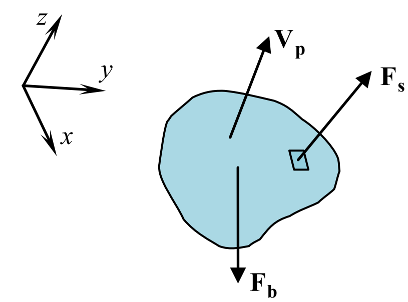
NAVIER-STOKES EQUATIONS DERIVATION

Forces on a fluid particle

Consider the fluid particle (a collection of molecules) shown to the right. It is moving with a velocity $\mathbf{V_p}$ with respect to the inertial reference frame (x,y,z). It is subjected to a body forces $\mathbf{F_b}$ and surface forces $\mathbf{F_s}$; the latter is typically the integral of all the surface stresses caused by interactions with its neighboring particles. Newton's Second Law then states that



$$m\frac{d\mathbf{V_p}}{dt} = \sum \mathbf{F_b} + \sum \mathbf{F_s} \tag{1}$$

where m is the mass of the fluid particle (a constant). The $\sum$ indicates that the forces need to be summed (or integrated) so that the right side of Eq. (1) is the net external force on the particle. In this course, the only body force we will consider is gravity, which can be written simply as

$$\mathbf{F_b} = m\mathbf{g}\text{, where } \mathbf{g} = \begin{bmatrix} g_x & g_y & g_z \end{bmatrix} \tag{2}$$

The velocity of the fluid particle $\mathbf{V_p}$ can be written in terms of the velocity field $\mathbf{V}(x,y,z,t)$ by using the substantial derivative as follows. First, define the trajectory (as a function of time) of the fluid particle as $(x_p(t), y_p(t), z_p(t))$ measured relative to the inertial reference frame; this trajectory provides the position of the particle at any time t. Then

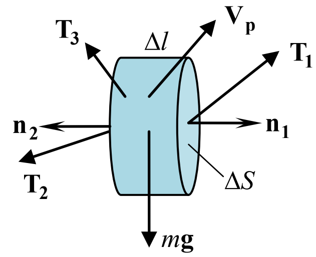
$$\begin{aligned}
\frac{d\mathbf{V_p}}{dt} &= \frac{d}{dt}\mathbf{V}(x_p(t), y_p(t), z_p(t), t) \\
&= \frac{\partial \mathbf{V}}{\partial t} + \frac{\partial \mathbf{V}}{\partial x}\frac{dx_p}{dt} + \frac{\partial \mathbf{V}}{\partial y}\frac{dy_p}{dt} + \frac{\partial \mathbf{V}}{\partial z}\frac{dz_p}{dt} \\
&= \frac{\partial \mathbf{V}}{\partial t} + \frac{\partial \mathbf{V}}{\partial x}u + \frac{\partial \mathbf{V}}{\partial y}v + \frac{\partial \mathbf{V}}{\partial z}w \\
&= \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V} \\
&\equiv \frac{D\mathbf{V}}{Dt}
\end{aligned} \tag{3}$$

where $\mathbf{V} = \begin{bmatrix} u & v & w \end{bmatrix}$ and $\nabla\mathbf{V} = \begin{bmatrix} \frac{\partial \mathbf{V}}{\partial x} & \frac{\partial \mathbf{V}}{\partial y} & \frac{\partial \mathbf{V}}{\partial z} \end{bmatrix}$. Substituting Eqs. (2-3) into Eq. (1) gives

$$m\frac{D\mathbf{V}}{Dt} = m\mathbf{g} + \sum \mathbf{F_s} \tag{4}$$

The rest of this document concerns finding a simple form for the surface forces $\mathbf{F_s}$.

Surface stresses as a matrix



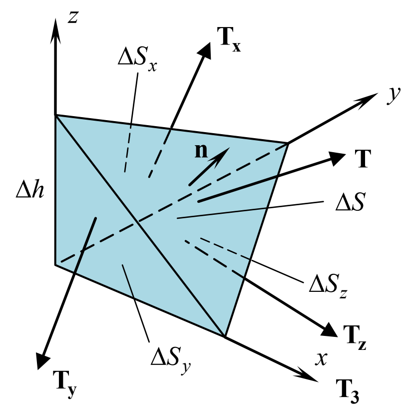
Consider the cylindrical fluid particle shown to the right. (For argument's sake, any shape particle can be chosen since it is infinitesimal in size.) Let the flat faces of the cylinder have normal vectors $\mathbf{n_1}, \mathbf{n_2}$ with applied surface stresses (force/area) $\mathbf{T_1}, \mathbf{T_2}$, respectively. Let $\mathbf{T_3}$ be the surface stress applied to the curved face, and let s be the circumference of the cylinder. If the density of this particle is ρ, then Eq. (4) for this particle takes the form

$$\rho \frac{D\mathbf{V}}{Dt} \Delta l \Delta S = \rho \mathbf{g} \Delta l \Delta S + \mathbf{T_1} \Delta S + \mathbf{T_2} \Delta S + \mathbf{T_3} s \Delta l \tag{5}$$

Taking the limit $\Delta l \to 0$ leaves

$$\mathbf{T_1} = -\mathbf{T_2} \tag{6}$$

which states that stresses on opposite faces of a surface are equal and opposite. This conclusion is equivalent to Newton's Third Law of action and reaction.



Now, consider the tetrahedral fluid particle shown to the right. Three of the faces are aligned with the coordinate planes; they have areas $\Delta S_x, \Delta S_y, \Delta S_z$ and applied stresses $\mathbf{T_x}, \mathbf{T_y}, \mathbf{T_z}$, respectively. Their outward normal vectors are $-\mathbf{e_x}, -\mathbf{e_y}, -\mathbf{e_z}$, the negatives of the standard basis vectors. The diagonal surface has area ΔS, applied stress $\mathbf{T}$, and outward normal vector $\mathbf{n}$. Applying Eq. (4) to this particle gives

$$\begin{aligned} \rho \frac{D\mathbf{V}}{Dt} \frac{\Delta h}{3} \Delta S_y &= \rho \mathbf{g} \frac{\Delta h}{3} \Delta S_y \\ &\quad + \mathbf{T_x} \Delta S_x + \mathbf{T_y} \Delta S_y + \mathbf{T_z} \Delta S_z + \mathbf{T} \Delta S \end{aligned} \tag{7}$$

The areas of the faces can be related using the dot products of their respective normal vectors:

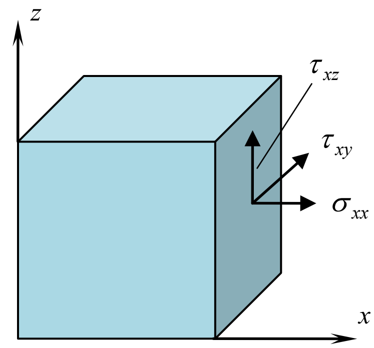
$$\begin{aligned} \Delta S_x &= (\mathbf{n} \cdot \mathbf{e_x}) \Delta S = n_x \Delta S \\ \Delta S_y &= (\mathbf{n} \cdot \mathbf{e_y}) \Delta S = n_y \Delta S \\ \Delta S_z &= (\mathbf{n} \cdot \mathbf{e_z}) \Delta S = n_z \Delta S \end{aligned} \tag{8}$$

where $\mathbf{n} = [n_x \quad n_y \quad n_z]$. Substituting Eq. (8) into Eq.(7) and taking the limit $h \to 0$ yields

$$0 = n_x \mathbf{T_x} + n_y \mathbf{T_y} + n_z \mathbf{T_z} + \mathbf{T} \tag{9}$$

It is standard practice to label the components of the stress vector on a surface facing the positive x-direction (hence with outward normal $\mathbf{e_x}$) as

$$\begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \\ \tau_{xz} \end{bmatrix} \qquad (10)$$



with similar notation for the y- and z-directions (see right). The σ_{xx} notation indicates that this is a normal stress component on the side of the cube facing the x-direction with the stress acting in the x-direction. The τ_{xy} notation indicates that this is a shear stress component on the side of the cube facing the x-direction with the stress acting in the y-direction. The τ_{xz} notation indicates that this is a shear stress component on the side of the cube facing the x-direction with the stress acting in the z-direction.

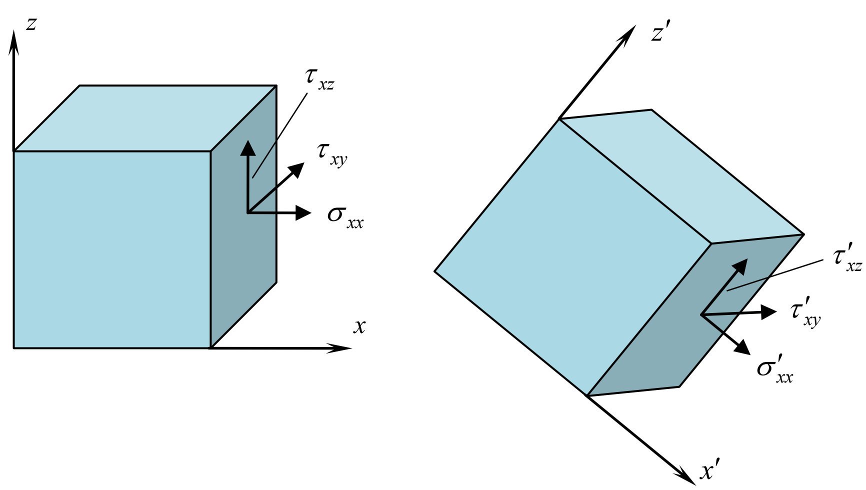
Substituting Eq. (10) into (9) (and using (6)) gives

$$\mathbf{T} = n_x \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \\ \tau_{xz} \end{bmatrix} + n_y \begin{bmatrix} \tau_{yx} \\ \sigma_{yy} \\ \tau_{yz} \end{bmatrix} + n_z \begin{bmatrix} \tau_{zx} \\ \tau_{zy} \\ \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} n_x & n_y & n_z \end{bmatrix} \underbrace{\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}}_{\boldsymbol{\sigma}} \qquad (11)$$

Eq. (11) shows that if the stresses are known on three orthogonal faces, then the stress on any diagonal face can be calculated using the stress tensor $\boldsymbol{\sigma}$, which is composed of the components of stress on these three orthogonal faces, and the coordinates of the normal vector of the diagonal face. Thus, the stresses at any point in the flow are completely characterized (i.e. determined) by the stresses on any three orthogonal faces. (The stress tensor is, however, generally a function of position and time in the flow and the orientation of the three orthogonal faces.)

Rotation of the reference frame

If the reference frame is rotated, as shown below, then the stress tensor in this new frame will generally differ from the original frame.



The surface stresses themselves do not change just because the coordinate system is rotated; however, their representation as column vectors does change, since each entry in the column represents the component of stress in the direction of the respective coordinate axis. Each face of the rotated cube is diagonal with respect to the original axes; thus, the discussions pertaining to the tetrahedral particle are relevant.

Let $\mathbf{e}_{\mathbf{x}'}, \mathbf{e}_{\mathbf{y}'}, \mathbf{e}_{\mathbf{z}'}$ be the standard basis vectors in the rotated coordinate system, and let $\mathbf{T}_{\mathbf{x}'}$ be the stress on the side of the rotated cube facing the x´-direction. Then, using Eq. (11),

$$\mathbf{T}_{\mathbf{x}'} = \begin{bmatrix} \mathbf{e}_{\mathbf{x}'} \cdot \mathbf{e}_{\mathbf{x}} & \mathbf{e}_{\mathbf{x}'} \cdot \mathbf{e}_{\mathbf{y}} & \mathbf{e}_{\mathbf{x}'} \cdot \mathbf{e}_{\mathbf{z}} \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \tag{12}$$

Similar equations can be found for $\mathbf{T}_{\mathbf{y}'}$ and $\mathbf{T}_{\mathbf{z}'}$. Stacking these row vectors gives the following matrix equation:

$$\begin{bmatrix} \mathbf{T}_{\mathbf{x}'} \\ \mathbf{T}_{\mathbf{y}'} \\ \mathbf{T}_{\mathbf{z}'} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_{\mathbf{x}'} \cdot \mathbf{e}_{\mathbf{x}} & \mathbf{e}_{\mathbf{x}'} \cdot \mathbf{e}_{\mathbf{y}} & \mathbf{e}_{\mathbf{x}'} \cdot \mathbf{e}_{\mathbf{z}} \\ \mathbf{e}_{\mathbf{y}'} \cdot \mathbf{e}_{\mathbf{x}} & \mathbf{e}_{\mathbf{y}'} \cdot \mathbf{e}_{\mathbf{y}} & \mathbf{e}_{\mathbf{y}'} \cdot \mathbf{e}_{\mathbf{z}} \\ \mathbf{e}_{\mathbf{z}'} \cdot \mathbf{e}_{\mathbf{x}} & \mathbf{e}_{\mathbf{z}'} \cdot \mathbf{e}_{\mathbf{y}} & \mathbf{e}_{\mathbf{z}'} \cdot \mathbf{e}_{\mathbf{z}} \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \tag{13}$$

The left side of Eq. (13) is almost the stress tensor in the rotated coordinate system; however, the components of $\mathbf{T}_{\mathbf{x}'}$, $\mathbf{T}_{\mathbf{y}'}$, and $\mathbf{T}_{\mathbf{z}'}$ are still given relative to the original coordinate system.

Consider $\mathbf{T}_{\mathbf{x}'}$, the stress on the face of the rotated cube facing the x´-direction, whose components are given relative to the original coordinate system. $\begin{bmatrix} \sigma'_{xx} & \tau'_{xy} & \tau'_{xz} \end{bmatrix}$ are the components of this vector in the rotated coordinate system. To relate the two, the same rotation used in Eq. (13) is employed:

$$\mathbf{T}_{\mathbf{x}'} = \begin{bmatrix} \sigma'_{xx} & \tau'_{xy} & \tau'_{xz} \end{bmatrix} \begin{bmatrix} \mathbf{e}_{\mathbf{x}'} \cdot \mathbf{e}_{\mathbf{x}} & \mathbf{e}_{\mathbf{x}'} \cdot \mathbf{e}_{\mathbf{y}} & \mathbf{e}_{\mathbf{x}'} \cdot \mathbf{e}_{\mathbf{z}} \\ \mathbf{e}_{\mathbf{y}'} \cdot \mathbf{e}_{\mathbf{x}} & \mathbf{e}_{\mathbf{y}'} \cdot \mathbf{e}_{\mathbf{y}} & \mathbf{e}_{\mathbf{y}'} \cdot \mathbf{e}_{\mathbf{z}} \\ \mathbf{e}_{\mathbf{z}'} \cdot \mathbf{e}_{\mathbf{x}} & \mathbf{e}_{\mathbf{z}'} \cdot \mathbf{e}_{\mathbf{y}} & \mathbf{e}_{\mathbf{z}'} \cdot \mathbf{e}_{\mathbf{z}} \end{bmatrix} \tag{14}$$

Similar equations can be found for $\mathbf{T}_{\mathbf{y}'}$ and $\mathbf{T}_{\mathbf{z}'}$. Stacking these row vectors gives the following matrix equation:

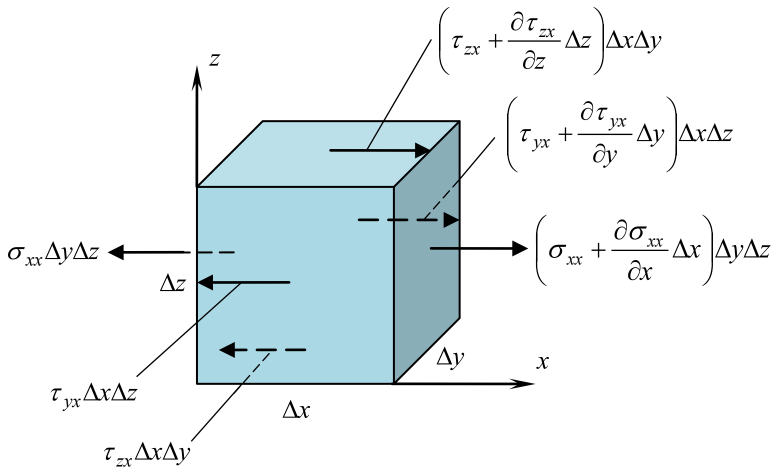
$$\begin{bmatrix} \mathbf{T}_{\mathbf{x}'} \\ \mathbf{T}_{\mathbf{y}'} \\ \mathbf{T}_{\mathbf{z}'} \end{bmatrix} = \begin{bmatrix} \sigma'_{xx} & \tau'_{xy} & \tau'_{xz} \\ \tau'_{yx} & \sigma'_{yy} & \tau'_{yz} \\ \tau'_{zx} & \tau'_{zy} & \sigma'_{zz} \end{bmatrix} \begin{bmatrix} \mathbf{e}_{\mathbf{x}'} \cdot \mathbf{e}_{\mathbf{x}} & \mathbf{e}_{\mathbf{x}'} \cdot \mathbf{e}_{\mathbf{y}} & \mathbf{e}_{\mathbf{x}'} \cdot \mathbf{e}_{\mathbf{z}} \\ \mathbf{e}_{\mathbf{y}'} \cdot \mathbf{e}_{\mathbf{x}} & \mathbf{e}_{\mathbf{y}'} \cdot \mathbf{e}_{\mathbf{y}} & \mathbf{e}_{\mathbf{y}'} \cdot \mathbf{e}_{\mathbf{z}} \\ \mathbf{e}_{\mathbf{z}'} \cdot \mathbf{e}_{\mathbf{x}} & \mathbf{e}_{\mathbf{z}'} \cdot \mathbf{e}_{\mathbf{y}} & \mathbf{e}_{\mathbf{z}'} \cdot \mathbf{e}_{\mathbf{z}} \end{bmatrix} \tag{15}$$

Combining Eqs. (13,15) gives

$$\begin{bmatrix} \sigma'_{xx} & \tau'_{xy} & \tau'_{xz} \\ \tau'_{yx} & \sigma'_{yy} & \tau'_{yz} \\ \tau'_{yz} & \tau'_{zy} & \sigma'_{zz} \end{bmatrix} = \begin{bmatrix} \mathbf{e_{x'}}\cdot\mathbf{e_x} & \mathbf{e_{x'}}\cdot\mathbf{e_y} & \mathbf{e_{x'}}\cdot\mathbf{e_z} \\ \mathbf{e_{y'}}\cdot\mathbf{e_x} & \mathbf{e_{y'}}\cdot\mathbf{e_y} & \mathbf{e_{y'}}\cdot\mathbf{e_z} \\ \mathbf{e_{z'}}\cdot\mathbf{e_x} & \mathbf{e_{z'}}\cdot\mathbf{e_y} & \mathbf{e_{z'}}\cdot\mathbf{e_z} \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \mathbf{e_{x'}}\cdot\mathbf{e_x} & \mathbf{e_{x'}}\cdot\mathbf{e_y} & \mathbf{e_{x'}}\cdot\mathbf{e_z} \\ \mathbf{e_{y'}}\cdot\mathbf{e_x} & \mathbf{e_{y'}}\cdot\mathbf{e_y} & \mathbf{e_{y'}}\cdot\mathbf{e_z} \\ \mathbf{e_{z'}}\cdot\mathbf{e_x} & \mathbf{e_{z'}}\cdot\mathbf{e_y} & \mathbf{e_{z'}}\cdot\mathbf{e_z} \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} \mathbf{e_{x'}}\cdot\mathbf{e_x} & \mathbf{e_{x'}}\cdot\mathbf{e_y} & \mathbf{e_{x'}}\cdot\mathbf{e_z} \\ \mathbf{e_{y'}}\cdot\mathbf{e_x} & \mathbf{e_{y'}}\cdot\mathbf{e_y} & \mathbf{e_{y'}}\cdot\mathbf{e_z} \\ \mathbf{e_{z'}}\cdot\mathbf{e_x} & \mathbf{e_{z'}}\cdot\mathbf{e_y} & \mathbf{e_{z'}}\cdot\mathbf{e_z} \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \mathbf{e_{x'}}\cdot\mathbf{e_x} & \mathbf{e_{y'}}\cdot\mathbf{e_x} & \mathbf{e_{z'}}\cdot\mathbf{e_x} \\ \mathbf{e_{x'}}\cdot\mathbf{e_y} & \mathbf{e_{y'}}\cdot\mathbf{e_y} & \mathbf{e_{z'}}\cdot\mathbf{e_y} \\ \mathbf{e_{x'}}\cdot\mathbf{e_z} & \mathbf{e_{y'}}\cdot\mathbf{e_z} & \mathbf{e_{z'}}\cdot\mathbf{e_z} \end{bmatrix} \tag{16}$$

where the second equality comes from the fact that rotation matrices are orthogonal (i.e. $\mathbf{A}^{-1} = \mathbf{A}^T$).

Equations of motion using the stress tensor



Consider the cubic volume of fluid shown above. For simplicity, only the surface forces in the x-direction are shown. Note the sign conventions used: σ_{xx} is positive outward (a tensile force), whereas τ_{yx} is positive in the positive x-direction on the side facing the positive y-axis. Since the stress tensor generally varies spatially throughout the fluid, different stresses appear on different faces of the cube. As the cube decreases in size, the limit will provide the variation in stress in the x-direction (much like secant lines provide the slope of a curve in the limit).

The x-component of Eq. (4) can hence be written as

$$\begin{aligned} m\frac{Du}{Dt} &= mg_x + \sum F_{s,x} \\ &= mg_x + \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\Delta x\right)\Delta y \Delta z - \sigma_{xx}\Delta y \Delta z + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y\right)\Delta x \Delta z - \tau_{yx}\Delta x \Delta z \\ &\quad + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\Delta z\right)\Delta x \Delta y - \tau_{zx}\Delta x \Delta y \\ &= mg_x + \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\Delta x \Delta y \Delta z \end{aligned} \tag{17}$$

Dividing through by $\Delta x \Delta y \Delta z$ gives

$$\rho\frac{Du}{Dt} = \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{18a}$$

Similar equations can be found for the y- and z-directions:

$$\rho\frac{Dv}{Dt} = \rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \tag{18b}$$

$$\rho\frac{Dw}{Dt} = \rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \tag{18c}$$

In vector form, these 3 equations can be combined as

$$\rho\frac{D\mathbf{V}}{Dt} = \rho\mathbf{g} + \nabla\cdot\boldsymbol{\sigma} \tag{19}$$

Eq. (19) is a statement of the Conservation of (Linear) Momentum for a fluid particle.

Angular momentum: symmetry of the stress tensor

The net change in angular momentum of a fluid particle is equal to the sum of the external torques applied to it. If $\mathbf{r_p}$ is the position vector of the fluid particle with respect to the inertial reference frame, this law can be written as

$$\mathbf{r_p}\times m\frac{d\mathbf{V_p}}{dt} + \mathbf{H} = \sum \mathbf{r_p}\times\mathbf{F_b} + \sum\left(\mathbf{r_p} + \Delta\mathbf{r}\right)\times\mathbf{F_s} \tag{20}$$

where $\mathbf{H}$ is the aggregate of the angular momentums of each molecule within the fluid particle about its own axis of rotation. In Eq. (20), $\Delta\mathbf{r}$ is the displacement of each surface force from the center of the fluid particle. (No corresponding term appears for the body forces, since they are assumed to act at the center of the particle.) Since $\mathbf{r_p}$ is constant for each surface force (only $\Delta\mathbf{r}$ varies), Eq. (20) can be rewritten as

$$\begin{gathered}\mathbf{r_p}\times m\frac{d\mathbf{V_p}}{dt} + \mathbf{H} = \mathbf{r_p}\times\sum\mathbf{F_b} + \mathbf{r_p}\times\sum\mathbf{F_s} + \sum\left(\Delta\mathbf{r}\times\mathbf{F_s}\right)\\ \mathbf{r_p}\times\left(m\frac{d\mathbf{V_p}}{dt} - \sum\mathbf{F_b} - \sum\mathbf{F_s}\right) + \mathbf{H} = \sum\left(\Delta\mathbf{r}\times\mathbf{F_s}\right)\end{gathered} \tag{21}$$

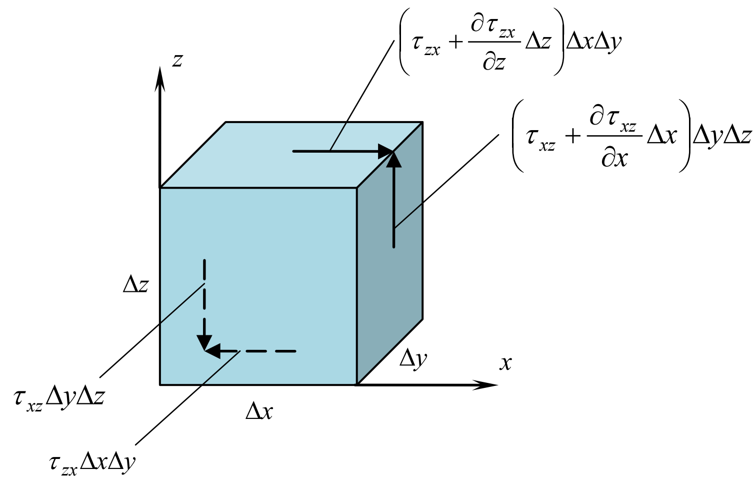
where the cross product rules $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = (\mathbf{a}\times\mathbf{b}) + (\mathbf{a}\times\mathbf{c})$ and $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = (\mathbf{a}\times\mathbf{c}) + (\mathbf{b}\times\mathbf{c})$ have been used. The terms in parentheses in Eq. (21) are equal to 0 by Eq. (1). Therefore,

$$\mathbf{H} = \sum\left(\Delta\mathbf{r}\times\mathbf{F_s}\right) \tag{22}$$

Eq. (22) states that the molecular angular momentum must be balanced by the sum of the torques of each surface force about the center of the fluid particle.

In most common fluids, the molecules are randomly oriented, and so there is no net angular momentum (i.e. the vector sum of the angular momentums of each molecule is 0.) Consequently,

$$\mathbf{0} = \sum \left(\Delta \mathbf{r} \times \mathbf{F_s} \right) \tag{23}$$



The figure above shows the surface forces that cause a rotation about the y-axis, i.e. that contribute to the y-component of Eq. (23). Writing out the y-component of Eq. (23) thus gives

$$\begin{aligned} 0 &= \frac{\Delta x}{2} \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial x} \Delta x \right) \Delta y \Delta z - \frac{\Delta z}{2} \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \Delta z \right) \Delta x \Delta y + \frac{\Delta x}{2} \tau_{xz} \Delta y \Delta z - \frac{\Delta z}{2} \tau_{zx} \Delta x \Delta y \\ &= \tau_{xz} \Delta x \Delta y \Delta z - \tau_{zx} \Delta x \Delta y \Delta z + \frac{1}{2} \frac{\partial \tau_{xz}}{\partial x} \Delta x^2 \Delta y \Delta z - \frac{1}{2} \frac{\partial \tau_{zx}}{\partial z} \Delta x \Delta y \Delta z^2 \end{aligned} \tag{24}$$

Dividing through by the common term $\Delta x \Delta y \Delta z$, then letting $\Delta x, \Delta z \to 0$ yields

$$\tau_{xz} = \tau_{zx} \tag{25}$$

Similarly, the x- and z-components of Eq. (23) show $\tau_{yz} = \tau_{zy}$ and $\tau_{xy} = \tau_{yx}$. Therefore, the stress tensor $\boldsymbol{\sigma}$ is symmetric, which means the surface forces are in balance and don't cause the fluid particles to spin about their own axes.

Newtonian fluids

The simplest form of the stress tensor $\boldsymbol{\sigma}$ for viscous fluids can be found by making the following reasonable assumptions:

1. Shear stress is proportional to the velocity gradient. This is a generalization of Newton's law of viscosity for 1-D flows: $\tau_{xy} = \mu \frac{\partial u}{\partial y}$. Note that shear stress cannot be a function of velocity itself; otherwise, shear stress would change in a coordinate system moving with constant velocity.
2. Viscosity is a function of the thermodynamic state of the fluid and is independent of its kinematics.

3. In a fluid static situation, the only stress is pressure, an inward normal stress.

   Mathematically, $\mathbf{V} = \mathbf{0} \Rightarrow \boldsymbol{\sigma} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} = -p\mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

4. Fluid is isotropic; therefore, the law relating stress to the velocity gradient should be independent of coordinate system rotation.

5. Furthermore, the aggregate molecular angular momentum in the fluid is negligible. This condition has already been shown to be equivalent to the symmetry of the stress tensor.

6. The average normal stress is equal to the thermodynamic pressure (Stokes' assumption). While this is only strictly true for static fluids, it is accurate for most fluids under most situations. Note that incompressible fluids do not have a thermodynamic pressure (it is undefined); therefore, the pressure is defined to be the average normal stress for these situations.

Stress-strain relationship for Newtonian fluids

The velocity gradient is a 3x3 matrix, just like the stress tensor:

$$\nabla\mathbf{V} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \tag{26}$$

Like any matrix, this matrix can be split into symmetric and antisymmetric components:

$$\nabla\mathbf{V} = \underbrace{\frac{1}{2}\begin{bmatrix} \frac{\partial u}{\partial x}+\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z} & \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z} & \frac{\partial w}{\partial z}+\frac{\partial w}{\partial z} \end{bmatrix}}_{\mathbf{e}} + \underbrace{\frac{1}{2}\begin{bmatrix} 0 & \frac{\partial u}{\partial y}-\frac{\partial v}{\partial x} & \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} & 0 & \frac{\partial v}{\partial z}-\frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x}-\frac{\partial u}{\partial z} & \frac{\partial w}{\partial y}-\frac{\partial v}{\partial z} & 0 \end{bmatrix}}_{\boldsymbol{\Omega}} \tag{27}$$

where $\mathbf{e}$, the rate of strain tensor, is symmetric, and $\boldsymbol{\Omega}$, the rate of rotation tensor, is antisymmetric. Since the stress tensor $\boldsymbol{\sigma}$ is symmetric (by assumption 5), it can only be a function of the symmetric rate of strain tensor $\mathbf{e}$. If it were also a function of $\boldsymbol{\Omega}$, then swapping the row and column indices in $\boldsymbol{\sigma}$ would give a different formula.

Furthermore, the principle axes of the stress tensor must be the same as the principle axes of the strain tensor by isotropy. In particular, for a 1-D flow, Newton's viscosity law $\tau_{xy} = \mu \frac{\partial u}{\partial y}$ shows that zero shear stress requires zero shear strain in the flow field.

Let (x', y', z') be the principle axes of the stress tensor. Then

$$\sigma_{x'x'} = -p + Ke_{x'x'} + C\left(e_{x'x'} + e_{y'y'} + e_{z'z'}\right) \tag{28}$$

which is the most generic linear relationship between stress and strain possible. The pressure term accounts for assumption 3, where the normal stress reduces to just the pressure in a static fluid. Additionally, the coefficients of the $e_{y'y'}, e_{z'z'}$ terms are the same by isotropy, since there should be no difference between the y´- and z´-directions. Eq. (28) must also apply to $\sigma_{y'y'}, \sigma_{z'z'}$ since the stress-strain relationships should be the same in every direction.

The rotation of coordinates formula, Eq. (16), is now used to write the stresses in an arbitrary coordinate system (x, y, z) in terms of the principal stresses. Swapping the primed and unprimed terms in Eq. (16), the equation for σ_{xx} reads

$$\sigma_{xx} = \left(\mathbf{e_x} \cdot \mathbf{e_{x'}}\right)^2 \sigma_{x'x'} + \left(\mathbf{e_x} \cdot \mathbf{e_{y'}}\right)^2 \sigma_{y'y'} + \left(\mathbf{e_x} \cdot \mathbf{e_{z'}}\right)^2 \sigma_{z'z'} \tag{29}$$

Similarly,

$$e_{xx} = \left(\mathbf{e_x} \cdot \mathbf{e_{x'}}\right)^2 e_{x'x'} + \left(\mathbf{e_x} \cdot \mathbf{e_{y'}}\right)^2 e_{y'y'} + \left(\mathbf{e_x} \cdot \mathbf{e_{z'}}\right)^2 e_{z'z'} \tag{30}$$

Therefore,

$$\begin{aligned}
\sigma_{xx} &= \left(\mathbf{e_x} \cdot \mathbf{e_{x'}}\right)^2 \left[-p + Ke_{x'x'} + C\left(e_{x'x'} + e_{y'y'} + e_{z'z'}\right)\right] \\
&\quad + \left(\mathbf{e_x} \cdot \mathbf{e_{y'}}\right)^2 \left[-p + Ke_{y'y'} + C\left(e_{x'x'} + e_{y'y'} + e_{z'z'}\right)\right] \\
&\quad + \left(\mathbf{e_x} \cdot \mathbf{e_{z'}}\right)^2 \left[-p + Ke_{z'z'} + C\left(e_{x'x'} + e_{y'y'} + e_{z'z'}\right)\right] \\
&= -p\left[\left(\mathbf{e_x} \cdot \mathbf{e_{x'}}\right)^2 + \left(\mathbf{e_x} \cdot \mathbf{e_{y'}}\right)^2 + \left(\mathbf{e_x} \cdot \mathbf{e_{z'}}\right)^2\right] \\
&\quad + K\left[\left(\mathbf{e_x} \cdot \mathbf{e_{x'}}\right)^2 e_{x'x'} + \left(\mathbf{e_x} \cdot \mathbf{e_{y'}}\right)^2 e_{y'y'} + \left(\mathbf{e_x} \cdot \mathbf{e_{z'}}\right)^2 e_{z'z'}\right] \\
&\quad + C\left[\left(\mathbf{e_x} \cdot \mathbf{e_{x'}}\right)^2 + \left(\mathbf{e_x} \cdot \mathbf{e_{y'}}\right)^2 + \left(\mathbf{e_x} \cdot \mathbf{e_{z'}}\right)^2\right]\left(e_{x'x'} + e_{y'y'} + e_{z'z'}\right) \\
&= -p + Ke_{xx} + C\left(\nabla \cdot \mathbf{V}\right)
\end{aligned} \tag{31a}$$

where $\left[\left(\mathbf{e_x} \cdot \mathbf{e_{x'}}\right)^2 + \left(\mathbf{e_x} \cdot \mathbf{e_{y'}}\right)^2 + \left(\mathbf{e_x} \cdot \mathbf{e_{z'}}\right)^2\right] = 1$ by the Pythagorean Theorem, and $\nabla \cdot \mathbf{V} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}$ is the divergence of the velocity field. Similar analyses for the other two directions show

$$\sigma_{yy} = -p + Ke_{yy} + C\left(\nabla \cdot \mathbf{V}\right) \tag{31b}$$

$$\sigma_{zz} = -p + Ke_{zz} + C\left(\nabla \cdot \mathbf{V}\right) \tag{31c}$$

The shear stresses in the arbitrary coordinate system are also found using Eq. (16):

$$\tau_{xy} = \left(\mathbf{e_x} \cdot \mathbf{e_{x'}}\right)\left(\mathbf{e_y} \cdot \mathbf{e_{x'}}\right)\sigma_{x'x'} + \left(\mathbf{e_x} \cdot \mathbf{e_{y'}}\right)\left(\mathbf{e_y} \cdot \mathbf{e_{y'}}\right)\sigma_{y'y'} + \left(\mathbf{e_x} \cdot \mathbf{e_{z'}}\right)\left(\mathbf{e_y} \cdot \mathbf{e_{z'}}\right)\sigma_{z'z'} \tag{32}$$

Similarly,

$$e_{xy} = (\mathbf{e_x}\cdot\mathbf{e_{x'}})(\mathbf{e_y}\cdot\mathbf{e_{x'}})e_{x'x'} + (\mathbf{e_x}\cdot\mathbf{e_{y'}})(\mathbf{e_y}\cdot\mathbf{e_{y'}})e_{y'y'} + (\mathbf{e_x}\cdot\mathbf{e_{z'}})(\mathbf{e_y}\cdot\mathbf{e_{z'}})e_{z'z'} \tag{33}$$

Therefore,

$$\begin{aligned}
\tau_{xy} &= (\mathbf{e_x}\cdot\mathbf{e_{x'}})(\mathbf{e_y}\cdot\mathbf{e_{x'}})\left[-p + Ke_{x'x'} + C(e_{x'x'} + e_{y'y'} + e_{z'z'})\right] \\
&\quad + (\mathbf{e_x}\cdot\mathbf{e_{y'}})(\mathbf{e_y}\cdot\mathbf{e_{y'}})\left[-p + Ke_{y'y'} + C(e_{x'x'} + e_{y'y'} + e_{z'z'})\right] \\
&\quad + (\mathbf{e_x}\cdot\mathbf{e_{z'}})(\mathbf{e_y}\cdot\mathbf{e_{z'}})\left[-p + Ke_{z'z'} + C(e_{x'x'} + e_{y'y'} + e_{z'z'})\right] \\
&= -p\left[(\mathbf{e_x}\cdot\mathbf{e_{x'}})(\mathbf{e_y}\cdot\mathbf{e_{x'}}) + (\mathbf{e_x}\cdot\mathbf{e_{y'}})(\mathbf{e_y}\cdot\mathbf{e_{y'}}) + (\mathbf{e_x}\cdot\mathbf{e_{z'}})(\mathbf{e_y}\cdot\mathbf{e_{z'}})\right] \\
&\quad + K\left[(\mathbf{e_x}\cdot\mathbf{e_{x'}})(\mathbf{e_y}\cdot\mathbf{e_{x'}})e_{x'x'} + (\mathbf{e_x}\cdot\mathbf{e_{y'}})(\mathbf{e_y}\cdot\mathbf{e_{y'}})e_{y'y'} + (\mathbf{e_x}\cdot\mathbf{e_{z'}})(\mathbf{e_y}\cdot\mathbf{e_{z'}})e_{z'z'}\right] \\
&\quad + C\left[(\mathbf{e_x}\cdot\mathbf{e_{x'}})(\mathbf{e_y}\cdot\mathbf{e_{x'}}) + (\mathbf{e_x}\cdot\mathbf{e_{y'}})(\mathbf{e_y}\cdot\mathbf{e_{y'}}) + (\mathbf{e_x}\cdot\mathbf{e_{z'}})(\mathbf{e_y}\cdot\mathbf{e_{z'}})\right](e_{x'x'} + e_{y'y'} + e_{z'z'}) \\
&= Ke_{xy}
\end{aligned} \tag{34a}$$

where $(\mathbf{e_x}\cdot\mathbf{e_{x'}})(\mathbf{e_y}\cdot\mathbf{e_{x'}}) + (\mathbf{e_x}\cdot\mathbf{e_{y'}})(\mathbf{e_y}\cdot\mathbf{e_{y'}}) + (\mathbf{e_x}\cdot\mathbf{e_{z'}})(\mathbf{e_y}\cdot\mathbf{e_{z'}}) = 0$. Similar analyses for the other two directions show

$$\tau_{xz} = Ke_{xz} \tag{34b}$$

$$\tau_{yz} = Ke_{yz} \tag{34c}$$

Examining Eq. (34a) and comparing it to Newton's viscosity law $\tau_{xy} = \mu\dfrac{\partial u}{\partial y}$ for 1-D flows shows that $K = 2\mu$ for these equations to be consistent (i.e. for Eq. (34a) to collapse to $\tau_{xy} = \mu\dfrac{\partial u}{\partial y}$ for a 1-D flow). The constant C is called Lamé's constant in linear elasticity and is given the symbol λ. Thus, combining Eqs. (31, 34) and making these substitutions for the constants results in

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{e} + \lambda(\nabla\cdot\mathbf{V})\mathbf{I} \tag{35}$$

which is the "simple" form of the stress tensor that satisfies assumptions 1-5.

Stokes' assumption (assumption 6) states that the average normal stress is equal to the thermodynamic pressure, i.e.

$$\begin{aligned}
-p &= \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \\
&= \frac{1}{3}\left[-p + 2\mu e_{xx} + \lambda(\nabla\cdot\mathbf{V}) + -p + 2\mu e_{yy} + \lambda(\nabla\cdot\mathbf{V}) + -p + 2\mu e_{zz} + \lambda(\nabla\cdot\mathbf{V})\right] \\
&= -p + \left(\frac{2}{3}\mu + \lambda\right)(\nabla\cdot\mathbf{V})
\end{aligned} \tag{36}$$

which implies $\lambda = -\frac{2}{3}\mu$. Therefore, Eq. (35) becomes

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{e} - \frac{2}{3}\mu(\nabla\cdot\mathbf{V})\mathbf{I} \tag{37}$$

Navier-Stokes equations

Substituting Eq. (37) into Eq. (19) gives the most general form of the Navier-Stokes equations:

$$\rho\frac{D\mathbf{V}}{Dt} = \rho\mathbf{g} + \nabla\cdot\left[-p\mathbf{I} + 2\mu\mathbf{e} - \frac{2}{3}\mu(\nabla\cdot\mathbf{V})\mathbf{I}\right] = \rho\mathbf{g} - \nabla p + 2\nabla\cdot(\mu\mathbf{e}) - \frac{2}{3}\nabla\left[\mu(\nabla\cdot\mathbf{V})\right] \tag{38}$$

Several special cases of Eq. (38) are common in practice.

1. Constant viscosity

$$\rho\frac{D\mathbf{V}}{Dt} = \rho\mathbf{g} - \nabla p + \mu\nabla^2\mathbf{V} + \frac{\mu}{3}\nabla(\nabla\cdot\mathbf{V}) \tag{39}$$

where the following formula is used:

$$2\mu\nabla\cdot(\mathbf{e}) - \frac{2\mu}{3}\nabla(\nabla\cdot\mathbf{V}) = \mu\begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}\begin{bmatrix}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\\ \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\\ \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z} & \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z} & \frac{\partial w}{\partial z}+\frac{\partial w}{\partial z}\end{bmatrix} - \frac{2\mu}{3}\nabla(\nabla\cdot\mathbf{V})$$

$$= \mu\begin{bmatrix}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y\partial x}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 w}{\partial z\partial x}+\frac{\partial^2 u}{\partial z^2}\\ \frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial z\partial y}+\frac{\partial^2 v}{\partial z^2}\\ \frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 w}{\partial z^2}\end{bmatrix}^T - \frac{2\mu}{3}\nabla(\nabla\cdot\mathbf{V})$$

$$= \mu\begin{bmatrix}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\\ \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\\ \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\end{bmatrix}^T + \begin{bmatrix}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\\ \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\\ \frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\end{bmatrix}^T - \frac{2\mu}{3}\nabla(\nabla\cdot\mathbf{V})$$

$$= \mu\nabla(\nabla\cdot\mathbf{V}) + \mu\nabla^2\mathbf{V} - \frac{2\mu}{3}\nabla(\nabla\cdot\mathbf{V})$$

$$= \mu\nabla^2\mathbf{V} + \frac{\mu}{3}\nabla(\nabla\cdot\mathbf{V})$$

2. Constant viscosity, incompressible flow

$$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{V} \tag{40}$$

3. Inviscid, incompressible flow

$$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{g} - \nabla p \tag{41}$$

4. Static flow

$$\mathbf{0} = \rho \mathbf{g} - \nabla p \tag{42}$$