

System identification based on Step and Impulse response considering first and second order transfer function models

These notes discuss system identification based on the step and impulse response. In the following we consider linear, time-invariant systems of first and second order, as they provide reasonable approximation for the description of the dominant behavior of most linear time-invariant systems.

I. First order system

A first order system is described in frequency domain by the transfer function

$$G(s) = \frac{k}{\tau s + 1}$$

where the parameters k and τ are the system gain (k) and the time constant of the system (τ).

The **step response** of this system, considering initial conditions equal with 0, is obtained as follows:

a. the Laplace transform of the system input is

$$U(s) = \frac{1}{s}$$

b. using the relation $Y(s) = H(s)U(s)$ one obtains the Laplace transform of the output signal

$$Y(s) = \frac{k}{\tau s + 1} \frac{1}{s}$$

c. we use partial fractions expansion and then inverse Laplace transform to obtain the system response in time domain

$$Y(s) = \frac{A}{\tau s + 1} + \frac{B}{s}$$

$$A = Y(s)(\tau s + 1) \Big|_{s=-\frac{1}{\tau}} = \frac{k}{s} \Big|_{s=-\frac{1}{\tau}} = -\tau k \quad \text{and} \quad B = Y(s)s \Big|_{s=0} = \frac{k}{\tau s + 1} \Big|_{s=0} = k$$

$$Y(s) = -\frac{\tau k}{\tau s + 1} + \frac{k}{s} = -\frac{k}{s + 1/\tau} + \frac{k}{s}$$

Inverse Laplace transform results in

$$y(t) = -ke^{-t/\tau} + ku_{-1}(t)$$

where $u_{-1}(t)$ denotes the step function $u_{-1}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$.

Figure 1 shows the step response of a system with the transfer function $G(s) = \frac{2}{3s + 1}$.

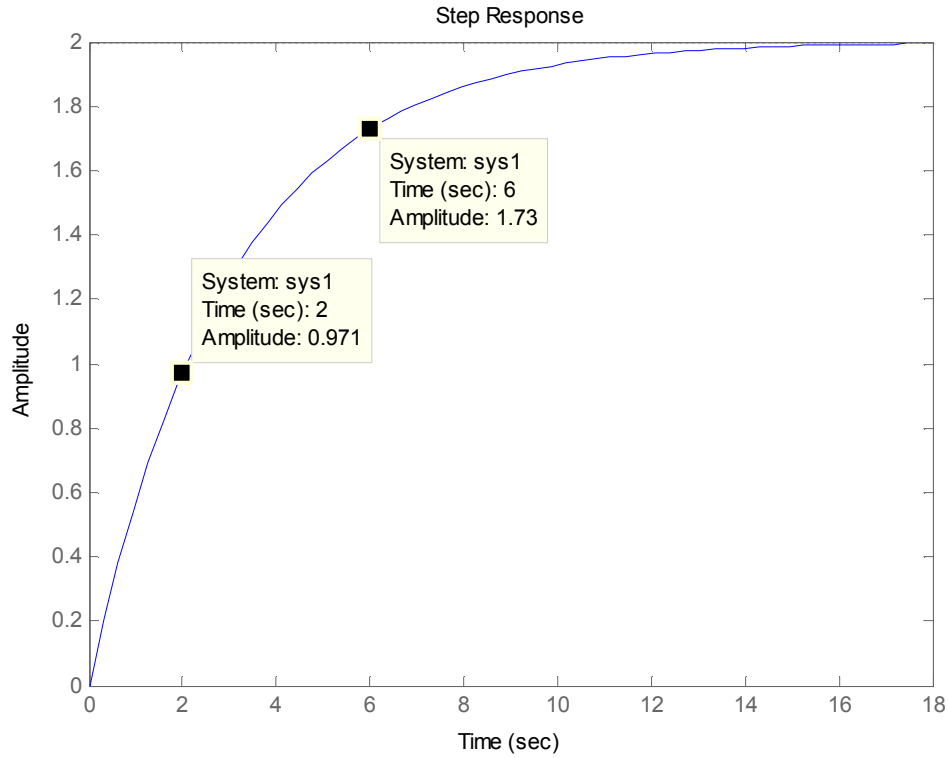


Figure 1. Step response of the system $G(s) = \frac{2}{3s+1}$

The identification procedure, i.e. finding the values of the two parameters k, τ describing the system dynamics, amounts to solving a system of two equations defined by two points on the system's step response $(t_1, y(t_1)), (t_2, y(t_2))$.

$$\begin{cases} y(t_1) = -ke^{-t_1 \frac{1}{\tau}} + ku_{-1}(t_1) \\ y(t_2) = -ke^{-t_2 \frac{1}{\tau}} + ku_{-1}(t_2) \end{cases}$$

Dividing the two equations, and considering that $u_{-1}(t) = 1, t > 0$, one gets

$$\frac{y(t_1) - k}{y(t_2) - k} = e^{-(t_1 - t_2) \frac{1}{\tau}}$$

which means

$$\frac{1}{t_2 - t_1} \ln \left(\frac{y(t_1) - k}{y(t_2) - k} \right) = \frac{1}{\tau}$$

or

$$\tau = (t_2 - t_1) \left(1 / \ln \left(\frac{y(t_1) - k}{y(t_2) - k} \right) \right).$$

This equation gives the value of the time constant of the system if $y(t_2), y(t_1), t_1, t_2, k$ are known.

In order to determine the value of k one notes that the steady state value of the step response is $\lim_{t \rightarrow \infty} y(t) = k$ which can be easily measured from the step response graph.

Question 1:

Use the graph in Fig. 1 and the relations given above to verify that the step response presented in the figure is indeed the step response of a system described by $G(s) = \frac{2}{3s+1}$.

Notes:

- Remember also the final value theorem $\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} Y(s)$, which means that the gain of the system at zero frequency, i.e. the DC gain, is k .
- If the step input is not of amplitude 1, say $U(s) = \frac{a}{s}$ where a denotes the amplitude of the step input, then the step response of the system is $y(t) = a(-ke^{-\frac{t}{\tau}} + ku_{-1}(t))$. Then the steady state value of the system's step response is $\lim_{t \rightarrow \infty} y(t) = ak$. In this case, in order to determine the value of the system's DC gain one needs to divide the measured steady state value by the amplitude of the step input.
- Let $t = \tau$ then $y(\tau) = k(1 - e^{-1}) = k * 0.6321$ thus the system's time constant is the time moment when the system step response reaches approximately 63% of the steady state value.
- Let $t = 5\tau$ then $y(5\tau) = k(1 - e^{-5}) = k * 0.993$ thus at $t = 5\tau$ the system step response reaches more than 99% of the steady state value.
- $G(s) = \frac{k}{\tau s + 1}$ can also be written as $G(s) = \frac{k/\tau}{s + 1/\tau} = \frac{q}{s + p}$ where $-p$ is the pole of the system.

The **impulse response** of this system, considering initial conditions equal with 0, is

$$y(t) = \frac{k}{\tau} e^{-\frac{t}{\tau}} u_{-1}(t)$$

Measuring the value of the impulse response at positive time moments, say t_1 and t_2 , one can determine the time constant of the system using

$$\frac{y(t_1)}{y(t_2)} = e^{(t_2 - t_1) \frac{1}{\tau}},$$

thus $\tau = (t_2 - t_1) / \ln \left(\frac{y(t_1)}{y(t_2)} \right)$.

The time constant being known the system gain can be determined from

$$k = \tau y(t_3) / e^{-t_3 \frac{1}{\tau}} \text{ for any } t_3 \geq 0.$$

Looking at the system's impulse response one sees that at $t = 0$ $y(0) = \frac{k}{\tau}$.

Figure 2 shows the impulse response of the system described by the transfer function

$$G(s) = \frac{2}{3s+1}.$$

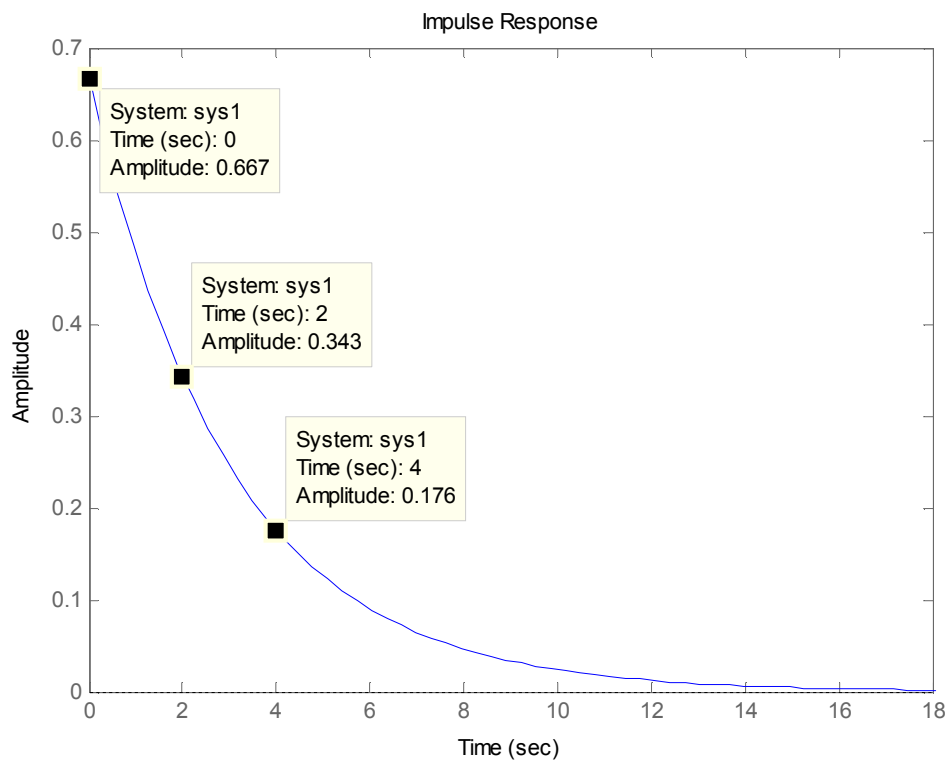


Figure 2. Impulse response of the system $G(s) = \frac{2}{3s+1}$.

Question 2:

Use the graph in Fig. 2 and the relations given above to verify that the impulse response presented in the figure is indeed the impulse response of the system $G(s) = \frac{2}{3s+1}$.

II. Second order system (complex pole pair)

A second order system with no finite zeros is described in frequency domain by the transfer function

$$H(s) = \frac{k_0 \omega_n^2}{s^2 + 2\alpha s + \omega_n^2} = \frac{k_0 \omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \equiv \frac{k_0 \omega_n^2}{\Delta(s)}$$

The numerator is chosen to scale the transfer function so that the DC gain (which can be calculated by $\lim_{s \rightarrow 0} H(s)$) is equal to k_0 .

The denominator is the *Characteristic polynomial* which can be written in several *canonical forms*, including

$$\Delta(s) = s^2 + 2\alpha s + \omega_n^2 = s^2 + 2\zeta \omega_n s + \omega_n^2.$$

One may write

$$\Delta(s) = s^2 + 2\alpha s + \omega_n^2 = (s + \alpha)^2 + \beta^2$$

where

$$\beta^2 + \alpha^2 = \omega_n^2.$$

Thus if $\alpha^2 < \omega_n^2$, the polynomial $\Delta(s)$ describes a complex pair of poles at $s = -\alpha \pm j\beta$.

Figure 3 shows the location of the two poles in the s-plane. The real part of the poles is $-\alpha$ and the imaginary part is $j\beta$. The norm of the vector from the origin to the pole is ω_n , which is known as the *natural frequency*.

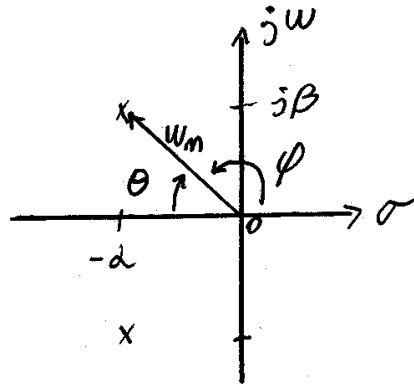


Figure 3. Location of the complex poles of a second order system in the s-plane

Notice that the system dynamics is completely described by the triplets (k_0, α, β) or (k_0, ζ, ω_n) .

To link the two forms of the characteristic polynomial we define the *damping ratio* as

$$\zeta = \frac{\alpha}{\omega_n},$$

which defines it as

$$\zeta = \cos \theta = -\cos \varphi = \frac{\alpha}{\omega_n}.$$

For complex poles in the left-half plane one has $0 < \zeta < 1$. If $0 > \zeta > -1$ then one has a complex pair in the right-half plane (e.g. unstable complex pair). Note that one may write

$$\beta = \sqrt{\omega_n^2 - \alpha^2} = \omega_n \sqrt{1 - \zeta^2}.$$

The **impulse response** of the system with transfer function

$$H(s) = \frac{k_0 \omega_n^2}{(s^2 + \alpha)^2 + \beta^2}$$

is given by

$$y(t) = k_0 \frac{\omega_n^2}{\beta} e^{-\alpha t} \sin \beta t u_{-1}(t),$$

which is plotted for $0 < \zeta < 1$ in the figure. This is known as the *underdamped case*. The figure clearly shows the meaning in the time domain of the real part $-\alpha$ of the poles, which provides the exponential decay term. Having in mind the standard form for sinusoids $\sin 2\pi t / T$, the period of the oscillation is given by

$$T = \frac{2\pi}{\beta}.$$

The variable β is known as the *oscillation frequency*.

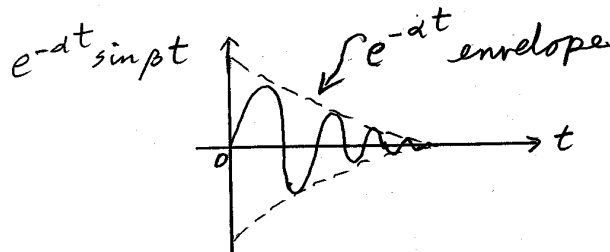


Figure 4. Sketch of the impulse response of a second order system with complex poles

If $\zeta = 0$ then $\alpha = 0, \beta = \omega_n$ and the poles are at $s = \pm j\beta$ on the imaginary axis. This is known as the *undamped case*. If $\zeta = 1$ then $\beta = 0, \alpha = \omega_n$ and the poles are on the real axis, both at $s = -\alpha$. In this *overdamped case*, the impulse response has the form $te^{-\alpha t}$. If $\zeta > 1$ then there are two real poles and we can split the quadratic factor $\Delta(s) = s^2 + 2\alpha s + \omega_n^2$ into two real linear factors.

In order to determine the parameters of the system, one can simply determine the period of oscillation T and subsequently use $T = \frac{2\pi}{\beta}$ to determine β .

Then by measuring the values of the impulse response at two moments in time when $\sin \beta t = 1$ one has

$$y(t_1) = k_0 \frac{\omega_n^2}{\beta} e^{-\alpha t_1}, \quad y(t_2) = k_0 \frac{\omega_n^2}{\beta} e^{-\alpha t_2}.$$

Dividing the two one gets $\frac{y(t_1)}{y(t_2)} = e^{\alpha(t_2 - t_1)}$ and from here

$$\alpha = \ln \left(\frac{y(t_1)}{y(t_2)} \right) / (t_2 - t_1).$$

Knowing the values α, β and $y(t_3) = k_0 \frac{\omega_n^2}{\beta} e^{-\alpha t_3} \sin \beta t_3$ one can also determine the system DC gain k_0 .

The **step response** of this second order system is given by inverse transforming

$$Y(s) = H(s)U(s) = H(s) \frac{1}{s} = \frac{k_0 \omega_n^2}{s [(s^2 + \alpha)^2 + \beta^2]}$$

to obtain

$$y(t) = k_0 (1 - e^{-\alpha t} [\cos \beta t + \frac{\alpha}{\beta} \sin \beta t]) u_{-1}(t) = k_0 (1 - \frac{\omega_n}{\beta} e^{-\alpha t} \sin(\beta t + \theta)) u_{-1}(t)$$

where the angle $\theta = \arctan \frac{\beta}{\alpha}$ is shown in Figure 3.

The step response of a underdamped second order system is presented in Figure 5.

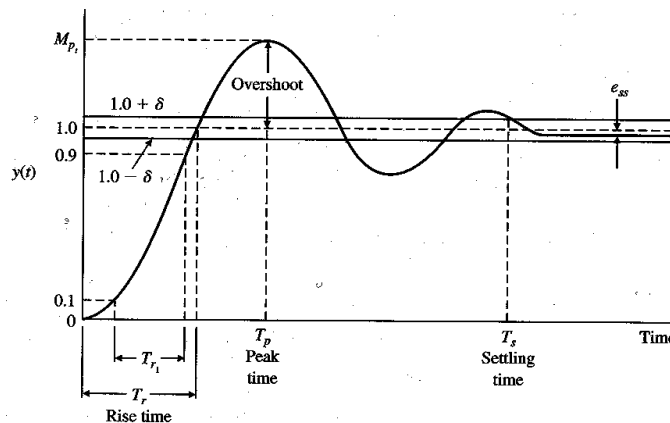


Figure 5. Sketch of the step response of a second order system with complex poles

The steady state value of the system's step response is $y_{ss} = \lim_{t \rightarrow \infty} y(t) = k_0$.

An important quantity for characterizing the performance of systems is the *percent overshoot (POV)* in the step response. This is defined as

$$POV = \frac{y_{\max} - y_{ss}}{y_{ss}} \times 100\%$$

where y_{\max} is the maximum value of the step response and y_{ss} is its steady-state value.

The *rise time*, t_r , is the time required for the step response to rise from 0.1 to 0.9 of its steady-state value.

The *settling time* t_s is the time required for the signal to effectively reach its steady-state value.

Note that for a first order system one has $t_r = 2.2\tau$ and $t_s = 5\tau$ (some take $t_s = 4\tau$).

For the underdamped pole pair, the time constant is $\tau = 1/\alpha$ and one may use $t_s = 5\tau$ to calculate the settling time. In this case however the signal rises faster and one may

approximate, for $0.3 \leq \zeta \leq 0.8$, using $t_r = \frac{2.16\zeta + 0.6}{\omega_n}$.

The percent overshoot is a function of damping ratio

$$POV = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

and conversely

$$\zeta = \left[\frac{\ln^2(POV/100)}{\ln^2(POV/100) + \pi^2} \right]^{1/2}.$$

To determine the parameters of the system, from the system's step response, one can simply calculate the POV and then determine the damping factor ζ . Then by measuring the settling time one can use $t_s = 5\tau$ to determine the time constant of the system and then α . Measuring the steady state value of the step response one gets the system DC gain $k_0 = y_{ss}$.

Note that in the case in which the system input is not a unit step, e.g. $U(s) = \frac{a}{s}$, then

$$y_{ss} = ak_0 \text{ and thus } k_0 = \frac{y_{ss}}{a}.$$

In Figure 6 are compared the step responses of the two systems $H_1(s) = \frac{1}{0.4s+1}$ and

$$H_2(s) = \frac{6}{s^2 + 5s + 6} \text{ which have the same time constant } \tau = 0.4. \text{ (Notice the difference.)}$$

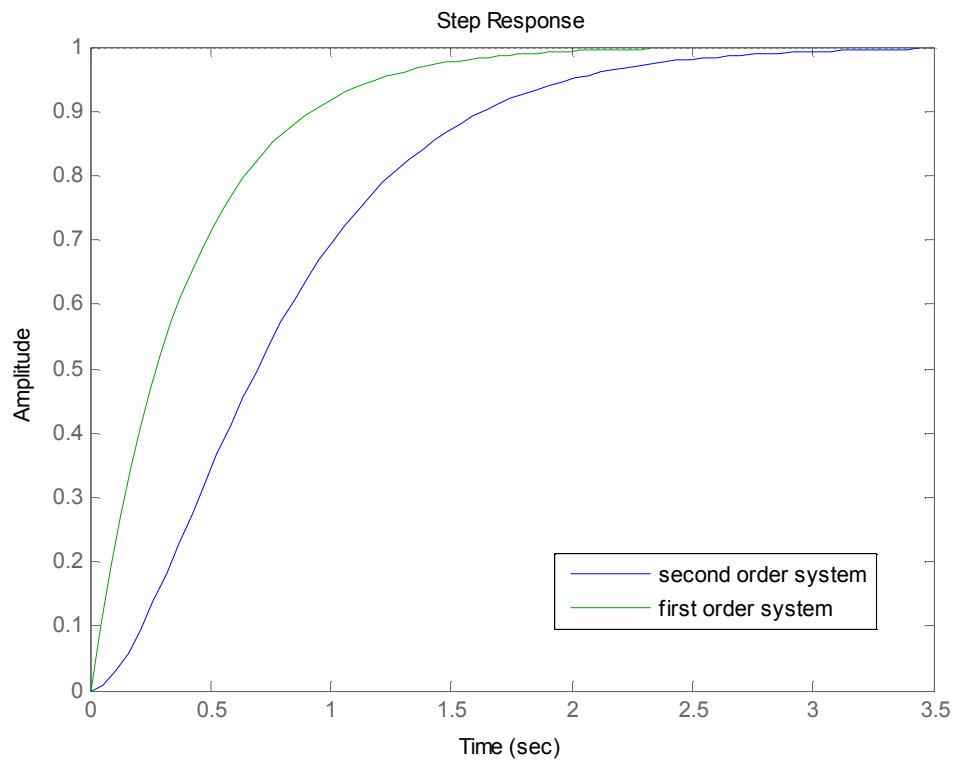


Figure 6. Step response comparison of a first order system and a second order, overdamped, system. Both systems have the same time constant thus the same settling time.

You can consider solving some of the following questions.

1. The following figures present the step responses (to a unit step) of two first order systems. Determine the transfer functions of the two systems. To help with the calculations, some points were given explicitly on the graph.

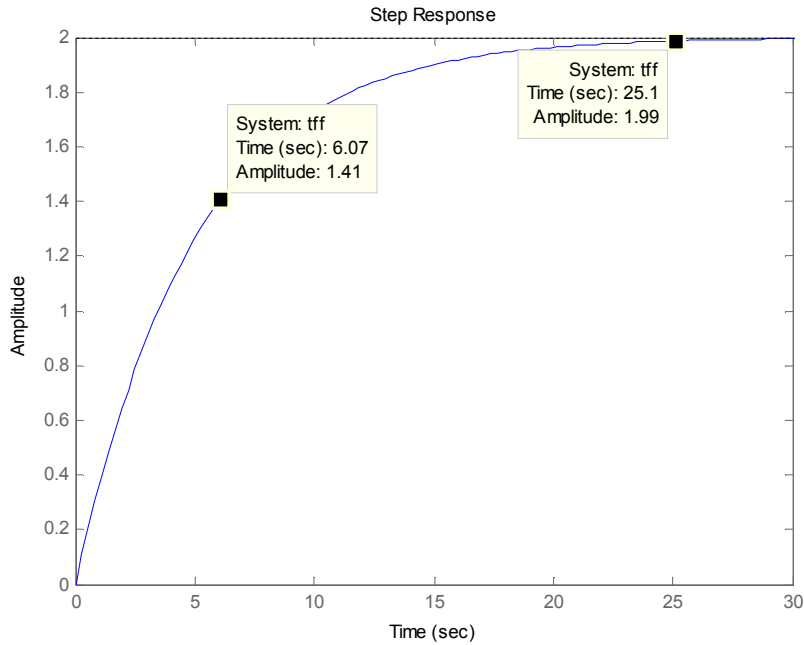


Figure 1. Step response of a system

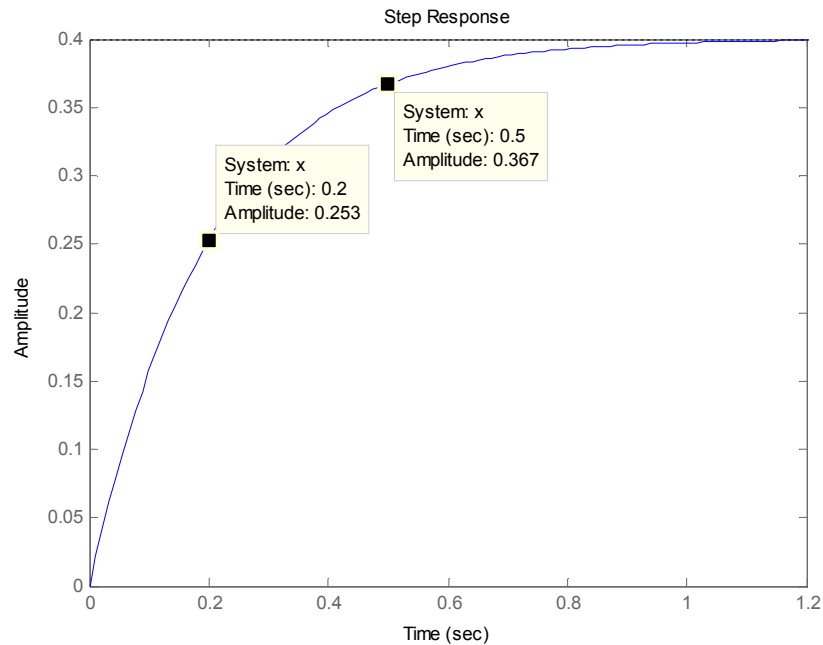


Figure 2. Step response of a system

2. The following figure presents the step responses (to a unit step) of a system. Determine the model transfer function of the system.
To help with the calculations, some points were given explicitly on the graph.

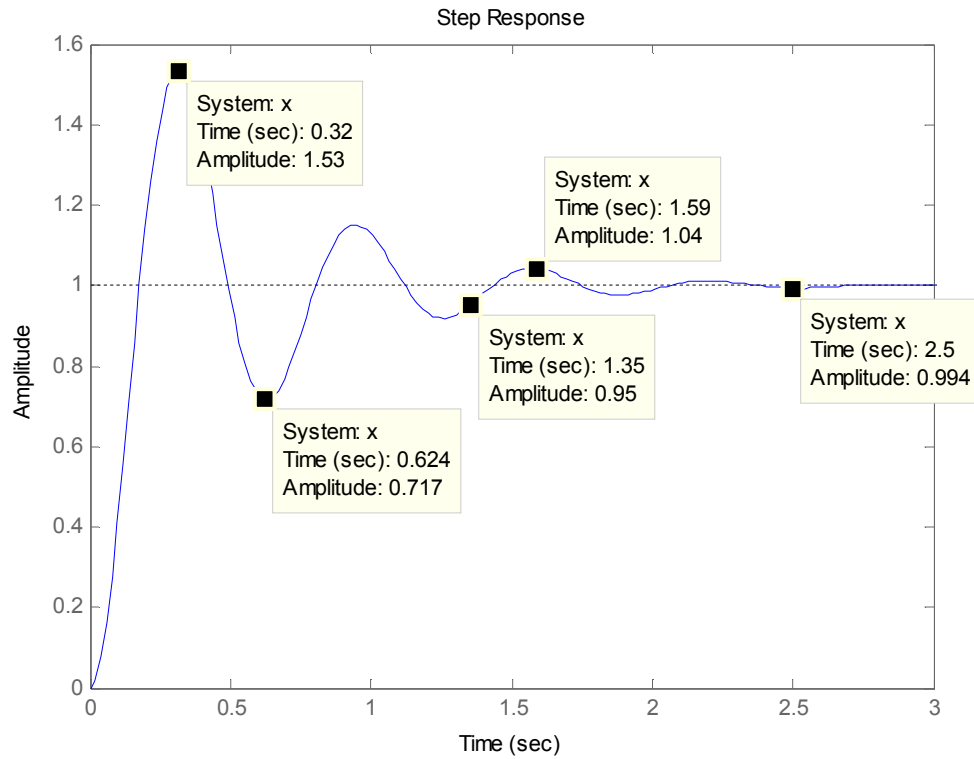


Figure 3. Step response of a system

Consider that the steady state is obtained when the signal enters in the $\pm 1\%$ of the steady state value (this is related with the value of the settling time).

3. The following figure presents the impulse response of a second order system. Determine the model transfer function of the system. To help with the calculations, some points were given explicitly on the graph.

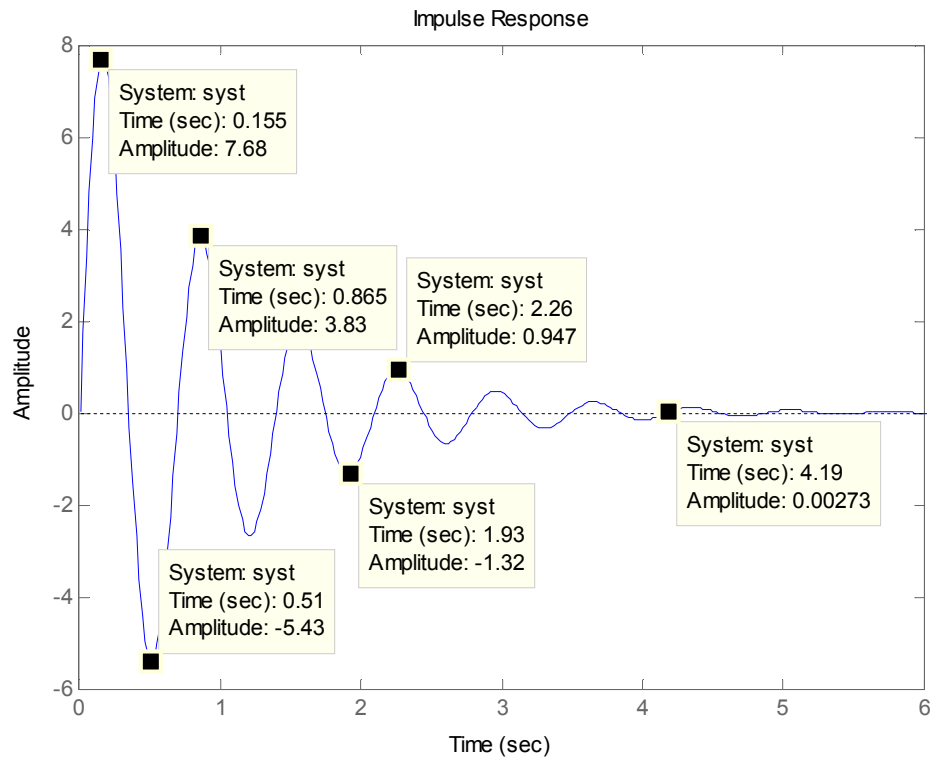


Figure 4. Impulse response of a system