Internal Model Principle

If the reference signal, or disturbance $d(t)$ satisfy some differential equation: e.g.

$$\frac{d^{n_d}}{dt^{n_d}}d(t) + \gamma_{n_d-1}\frac{d^{n_d-1}}{dt^{n_d-1}}d(t) + \ldots \gamma_1\frac{d}{dt}d(t) + \gamma_0 d(t) = 0$$

then, taking Laplace transform,

$$\underbrace{\left[s^{n_d} + \gamma_{n_d-1}s^{n_d-1} + \ldots + \gamma_0\right]}_{\Gamma_d(s)} D(s) = f(0,s)$$

where $f(0,s)$ is a polynomial in s arises because of initial conditions, $d(0)$, $\dot{d}(0)$, $\ddot{d}(0)$ etc.

We call $\Gamma_d(s)$ the disturbance generating polynomial.

Example:

- $d(t) = sin(\omega t)$: $\Gamma_d(s) = (s^2 + \omega^2)$.

- $d(t) = d_0$ a constant: $\Gamma_d(s) = s$

- $d(t) = e^{at}$: $\Gamma_d(s) = (s - a)$.

- $d(t) = d_0 + d_1 e^{at}$: $\Gamma_d(s) = s(s - a)$.

From the last example, we see that we can form disturbance generating polynomials for $d(t) = \alpha(t) + \beta(t)$ by combining (multiplying) the disturbance generating polynomials for $\alpha(t)$ and $\beta(t)$.

The internal model principle says *that if the input disturbance, $d_i(t)$, the output disturbance, $d_o(t)$, or a reference $r(t)$ has $\Gamma_d(s)$ as the generating polynomial, then using a controller of the form:*

$$C(s) = \frac{P(s)}{\Gamma_d(s)\bar{L}(s)} \tag{1}$$

in the standard one degree-of-freedom control architecture can asymptotically reject the effect of the disturbance and cause the output to track the reference.

Note that only the generating polynomial is needed. The magnitude of the disturbances or of the reference is not needed.

To see why IMP works, let the plant model be

$$G_o(s) = \frac{B_o(s)}{A_o(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_0}.$$

Assume that $\Gamma_d(s)$ is not a factor of $B_o(s)$.

The sensitivity, input sensitivity, and complementary sensitivity functions of the closed loop system $S_o(s)$, $S_{io}(s)$ and $T_o(s)$ are given by:

$$S_o(s) = \frac{\Gamma_d(s)\bar{L}(s)A_o(s)}{\Gamma_d(s)\bar{L}(s)A_o(s) + P(s)B_o(s)}$$

$$S_{io}(s) = \frac{\Gamma_d(s)\bar{L}(s)B_o(s)}{\Gamma_d(s)\bar{L}(s)A_o(s) + P(s)B_o(s)}$$

$$T_o(s) = \frac{P(s)B_o(s)}{\Gamma_d(s)\bar{L}(s)A_o(s) + P(s)B_o(s)}$$

Suppose that $\bar{L}(s)$ and $P(s)$ have been chosen such that the closed loop characteristic equation

$$A_{cl}(s) = \Gamma_d(s)\bar{L}(s)A_o(s) + P(s)B_o(s)$$

has roots that have negative real parts.

The response of the system to the output disturbance $d_0(t)$ with generating polynomial $\Gamma_d(s)$ is:

$$Y(s) = S_o(s)D_o(s) = S_o(s)\frac{f(0,s)}{\Gamma_d(s)} = \frac{\bar{L}(s)A_o(s)}{A_{cl}(s)}f(0,s)$$

The last equality is because of the cancellation of $\Gamma_d(s)$ term. Since $A_{cl}(s)$ is assumed to have stable roots,

the inverse Laplace transform of $Y(s)$ converges to 0. Thus, $y(t \to \infty) = 0$.

Similarly with input disturbance, $d_i(t)$,

$$Y(s) = S_{io}(s)D_i(s) = \frac{\bar{L}(s)B_o(s)}{A_{cl}(s)}f(0,s)$$

so that $y(t \to \infty) = 0$.

Finally for a reference $r(t)$ with $\Gamma_d(s)$ as its generating polynomial. Let $e(t) = r(t) - y(t)$.

$$E(s) = (1 - T_o(s))\frac{f(0,s)}{\Gamma_d(s)} = S_o(s)\frac{f(0,s)}{\Gamma_d(s)}$$

so that $e(t \to \infty) = 0$ and $y(t) \to r(t)$.

Therefore, 1) in the presence of output or input disturbances ($d_o(t)$ or $d_i(t)$), $y(t) \to 0$; and 2) in the presence of reference signal, $y(t) \to r(t)$ whenever $d_o(t)$, $d_i(t)$ and $r(t)$ have generating polynomial $\Gamma_d(s)$.

The application of IMP is straightforward. However, we need to design $\bar{L}(s)$ and $P(s)$ in the controller (1) so that the closed loop characteristic equation $A_{cl}(s)$ has stable roots. One way is by assigning the poles of $A_{cl}(s)$.

Pole placement Problem

Plant: Strcitly proper and of order n

$$G_o(s) = \frac{B_o(s)}{A_o(s)}$$

$$B_o(s) = b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_1 s + b_0$$
$$A_o(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$$

Controller: Proper and of degree (i.e. order) n_l

$$C(s) = \frac{P(s)}{L(s)}$$

$$P(s) = p_{n_l}s^{n_l} + p_{n_l-1}s^{n-1} + \ldots + p_1 s + p_0$$
$$L(s) = s^{n_l} + l_{n_l-1}s^{n-1} + \ldots + l_1 s + l_0$$

We assume that $L(s)$ and $A_o(s)$ are monic (i.e. the leading coefficients are 1).

The closed loop characteristic polynomial is of degree $n_{cl} = n + n_l$

$$A_{cl}(s) = A_o(s)L(s) + B_o(s)P(s)$$
$$= s^{n+n_l} + \gamma_{n+n_l-1}s^{n+n_l-1} + \ldots + \gamma_0$$

Question: Under what circumstance can we use $C(s)$ to assign the roots of $A_{cl}(s)$ (i.e. the poles of the closed loop system) ?

Theorem: (Pole placement) *If $A_o(s)$ and $B_o(s)$ have no common factors (co-prime), then using a controller of order $n_l = n - 1$ can be used to assign the $n_c = n_l + n$-th order closed loop characteristic polynomial $A_{cl}(s)$ arbitrarily.*

Thus, in order for roots of $A_{cl}(s)$ to be arbitrarily assignable, we need at least $n_{cl} = 2n - 1$.

Idea of proof: Count the number of parameters

- The number of free coefficients in $L(s)$ and $P(s)$ available is: $n_l + n_l + 1 = 2n_l + 1$.

- The number of equations that need to be solved equals number of assignable coefficients in $A_{cl}(s)$.

Since leading coefficient in $A_{cl}(s)$ is always 1, there are $n_{cl} = n_l + n$ coefficients to assign.

- For problem to be solvable, number of free parameters $\geq$ number of equations

$$2n_l + 1 \geq n_l + n \quad \Rightarrow n_l \geq n - 1$$

or $n_{cl} = n + n_l \geq 2n - 1$.

- Notice that coefficients in $L(s)$ and $P(s)$ enter into coefficient is $A_{cl}(s)$ linearly. In the case $n_l = n - 1$, then matrix that relate the coefficients of $L(s)$ and $P(s)$ to the coefficients of $A_{cl}(s)$ is related to the Sylvester matrix (see Goodwin 7.2) which is non-singular when $A_o(s)$ and $B_o(s)$ are co-prime.

Typically, if n_l is greater than what is stated in the Theorem, $A_{cl}(s)$ is still assignable, but the solution is not unique.

Example: $G_o(s) = 1/s$ and $C(s) = \frac{p_1 s + p_0}{s + l_0}$.

$$A_{cl}(s) = s^2 + (p_1 + l_0)s + p_0$$

The roots of $A_{cl}(s)$ can be assigned arbitrarily, but $L(s)$ and $P(s)$ are not unique.

Pole Assignment for IMP

For solving the IMP problem, $C(s)$ has an additional constraint:
$$C(s) = \frac{P(s)}{\Gamma_d(s)\bar{L}(s)}$$
where the disturbance generating polynomial $\Gamma_d(s)$ is monic and of order n_d.

To make sure that the controller is implementable, $C(s)$ must be proper, we ensure that the degree (or order) of $P(s)$ is at most the degree of $\bar{L}(s)\Gamma_d(s)$.

$$\bar{L}(s) = s^{n_l} + l_{n_l-1}s^{n-1} + \ldots + l_1 s + l_0$$
$$P(s) = p_{n_l+n_d}s^{n_l+n_d} + p_{n_l+d-1}s^{n_l+n_d-1} + \ldots + p_1 s + p_0.$$

Note: n_l here denotes the degree of $\bar{L}(s)$. The total degree of the controller is $n_l + n_d$.

The closed loop characteristic polynomial $A_{cl}(s)$ will be of order $n_{cl} = n_l + n_d + n$:

$$A_{cl}(s) = A_o(s)\bar{L}(s)\Gamma_d(s) + B_o(s)P(s)$$

Question: When is the roots of $A_{cl}(s)$ arbitrarily assignable with the IMP constraint?

Theorem: (Pole placement for IMP) *Let $\Gamma_d(s)$ be a n_d degree disturbance generating polynomial. Assume that $A_o(s)$ and $B_o(s)$, nor $\Gamma_d(s)$ and $B_o(s)$ has common factors and $A_o(s)$ is of degree n. Then, using an IMP controller of the form:*

$$C(s) = \frac{P(s)}{\bar{L}(s)\Gamma_d(s)}$$

where $\bar{L}(s)$ is of the order $n_l = n-1$, $P(s)$ is of degree $n_l + n_d$ can be used to assign the $n_{cl} = n_l + n_d + n = 2n + n_d - 1$-th order closed loop characteristics polynomial $A_{cl}(s)$ arbitrarily.

Thus, in order for roots of $A_{cl}(s)$ to be arbitrarily assignable, $n_{cl} = 2n + n_d - 1$.

Idea of Proof:

- Number of free parameters is $n_l + n_d + 1$ (from $P(s)$) plus n_l (from $\bar{L}(s)$), i.e. $2n_l + n_d + 1$ in total.

- Number of coefficients in $A_{cl}(s)$ is $n_l + n_d + n$.

- So, we need $2n_l + n_d + 1 \geq n_l + n_d + n$. This gives the condition, $n_l \geq n - 1$.

- Total degree of $A_{cl}(s)$ is $n_c = n_l + n + n_d$, so it must be at least $n_c \geq 2n + n_d - 1$. In particular, $n_{cl} = 2n + n_d - 1$ will work.

Typically, if n_l is greater than what is stated in the Theorems, the solution is not unique.

Internal model principle in states space

We develop IMP based control concepts in the states space formulation. There are two methods:

- disturbance estimate feedback

- output filtering

Disturbance Exo-system:
Suppose that disturbance $d(t)$ is unknown, but we know that it satisfies some differential equation. This implies that $d(t)$ is generated by an exo-system:

$$\dot{x}_d = A_d x_d$$

$$d = C_d x_d$$

Since,

$$D(s) = C_d(sI - A_d)^{-1} x_d(0) = C_d \frac{Adj(sI - A_d)}{det(sI - A_d)} x_d(0)$$

where $x_d(0)$ is the inital value of $x_d(t = 0)$. Thus, the disturbance generating polymial is nothing but the characteristic polynomial of A_d,

$$\Gamma_d(s) = det(sI - A_d)$$

For example, if $d(t) = d_0 + \alpha sin(\omega t + \phi)$,

$$\begin{pmatrix} \dot{x}_{d1} \\ \dot{x}_{d2} \\ \dot{x}_{d3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\omega^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{d1} \\ x_{d2} \\ x_{d3} \end{pmatrix}$$

$$d = x_{d1} + x_{d3}$$

The characeteristic polynomial, as expected, is:

$$\Gamma_d(s) = det(sI - A_d) = s(s^2 + \omega^2)$$

Method 1: Disturbance-estimate feedback

Suppose that disturbance enters a state space system:

$$\dot{x} = Ax + B(u + d)$$
$$y = Cx$$

If we *knew* $d(t)$ then an obvious control is:

$$u = -d + v - Kx$$

where K is the state feedback gain. However, $d(t)$ is generally unknown. Thus, we estimate it using an observer. First, augment the plant model.

$$\begin{pmatrix} \dot{x} \\ \dot{x}_d \end{pmatrix} = \begin{pmatrix} A & BC_d \\ 0 & A_d \end{pmatrix} \begin{pmatrix} x \\ x_d \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} x \\ x_d \end{pmatrix}$$

Notice that the augmented system is not controllable from u. Nevertheless, if d has effect on y, it is observable from y.

Thus, we can design an observer for the augmented system, and use the observer state for feedback:

$$\frac{d}{dt}\begin{pmatrix} \hat{x} \\ \hat{x}_d \end{pmatrix} = \begin{pmatrix} A & BC_d \\ 0 & A_d \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{x}_d \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} (y - C\hat{x})$$

$$u = -\underbrace{C_d \hat{x}_d}_{\hat{d}} + v - K\hat{x} = v - \begin{pmatrix} K & C_d \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{x}_d \end{pmatrix}$$

where $L = [L_1^T, L_2^T]^T$ is the observer gain. The controller can be simplified to be:

$$\frac{d}{dt}\begin{pmatrix} \hat{x} \\ \hat{x}_d \end{pmatrix} = \begin{pmatrix} A - BK - L_1C & 0 \\ -L_2C & A_d \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{x}_d \end{pmatrix}$$

$$- \begin{pmatrix} -B & L_1 \\ 0 & L_2 \end{pmatrix} \begin{pmatrix} v \\ y \end{pmatrix}$$

$$u = -\begin{pmatrix} K & C_d \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{x}_d \end{pmatrix} + v$$

The $y(t) \to u(t)$ controller transfer function $C_{yu}(s)$ has eigenvalues of $A - BK - L_1C$ and of A_d as poles.

To see this, notice that the transfer function of the

controller from $y \to u$ is: $C_{yu}(s) =$

$$\begin{pmatrix} K & C_d \end{pmatrix} \begin{pmatrix} sI - A + BK + LC & 0 \\ L_2C & sI - A_d \end{pmatrix}^{-1} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$$

And the determinant of the block lower triangular matrix is:

$$det(sI - A + BK + LC)det(sI - A_d)$$
$$= det(sI - A + BK + LC)\Gamma_d(s)$$

Hence, the controller $C_{yu}(s)$ has $\Gamma_d(s)$ in its denominator.

This is exactly the **Internal Model Principle**.

Method 2: Augmenting plant dynamics by filtering the output

In this case, the goal is to introduce the disturbance generating polynomial into the controller dynamics by filtering the output $y(t)$.

Let $\dot{x}_d = A_d x_d$, $d = C_d x_d$ be the disturbance exosystem.

Nominal plant

$$\dot{x} = Ax + Bu + Bd$$
$$y = Cx$$

Output filter:

$$\dot{x}_a = A_d^T x_a + C_d^T \cdot y$$

Stabilize the augmented system using (observer) state feedback:

$$u = -[K_o \quad K_a]\begin{pmatrix} \hat{x} \\ x_a \end{pmatrix}$$

where $\hat{x}$ is the observer estimate of the original plant itself.

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}).$$

Notice that x_a need not be estimated since it is generated by the controller itself!

The transfer function of the controller is: $C(s) =$

$$\begin{pmatrix} K_o & K_a \end{pmatrix} \begin{pmatrix} sI - A + BK + LC & BK \\ 0 & sI - A_d \end{pmatrix}^{-1} \begin{pmatrix} L \\ C_d^T \end{pmatrix}$$

from which it is clear that its denominator has $\Gamma_d(s) = det(sI - A_d)$ in it. i.e. the **Internal Model Principle**.

The following is an **intuitive** way of understanding how the output filtering approach works.

For concreteness, assume that the disturbance $d(t)$ is a sinusoid with frequency ω.

- Suppose that the closed loop system is stable. This means that for any bounded input, any internal signals will also be bounded.

- For the sake of contradiction, if some residual sinuoidal response in $y(t)$ still remains, then $Y(s)$ is of the form:
$$Y(s) = \frac{\alpha(s, 0)}{s^2 + \omega^2}$$

- The augmented state is the filtered version of $Y(s)$,

$$X_a(s) = \frac{\beta(s, 0)}{(s^2 + \omega)^2}$$

for some polynomial $\beta(0, s)$. The time response of $x_a(t)$ is of the form

$$x_a(t) = \gamma sin(\omega t + \phi_1) + \delta \cdot t \cdot sin(\omega t + \phi_2).$$

The second term will be unbounded.

- In general, for other types of disturbances, system output $y(t)$ will have the same mode as specified by $\Gamma_d(s)$. By filtering the output by a filter with $\Gamma_d(s)$ in its denominator, one would always have $\Gamma_d(s)^2$ in the denominator. This leads to "t" term. Since the Laplace transform relationship is:

$$\mathcal{L}(f(t)) = F(s) \Rightarrow \mathcal{L}(tf(t)) = -\frac{d}{ds}F(s)$$

For example,

$$\frac{1}{(s+a)^2} = -\frac{d}{ds}\left(\frac{1}{s+a}\right) \Rightarrow \mathcal{L}^{-1}\left(\frac{1}{(s+a)^2}\right) = te^{-at}$$

For the sinusoidal case, $a = j\omega$. In this case, $\left|te^{-at}\right| \to \infty$.

- Since $d(t)$ is a bounded sinuosoidal signal, $x_a(t)$ must also be bounded. This must mean that $y(t)$ does not contain sinuosoidal components with frequency ω.

The most usual case concerns combating constant disturbances using integral control. In this case, the augmented state is:

$$x_a(t) = \int_0^t y(\tau)d\tau.$$

It is clear that if the output converges to some steady value, $y(t) \to y_\infty$, y_∞ must be 0. Or otherwise $x_a(t)$ will be unbounded.

State space formulation of discrete time repetitive control

The goal of repetitive control is to eliminate the effect of periodic disturbance or to track a periodic reference input. The mentality if that of *learning* the disturbance or the required control signal that cancels it. In implementation, it is basically an IMP controller.

Consider periodic discrete time disturbance, $d(k - N) = d(k)$ where N is the period. We can formulate a disturbance generating exo-system:

$$\begin{pmatrix} x_{d1} \\ x_{d2} \\ x_{d3} \\ \vdots \\ x_{dN} \end{pmatrix}(k+1) = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & \dots & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{d1} \\ x_{d2} \\ x_{d3} \\ \vdots \\ x_{dN} \end{pmatrix}(k)$$

$$d(k) = x_{d1}(k).$$

This can be written as

$$x_d(k+1) = A_d x_d(k)$$
$$d(k) = C_d x_d(k)$$

If the plant to be controlled is:

$$e(k+1) = Ae(k) + B(u(k) + d(k))$$
$$y(k) = Ce(k)$$

We proceed just like the disturbance-estimate feedback approach to IMP. A control strategy would be:

$$u(k) = -Ke(k) - \hat{d}(k)$$

where $\hat{d}(k)$ is the estimate of $d(k)$ obtained using an observer:

$$\begin{pmatrix} \hat{e} \\ \hat{x}_d \end{pmatrix}(k+1) = \underbrace{\begin{pmatrix} A & BC_d \\ 0 & A_d \end{pmatrix}}_{A_{com}} \begin{pmatrix} \hat{x} \\ \hat{x}_d \end{pmatrix}(k+1) + \begin{pmatrix} B \\ 0 \end{pmatrix} u(k)$$
$$- L(C\hat{e}(k) - y(k))$$
$$\hat{d}(k) = C_d \hat{x}_d(k)$$

where L is the observer gain chosen such that all the eigenvalues of $A_{com} - LC$ have absolute values less than 1. i.e. the eigenvalues should lie within the unit disk centered at the origin. This is the stability criterion for discrete time systems.

A difficulty in this approach is that N, the dimension of $x_d(k)$ is typically very large. The dimension, N is the ratio of period of the periodic disturbance to the sampling time. This makes designing a stable observer difficult.

Polynomial Approach to Repetitive Control

Here we consider a polynomial (transfer function) approach to solving the repetitive control problem. The approach is to utilize the Internal Model Principle and design a discrete time controller that contains the generating polynomial for the discrete time disturbance in the denominator.

First we discuss the transfer function representation for discrete time system. ...

Discrete Time Transfer Functions

Let $u(k)$ be a discrete time sequence, i.e. the sequence $u(0)$, $u(1)$, $u(2)$, ...

The shift or the delay operator is denoted by q^{-1}. $y(k) = q^{-1}[u(k)]$ denotes the 1 step delay version of the sequence $u(k)$, i.e. $y(k) = u(k-1)$ for any $k \geq 0$.

A discrete time system can be described by a difference equation:

$$y(k) + a_1 y(k-1) + a_2 y(k-2) + \ldots a_n y(k-n)$$
$$= b_0 u(k-\delta) + b_1 u(k-\delta-1) + \ldots b_m u(k-\delta-m),$$

so that the response at time index k, $y(k)$ is a function of previous output as well as present (if the relative degree (or delay) $\delta = 0$) and past inputs.

Using the shift operator notation, the system can be written as:

$$\left(1 + a_1 q^{-1} + a_2 q^{-2} + \ldots a_n q^{-n}\right)[y(k)]$$
$$= q^{-\delta}\left(b_0 + b_1 q^{-1} + \ldots + b_m q^{-m}\right)[u(k)]$$

We can define the transfer function

$$
\begin{aligned}
G_o(q^{-1}) &= \frac{B_o(q^{-1})}{A_o(q^{-1})} \\
&= \frac{q^{-\delta}\left(b_0 + b_1 q^{-1} + \ldots + b_m q^{-m}\right)}{\left(1 + a_1 q^{-1} + a_2 q^{-2} + \ldots a_n q^{-n}\right)}
\end{aligned}
$$

to represent the difference equation. Sometimes, the delay (or relative degree term) $q^{-\delta}$ is factored out of $B_o(q^{-1})$ to make the structure more clear.

Stability of discrete time transfer function

If we factor $A_o(q^{-1})$ into

$$A_o(q^{-1}) = (1 - \alpha_1 q^{-1})(1 - \alpha_2 q^{-1}) \ldots (1 - \alpha_n q^{-1})$$

then $q = \alpha_i$ are the roots of the $A_o(q^{-1})$. Notice that α_i can be in complex conjugate pairs. Then, we say that $A_o(q^{-1})$ is stable if all $|\alpha_i| < 1$.

To see why stability is related to the roots of $A_o(q^{-1})$ this way, consider the zero input response, $e(k)$, of the system (i.e. no input):

$$A_o(q^{-1})[e(k)] = 0$$

If input is considered, then the RHS will have present and past inputs $u(k)$, $u(k - 1)$ etc. in it.

Now factor $A_o(q^{-1})$ into:

$$A_o(q^{-1})[e(k)] = \left((1 - \alpha_1 q^{-1})\Pi_{i=2}^n (1 - \alpha_i q^{-1}) \right)[e(k)] = 0$$

Let

$$g_1(k) := \Pi_{i=2}^n (1 - \alpha_i q^{-1})[e(k)]$$

Then,

$$(1 - \alpha_1 q^{-1})[g_1(k)] = 0 \quad \Rightarrow \quad g_1(k + 1) = \alpha_1 g_1(k).$$

Now for $m = 2, \ldots, n - 1$.

$$g_m(k) := \Pi^n_{i=m+1}(1 - \alpha_i q^{-1})[e(k)]$$

so that $(1 - \alpha_m q^{-1})[g_m(k)] = g_{m-1}(k)$, i.e.

$$g_m(k+1) = \alpha_m g_m(k) + g_{m-1}(k+1)$$

Thus, $e(k) = g_n(k)$. We can rewrite this in state-space form:

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_n \end{pmatrix}(k+1) = \begin{pmatrix} \alpha_1 & 0 & 0 & \ldots & \\ \alpha_1 & \alpha_2 & 0 & 0 & \ldots \\ \alpha_1 & \alpha_2 & \alpha_3 & 0 & \ldots \\ \vdots & \vdots & \vdots & & 0 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_{n-1} & \alpha_n \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_n \end{pmatrix}(k).$$

So that

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_n \end{pmatrix}(k+1) = \begin{pmatrix} \alpha_1 & 0 & 0 & \ldots & \\ \alpha_1 & \alpha_2 & 0 & 0 & \ldots \\ \alpha_1 & \alpha_2 & \alpha_3 & 0 & \ldots \\ \vdots & \vdots & \vdots & & 0 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_{n-1} & \alpha_n \end{pmatrix}^k\begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_n \end{pmatrix}(0).$$

Notice that the eigen values of this system are α_i, $i = 1, \ldots, n$. So that if $|\alpha_i| < 1$ for all i, we have $g_i(k) \to 0$ as $k \to 0$.

Hence $|\alpha_i|$, $i = 1, \ldots, n$, the magnitude of the roots of $A_o(q^{-1}) = \Pi_{i=1}^{n}(1 - \alpha_i q^{-1})$, determine the stability of the transfer function.

Periodic Disturbance and IMP

If a discrete time disturbance $d(k)$ satisfies:

$$d(k) + \gamma_1 d(k-1) + \ldots + \gamma_N d(k-N) = 0$$

we have $(1 + \gamma_1 q^{-1} + \ldots + \gamma_N q^{-N})[d(k)] = 0$. We call

$$\Gamma_d(q^{-1}) := (1 + \gamma_1 q^{-1} + \ldots + \gamma_N q^{-N})$$

the disturbance generating polynomial for $d(k)$.

In the case of a $N-$periodic disturbance, we have $d(k-N) = d(k)$. So, $d(k)$ satisfies:

$$q^{-N}[d(k)] = d(k), \quad \Rightarrow (1 - q^{-N})[d(k)] = 0.$$

The periodic disturbance generating polynomial is therefore:
$$\Gamma_d(q^{-1}) = 1 - q^{-N}.$$

IMP in the discrete time domain works the same way as in the continuous time. Specifically, if the controller is of the form

$$C(q^{-1}) = \frac{P(q^{-1})}{\Gamma_d(q^{-1})\bar{L}(q^{-1})}$$

The closed loop characteristic polynomial is

$$A_{cl}(q^{-1}) = \Gamma_d(q^{-1})\bar{L}(q^{-1})A_o(q^{-1}) + P(q^{-1})B_o(q^{-1}).$$

Then, the sensitivities are:

$$S_o(q^{-1}) = \frac{\Gamma_d(q^{-1})\bar{L}(q^{-1})A_o(q^{-1})}{A_{cl}(q^{-1})}$$

$$S_{io}(q^{-1}) = \frac{\Gamma_d(q^{-1})\bar{L}(q^{-1})B_o(q^{-1})}{A_{cl}(q^{-1})}$$

$$T_o(q^{-1}) = \frac{P(q^{-1})B_o(q^{-1})}{A_{cl}(q^{-1})}$$

In the case of output disturbance, the response is

$$y(k) = S_o(q^{-1})d(k).$$

$$A_{cl}(q^{-1})[y(k)] = \bar{L}(q^{-1})A_o(q^{-1})\Gamma_d(q^{-1})d(k)$$
$$= (\bar{L}(q^{-1})A_o(q^{-1}))[\Gamma_d(q^{-1})[d(k)]] = 0.$$

If $A_{cl}(q^{-1})$ is designed so that it is stable, $y(k) \to 0$ as $k \to \infty$.

Similarly for input disturbance, $y(k) = S_{io}(q^{-1})d_i(k)$

$$A_{cl}(q^{-1})[y(k)] = \bar{L}(q^{-1})B_o(q^{-1})[\Gamma_d(q^{-1})d(k)] = 0.$$

Hence, $y(k) \to 0$.

In the presence of a reference $r(k)$,

$$y(k) = T_o(q^{-1})[r(k)] = r(k) - S_o(q^{-1})[r(k)].$$

Since $S_o(q^{-1})[r(k)] \to 0$, we have $y(k) \to r(k)$.

Prototype Repetitive Controller

Since the disturbance generating polynomial is of very high order, it is difficult to assign the poles of the closed loop system. The prototype repetitive controller which is of very simple structure is therefore proposed.

For simplicity, we assume that $A_o(q^{-1})$ and $B_o(q^{-1})$ in $G_o(q^{-1})$ have roots within the unit disk (stable and minimum phase).

$$C(q^{-1}) = \frac{k_r G_o^{-1}(q^{-1})q^{-N}}{1 - q^{-N}} = k_r \frac{A_o(q^{-1})q^{\delta - N}}{B_o(q^{-1})(1 - q^{-N})}$$

Notice that $C(q^{-1})$ consists of

- inverting the plant $G_o(q^{-1})$,

- incorporating $1 - q^{-N}$ in the denominator (IMP), and

- in delaying the control by one cycle q^{-N}.

Since N is typically much larger than the excess degree of $G_o(q^{-1})$, $C(q^{-1})$ will be proper.

With periodic (output) disturbance (input disturbance / reference cases are similar), we have:

$$A_{cl}(q^{-1})[e(k)] = (1 - q^{-N})d(k) = 0$$
$$\Rightarrow 0 = \left((1 - q^{-N}) + k_r q^{-N}\right)[e(k)]$$
$$\Rightarrow 0 = (1 - (1 - k_r)q^{-N})[e(k)]$$
$$\Rightarrow e(k) = (1 - k_r)e(k - N).$$

Let $\lambda := (1 - k_r)$. If $0 < k_r < 2$, then $|\lambda| < 1$. For any $k \geq 0$, we can write $k = n + iN$ where $0 \leq n \leq N - 1$, and $i \geq 0$ is an integer. We then have

$$e(k) = e(n + iN) = \lambda^i e(n) \to 0.$$

Remarks:

- In the case when $A_o(q^{-1})$ has unstable roots, the system can be stabilized first. If $B_o(q^{-1})$ has unstable factors, these should not be cancelled (else internal instability). The repetitive controller needs to be modified by doing a zero phase compensation and by modification of the gain. See (Tomizuka, Tsao, Chew 1989) paper for details.

- Having exactly $1-q^{-N}$ as the generating polynomial may present some robustness problems, since this implies the controller has very high gain at *all* harmonics of the disturbance frequency. One idea is to limit the bandwidth by modifying $1 - q^{-N}$ to

$$1 - Q(q, q^{-1})q^{-N}$$

where $Q(q, q^{-1})$ is a unity gain zero phase filter (known as Q-filter). For example,

$$Q(q, q^{-1}) = 0.1q^2 + 0.15q + 0.5 + 0.15q^{-1} + 0.1q^{-2}$$

This smoothes out the generating polynomial and has the effect of reducing the gain at high order harmonics.

- Normally, if Q-filter modification is not used, the sensitivity function $S_o(q^{-1})$ will have $1 - q^{-N}$ in its numerator. The sensitivity vanishes at $q = e^{j\Omega T_s}$, for Ω are harmonics of the fundamental: $\omega_{fund} = \frac{2\pi}{NT_s}$ where T_s is the sampling period. Using the Q-filter, if $Q(q, q^{-1})$ is low pass, the sensitivity is small only at the 1st few harmonics. High frequency harmonics are ignored to preserve robustness.

Repetitive Control and IMP Example

Consider a plant given by:

$$\dot{x} = u$$

$$e = x + d(t)$$

$d(t) = d(t - 2)$ is a periodic disturbance of period 2 given by:

$$d(t) = 0.5 * sin(\omega t) + 0.3cos(2\omega t) + d_1(t)$$

where $\omega = 2\pi/2$ is the fundamental frequency, $d_1(t) = d_1(t - 2)$ is a random, piecewise constant (with sampling period $T_s = 0.02$s) periodic distrbance between ± 0.1.

We develop 3 controllers:

- IMP based continuous time controller that compensates for the fundamental and harmonics.

- A discrete time repetitive controller based on disturbance estimate cancellation using an observer.

- A Prototype Repetitive Controller.

The latter two will be based on discrete time formulation with sampling time of $T_s = 0.02s$.

Continuous time IMC

The continuous time IMC Controller that takes care of the fundamental and 1st harmonics:

$$C(s) = \frac{p_4 s^4 + p_3 s^3 + p_2 s^2 + p_1 s + p_0}{(s^2 + \omega^2)(s^2 + 4\omega^2)}$$

Coefficients are chosen such that closed loop poles are at -1.

To develop the repetitive controller, we discretize the system using sampling period of $T_s = 0.02$.

Discrete time Repetitive Control

The time discretized system is:

$$e(k + 1) = e(k) + T_s(u(k) + \bar{d}(k))$$

where $\bar{d}(k) = \frac{1}{T_s}[d(k+1) - d(k)]$. We assume that $\bar{d}(k)$ is generated by a disturbance generating exo-system, $x_d(k + 1) = A_d x_d(k)$, $\bar{d}(k) = C_d x_d(k)$.

We consider the repetitive control law using the disturbance cancellation method:

$$u(k) = -\hat{\bar{d}}(k) - 0.1e(k).$$

The disturbance observer is designed to be:

$$\begin{pmatrix} \hat{e} \\ \hat{x}_d \end{pmatrix}(k+1) = \begin{pmatrix} 1 & T_s C_d \\ 0 & A_d \end{pmatrix} \begin{pmatrix} \hat{e} \\ \hat{x}_d \end{pmatrix}(k) + \begin{pmatrix} T_s \\ 0 \end{pmatrix} u(k)$$
$$-L(\hat{e} - e)$$

where L is chosen using the LQ design method via Matlab.

```
>> [K, S, E] = lqr(A', C', Q, R);
>> L = K';
```

where Q is a positive definite matrix (e.g. identity) with dimension of (e, x_d^T), and R (small scalar) is of the dimension of e. This ensures that $A - LC$ is stable.

Prototype Repetitive Controller

To design the prototype repetitive control law, the transfer function of the plant is:

$$G_o(q^{-1}) = \frac{T_s q^{-1}}{1 - q^{-1}}$$

i.e. $B(q^{-1}) = T_s$, $\delta = 1$, and $A_o(q^{-1}) = 1 - q^{-1}$.
Therefore, prototype repetitive control law is:

$$C(q^{-1}) = \frac{k_r}{T_s} \frac{q^{-(N-1)}(1 - q^{-1})}{1 - q^{-N}} = \frac{k_r}{T_s} \frac{q^{-(N-1)} - q^{-(N-2)}}{1 - q^{-N}}$$

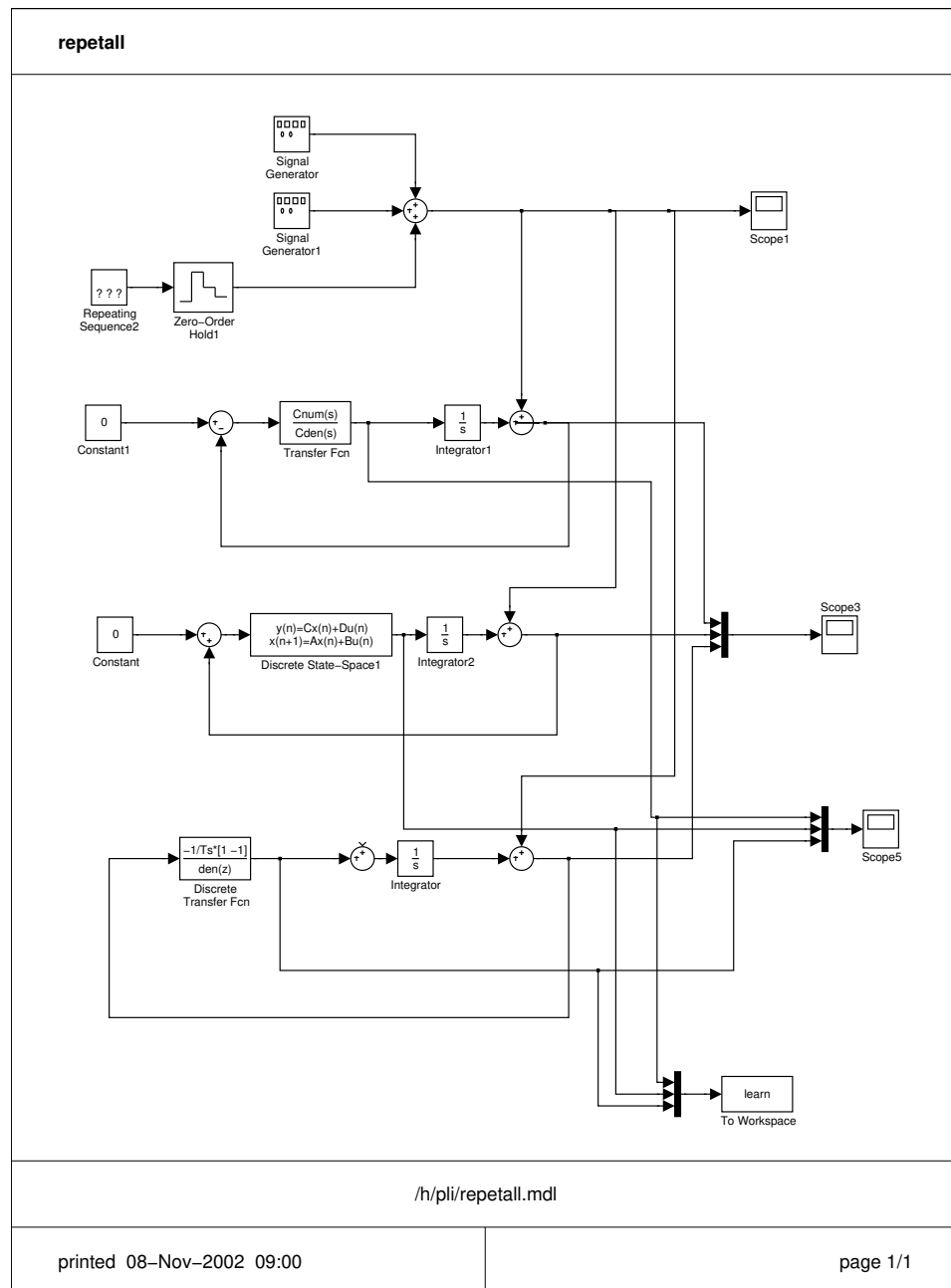The control action is given by:

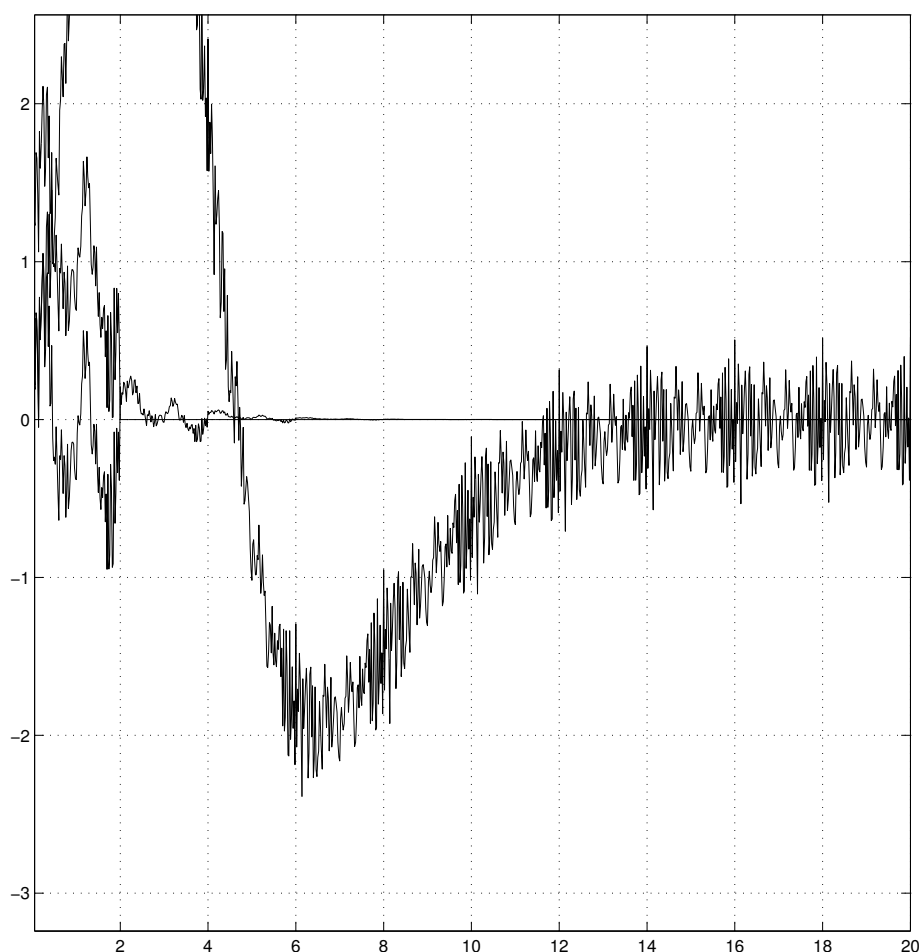$$(1 - q^{-N})[u(k)] = \frac{k_r}{T_s}(q^{-(N-1)} - q^{-(N-2)})[-e(k)]$$

Hence,

$$u(k) = u(k - N) - \frac{k_r}{T_s}(e(k + 1 - N) - e(k + 2 - N)).$$

Notice that the control action is based on modifying the previous control input by the error in the previous cycle. Hence, the control is being learned.

The SIMULINK diagram is shown below:

repetall

Signal
Generator

Signal
Generator1

Scope1

? ? ?
Repeating
Sequence2

Zero–Order
Hold1

0
Constant1

$\dfrac{Cnum(s)}{Cden(s)}$
Transfer Fcn

$\dfrac{1}{s}$
Integrator1

0
Constant

y(n)=Cx(n)+Du(n)
x(n+1)=Ax(n)+Bu(n)
Discrete State–Space1

$\dfrac{1}{s}$
Integrator2

Scope3

$\dfrac{-1/Ts^*[1\ -1]}{den(z)}$
Discrete
Transfer Fcn

$\dfrac{1}{s}$
Integrator

Scope5

learn
To Workspace

/h/pli/repetall.mdl

printed 08–Nov–2002 09:00   page 1/1
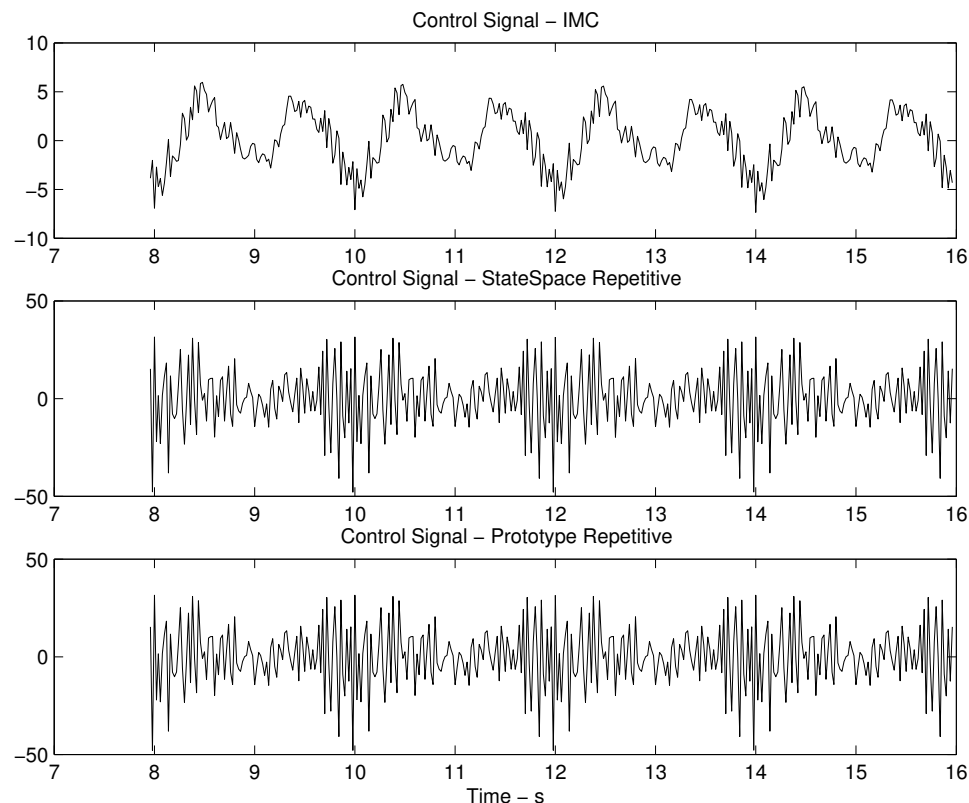
The error shown is below.

Time offset: 0

Error using IMC, state-space type repetitive control and prototype repetitive control

Notice that both repetitive controllers are able to make the error to completely vanish, whereas the random part of the error passes straight through the IMC unattenuated. With $k_r = 1$ in the prototype repetitive control, the control action is learned in 1 cycle. Thus,

the error converges pretty much after the 1st cycle.

Control action using IMC, state-space type repetitive control and prototype repetitive control are shown above.

Notice from the control actions that all three cases learn a periodic control action, with the IMC mainly having the fundamental and harmonics. The learned control action by both the repetitive controllers are similar.