Numerical analysis of Debye sheath physics

github.com/tobiasschuett/DebyeSheath

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1 Abstract

The ability to form Debye sheath is a fundamental characteristic of a plasma. In this letter the ion velocity across the sheath is compared for different numerical and analytical models. Numerical solutions are obtained assuming no collisions as well as assuming finite collision lengths between ions and neutrals. A comparison with the analytical approximation presented in the book by *J.P. Freidberg* shows qualitatively different behaviour. Numerical results show that the ion wall velocity increases strongly with collision length for low collision lengths while it saturates for collision lengths that are much larger than the Debye length.

2 Introduction

Debye sheaths are a fundamental characteristic of a plasma and are formed when the plasma comes into contact with a material surface (wall). If one assumes that the ions and electrons are at the same temperature, i.e $T_e \approx T_i$, the thermal velocity of the electrons will generally be much larger than the thermal velocity of the ions due to their mass difference. As the plasma comes into contact with a material wall, initially there will be a higher electron flux into into the wall. As a result the wall will charge up negatively, hence creating an electric field that reduces the electron flux [1]. The electron charge inside the wall and the electric field will continue to build up until the flux of electrons and ions into the wall is equal as the ions are now accelerated into the wall due to the electric field. It is therefore clear that in order for the plasma to exist in steady state without a constant loss of negative charge, this electric field is required [1]. This opposing electric field is also created when a voltage is applied to the material wall and hence always shields the main plasma from any externally applied DC voltage [2]. A Langmuir probe utilises the principle of the Debye sheath to infer the plasma density and temperature by enforcing a strong negative or positive voltage and measuring the resulting wall current as a function of the applied voltage [1]. It is therefore of great importance to have a good understanding of the Debye sheath physics in order to achieve better estimates of plasma parameters.

3 Results

In order to obtain numerical results about the Debye sheath characteristics it is most useful to treat the ions and electrons as separate fluids with the steady state equilibrium governed by the 1D momentum equation of fluid dynamics [2]:

$$m_j n_j \left(\partial_t v_j + v_j \partial_x v_j\right) = \underbrace{-T_j \partial_x n_j}_{\text{pressure force}} + \underbrace{q n_j E}_{\text{Lorentz force}} \tag{1}$$

where j represents either ions or electrons [2]. The force term on the RHS is given by the general pressure gradient force and by the force acting on the fluid due to the emerging electric field E mentioned in 2. Different assumptions of the nature of the fluids can be made at this point leading to different models and results.

In order to obtain an *analytical approximation* of the steady state electrostatic potential, it is useful to set $v_j = 0$, i.e. assuming that there is no net fluid motion in steady state [2]. This results in a vanishing LHS in equation 1. Integrating the RHS and assuming constant ions and electron temperatures, leads to the following Boltzmann distributions for ion and electron densities.

$$n_j = n_0 \exp(q\phi/T_j) \tag{2}$$

By closing the system of equations with Poisson's equation:

$$\nabla^2\phi = \frac{e}{\epsilon}(n_e - n_i) \tag{3}$$

and linearizing the exponential functions in equation 2, an analytical result can be obtained:

$$\phi(x) = \begin{cases} \phi_{wall} \exp\left(\frac{x}{\lambda_D}\right) & \text{for } x \leq 0 \\ \phi_{wall} & \text{for } x > 0 \end{cases} \tag{4}$$

However, the assumption $v_j = 0$ is rather strong as it ignores the net ion fluid velocity necessary to keep the ion and electron flux into the wall constant. Still equation 4 shall serve as a qualitative analytical comparison to the obtained results. It should be noted however that its parameters are determined by the numerical integration and it therefore serves as a qualitative comparison.

A *weaker assumption* is to assume that $T_i \ll T_e$ and $v_e \ll v_i$ as is justified in [1]. This implies that the pressure term vanishes for the ions in equation 1 and that the LHS vanishes for the electrons in equation 1. Integrating the resulting momentum equations yields the Boltzmann distribution for the electron density and the conservation of energy equation for the ions [1]:

$$n_e = n_s exp(e\phi/T_e) \tag{5}$$

$$\frac{1}{2}m_i v_i^2 + e\phi(x) = \frac{1}{2}m_i v_s^2 \tag{6}$$

Equations 3, 5 and 6 together with the continuity equation for the ions

$$v_i n_i = v_s n_s \tag{7}$$

build the system of coupled ordinary differential equations which were integrated using `scipy.odeint` [3] to determine $\phi(x)$[1]. After some further derivation it can be shown can the current depends on ϕ in the following way [1]:

$$j = en_s \left(\frac{T_e}{m_i}\right)^{1/2} \left\{ \left(\frac{m_i}{2\pi m_e}\right)^{1/2} \exp\left(\frac{e\phi}{T_e}\right) - 1 \right\} \tag{8}$$

This model assumes no collisions in the sheath, i.e. a collision length $L = \infty$, and therefore the ion velocity across the sheath can easily be obtained with equation 6 once ϕ has been successfully integrated. The resulting ion velocity is shown in figure 1.
A further attempt to *introduce a more realistic model* is to introduce an average collision length L after which the ions lose their momentum due to collisions with neutrals while being accelerated across the sheath. This means that the ion energy is not conserved anymore and equation 6 is replaced by the following momentum equation:

$$\frac{dv_i}{dx} = \underbrace{\frac{e}{m_i v_i}E}_{\text{growth}} - \underbrace{\frac{v_i}{L}}_{\text{loss}} \tag{9}$$

[1] Note that equation 3 was transformed into two coupled first order differential equations in order to be solvable with `scipy.odeint`.

which is integrated together with equations 3, 5, 7 to yield $v_i(x, L)$. The result is shown in figure 1.

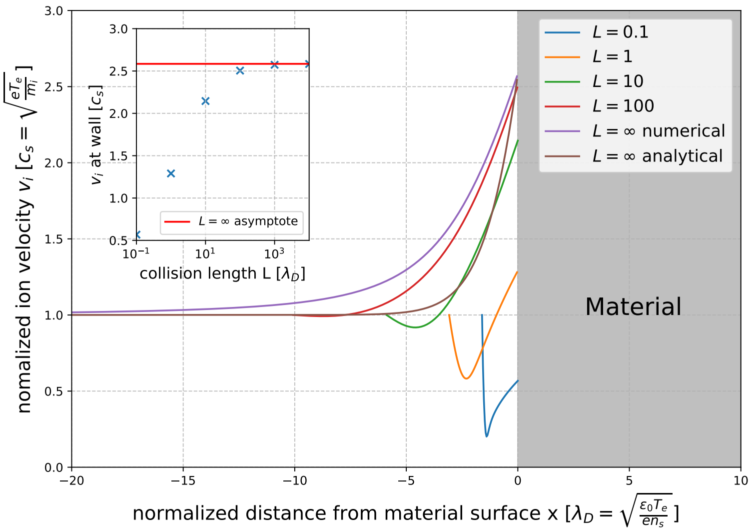

Figure 1: The ion velocity while travelling towards the material wall for different collision lengths L. Also shown is the ion velocity at the wall for different collision lengths of the model. Note that $x = 0$ corresponds to the material surface and the sheath and plasma region correspond to $x < 0$. The material surface was determined for each result by determining x where $j = 0$. The solution of the numerical model and analytical approximation without collisions is shown for comparison.

Figure 1 shows that ion velocity at the wall increases with L as a result of less collisions occurring to each ion while being accelerated towards the wall. Figure 1 also shows that the increase is much more significant for low values of L as v_i increases by a factor of ~ 2.5 between $L = 0.1\lambda_D$ and $L = 100\lambda_D$. For larger L the increase in ion wall velocity is comparably small and quickly approaches the threshold given by the numerical model with no collisions.
It also shows that for $L \leq 10$ the ion velocity initially decreases at the beginning of the sheath before it then increases again, meaning that there is a sheath region for low L where the loss term in equation 9 dominates over the growth term. This implies that the applied model starts to become less valid for these cases where $L \simeq \lambda_D$ as the two fluid approach assumes collisionless fluids [1]. The ions can only bet treated as a collisionless fluid if the mean free path due to collisions is much greater than the Debye length, i.e. $L \ll \lambda_D$. Only in this regime the long-range collective effects such as the response to an electromagnetic field E dominate over the short-range collisional effects [2, p.130]. The assumption made by the model of a smooth, continuous charge distribution breaks down in this regime as can be seen in figure 2 where the discontinuity becomes visible in the transition from $L = 10$ to $L = 0.1$. This demonstrates the consistency of the model with its assumptions. In addition, figure 1 shows that the analytical solution 4 shows a qualitatively different behaviour of the ion velocity across the sheath compared to the numerical solution for no collisions denoted by "$L = \infty$ numerical" in figure 1. It is important to note that while the ion velocity at the wall varies with collision length L, the ion flux will be independent of L. This must be the case as the ion continuity equation 7 in this model implies that the flux is a constant across the entire sheath set by the initial flux $\Phi = n_s v_s$.

4 Discussion

In this letter we have presented two different numerical models to compare the ion velocity at the material wall for different collision lengths between ions and neutrals. The results show

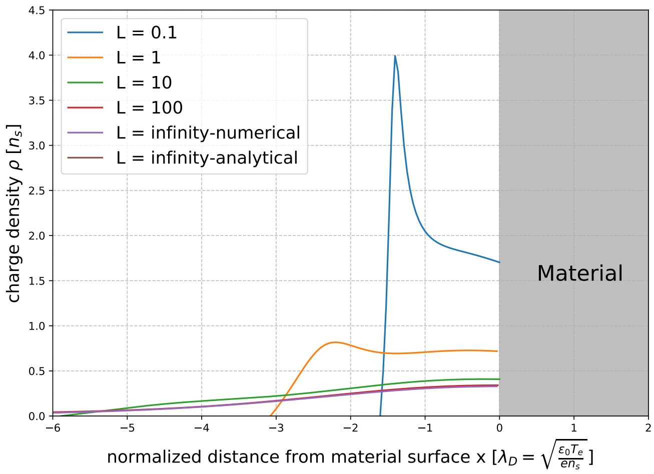

Figure 2: The charge density ρ across the sheath region for different L. The discontinuity as L normalised to λ_D approaches unity becomes apparent and displays the limitations of the collisionless fluid model in this regime.

that the ion wall velocity increases with collision length, especially in the regime where the collision length is of similar length as the Debye length. However, it has also been demonstrated that the model starts to break down and conflicts its assumptions in this regime due to the emerging discontinuity of the charge density [1]. The ion wall velocity saturates for longer collision lengths and approaches the solution of the model which does not include collisions. An analytical comparison was performed with the sheath model presented by *J.P. Freidberg* [2] and shows qualitatively different behaviour to the results obtained by both numerical models. This shows that the different assumptions going into each model do have a significant impact on the resulting electrostatic potential and should therefore be chosen with care. Note that the solutions obtained in this letter do not enforce a wall potential. An often found law is the Child-Langmuir law which only becomes valid for highly negative enforced wall potentials in order to measure the ion saturation current in Langmuir probes [4]. The model for the current density presented in this letter does not depend on the collision length L of the ions as it solely uses the Boltzmann distribution of the electron density and the ion continuity equation which are both independent of L. The current density equation is the basis on which a Langmuir probe operates as discussed in the lecture notes of *R. Fitzpatrick* [1]. Hence the Langmuir probe measurements should be unaffected by L as long as L is much larger than the Debye length. If however, the recombination of ions and electrons to form neutrals is included in the model, this does the effect the ion continuity equation. Hence, the current density model for the Langmuir probe must be adjusted in the case of significant recombination rates within the sheath region in order to obtain more accurate results.

References

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