Nonlinear Sliding Mode Control of The Furuta Pendulum

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***Abstract*— This paper deals with the control design of a nonlinear Sliding Mode Controller (SMC) for an underactuated mechanical system under external disturbance. The Furuta Pendulum represents a testbed on which control methodology is tested because it exhibits chaotic behavior, it involves highly nonlinear dynamics, and it is underactuated. The aforementioned remarks make the control of the Furuta pendulum a nontrivial task. The SMC methodology which uses the full model of the system, applied here, is pegged against a linear state feedback control law which is based on the linearized model around the control objective. The control laws are tested in a simulated environment in which external disturbances are injected, and the performance of the approaches is then compared.**

Keywords—Furuta pendulum, Rotary inverted pendulum, Sliding Mode Control, State Feedback

I. Introduction

The Furuta pendulum (also known as rotary inverted pendulum) is an under-actuated system, which is formed by two bodies in series. The first one is rotating around a vertical axis followed by a pendulum rotating around a horizontal axis. It was first proposed by Furuta [1] mainly to test different control laws [2]. This mechanism is a popular one that can be found in most control laboratories. The Furuta pendulum is mainly used as a testbed of nonlinear control strategies or as an educational apparatus. Most of the works in the literature use the Furuta pendulum to illustrate their proposed control law [1] − [4]. However, these works did not elaborate on the dynamic model of the Furuta pendulum and they limited their work to a simplified model [5], [6].

Few papers dealt with the modeling of the Furuta pendulum. They highlighted the complexity of the dynamic model and the necessity to simplify it based on several assumptions. The authors of [1] and [7] proposed a linearized model based on a small angle assumption. Others neglected one or more of the cross-coupling terms relating the two rotations [1], [4] −[6], [8], [9]. The authors of [3] and [7] showed, through simulations, that this assumption is not acceptable and it could have an important effect on the dynamic response. The authors of [8] derived a significantly simplified system and carried a brief 3-second test. All the experimental works on the Furuta pendulum are directed at testing various control strategies [10], [11] − [12]. The nonlinear dynamic behavior of the system, which includes Coriolis and centrifugal forces in the two rotating frames of motion, is often overlooked or linearized. However, the knowledge of the dynamic behavior of a system is highly desirable to design advanced nonlinear controllers for the given nonlinear system.

Literature is rich with works dealing with the control of the Furuta Pendulum. Many approaches are linear control methodologies that utilize the linearized equations of motion in the control design of Proportional Derivative Integral (PID), State Feedback, Linear Quadratic Regulator (LQR), and Linear Quadratic Gaussian controllers [12]−[14]. Other approaches include nonlinear control laws based on exploiting the relationship between the coupled degrees of freedom of the underactuated pendulum setup; however, most works here do not use the correct model for the Furuta Pendulum [15]−[18]. Some works even introduce adaptive control laws that give some robustness against parameter uncertainties and external disturbances [13].

In this paper, the dynamics of the Furuta Pendulum are derived through a Euler-Lagrange Formulation. The coupling between the two serial linkages of the pendulum is also exploited in the control design of a sliding mode controller to stabilize the underactuated system around its unstable equilibrium position represented by the second link in its upward position. Sliding mode control was chosen for its robustness against uncertainties and disturbances that can affect the system.

The rest of the paper is organized as follows: Section II introduces the developed model. The control approaches are presented in Section III. The comparison of the results, along with a discussion, is presented in Section IV. Some concluding remarks and future work are presented in Section V.

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TABLE I. Parameters of the two Links

Parameter	Arm (Link 1)	Pendulum (Link 2)
Mass (Kg)	$m_1 = 0.370$	$\mathrm{m}_2 = 0.128$
Moments of Inertia $(Kg.m^2)$	$j_{1z} = 3.09 \times 10^{-3}$	$j_{2xx} = j_{2yy} = 5.25 \times 10^{-3}$ $j_{2zz} = 2.91 \times 10^{-6}$
Distance from pivot point to center of gravity (m)	$l_1 = \mathrm{OG}_1 = 0.0620$	$l_2 = \mathrm{AG}_2 = 0.185$
Total length (m)	$\mathrm{L}_1 = \mathrm{OA} = 0.216$	$\mathrm{L}_2 = \mathrm{AB} = 0.316$

TABLE II. Parameters of the two Joints

Parameter	Joint 1	Joint 2
Radius of the shaft (mm)	$r_1 = 4.00$	$r_2 = 4.00$
Viscous coefficient of friction $\left(\frac{N.s}{m}\right)$	$\eta_1 = 3.09 \times 10^{-3}$	$\eta_2 = 1.1 \times 10^{-3}$
Dry coefficient of friction	$\mu_1 = 17.0 \times 10^{-3}$	$\mu_2 = 2.0 \times 10^{-3}$

II. DYNAMIC MODEL OF THE SYSTEM

A. *The Furuta Pendulum*

A schematic of the Furuta pendulum is shown in Fig. 1 shows its tilted position. The base reference frame is $\mathrm{x}_0 - \mathrm{y}_0 - \mathrm{z}_0$, where z_0 is now the axis of the first joint between the ground and the arm (1). The joint is disconnected from the existing motor and the pendulum is subjected only to its weight. Fig. 1 shows the schematic of the Furuta pendulum in its tilted position. The arm is rotating around the fixed axis z_0 by an angle θ_0 and the pendulum is rotating with respect to the moving axis x_1 by an angle θ_1. The different parameters of the two links are defined in TABLE I., whereas those defining the joints are listed in TABLE II. All these values are provided by the manufacturer of the system.

B. *Dynamic Model of the Furuta Pendulum*

Based on the schematic of the Furuta pendulum, shown in Fig. 1, the dynamic model is elaborated. The Euler-Lagrange Formulation is used to derive the equations of motion of the system. Two generalized coordinates are required to fully describe the dynamics of the two-degrees-of-freedom system, which are the angular displacements of the Arm and the Pendulum links:

$$\mathbf{q} = [\theta_0 \quad \theta_1]^T$$

The Euler-Lagrange equation, for the i^{th} generalized coordinate of the system can be written as follows:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \quad (i = 0,1) \tag{1}$$

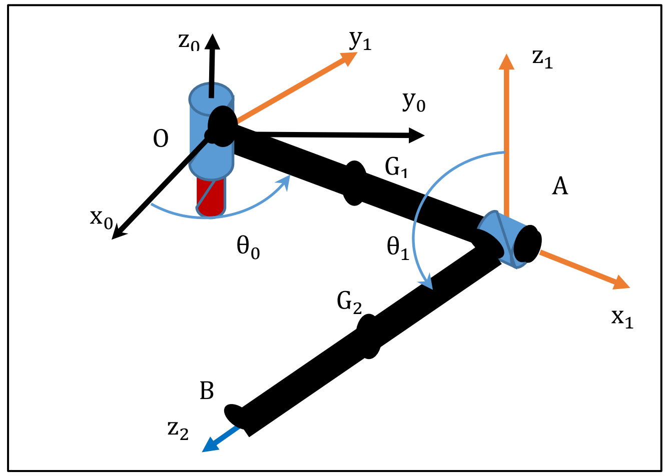

Fig. 1. Schematic of the tilted Furuta pendulum.

The kinetic energy, T, is written as the sum of the linear and angular kinetic energies of the two links, which is given by

$$T = \sum_{i=1}^{2} \frac{1}{2} m_i \mathbf{v}_{Gi}^2 + \sum_{i=1}^{2} \frac{1}{2} \boldsymbol{\omega}_{i-1}^{\mathrm{T}} (\mathbf{J}_i \boldsymbol{\omega}_{i-1}) \tag{2}$$

where $\boldsymbol{\omega}_0 = \dot{\theta}_0 \mathbf{z}_0$, $\boldsymbol{\omega}_1 = \dot{\theta}_0 \mathbf{z}_0 + \dot{\theta}_1 \mathbf{x}_1$ and $\mathbf{v}_{Gi}$ is the velocity of G_i ($i = 1$ for the Arm and $i = 2$ for the Pendulum).

The potential energy of the gravity is obtained following the scalar product between the global vertical axis and the center of gravity position vector in the local tilted frame of reference.

$$V = -m_1\, g\, (\mathbf{z}\,.\,\mathbf{OG_1}) - m_2\, g\, (\mathbf{z}\,.\,\mathbf{OG_2}) \tag{3}$$

The generalized forces of the viscous and dry friction in the joints are written in Eq. (4) as the sum of frictional torques acting on the links. This simple model is chosen because it only has two parameters to be identified per joint.

$$Q_j = -\eta_j\, \dot{\theta}_j - \mu_j\, r_j\, F_{nj}\, \mathrm{sign}(\theta_j) \tag{4}$$

where η_j and μ_j are, respectively, the viscous and dry friction coefficients of joint j. F_{nj} is the normal force in the joint j. A dynamic force analysis yielded the reaction forces at the joints of interest and, consequently, the normal forces, F_{nj}.

The final model is given in the following form:

$$\mathbf{H}\,\ddot{\mathbf{q}} + \mathbf{B}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}) = \mathbf{Q} \tag{5}$$

Where the generalized inertia 2×2 matrix is given by:

$$\mathbf{H} = \begin{bmatrix} \begin{pmatrix} \mathrm{j}_{1z} + \mathrm{m}_1 l_1^2 + \mathrm{m}_2 \mathrm{L}_1^2 + \\ (\mathrm{m}_2 l_2^2 + \mathrm{j}_{2y}) \sin(\theta_1)^2 + \\ \mathrm{j}_{2z} \cos(\theta_2)^2 \end{pmatrix} & -m_2 l_2 L_1 \cos(\theta_1) \\ -m_2 l_2 L_1 \cos(\theta_1) & j_{2x} + m_2 l_2^2 \end{bmatrix}$$

the centrifugal and Coriolis forces are given by the following vector:

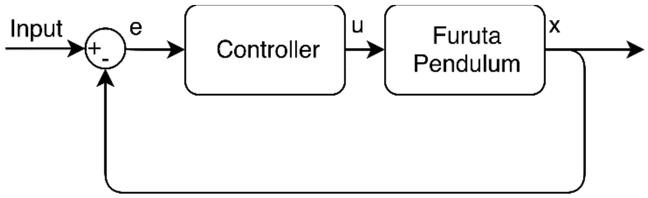

Fig. 2. Control Block Diagram

$$\mathbf{B}(\mathbf{q},\dot{\mathbf{q}}) = \begin{bmatrix} -\left(\begin{gathered}\left(\left(\begin{matrix} m_2 l_2^2 \\ -j_{2y}+j_{2x}\end{matrix}\right)\sin(2\theta_1)\right)\dot{\theta}_0\dot{\theta}_1 \\ +(m_2 l_2 L_1\sin(\theta_1))\,\dot{\theta}_1^2\end{gathered}\right) \\ \left(\left(\begin{matrix}-m_2 l_2^2 \\ -j_{2y}+j_{2z}\end{matrix}\right)\sin(\theta_1)\cos(\theta_1)\right)\dot{\theta}_0^2 \end{bmatrix}$$

the effect of gravity is given by the following vector

$$\mathbf{G}(\mathbf{q}) = \begin{bmatrix} 0 \\ m_2\, g\, l_2 \sin(\theta_1)\end{bmatrix}$$

the generalized forces due to friction are given by the following vector:

$$\mathbf{Q} = \begin{bmatrix} -\eta_0\,\dot{\theta}_0 - \mu_0\, r_0\, F_{n0}\,\mathrm{sign}(\theta_0) \\ -\eta_1\,\dot{\theta}_1 - \mu_1\, r_1\, F_{n1}\,\mathrm{sign}(\theta_1)\end{bmatrix}$$

III. Control Design

The block diagram in Fig. 2 serves to explain the control architecture behind the two control approaches. The Furuta Pendulum states of angular positions and velocities are assumed to be observable, the negative feedback serves as the input to both controllers, and the controller-generate control signal, u, represents the torque applied on the Arm link.

A. State Feedback Control Design

To implement a state feedback control on the Furuta pendulum, a linearized model is required. The derived model in equation (5) is linearized around the target positions and angular velocities for the pendulum links of $\dot{\theta}_0 = 0$, $\dot{\theta}_1 = 0, \theta_0 = 0, \theta_1 = 0$, which correspond to the upwards unstable equilibrium position. Starting with the state vector

$$\mathbf{X} = [\dot{\theta}_0 \quad \dot{\theta}_1 \quad \theta_0 \quad \theta_1]^T \tag{6}$$

the linearized model can be shown to take the form:

$$\dot{\boldsymbol{X}} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}\boldsymbol{u} \tag{7}$$

where the matrices of the model are given as

$$\boldsymbol{A} = \frac{1}{\zeta}\begin{bmatrix} \begin{pmatrix} j_{2x} + \\ m_2 l_2^2 \end{pmatrix}\eta_0 & m_2 l_2 L_1 \eta_1 & 0 & -(m_2 l_2)^2 L_1\, g \\ (m_2 l_2 L_1)\eta_0 & \begin{pmatrix} j_{1z} + m_1 l_1^2 \\ + m_2 L_1^2 \end{pmatrix}\eta_1 & 0 & -\begin{pmatrix} j_{1z} + m_1 l_1^2 \\ + m_2 L_1^2 \end{pmatrix} m_2\, g\, l_2 \\ \zeta & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{B} = \frac{1}{\zeta}\begin{bmatrix} j_{2x} + m_2 l_2^2 \\ m_2 l_2 L_1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\zeta = (j_{2x} + m_2 l_2^2)(j_{1z} + m_1 l_1^2 + m_2 L_1^2) - (m_2 l_2 L_1)^2$$

The goal of state feedback is to design a state feedback gain, $\boldsymbol{K}$, such that the poles of the closed loop dynamics are located at desired locations. Let u be defined as:

$$u = -\boldsymbol{K}\boldsymbol{X} \tag{8}$$

Designing the state feedback control law to have the two dominant poles of $-2.8 \pm 3j$ yields the following gain matrix:

$$\boldsymbol{K} = [-1 \quad 1.2 \quad -0.5 \quad 5] \tag{9}$$

B. Sliding Mode Control Design

Starting with the derived model in equation (5), the following state vector is defined:

$$\boldsymbol{X} = [\dot{\theta}_0 \quad \dot{\theta}_1 \quad \theta_0 \quad \theta_1]^T = [\boldsymbol{X}_1^T \quad \boldsymbol{X}_2^T]^T \tag{10}$$

The model can then be rewritten as:

$$\dot{\boldsymbol{X}} = \begin{bmatrix} \ddot{\theta}_0 \\ \ddot{\theta}_1 \\ \dot{\theta}_0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} f_1(X_2) \\ f_2(X_1, X_2) \end{bmatrix} + \begin{bmatrix} B \\ 0_{2x1} \end{bmatrix} u \tag{11}$$

In light of the objective of stabilizing the pendulum link in the unstable upward equilibrium, the sliding mode variable s is chosen such that

$$s = C^T X = [C_1 \quad C_2]\begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix} = C_1\boldsymbol{X}_1 + C_2\boldsymbol{X}_2 \tag{12}$$

where $C^T = [\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4]$ with the design constants, λ_i, chosen to be:

$$C^T = [-0.03 \quad 0.06 \quad -0.06 \quad 0.6] \tag{13}$$

The derivative of the sliding variable is then

$$\dot{s} = C_1\dot{\boldsymbol{X}}_1 + C_2\dot{\boldsymbol{X}}_2 = C_1 f_1 + C_2 f_2 + C_1 B u \tag{14}$$

With the Lyapunov function candidate given to be

$$V = \frac{1}{2}s^2 \tag{15}$$

which once differentiated with respect to time, yields

$$\begin{aligned}\dot{V} &= s\dot{s} \\ &= s(C_1 f_1 + C_2 f_2 + C_1 B u)\end{aligned} \tag{16}$$

From equation (16), the following control law is devised:

$$u = -\frac{1}{C_1 B}\left(C_1 f_1 + C_2 f_2 + \tau \operatorname{sign}(s)\right) \quad (17)$$

where the sign function serves to provide robustness against disturbances and the rest of the control law terms serve to compensate for the plant dynamics.

When the control law is substituted in equation (16), the resulting dynamics are as follows:

$$\dot{V} = -\tau\ |s| < 0\ for\ s \neq 0, \tau > 0 \quad (18)$$

where τ is a positive design parameter.

According to Lyapunov stability theory, the sliding variable should reach the sliding mode in a finite time. To eliminate chattering, replacing the signum function with a sigmoid function is possible. The hyperbolic tangent and the saturation function are possible choices, of which we apply the first.

IV. Results and Discussion

The proposed controllers were tested in a simulated environment under Simulink. Zero mean white Gaussian noise was injected into all the states to stress test the controllers against disturbances and uncertainties present in the mathematical model and measurements apparatus. The results are obtained for the initial positions of $\theta_0 = 0.1\, rad\ (5.7^o)$ and $\theta_1 = 0.4\, rad\ (22.9^o)$. The equilibrium positions are the zero angular displacement position for the arm link and the upward position for the pendulum link. Both links start with a zero initial angular velocity. The response of the links is simulated as well as the controller effort.

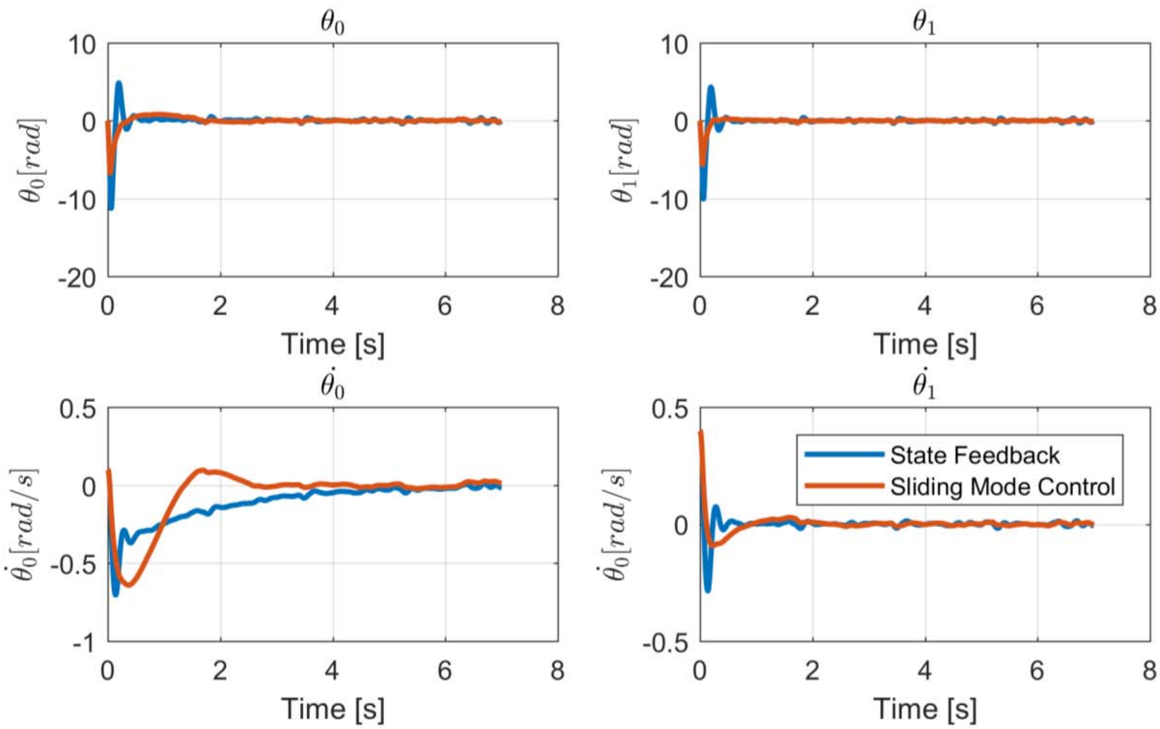

Fig. 3. Stabilization performance of the two controllers

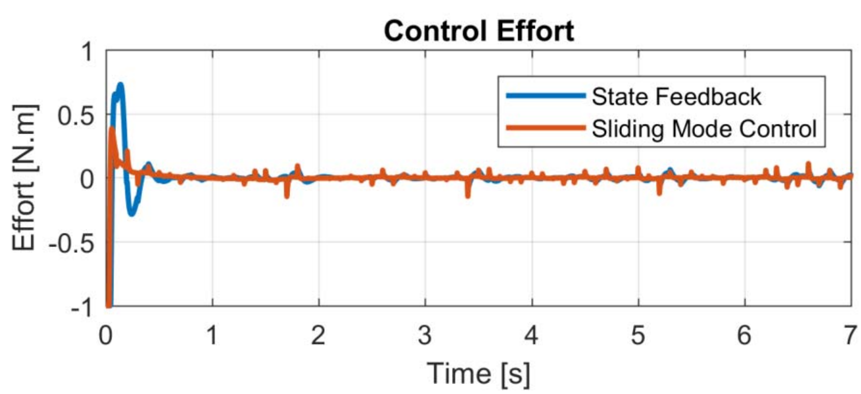

Fig. 4. Controller effort performance of the two controllers

Fig. 3 and Fig. 4 showcase the controller performance when it comes to stabilizing the Furuta pendulum around its unstable equilibrium position. The time response of the four states is shown in Fig. 3. The two controllers successfully stabilize the pendulum links; however, the SMC shows a faster response while managing to achieve smaller overshoot in the angular position states. The settling time is shown to be almost identical with the two controllers. Fig. 4 shows the required controller effort in terms of the applied torque on the Arm link. The SMC requires less effort to achieve better performance results when compared to the state feedback approach. It is also evident that the SMC successfully mitigates the adverse effects of external disturbances.

V. Conclusion

This paper applies a nonlinear control methodology to a tilted Furuta pendulum and compares the stabilization performance with a linear state feedback control approach. The pendulum was tilted to ensure the existence of a stable equilibrium configuration with the weight of each link is the only external force applied to the system. A complete analytical model was derived where full inertia of the pendulum is taken into account. First, the linearized equations of motion were exploited in the design of a state feedback controller, where the linearization was done around the unstable equilibrium position. A nonlinear Sliding Mode control law was then devised based on the equations of motion of the underactuated mechanical system. The coupling between the two degrees of freedom allows one actuator to control both the Arm link and the Pendulum link. The controllers were tested in a simulated environment under Simulink, and white gaussian noise was injected in all states to test the controllers against disturbances. Compared to the linearization-based state feedback control law, the nonlinear SMC was able to realize the control with considerable decrease in overshoot, roughly the same settling time, and a considerable reduction in control effort. This shows the benefit of applying nonlinear control approaches to nonlinear plants.

As a future effort, the controllers are to be tested on the instrumented hardware, and experimental validation is to be carried. Also, it is of interest to devise an adaptive gain tuning law to update the SMC gains online and eliminate the need of extensive testing before applying the controllers on experimental hardware.

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