering length limit $a_{1D} \to -\infty$ of a 1D Fermi gas with zero-range attractive interactions leading to a 1D scattering length defined in terms of the ratio of the derivative Ψ'_F of the wave function to its value at contact: $\Psi_F(x_{jk} = 0+) = -\Psi_F(x_{jk} = 0-) = -a_{1D}\Psi'_F(x_{jk} = 0\pm)$ [7,9,10]. The limit $a_{1D} \to -\infty$ corresponds to a zero-energy scattering resonance reachable by Feshbach resonance tuning to a 1D confinement-induced resonance [4–7], where the exterior 1D scattering wave function is constant. The contact discontinuities of Ψ_F [8] can be understood as a zero-range limit $x_0 \to 0+$ and $V_0 \to \infty$ of the two-body solution for a square well of width $2x_0$ and depth V_0, where the limit is carried out such that $V_0 x_0^2$ approaches a finite, nonzero limit [9,10]. For $a_{1D} \to -\infty$ the exterior solution is constant ($+1$ for $x_{12} > 0$ and -1 for $x_{12} < 0$) and the interior solution is $\sin(\kappa x_{12})$ with $\kappa = \sqrt{mV_0/\hbar^2} = \pi/2x_0$. In the zero-range limit the interior kinetic energy $\to +\infty$ and potential energy $\to -\infty$, but their sum remains zero, the ground state energy. The N-body problem is solved by mapping to the ideal Bose gas [9,10]. For periodic boundary conditions this Bose ground state is a trivial constant. However, by looking at the square well solution one sees that there is a nontrivial interior Bose wave function $\sin(\kappa|x_{12}|)$ vanishing with a cusp at $x_{12} = 0$. Therefore, physical consistency requires the presence of a zero-diameter hard-core interaction added to the square well. The mapped Bose gas is then not truly ideal, but rather a *TG gas with superimposed attractive well*, whose nontrivial interior wave function becomes invisible in the zero-range limit, simulating an ideal Bose gas insofar as the energy and exterior wave function are concerned. The required impenetrable core is physically quite reasonable, since the atoms have a strong short-range Pauli exclusion repulsion of their inner shells, whose diameter is effectively zero and strength infinite on length and energy scales appropriate to ultracold gas experiments.

This approach is easily generalized to the anyonic case. In the resonant case $a_{1D} \to -\infty$ the exterior two-body wave function is $\epsilon(x_{12})e^{-(1/2)i\theta\epsilon(x_{12})}$ and the interior wave function is $e^{-(1/2)i\theta\epsilon(x_{12})}\sin(\kappa x_{12})$ with the same value of κ as for the FTG gas. This generalizes to $N > 2$ giving an almost trivial ground state: $\Psi_{A0}(x_1, \ldots, x_N) = \prod_{1 \le j < k \le N}\epsilon(x_{jk})e^{-(1/2)i\theta\epsilon(x_{jk})}\prod_{j=1}^N \phi_0(x_j)$, where ϕ_0 is the lowest ideal Bose gas orbital for given boundary conditions. The ground state energy and all properties depending only on $|\Psi_{A0}|^2$ are the same as those of the ideal Bose gas.

A very interesting odd/even N effect occurs if the system is contained on a ring of circumference L. For the FTG gas we found [35] that for even N the mapped bosonic ground state Ψ_{B0} is an antiperiodic Bose-Einstein condensate fragmented between the k-space sites $k = \pm\pi/L$, thus forcing periodicity of Ψ_{F0} in view of the antiperiodicity of the mapping for even N. However, for anyonic symmetry one can instead choose the lower-energy Bose ground state with all particles condensed at $k = 0$, and force periodicity by requiring that θ be an odd integral multiple of $\pi/(N - 1)$. For $N = 2$ and $\theta = \pi$, Ψ_{A0} reduces to a "bosonic FTG gas," but for $N \ge 4$ there are true zero-energy anyonic ground states of the form $\Psi_{A0} = \prod_{1 \le j < k \le N}\epsilon(x_{jk})e^{-(1/2)i\theta\epsilon(x_{jk})}$. For even N, choosing θ to be an odd integral multiple of $\pi/(N - 1)$ avoids discontinuities at boundaries between adjacent periodicity cells [36]. From this one proves the following: *Theorem 1: For even $N \ge 4$ there is a single-valued, continuous, and periodic anyonic ground state of this system on a ring, with anyonic phase slips which are odd integral multiples of $\pi/(N - 1)$, with energy lower by an amount $Nh^2/8mL^2$ than that of the fermionic ground state with the same interaction.*

Spinor fermions with anyonic spatial symmetry.—Transitions from spin-aligned fermion states to anyonic states cannot be generated by realistic interactions, but by generalizing to a spinor (spin-free) Fermi gas one can obtain a gas of spin-$\frac{1}{2}$ fermions with anyonic *spatial* symmetry by generalizing the previous state Ψ_{A0} to $\Phi_{F0} = \prod_{1 \le j < k \le N}\epsilon(x_{jk})e^{-(1/2)i\theta\epsilon(x_{jk})(\delta_{\sigma_j 1}\delta_{\sigma_k 1} - \delta_{\sigma_j 1}\delta_{\sigma_k 1})}$. Φ_{F0} is fermionic under combined space-spin exchange, but undergoes a fermionic sign change plus a phase slip $\pm\theta$ under exchange of only space coordinates (x_j, x_k) if $\sigma_j \ne \sigma_k$, but only the fermionic sign change if $\sigma_j = \sigma_k$. In general there are both even and odd-wave interactions, in which case the exact ground state will not have this simple form [10,37], but if the even-wave repulsion is weak (dimensionless even-wave coupling constant $\gamma_e \ll 1$) and odd-wave attraction infinite as in the FTG gas ($\gamma_o \to \infty$), then the ground state will be approximately of this form. Φ_{F0} is an *exact energy eigenstate of energy zero* if $\gamma_e = 0$ and under the FB mapping it maps to the ideal Bose gas ground state which is totally condensed at $k = 0$, whereas the

degenerate spin-aligned FTG ground states on a ring map to an antiperiodic Bose condensate fragmented between $k = \pm \pi/L$ [35]. Φ_{F0} is connected to these spin-aligned states by dipolar interactions, enabling energy lowering by spin flips and anyonic phase slips. Φ_{F0} is not an exact eigenstate of $\hat{S}_z$, and in fact the ground state is degenerate with respect to both S_z and total spin S along the hyperbola $\gamma_e \gamma_0 = 4$ in the (γ_e, γ_o) plane; see pp. 19 and 20 of [10]. This is consistent with our assumptions $\gamma_e \to 0$ and $\gamma_o \to \infty$ if the limits are taken along this line. Φ_{F0} has $\langle \hat{S}_z \rangle = 0$, and furthermore, the projected state $\Psi_{F0} = \hat{P}_0 \Phi_{F0}$ is an eigenstate of $\hat{S}_z$ with eigenvalue zero and the same (zero) energy, assuming no spin-dependent interactions; here $\hat{P}_0 = (2\pi)^{-1} \int_0^{2\pi} d\phi \, e^{i\phi \hat{S}_z}$ is the $S_z = 0$ projector. Then Ψ_{F0} satisfies periodicity exactly if θ is chosen to be an odd multiple of $2\pi/N$ [38], implying the following:

Theorem 2: Ψ_{F0} *is an exact ground state of the spinor FTG gas with even N on a ring if θ is an odd multiple of $2\pi/N$. It lies lower than the lowest spin-aligned state by an amount $Nh^2/8mL^2$.*

I close by pointing out that Φ_{F0} has superconductive off-diagonal long-range order as in [35], but now for $(x_1, \uparrow; x_2, \downarrow)$ pairs.

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[1] M. Girardeau, J. Math. Phys. (N.Y.) **1**, 516 (1960).
[2] B. Paredes *et al.*, Nature (London) **429**, 277 (2004).
[3] T. Kinoshita, T. Wenger, and D. S. Weiss, Science **305**, 1125 (2004).
[4] J. L. Roberts *et al.*, Phys. Rev. Lett. **86**, 4211 (2001).
[5] M. Olshanii, Phys. Rev. Lett. **81**, 938 (1998).
[6] T. Bergeman, M. G. Moore, and M. Olshanii, Phys. Rev. Lett. **91**, 163201 (2003).
[7] B. E. Granger and D. Blume, Phys. Rev. Lett. **92**, 133202 (2004).
[8] T. Cheon and T. Shigehara, Phys. Lett. A **243**, 111 (1998); Phys. Rev. Lett. **82**, 2536 (1999).
[9] M. D. Girardeau and M. Olshanii, cond-mat/0309396.
[10] M. D. Girardeau, Hieu Nguyen, and M. Olshanii, Opt. Commun. **243**, 3 (2004).
[11] Elliott H. Lieb and Werner Liniger, Phys. Rev. **130**, 1605 (1963).
[12] V. I. Yukalov and M. D. Girardeau, Laser Phys. Lett. **2**, 375 (2005).

[13] A. M. L. Messiah and O. Greenberg, Phys. Rev. **136**, B248 (1964).
[14] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That* (W. A. Benjamin, New York, 1964), p. 146.
[15] E. M. Corson, *Perturbation Methods in the Quantum Mechanics of n-Electron Systems* (Blackie and Son, Glascow, 1951), p. 113.
[16] M. D. Girardeau, Phys. Rev. **139**, B500 (1965).
[17] J. M. Leinaas and J. Myrheim, Nuovo Cimento Soc. Ital. Fis. **37B**, 1 (1977).
[18] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[19] B. I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984); **52**, 2390(E) (1984).
[20] F. E. Camino, Wei Zhou, and V. J. Goldman, Phys. Rev. B **72**, 075342 (2005).
[21] R. B. Laughlin, Phys. Rev. Lett. **60**, 2677 (1988).
[22] Frank Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific, Singapore, 1990).
[23] N. K. Wilkin and J. M. F. Gunn, Phys. Rev. Lett. **84**, 6 (2000).
[24] N. R. Cooper and N. K. Wilkin, Phys. Rev. B **60**, R16279 (1999).
[25] B. Paredes, P. Fedichev, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **87**, 010402 (2001).
[26] Jian-Xin Zhu and Z. D. Wang, Phys. Rev. A **53**, 600 (1996).
[27] Sankar Das Sarma, Michael Freedman, and Chetan Nayak, Phys. Rev. Lett. **94**, 166802 (2005).
[28] A. Kundu, Phys. Rev. Lett. **83**, 1275 (1999).
[29] Yue Yu, cond-mat/0603340.
[30] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, Cambridge, 1993).
[31] Bill Sutherland, Phys. Rev. A **4**, 2019 (1971).
[32] A. Griesmaier, J. Werner, S. Hensler, J. Stuhler, and T. Pfau, Phys. Rev. Lett. **94**, 160401 (2005).
[33] M. D. Girardeau, E. M. Wright, and J. M. Triscari, Phys. Rev. A **63**, 033601 (2001).
[34] A. del Campo and J. G. Muga, cond-mat/0511747.
[35] M. D. Girardeau and A. Minguzzi, Phys. Rev. Lett. **96**, 080404 (2006).
[36] This expression for Ψ_{A0} is valid when all x_j are in the fundamental periodicity cell $\mathfrak{C}_1 = (0, L)$, and is to be extended outside $\mathfrak{C}_1$ by replacing $\epsilon(x_{jk})$ by $\epsilon[\xi(x_j) - \xi(x_k)]$, where $\xi(x)$ is the periodic map from x to $\mathfrak{C}_1$.
[37] M. D. Girardeau and M. Olshanii, Phys. Rev. A **70**, 023608 (2004).
[38] If $S_z = 0$, then there are $\frac{N}{2}$ particles with $\sigma_j = \uparrow$ and $\frac{N}{2}$ particles with $\sigma_j = \downarrow$. On traveling once around the ring a particle passes $(N - 1)$ other particles, giving a factor -1 for even N, but only $\frac{N}{2}$ of opposite spin, giving a total phase slip $N\theta/2$. The net phase change is then a multiple of 2π, ensuring periodicity, if θ is an odd multiple of $2\pi/N$.