Modeling an Optical Spring

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Abstract

The basic physics of an optical spring is developed analytically and modeled in Optickle.

1 Optical spring (longitudinal)

When detuned from resonance, the power circulating within a Fabry-Perot cavity varies linearly with small deviations from that detuning. This gives rise to a displacement-dependent force, which can be described via a spring constant. This effect is called the *optical spring*[1].

For frequencies that are slow compared to the cavity pole, we can calculate the behavior of the spring using a quasi-static approximation, simply using the derivative of the power buildup versus cavity detuning.

The power circulating in a cavity is:

$$\frac{P_+}{P_{IN}} = \frac{g^2}{1 + F \sin^2 \phi} \tag{1}$$

where P_{IN} is the incident power, P_+ is the forward-circulating power, $g^2 = (t_1)^2 / (1 - r_1 r_2)$ is the power buildup on resonance, $F = 4 r_1 r_2 / (1 - r_1 r_2)^2$ is the coefficient of finesse[2], and ϕ is the one-way phase detuning of the cavity, which is related to cavity length x as $\phi = (2\pi/\lambda) x$.

For a given power circulating in the cavity, the radiation pressure force due to the intracavity power on each of the mirrors is $f = 2P/c$. We can find the spring constant by taking the derivative:

$$k \equiv -\frac{\partial f}{\partial x} = -\frac{\partial}{\partial x}\frac{2P}{c} = -\frac{2}{c}\frac{\partial \phi}{\partial x}\frac{\partial P}{\partial \phi}$$

Working out the derivative, we find:

[1] This is the longitudinal optical spring; the angular optical spring arises due to other effects.

[2] The finesse ($\mathcal{F}$) is related to the coefficient of finesse (F) via $\mathcal{F} \approx \frac{\pi}{2}\sqrt{F}$.

$$\frac{\partial}{\partial \phi} P_+ = -2Fg^2 \frac{\cos(\phi)\sin(\phi)}{\left(1+F\sin^2\phi\right)^2} P_{IN} \tag{2}$$

$$= -2Fg^2 P_{IN}\phi + O\left(\phi^3\right) \tag{3}$$

Putting it all together, we get:

$$k = 2Fg^2 \left(\frac{2P_{IN}}{c}\right)\left(\frac{2\pi}{\lambda}\right)\frac{\cos(\phi)\sin(\phi)}{\left(1+F\sin^2\phi\right)^2} \tag{4}$$

$$\approx 2Fg^2 \left(\frac{2P_{IN}}{c}\right)\left(\frac{2\pi}{\lambda}\right)\frac{\phi}{\left(1+F\phi^2\right)^2} \tag{5}$$

$$\approx 2Fg^2 \left(\frac{2P_{IN}}{c}\right)\left(\frac{2\pi}{\lambda}\right)\phi + O\left(\phi^3\right) \tag{6}$$

where, of course, $\phi = (2\pi/\lambda)\delta x$, where x is the (one-way) detuning length. If a mirror is displaced by (δx), the spring constant is:

$$k \approx \frac{64\mathcal{F}^2 g^2 P_{IN}}{c\lambda^2}(\delta x)$$

Putting in some numbers for the Enhanced LIGO arms:

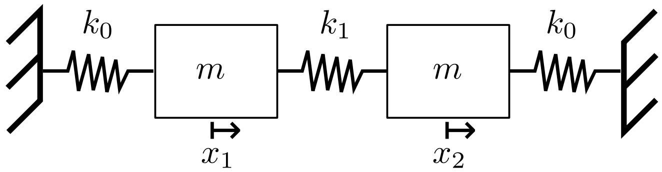
$$\begin{aligned}
\mathcal{F} &= 220 \\
g^2 &= 137 \\
P_{IN} &= 400 \text{ Watts} \\
\lambda &= 1064 \text{ nm} \\
\delta x &= 5 \text{ pm} \\
\hline
k &\approx 624 \text{ N/m}
\end{aligned}$$

It can also be handy to put Eq. 5 into terms of the unitless detuning parameter $\delta_\gamma = \sqrt{F}\phi$, where $\delta_\gamma \equiv \frac{\delta}{\gamma}$, where δ is the cavity detuning (in radians/sec), and γ is the line-width (cavity pole) in the same units. If we further assume that the cavity is strongly-overcoupled, we can use the relations $g^2 = \sqrt{F} = \frac{2}{\pi}\mathcal{F} = 4/T_1$. With these substitutions (and $\lambda = 2\pi c/w_0$), we recover expression (3.14) given in Thomas Corbitt's thesis [1]:

$$K_0 \approx \frac{64 P_{IN} w_0}{T^2 c^2}\frac{\delta_\gamma}{\left(1+\delta_\gamma^2\right)^2} \tag{7}$$

2 Coupled oscillators

Consider a system of two masses, connected to each other via a spring with spring constant k_1 and each connected to the wall via a spring of spring constant k_0. (Later, k_0 will represent the pendula by which the optics are suspended, and k_1 will represent the optical spring.)



By inspection, the equations of motion are:

$$m\ddot{x}_1 = -k_0 x_1 + k_1(x_2 - x_1) \tag{8}$$
$$m\ddot{x}_2 = -k_0 x_2 - k_1(x_2 - x_1) \tag{9}$$

which may be written in matrix form as

$$\ddot{\mathbf{x}} = \frac{1}{m}\begin{bmatrix} -(k_0+k_1) & k_1 \\ k_1 & -(k_0+k_1) \end{bmatrix}\mathbf{x} \tag{10}$$

Because of the form of the matrix[3], we can immediately see that it has eigenvectors corresponding to common and differential motion, with eigenvalues $\{-k_0, -(k_0+2k_1)\}$.

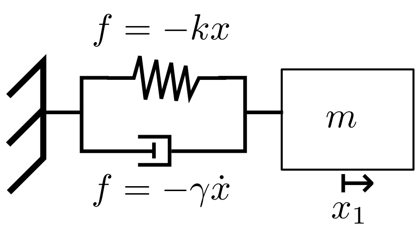
Applying this diagonalization, we find:

$$\ddot{\mathbf{x}}' = \frac{1}{m}\begin{bmatrix} -k_0 & 0 \\ 0 & -(k_0+2k_1) \end{bmatrix}\mathbf{x}' \text{ where } \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\mathbf{x}$$

The presence of the coupling k_1 only affects the differential mode.

3 Damped oscillators

Now consider a mass connected to the wall via a spring with spring constant k and a velocity damper with damping constant γ:



The equation of motion of the mass is:

$$m\ddot{x} = -kx - \gamma\dot{x} + f_{external} \tag{11}$$

with Laplace transform

$$ms^2X = -kX - \gamma sX + F_{external} \tag{12}$$

[3]The matrix $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ has eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with eigenvalues $(a+b)$ and $(a-b)$.

giving rise to a transfer function of

$$\begin{aligned}\frac{X}{F} &= \frac{1}{ms^2+\gamma s+k} \\ &= \left(\frac{1}{m}\right)\frac{1}{(s-s_+)(s-s_-)} \text{ with } s_\pm = -\frac{1}{2}\frac{\gamma}{m} \pm \frac{1}{2}\sqrt{\left(\frac{\gamma}{m}\right)^2 - 4\frac{k}{m}}\end{aligned}$$

4 Optickle

In the attached Matlab code, I construct a very simple model in Optickle[2] consisting of only a laser and two mirrors, forming a resonant cavity. Optickle is also supplied with the mechanical (force to position) transfer functions of each optic (in isolation). Together these transfer functions compose the *reaction matrix*, which is diagonal in the sense that force applied to one optic only affects the position of that same optic.

After constructing the model, we call Optickle via the tickle function:

```
[fDC, sigDC, sigAC, mMech, noiseAC, noiseMech] = tickle(opt, pos, f);
```

The output we are interested in here is "`mMech`," which gives the modifications to the mechanical transfer functions due to the radiation pressure couplings. Its units are "meters per meter"–a bit perplexing at first but some intuition may be gained by considering it as (meters in the presense of radiation pressure)/(meters in the absense of radiation pressure). A more sensible approach is to multiply `mMech` with the reaction matrix, which gives the mechanical transfer functions (from force applied at some optic to displacement of every other optic) including all optomechanical couplings.

To extract the optical spring transfer function, I further transform `mMech` to the basis of common and differential degrees of freedom.

Because we know the mass, resonant frequency, and damping coefficient (and thus also the equivalent spring constant) of the mechanical transfer functions supplied to Optickle, we can compute the expected differential mode transfer function in the presence of radiation pressure by adjusting the spring constant from k to $k + 2k_{opt}$ where k_{opt} is the calculated optical spring constant, or simply shifting the resonance to $w' = w - 2k_{opt}/m$.

This approach shows good agreement (see Figure 1 on the following page) with the results returned by Optickle with the caveat that I've omitted that factor of two and am instead using $k' = k + k_{opt}$. I currently believe this is an error in Optickle.

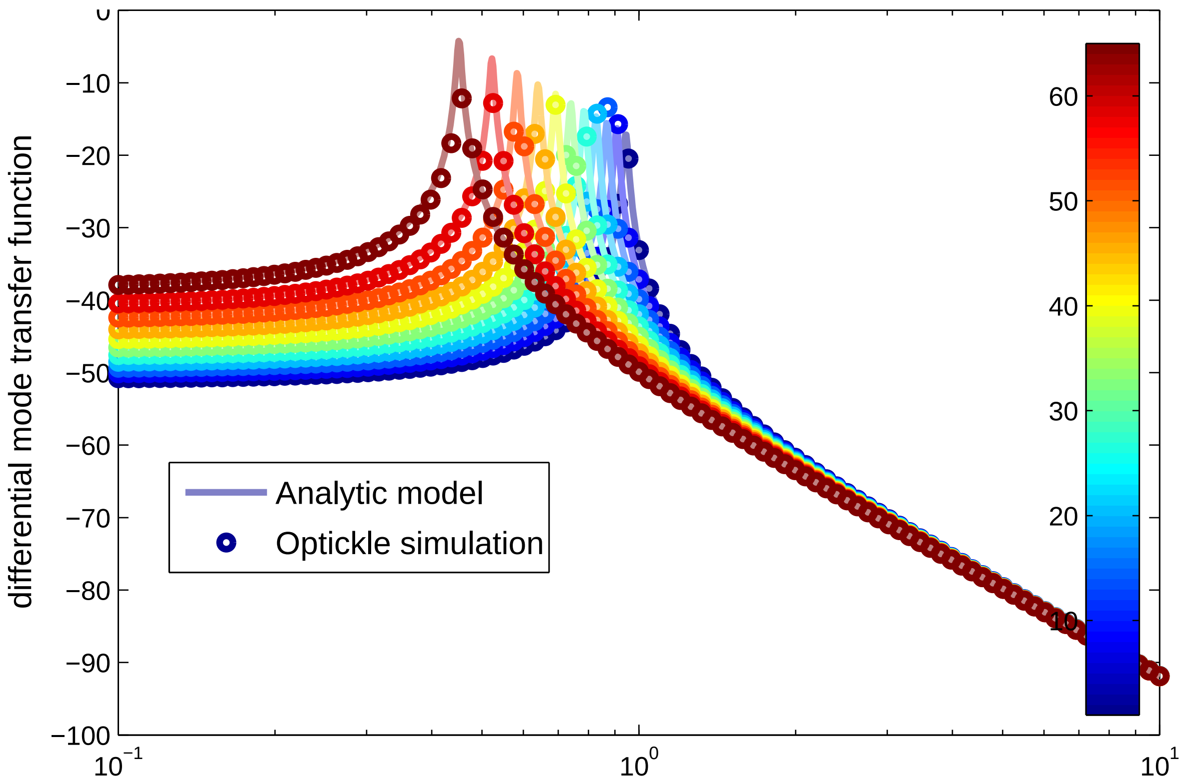


Figure 1: Comparison of the Optickle model with the analytically-derived results. **Note:** the excellent agreement only occurs when an unphysical factor of two is put into the analytic form of the optical spring constant! The units of the y-axis are $20\log_{10}(\text{displacement}/\text{force})$; the x-axis is Hz; color indicates cavity detuning in picometers.

References

[1] Thomas R. Corbitt. *Quantum Noise and Radiation Pressure Effects in High Power Optical Interferometers*. PhD thesis, Massachusetts Institute of Technology, August 2008. Available from: `http://hdl.handle.net/1721.1/45452`.

[2] Matthew Evans. Optickle. Technical report, October 2007. Available from: `https://dcc.ligo.org/cgi-bin/private/DocDB/ShowDocument?docid=27900`.