YEAR **10**

CambridgeMATHS NSW

STAGE 5.1 / 5.2 / 5.3

SECOND EDITION

STUART PALMER, DAVID GREENWOOD
SARA WOOLLEY, JENNY GOODMAN
JENNIFER VAUGHAN

INCLUDES INTERACTIVE TEXTBOOK POWERED BY CAMBRIDGE HOTMATHS

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About the authors

Stuart Palmer was born and educated in NSW. He is a high school mathematics teacher with more than 25 years' experience teaching students from all walks of life in a variety of schools. He has been a head of department in two schools and is now an educational consultant who conducts professional development workshops for teachers all over NSW and beyond. He also works with pre-service teachers at The University of Sydney.

David Greenwood is the Head of Mathematics at Trinity Grammar School in Melbourne and has over 20 years' experience teaching mathematics from Years 7 to 12. He has run workshops within Australia and overseas regarding the implementation of the Australian Curriculum and the use of technology for the teaching of mathematics. He has written more than 30 mathematics titles and has a particular interest in the sequencing of curriculum content and working with the Australian Curriculum proficiency strands.

Sara Woolley was born and educated in Tasmania. She completed an Honours degree in Mathematics at the University of Tasmania before completing her education training at the University of Melbourne. She has taught mathematics in Victoria from Years 7 to 12 since 2006, has written more than 10 mathematics titles and specialises in lesson design and differentiation.

Jenny Goodman has worked for over 20 years in comprehensive State and selective high schools in NSW and has a keen interest in teaching students of differing ability levels. She was awarded the Jones medal for education at Sydney University and the Bourke prize for Mathematics. She has written for Cambridge NSW and was involved in the Spectrum and Spectrum Gold series.

Jennifer Vaughan has taught secondary mathematics for over 30 years in NSW, WA, Queensland and New Zealand and has tutored and lectured in mathematics at Queensland University of Technology. She is passionate about providing students of all ability levels with opportunities to understand and to have success in using mathematics. She has taught special needs students and has had extensive experience in developing resources that make mathematical concepts more accessible; hence, facilitating student confidence, achievement and an enjoyment of math

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Introduction and guide to this book

The **second edition** of this popular resource features a new interactive digital platform powered by Cambridge HOTmaths, together with improvements and updates to the textbook, and additional online resources such as video demonstrations of all the worked examples, Desmos-based interactives, carefully chosen HOTmaths resources including widgets and walkthroughs, and worked solutions for all exercises, with access controlled by the teacher. The Interactive Textbook also includes the ability for students to complete textbook work, including full working-out online, where they can self-assess their own work and alert teachers to particularly difficult questions. Teachers can see all student work, the questions that students have 'red-flagged', as well as a range of reports. As with the first edition, the complete resource is structured on detailed teaching programs for teaching the NSW Syllabus, now found in the Online Teaching Suite.

The chapter and section structure has been retained, and remains based on a logical teaching and learning sequence for the syllabus topic concerned, so that chapter sections can be used as ready-prepared lessons. Exercises have questions graded by level of difficulty and are grouped according to the **Working Mathematically components** of the NSW Syllabus, with enrichment questions at the end. Working programs for three ability levels (Building Progressing and Mastering) have been subtly embedded inside the exercises to facilitate the management of differentiated learning and reporting on students' achievement (see page x for more information on the Working Programs). In the second edition, the *Understanding* and *Fluency* components have been combined, as have *Problem-Solving* and *Reasoning*. This has allowed us to better order questions according to difficulty and better reflect the interrelated nature of the Working Mathematically components, as described in the NSW Syllabus.

Topics are aligned exactly to the NSW Syllabus, as indicated at the start of each chapter and in the teaching program, except for topics marked as:

- REVISION — prerequisite knowledge
- EXTENSION — goes beyond the Syllabus
- FRINGE — topics treated in a way that lies at the edge of the Syllabus requirements, but which provide variety and stimulus.

See the Stage 5 books for their additional curriculum linkage.

The parallel ***CambridgeMATHS Gold*** series for Years 7–10 provides resources for students working at Stages 3, 4, and 5.1. The two series have a content structure designed to make the teaching of mixed ability classes smoother.

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Guide to the working programs

It is not expected that any student would do every question in an exercise. The print and online versions contain working programs that are subtly embedded in every exercise. The suggested working programs provide three pathways through each book to allow differentiation for Building, Progressing and Mastering students.

Each exercise is structured in subsections that match the Working Mathematically strands, as well as Enrichment (Challenge).

The questions suggested for each pathway are listed in three columns at the top of each subsection:

- The left column (lightest-shaded colour) is the Building pathway
- The middle column (medium-shaded colour) is the Progressing pathway
- The right column (darkest-shaded colour) is the Mastering pathway.

	Building	Progressing	Mastering
UNDERSTANDING AND FLUENCY	1–3, 4, 5	3, 4–6	4–6
PROBLEM-SOLVING AND REASONING	7, 8, 11	8–12	8–13
ENRICHMENT	—	—	14

Gradients within exercises and question subgroups

The working programs make use of the gradients that have been seamlessly integrated into the exercises. A gradient runs through the overall structure of each exercise, where there is an increasing level of mathematical sophistication required in the Problem-solving and Reasoning group of questions than in the Understanding and Fluency group, and within each group the first few questions are easier than the last.

The right mix of questions

Questions in the working programs are selected to give the most appropriate mix of *types* of questions for each learning pathway. Students going through the Building pathway will likely need more practice at Understanding and Fluency but should also attempt the easier Problem-Solving and Reasoning questions.

Choosing a pathway

There are a variety of ways of determining the appropriate pathway for students through the course. Schools and individual teachers should follow the method that works for them if the chapter pre-tests can be used as a diagnostic tool.

For classes grouped according to ability, teachers may wish to set one of the Building, Progressing or Mastering pathways as the default setting for their entire class and then make individual alterations, depending on student need. For mixed-ability classes, teachers may wish to set a number of pathways within the one class, depending on previous performance and other factors.

The nomenclature used to list questions is as follows:

- 3, 4: complete all parts of questions 3 and 4
- 1–4: complete all parts of questions 1, 2, 3 and 4
- 10(½): complete half of the parts from question 10 (a, c, e … or b, d, f, …)
- 2–4(½): complete half of the parts of questions 2, 3 and 4
- 4(½), 5: complete half of the parts of question 4 and all parts of question 5
- —: do not complete any of the questions in this section.

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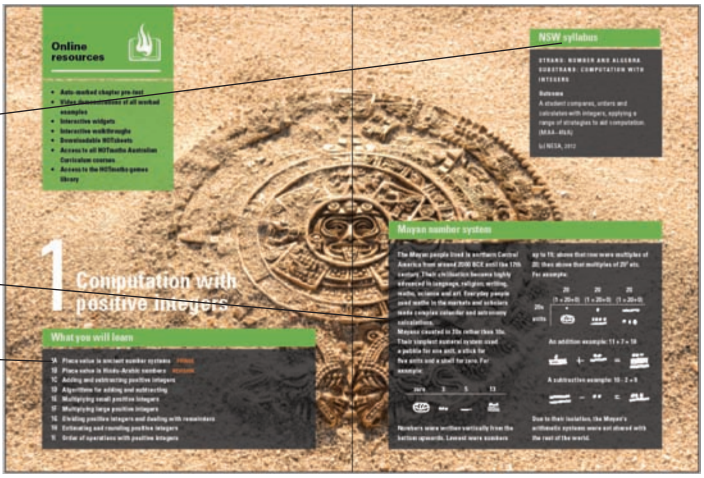
Guide to this book

Features:

NSW Syllabus: strands, substrands and content outcomes for chapter (see teaching program for more detail)

Chapter introduction: use to set a context for students

What you will learn: an overview of chapter contents



Pre-test: establishes prior knowledge (also available as an auto-marked quiz in the Interactive Textbook as well as a printable worksheet)

Pre-test

2 Which of the following is *not* equivalent to one whole?

A $\frac{2}{2}$ B $\frac{6}{6}$ C $\frac{1}{4}$ D $\frac{12}{12}$

3 Which of the following is *not* equivalent to one-half?

Topic introduction: use to relate the topic to mathematics in the wider world

Let's start: an activity (which can often be done in groups) to start the lesson

5A Describing probability

Often, there are times when you may wish to describe how likely it is that an event will occur. For example, you may want to know how likely it is that it will rain tomorrow, or how likely it is that your sporting team will win this year's premiership, or how likely it is that you will win a lottery. Probability is the study of chance.

Let's start: Likely or unlikely?

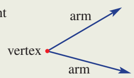
Try to rank these events from least likely to most likely. Compare your answers with other students in the class and discuss any differences.

- It will rain tomorrow.
- Australia will win the soccer World Cup.
- Tails landing uppermost when a 20-cent coin is tossed.

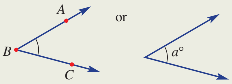
Key ideas: summarises the knowledge and skills for the lesson

Key ideas

- When two rays (or lines) meet, an angle is formed at the intersection point called the **vertex**. The two rays are called **arms** of the angle.



- An **angle** is named using three points, with the vertex as the middle point. A common type of notation is $\angle ABC$ or $\angle CBA$. The measure of the angle is $a°$, where a represents an unknown number.



Examples: solutions with explanations and descriptive titles to aid searches. Video demonstrations of every example are included in the Interactive Textbook.

Example 1 Using measurement systems

a How many feet are there in 1 mile, using the Roman measuring system?
b How many inches are there in 3 yards, using the imperial system?

SOLUTION	EXPLANATION
a 1 mile = 1000 paces = 5000 feet	There are 1000 paces in a R in a pace.
b 3 yards = 9 feet = 108 inches	There are 3 feet in an impe in a foot.

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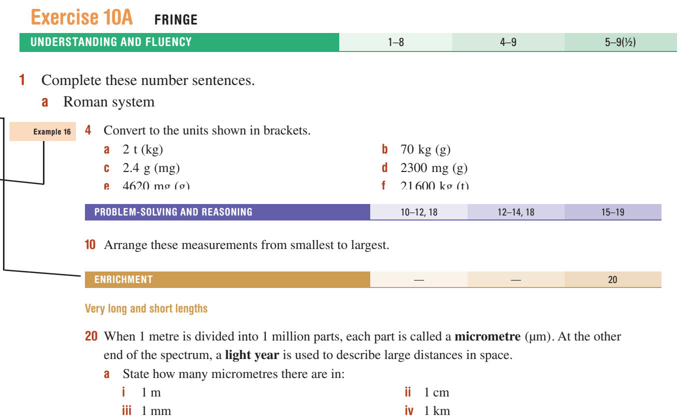
Exercise questions categorised by the **working mathematically components** and **enrichment**

Example references link exercise questions to worked examples.


Exercise 10A FRINGE

UNDERSTANDING AND FLUENCY	1–8	4–9	5–9(½)

1 Complete these number sentences.
a Roman system

Example 16 4 Convert to the units shown in brackets.
a 2 t (kg) b 70 kg (g)
c 2.4 g (mg) d 2300 mg (g)
e 4620 mg (g) f 21 600 kg (t)

PROBLEM-SOLVING AND REASONING	10–12, 18	12–14, 18	15–19

10 Arrange these measurements from smallest to largest.

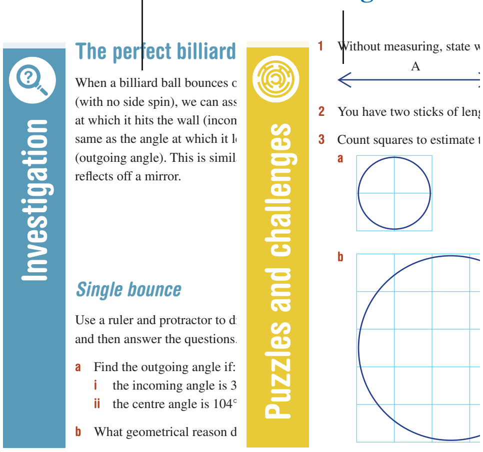
ENRICHMENT	—	—	20

Very long and short lengths

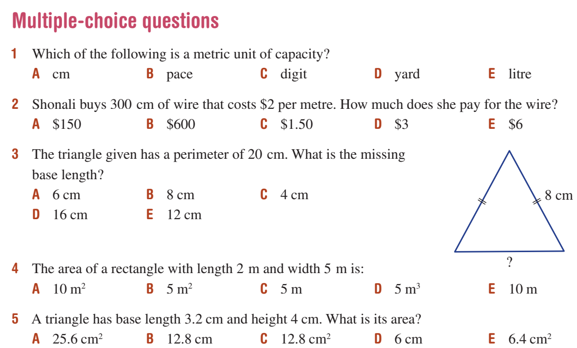
20 When 1 metre is divided into 1 million parts, each part is called a **micrometre** (μm). At the other end of the spectrum, a **light year** is used to describe large distances in space.
a State how many micrometres there are in:
i 1 m ii 1 cm
iii 1 mm iv 1 km

Investigations: inquiry-based activities

Puzzles and **challenges**



Chapter reviews with **multiple-choice, short-answer** and **extended-response questions**


Multiple-choice questions

1 Which of the following is a metric unit of capacity?
A cm B pace C digit D yard E litre

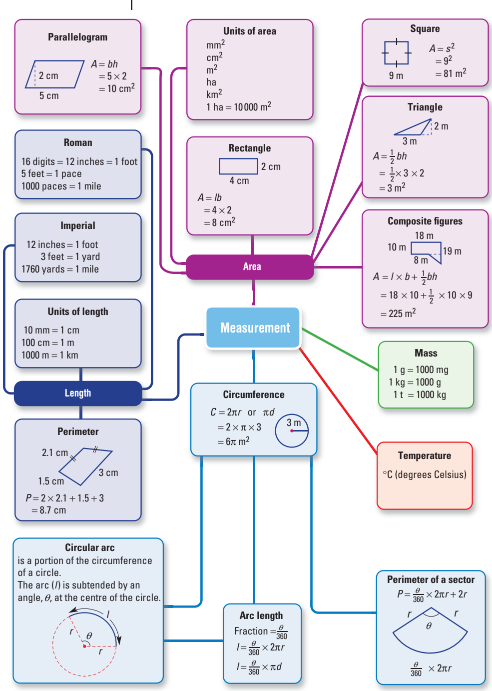
2 Shonali buys 300 cm of wire that costs $2 per metre. How much does she pay for the wire?
A $150 B $600 C $1.50 D $3 E $6

3 The triangle given has a perimeter of 20 cm. What is the missing base length?
A 6 cm B 8 cm C 4 cm
D 16 cm E 12 cm

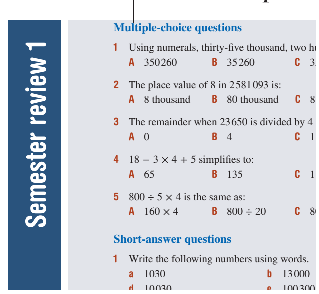
4 The area of a rectangle with length 2 m and width 5 m is:
A 10 m² B 5 m² C 5 m D 5 m³ E 10 m

5 A triangle has base length 3.2 cm and height 4 cm. What is its area?
A 25.6 cm² B 12.8 cm C 12.8 cm² D 6 cm E 6.4 cm²

Chapter summary: mind map of key concepts & interconnections



Two **Semester reviews** per book



Textbooks also include:

- Complete **answers**
- **Index**

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Stage Ladder icons

Shading on the ladder icons at the start of each section indicate the Stage covered by most of that section.

This key explains what each rung on the ladder icon means in practical terms. For more information see the teaching program and teacher resource package:

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Stage	Past and present experience in Stages 4 and 5	Future direction for Stage 6 and beyond
5.3#	These are optional topics which contain challenging material for students who will complete all of Stage 5.3 during Years 9 and 10.	These topics are intended for students who are aiming to study Mathematics at the very highest level in Stage 6 and beyond.
5.3	Capable students who rapidly grasp new concepts should go beyond 5.2 and study at a more advanced level with these additional topics.	Students who have completed 5.1, 5.2 and 5.2 and 5.3 are generally well prepared for a calculus-based Stage 6 Mathematics course.
5.3§	These topics are recommended for students who will complete all the 5.1 and 5.2 content and have time to cover some additional material.	These topics are intended for students aiming to complete a calculus-based Mathematics course in Stage 6.
5.2	A typical student should be able to complete all the 5.1 and 5.2 material by the end of Year 10. If possible, students should also cover some 5.3 topics.	Students who have completed 5.1 and 5.2 without any 5.3 material typically find it difficult to complete a calculus-based Stage 6 Mathematics course.
5.2◊	These topics are recommended for students who will complete all the 5.1 content and have time to cover some additional material.	These topics are intended for students aiming to complete a non-calculus course in Stage 6, such as Mathematics Standard.
5.1	Stage 5.1 contains compulsory material for all students in Years 9 and 10. Some students will be able to complete these topics very quickly. Others may need additional time to master the basics.	Students who have completed 5.1 without any 5.2 or 5.3 material have very limited options in Stage 6 Mathematics.
4	Some students require revision and consolidation of Stage 4 material prior to tackling Stage 5 topics.	

Overview of the digital resources

Interactive Textbook: general features

The **Interactive Textbook (ITB)** is an online HTML version of the print textbook powered by the HOTmaths platform, included with the print book or available separately. (A **downloadable PDF textbook** is also included for offline use). These are its features, including those enabled when the students' ITB accounts are linked to the teacher's **Online Teaching Suite (OTS)** account.

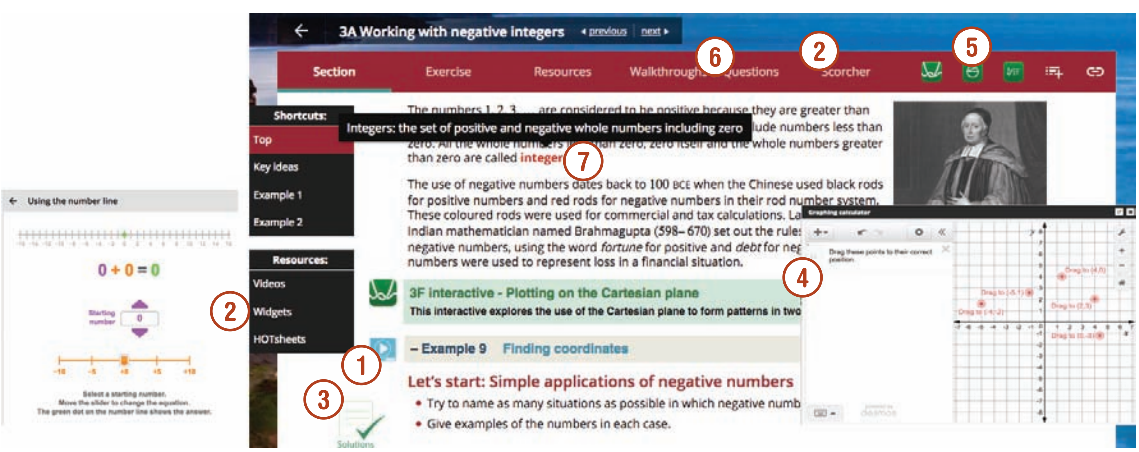
The features described below are illustrated in the screenshot below.

1 Every worked example is linked to a high-quality video demonstration, supporting both in-class learning and the 'flipped classroom'
2 Seamlessly blend with Cambridge HOTmaths, including hundreds of interactive widgets, walkthroughs and games and access to Scorcher
3 **Worked solutions** are included and can be enabled or disabled in the student accounts by the teacher
4 **Desmos interactives** based on embedded graphics calculator and geometry tool windows demonstrate key concepts and enable students to visualise the mathematics
5 The **Desmos scientific calculator** is also available for students to use (as well as the graphics calculator and geometry tools)
6 Auto-marked practice quizzes in each section with saved scores
7 **Definitions** pop up for key terms in the text, and access to the Hotmaths **dictionary**

Not shown but also included:
- Access to alternative HOTmaths lessons is included, including content from previous year levels.
- Auto-marked pre-tests and multiple-choice review questions in each chapter.

INTERACTIVE TEXTBOOK POWERED BY THE HOTmaths PLATFORM

Note: *HOTmaths platform features are updated regularly.*



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Interactive Textbook: Workspaces and self-assessment tools

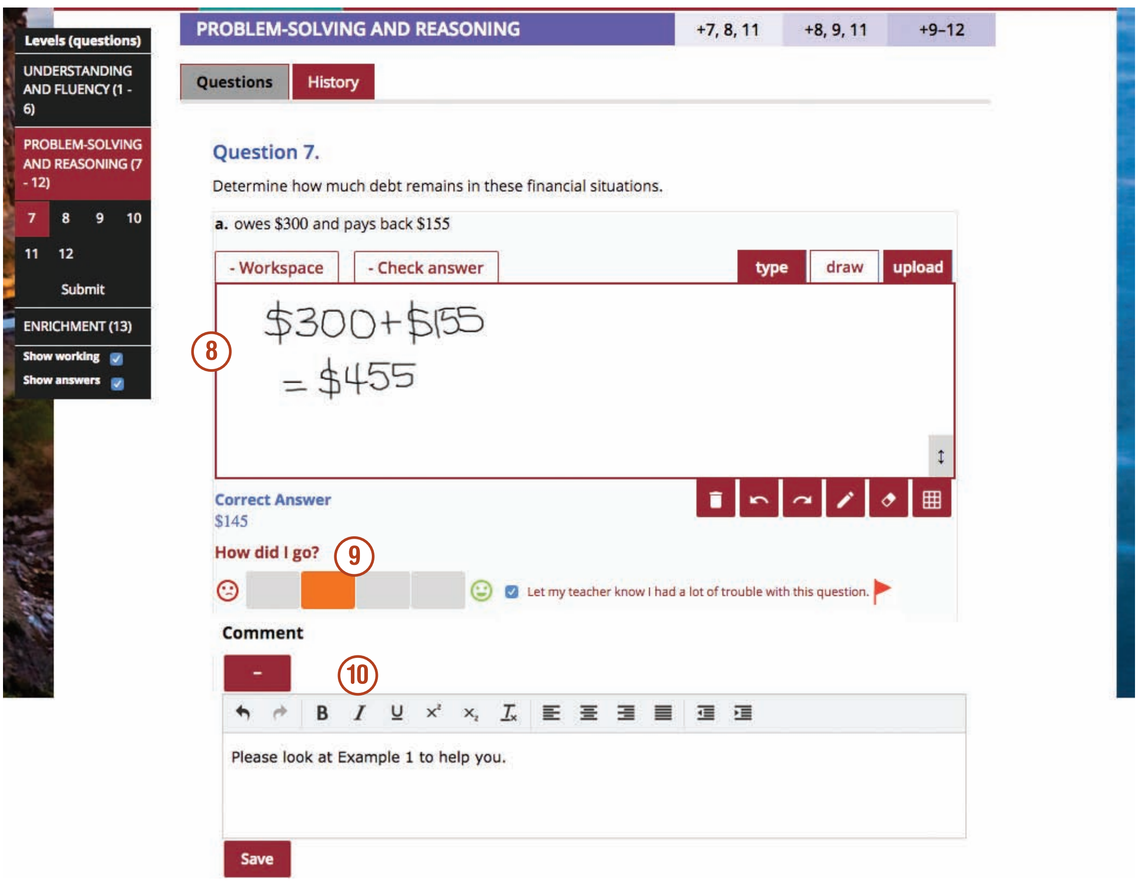
Almost every question in ***CambridgeMATHS NSW Second Edition*** can be completed and saved by students, including showing full working-out and students critically assessing their own work. This is done via the workspaces and self-assessment tools that are found below every question in the Interactive Textbook.

8 The new **Workspaces** enable students to enter working and answers online and to save them. Input is by typing, with the help of a symbol palette, handwriting and drawing on tablets, or by uploding images of writing or drawing.

9 The new **self-assessment tools** enable students to check answers including questions that have been red-flagged, and can rate their confidence level in their work, and alert teachers to questions the student has had particular trouble with. This self-assessment helps develop responsibility for learning and communicates progress and performance to the teacher.

10 Teachers can view the students' self-assessment individually or provide feedback. They can also view results by class.

WORKSPACES AND SELF-ASSESSMENT



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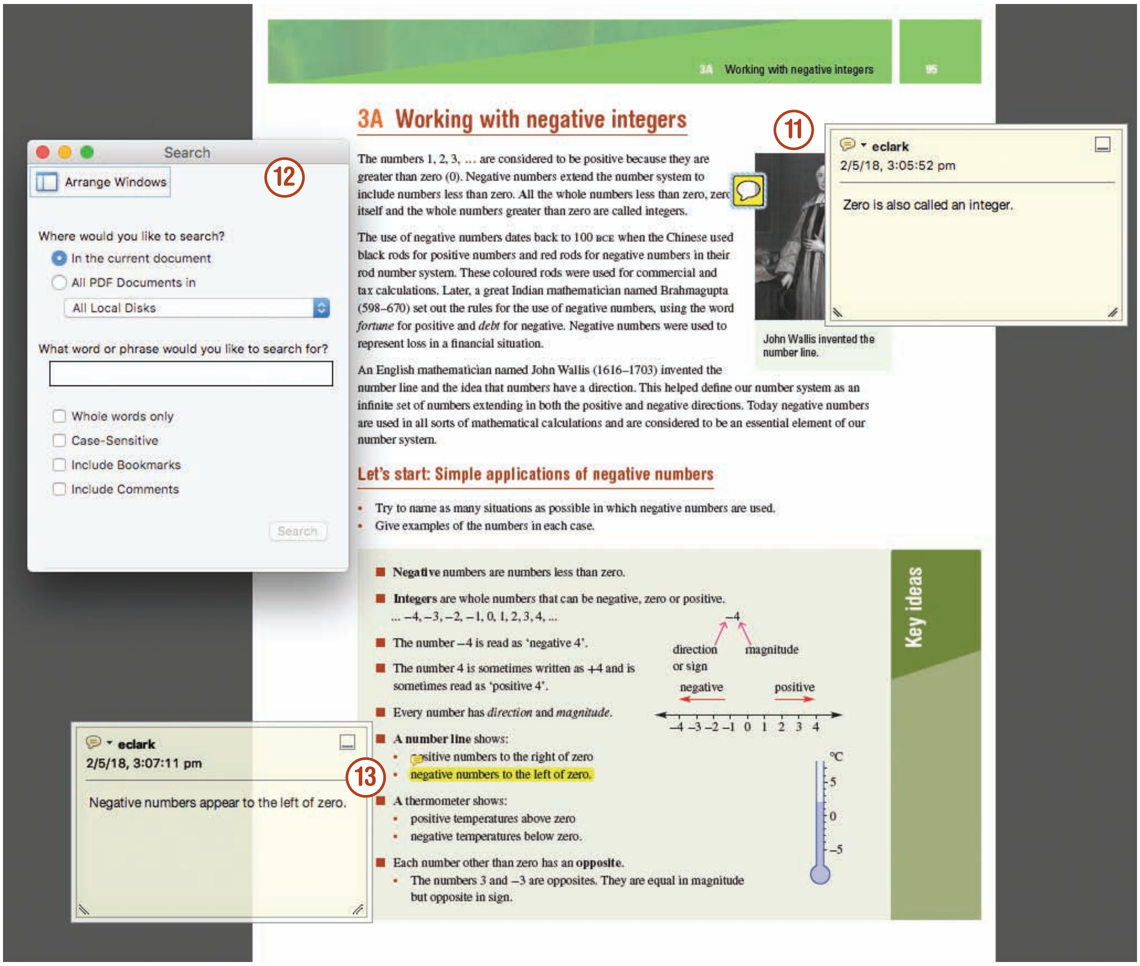
Downloadable PDF Textbook

The convenience of a downloadable PDF textbook has been retained for times when users cannot go online.

The features include:

11 PDF note-taking
12 PDF search features are enabled
13 highlighting functionality.

PDF TEXTBOOK



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Online Teaching Suite

The Online Teaching Suite is automatically enabled with a teacher account and is integrated with the teacher's copy of the Interactive Textbook. All the assets and resources are in one place for easy access.

The features include:

14 The HOTmaths learning management system with class and student analytics and reports, and communication tools

15 Teacher's view of a student's working and self-assessment, including multiple progress and completion reports viewable at both student and class level, as well as seeing the questions that a class has flagged as being difficult

16 A HOTmaths-style test generator

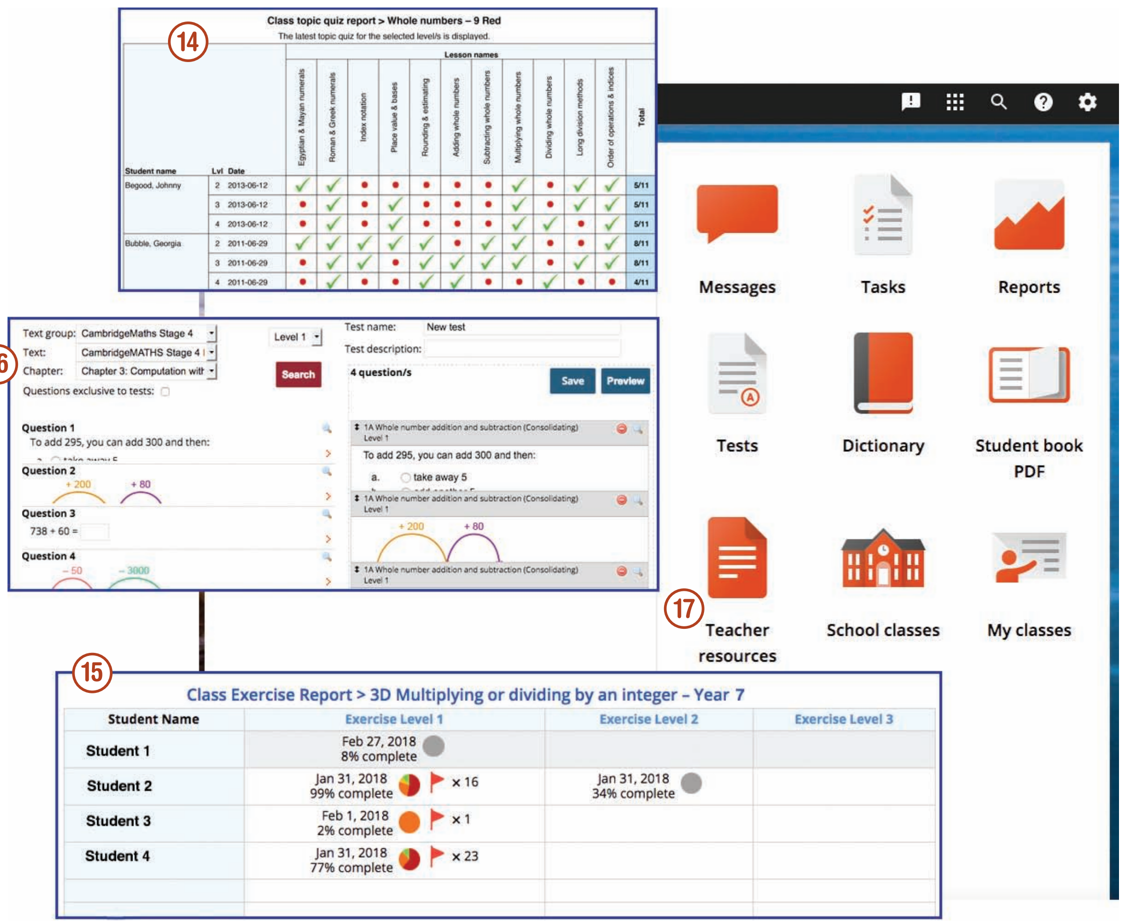
17 Chapter tests and worksheets

Not shown but also available:

- Editable teaching programs and curriculum grids.

ONLINE TEACHING SUITE POWERED BY THE HOTmaths PLATFORM

Note: *HOTmaths platform features are updated regularly.*



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Online resources

- Auto-marked chapter pre-test
- Video demonstrations of all worked examples
- Interactive widgets
- Interactive walkthroughs
- Downloadable HOTsheets
- Access to all HOTmaths Australian Curriculum courses
- Access to the HOTmaths games library

1 Measurement

What you will learn

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NSW syllabus

STRAND: MEASUREMENT AND GEOMETRY

SUBSTRANDS: NUMBERS OF ANY MAGNITUDE

AREA AND SURFACE AREA

VOLUME

Outcomes

A student interprets very small and very large units of measurement, uses scientific notation, and rounds to significant figures.
(MA5.1–9MG)

A student calculates the areas of composite shapes, and the surface areas of rectangular and triangular prisms.
(MA5.1–8MG)

A student calculates the surface areas of right prisms, cylinders and related composite solids.
(MA5.2–11MG)

A student applies formulas to find the surface areas of right pyramids, right cones, spheres and related composite solids.
(MA5.3–13MG)

A student applies formulas to calculate the volumes of composite solids composed of right prisms and cylinders.
(MA5.2–12MG)

A student applies formulas to find the volumes of right pyramids, right cones, spheres and related composite solids.
(MA5.3–14MG)

Tiny lead spheres and a very large glass cone

Melbourne Central shopping centre houses the historic Coop's Shot Tower, which made spherical lead shot for use as pellets in shotguns. The tower was built around 1890 and owned and run by the Coops family. It is 50 m high or nine storeys and has 327 steps, which were ascended by the shot maker carrying up his heavy load of lead bars. To produce the spherical lead shot, the lead bars were heated until molten and then dropped through a sieve or colander from near the top of the tower. The drops of falling molten lead formed into small spherical balls due to the forces of surface tension. The shot also cooled as it fell and was collected in a water trough at the bottom. Six tonnes of lead shot were produced weekly until 1961.

Above the Shot Tower is the largest glass cone of its type in the world. It is built of steel and glass with a total weight of 490 tonnes. The cone itself has a base diameter of 44 m and a height of 48 m. It reaches 20 storeys high and has 924 glass panes.

1 Evaluate the following.

a 2×100 **b** $5 \div 10^2$ **c** $230 \div 10^2$

d 0.043×100^2 **e** $62\,900 \div 1000$ **f** 1.38×1000^2

2 Evaluate the following.

a $A = l \times b$ when $l = 3$ *and* $b = 7$

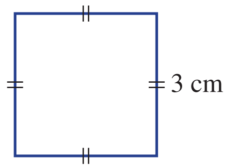
b $A = b \times h$ when $b = 10$ and $h = 3$

c $A = \frac{1}{2}b \times h$ when $b = 2$ and $h = 3.8$

d $A = \frac{1}{2}h(a + b)$ if $a = 2$, $b = 3$ and $h = 4$

3 Find the perimeter of these shapes.

a



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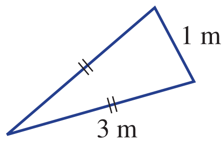

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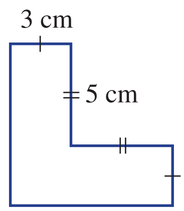
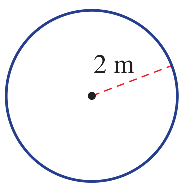


4 Use these rules to find the circumference (C) and area (A) of the circles, correct to 1 decimal place: $C = 2\pi r$ and $A = \pi r^2$, where r is the radius.

a



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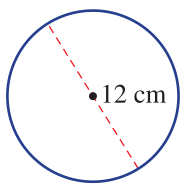

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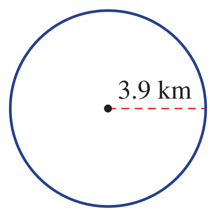
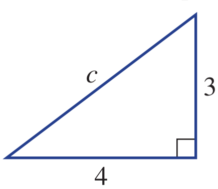


5 Use Pythagoras' theorem $a^2 + b^2 = c^2$ to find the value of the pronumeral in these triangles. Round to 1 decimal place where necessary.

a



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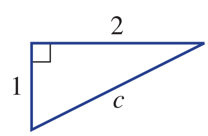
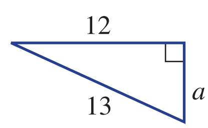
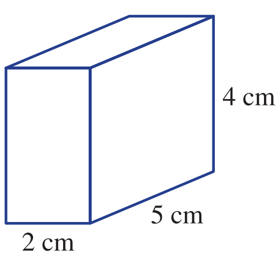


c



6 Find the surface area and volume of these solids. Round to 1 decimal place where necessary.

a



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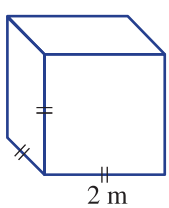

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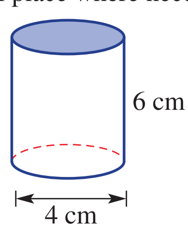


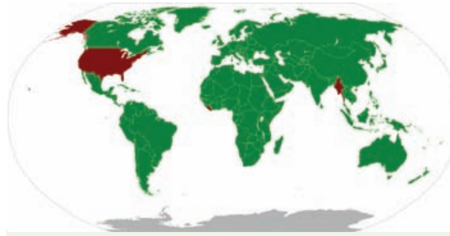
ISBN 978-1-108-46847-3

1A Converting units of measurement

Interactive
Widgets
HOTsheets
Walkthrough

The international system of units (SI) is from the French Système International d'Unités that has been adapted by most countries since its development in the 1960s. It is called the Metric system.

As our technology expands and our knowledge of the universe widens, the need increases for even larger and smaller units of measurements to exist. The common prefixes of milli $\left(\frac{1}{1000}\text{th}\right)$, centi $\left(\frac{1}{100}\text{th}\right)$ and kilo (1000) are joined by others. Giga (10^9) and nano (10^{-9}) are just two of the prefixes added to our everyday usage.



A world map showing countries who have adopted the Metric system. The only countries not to have adopted the system are the United States, Myanmar and Liberia.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Powers of 10

In your workbook, write each of the following as a power of 10. The first line has been done for you.

$10 = 10^1$	$0.1 = 10^{-1}$
100	0.01
1000	0.001
10 000	0.0001
100 000	0.00001
1 000 000	0.000001
10 000 000	0.0000001
100 000 000	0.00000001
1 000 000 000	0.000000001

Key ideas

- Metric prefixes in everyday use

Prefix	Symbol	Factor of 10	Standard form	
tera	T	1 000 000 000 000	10^{12}	1 trillion
giga	G	1 000 000 000	10^9	1 billion
mega	M	1 000 000	10^6	1 million
kilo	k	1000	10^3	1 thousand
hecto	h	100	10^2	1 hundred
deca	da	10	10	1 ten
deci	d	0.1	10^{-1}	1 tenth
centi	c	0.01	10^{-2}	1 hundredth
milli	m	0.001	10^{-3}	1 thousandth
micro	μ	0.000001	10^{-6}	1 millionth
nano	n	0.000000001	10^{-9}	1 billionth

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Example 1 Converting units of time

Convert 3 minutes to:

a microseconds (μs) **b** nanoseconds (ns)

SOLUTION	EXPLANATION
a 3 minutes $= 180$ s	1 min $= 60$ s
$= 180 \times 10^6$ μs	1 s $= 1\,000\,000$ μs
$= 1.8 \times 10^8$ μs	Express the answer in scientific notation.
b 3 minutes $= 180$ s	1 min $= 60$ s
$= 180 \times 10^9$ ns	1 s $= 1\,000\,000\,000 (10^9)$ ns
$= 1.8 \times 10^{11}$ ns	Express the answer in scientific notation.

Example 2 Converting units of mass

Convert $4\,000\,000\,000$ milligrams (mg) to tonnes (t).

SOLUTION	EXPLANATION
$4\,000\,000\,000$ mg $= 4\,000\,000$ g	mg means milligrams 1000 mg $= 1$ g
$= 4000$ kg	1000 g $= 1$ kg
$= 4$ t	1000 kg $= 1$ t
	Dividing by the conversion factor converts a small unit to a larger unit, as there are fewer of them.

Exercise 1A

UNDERSTANDING AND FLUENCY	1–5	4, 5	5

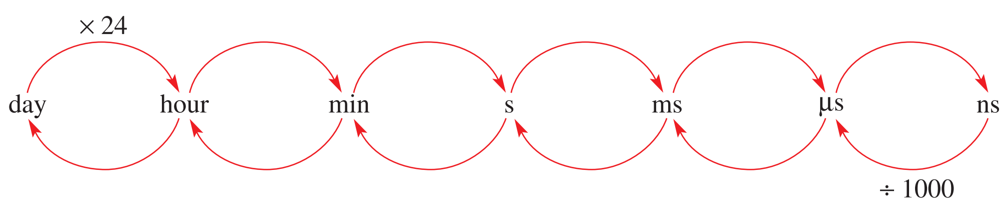
1 What is meant by each of the following abbreviations?

a ms **b** mm **c** km **d** Mt
e Mg **f** mg **g** ns **h** dm

2 Write down what is meant by each of the following expressions.

a an income of 40K
b an 8 GB USB drive
c 2 Mt of sand
d 3 ns

3 Complete this flow diagram, which could be used to convert units of time.



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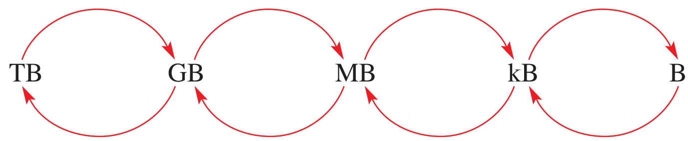
4 Complete this flow diagram, which could be used to convert between measurements of digital information. (Note: B stands for 'byte'.)



Examples 1 & 2

5 Complete these conversions.

a 2000 t = ____________ Mt
b 5 s = ____________ ms
c 5 km = ____________ m
d 20 dm = ____________ m
e 10 hectometres = ____________ km
f 3 MB = ____________ B
g 32 GB = ____________ MB = ____________ B
h 60 mm = ____________ m
i 200 cm = ____________ m
j 5 kg = ____________ g
k 0.06 g = ____________ mg
l $\frac{1}{2}$ t = ____________ g
m 1 s = ____________ milliseconds (ms)
n 1 s = ____________ microseconds (μs)
o 2 terabytes (TB) = ____________ GB
p 35 mg = ____________ g
q 2×10^{30} kg = ____________ Mt
r 1 GHz (gigahertz) = ____________ Hz
s 1 TB = ____________ GB = ____________ MB = ____________ kB
t 4 hectometres = ____________ m

PROBLEM-SOLVING AND REASONING	6, 7, 12	6–9, 12, 13	8–15

6 The Hatton family have a monthly internet and pay TV 5 GB home bundle, costing $219.12 per month.
a What is the cost over the 24 month contract?
b What is the monthly cost for the bundle per gigabyte?
c What is the average daily allowance for this bundle?

7 Olympic swimmer Michael Phelps won a gold medal at the 2012 London Olympic Games for the 200 m medley in a time of 1:54.27. What does this mean and what is this time when converted to milliseconds?

8 In the Olympic Games, the time-keeping devices are said to measure correctly to the nearest thousandth of a second. Which SI prefix is used for this purpose?

9 How many milliseconds are there in one day? Express this value in scientific notation.

10 The chemical element copernicium 277 (^{277}Cn) has a half-life of 240 microseconds. How many seconds is this?

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11 The average human eye blink takes 350000 microseconds. How many times can the average person blink consecutively, at this rate, in 1 minute?

12 How many times greater is the prefix M to the prefix m?

13 Complete this comparison:

1 microsecond is to 1 second as what 1 second is to _______ days.

14 Stored on her computer, Hannah has photos of her recent weekend away. She has filed them according to various events. The files have the following sizes: 1.2 MB, 171 KB, 111 KB, 120 KB, 5.1 MB and 2.3 MB. (Note that some computers use KB instead of kB in their information on each file.)

a What is the total size of the photos of her weekend, in kilobytes?

b What is the total in megabytes?

c Hannah wishes to email these photos to her mum. However, her mum's file server can only receive email attachments no bigger than 8 MB. Is it possible for Hannah to send all of her photos from the weekend in one email?

15 Complete the tables for these time conversion scales.

a 1 second is equivalent to:

millisecond	
microsecond	
nanosecond	
minute	
hour	
day	
week	
month	
year	
century	
millennium	

b A year containing 365 days is equivalent to:

millisecond	
microsecond	
nanosecond	
minute	
hour	
day	
week	
month	
year	
century	
millennium	

ENRICHMENT	–	–	16

Half-life

16 The half-life of an isotope (a form) of a chemical element is the time taken for half of its atoms to decay into another form. Different isotopes have different half-lives. They can range from billions of years to microseconds. The table shows some isotopes and their half-lives.

a Place the isotopes listed in the table in order from the shortest half-life to the longest.

b How many times greater is the half-life of ^{24}Na to that of:

i ^{216}Po?

ii ^{15}O?

c If, initially, there is 20 mg of gold-198 (^{198}Au), how many days does it take for less than 1 mg to remain?

d Research half-lives for other isotopes of elements in the periodic table.

Isotope	Half-life
argon-41 (^{41}Ar)	1.827 hours
barium-142 (^{142}Ba)	10.6 minutes
calcium-41 (^{41}Ca)	130000 years
carbon-14 (^{14}C14)	5730 years
chromium-51 (^{51}Cr)	27.704 days
gold-198 (^{198}Au)	2.696 days
hydrogen-3 (^{3}H)	12.35 years
oxygen-15 (^{15}O)	122.24 seconds
polonium-213 (^{213}Po)	4.2 microseconds
polonium-216 (^{216}Po)	0.15 seconds
sodium-24 (^{24}Na)	15 hours
uranium-232 (^{232}U)	72 years
zinc-69 (^{69}Zn)	57 minutes

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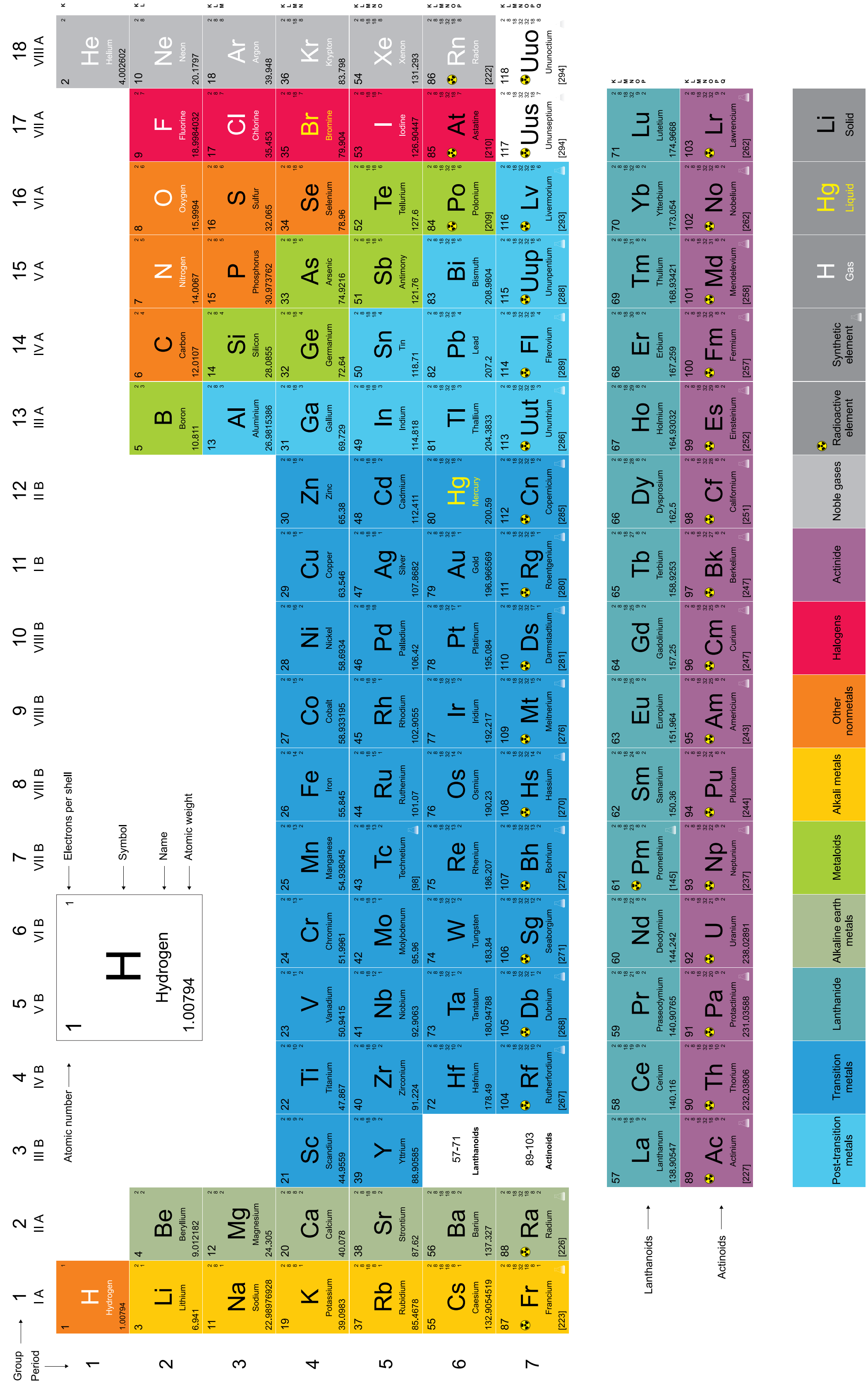
PERIODIC TABLE OF THE ELEMENTS



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1B Accuracy of measuring instruments

Interactive
Widgets
HOTsheets
Walkthrough

Humans and machines measure many different things, such as the time taken to swim a race, the length of timber needed for a building and the volume of cement needed to lay a concrete path around a swimming pool. The degree or level of accuracy required usually depends on the intended purpose of the measurement.

For example, using a ruler marked in millimetres is accurate enough for building a house, but not accurate enough for measuring microscopic objects.

During many sporting events, a high degree of accuracy is required for time-keeping.

Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Rounding a decimal

1 A piece of timber is measured to be 86 cm, correct to the nearest centimetre.
- **a** What is the smallest decimal that it could be rounded from?
- **b** What is the largest decimal that is recorded as 86 cm when rounded to the nearest whole?

2 If a measurement is recorded as 6.0 cm, correct to the nearest millimetre, then:
- **a** What units were used when measuring?
- **b** What is the smallest decimal that could be rounded to this value?
- **c** What is the largest decimal that would have resulted in 6.0 cm?

3 Consider a square with sides of length 7.8941 cm.
- **a** What is the perimeter of the square if the side length is:
  - **i** left with 4 decimal places?
  - **ii** rounded to 1 decimal place?
  - **iii** truncated (i.e. chopped off) at 1 decimal place?
- **b** What is the difference between the perimeters if the decimal is rounded to 2 decimal places or truncated at 2 decimal places or written with two significant figures?

Key ideas

- All measurements are approximate. Errors can happen as a result of the equipment being used or the person using the measuring device.
- **Accuracy** is the measure of how true the measure is to the 'real' one, whereas **precision** is the ability to obtain the same result over and over again
- The limits of accuracy tell you what the upper and lower boundaries are for the measured value. The limits of accuracy depend on the instrument used for the measurement.
  - The limits of accuracy are given by:

    Upper limit = Measurement + 0.5 × smallest unit of measurement

    Lower limit = Measurement − 0.5 × smallest unit of measurement

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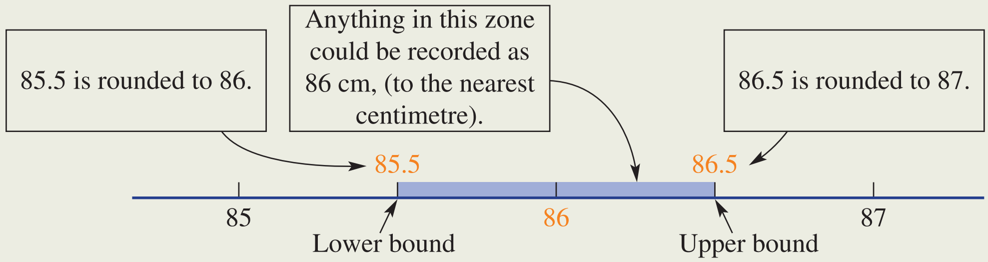
Key ideas

For example, when measuring to the nearest centimetre, 86 cm has limits from 85.5 cm up to 86.5 cm.



Example 3 Finding limits of accuracy

Give the limits of accuracy for these measurements.

a 72 cm **b** 86.6 mm

SOLUTION	EXPLANATION
a $72 \pm 0.5 \times 1$ cm $= 72 - 0.5$ cm to $72 + 0.5$ cm $= 71.5$ cm to 72.5 cm	Smallest unit of measurement is one whole cm. Error $= 0.5 \times 1$ cm This error is subtracted and added to the given measurement to find the limits of accuracy.
b $86.6 \pm 0.5 \times 0.1$ mm $= 86.6 \pm 0.05$ mm $= 86.6 - 0.05$ mm to $86.6 + 0.05$ mm $= 86.55$ mm to 86.65 mm	Smallest unit of measurement is 0.1 mm. Error $= 0.5 \times 0.1$ mm $= 0.05$ mm This error is subtracted and added to the given measurement to find the limits of accuracy.

Example 4 Considering the limits of accuracy

Janis measures each side of a square as 6 cm. Find:

a the upper and lower limits for the sides of the square
b the upper and lower limits for the perimeter of the square
c the upper and lower limits for the square's area

SOLUTION	EXPLANATION
a $6 \pm 0.5 \times 1$ cm $= 6 - 0.5$ cm to $6 + 0.5$ cm $= 5.5$ cm to 6.5 cm	Smallest unit of measurement is one whole cm. Error $= 0.5 \times 1$ cm
b Lower limit $P = 4 \times 5.5 = 22$ cm Upper limit $P = 4 \times 6.5 = 26$ cm	The lower limit for the perimeter uses the lower limit for the measurement taken and the upper limit for the perimeter uses the upper limit of 6.5 cm.
c Lower limit $= 5.5^2 = 30.25$ cm^2 Upper limit $= 6.5^2 = 42.25$ cm^2	The lower limit for the area is 5.5^2, whereas the upper limit will be 6.5^2.

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Exercise 1B

UNDERSTANDING AND FLUENCY	1–5	3, 4–5(½)	4–5(½)

1 Write down 3 decimals that round off to 3.4.

2 Write down a measurement of 3467 mm, correct to the nearest:
a centimetre
b metre

3 What is the smallest decimal that could result in an answer of 6.7 when rounded to 1 decimal place?

Example 3

4 For each of the following:
i Give the smallest unit of measurement (e.g. 0.1 cm is the smallest unit in 43.4 cm).
ii Give the limits of accuracy.
a 45 cm
b 6.8 mm
c 12 m
d 15.6 kg
e 56.8 g
f 10 m
g 673 h
h 9.84 m
i 12.34 km
j 0.987 km

5 Give the limits of accuracy for the following measurements.
a 5 m
b 8 cm
c 78 mm
d 5 ns
e 2 km
f 34.2 cm
g 3.9 kg
h 19.4 kg
i 457.9 t
j 18.65 m
k 7.88 km
l 5.05 s

PROBLEM-SOLVING AND REASONING	6, 7, 12	6–9, 12	8–13

6 What are the limits of accuracy for the amount \$4500 when it is written:
a to two significant figures?
b to three significant figures?
c to four significant figures?

7 Write the following as a measurement, given that the lower and upper limits of these measurements are as follows.
a 29.5 m to 30.5 m
b 140 g to 150 g
c 4.55 km to 4.65 km
d 8.95 km to 9.05 km
e 985 g to 995 g
f 989.5 g to 990.5 g

8 Martha writes down the length of her fabric as 150 cm. As Martha does not give her level of accuracy, give the limits of accuracy of her fabric if it was measured correct to the nearest:
a centimetre
b 10 centimetres
c millimetre

9 A length of copper pipe is given as 25 cm, correct to the nearest centimetre.
a What are the limits of accuracy for this measurement?
b If 10 pieces of copper each with a given length of 25 cm are joined end to end, what is the minimum length that it could be?
c What is the maximum length for the 10 pieces of pipe?

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Example 4

10 The side of a square is recorded as 9.2 cm, correct to two significant figures.
- **a** What is the minimum length that the side of this square could be?
- **b** What is the maximum length that the side of this square could be?
- **c** Find the upper and lower boundaries for this square's perimeter.
- **d** Find the upper and lower limits for the area of this square.

11 The side of a square is recorded as 9.20 cm, correct to three significant figures.
- **a** What is the minimum length that the side of this square could be?
- **b** What is the maximum length that the side of this square could be?
- **c** Find the upper and lower boundaries for this square's perimeter.
- **d** Find the upper and lower limits for the area of this square.
- **e** How has changing the level of accuracy from 9.2 cm (see Question **10**) to 9.20 cm affected the calculation of the square's perimeter and area?

12 Cody measures the mass of an object to be 6 kg. Jacinta says the same object is 5.8 kg and Lachlan gives his answer as 5.85 kg.
- **a** Explain how all three people could have different answers for the same measurement.
- **b** Write down the level of accuracy being used by each person.
- **c** Are all their answers correct? Discuss.

13 Write down a sentence explaining the need to accurately measure items in our everyday lives and the accuracy required for each of your examples. Give three examples of items that need to be measured correct to the nearest:

a kilometre **b** millimetre **c** millilitre **d** litre

ENRICHMENT	–	–	14

Percentage error

To calculate the percentage error of any measurement, the error (i.e. $\pm$ the smallest unit of measurement) is compared to the given or recorded measurement and then converted to a percentage.

For example: 5.6 cm

Error $= \pm 0.5 \times 0.1 = \pm 0.05$

$$\text{Percentage error} = \frac{0.05}{5.6} \times 100\%$$
$$= 0.89\% \text{ (to two siginificant figures)}$$

14 Find the percentage error for each of the following. Round to two significant figures.

a 28 m **b** 9 km **c** 8.9 km **d** 8.90 km

e 178 mm **f** \$8.96 **g** 4.25 ms **h** 700 mL

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1C Pythagoras' theorem in three-dimensional problems

Interactive
Widgets
HOTsheets
Walkthrough

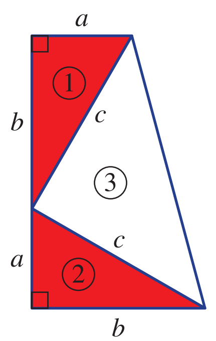
Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

You will recall that for any right-angled triangle we can link the length of the three sides using Pythagoras' theorem. Given two of the sides, we can work out the length of the remaining side. This has applications in all sorts of two- and three-dimensional problems.

US President Garfield discovered a proof of Pythagoras' theorem.

Let's start: President Garfield's proof

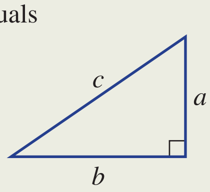
Five years before he became president of the United States of America in 1881, James Garfield discovered a proof of Pythagoras' theorem. It involves arranging two identical right-angled triangles (① and ②) to form a trapezium, as shown.



- Use the formula for the area of a trapezium $A = \frac{1}{2}h(a + b)$ to find an expression for the area of the entire shape.
- Explain why the third triangle ③ is right-angled.
- Find an expression for the sum of the areas of the three triangles.
- Hence, prove $c^2 = a^2 + b^2$.

Key ideas

- **Pythagoras' theorem** states that:
  The sum of the squares of the two shorter sides of a right-angled triangle equals the square of the hypotenuse.
  $a^2 + b^2 = c^2$



- The exact length of a side is sometimes written as $\sqrt{5}$, for example, rather than a non-repeating, never-ending decimal such as 2.236….

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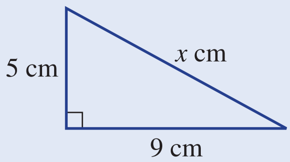
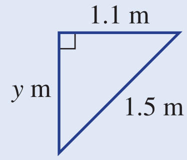
Example 5 Finding side lengths using Pythagoras' theorem

Find the length of the unknown side in these right-angled triangles, correct to 2 decimal places.

a



b



SOLUTION	EXPLANATION
a $c^2 = a^2 + b^2$	x cm is the length of the hypotenuse.
$\therefore x^2 = 5^2 + 9^2$	Substitute the two shorter sides
$= 106$	$a = 5$ and $b = 9$ (or $a = 9$ and $b = 5$).
$\therefore x = \sqrt{106}$	Find the square root of both sides and round your answer as required.
$= 10.2956\ldots$	
The length of the unknown side is 10.30 cm (to 2 decimal places).	
b $a^2 + b^2 = c^2$	Substitute the shorter side $b = 1.1$ and the
$y^2 + 1.1^2 = 1.5^2$	hypotenuse $c = 1.5$.
$y^2 = 1.5^2 - 1.1^2$	Subtract 1.1^2 from both sides.
$= 2.25 - 1.21$	
$= 1.04$	
$\therefore y = \sqrt{1.04}$	Find the square root of both sides and evaluate.
$= 1.0198\ldots$	
The length of the unknown side is 1.02 m (to 2 decimal places).	

For centuries builders, carpenters and landscapers have used Pythagoras' theorem to construct right angles for their foundations and plots. The ancient Egyptians used three stakes joined by a rope to make a triangular shape with side lengths of 3, 4 and 5 units, which forms a right angle when the rope is taut.

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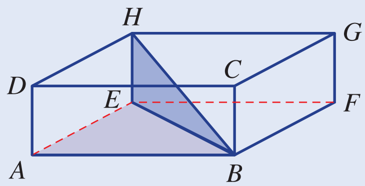
Example 6 Using Pythagoras' theorem in 3D

Consider a rectangular prism $ABCDEFGH$ with the side lengths $AB = 7$, $AE = 4$ and $EH = 2$. Find:

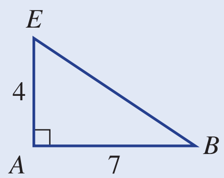
a BE, leaving your answer in exact form

b BH, correct to 2 decimal places



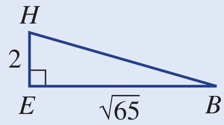
SOLUTION

a



$$c^2 = a^2 + b^2$$
$$\therefore BE^2 = 4^2 + 7^2$$
$$= 65$$
$$\therefore BE = \sqrt{65}$$

b



$$BH^2 = HE^2 + EB^2$$
$$= 2^2 + (\sqrt{65})^2$$
$$= 4 + 65$$
$$= 69$$
$$\therefore BH = \sqrt{69}$$
$$= 8.31 \text{ (to 2 decimal places)}$$

EXPLANATION

Draw the appropriate triangle.

Substitute $a = 4$ and $b = 7$.
Solve for BE exactly.
Leave intermediate answers in surd form to reduce the chance of accumulating errors in further calculations.

Draw the appropriate triangle.

Substitute $HE = 2$ and $EB = \sqrt{65}$.
Note $(\sqrt{65})^2 = \sqrt{65} \times \sqrt{65} = 65$.

Exercise 1C

UNDERSTANDING AND FLUENCY	1–5	2–6	3–4(½), 5, 6(½)

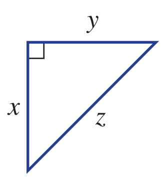
1 Solve for a in these equations, leaving your answer in exact form. Assume $a > 0$.

a $a^2 + 3^2 = 8^2$ **b** $a^2 + 5^2 = 6^2$ **c** $2^2 + a^2 = 9^2$

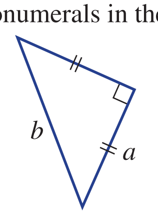
d $a^2 + a^2 = 2^2$ **e** $a^2 + a^2 = 4^2$ **f** $a^2 + a^2 = 10^2$

2 Write an equation connecting the pronumerals in these right-angled triangles.

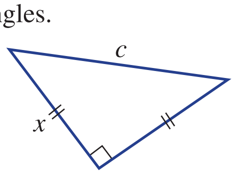
a



b



c



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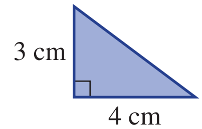
Example 5a

3 Use Pythagoras' theorem to find the length of the hypotenuse for these right-angled triangles. Round your answers to 2 decimal places where necessary.

a


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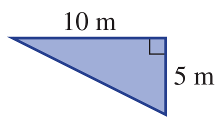

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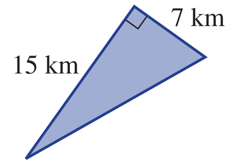

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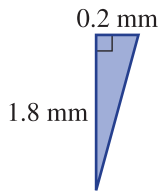
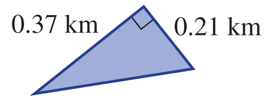


e


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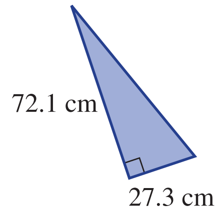
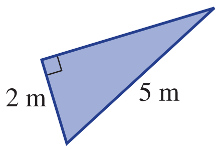


Example 5b

4 Find the length of the unknown side in these right-angled triangles, correct to 2 decimal places.

a


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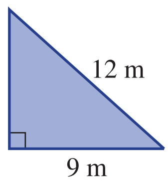

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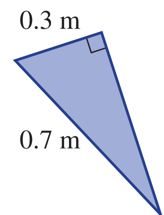

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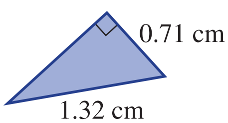

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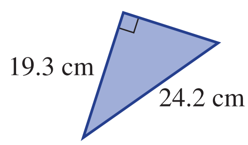

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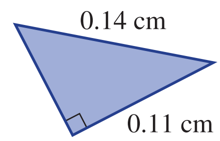
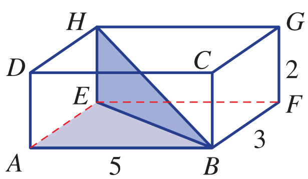


Example 6

5 For each of these rectangular prisms, find:

i *BE*, leaving your answer in exact form

ii *BH*, correct to 2 decimal places

a


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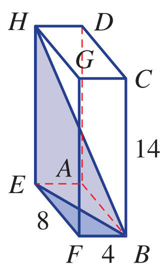

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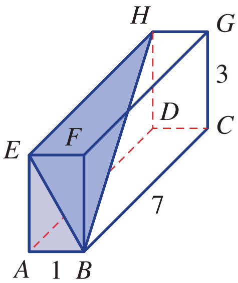

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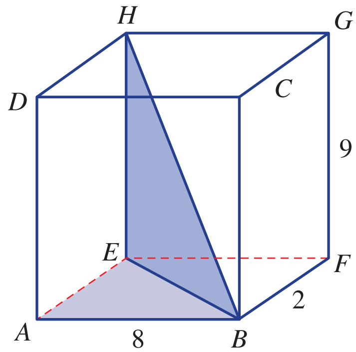

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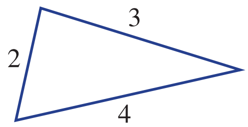
6 Use Pythagoras' theorem to help decide whether these triangles are right angled. They may not be drawn to scale.

a


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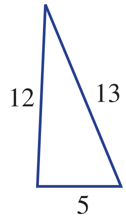

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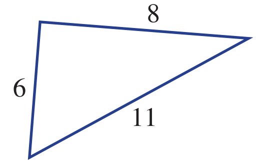

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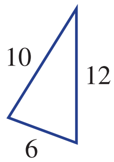

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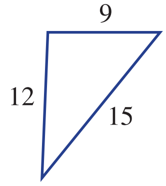

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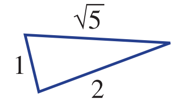
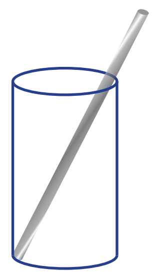
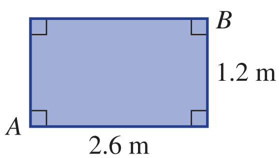


PROBLEM-SOLVING AND REASONING	7, 8, 12	8–10, 12, 13	9(½), 10, 11, 13, 14

7 A 20 cm drinking straw sits diagonally in a glass of radius 3 cm and height 10 cm. What length of straw protrudes from the glass? Round your answer to 1 decimal place.



8 Use Pythagoras' theorem to find the distance between points A and B in these diagrams, correct to 2 decimal places.

a


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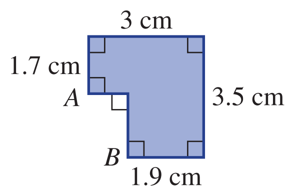

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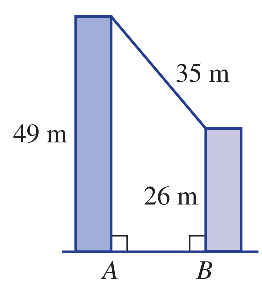

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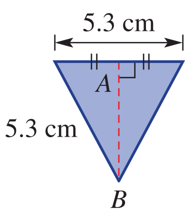

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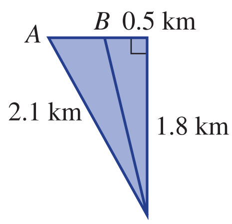

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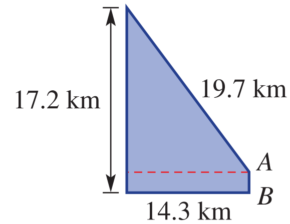

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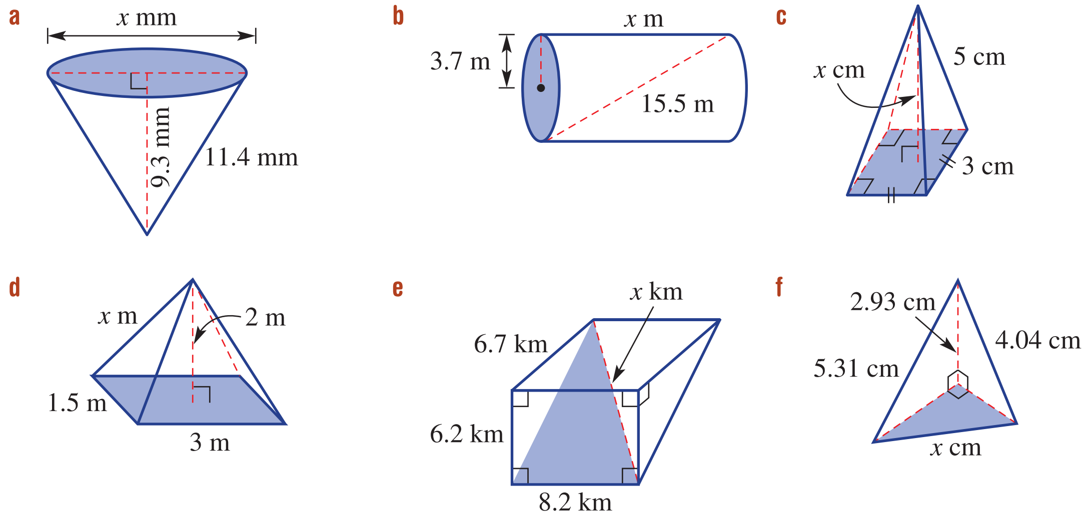
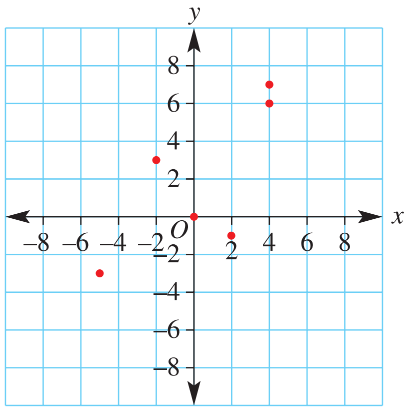
9 Find the value of x, correct to 2 decimal places, in these three-dimensional diagrams.



10 Find the exact distance between these pairs of points on a number plane.



a (0, 0) and (4, 6)
b (−2, 3) and (2, −1)
c (−5, −3) and (4, 7)

11 **a** Find the length of the longest rod that will fit inside these objects. Give your answer correct to 1 decimal place.
  i a cylinder with diameter 10 cm and height 20 cm
  ii a rectangular prism with side lengths 10 cm, 20 cm and 10 cm

b Investigate the length of the longest rod that will fit in other solids, such as triangular prisms, pentagonal prisms, hexagonal prisms and truncated rectangular pyramids. Include some three-dimensional diagrams.

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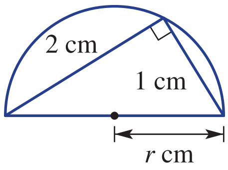
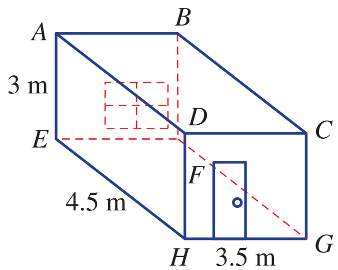
12 Two joining chords in a semicircle have lengths 1 cm and 2 cm, as shown.
Find the exact radius, r cm, of the semicircle.



13 The diagonals of a rectangle are 10 cm long. Find the exact dimensions of the rectangle if:

a the length is twice the width
b the length is three times the width
c the length is 10 times the width

14 Streamers are used to decorate the interior of a room that is 4.5 m long, 3.5 m wide and 3 m high, as shown.



a Find the length of streamer, correct to 2 decimal places, required to connect from:
  i A to H
  ii E to B
  iii A to C
  iv A to G via C
  v E to C via D
  vi E to C directly

b Find the shortest length of streamer required, correct to 2 decimal places, to reach from A to G if the streamer is not allowed to reach across open space.

ENRICHMENT	–	–	15

How many proofs?

15 There are hundreds of proofs of Pythagoras' theorem.

a Research some of these proofs using the internet and pick one you understand clearly.
b Write up the proof, giving full reasons.
c Present your proof to a friend or the class. Show all diagrams, algebra and reasons.

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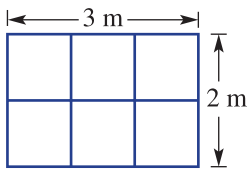
1D Area of triangles, quadrilaterals, circles and sectors REVISION

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Area is a measure of surface and is expressed as a number of square units.

By the inspection of a simple diagram like the one shown below, a rectangle with side lengths 2 m and 3 m has an area of 6 square metres or 6 m^2.



Area = 6 m^2

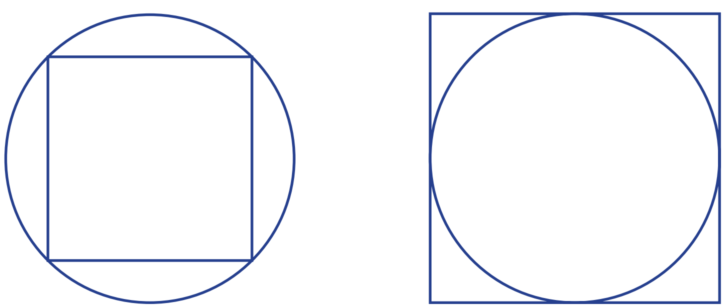
The appropriate unit for the area of a desert is the square kilometre.

For rectangles and other basic shapes, we can use area formulas to help us calculate the number of square units.

Some common metric units for area include square kilometres (km^2), square metres (m^2), square centimetres (cm^2) and square millimetres (mm^2).

Let's start: Pegs in holes

Discuss with reasons relating to the area of the shapes, which is the better fit:
- a square peg in a round hole?
- a round peg in a square hole?



Key ideas

- Conversion of units of area

$$\mathrm{km}^2 \underset{\div 1000^2}{\overset{\times 1000^2}{\rightleftarrows}} \mathrm{m}^2 \underset{\div 100^2}{\overset{\times 100^2}{\rightleftarrows}} \mathrm{cm}^2 \underset{\div 10^2}{\overset{\times 10^2}{\rightleftarrows}} \mathrm{mm}^2$$

$$\text{hectare (ha)} \underset{\div 10\,000}{\overset{\times 10\,000}{\rightleftarrows}} \text{square metre } (\mathrm{m}^2)$$

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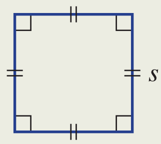
Key ideas

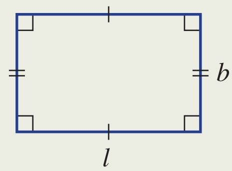
- The area of a two-dimensional shape can be defined as the number of square units contained within its boundaries. Some common area formulas are given.

Square



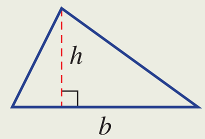
Area $= s^2$

Rectangle



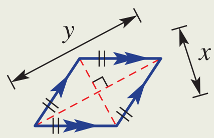
Area $= l \times b$

Triangle



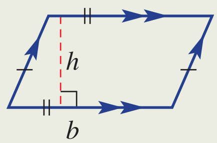
Area $= \frac{1}{2}bh$

Rhombus



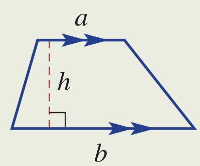
Area $= \frac{1}{2}xy$

Parallelogram



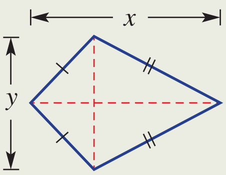
Area $= bh$

Triangle



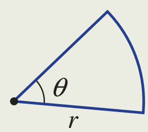
Area $= \frac{1}{2}h(a + b)$

Kite



Area $= \frac{1}{2}xy$

- The formula for the area of a circle is:
  Area $= \pi r^2$, where r is the radius.
- The formula for the area of a sector is $A = \frac{\theta}{360}\pi r^2$.



Example 7 Converting between units of area

Convert these areas to the units shown in the brackets.

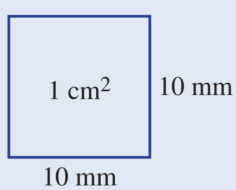
a 2.5 cm^2 (mm^2)

b 2000000 cm^2 (km^2)

SOLUTION

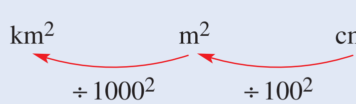
a $2.5\text{ cm}^2 = 2.5 \times 100\text{ mm}^2$
$= 250\text{ mm}^2$

EXPLANATION



$1\text{ cm}^2 = 10 \times 10\text{ mm}^2 = 100\text{ mm}^2$

b $2000000\text{ cm}^2 = 2000000 \div 100^2\text{ m}^2$
$= 200\text{ m}^2$
$= 200 \div 1000^2\text{ km}^2$
$= 0.0002\text{ km}^2$



$1000^2 = 1000000$ and $100^2 = 10000$

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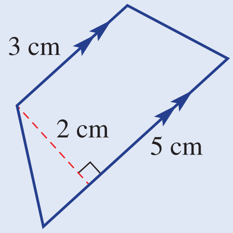
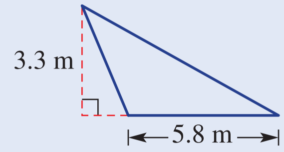
Example 8 Finding the area of basic shapes

Find the area of these basic shapes, correct to 2 decimal places where necessary.

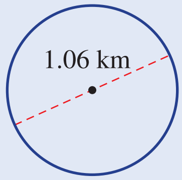
a



b



c



SOLUTION / **EXPLANATION**

a $A = \frac{1}{2}h(a + b)$ — The shape is a trapezium, so use this formula.

$= \frac{1}{2} \times 2(3 + 5)$ — Substitute $a = 3,\ b = 5$ and $h = 2$.

$= 8 \text{ cm}^2$ — Simplify and include the correct units.

b $A = \frac{1}{2}bh$ — The shape is a triangle.

$= \frac{1}{2}(5.8)(3.3)$ — Substitute $b = 5.8$ and $h = 3.3$.

$= 9.57 \text{ m}^2$ — Simplify and include the correct units.

c $A = \pi r^2$ — The shape is a circle.

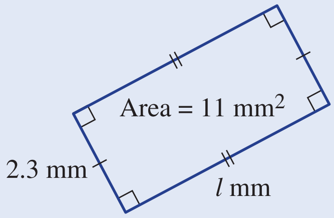
$= \pi(0.53)^2$ — The radius r is half the diameter.

$= 0.88 \text{ km}^2$ (to 2 decimal places) — Round your answer to the required number of decimal places.

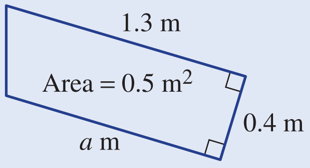
Example 9 Using area to find unknown lengths

Find the value of the pronumeral for these basic shapes, rounding your answer to 2 decimal places where necessary.

a



b



Example continued on next page

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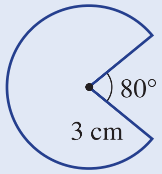
SOLUTION	EXPLANATION
a $A = l \times b$	Use the rectangle area formula.
$11 = l \times 2.3$	Substitute $A = 11$ and $b = 2.3$.
$\therefore l = \dfrac{11}{2.3}$	Divide both sides by 2.3 to solve for l.
$= 4.78$	
b $A = \dfrac{1}{2}h(a + b)$	Use the trapezium area formula.
$0.5 = \dfrac{1}{2} \times 0.4(a + 1.3)$	Substitute $A = 0.5,\ b = 1.3$ and $h = 0.4$.
$0.5 = 0.2(a + 1.3)$	Simplify, then divide both sides by 0.2 and solve for a.
$2.5 = a + 1.3$	
$\therefore a = 1.2$	

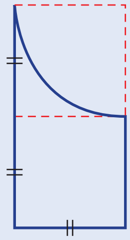
Example 10 Finding areas of sectors and composite figures

Find the area of this sector and composite figure. Write your answer as an exact value and as a decimal, correct to 2 places.

a



b



SOLUTION	EXPLANATION
a $A = \dfrac{\theta}{360} \times \pi r^2$	Write the formula for the area of a sector.
$= \dfrac{280}{360} \times \pi \times 3^2$	Sector angle $= 360° - 80° = 280°$
$= 7\pi$	Simplify to express as an exact value (7π), then round your answer as required.
$= 21.99\text{ cm}^2$ (to 2 decimal place)	
b $A = 2 \times 5^2 - \dfrac{1}{4} \times \pi \times 5^2$	The area consists of two squares minus a quarter circle with radius 5 m.
$= 50 - \dfrac{25\pi}{4}$	$50 - \dfrac{25\pi}{4}$ is the exact value.
$= 30.37\text{ m}^2$ (to 2 decimal places)	

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Exercise 1D REVISION

UNDERSTANDING AND FLUENCY	1–4	2, 3–4(½), 5	3–5(½)

1 Write the formula for the area of these shapes.

a circle b sector c square d rectangle
e kite f trapezium g triangle h rhombus
i parallelogram j semicircle k quadrant (quarter circle)

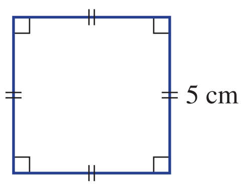
Example 7

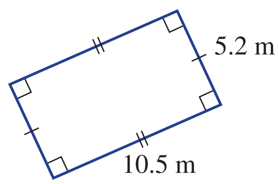
2 Convert the following area measurements to the units given in brackets.

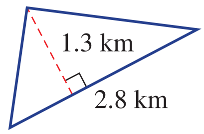
a 3000 mm^2 (cm^2) b $29\,800\text{ cm}^2$ (m^2) c $205\,000\text{ m}^2$ (km^2)
d 0.5 m^2 (cm^2) e 5 km^2 (m^2) f 0.0001 km^2 (m^2)
g 0.023 m^2 (cm^2) h 537 cm^2 (mm^2) i 0.0027 km^2 (m^2)
j 10 m^2 (mm^2) k 0.00022 km^2 (cm^2) l $145\,000\,000\text{ mm}^2$ (cm^2)

Example 8

3 Find the area of these basic shapes, rounding your answer to 2 decimal places where necessary.

a


b


c


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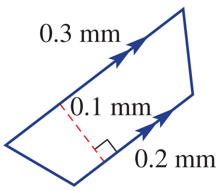

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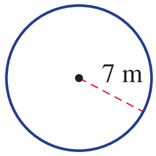

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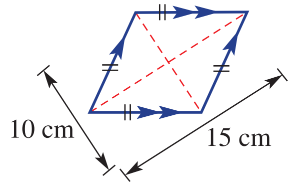

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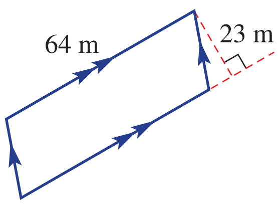

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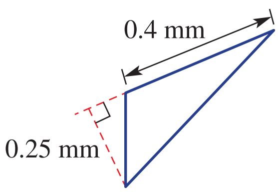

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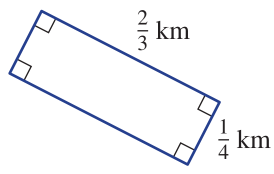

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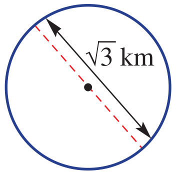
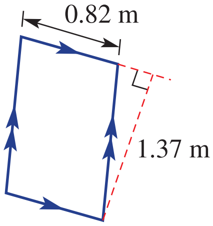
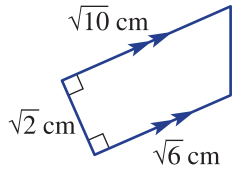


k


l


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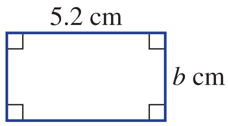
Example 9

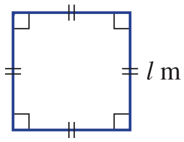
4 Find the value of the pronumeral for these basic shapes, rounding your answer to 2 decimal places where necessary.

a



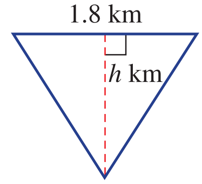
Area = 15 cm^2

b



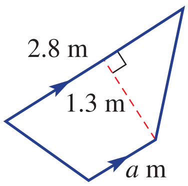
Area = 206 m^2

c



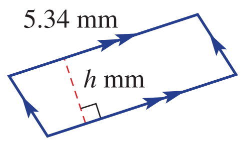
Area = 1.3 km^2

d



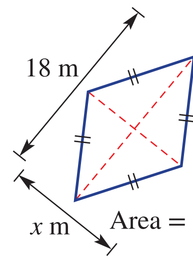
Area = 2.5 m^2

e



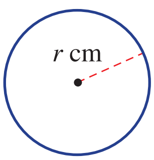
Area = 10.21 mm^2

f



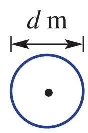
Area = 80 m^2

g



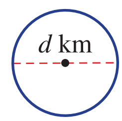
Area = 5 cm^2

h



Area = 0.21 m^2

i

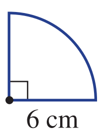


Area = 26 km^2

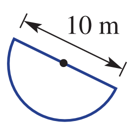
Example 10a

5 Find the area of each sector. Write your answer as an exact value and as a decimal rounded to 2 places.

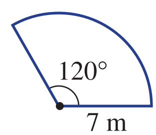
a



b



c



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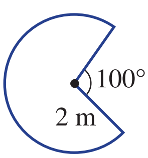

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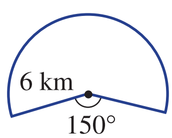

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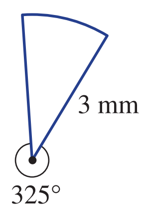
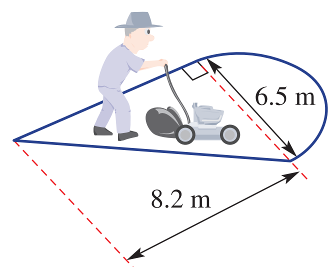


PROBLEM-SOLVING AND REASONING	6, 7, 9	7(½), 8–10	7(½), 8, 10–12

6 A lawn area is made up of a semicircular region with diameter 6.5 m and a triangular region of length 8.2 m, as shown. Find the total area of lawn, to 1 decimal place.



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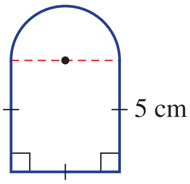
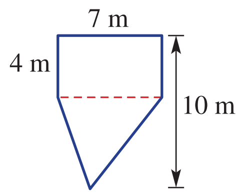
Example 10b

7 Find the area of these composite figures. Write your answers as exact values and as decimals, correct to 2 places.

a



b



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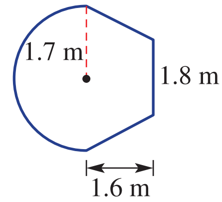

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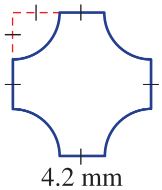

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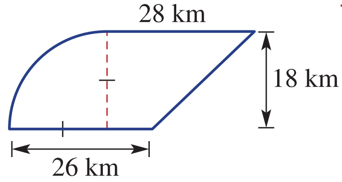

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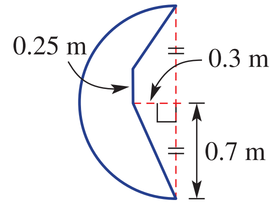
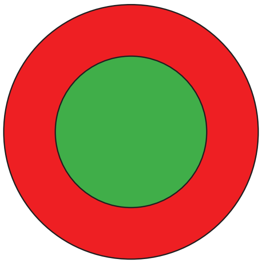


8 An L-shaped concrete slab being prepared for the foundation of a new house is made up of two rectangles with dimensions 3 m by 2 m and 10 m by 6 m.

a Find the total area of the concrete slab.

b If two bags of cement are required for every 5 m^2 of concrete, how many whole bags of cement will need to be purchased for the job?

9 Consider the green circle and the red annulus.



a The radius of the smaller circle is half the radius of the larger circle. Copy and complete:
green area : red area = 1: $\boxed{}$

b The green area is then increased to be equal to the red area. Copy and complete:
smaller circle radius : larger circle radius = 1: $\boxed{}$

10 1 hectare (1 ha) is 10 000 m^2 and an acre is $\frac{1}{640}$ square miles (1 mile ≈ 1.61 km). Find how many:

a hectares in 1 km^2

b square metres in 20 hectares

c hectares in 1 acre (round to 1 decimal place)

d acres in 1 hectare (round to 1 decimal place)

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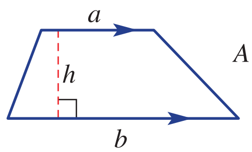
11 Consider a trapezium with area A, parallel side lengths a and b, and height h.



$$A = \frac{1}{2}h(a + b)$$

a Rearrange the area formula to express a in terms of A, b and h.

b Hence, find the value of a for these given values of A, b and h.

i $A = 10,\ b = 10,\ h = 1.5$ **ii** $A = 0.6,\ b = 1.3,\ h = 0.2$ **iii** $A = 10,\ b = 5,\ h = 4$

c Sketch the trapezium with the dimensions found in part **c iii**. What shape have you drawn?

12 Provide a proof of the following area formulas, using only the area formulas for rectangles and triangles.

a parallelogram

b kite

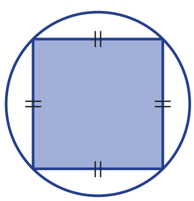
c trapezium

ENRICHMENT	–	–	13

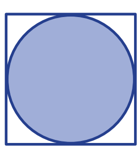
Percentage areas

13 Find, correct to 1 decimal place, the percentage areas for these situations.

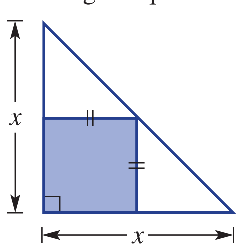
a The largest square inside a circle.



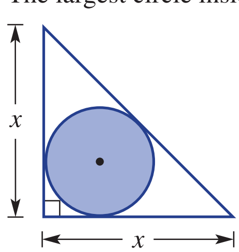
b The largest circle inside a square.



c The largest square inside a right isosceles triangle.



d The largest circle inside a right isosceles triangle.



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1E Surface area of prisms and cylinders

Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Knowing how to find the area of simple shapes combined with some knowledge about three-dimensional objects helps us to find the total surface area of a range of solids.

Interactive
Widgets
HOTsheets
Walkthrough

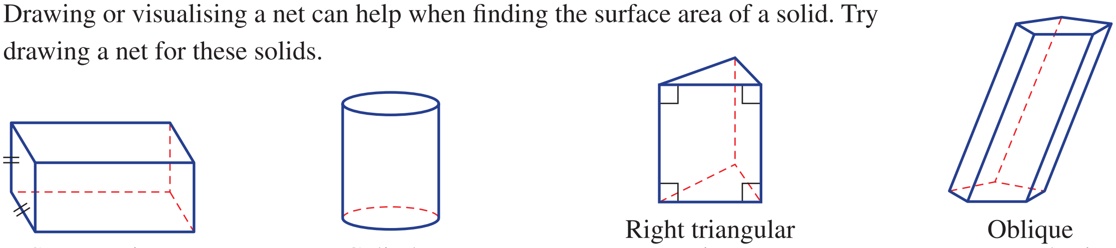
A cylindrical can, for example, has two circular ends and a curved surface that could be rolled out to form a rectangle. Finding the sum of the two circles and the rectangle will give the total surface area of the cylinder.

You will recall the following information about prisms and cylinders.

- A **prism** is a polyhedron with a uniform cross-section and two congruent ends.
  - A prism is named by the shape of the cross-section.
  - The remaining sides are parallelograms.
- A **cylinder** has a circular cross-section.
  - A cylinder is similar to a prism in that it has a uniform cross-section and two congruent ends.

Let's start: Drawing nets

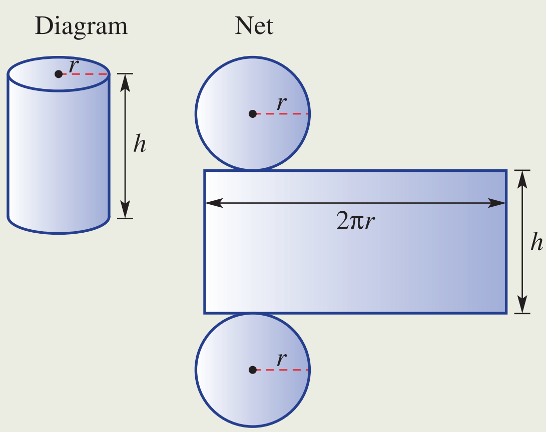
Drawing or visualising a net can help when finding the surface area of a solid. Try drawing a net for these solids.



Square prism · Cylinder · Right triangular prism · Oblique pentagonal prism

Key ideas

- The **surface area** (A) of a three-dimensional object can be found by finding the sum of the areas of each of the shapes that make up the surface of the object.
- A **net** is a two-dimensional illustration of all the surfaces of a solid object.
- Given below are the net and surface area of a **cylinder**.



$$\begin{aligned} A &= 2 \text{ circle} + 1 \text{ rectangle} \\ &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r(r + h) \end{aligned}$$

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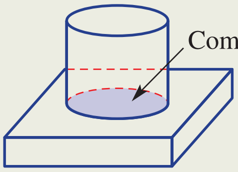
Key ideas

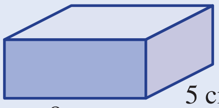
- **Composite solids** are solids made up of two or more basic solids.
  - To find the surface area do not include any common faces.



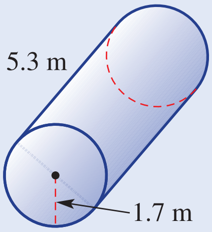
Example 11 Finding the surface area of prisms and cylinders

Find the surface area of these objects. Round your answer to 2 decimal places where necessary.

a



b

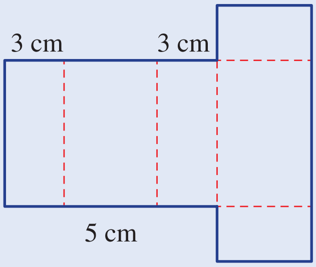


SOLUTION

a $A = 2 \times (8 \times 3) + 2 \times (5 \times 3) + 2 \times (8 \times 5)$
$= 158\ \text{cm}^2$

EXPLANATION

Draw the net of the solid. Sum the area of the rectangles.

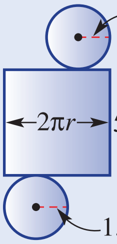


SOLUTION

b $A = 2\pi r^2 + 2\pi rh$
$= 2\pi(1.7)^2 + 2\pi(1.7) \times 5.3$
$= 74.77\ \text{m}^2$ (to 2 decimal places)

EXPLANATION

Write the formula and substitute the radius and height.



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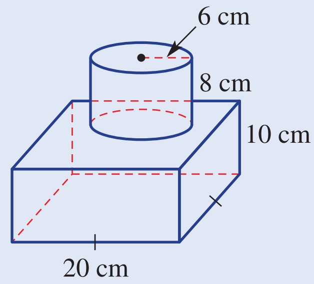
Example 12 Finding the surface area of composite solids

A composite object consists of a square-based prism and a cylinder, as shown. Find the surface area, correct to 1 decimal place.



SOLUTION

$A = 4 \times (20 \times 10) + 2 \times (20 \times 20) + 2 \times \pi \times 6 \times 8$
$\quad = 1600 + 96\pi$
$\quad = 1901.6 \text{ cm}^2$ (to 1 decimal place)

EXPLANATION

The common circular area, which should not be included, is added back on with the top of the cylinder. So the surface area of the prism is added to only the curved area of the cylinder.

Exercise 1E

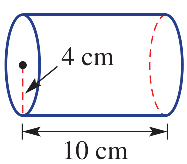
UNDERSTANDING AND FLUENCY	1–5	2, 3–4(½), 5, 6	4(½), 5, 6

1 Draw an example of these solids.

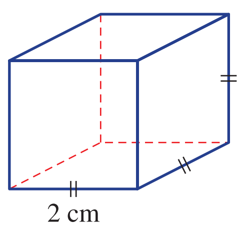
a cylinder **b** rectangular prism **c** triangular prism

2 Draw a net for each of these solids.

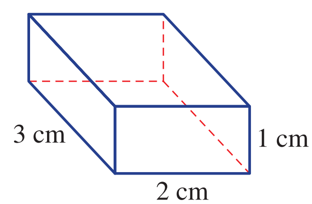
a



b



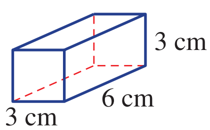
c



Example 11

3 Find the surface area of these solids. Round your answer to 2 decimal places where necessary.

a



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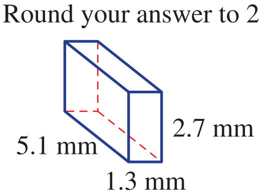

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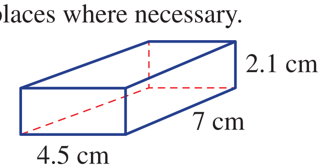

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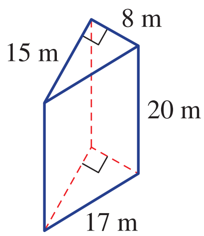

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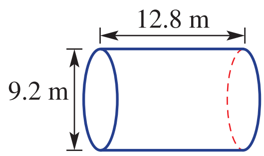

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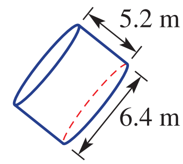

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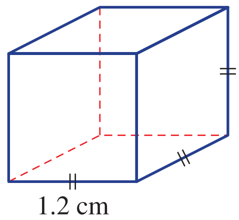
4 Find the surface area of these solids.

a


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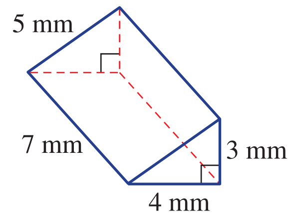

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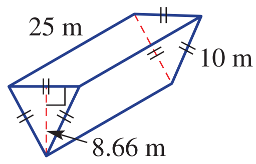

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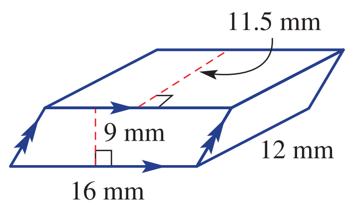

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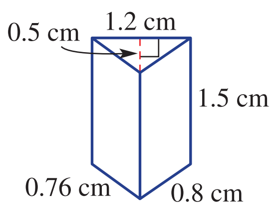

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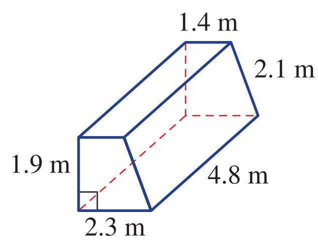
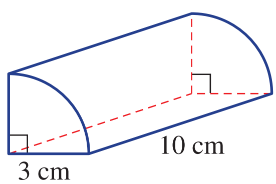


5 Find the surface area, in square metres, of the outer surface of an open pipe with radius 85 cm and length 4.5 m, correct to 2 decimal places.

6 What is the minimum area of paper required to wrap a box with dimensions 25 cm wide, 32 cm long and 20 cm high?

PROBLEM-SOLVING AND REASONING	7, 8, 10	8, 9, 11, 12	8(½), 9, 10, 12–14

7 The cross-sections of these solids are sectors. Find the surface area, rounding your answer to 1 decimal place.

a


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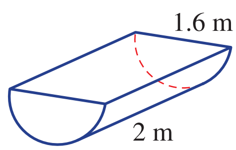

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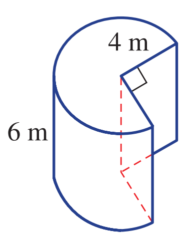

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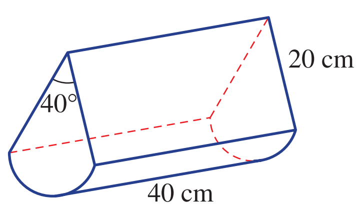

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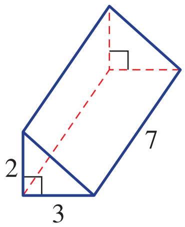
8 Use Pythagoras' theorem to determine any unknown side lengths and find the surface area of these solids, correct to 1 decimal place.

a



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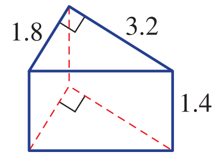

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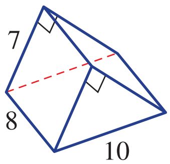

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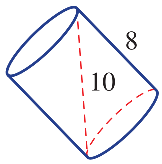

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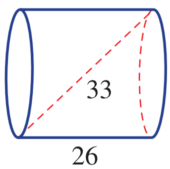

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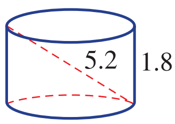
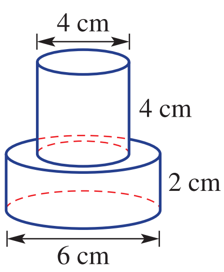


Example 12

9 Find the surface area of these composite solids. Answer correct to 1 decimal place.

a



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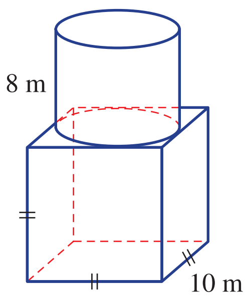

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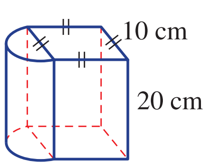

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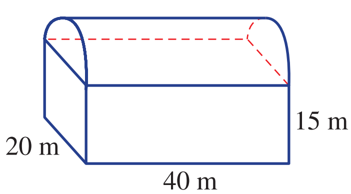
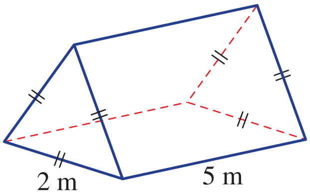


10 Find the surface area of this triangular prism, correct to 1 decimal place.



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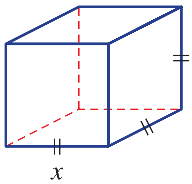
11 Find a formula for the surface area of these solids, using the given pronumerals.

a



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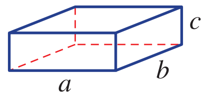

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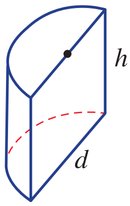
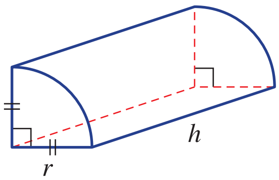


d



12 Find the exact surface area for a cylinder with the dimensions given. Your exact answer will be in terms of π.

a $r = 1$ and $h = 2$

b $r = \frac{1}{2}$ and $h = 5$

13 If the surface area of a cylinder is given by the rule $A = 2\pi r(r + h)$, find the height, to 2 decimal places, of a cylinder that has a radius of 2 m and a surface area of:

a 35 m^2

b 122 m^2

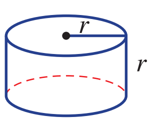
14 Can you find the exact radius of the base of a cylinder when its surface area is 8π cm^2 and its height is 3 cm?

ENRICHMENT	–	–	15

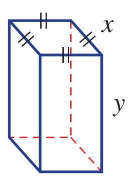
Deriving formulas for special solids

15 Derive the formulas for the surface area of the following solids.

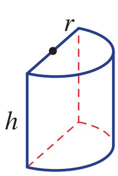
a a cylinder with its height equal to its radius r



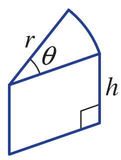
b a square-based prism with square side length x and height y



c a half cylinder with radius r and height h



d a solid with a sector cross-section, radius r, sector angle θ and height h



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1F Surface area of pyramids and cones

Interactive

Widgets

HOTsheets

Walkthrough

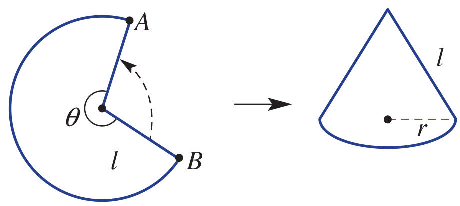
Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Pyramids and cones are solids for which we can also calculate the surface area by finding the sum of the areas of all the outside surfaces.

The surface area of a pyramid involves finding the sum of the areas of the base and its triangular faces. The rule for the surface area of a cone can be developed after drawing a net that includes a circle (base) and sector (curved surface), as you will see in the following examples.

Let's start: The cone formula

Use a pair of compasses to construct a large sector. Use any sector angle θ that you like. Cut out the sector and join the points A and B to form a cone of radius r.

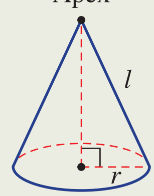


- Give the rule for the area of the base of the cone.
- Give the rule for the circumference of the base of the cone.
- Give the rule for the circumference of a circle with radius l.
- Use the above to find an expression for the area of the base of the cone as a fraction of the area πl^2.
- Hence, explain why the rule for the surface area of a cone is given by $A = \pi r^2 + \pi r l$.

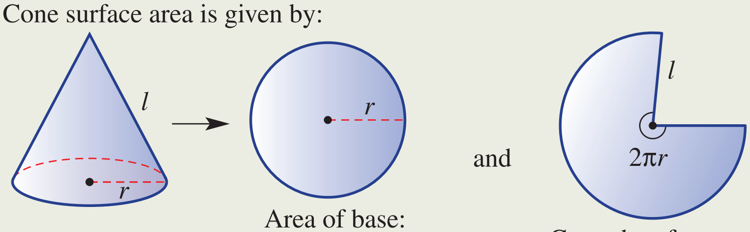
Key ideas

- A **cone** is a solid with a circular base and a curved surface that reaches from the base to a point called the **apex**.
  - A right cone has its apex directly above the centre of the base.
  - The pronumeral l is used for the slant height and r is the radius of the base.
  - Cone surface area is given by:

Apex



Right cone



Area of base:
$A = \pi r^2$

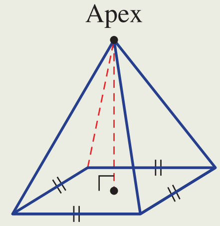
Curved surface area:

$$A = \frac{2\pi r}{2\pi l} \times \pi l^2$$
$$\therefore A = \pi r l$$

$$\therefore A = \pi r^2 + \pi r l = \pi r(r + l)$$

- A **pyramid** has a base that is a polygon and its remaining faces are triangles that meet at the apex.
  - A pyramid is named by the shape of its base.
  - A right pyramid has its apex directly above the centre of the base.

Apex



Right square pyramid

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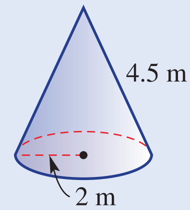
Example 13 Finding the surface area of a cone and pyramid

Find the surface area of these solids, using 2 decimal places for part **a**.

a cone with radius 2 m and slant height 4.5 m

b square pyramid with base length 25 mm and triangular face height 22 mm



SOLUTION

a $A = \pi r^2 + \pi r l$

$= \pi(2)^2 + \pi(2) \times (4.5)$

$= 40.84\text{ m}^2$ (to 2 decimal places)

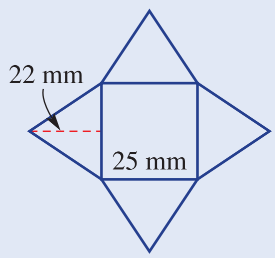
b $A = s^2 + 4 \times \frac{1}{2}bh$

$= 25^2 + 4 \times \frac{1}{2} \times 25 \times 22$

$= 1725\text{ mm}^2$

EXPLANATION

The cone includes the circular base plus the curved part. Substitute $r = 2$ and $l = 4.5$.



Example 14 Finding the slant height and vertical height of a cone

A cone with radius 3 cm has a curved surface area of 100 cm^2.

a Find the slant height of the cone, correct to 1 decimal place.

b Find the height of the cone, correct to 1 decimal place.

SOLUTION

a $A = \pi r l$

$100 = \pi \times 3 \times l$

$l = \frac{100}{3\pi}$

$= 10.6\text{ cm}$ (to 1 decimal place)

b $h^2 + r^2 = l^2$

$h^2 + 3^2 = \left(\frac{100}{3\pi}\right)^2$

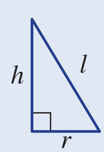
$h^2 = \left(\frac{100}{3\pi}\right)^2 - 9$

$h = \sqrt{\left(\frac{100}{3\pi}\right)^2 - 9}$

$= 10.2\text{ cm}$ (to 1 decimal place)

EXPLANATION

Substitute the given information into the formula for the curved surface area of a cone and solve for l.



Identify the right-angled triangle within the cone and use Pythagoras' theorem to find the height, h. Use the exact value of l from part **a** to avoid accumulating errors.

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Exercise 1F

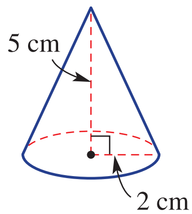
UNDERSTANDING AND FLUENCY	1–6(a, b)	3–7	4–7

1 Write the formula for the following.

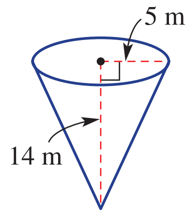
- **a** the area of a triangle
- **b** the surface area of the base of a cone with radius r
- **c** the surface area of the curved part of a cone with slant height l and radius r

2 Find the exact slant height for these cones, using Pythagoras' theorem. Answer exactly, using a square root sign.

a



b



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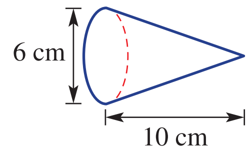
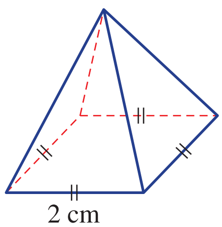


3 Draw a net for each of these closed solids.

a



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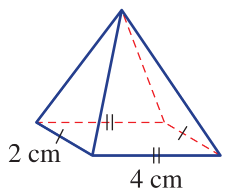

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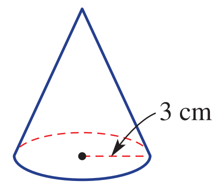
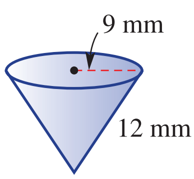


Example 13a

4 Find the surface area of these cones, correct to 2 decimal places.

a



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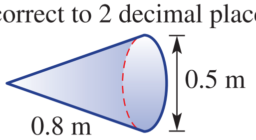

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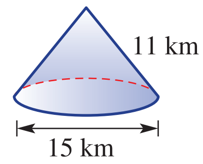
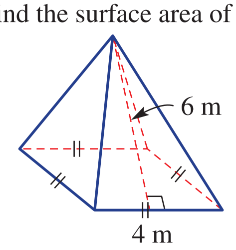


Example 13b

5 Find the surface area of these pyramids.

a



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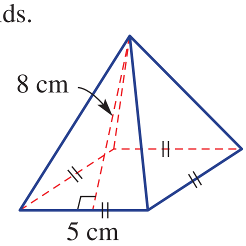

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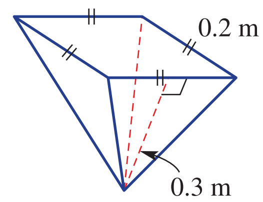
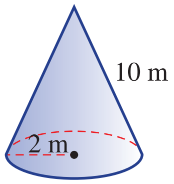
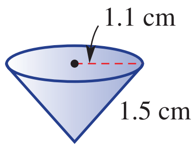


6 For each cone, find the area of the *curved surface* only, correct to 2 decimal places.

a



b



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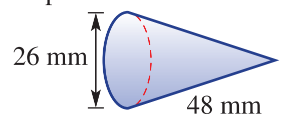

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7 A cone has height 10 cm and radius 3 cm.

a Use Pythagoras' theorem to find the slant height of the cone, rounding your answer to 2 decimal places.

b Find the surface area of the cone, correct to 1 decimal place.

PROBLEM-SOLVING AND REASONING	8, 9, 13	9–11, 13, 14	10, 11, 12(½), 14, 15

Example 14

8 A cone with radius 5 cm has a curved surface area of 400 cm^2.

a Find the slant height of the cone, correct to 1 decimal place.

b Find the height of the cone, correct to 1 decimal place.

9 A cone with radius 6.4 cm has a curved surface area of 380 cm^2.

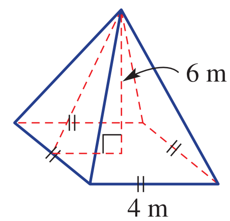
a Find the slant height of the cone, correct to 1 decimal place.

b Find the height of the cone, correct to 1 decimal place.

10 This right square pyramid has base side length 4 m and vertical height 6 m.

a Find the height of the triangular faces, correct to 1 decimal place.

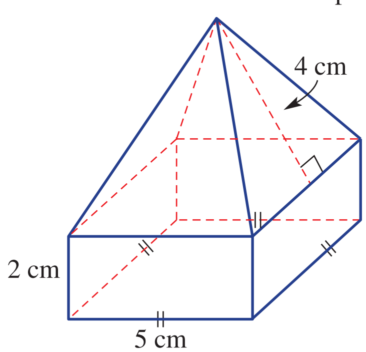
b Find the surface area, correct to 1 decimal place.



11 Party hats A and B are in the shape of open cones with no base. Hat A has radius 7 cm and slant height 25 cm, and hat B has radius 9 cm and slant height 22 cm. Which hat has the greater surface area?

12 Find the surface area of these composite solids, correct to 1 decimal place as necessary.

a



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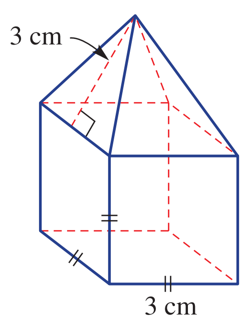

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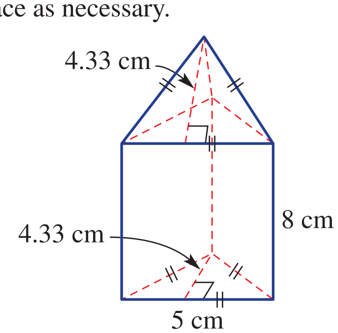

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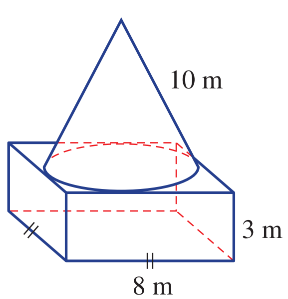

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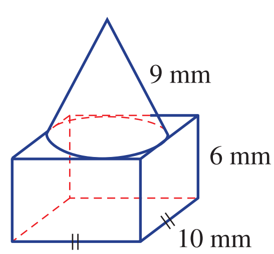

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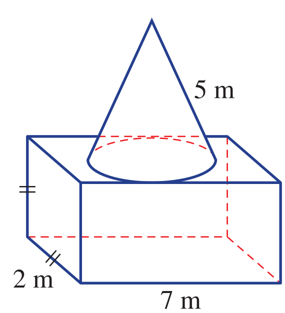
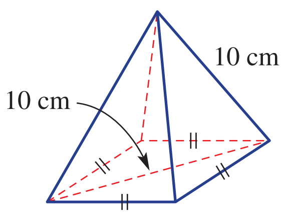


13 Explain why the surface area of a cone with radius r and height h is given by the expression $\pi r(r + \sqrt{r^2 + h^2})$.

14 A cone has a height equal to its radius ($h = r$). Show that its surface area is given by the expression $\pi r^2(1 + \sqrt{2})$.

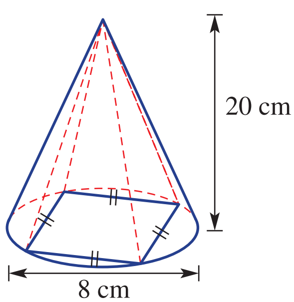
15 There is enough information in this diagram to find the surface area, although the side length of the base and the height of the triangular faces are not given. Find the surface area, correct to 1 decimal place.



ENRICHMENT	–	–	16

Carving pyramids from cones

16 A woodworker uses a rotating lathe to produce a cone with radius 4 cm and height 20 cm. From that cone the woodworker then cut slices off the sides of the cone to produce a square pyramid of the same height.



a Find the exact slant height of the cone.
b Find the surface area of the cone, correct to 2 decimal places.
c Find the exact side length of the base of the square pyramid.
d Find the height of the triangular faces of the pyramid, correct to 3 decimal places.
e Find the surface area of the pyramid, correct to 2 decimal places.
f Express the surface area of the pyramid as a percentage of the surface area of the cone. Give the answer correct to the nearest whole percentage.

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1G Volume of prisms and cylinders

Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Interactive

Widgets

HOTsheets

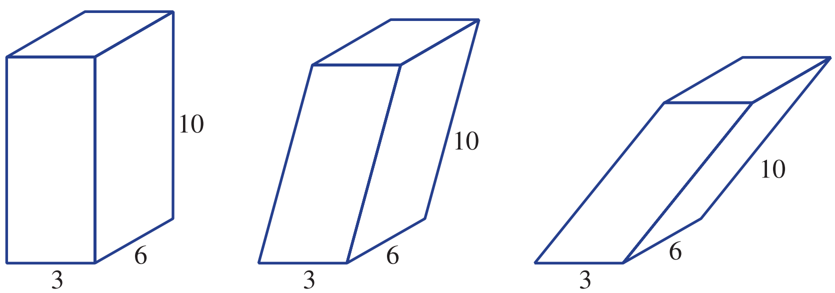
Walkthrough

Volume is the amount of space contained within the outer surfaces of a three-dimensional object and is measured in cubic units.

The common groups of objects considered in this section are the prisms and the cylinders.

Let's start: Right and oblique prisms

Recall that to find the volume of a right prism, you would first find the area of the base and multiply by the height. Here is a right rectangular prism and two oblique rectangular prisms with the same side lengths.



- What is the volume of the right rectangular prism?
- Do you think the volume of the two oblique prisms would be calculated in the same way as that for the right rectangular prism and using exactly the same lengths?
- Would the volume of the oblique prisms be equal to or less than that of the rectangular prism?
- Discuss what extra information is required to find the volume of the oblique prisms.
- How does finding the volume of oblique prisms (instead of a right prism) compare with finding the area of a parallelogram (instead of a rectangle)?

Key ideas

- Metric units for **volume** include cubic kilometres (km^3), cubic metres (m^3), cubic centimetres (cm^3) and cubic millimetres (mm^3).

$$\begin{tikzcd} km^3 \arrow[r, bend left, "\times 1000^3"] & m^3 \arrow[l, bend left, "\div 1000^3"] \arrow[r, bend left, "\times 100^3"] & cm^3 \arrow[l, bend left, "\div 100^3"] \arrow[r, bend left, "\times 10^3"] & mm^3 \arrow[l, bend left, "\div 10^3"] \end{tikzcd}$$

- Units for **capacity** include megalitres (ML), kilolitres (kL), litres (L) and millilitres (mL).
  - $1\ cm^3 = 1\ mL$

$$\begin{tikzcd} ML \arrow[r, bend left, "\times 1000"] & kL \arrow[l, bend left, "\div 1000"] \arrow[r, bend left, "\times 1000"] & L \arrow[l, bend left, "\div 1000"] \arrow[r, bend left, "\times 1000"] & mL \arrow[l, bend left, "\div 1000"] \end{tikzcd}$$

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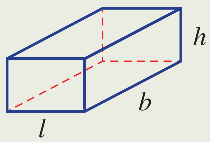
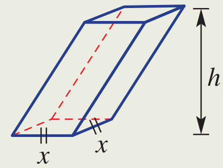
Key ideas

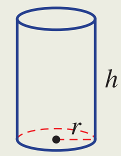
- For right and oblique prisms and cylinders the volume is given by:
  $V = Ah$ where
  - A is the area of the base
  - h is the perpendicular height.



Right rectangular prism
$V = Ah$
$= lbh$



Oblique square prism
$V = Ah$
$= x^2h$

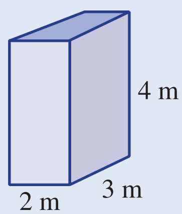


Right cylinder
$V = Ah$
$= \pi r^2 h$

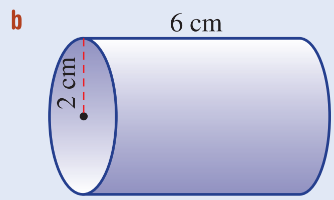
Example 15 Finding the volume of right prisms and cylinders

Find the volume of these solids, rounding your answer to 2 decimal places for part **b**.

a



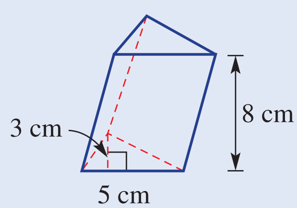
b



SOLUTION	EXPLANATION
a $V = lbh$	Write the volume formula for a rectangular prism.
$= 2 \times 3 \times 4$	Substitute $l = 2,\ b = 3$ and $h = 4$.
$= 24\text{ m}^3$	
b $V = \pi r^2 h$	The prism is a cylinder with base area πr^2.
$= \pi(2)^2 \times 6$	Substitute $r = 2$ and $h = 6$.
$= 75.40\text{ cm}^3$ (to 2 decimal places)	Evaluate and round your answer.

Example 16 Finding the volume of an oblique prism

Find the volume of this oblique prism.



SOLUTION

$$V = \frac{1}{2}bh \times 8$$
$$= \frac{1}{2} \times 5 \times 3 \times 8$$
$$= 60\text{ cm}^3$$

EXPLANATION

The base is a triangle, so multiply the area of the base triangle by the perpendicular height.

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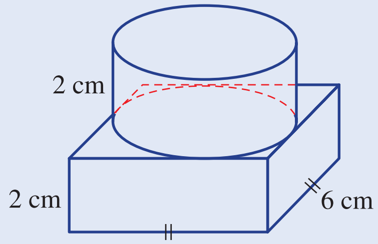
Example 17 Finding the volume of a composite solid

Find the volume of this composite solid, correct to 1 decimal place.



SOLUTION

Radius of cylinder $= \frac{6}{2} = 3$ cm

$V = lbh + \pi r^2 h$
$= 6 \times 6 \times 2 + \pi \times 3^2 \times 2$
$= 72 + 18\pi$
$= 128.5$ cm^3 (to 1 decimal place)

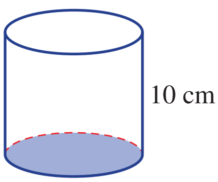
EXPLANATION

First, find the radius length, which is half the side length of the square base.
Add the volume of the square prism and the volume of the cylinder.

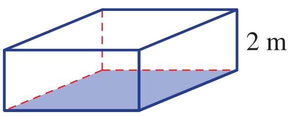
Exercise 1G

UNDERSTANDING AND FLUENCY	1, 2, 3–5(a, b)	2(½), 3–5(a, c), 6(½)	3–5(b, c), 6(½)

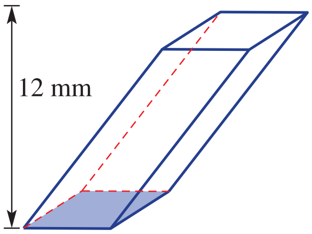
1 Find the volume of these solids with the given base areas.

a


$A = 8$ cm^2

b


$A = 16$ m^2

c


$A = 9$ mm^2

2 Convert these volume measurements to the units given in brackets.

a 2 cm^3 (mm^3) **b** 0.2 m^3 (cm^3) **c** 0.015 km^3 (m^3)
d 5700 mm^3 (cm^3) **e** 28 300 000 m^3 (km^3) **f** 762 000 cm^3 (m^3)
g 0.13 m^3 (cm^3) **h** 0.000001 km^3 (m^3) **i** 2.094 cm^3 (mm^3)
j 2.7 L (mL) **k** 342 kL (ML) **l** 35 L (kL)
m 5.72 ML (kL) **n** 74 250 mL (L) **o** 18.44 kL (L)

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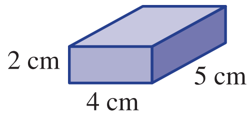
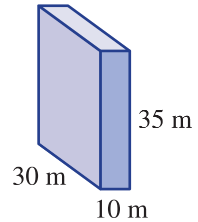
Example 15a

3 Find the volume of each rectangular prism.

a



b



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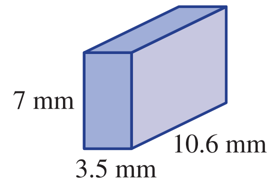
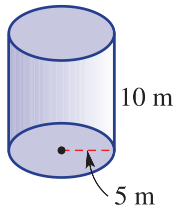


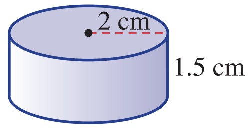
Example 15b

4 Find the volume of each cylinder, correct to 2 decimal places.

a



b



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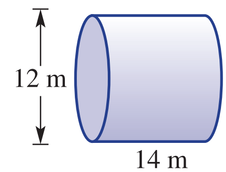
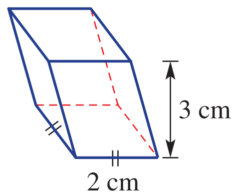


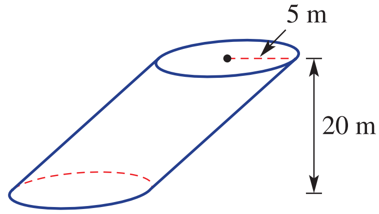
Example 16

5 Find the volume of these oblique solids. Round your answer to 1 decimal place for part **b**.

a



b



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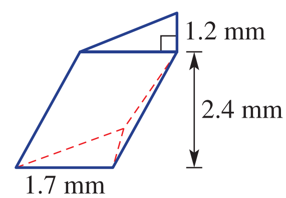
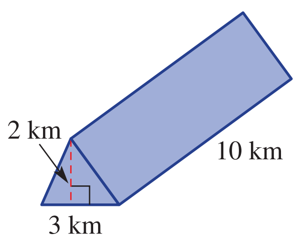


6 Find the volume of these solids, rounding your answers to 3 decimal places where necessary.

a



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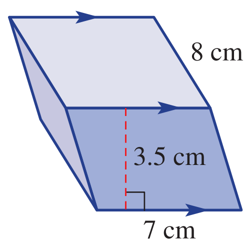

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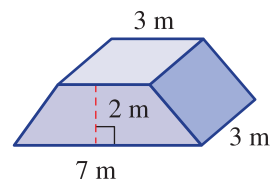

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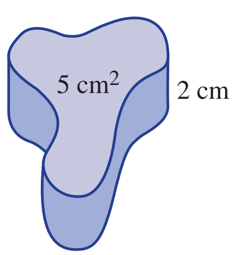

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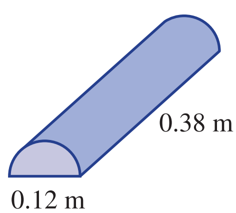

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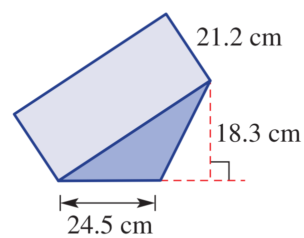

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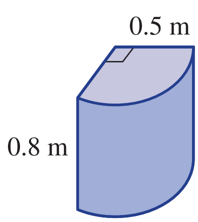

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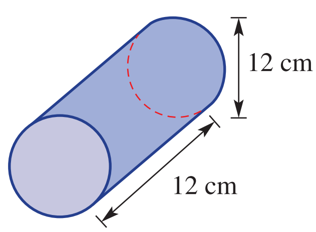

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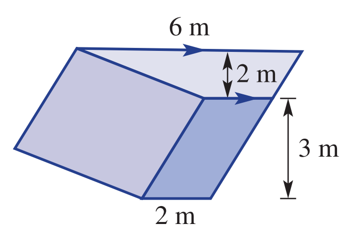

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PROBLEM-SOLVING AND REASONING	7, 8, 11	9–12	12–15

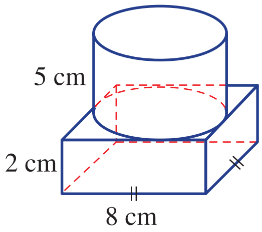
7 How many containers holding $1000\ \text{cm}^3$ (1 L) of water are needed to fill $1\ \text{m}^3$?

8 How many litres of water are required to fill a rectangular fish tank that is 1.2 m long, 80 cm wide and 50 cm high?

Example 17

9 Find the volume of these composite objects, rounding your answer to 2 decimal places where necessary.

a



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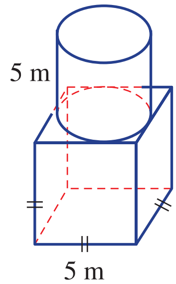

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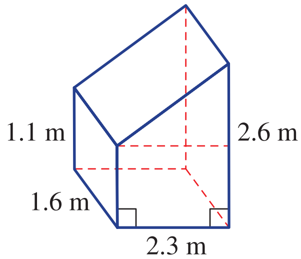

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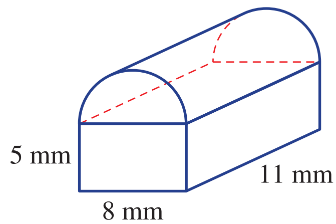

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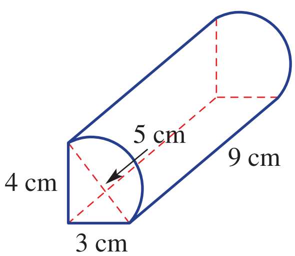

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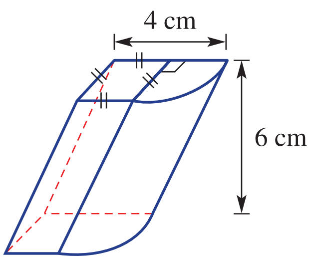

10 Find the exact volume of a cube if its surface area is:

a $54\ \text{cm}^2$

b $18\ \text{m}^2$

11 Juice is sold in a rectangular prism measuring 11 cm by 6 cm by 4 cm. A cube is built to hold the same volume. By how much does the surface area decrease? Give your answer to 1 decimal place.

12 Use the formula $V = \pi r^2 h$ to find the height of a cylinder, to 1 decimal place, with radius 6 cm and volume $62\ \text{cm}^3$.

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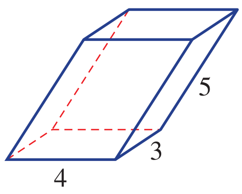
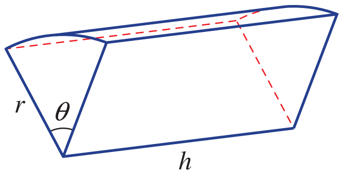
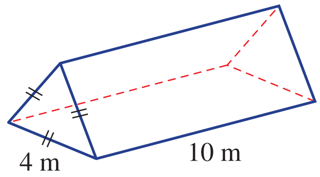
13 A fellow student says the volume of this prism is given by $V = 4 \times 3 \times 5$. Explain the student's error.



14 Find a formula for the volume of a cylindrical portion with angle θ, radius r and height h, as shown.



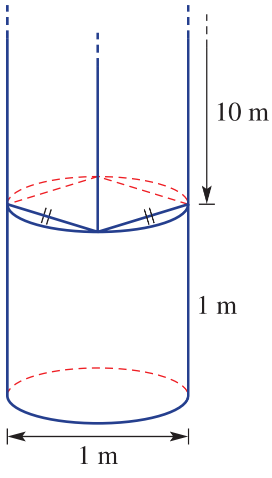
15 Decide whether there is enough information in this diagram of a triangular prism to find its volume. If so, find the volume, correct to 1 decimal place.



ENRICHMENT	–	–	16

Concrete poles

16 A concrete support structure for a building is made up of a cylindrical base and a square prism as the main column. The cylindrical base is 1 m in diameter and 1 m high, and the square prism is 10 m long and sits on the cylindrical base, as shown.



a Find the exact side length of the square base of the prism.

b Find the volume of the entire support structure, correct to 1 decimal place.

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1H Volume of pyramids and cones

Interactive

Widgets

HOTsheets

Walkthrough

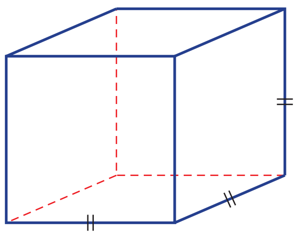
The volume of a cone or pyramid is a certain fraction of the volume of a prism with the same base area. This particular fraction is the same for both cones and pyramids and will be explored in this section.

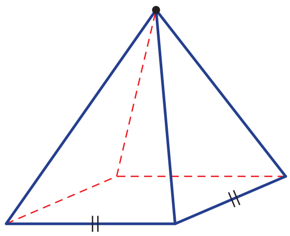
Stage

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

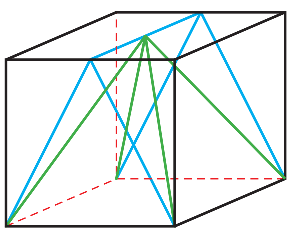
Let's start: Is a pyramid half the volume of a prism?

Here is a cube and a square pyramid with equal base side lengths and equal heights.





- Discuss whether or not you think the pyramid is half the volume of the cube.
- Now consider this diagram of the cube with the pyramid inside.



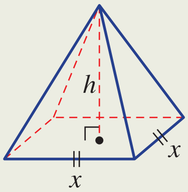
The cube is black.
The pyramid is green.
The triangular prism is blue.

- Compared to the cube, what is the volume of the triangular prism (blue)? Give reasons.
- Is the volume of the pyramid (green) more or less than the volume of the triangular prism (blue)?
- Do you know what is the volume of the pyramid as a fraction of the volume of the cube?

Key ideas

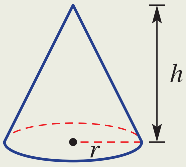
■ For pyramids and cones, the volume is given by $V = \frac{1}{3}Ah$, where A is the area of the base and h is the perpendicular height.

Right square pyramid



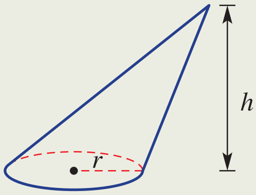
$V = \frac{1}{3}Ah$
$= \frac{1}{3}x^2h$

Right cone



$V = \frac{1}{3}Ah$
$= \frac{1}{3}\pi r^2h$

Oblique cone



$V = \frac{1}{3}Ah$
$= \frac{1}{3}\pi r^2h$

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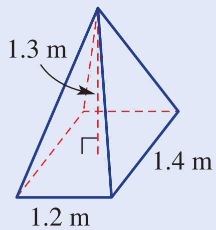
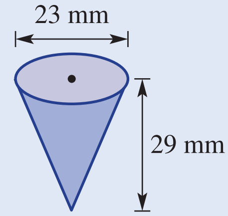
Example 18 Finding the volume of pyramids and cones

Find the volume of this pyramid and cone. Give the answer for part **b** correct to 2 decimal places.

a



b



SOLUTION

a $V = \frac{1}{3}Ah$

$= \frac{1}{3}(l \times b) \times h$

$= \frac{1}{3}(1.4 \times 1.2) \times 1.3$

$= 0.728\text{ m}^3$

b $V = \frac{1}{3}Ah$

$= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(11.5)^2 \times 29$

$= 4016.26\text{ mm}^3$ (to 2 decimal places)

EXPLANATION

The pyramid has a rectangular base with area $l \times b$.

Substitute $l = 1.4,\ b = 1.2$ and $h = 1.3$.

Evaluate the answer.

The cone has a circular base of area πr^2.

Substitute $r = \frac{23}{2} = 11.5$ and $h = 29$.

Evaluate and round the answer.

Exercise 1H

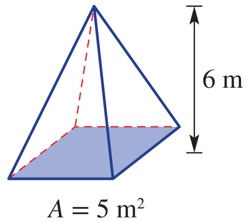
UNDERSTANDING AND FLUENCY	1–4	3(a, c), 4	3(c), 4(½)

1 A cylinder has volume 12 cm^3. What will be the volume of a cone with the same base area and perpendicular height?

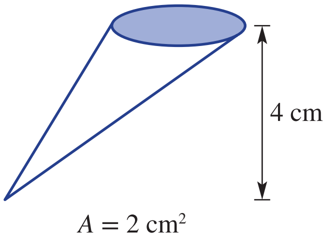
2 A pyramid has volume 5 m^3. What will be the volume of a prism with the same base area and perpendicular height?

3 Find the volume of these solids with the given base areas.

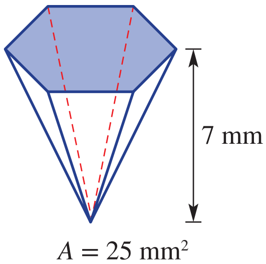
a



b



c



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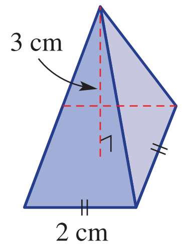
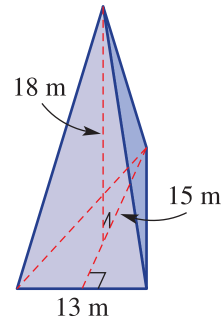
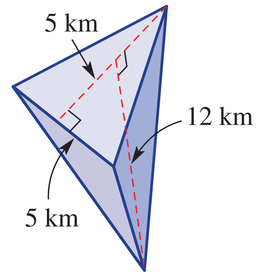
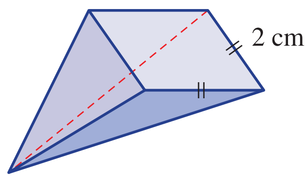
Example 18a

4 Find the volume of the following pyramids. For the oblique pyramids (parts **d**, **e**, **f**), use the given perpendicular height.

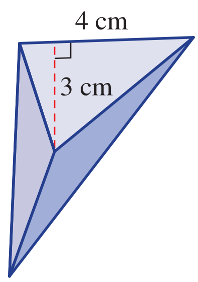
a


b


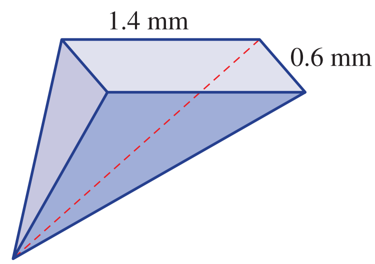
c


d


Perpendicular height = 2 cm

e


Perpendicular height = 4 cm

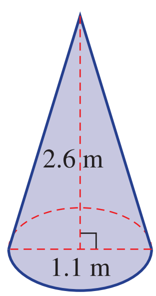
f


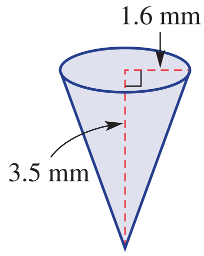
Perpendicular height = 1.2 mm

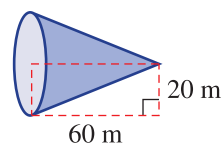
PROBLEM-SOLVING AND REASONING	5(½), 6, 7(½), 8	6, 7(½), 8–10	7(½), 8–11

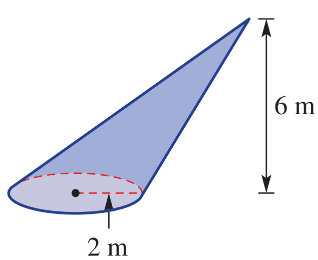
Example 18b

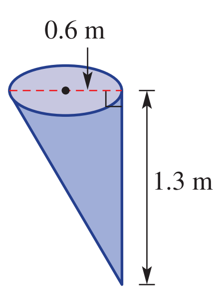
5 Find the volume of the following cones, correct to 2 decimal places.

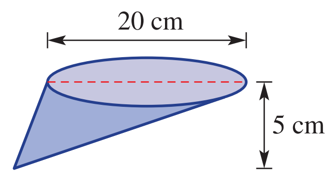
a


b


c


d


e


f


6 A medicine measuring cup is in the shape of a cone with base radius 3 cm and height 5 cm. Find its capacity in mL, correct to the nearest mL.

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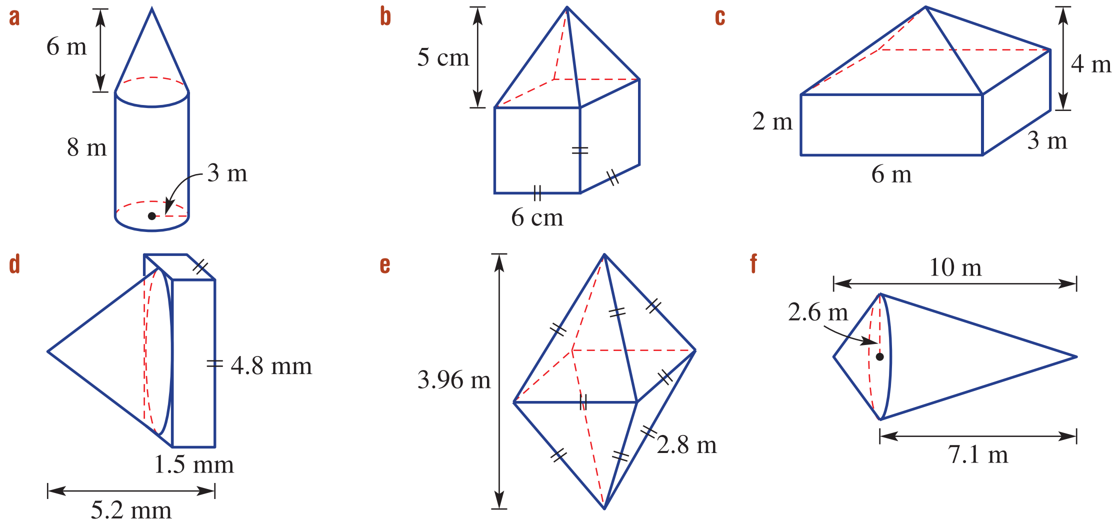
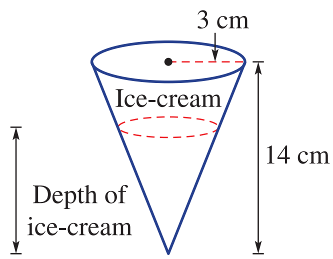
7 Find the volume of these composite objects, rounding to 2 decimal places where necessary.



8 The volume of ice-cream in the cone is half the volume of the cone. The cone has a 3 cm radius and height of 14 cm, as shown at right. What is the depth of the ice-cream, correct to 2 decimal places?



9 A wooden cylinder is carved to form a cone that has the same base area and same height as the original cylinder. What fraction of the wooden cylinder is wasted? Give a reason for your answer.

10 A square pyramid and a cone are such that the diameter of the cone is equal to the length of the side of the square base of the pyramid. They also have the same height.

a Using x as the side length of the pyramid and h as its height, write a rule for:

i the volume of the pyramid in terms of x and h

ii the volume of the cone in terms of x and h

b Express the volume of the cone as a fraction of the volume of the pyramid. Give an exact answer.

11 **a** Use the rule $V = \frac{1}{3}\pi r^2 h$ to find the base radius of a cone, to 1 decimal place, with height 23 cm and volume 336 cm^3.

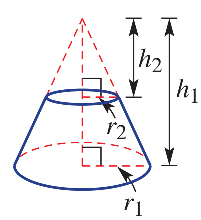
b Rearrange the rule $V = \frac{1}{3}\pi r^2 h$ to write:

i h in terms of V and r **ii** r in terms of V and h

ENRICHMENT	–	–	12

Truncated cones

12 A truncated cone is a cone that has its apex cut off by an intersecting plane. In this example, the top has radius r_2, the base has radius r_1 and the two circular ends are parallel.



a Give reasons why $\frac{r_1}{r_2} = \frac{h_1}{h_2}$.

b Find a rule for the volume of a truncated cone.

c Find the volume, to 1 decimal place, of a truncated cone if $r_1 = 2$ cm, $h_1 = 5$ cm and h_2 equals:

i $\frac{1}{2}h_1$ **ii** $\frac{2}{3}h_1$

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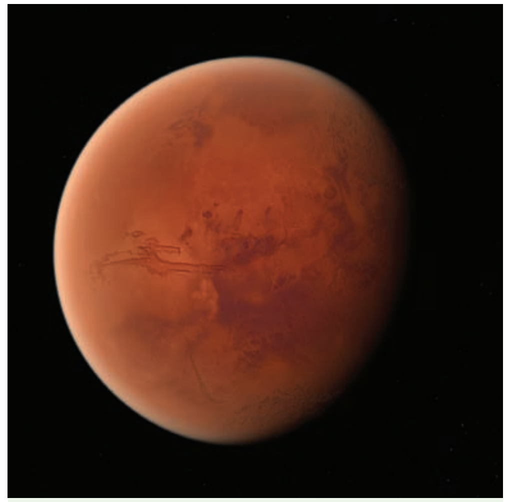
1I Volume and surface area of spheres

Interactive
Widgets
HOTsheets
Walkthrough

Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Planets are spherical in shape due to the effects of gravity. This means that we can describe a planet's size using only one measurement: its diameter or radius. Mars, for example, has a diameter of about half that of the Earth, which is about 12 756 km. The Earth's volume is about nine times that of Mars and this is because the volume of a sphere varies with the cube of the radius. The surface area of the Earth is about 3.5 times that of Mars because the surface of a sphere varies with the square of the radius.



Mars is approximately spherical in shape.

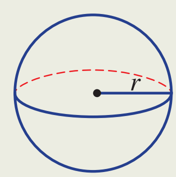
Let's start: What percentage of a cube is a sphere?

A sphere of radius 1 unit just fits inside a cube.

- First, guess the percentage of space occupied by the sphere.
- Draw a diagram showing the sphere inside the cube.
- Calculate the volume of the cube and the sphere. For the sphere, use the formula $V = \frac{4}{3}\pi r^3$.
- Now calculate the percentage of space occupied by the sphere. How close was your guess?

Key ideas

- The surface area of a **sphere** depends on its radius, r, and is given by $V = 4\pi r^2$.
- The volume of a sphere depends on its radius, r, and is given by $V = \frac{4}{3}\pi r^3$.



Example 19 Finding the surface area and volume of a sphere

Find the surface area and volume of a sphere of radius 7 cm, correct to 2 decimal places.

SOLUTION	EXPLANATION
$A = 4\pi r^2$	Write the formula for the surface area of a sphere
$= 4\pi(7)^2$	and substitute $r = 7$.
$= 615.75$ cm^2 (to 2 decimal places)	Evaluate and round the answer.
$V = \frac{4}{3}\pi r^3$	Write the formula for the volume of a sphere and
$= \frac{4}{3}\pi(7)^3$	substitute $r = 7$.
$= 1436.76$ cm^3 (to 2 decimal places)	Evaluate and round the answer.

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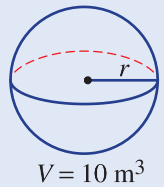
Example 20 Finding the radius of a sphere

Find the radius of a sphere with volume 10 m³, correct to 2 decimal places.



SOLUTION

$$V = \frac{4}{3}\pi r^3$$

$$10 = \frac{4}{3}\pi r^3$$

$$30 = 4\pi r^3$$

$$\frac{15}{2\pi} = r^3$$

$$\therefore r = \sqrt[3]{\frac{15}{2\pi}}$$

$$= 1.336\ldots$$

$\therefore$ The radius is 1.34 m (to 2 decimal places).

EXPLANATION

Substitute $V = 10$ into the formula for the volume of a sphere.

Solve for r^3 by multiplying both sides by 3 and then dividing both sides by 4π. Simplify $\frac{30}{4\pi} = \frac{15}{2\pi}$.

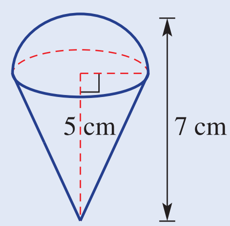
Take the cube root of both sides to make r the subject, then evaluate.

Example 21 Finding the surface area and volume of composite solids with sphere portions

This composite object includes a hemisphere and cone, as shown.

a Find the surface area, rounding to 2 decimal places.

b Find the volume, rounding to 2 decimal places.



SOLUTION

a Radius $r = 7 - 5 = 2$

$$l^2 = 5^2 + 2^2$$

$$l^2 = 29$$

$$l = \sqrt{29}$$

$$A = \frac{1}{2} \times 4\pi r^2 + \pi rl$$

$$= \frac{1}{2} \times 4\pi(2)^2 + \pi(2)(\sqrt{29})$$

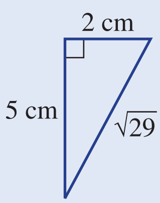
$$= 8\pi + 2\sqrt{29}\pi$$

$$= 58.97 \text{ cm}^2 \text{ (to 2 decimal places)}$$

EXPLANATION

First, find the radius, r cm, of the hemisphere.

Calculate the slant height, l, of the cone using Pythagoras' theorem.



Write the rules for the surface area of each component and note that the top shape is a hemisphere (i.e. half sphere). Only the curved surface of the cone is required.

Substitute $r = 2$ and $h = 5$.

Simplify and then evaluate, rounding as required.

Example continued on next page

b $V = \frac{1}{2} \times \frac{4}{3}\pi r^3 + \frac{1}{3}\pi r^2 h$ — Volume (object) $= \frac{1}{2}$Volume (sphere) + Volume (cone)

$= \frac{1}{2} \times \frac{4}{3}\pi(2)^3 + \frac{1}{3}\pi(2)^2(5)$ — Substitute $r = 2$ and $h = 5$.

$= \frac{16\pi}{3} + \frac{20\pi}{3}$ — Simplify and then evaluate, rounding as required.

$= \frac{36\pi}{3}$

$= 12\pi$

$= 37.70 \text{ cm}^3$ (to 2 decimal places)

Exercise 1I

UNDERSTANDING AND FLUENCY	1–6	4–7	5(½), 6, 7

1 Evaluate and round your answer to 2 decimal places.

a $4 \times \pi \times 5^2$ **b** $4 \times \pi \times 2.2^2$ **c** $4 \times \pi \times \left(\frac{1}{2}\right)^2$

d $\frac{4}{3} \times \pi \times 2^3$ **e** $\frac{4}{3} \times \pi \times 2.8^3$ **f** $\frac{4\pi(7)^3}{3}$

2 For each part of Question **1**, write a question to match these calculations.

3 Find the surface area and volume of a sphere with the given dimensions. Give the answer correct to 2 decimal places.

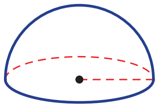
a radius 3 cm **b** radius 4 m **c** radius 7.4 m

d diameter $\sqrt{5}$ mm **e** diameter $\sqrt{7}$ m **f** diameter 2.2 km

4 **a** What fraction of a sphere is shown in this diagram?

b What is the formula for the volume of a hemisphere?

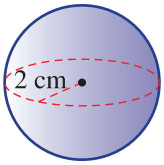
c What is the formula for the curved surface area of a hemisphere?



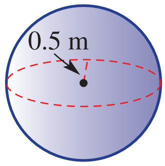
Example 19

5 Find the surface area and volume of the following spheres, correct to 2 decimal places.

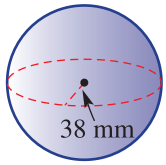
a



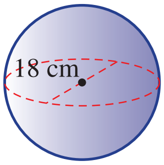
b



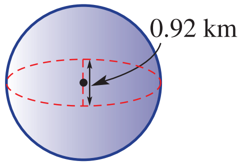
c



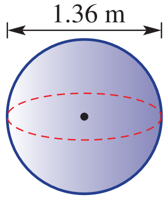
d



e



f



6 **a** Rearrange $A = 4\pi r^2$ to write r in terms of A.

b Rearrange $V = \frac{4}{3}\pi r^3$ to write r in terms of V.

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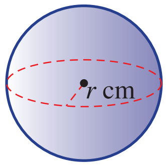
Example 20

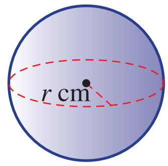
7 **a** Find the radius of these spheres with the given volumes, correct to 2 decimal places.

i



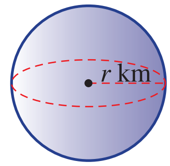
$V = 15\text{ cm}^3$

ii



$V = 180\text{ cm}^3$

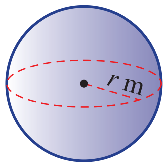
iii



$V = 0.52\text{ km}^3$

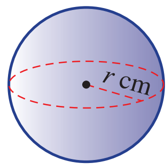
b Find the radius of these spheres with the given surface area, correct to 2 decimal places.

i



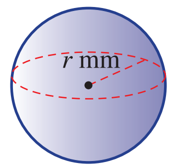
$A = 10\text{ m}^2$

ii



$A = 120\text{ cm}^2$

iii



$A = 0.43\text{ mm}^2$

PROBLEM-SOLVING AND REASONING	8–10, 18	11–13, 14(½), 18, 19	14–15(½), 16, 17, 19–21

8 A box with dimensions 30 cm long, 30 cm wide and 30 cm high holds 50 tennis balls of radius 3 cm. Find:

a the volume of one tennis ball, correct to 2 decimal places

b the volume of 50 tennis balls, correct to 1 decimal place

c the volume of the box not taken up by the tennis balls, correct to 1 decimal place

9 An expanding spherical storage bag has 800 cm^3 of water pumped into it. Find the diameter of the bag, correct to 1 decimal place, after all the water has been pumped in.

10 A sphere just fits inside a cube. What is the surface area of the sphere as a percentage of the surface area of the cube? Round your answer to the nearest whole percentage.

11 Consider a closed hemisphere with radius 2.5 cm. Calculate, correct to 1 decimal place:

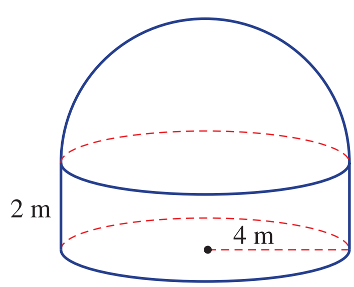
a the volume

b the curved surface area

c the total surface area

12 Two sports balls have radii 10 cm and 15 cm. Find the difference in their surface areas, correct to 1 decimal place.

13 A monolithic structure has a cylindrical base of radius 4 m and height 2 m and a hemispherical top.



a What is the radius of the hemispherical top?

b Find the total volume of the entire monolithic structure, correct to 1 decimal place.

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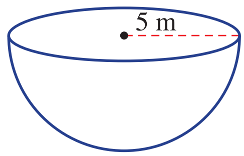
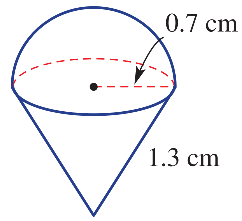
Example 21a

14 Find the surface area for these solids, correct to 2 decimal places.

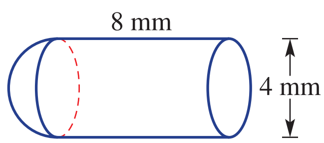
a



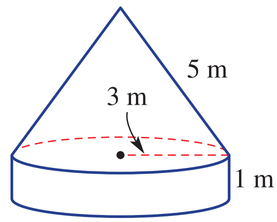
b



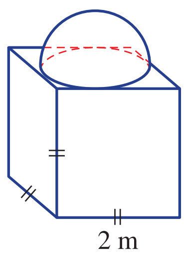
c



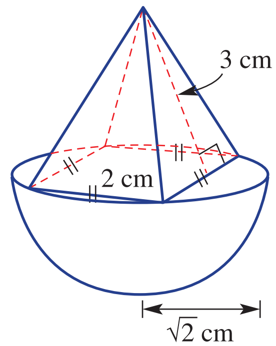
d



e



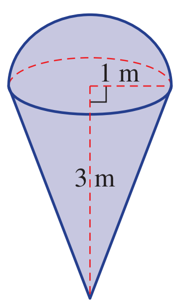
f



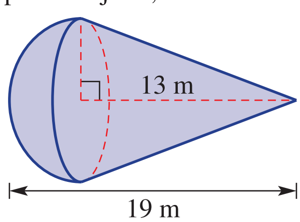
Example 21b

15 Find the volume of the following composite objects, correct to 2 decimal places.

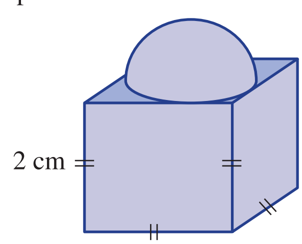
a



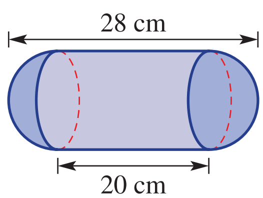
b



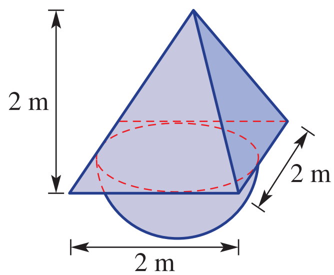
c



d



e



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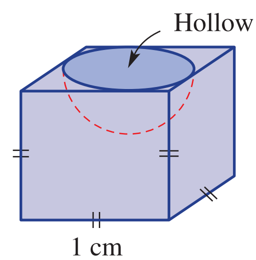

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16 A spherical party balloon is blown up to help decorate a room.

a Find the volume of air, correct to 2 decimal places, needed for the balloon to be:

i 10 cm wide **ii** 20 cm wide **iii** 30 cm wide

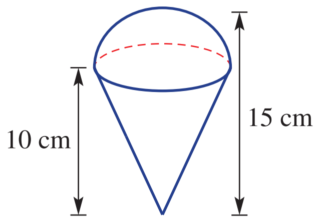
b If the balloon pops when the volume of air reaches 120 000 cm^3, find the diameter of the balloon at that point, correct to 1 decimal place.

17 A hemisphere sits on a cone and two height measurements are given as shown. Find:

a the radius of the hemisphere

b the exact slant height of the cone, in surd form

c the surface area of the solid, correct to 1 decimal place



18 **a** Find a formula for the radius of a sphere with surface area A.

b Find a formula for the radius of a sphere with volume V.

19 A ball's radius is doubled.

a By how much does its surface area change?

b By how much does its volume change?

20 Show that the volume of a sphere is given by $V = \frac{1}{6}\pi d^3$, where d is the diameter.

21 A cylinder and a sphere have the same radius, r, and volume, V. Find a rule for the height of the cylinder in terms of r.

ENRICHMENT	–	–	22

Comparing surface areas

22 Imagine a cube and a sphere that have the same volume.

a If the sphere has volume 1 $unit^3$, find:

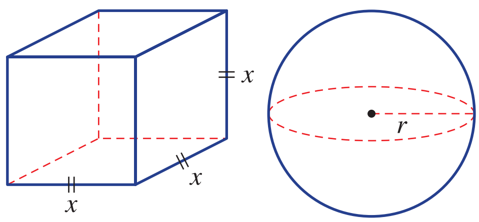
i the exact radius of the sphere

ii the exact surface area of the sphere

iii the value of x (the side length of the cube)

iv the surface area of the cube

v the surface area of the sphere as a percentage of the surface area of the cube, correct to 1 decimal place.



b Now take the radius of the sphere to be r units. Write:

i the formula for the surface area of the sphere

ii the formula for x in terms of r, given the volumes are equal

iii the surface area of the cube in terms of r

c Now write the surface area of the sphere as a fraction of the surface area of the cube, using your results from part **b**.

Simplify to show that the result is $\sqrt[3]{\frac{\pi}{6}}$.

d Compare your answer from part **a v** with that of part **c**.

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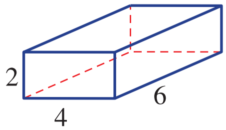
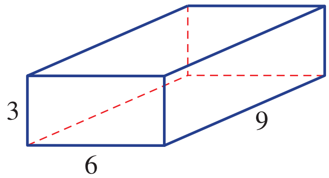
Investigation

1 Similar objects

Enlarging a rectangular prism

The larger rectangular prism shown here is an enlargement of the smaller prism by a factor of 1.5.





a Write down the ratios of the side lengths.
b Find the surface areas of the prisms and write these as a ratio in simplest form.
c Find the volumes of the prisms and write these as a ratio in simplest form.
d What do you notice about the ratios for length, area and volume? Explain.
e Can you write all three ratios using indices with the same base?

Using similarity

From the previous section, we note that if two similar objects have a length ratio $a : b$, then the following apply.

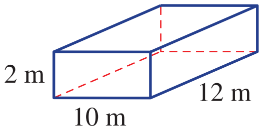
Length ratio $= a : b$ Scale factor $= \frac{b}{a}$

Area ratio $= a^2 : b^2$ Scale factor $= \frac{b^2}{a^2}$

Volume ratio $= a^3 : b^3$ Scale factor $= \frac{b^3}{a^3}$

a A sphere of radius 1 cm is enlarged by a length scale factor of 3.
- **i** Find the ratio of the surface areas of the two spheres.
- **ii** Find the ratio of the volumes of the two spheres.
- **iii** Use your ratio to find the surface area of the larger sphere.
- **iv** Use your ratio to find the volume of the larger sphere.

b This rectangular prism is enlarged by a factor of $\frac{1}{2}$, which means that the prism is reduced in size.



- **i** Find the ratio large : small of the surface areas of the prisms.
- **ii** Find the ratio large : small of their volumes.
- **iii** Use your ratios to find the surface area of the smaller prism.
- **iv** Use your ratios to find the volume of the smaller prism.

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2 Density

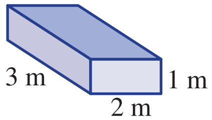
Density is defined as the mass or weight of a substance per cubic unit of volume.

$$\text{Density} = \frac{\text{mass}}{\text{volume}} \qquad \textit{or} \qquad \text{Mass} = \text{density} \times \text{volume}$$

Finding mass

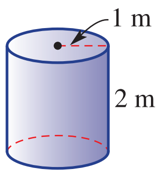
Find the total mass of these objects with the given densities, correct to 1 decimal place where necessary.

a



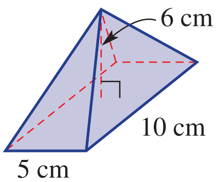
Density = 50 kg per m^3

b



Density = 100 kg per m^3

c

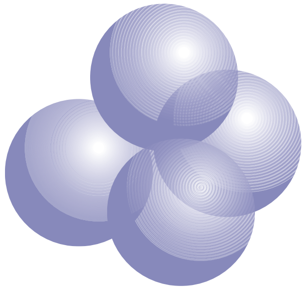


Density = 0.05 kg per m^3

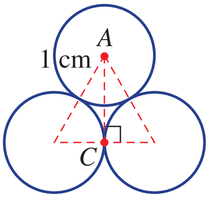
Finding density

a Find the density of a compound with the given mass and volume measurements, rounding your answer to 2 decimal places where necessary.
- **i** mass 30 kg, volume 0.4 m^3
- **ii** mass 10 g, volume 2 cm^3
- **iii** mass 550 kg, volume 1.8 m^3

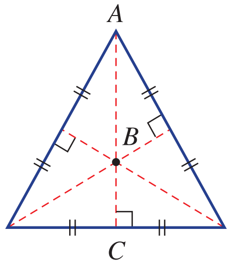
b The density of a solid depends somewhat on how its molecules are packed together. Molecules represented as spheres are packed tightly if they are arranged in a triangular form. The diagram at right relates to this packing arrangement.



- **i** Find the length AC for three circles, each of radius 1 cm, as shown. Use exact values.
- **ii** Find the total height of four spheres, each of radius 1 cm, if they are packed to form a triangular-based pyramid. Use exact values.



First, note that $AB = 2BC$ for an equilateral triangle (shown at right). Pythagoras' theorem can be used to prove this, but this is trickier.



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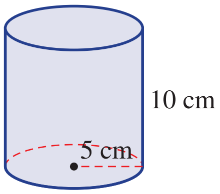
3 Concentration

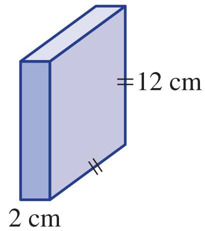
Concentration is associated with the purity of dissolved substances and will be considered here using percentages.

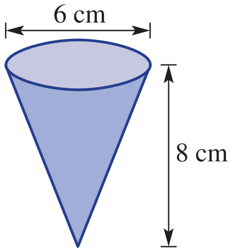
$$\text{Concentration}\,(\%) = \frac{\text{volume of substance}}{\text{total volume}} \times \frac{100}{1}$$

Finding concentration

Find the concentration of acid as a percentage when 10 cm^3 of pure acid is mixed into the given containers, which are *half full* (i.e. half the volume) of water. Give your answers correct to 2 decimal places.

a


b


c


Finding volumes

a Find the volume of pure acid required to give the following concentrations and total volumes.
- **i** concentration 25%, total volume 100 cm^3
- **ii** concentration 10%, total volume 90 m^3
- **iii** concentration 32%, total volume 1.8 L

b A farmer mixes 5 litres of a chemical herbicide into a three-quarters full tank of water. The tank is cylindrical with diameter 1 m and height 1.5 m. Find:
- **i** the number of litres of water in the tank, correct to 3 decimal places
- **ii** the concentration of the herbicide after it has been added to the water, correct to 2 decimal places
- **iii** the possible diameter and height measurements of a tank that would make the concentration of herbicide 1%

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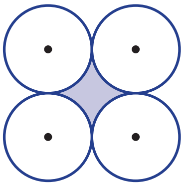
1 A cube has a surface area that has the same value as its volume. What is the side length of this cube?

2 The wheels of a truck travelling at 60 km/h make 4 revolutions per second. What is the diameter of each wheel in metres, correct to 1 decimal place?

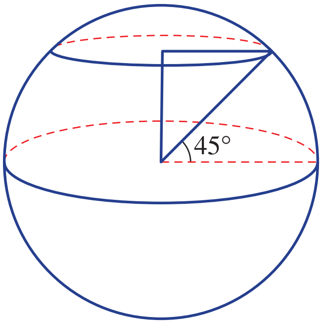
3 A sphere fits exactly inside a cylinder, and just touches the top, bottom and curved surface. Show that the surface area of the sphere equals the curved surface area of the cylinder.

4 A sphere and cone with the same radius, r, have the same volume. Find the height of the cone in terms of r.

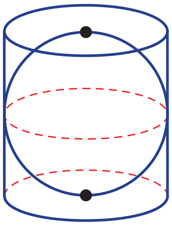
5 Four of the same circular coins of radius r are placed such that they are just touching, as shown. What is the area of the shaded region enclosed by the coins, in terms of r?



6 Find the exact ratio of the equator to the distance around the Earth at latitude 45° north. (Assume that the Earth is a perfect sphere.)



7 A ball just fits inside a cylinder. What percentage of the volume of the cylinder is taken up by the ball? Round your answer to the nearest whole percentage.



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Chapter summary

Measurement

Area

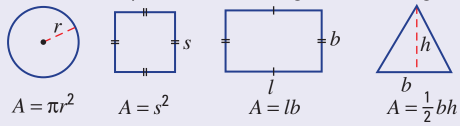
Units of area

$$\text{km}^2 \xrightarrow{\times 1000^2} \text{m}^2 \xrightarrow{\times 100^2} \text{cm}^2 \xrightarrow{\times 10^2} \text{mm}^2$$

$$\text{mm}^2 \xrightarrow{\div 10^2} \text{cm}^2 \xrightarrow{\div 100^2} \text{m}^2 \xrightarrow{\div 1000^2} \text{km}^2$$

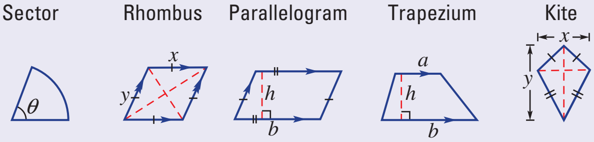
Formulas

Circle, Square, Rectangle, Triangle



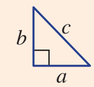
$A = \pi r^2$ | $A = s^2$ | $A = lb$ | $A = \frac{1}{2}bh$

Sector, Rhombus, Parallelogram, Trapezium, Kite



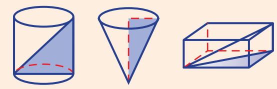
$A = \frac{\theta}{360} \times \pi r^2$ | $A = \frac{1}{2}xy$ | $A = bh$ | $A = \frac{1}{2}h(a+b)$ | $A = \frac{1}{2}xy$

Pythagoras' theorem



$c^2 = a^2 + b^2$

Can occur in 3D shapes



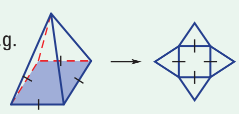
Limits of accuracy

Usually $\pm 0.5 \times$ the smallest unit of measurement

Surface area

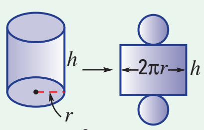
For prisms and pyramids draw the net and add the areas of all the faces.

e.g.



$A = 4 \times \text{triangles} + \text{square base}$

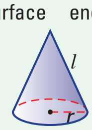
Cylinder



$A = 2\pi rh + 2\pi r^2$

curved surface + base ends

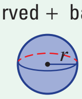
Cone



$A = \pi rl + \pi r^2$

curved + base

Sphere



$A = 4\pi r^2$

For composite solids consider which surfaces are exposed.

Volume

Units of volume

$$\text{km}^3 \xrightarrow{\times 1000^3} \text{m}^3 \xrightarrow{\times 100^3} \text{cm}^3 \xrightarrow{\times 10^3} \text{mm}^3$$

$$\text{mm}^3 \xrightarrow{\div 10^3} \text{cm}^3 \xrightarrow{\div 100^3} \text{m}^3 \xrightarrow{\div 1000^3} \text{km}^3$$

Units of capacity

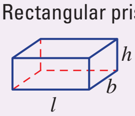
megalitres (ML) $\xrightarrow{\times 1000}$ kilolitres (kL) $\xrightarrow{\times 1000}$ litres (L) $\xrightarrow{\times 1000}$ millilitres (mL)

millilitres (mL) $\xrightarrow{\div 1000}$ litres (L) $\xrightarrow{\div 1000}$ kilolitres (kL) $\xrightarrow{\div 1000}$ megalitres (ML)

$1 \text{ cm}^3 = 1 \text{ mL}$

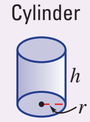
For right and oblique prisms and cylinders $V = Ah$, where A is the area of the base and h is the perpendicular height.

Rectangular prism



$V = lbh$

Cylinder

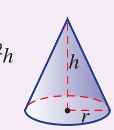


$V = \pi r^2 h$

For pyramids and cones:

$V = \frac{1}{3}Ah$, where A is the area of the base

Cone: $V = \frac{1}{3}\pi r^2 h$



Sphere: $V = \frac{4}{3}\pi r^3$

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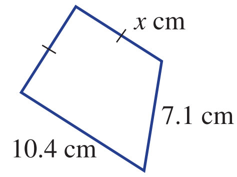
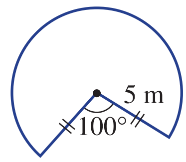
Multiple-choice questions

1 If the perimeter of this shape is 30.3 cm, then the value of x is:



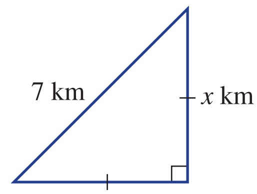
A 12.8 B 6.5
C 5.7 D 6.4
E 3.6

2 The perimeter of the sector shown, rounded to 1 decimal place, is:



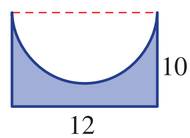
A 8.7 m B 22.7 m
C 18.7 m D 56.7 m
E 32.7 m

3 The value of x in this triangle is closest to:



A 4.9 B 3.5 C 5.0
D 4.2 E 3.9

4 The exact shaded area, in square units, is:



A $32 - 72\pi$ B $120 - 36\pi$
C $32 + 6\pi$ D $120 - 18\pi$
E $48 + 18\pi$

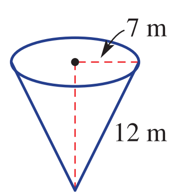
5 $0.128\ \text{m}^2$ is equivalent to:
A $12.8\ \text{cm}^2$ B $128\ \text{mm}^2$ C $1280\ \text{cm}^2$ D $0.00128\ \text{cm}^2$ E $1280\ \text{mm}^2$

6 A cube has a surface area of $1350\ \text{cm}^2$. The side length of the cube is:
A 15 cm B 11 cm C 18 cm D 12 cm E 21 cm

7 A cylindrical tin of canned food has a paper label glued around its curved surface. If the can is 14 cm high and has a radius of 4 cm, the area of the label is closest to:
A $452\ \text{cm}^2$ B $352\ \text{cm}^2$ C $126\ \text{cm}^2$ D $704\ \text{cm}^2$ E $235\ \text{cm}^2$

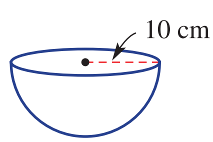
8 The exact surface area of a cone of diameter 24 cm and slant height 16 cm is:
A $216\pi\ \text{cm}^2$ B $960\pi\ \text{cm}^2$ C $528\pi\ \text{cm}^2$ D $336\pi\ \text{cm}^2$ E $384\pi\ \text{cm}^2$

9 A cone has a radius of 7 m and a slant height of 12 m. The cone's exact height, in metres, is:



A $\sqrt{52}$ B $\sqrt{193}$ C $\sqrt{85}$
D $\sqrt{137}$ E $\sqrt{95}$

10 The volume of the hemisphere shown, correct to the nearest cubic centimetre, is:



A $1257\ \text{cm}^3$ B $628\ \text{cm}^3$ C $2094\ \text{cm}^3$
D $1571\ \text{cm}^3$ E $4189\ \text{cm}^3$

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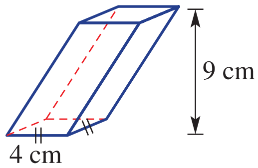
11 The volume of this oblique square prism is:



A 72 cm^3 **B** 48 cm^3
C 176 cm^3 **D** 144 cm^3
E 120 cm^3

12 The volume of air in a sphere is 100 cm^3. The radius of the sphere, correct to 2 decimal places, is:

A 1.67 cm **B** 10.00 cm **C** 2.82 cm **D** 23.87 cm **E** 2.88 cm

Short-answer questions

1 Convert the given measurements to the units in the brackets.

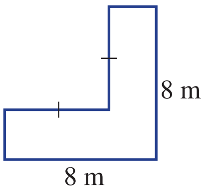
- **a** 0.23 m (cm)
- **b** 270 mm^2 (cm^2)
- **c** 2.6 m^3 (cm^3)
- **d** 8.372 litres (mL)
- **e** 638 250 mm^2 (m^2)
- **f** 0.0003 km^2 (cm^2)
- **g** 6000 ms (s)
- **h** 9 days (s)
- **i** 10^{12} ns (s)
- **j** 1 kt (kg)
- **k** 0.125 s (ms)
- **l** 89 000 000 KB (TB)

2 Write down the limits of accuracy for each of the following measurements.

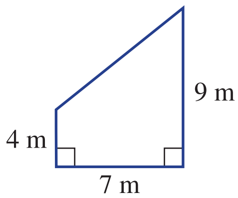
a 6 m **b** 8.9 g **c** 12.05 min

3 Find the perimeter of these shapes, correct to 1 decimal place where necessary. Note Pythagoras' theorem may be required in part **b**.

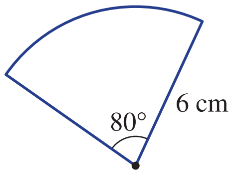
a



b



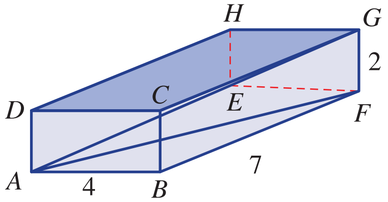
c



4 A floral clock at the Botanic Gardens is in the shape of a circle and has a circumference of 14 m.

- **a** Find the radius of the clock, in exact form.
- **b** Hence, find the area occupied by the clock. Answer to 2 decimal places.

5 For the rectangular prism with dimensions as shown, use Pythagoras' theorem to find:



- **a** AF, leaving your answer in exact form
- **b** AG, to 2 decimal places

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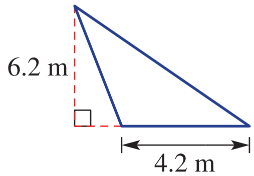
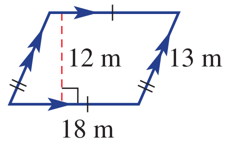
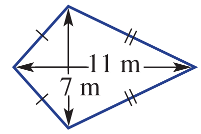
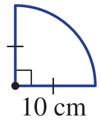
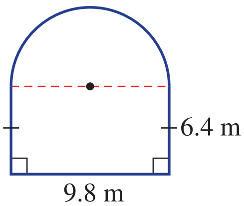
6 Find the area of these shapes. Round your answer to 2 decimal places where necessary.

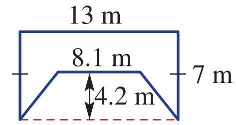
a


b


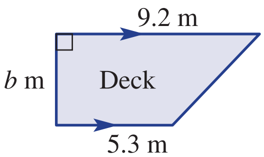
c


d


e


f


7 A backyard deck, as shown, has an area of 34.8 m^2.



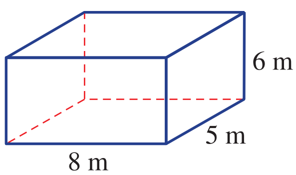
a Find the breadth, *b* metres, of the deck.

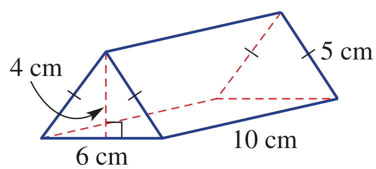
b Calculate the perimeter of the deck, correct to 2 decimal places. Pythagoras' theorem will be required to calculate the missing length.

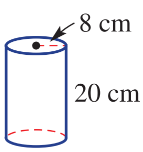
8 For each of these solids find, correct to 2 decimal places where necessary:

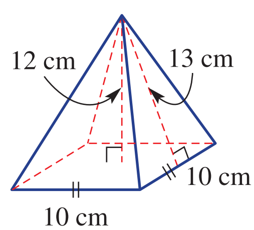
i the surface area

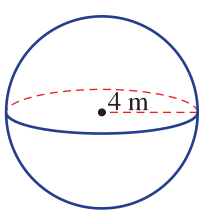
ii the volume

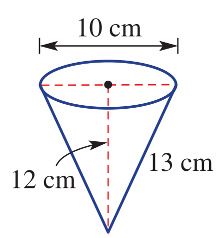
a


b


c


d


e


f


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9 A cone has a radius of 6 cm and a curved surface area of 350 cm^2. Find:

a the slant height of the cone, in exact form

b the height of the cone, correct to 1 decimal place

10 A papier mâché model of a square pyramid with base length 30 cm has a volume of 5400 cm^3.

a What is the height of the pyramid?

b Use Pythagoras' theorem to find the exact height of the triangular faces.

c Hence, find the surface area of the model, correct to 1 decimal place.

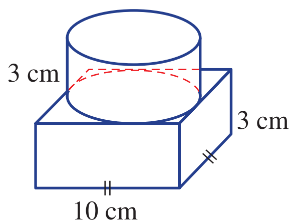
11 A cylinder and a cone each have a base radius of 1 m. The cylinder has a height of 4 m. Determine the height of the cone if the cone and cylinder have the same volume.

12 For each of the following composite solids find, correct to 2 decimal places where necessary:

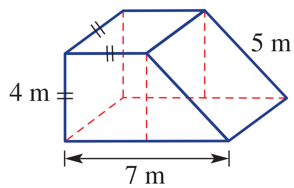
i the surface area

ii the volume

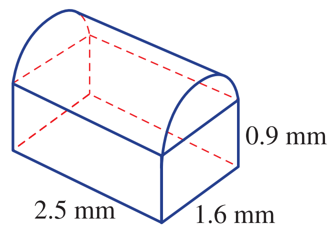
a



b



c

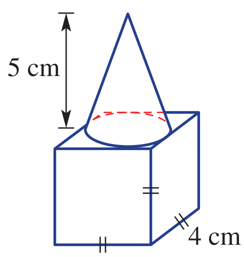


13 For each of the following composite solids find, correct to 2 decimal places where necessary:

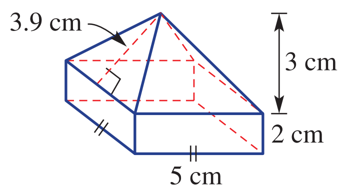
i the surface area

ii the volume

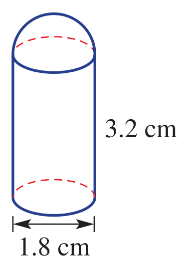
a



b



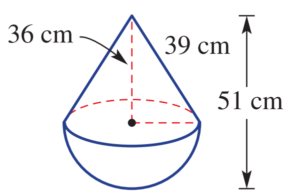
c



14 A water buoy is in the shape shown. Find:

a the volume of air inside the buoy, in exact form

b the surface area of the buoy, in exact form



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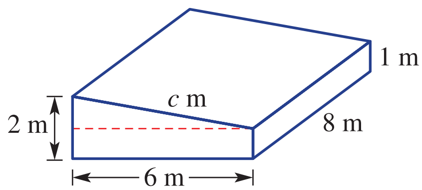
Extended-response questions

1 A waterski ramp consists of a rectangular flotation container and a triangular angled section, as shown.



a What volume of air is contained within the entire ramp structure?

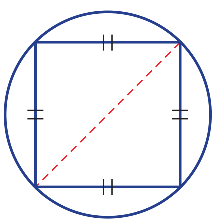
b Find the length of the angled ramp (c metres), in exact surd form.

The entire structure is to be painted with a waterproof paint costing $20 per litre. One litre of paint covers 25 square metres.

c Find the surface area of the ramp, correct to 1 decimal place.

d Find the number of litres and the cost of paint required for the job. Assume you can purchase only 1 litre tins of paint.

2 A circular school oval of radius 50 metres is marked with spray paint to form a square pitch, as shown.



a State the diagonal length of the square.

b Use Pythagoras' theorem to find the side length of the square, in exact surd form.

c Find the area of the square pitch.

d Find the percentage area of the oval that is not part of the square pitch. Round your answer to the nearest whole percentage.

Two athletes challenge each other to a one-lap race around the oval. Athlete A runs around the outside of the oval at an average rate of 10 metres per second. Athlete B runs around the outside of the square at an average rate of 9 metres per second. Athlete B's average running speed is less because of the need to slow down at each corner.

e Find who comes first and the difference in times, correct to the nearest hundredth of a second.

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Online resources

- Auto-marked chapter pre-test
- Video demonstrations of all worked examples
- Interactive widgets
- Interactive walkthroughs
- Downloadable HOTsheets
- Access to all HOTmaths Australian Curriculum courses
- Access to the HOTmaths games library

2 Indices and surds

What you will learn

NSW syllabus

STRAND: NUMBER AND ALGEBRA

SUBSTRANDS: INDICES

SURDS AND INDICES

Outcomes

A student operates with algebraic expressions involving positive-integer and zero indices, and establishes the meaning of negative indices for numerical bases.
(MA5.1–5NA)

A student applies index laws to operate with algebraic expressions involving integer indices.
(MA5.2–7NA)

A student performs operations with surds and indices.
(MA5.3–6NA)

Investment returns

Does an average investment return of 15% sound good to you? Given this return is compounded annually, a \$10 000 investment would grow to over \$40 000 after 10 years. This is calculated by multiplying the investment total by 1.15 (to return the original amount plus the 15%) for each year. Using indices, the total investment value after n years would be given by Value $= 10\,000 \times 1.15^n$.

This is an example of an exponential relation that uses indices to link variables.

We can use such a rule to introduce the set of special numbers called surds. If, for example, you wish to calculate the average investment return that delivers \$10 000 after 10 years from a \$10 000 investment, then you need to calculate x in $100\,000 = 10\,000 \times \left(1 + \frac{x}{100}\right)^{10}$. This gives $x = \left(\sqrt[10]{10} - 1\right) \times 100 \approx 25.9$, representing an annual return of 25.9%.

Indices and surds are commonplace in the world of numbers, especially where money is involved!

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Pre-test

1 Evaluate the following.

a 5^2 **b** $(-5)^2$ **c** -5^2
d $-(-5)^2$ **e** 3^3 **f** 3^{-2}
g 2^{-4} **h** -2×2^{-2} **i** $\sqrt{16}$
j $\sqrt{169}$ **k** $\sqrt[3]{8}$ **l** $\sqrt[3]{64}$

2 Write the following in index form.

a $2 \times 2 \times 2 \times 2 \times 2$ **b** $3 \times 3 \times 4 \times 4 \times 4$ **c** $2 \times a \times a \times b$

3 Write the following in expanded form.

a 5^3 **b** $4^2 3^1$ **c** m^4 **d** $7x^3y^2$

4 Simplify the following, using index laws.

a $y^4 \times y^3$ **b** $b^8 \div b^5$
c $(a^3)^5$ **d** t^0
e $2d^3e \times 5d^4e^2$ **f** $6s^3t \div (4st)$
g $(2gh^4)^3$ **h** $5(mn^7)^0$
i $2x \div (2x)$

5 State whether the following are rational (fractions) or irrational numbers.

a 7 **b** $3.3\dot{3}$ **c** π
d 4.873 **e** $\frac{3}{7}$ **f** $\sqrt{5}$

6 Simplify by collecting like terms.

a $7x - 11x + 5x$ **b** $7a - 4b - 3a + 2b$ **c** $-3ab - 7ba + a$

7 Expand the following.

a $3(x + 4)$ **b** $-2(x + 3)$
c $2(-x - 5)$ **d** $-4(x - 7)$

8 Expand and simplify.

a $(x + 3)(x + 1)$ **b** $(x - 1)(x + 5)$
c $(2x + 3)(x - 4)$ **d** $(x - 2)^2$

9 A sum of \$2000 is deposited into an account paying interest at 5% p.a., compounded annually. The investment balance (\$$A$) is given by $A = 2000(1.05)^t$, where t is the time in years.

a Find the investment balance, correct to the nearest dollar, after:
  i 2 years
  ii 10 years
b How many whole years does it take for the investment to at least double in value. Use a trial-and-error approach.

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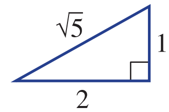
2A Rational numbers and irrational numbers

Interactive
Widgets
HOTsheets
Walkthrough

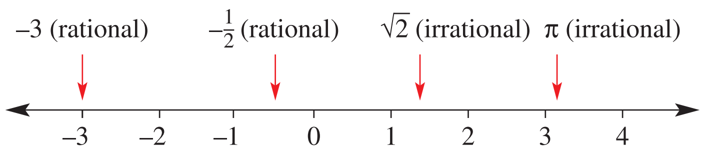
Stage

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

You will recall that when using Pythagoras' theorem to find unknown lengths in right-angled triangles, many answers expressed in exact form are surds. The length of the hypotenuse in this triangle, for example, is $\sqrt{5}$, which is a surd.

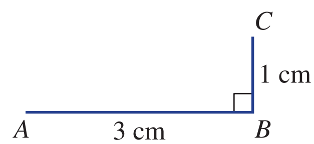


A surd is a number that uses a root sign ($\sqrt{\ }$), sometimes called a radical sign. They are irrational numbers, meaning that they cannot be expressed as a fraction in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Surds, together with other irrational numbers such as pi (π), and all rational numbers (fractions) make up the entire set of real numbers, which can be illustrated as a point on a number line.



Let's start: Constructing surds

Someone asks you: 'How do you construct a line $\sqrt{10}$ cm long?'
Use these steps to construct a line $\sqrt{10}$ cm long.



- First, draw a line segment AB that is 3 cm in length.
- Construct segment BC so that $BC = 1$ cm and $AB \perp BC$. You may wish to use a set square or pair of compasses.
- Now connect point A and point C and measure the length of the segment.
- Use Pythagoras' theorem to check the length AC.

Use this idea to construct line segments with the following lengths. You may need more than one triangle for parts **d** to **f**.

a $\sqrt{2}$ **b** $\sqrt{17}$ **c** $\sqrt{20}$
d $\sqrt{3}$ **e** $\sqrt{6}$ **f** $\sqrt{22}$

Key ideas

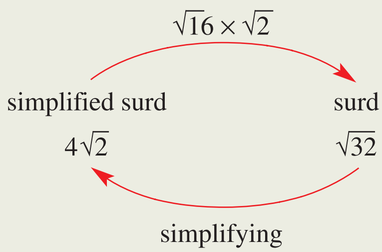
- All **real** numbers can be located as a point on a number line. Real numbers include:
  - **rational numbers** (i.e. numbers that can be expressed as fractions)
    For example: $\frac{3}{7}, -\frac{4}{39}, -3, 1.6, 2.\dot{7}, 0.\overline{19}$
    The decimal representation of a rational number is either a **terminating** or **recurring decimal**.
  - **irrational numbers** (i.e. numbers that cannot be expressed as fractions)
    For example: $\sqrt{3}, -2\sqrt{7}, \sqrt{12} - 1, \pi, 2\pi - 3$
    The decimal representation of an irrational number is an **infinite non-recurring decimal**.
- **Surds** are irrational numbers that use a root sign ($\sqrt{\ }$).
  - For example: $\sqrt{2}, 5\sqrt{11}, -\sqrt{200}, 1 + \sqrt{5}$
  - These numbers are not surds: $\sqrt{4} (= 2), \sqrt[3]{125}\ (= 5), -\sqrt[4]{16}\ (= -2)$

Key ideas

- The following rules apply to surds.
  - $(\sqrt{x})^2 = x$ and $\sqrt{x^2} = x$ when $x \geqslant 0$. For example: $(\sqrt{3})^2 = \sqrt{3^2} = 3$
  - $\sqrt{xy} = \sqrt{x} \times \sqrt{y}$ when $x \geqslant 0$ and $y \geqslant 0$. For example: $\sqrt{6 \times 2} = \sqrt{6} \times \sqrt{2}$
  - $\sqrt{\dfrac{x}{y}} = \dfrac{\sqrt{x}}{\sqrt{y}}$ when $x \geqslant 0$ and $y \geqslant 0$. For example: $\sqrt{\dfrac{5}{3}} = \dfrac{\sqrt{5}}{\sqrt{3}}$
- When a factor of a number is a perfect square we call that factor a square factor. Examples of perfect squares are: 4, 9, 16, 25, 36, 49, 64, 81, 100, …
- When simplifying surds, look for square factors of the number under the root sign.

$$\sqrt{a^2 \times b} = \sqrt{a^2} \times \sqrt{b} = a\sqrt{b}$$

For example:
$$\begin{aligned}\sqrt{32} &= \sqrt{16 \times 2}\\ &= \sqrt{4^2 \times 2}\\ &= \sqrt{4^2} \times \sqrt{2}\\ &= 4\sqrt{2}\end{aligned}$$

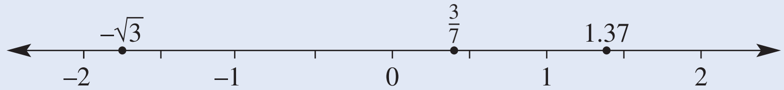


Example 1 Defining and locating surds

Express each number as a decimal and decide if it is rational or irrational. Then locate all the numbers on the same number line.

a $-\sqrt{3}$ **b** 137% **c** $\dfrac{3}{7}$

SOLUTION	EXPLANATION
a $-\sqrt{3} = -1.732050807\ldots$ $-\sqrt{3}$ is irrational.	Use a calculator to express as a decimal. The decimal does not terminate and there is no recurring pattern.
b $137\% = \dfrac{137}{100} = 1.37$ 137% is rational.	137% is a fraction and can be expressed using a terminating decimal.
c $\dfrac{3}{7} = 0.\overline{428571}$ or $0.\dot{4}2857\dot{1}$ $\dfrac{3}{7}$ is rational.	$\dfrac{3}{7}$ is an infinitely recurring decimal. Use the decimal equivalents to locate each number on the real number line.



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Example 2 Converting recurring decimals to fractions

Convert the following decimals to fractions.

a $0.\dot{2}$ **b** $0.\dot{2}\dot{3}$ **c** $0.2\dot{3}$

SOLUTION	EXPLANATION
a $0.\dot{2} = 0.22222\ldots$	
Let $x = 0.22222\ldots$ ①	There is only one digit that repeats, so multiply by 10.
$\therefore 10x = 2.22222\ldots$ ②	
② − ① gives $9x = 2$	Subtract ① from ②.
$\therefore x = \dfrac{2}{9}$	Divide both sides by 9. Note: Some calculators are able to do this. Fill your screen with 0.22222… and then press ⊟.
b $0.\dot{2}\dot{3} = 0.232323\ldots$	
Let $x = 0.232323\ldots$ ①	In this example, two digits repeat, so multiply by 100.
$\therefore 100x = 23.232323\ldots$ ②	
② − ① gives $99x = 23$	Divide both sides by 99.
$\therefore x = \dfrac{23}{99}$	Use a calculator to check your answer.
c $0.2\dot{3} = 0.2333333\ldots$	
Let $x = 0.2333333\ldots$ ①	Only one digit repeats, so multiply by 10.
$\therefore 10x = 2.3333333\ldots$ ②	
② − ① gives $9x = 2.1$	
$x = \dfrac{2.1}{9}$	Multiply the numerator and denominator by 10.
$x = \dfrac{21}{90}$	Use a calculator to check your answer.

Example 3 Simplifying surds

Simplify the following.

a $\sqrt{32}$ **b** $3\sqrt{200}$ **c** $\dfrac{5\sqrt{40}}{6}$ **d** $\sqrt{\dfrac{75}{9}}$

SOLUTION	EXPLANATION
a $\sqrt{32} = \sqrt{16 \times 2}$ $= \sqrt{16} \times \sqrt{2}$ $= 4\sqrt{2}$	When simplifying, choose the highest square factor of 32 (i.e. 16 rather than 4), as there is less work to do to arrive at the same answer. Compare with $\sqrt{32} = \sqrt{4 \times 8} = 2\sqrt{8} = 2\sqrt{4 \times 2} = 4\sqrt{2}$
b $3\sqrt{200} = 3\sqrt{100 \times 2}$ $= 3 \times \sqrt{100} \times \sqrt{2}$ $= 3 \times 10 \times \sqrt{2} = 30\sqrt{2}$	Select the appropriate factors of 200 by finding its highest square factor: 100. Use $\sqrt{x \times y} = \sqrt{x} \times \sqrt{y}$ and simplify.

Example continued on next page

c $\frac{5\sqrt{40}}{6} = \frac{5\sqrt{4 \times 10}}{6}$

$= \frac{5 \times \sqrt{4} \times \sqrt{10}}{6}$

$= \frac{{}^{5}\cancel{10}\sqrt{10}}{\cancel{6}_{3}}$

$= \frac{5\sqrt{10}}{3}$

Select the appropriate factors of 40. The highest square factor is 4.

Cancel and simplify.

d $\sqrt{\frac{75}{9}} = \frac{\sqrt{75}}{\sqrt{9}}$

$= \frac{\sqrt{25 \times 3}}{\sqrt{9}} = \frac{5\sqrt{3}}{3}$

Use $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$.

Then select the factors of 75 that include a square number and simplify.

Example 4 Expressing as a single square root of a positive integer

Express these surds as a square root of a positive integer.

a $2\sqrt{5}$ **b** $7\sqrt{2}$

SOLUTION

a $2\sqrt{5} = \sqrt{4} \times \sqrt{5}$

$= \sqrt{20}$

b $7\sqrt{2} = \sqrt{49} \times \sqrt{2}$

$= \sqrt{98}$

EXPLANATION

Write 2 as $\sqrt{4}$ and then combine the two surds using $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$.

Write 7 as $\sqrt{49}$ and combine.

Exercise 2A

UNDERSTANDING AND FLUENCY	1–5, 6–8(½)	2(½), 3–9(½)	3–9(½)

1 Choose the correct word from the words given in italic to make the sentence true.

a A number that cannot be expressed as a fraction is a *rational/irrational* number.

b A surd is an irrational number that uses a *root/square* symbol.

c The decimal representation of a surd is a *terminating/recurring/non-recurring* decimal.

d $\sqrt{25}$ is a *surd/rational number*.

Example 1

2 Express each number as a decimal and decide if it is rational or irrational. Then locate all the numbers on the same number line.

a $\sqrt{5}$ **b** 18% **c** $\frac{2}{5}$ **d** −124%

e $1\frac{5}{7}$ **f** $-\sqrt{2}$ **g** $2\sqrt{3}$ **h** π

Example 2

3 Convert these recurring decimals to fractions. Use your calculator to check your answers.

a $0.\dot{8}$ **b** $0.1\dot{8}$ **c** $0.\dot{1}\dot{8}$ **d** $0.41\dot{6}$

4 Decide if these numbers are surds.

a $\sqrt{7}$ **b** $2\sqrt{11}$ **c** $2\sqrt{25}$ **d** $-5\sqrt{144}$

e $\frac{3\sqrt{9}}{2}$ **f** $\frac{-5\sqrt{3}}{2}$ **g** $1 - \sqrt{3}$ **h** $2\sqrt{1} + \sqrt{4}$

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5 Write down the highest square factor of these numbers. For example, the highest square factor of 45 is 9.

a 20 **b** 18 **c** 125 **d** 24
e 48 **f** 96 **g** 72 **h** 108

Example 3a

6 Simplify the following surds.

a $\sqrt{12}$ **b** $\sqrt{45}$ **c** $\sqrt{24}$
d $\sqrt{48}$ **e** $\sqrt{75}$ **f** $\sqrt{500}$
g $\sqrt{98}$ **h** $\sqrt{90}$ **i** $\sqrt{128}$
j $\sqrt{360}$ **k** $\sqrt{162}$ **l** $\sqrt{80}$

Example 3b, c

7 Simplify the following.

a $2\sqrt{18}$ **b** $3\sqrt{20}$
c $4\sqrt{48}$ **d** $2\sqrt{63}$
e $3\sqrt{98}$ **f** $4\sqrt{125}$
g $\frac{\sqrt{45}}{3}$ **h** $\frac{\sqrt{28}}{2}$
i $\frac{\sqrt{24}}{4}$ **j** $\frac{\sqrt{54}}{12}$
k $\frac{\sqrt{80}}{20}$ **l** $\frac{\sqrt{99}}{18}$
m $\frac{3\sqrt{44}}{2}$ **n** $\frac{5\sqrt{200}}{25}$
o $\frac{2\sqrt{98}}{7}$ **p** $\frac{3\sqrt{68}}{21}$
q $\frac{6\sqrt{75}}{20}$ **r** $\frac{4\sqrt{150}}{5}$
s $\frac{2\sqrt{108}}{18}$ **t** $\frac{3\sqrt{147}}{14}$

Example 3d

8 Simplify the following.

a $\sqrt{\frac{8}{9}}$ **b** $\sqrt{\frac{12}{49}}$ **c** $\sqrt{\frac{18}{25}}$ **d** $\sqrt{\frac{11}{25}}$
e $\sqrt{\frac{10}{9}}$ **f** $\sqrt{\frac{21}{144}}$ **g** $\sqrt{\frac{26}{32}}$ **h** $\sqrt{\frac{28}{50}}$
i $\sqrt{\frac{15}{27}}$ **j** $\sqrt{\frac{27}{4}}$ **k** $\sqrt{\frac{45}{72}}$ **l** $\sqrt{\frac{56}{76}}$

Example 4

9 Express these surds as a square root of a positive integer.

a $2\sqrt{3}$ **b** $4\sqrt{2}$ **c** $5\sqrt{2}$
d $3\sqrt{3}$ **e** $3\sqrt{5}$ **f** $6\sqrt{3}$
g $8\sqrt{2}$ **h** $10\sqrt{7}$ **i** $9\sqrt{10}$
j $5\sqrt{5}$ **k** $7\sqrt{5}$ **l** $11\sqrt{3}$

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PROBLEM-SOLVING AND REASONING	10, 11, 14	10, 11, 13(½), 14–16	12, 13(½), 14–17

10 Simplify by searching for the highest square factor.

a $\sqrt{675}$ **b** $\sqrt{1183}$ **c** $\sqrt{1805}$ **d** $\sqrt{2883}$

11 Determine the exact side length, in simplest form, of a square with the given area.

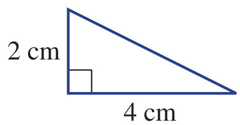
a 32 m^2 **b** 120 cm^2 **c** 240 mm^2

12 Determine the exact radius and diameter of a circle, in simplest form, with the given area.

a $24\pi\text{ cm}^2$ **b** $54\pi\text{ m}^2$ **c** $128\pi\text{ m}^2$

13 Use Pythagoras' theorem to find the unknown length in these triangles, in simplest form.

a



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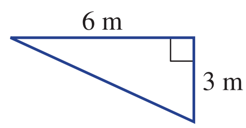

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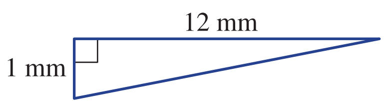

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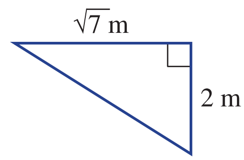

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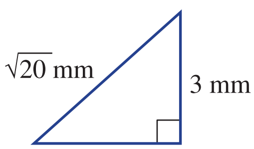
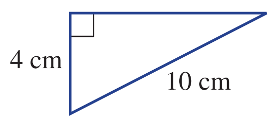


f



14 Ricky uses the following working to simplify $\sqrt{72}$. Show how Ricky could have simplified $\sqrt{72}$ using fewer steps.

$$\begin{aligned}\sqrt{72} &= \sqrt{9 \times 8}\\ &= 3\sqrt{8}\\ &= 3\sqrt{4 \times 2}\\ &= 3 \times 2 \times \sqrt{2}\\ &= 6\sqrt{2}\end{aligned}$$

15 **a** List all the factors of 450 that are perfect squares.

b Now simplify $\sqrt{450}$ using the highest of these factors.

16 Use Pythagoras' theorem to construct a line segment with the given lengths. You can use only a ruler and a set square or pair of compasses. Do not use a calculator.

a $\sqrt{10}$ cm **b** $\sqrt{29}$ cm

c $\sqrt{6}$ cm **d** $\sqrt{22}$ cm

17 Is $0.\dot{9} = 1$? Search the internet and write an explanation.

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ENRICHMENT	–	–	18

Proving $\sqrt{2}$ is irrational

18 We will prove that $\sqrt{2}$ is irrational by the method called 'proof by contradiction'. Your job is to follow and understand the proof, then copy it out and try explaining it to a friend or teacher.

a Before we start, we first need to show that if a perfect square a^2 is even, then a is even. We do this by showing that if a is even then a^2 is even, and if a is odd then a^2 is odd.

If a is even, then $a = 2k$, where k is an integer.

$$\begin{aligned} \text{So } a^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2 \times 2k^2, \text{ which must be even.} \end{aligned}$$

If a is odd then $a = 2k + 1$, where k is an integer.

$$\begin{aligned} \text{So } a^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2 \times (2k^2 + 2k) + 1, \text{ which must be odd.} \end{aligned}$$

$\therefore$ If a^2 is even, then a is even.

b Now, to prove $\sqrt{2}$ is irrational let's suppose that $\sqrt{2}$ is instead rational and can be written in the form $\frac{a}{b}$ in simplest form, where a and b are integers ($b \neq 0$) and at least one of a or b is odd.

$\therefore \sqrt{2} = \frac{a}{b}$

So $2 = \frac{a^2}{b^2}$ (squaring both sides)

$a^2 = 2b^2$

$\therefore$ a^2 is even and, from part **a** above, a must be even.

If a is even then $a = 2k$, where k is an integer.

$\therefore$ If $a^2 = 2b^2$

Then $(2k)^2 = 2b^2$

$4k^2 = 2b^2$

$2k^2 = b^2$

$\therefore$ b^2 is even and therefore b is even.

This is a contradiction because at least one of a or b must be odd. Therefore, the assumption that $\sqrt{2}$ can be written in the form $\frac{a}{b}$ must be incorrect and so $\sqrt{2}$ is irrational.

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2B Adding and subtracting surds

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We can apply our knowledge of like terms in algebra to help simplify expressions involving the addition and subtraction of surds. Recall that $7x$ and $3x$ are like terms, so $7x + 3x = 10x$. The pronumeral x represents any number. When $x = 5$, then $7 \times 5 + 3 \times 5 = 10 \times 5$ and when $x = \sqrt{2}$, then $7\sqrt{2} + 3\sqrt{2} = 10\sqrt{2}$. Multiples of the same surd are called 'like surds' and can be collected (i.e. counted) in the same way as we collect like terms in algebra.

Let's start: Can $3\sqrt{2} + \sqrt{8}$ be simplified?

To answer this question, first discuss these points.

- Are $3\sqrt{2}$ and $\sqrt{8}$ like surds?
- How can $\sqrt{8}$ be simplified?
- Now decide whether $3\sqrt{2} + \sqrt{8}$ can be simplified. Discuss why $3\sqrt{2} - \sqrt{7}$ cannot be simplified.

Key ideas

- **Like surds** are multiples of the same surd.
  For example: $\sqrt{3}, -5\sqrt{3}, \sqrt{12}, (= 2\sqrt{3}), 2\sqrt{75}, (= 10\sqrt{3})$
- Like surds can be added and subtracted.
- Simplify all surds before attempting to add or subtract them.

Example 5 Adding and subtracting surds

Simplify the following.

a $2\sqrt{3} + 4\sqrt{3}$

b $4\sqrt{6} + 3\sqrt{2} - 3\sqrt{6} + 2\sqrt{2}$

SOLUTION

a $2\sqrt{3} + 4\sqrt{3} = 6\sqrt{3}$

b $4\sqrt{6} + 3\sqrt{2} - 3\sqrt{6} + 2\sqrt{2} = \sqrt{6} + 5\sqrt{2}$

EXPLANATION

Collect like surds by adding the coefficients: $2 + 4 = 6$.

Collect like surds involving $\sqrt{6}$:
$4\sqrt{6} - 3\sqrt{6} = 1\sqrt{6} = \sqrt{6}$
Then collect those with $\sqrt{2}$.

Example 6 Simplifying surds to add or subtract

Simplify these surds.

a $5\sqrt{2} - \sqrt{8}$

b $2\sqrt{5} - 3\sqrt{20} + 6\sqrt{45}$

SOLUTION

a
$$\begin{aligned} 5\sqrt{2} - \sqrt{8} &= 5\sqrt{2} - \sqrt{4 \times 2} \\ &= 5\sqrt{2} - 2\sqrt{2} \\ &= 3\sqrt{2} \end{aligned}$$

b
$$\begin{aligned} 2\sqrt{5} - 3\sqrt{20} + 6\sqrt{45} &= 2\sqrt{5} - 3\sqrt{4 \times 5} + 6\sqrt{9 \times 5} \\ &= 2\sqrt{5} - 6\sqrt{5} + 18\sqrt{5} \\ &= 14\sqrt{5} \end{aligned}$$

EXPLANATION

First look to simplify surds: $\sqrt{8}$ has a highest square factor of 4 and can be simplified to $2\sqrt{2}$. Then subtract like surds.

Simplify the surds and then collect like surds.
Note that $3\sqrt{4 \times 5} = 3 \times \sqrt{4} \times \sqrt{5} = 6\sqrt{5}$.

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Exercise 2B

UNDERSTANDING AND FLUENCY	1–4, 5–7(½)	4, 5–7(½)	5–7(½)

1 Decide if the following pairs of numbers are like surds.

a $\sqrt{3}, 2\sqrt{3}$ **b** $5, \sqrt{5}$ **c** $2\sqrt{2}, 2$ **d** $4\sqrt{6}, \sqrt{6}$

e $2\sqrt{3}, 5\sqrt{3}$ **f** $3\sqrt{7}, 3\sqrt{5}$ **g** $-2\sqrt{5}, 3\sqrt{5}$ **h** $-\sqrt{7}, -2\sqrt{7}$

2 Copy and complete:

a $11x - 5x =$ ______, so $11\sqrt{2} - 5\sqrt{2} =$ ______.

b $8y - y =$ ______, so $8\sqrt{2} - \sqrt{2} =$ ______.

c $8y - 7y =$ ______, so $8\sqrt{2} - 7\sqrt{2} =$ ______.

3 **a** Simplify the surd $\sqrt{48}$.

b Hence, simplify the following.

i $\sqrt{3} + \sqrt{48}$ **ii** $\sqrt{48} - 7\sqrt{3}$ **iii** $5\sqrt{48} - 3\sqrt{3}$

4 **a** Simplify the surd $\sqrt{200}$.

b Hence, simplify the following.

i $\sqrt{200} - 5\sqrt{2}$ **ii** $-2\sqrt{200} + 4\sqrt{2}$ **iii** $3\sqrt{200} - 30\sqrt{2}$

Example 5a

5 Simplify the following.

a $2\sqrt{5} + 4\sqrt{5}$

b $5\sqrt{3} - 2\sqrt{3}$

c $7\sqrt{2} - 3\sqrt{2}$

d $8\sqrt{2} - 5\sqrt{2}$

e $7\sqrt{5} + 4\sqrt{5}$

f $6\sqrt{3} - 5\sqrt{3}$

g $4\sqrt{10} + 3\sqrt{10} - \sqrt{10}$

h $6\sqrt{2} - 4\sqrt{2} + 3\sqrt{2}$

i $\sqrt{21} - 5\sqrt{21} + 2\sqrt{21}$

j $3\sqrt{11} - 8\sqrt{11} - \sqrt{11}$

k $-2\sqrt{13} + 5\sqrt{13} - 4\sqrt{13}$

l $10\sqrt{30} - 15\sqrt{30} - 2\sqrt{30}$

Example 5b

6 Simplify the following.

a $2\sqrt{3} + 3\sqrt{2} - \sqrt{3} + 2\sqrt{3}$

b $5\sqrt{6} + 4\sqrt{11} - 2\sqrt{6} + 3\sqrt{11}$

c $3\sqrt{5} - 4\sqrt{2} + \sqrt{5} - 3\sqrt{2}$

d $5\sqrt{2} + 2\sqrt{5} - 7\sqrt{2} - \sqrt{5}$

e $2\sqrt{3} + 2\sqrt{7} + 2\sqrt{3} - 2\sqrt{7}$

f $5\sqrt{11} + 3\sqrt{6} - 3\sqrt{6} - 5\sqrt{11}$

g $2\sqrt{2} - 4\sqrt{10} - 5\sqrt{2} + \sqrt{10}$

h $-4\sqrt{5} - 2\sqrt{15} + 5\sqrt{15} + 2\sqrt{5}$

Example 6a

7 Simplify the following.

a $\sqrt{8} - \sqrt{2}$ **b** $\sqrt{8} + 3\sqrt{2}$ **c** $\sqrt{27} + \sqrt{3}$

d $\sqrt{20} - \sqrt{5}$ **e** $4\sqrt{18} - 5\sqrt{2}$ **f** $2\sqrt{75} + 2\sqrt{3}$

g $3\sqrt{44} + 2\sqrt{11}$ **h** $3\sqrt{8} - \sqrt{18}$ **i** $\sqrt{24} + \sqrt{54}$

j $2\sqrt{125} - 3\sqrt{45}$ **k** $3\sqrt{72} + 2\sqrt{98}$ **l** $3\sqrt{800} - 4\sqrt{200}$

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PROBLEM-SOLVING AND REASONING	8(½), 11	8(½), 10(½), 11, 12(½)	8–10(½), 12(½), 13

Example 6b

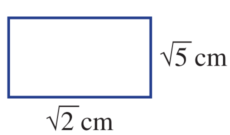
8 Simplify the following.

- **a** $\sqrt{2} + \sqrt{50} + \sqrt{98}$
- **b** $\sqrt{6} - 2\sqrt{24} + 3\sqrt{96}$
- **c** $5\sqrt{7} + 2\sqrt{5} - 3\sqrt{28}$
- **d** $2\sqrt{80} - \sqrt{45} + 2\sqrt{63}$
- **e** $\sqrt{150} - \sqrt{96} - \sqrt{162} + \sqrt{72}$
- **f** $\sqrt{12} + \sqrt{125} - \sqrt{50} + \sqrt{180}$
- **g** $7\sqrt{3} - 2\sqrt{8} + \sqrt{12} + 3\sqrt{8}$
- **h** $\sqrt{36} - \sqrt{108} + \sqrt{25} - 3\sqrt{3}$
- **i** $3\sqrt{49} + 2\sqrt{288} - \sqrt{144} - 2\sqrt{18}$
- **j** $2\sqrt{200} + 3\sqrt{125} + \sqrt{32} - 3\sqrt{242}$

9 Simplify these surds that involve fractions. Remember to use the LCD (lowest common denominator).

- **a** $\dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{3}}{3}$
- **b** $\dfrac{\sqrt{5}}{4} + \dfrac{\sqrt{5}}{3}$
- **c** $\dfrac{\sqrt{2}}{5} - \dfrac{\sqrt{2}}{6}$
- **d** $\dfrac{\sqrt{7}}{4} - \dfrac{\sqrt{7}}{12}$
- **e** $\dfrac{2\sqrt{2}}{5} - \dfrac{\sqrt{2}}{2}$
- **f** $\dfrac{3\sqrt{3}}{7} + \dfrac{\sqrt{3}}{2}$
- **g** $\dfrac{7\sqrt{5}}{6} - \dfrac{4\sqrt{5}}{9}$
- **h** $\dfrac{3\sqrt{3}}{10} - \dfrac{8\sqrt{3}}{15}$
- **i** $\dfrac{-5\sqrt{10}}{6} + \dfrac{3\sqrt{10}}{8}$

10 Find the perimeter of these rectangles and triangles, in simplest form.

a


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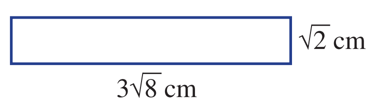

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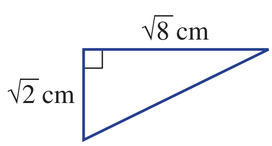

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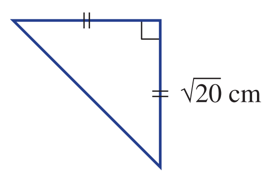

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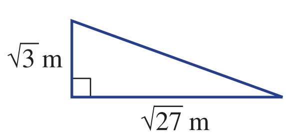

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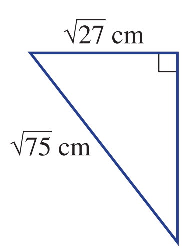

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11 **a** Explain why $\sqrt{5}$ and $\sqrt{20}$ can be thought of as like surds.
b Explain why $3\sqrt{72}$ and $\sqrt{338}$ can be thought of as like surds.

12 Prove that each of the following simplifies to zero by showing all steps.

a $5\sqrt{3} - \sqrt{108} + \sqrt{3}$
b $\sqrt{6} + \sqrt{24} - 3\sqrt{6}$
c $6\sqrt{2} - 2\sqrt{32} + 2\sqrt{2}$
d $\sqrt{8} - \sqrt{18} + \sqrt{2}$
e $2\sqrt{20} - 7\sqrt{5} + \sqrt{45}$
f $3\sqrt{2} - 2\sqrt{27} - \sqrt{50} + 6\sqrt{3} + \sqrt{8}$

13 Prove that the surds in these expressions cannot be added or subtracted.

a $3\sqrt{12} - \sqrt{18}$
b $4\sqrt{8} + \sqrt{20}$
c $\sqrt{50} - 2\sqrt{45}$
d $5\sqrt{40} + 2\sqrt{75}$
e $2\sqrt{200} + 3\sqrt{300}$
f $\sqrt{80} - 2\sqrt{54}$

ENRICHMENT	–	–	14

Simplifying both surds and fractions

14 To simplify the following, you will need to simplify surds and combine using a common denominator.

a $\frac{\sqrt{8}}{3} - \frac{\sqrt{2}}{5}$
b $\frac{\sqrt{12}}{4} + \frac{\sqrt{3}}{6}$
c $\frac{3\sqrt{5}}{4} - \frac{\sqrt{20}}{3}$
d $\frac{\sqrt{98}}{4} - \frac{5\sqrt{2}}{2}$
e $\frac{2\sqrt{75}}{5} - \frac{3\sqrt{3}}{2}$
f $\frac{\sqrt{63}}{9} - \frac{4\sqrt{7}}{5}$
g $\frac{2\sqrt{18}}{3} - \frac{\sqrt{72}}{2}$
h $\frac{\sqrt{54}}{4} + \frac{\sqrt{24}}{7}$
i $\frac{\sqrt{27}}{5} - \frac{\sqrt{108}}{10}$
j $\frac{5\sqrt{48}}{6} + \frac{2\sqrt{147}}{3}$
k $\frac{2\sqrt{96}}{5} - \frac{\sqrt{600}}{7}$
l $\frac{3\sqrt{125}}{14} - \frac{2\sqrt{80}}{21}$

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2C Multiplying and dividing surds

Interactive Widgets HOTsheets Walkthrough

Stage

5.3#
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5.2
5.2◊
5.1
4

When simplifying surds such as $\sqrt{18}$, we write $\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$, where we use the fact that $\sqrt{xy} = \sqrt{x} \times \sqrt{y}$. This can be used in reverse to simplify the product of two surds. A similar process is used for division.

Let's start: Exploring products and quotients

When adding and subtracting surds we can combine like surds. Do you think this is true for multiplying and dividing surds?

- Use a calculator to find a decimal approximation for $\sqrt{5} \times \sqrt{3}$ and for $\sqrt{15}$.
- Use a calculator to find a decimal approximation for $2\sqrt{10} \div \sqrt{5}$ and for $2\sqrt{2}$.
- What do you notice about the results from above? Try other pairs of surds to see if your observations are consistent.

Key ideas

- When multiplying surds, use the following result.
  - $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$
  - More generally: $a\sqrt{x} \times b\sqrt{y} = ab\sqrt{xy}$
- When dividing surds, use the following result.
  - $\dfrac{\sqrt{x}}{\sqrt{y}} = \sqrt{\dfrac{x}{y}}$
  - More generally: $\dfrac{a\sqrt{x}}{b\sqrt{y}} = \dfrac{a}{b}\sqrt{\dfrac{x}{y}}$
- Use the distributive law to expand brackets.
  - $a(b + c) = ab + ac$, so $\sqrt{2}(\sqrt{3} + 1) = \sqrt{6} + \sqrt{2}$.

Example 7 Simplifying a product of two surds

Simplify the following.

a $\sqrt{2} \times \sqrt{3}$

b $2\sqrt{3} \times 3\sqrt{15}$

SOLUTION

a $\sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3}$
$= \sqrt{6}$

EXPLANATION

Use $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$.

SOLUTION

b
$$\begin{aligned} 2\sqrt{3} \times 3\sqrt{15} &= 2 \times 3 \times \sqrt{3 \times 15} \\ &= 6\sqrt{45} \\ &= 6\sqrt{9 \times 5} \\ &= 6 \times \sqrt{9} \times \sqrt{5} \\ &= 18\sqrt{5} \end{aligned}$$

EXPLANATION

Use $a\sqrt{x} \times b\sqrt{y} = ab\sqrt{xy}$.
Then simplify the surd $\sqrt{45}$, which has a highest square factor of 9, using $\sqrt{9} = 3$.
Alternatively, using $\sqrt{15} = \sqrt{3} \times \sqrt{5}$:

$$\begin{aligned} 2\sqrt{3} \times 3\sqrt{15} &= 2 \times 3 \times \sqrt{3} \times \sqrt{3} \times \sqrt{5} \\ &= 2 \times 3 \times 3 \times \sqrt{5} \\ &= 18\sqrt{5} \end{aligned}$$

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Example 8 Simplifying surds using division

Simplify these surds.

a $-\sqrt{10} \div \sqrt{2}$

b $\dfrac{12\sqrt{18}}{3\sqrt{3}}$

SOLUTION

a $-\sqrt{10} \div \sqrt{2} = -\sqrt{\dfrac{10}{2}} = -\sqrt{5}$

b $\dfrac{12\sqrt{18}}{3\sqrt{3}} = \dfrac{12}{3}\sqrt{\dfrac{18}{3}} = 4\sqrt{6}$

EXPLANATION

Use $\sqrt{x} \div \sqrt{y} = \sqrt{\dfrac{x}{y}}$.

Use $\dfrac{a\sqrt{x}}{b\sqrt{y}} = \dfrac{a}{b}\sqrt{\dfrac{x}{y}}$.

Example 9 Using the distributive law

Use the distributive law to expand the following and then simplify the surds where necessary.

a $\sqrt{3}(3\sqrt{5} - \sqrt{6})$

b $3\sqrt{6}(2\sqrt{10} - 4\sqrt{6})$

SOLUTION

a
$$\begin{aligned}\sqrt{3}(3\sqrt{5} - \sqrt{6}) &= 3\sqrt{15} - \sqrt{18}\\ &= 3\sqrt{15} - \sqrt{9 \times 2}\\ &= 3\sqrt{15} - 3\sqrt{2}\end{aligned}$$

b
$$\begin{aligned}3\sqrt{6}(2\sqrt{10} - 4\sqrt{6}) &= 6\sqrt{60} - 12\sqrt{36}\\ &= 6\sqrt{4 \times 15} - 12 \times 6\\ &= 12\sqrt{15} - 72\end{aligned}$$

EXPLANATION

Expand the brackets, giving $\sqrt{3} \times 3\sqrt{5} = 3\sqrt{15}$ and $\sqrt{3} \times \sqrt{6} = \sqrt{18}$.
Then simplify $\sqrt{18}$.

Expand the brackets and simplify the surds.
Recall that $\sqrt{6} \times \sqrt{6} = 6$.

Exercise 2C

UNDERSTANDING AND FLUENCY	1–6(½)	2–7(½)	4–7(½)

1 Express the result of these computations as whole numbers.

a $\sqrt{3} \times \sqrt{3}$ b $(\sqrt{5})^2$ c $\dfrac{\sqrt{7}}{\sqrt{7}}$

d $\dfrac{5\sqrt{7}}{\sqrt{7}}$ e $(2\sqrt{3})^2$ f $(3\sqrt{2})^2$

Example 7a

2 Simplify the following.

a $\sqrt{3} \times \sqrt{5}$ b $\sqrt{7} \times \sqrt{3}$ c $\sqrt{2} \times \sqrt{13}$

d $\sqrt{5} \times \sqrt{7}$ e $\sqrt{2} \times (-\sqrt{15})$ f $-\sqrt{6} \times \sqrt{5}$

g $-\sqrt{6} \times (-\sqrt{11})$ h $-\sqrt{3} \times (-\sqrt{2})$ i $\sqrt{10} \times \sqrt{7}$

Example 8a

3 Simplify the following.

a $\sqrt{20} \div \sqrt{2}$ b $\sqrt{18} \div \sqrt{3}$ c $\sqrt{33} \div (-\sqrt{11})$

d $\sqrt{30} \div (-\sqrt{6})$ e $\dfrac{\sqrt{15}}{\sqrt{5}}$ f $\dfrac{\sqrt{30}}{\sqrt{3}}$

g $\dfrac{\sqrt{40}}{\sqrt{8}}$ h $-\dfrac{\sqrt{26}}{\sqrt{2}}$ i $-\dfrac{\sqrt{50}}{\sqrt{10}}$

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4 Simplify the following.

a $\sqrt{7} \times \sqrt{3}$ **b** $\sqrt{2} \times \sqrt{5}$ **c** $\sqrt{10} \times \sqrt{3}$
d $\sqrt{3} \times \sqrt{3}$ **e** $\sqrt{5} \times \sqrt{5}$ **f** $\sqrt{9} \times \sqrt{9}$
g $\sqrt{14} \times \sqrt{7}$ **h** $\sqrt{2} \times \sqrt{22}$ **i** $\sqrt{3} \times \sqrt{18}$
j $\sqrt{10} \times \sqrt{5}$ **k** $\sqrt{12} \times \sqrt{8}$ **l** $\sqrt{5} \times \sqrt{20}$

Example 7b

5 Simplify the following.

a $2\sqrt{5} \times \sqrt{15}$ **b** $3\sqrt{7} \times \sqrt{14}$ **c** $4\sqrt{6} \times \sqrt{21}$
d $-5\sqrt{10} \times \sqrt{30}$ **e** $3\sqrt{6} \times (-\sqrt{18})$ **f** $5\sqrt{3} \times \sqrt{15}$
g $3\sqrt{14} \times 2\sqrt{21}$ **h** $-4\sqrt{6} \times 5\sqrt{15}$ **i** $2\sqrt{10} \times (-2\sqrt{25})$
j $-2\sqrt{7} \times (-3\sqrt{14})$ **k** $4\sqrt{15} \times 2\sqrt{18}$ **l** $9\sqrt{12} \times 4\sqrt{21}$

Example 8b

6 Simplify the following.

a $\dfrac{6\sqrt{14}}{3\sqrt{7}}$ **b** $\dfrac{15\sqrt{12}}{5\sqrt{2}}$ **c** $\dfrac{4\sqrt{30}}{8\sqrt{6}}$
d $-\dfrac{8\sqrt{2}}{2\sqrt{26}}$ **e** $-\dfrac{3\sqrt{3}}{9\sqrt{21}}$ **f** $\dfrac{12\sqrt{70}}{18\sqrt{14}}$

Example 9

7 Use the distributive law to expand the following and simplify the surds where necessary.

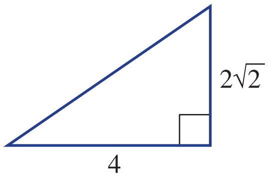
a $\sqrt{3}(\sqrt{2} + \sqrt{5})$ **b** $\sqrt{2}(\sqrt{7} - \sqrt{5})$ **c** $-\sqrt{5}(\sqrt{11} + \sqrt{13})$
d $-2\sqrt{3}(\sqrt{5} + \sqrt{7})$ **e** $3\sqrt{2}(2\sqrt{13} - \sqrt{11})$ **f** $4\sqrt{5}(\sqrt{5} - \sqrt{10})$
g $5\sqrt{3}(2\sqrt{6} + 3\sqrt{10})$ **h** $-2\sqrt{6}(3\sqrt{2} - 2\sqrt{3})$ **i** $3\sqrt{7}(2\sqrt{7} + 3\sqrt{14})$
j $6\sqrt{5}(3\sqrt{15} - 2\sqrt{8})$ **k** $-2\sqrt{8}(2\sqrt{2} - 3\sqrt{20})$ **l** $2\sqrt{3}(7\sqrt{6} + 5\sqrt{3})$

PROBLEM-SOLVING AND REASONING	9, 11	8(½), 9, 11, 12	8(½), 9, 10, 12, 13

8 Simplify the following.

a $(2\sqrt{7})^2$ **b** $(-3\sqrt{2})^2$ **c** $-(5\sqrt{3})^2$
d $\sqrt{2}(3 - \sqrt{3}) - \sqrt{8}$ **e** $\sqrt{8}(\sqrt{6} + \sqrt{2}) - \sqrt{3}$ **f** $\sqrt{5}(\sqrt{2} + 1) - \sqrt{40}$
g $\sqrt{44} - 2(\sqrt{11} - 1)$ **h** $\sqrt{24} - 2\sqrt{2}(\sqrt{3} - 4)$ **i** $2\sqrt{3}(\sqrt{6} - \sqrt{3}) - \sqrt{50}$

9 Determine the unknown side of the following right-angled triangles.

a


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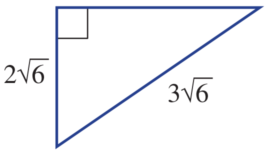

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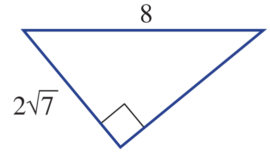

10 **a** A square has a perimeter of $2\sqrt{3}$ cm. Find its area.
b Find the length of the diagonal of a square, which has an area of $12\,\text{cm}^2$.

11 Use $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$ to prove the following results.

a $\sqrt{6} \times \sqrt{6} = 6$ **b** $-\sqrt{8} \times \sqrt{8} = -8$ **c** $-\sqrt{5} \times (-\sqrt{5}) = 5$

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12 $\sqrt{8} \times \sqrt{27}$ can be simplified in two ways, as shown.

Method A

$$\begin{aligned}\sqrt{8} \times \sqrt{27} &= \sqrt{4 \times 2} \times \sqrt{9 \times 3}\\ &= 2\sqrt{2} \times 3\sqrt{3}\\ &= 2 \times 3 \times \sqrt{2 \times 3}\\ &= 6\sqrt{6}\end{aligned}$$

Method B

$$\begin{aligned}\sqrt{8} \times \sqrt{27} &= \sqrt{8 \times 27}\\ &= \sqrt{216}\\ &= \sqrt{36 \times 6}\\ &= 6\sqrt{6}\end{aligned}$$

a Describe the first step in method **A**.

b Why is it useful to simplify surds before multiplying, as in method **A**?

c Multiply the following by first simplifying each surd.

i $\sqrt{18} \times \sqrt{27}$ **ii** $\sqrt{24} \times \sqrt{20}$ **iii** $\sqrt{50} \times \sqrt{45}$

iv $\sqrt{54} \times \sqrt{75}$ **v** $2\sqrt{18} \times \sqrt{48}$ **vi** $\sqrt{108} \times (-2\sqrt{125})$

vii $-4\sqrt{27} \times (-\sqrt{28})$ **viii** $\sqrt{98} \times \sqrt{300}$ **ix** $2\sqrt{72} \times 3\sqrt{80}$

13 $\dfrac{\sqrt{12}}{\sqrt{3}}$ can be simplified in two ways.

Method A

$$\begin{aligned}\frac{\sqrt{12}}{\sqrt{3}} &= \sqrt{\frac{12}{3}}\\ &= \sqrt{4}\\ &= 2\end{aligned}$$

Method B

$$\begin{aligned}\frac{\sqrt{12}}{\sqrt{3}} &= \sqrt{\frac{4 \times 3}{3}}\\ &= \frac{2\sqrt{3}}{\sqrt{3}}\\ &= 2\end{aligned}$$

Use method **B** to simplify these surds.

a $\dfrac{\sqrt{27}}{\sqrt{3}}$ **b** $\dfrac{\sqrt{20}}{\sqrt{5}}$ **c** $-\dfrac{\sqrt{162}}{\sqrt{2}}$

d $-\dfrac{2\sqrt{2}}{5\sqrt{8}}$ **e** $\dfrac{2\sqrt{45}}{15\sqrt{5}}$ **f** $\dfrac{5\sqrt{27}}{\sqrt{75}}$

ENRICHMENT	–	–	14

Higher powers

14 Look at this example before simplifying the following.

$$\begin{aligned}(2\sqrt{3})^2 &= 2^3(\sqrt{3})^3\\ &= 2 \times 2 \times 2 \times \sqrt{3} \times \sqrt{3} \times \sqrt{3}\\ &= 8 \times 3 \times \sqrt{3}\\ &= 24\sqrt{3}\end{aligned}$$

a $(3\sqrt{2})^3$ **b** $(5\sqrt{3})^3$ **c** $2(3\sqrt{3})^3$

d $(\sqrt{5})^4$ **e** $(-\sqrt{3})^4$ **f** $(2\sqrt{2})^5$

g $-3(2\sqrt{5})^3$ **h** $2(-3\sqrt{2})^3$ **i** $5(2\sqrt{3})^4$

j $\dfrac{(2\sqrt{7})^3}{4}$ **k** $\dfrac{(3\sqrt{2})^3}{4}$ **l** $\dfrac{(3\sqrt{2})^4}{4}$

m $\dfrac{(5\sqrt{2})^2}{4} \times \dfrac{(2\sqrt{3})^3}{3}$ **n** $\dfrac{(2\sqrt{3})^2}{9} \times \dfrac{(-3\sqrt{2})^4}{3}$ **o** $\dfrac{(2\sqrt{5})^3}{5} \times \dfrac{(-2\sqrt{3})^5}{24}$

p $\dfrac{(3\sqrt{3})^3}{2} \div \dfrac{(5\sqrt{2})^2}{4}$ **q** $\dfrac{(2\sqrt{5})^4}{50} \div \dfrac{(2\sqrt{3})^3}{5}$ **r** $\dfrac{(2\sqrt{2})^3}{9} \div \dfrac{(2\sqrt{8})^2}{(\sqrt{27})^3}$

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2D Binomial products

Interactive
Widgets
HOTsheets
Walkthrough

In the previous section we used the distributive law to expand surds of the form $a(b + c)$. Now we will extend this to expand and simplify the product of two binomial terms, including perfect squares and the difference of two squares. We expand binomial products in the same way that we expand the product of binomial expressions in algebra, such as $(x + 1)(x + 3)$.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Show the missing steps

You are told that the following three equations are all true. Provide all the missing steps to show how to obtain the right-hand side from each given left-hand side.

- $(2 - \sqrt{3})(5 + \sqrt{3}) = 7 - 3\sqrt{3}$
- $(3 + 2\sqrt{2})^2 = 17 + 12\sqrt{2}$
- $(5 + 3\sqrt{5})(5 - 3\sqrt{5}) = -20$

Key ideas

- Use the **distributive law** to expand **binomial products** and simplify by collecting like terms where possible.
  - $(a + b)(c + d) = ac + ad + bc + bd$
- Expanding perfect squares
  - $\begin{aligned}(a + b)^2 &= (a + b)(a + b)\\ &= a^2 + ab + ba + b^2\\ &= a^2 + 2ab + b^2\end{aligned}$ For example: $\begin{aligned}(\sqrt{2} + 1)^2 &= (\sqrt{2} + 1)(\sqrt{2} + 1)\\ &= 2 + \sqrt{2} + \sqrt{2} + 1\\ &= 3 + 2\sqrt{2}\end{aligned}$
  - $\begin{aligned}(a - b)^2 &= (a - b)(a - b)\\ &= a^2 - ab - ba + b^2\\ &= a^2 - 2ab + b^2\end{aligned}$ For example: $\begin{aligned}(\sqrt{2} - 1)^2 &= (\sqrt{2} - 1)(\sqrt{2} - 1)\\ &= 2 - \sqrt{2} - \sqrt{2} + 1\\ &= 3 - 2\sqrt{2}\end{aligned}$
- Forming a difference of two squares
  - $\begin{aligned}(a + b)(a - b) &= a^2 - ab + ba - b^2\\ &= a^2 - b^2\end{aligned}$ For example: $\begin{aligned}(\sqrt{2} + 1)(\sqrt{2} - 1) &= 2 - \sqrt{2} + \sqrt{2} - 1\\ &= 2 - 1\\ &= 1\end{aligned}$
- Note: $(\sqrt{2})^2 = \sqrt{2} \times \sqrt{2} = 2$ and $(3\sqrt{2})^2 = 3\sqrt{2} \times 3\sqrt{2} = 9 \times 2 = 18$.

Example 10 Expanding binomial products

Expand and simplify.

a $(4 + \sqrt{3})(\sqrt{3} - 2)$

b $(2\sqrt{5} - 1)(3\sqrt{5} + 4)$

SOLUTION	EXPLANATION
a $\begin{aligned}(4 + \sqrt{3})(\sqrt{3} - 2) &= 4\sqrt{3} - 8 + 3 - 2\sqrt{3}\\ &= 2\sqrt{3} - 5\end{aligned}$	Use $(a + b)(c + d) = ac + ad + bc + bd$ and note that $\sqrt{3} \times \sqrt{3} = 3$. Simplify by collecting like surds.
b $\begin{aligned}&(2\sqrt{5} - 1)(3\sqrt{5} + 4)\\ &= 6 \times 5 + 8\sqrt{5} - 3\sqrt{5} - 4\\ &= 30 + 5\sqrt{5} - 4\\ &= 26 + 5\sqrt{5}\end{aligned}$	Use the distributive law and collect like surds. Recall that $2\sqrt{5} \times 3\sqrt{5} = 2 \times 3 \times \sqrt{5} \times \sqrt{5} = 6 \times 5$.

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Example 11 Expanding perfect squares

Expand and simplify.

a $(2 - \sqrt{7})^2$

b $(3\sqrt{2} + 5\sqrt{3})^2$

SOLUTION

a $(2 - \sqrt{7})^2 = (2 - \sqrt{7})(2 - \sqrt{7})$
$= 4 - 2\sqrt{7} - 2\sqrt{7} + 7$
$= 11 - 4\sqrt{7}$

Alternatively:
$(2 - \sqrt{7})^2 = 4 - 2 \times 2\sqrt{7} + 7$
$= 11 - 4\sqrt{7}$

b $(3\sqrt{2} + 5\sqrt{3})^2$
$= (3\sqrt{2} + 5\sqrt{3})(3\sqrt{2} + 5\sqrt{3})$
$= 9 \times 2 + 15 \times \sqrt{6} + 15 \times \sqrt{6} + 25 \times 3$
$= 18 + 30\sqrt{6} + 75$
$= 93 + 30\sqrt{6}$

Alternatively:
$(3\sqrt{2} + 5\sqrt{3})^2 = 9 \times 2 + 2 \times 15\sqrt{6} + 25 \times 3$
$= 18 + 30\sqrt{6} + 75$
$= 93 + 30\sqrt{6}$

EXPLANATION

Note that $(a - b)^2 = (a - b)(a - b)$.
Expand and simplify using the distributive law.
Recall that $\sqrt{7} \times \sqrt{7} = 7$.

Use $(a - b)^2 = a^2 - 2ab + b^2$.

Use the distributive law to expand and simplify.

Use $(a + b)^2 = a^2 + 2ab + b^2$.

Example 12 Expanding to form a difference of two squares

Expand and simplify.

a $(2 + \sqrt{5})(2 - \sqrt{5})$

b $(7\sqrt{2} - \sqrt{3})(7\sqrt{2} + \sqrt{3})$

SOLUTION

a $(2 + \sqrt{5})(2 - \sqrt{5}) = 4 - 2\sqrt{5} + 2\sqrt{5} - 5$
$= -1$

Alternatively:
$(2 + \sqrt{5})(2 - \sqrt{5}) = 2^2 - (\sqrt{5})^2$
$= 4 - 5$
$= -1$

b $(7\sqrt{2} - \sqrt{3})(7\sqrt{2} + \sqrt{3})$
$= 49 \times 2 + 7\sqrt{6} - 7\sqrt{6} - 3$
$= 98 - 3$
$= 95$

Alternatively:
$(7\sqrt{2} - \sqrt{3})(7\sqrt{2} + \sqrt{3}) = (7\sqrt{2})^2 - (\sqrt{3})^2$
$= 49 \times 2 - 3$
$= 95$

EXPLANATION

Expand using the distributive law and cancel the two middle terms.

Use $(a + b)(a - b) = a^2 - b^2$.

Use the distributive law and then cancel the two middle terms.

Use $(a - b)(a + b) = a^2 - b^2$.

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Exercise 2D

UNDERSTANDING AND FLUENCY	1–7(½)	3–7(½)	4–7(½)

1 Simplify the following.

a $\sqrt{3} \times \sqrt{7}$ b $-\sqrt{2} \times \sqrt{5}$ c $2\sqrt{3} \times 3\sqrt{2}$
d $(\sqrt{11})^2$ e $(\sqrt{13})^2$ f $(2\sqrt{3})^2$
g $(5\sqrt{5})^2$ h $(7\sqrt{3})^2$ i $(9\sqrt{2})^2$

2 Simplify the following.

a $5\sqrt{2} - 5\sqrt{2}$ b $-2\sqrt{3} + 2\sqrt{3}$ c $6 \times \sqrt{7} - \sqrt{7} \times 6$
d $2\sqrt{2} - \sqrt{8}$ e $2\sqrt{27} - 4\sqrt{3}$ f $5\sqrt{12} - \sqrt{48}$

3 Use the distributive law $(a + b)(c + d) = ac + ad + bc + bd$ to expand and simplify these algebraic expressions.

a $(x + 2)(x + 3)$ b $(x - 5)(x + 1)$ c $(x + 4)(x - 3)$
d $(2x + 1)(x - 5)$ e $(3x + 2)(2x - 5)$ f $(6x + 7)(x - 4)$
g $(x + 4)(x - 4)$ h $(2x - 3)(2x + 3)$ i $(5x - 6)(5x + 6)$
j $(x + 2)^2$ k $(2x - 1)^2$ l $(3x - 7)^2$

Example 10a

4 Expand and simplify.

a $(2 + \sqrt{2})(\sqrt{2} - 3)$ b $(4 + \sqrt{5})(\sqrt{5} - 2)$ c $(\sqrt{6} + 2)(\sqrt{6} - 1)$
d $(5 - \sqrt{3})(2 + \sqrt{3})$ e $(3 + \sqrt{7})(4 - \sqrt{7})$ f $(\sqrt{2} - 5)(3 + \sqrt{2})$
g $(\sqrt{5} - 2)(4 - \sqrt{5})$ h $(4 - \sqrt{10})(5 - \sqrt{10})$ i $(\sqrt{7} - 4)(\sqrt{7} - 4)$

Example 10b

5 Expand and simplify.

a $(5\sqrt{2} - 1)(3\sqrt{2} + 3)$ b $(4\sqrt{3} + 3)(2\sqrt{3} - 1)$ c $(6\sqrt{5} - 5)(2\sqrt{5} + 7)$
d $(7\sqrt{6} + 4)(2 - 3\sqrt{6})$ e $(2\sqrt{10} - 3)(\sqrt{10} - 5)$ f $(2\sqrt{7} - 3)(3 - 4\sqrt{7})$
g $(4\sqrt{3} - 5)(2 - 3\sqrt{3})$ h $(1 - 5\sqrt{2})(3 - 4\sqrt{2})$ i $(4\sqrt{5} - 3)(3 - 4\sqrt{5})$

Example 11a

6 Expand and simplify these perfect squares.

a $(3 - \sqrt{5})^2$ b $(2 - \sqrt{6})^2$ c $(4 + \sqrt{7})^2$
d $(\sqrt{11} + 2)^2$ e $(\sqrt{3} + 5)^2$ f $(\sqrt{5} - 7)^2$
g $(\sqrt{7} + \sqrt{2})^2$ h $(\sqrt{11} - \sqrt{2})^2$ i $(\sqrt{10} - \sqrt{3})^2$
j $(\sqrt{13} + \sqrt{19})^2$ k $(\sqrt{17} + \sqrt{23})^2$ l $(\sqrt{31} - \sqrt{29})^2$

Example 12a

7 Expand and simplify these difference of two squares.

a $(3 - \sqrt{2})(3 + \sqrt{2})$ b $(5 - \sqrt{6})(5 + \sqrt{6})$ c $(4 + \sqrt{3})(4 - \sqrt{3})$
d $(\sqrt{7} - 1)(\sqrt{7} + 1)$ e $(\sqrt{8} - 2)(\sqrt{8} + 2)$ f $(\sqrt{10} - 4)(\sqrt{10} + 4)$
g $(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2})$ h $(\sqrt{11} + \sqrt{5})(\sqrt{11} - \sqrt{5})$ i $(\sqrt{3} - \sqrt{7})(\sqrt{3} + \sqrt{7})$

PROBLEM-SOLVING AND REASONING	8(½), 10(½), 11	8–10(½), 11–12(½)	8–10(½), 12(½), 13, 14

Example 11b

8 Expand and simplify these perfect squares.

a $(2\sqrt{5} + 7\sqrt{2})^2$ b $(3\sqrt{3} + 4\sqrt{7})^2$ c $(5\sqrt{6} + 3\sqrt{5})^2$
d $(8\sqrt{2} - 4\sqrt{3})^2$ e $(6\sqrt{3} - 3\sqrt{11})^2$ f $(3\sqrt{7} - 2\sqrt{6})^2$
g $(2\sqrt{3} + 3\sqrt{6})^2$ h $(3\sqrt{10} + 5\sqrt{2})^2$ i $(5\sqrt{3} - 2\sqrt{8})^2$

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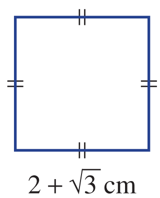
Example 12b

9 Expand and simplify these difference of two squares.

a $(3\sqrt{11} - \sqrt{2})(3\sqrt{11} + \sqrt{2})$
b $(2\sqrt{5} - \sqrt{3})(2\sqrt{5} + \sqrt{3})$
c $(4\sqrt{3} + \sqrt{7})(4\sqrt{3} - \sqrt{7})$
d $(5\sqrt{7} - 2\sqrt{3})(5\sqrt{7} + 2\sqrt{3})$
e $(4\sqrt{5} - 3\sqrt{6})(4\sqrt{5} + 3\sqrt{6})$
f $(4\sqrt{10} - 5\sqrt{6})(4\sqrt{10} + 5\sqrt{6})$
g $(5\sqrt{8} - 10\sqrt{2})(5\sqrt{8} + 10\sqrt{2})$
h $(3\sqrt{7} - 4\sqrt{6})(3\sqrt{7} + 4\sqrt{6})$
i $(2\sqrt{10} + 4\sqrt{5})(2\sqrt{10} - 4\sqrt{5})$

10 Find the area of these rectangles and triangles in expanded and simplified form.

a


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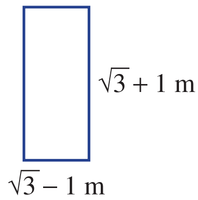

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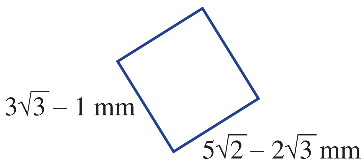

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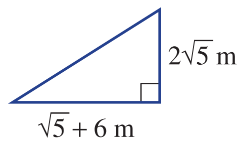
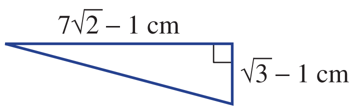


e


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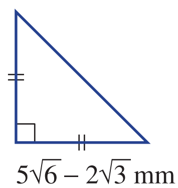

11 Show that the following simplify to an integer.

a $(\sqrt{2} + \sqrt{7})(\sqrt{2} - \sqrt{7})$
b $(2\sqrt{3} - \sqrt{5})(2\sqrt{3} + \sqrt{5})$
c $(5\sqrt{11} - 7\sqrt{3})(5\sqrt{11} + 7\sqrt{3})$

12 Use your knowledge of the simplification of surds to fully simplify the following.

a $(3 - 2\sqrt{7})(\sqrt{21} - 4\sqrt{3})$
b $(2\sqrt{6} + 5)(\sqrt{30} - 2\sqrt{5})$
c $(3\sqrt{5} + 1)(\sqrt{7} + 2\sqrt{35})$
d $(4\sqrt{2} + \sqrt{7})(3\sqrt{14} - 5)$
e $(3\sqrt{3} + 4)(\sqrt{6} - 2\sqrt{2})$
f $(5 - 3\sqrt{2})(2\sqrt{10} + 3\sqrt{5})$

13 Find the value of a and b.

a $(\sqrt{3} + 1)^2 = a + b\sqrt{3}$
b $(4 - \sqrt{5})^2 = a + b\sqrt{5}$
c $(2\sqrt{3} + \sqrt{2})^2 = a + b\sqrt{6}$
d $(2\sqrt{2} + 1)^2 = a + \sqrt{b}$

14 Is it possible for $(a + b)^2$ to simplify to an integer when at least a or b is a surd? If your answer is yes, give an example.

ENRICHMENT	–	–	15

Expansion challenge

15 Fully expand and simplify these surds.

a $(2\sqrt{3} - \sqrt{2})^2 + (\sqrt{3} + \sqrt{2})^2$
b $(\sqrt{5} - \sqrt{3})^2 + (\sqrt{5} + \sqrt{3})^2$
c $(\sqrt{3} - 4\sqrt{5})(\sqrt{3} + 4\sqrt{5}) - (\sqrt{3} - \sqrt{5})^2$
d $-10\sqrt{3} - (2\sqrt{3} - 5)^2$
e $(\sqrt{3} - 2\sqrt{6})^2 + (1 + \sqrt{2})^2$
f $(2\sqrt{7} - 3)^2 - (3 - 2\sqrt{7})^2$
g $(2\sqrt{3} - 3\sqrt{2})(2\sqrt{3} + 3\sqrt{2}) - (\sqrt{6} - \sqrt{2})^2$
h $\sqrt{2}(2\sqrt{5} - 3\sqrt{3})^2 + (\sqrt{6} + \sqrt{5})^2$

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2E Rationalising the denominator

Interactive
Widgets
HOTsheets
Walkthrough

Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

As you know, it is easier to add or subtract fractions when the fractions are expressed with the same denominator. In a similar way, it is easier to work with surds such as $\frac{1}{\sqrt{2}}$ and $\frac{\sqrt{3}-1}{\sqrt{5}}$ when they are expressed using a whole number in the denominator. The process that removes a surd from the denominator is called 'rationalising the denominator' because the denominator is being converted from an irrational number to a rational number.

Let's start: What do I multiply by?

When trying to rationalise the denominator in a surd like $\frac{1}{\sqrt{2}}$, you must multiply the surd by a chosen number so that the denominator is converted to a whole number.

- First, decide what each of the following is equivalent to.
  - i $\frac{\sqrt{3}}{\sqrt{3}}$
  - ii $\frac{\sqrt{2}}{\sqrt{2}}$
  - iii $\frac{\sqrt{21}}{\sqrt{21}}$
- Recall that $\sqrt{x} \times \sqrt{x} = x$ and simplify the following.
  - i $\sqrt{5} \times \sqrt{5}$
  - ii $2\sqrt{3} \times \sqrt{3}$
  - iii $4\sqrt{7} \times \sqrt{7}$
- Now, decide what can you multiply $\frac{1}{\sqrt{2}}$ by so that:
  - – the value of $\frac{1}{\sqrt{2}}$ does not change?
  - – the denominator becomes a whole number?
- Repeat this for:
  - i $\frac{1}{\sqrt{5}}$
  - ii $\frac{3}{2\sqrt{3}}$

Key ideas

- **Rationalising a denominator** involves multiplying by a number that is equivalent to 1. This changes the denominator from a surd to a whole number.

$$\frac{x}{\sqrt{y}} = \frac{x}{\sqrt{y}} \times \frac{\sqrt{y}}{\sqrt{y}} = \frac{x\sqrt{y}}{y}$$

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Example 13 Rationalising the denominator

Rationalise the denominator in the following.

a $\frac{2}{\sqrt{3}}$ **b** $\frac{3\sqrt{2}}{\sqrt{5}}$ **c** $\frac{2\sqrt{6}}{5\sqrt{2}}$ **d** $\frac{1-\sqrt{3}}{\sqrt{3}}$

SOLUTION	EXPLANATION
a $\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$ $= \frac{2\sqrt{3}}{3}$	Choose the appropriate fraction equivalent to 1 to multiply by. In this case, choose $\frac{\sqrt{3}}{\sqrt{3}}$ since $\sqrt{3} \times \sqrt{3} = 3$.
b $\frac{3\sqrt{2}}{\sqrt{5}} = \frac{3\sqrt{2}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$ $= \frac{3\sqrt{10}}{5}$	Choose the appropriate fraction. In this case, use $\frac{\sqrt{5}}{\sqrt{5}}$ since $\sqrt{5} \times \sqrt{5} = 5$. Recall that $\sqrt{2} \times \sqrt{5} = \sqrt{2 \times 5} = \sqrt{10}$.
c $\frac{2\sqrt{6}}{5\sqrt{2}} = \frac{2\sqrt{6}}{5\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$ $= \frac{2\sqrt{12}}{10}$	Choose the appropriate fraction. In this case, use $\frac{\sqrt{2}}{\sqrt{2}}$. $5 \times \sqrt{2} \times \sqrt{2} = 5 \times 2 = 10$
$= \frac{{}^{1}\cancel{2}\sqrt{4 \times 3}}{\cancel{10}_{5}}$ $= \frac{2\sqrt{3}}{5}$	Simplify the surd $\sqrt{12}$ and cancel.
d $\frac{1-\sqrt{3}}{\sqrt{3}} = \frac{1-\sqrt{3}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$ $= \frac{\sqrt{3}-3}{3}$	Expand using the distributive law: $(1-\sqrt{3}) \times \sqrt{3} = 1 \times \sqrt{3} - \sqrt{3} \times \sqrt{3} = \sqrt{3} - 3$

Exercise 2E

UNDERSTANDING AND FLUENCY	1–3, 4–7(½)	3, 4–7(½)	4–7(½)

1 Simplify.

a $\frac{\sqrt{6}}{\sqrt{6}}$ **b** $\frac{\sqrt{11}}{\sqrt{11}}$ **c** $\frac{2\sqrt{5}}{4\sqrt{5}}$ **d** $-\frac{7\sqrt{3}}{14\sqrt{3}}$

e $-\frac{\sqrt{8}}{\sqrt{2}}$ **f** $-\frac{3\sqrt{27}}{\sqrt{3}}$ **g** $\frac{\sqrt{72}}{\sqrt{2}}$ **h** $-\frac{3\sqrt{45}}{9\sqrt{5}}$

2 Write the missing number.

a $\sqrt{3} \times \_\_\_ = 3$ **b** $\_\_\_ \times \sqrt{5} = 5$ **c** $\sqrt{10} \times \sqrt{10} = \_\_\_$

d $2\sqrt{5} \times \_\_\_ = 10$ **e** $4\sqrt{3} \times \_\_\_ = 12$ **f** $\_\_\_ \times 3\sqrt{7} = 21$

g $\frac{1}{\sqrt{3}} \times \_\_\_ = \frac{\sqrt{3}}{3}$ **h** $\frac{1}{\sqrt{7}} \times \_\_\_ = \frac{\sqrt{7}}{7}$ **i** $\frac{2}{\sqrt{13}} \times \_\_\_ = \frac{2\sqrt{13}}{13}$

3 Use a calculator to confirm the following.

a $\frac{1}{\sqrt{7}} = \frac{\sqrt{7}}{7}$ **b** $\frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$ **c** $\frac{11\sqrt{11}}{\sqrt{5}} = \frac{11\sqrt{55}}{5}$

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Example 13a

4 Rationalise the denominators.

a $\dfrac{1}{\sqrt{2}}$ **b** $\dfrac{1}{\sqrt{7}}$ **c** $\dfrac{3}{\sqrt{11}}$ **d** $\dfrac{4}{\sqrt{5}}$

e $\dfrac{5}{\sqrt{3}}$ **f** $\dfrac{8}{\sqrt{2}}$ **g** $\dfrac{\sqrt{5}}{\sqrt{3}}$ **h** $\dfrac{\sqrt{2}}{\sqrt{7}}$

5 Rewrite each of the following in the form $\dfrac{\sqrt{a}}{\sqrt{b}}$ and then rationalise the denominators.

a $\sqrt{\dfrac{2}{3}}$ **b** $\sqrt{\dfrac{5}{7}}$ **c** $\sqrt{\dfrac{6}{11}}$ **d** $\sqrt{\dfrac{2}{5}}$

e $\sqrt{\dfrac{7}{3}}$ **f** $\sqrt{\dfrac{6}{7}}$ **g** $\sqrt{\dfrac{10}{3}}$ **h** $\sqrt{\dfrac{17}{2}}$

Example 13b

6 Rationalise the denominators.

a $\dfrac{4\sqrt{2}}{\sqrt{7}}$ **b** $\dfrac{5\sqrt{2}}{\sqrt{3}}$ **c** $\dfrac{3\sqrt{5}}{\sqrt{2}}$

d $\dfrac{3\sqrt{6}}{\sqrt{7}}$ **e** $\dfrac{7\sqrt{3}}{\sqrt{10}}$ **f** $\dfrac{2\sqrt{7}}{\sqrt{15}}$

Example 13c

7 Rationalise the denominators.

a $\dfrac{4\sqrt{7}}{5\sqrt{3}}$ **b** $\dfrac{2\sqrt{3}}{3\sqrt{2}}$ **c** $\dfrac{5\sqrt{7}}{3\sqrt{5}}$ **d** $\dfrac{4\sqrt{5}}{5\sqrt{10}}$

e $\dfrac{2\sqrt{7}}{3\sqrt{35}}$ **f** $\dfrac{5\sqrt{12}}{3\sqrt{27}}$ **g** $\dfrac{9\sqrt{6}}{2\sqrt{3}}$ **h** $\dfrac{7\sqrt{90}}{2\sqrt{70}}$

PROBLEM-SOLVING AND REASONING	8(½), 9, 10	8(½), 9, 10(½), 11, 12(½)	8(½), 9, 10(½), 12(½), 13

Example 13d

8 Rationalise the denominators.

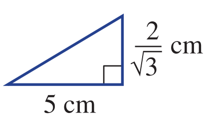
a $\dfrac{1+\sqrt{2}}{\sqrt{3}}$ **b** $\dfrac{3+\sqrt{5}}{\sqrt{7}}$ **c** $\dfrac{2-\sqrt{3}}{\sqrt{5}}$ **d** $\dfrac{\sqrt{3}-\sqrt{5}}{\sqrt{2}}$

e $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{7}}$ **f** $\dfrac{\sqrt{10}-\sqrt{7}}{\sqrt{3}}$ **g** $\dfrac{\sqrt{2}+\sqrt{7}}{\sqrt{6}}$ **h** $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{10}}$

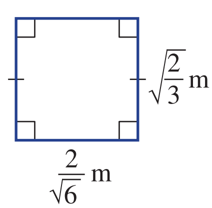
i $\dfrac{\sqrt{6}-\sqrt{10}}{\sqrt{5}}$ **j** $\dfrac{4\sqrt{2}-5\sqrt{3}}{\sqrt{6}}$ **k** $\dfrac{3\sqrt{5}+5\sqrt{2}}{\sqrt{10}}$ **l** $\dfrac{3\sqrt{10}+5\sqrt{3}}{\sqrt{2}}$

9 Determine the exact value of the area of the following shapes. Express your answers using a rational denominator.

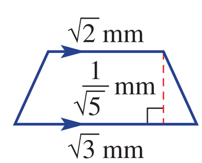
a



b



c



10 Simplify the following by first rationalising denominators and then using a common denominator.

a $\dfrac{1}{\sqrt{3}}+\dfrac{1}{\sqrt{2}}$ **b** $\dfrac{3}{\sqrt{5}}+\dfrac{1}{\sqrt{2}}$ **c** $\dfrac{3}{\sqrt{7}}-\dfrac{2}{\sqrt{3}}$

d $\dfrac{5}{2\sqrt{3}}-\dfrac{2}{3\sqrt{2}}$ **e** $\dfrac{1}{3\sqrt{2}}+\dfrac{5}{4\sqrt{3}}$ **f** $\dfrac{3}{2\sqrt{5}}+\dfrac{2}{5\sqrt{3}}$

g $\dfrac{7\sqrt{2}}{5\sqrt{7}}-\dfrac{2\sqrt{7}}{3\sqrt{2}}$ **h** $\dfrac{10\sqrt{6}}{3\sqrt{5}}+\dfrac{4\sqrt{2}}{3\sqrt{3}}$ **i** $\dfrac{5\sqrt{2}}{3\sqrt{5}}-\dfrac{4\sqrt{7}}{3\sqrt{6}}$

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11 Explain why multiplying a number by $\frac{\sqrt{x}}{\sqrt{x}}$ does not change its value.

12 Rationalise the denominators and simplify the following.

a $\frac{\sqrt{3}+a}{\sqrt{7}}$ **b** $\frac{\sqrt{6}+a}{\sqrt{5}}$ **c** $\frac{\sqrt{2}+a}{\sqrt{6}}$

d $\frac{\sqrt{3}-3a}{\sqrt{3}}$ **e** $\frac{\sqrt{5}-5a}{\sqrt{5}}$ **f** $\frac{\sqrt{7}-7a}{\sqrt{7}}$

g $\frac{4a+\sqrt{5}}{\sqrt{10}}$ **h** $\frac{3a+\sqrt{3}}{\sqrt{6}}$ **i** $\frac{2a+\sqrt{7}}{\sqrt{14}}$

13 To explore how to simplify a number such as $\frac{3}{4-\sqrt{2}}$, first answer these questions.

a Simplify:

i $(4-\sqrt{2})(4+\sqrt{2})$

ii $(3-\sqrt{7})(3+\sqrt{7})$

iii $(5\sqrt{2}-\sqrt{3})(5\sqrt{2}+\sqrt{3})$

b What do you notice about each question and answer in part **a** above?

c Now decide what to multiply $\frac{3}{4-\sqrt{2}}$ by to rationalise the denominator.

d Rationalise the denominator in these expressions.

i $\frac{3}{4-\sqrt{2}}$ **ii** $\frac{-3}{\sqrt{3}-1}$ **iii** $\frac{\sqrt{2}}{\sqrt{4}-\sqrt{3}}$ **iv** $\frac{2\sqrt{6}}{\sqrt{6}-2\sqrt{5}}$

ENRICHMENT	–	–	14

Binomial denominators

14 Rationalise the denominators in the following, using the difference of two squares.

For example:
$$\begin{aligned}\frac{2}{\sqrt{2}+1} &= \frac{2}{\sqrt{2}+1}\times\frac{\sqrt{2}-1}{\sqrt{2}-1}\\ &= \frac{2(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}\\ &= \frac{2\sqrt{2}-2}{2-1} = 2\sqrt{2}-2\end{aligned}$$

a $\frac{5}{\sqrt{3}+1}$ **b** $\frac{4}{\sqrt{3}-1}$ **c** $\frac{3}{\sqrt{5}-2}$ **d** $\frac{4}{1-\sqrt{2}}$

e $\frac{3}{1-\sqrt{3}}$ **f** $\frac{7}{6-\sqrt{7}}$ **g** $\frac{4}{3-\sqrt{10}}$ **h** $\frac{7}{2-\sqrt{5}}$

i $\frac{2}{\sqrt{11}-\sqrt{2}}$ **j** $\frac{6}{\sqrt{2}+\sqrt{5}}$ **k** $\frac{4}{\sqrt{3}+\sqrt{7}}$ **l** $\frac{\sqrt{2}}{\sqrt{7}+1}$

m $\frac{\sqrt{6}}{\sqrt{6}-1}$ **n** $\frac{3\sqrt{2}}{\sqrt{7}-2}$ **o** $\frac{2\sqrt{5}}{\sqrt{5}+2}$ **p** $\frac{b}{\sqrt{a}+\sqrt{b}}$

q $\frac{a}{\sqrt{a}-\sqrt{b}}$ **r** $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ **s** $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$ **t** $\frac{\sqrt{ab}}{\sqrt{a}-\sqrt{b}}$

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2F Review of index laws REVISION

Interactive
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From your work in Year 9, you will recall that powers (numbers with indices) can be used to represent repeated multiplication of the same factor. For example, $2 \times 2 \times 2 = 2^3$ and $5 \times x \times x \times x \times x = 5x^4$.

The five basic index laws and the zero index will be revised in this section.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Recall the laws

Try to recall how to simplify each expression and use words to describe the index law used.

- $5^3 \times 5^7$
- $x^4 \div x^2$
- $(a^7)^2$
- $(2a)^3$
- $\left(\dfrac{x}{3}\right)^4$
- $(4x^2)^0$

Key ideas

- The index laws
  - $a^m \times a^n = a^{m+n}$ Retain the base and add the indices.
  - $a^m \div a^n = \dfrac{a^m}{a^n} = a^{m-n}$ Retain the base and subtract the indices.
  - $(a^m)^n = a^{m \times n}$ Retain the base and multiply the indices.
  - $(a \times b)^m = a^m \times b^m$ Distribute the index number across the bases.
  - $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$ Distribute the index number across the bases.
- The zero index: $a^0 = 1$ Any number (except zero) to the power of zero is equal to 1.
- Power of 1: $a^1 = a$ Any number to the power of 1 is equal to itself.

Example 14 Using the index law for multiplication

Simplify the following using the index law for multiplication.

a $x^5 \times x^4$

b $3a^2b \times 4ab^3$

SOLUTION	EXPLANATION
a $x^5 \times x^4 = x^9$	There is a common base of x, so add the indices.
b $3a^2b \times 4ab^3 = 12a^3b^4$	Multiply the coefficients and add indices for each base a and b. Recall that $a = a^1$.

Example 15 Using the index law for division

Simplify the following using the index law for division.

a $m^7 \div m^5$

b $4x^2y^5 \times (8xy^2)$

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SOLUTION

a $\dfrac{m^7}{m^5} = m^2$

b $4x^2y^5 \div (8xy^2) = \dfrac{4}{8}xy^3$
$= \dfrac{1}{2}xy^3$

EXPLANATION

Subtract the indices when dividing terms with the same base.

Divide the coefficients and subtract the indices of x and y (i.e. $x^{2-1}y^{5-2}$).

Example 16 Combining index laws

Simplify the following using the index laws.

a $(a^3)^4$

b $(2y^5)^3$

c $\left(\dfrac{3x^2}{5y^2z}\right)^3$

d $\dfrac{3(xy^2)^3 \times 4x^4y^2}{8x^2y}$

SOLUTION

a $(a^3)^4 = a^{12}$

b $(2y^5)^3 = 2^3y^{15}$
$= 8y^{15}$

c $\left(\dfrac{3x^2}{5y^2z}\right)^3 = \dfrac{3^3x^6}{5^3y^6z^3}$
$= \dfrac{27x^6}{125y^6z^3}$

d $\dfrac{3(xy^2)^3 \times 4x^4y^2}{8x^2y} = \dfrac{3x^3y^6 \times 4x^4y^2}{8x^2y}$
$= \dfrac{12x^7y^8}{8x^2y}$
$= \dfrac{3x^5y^7}{2}$

EXPLANATION

Multiply the indices.

Multiply the indices for each base 2 and y.
Note that $2 = 2^1$.

Raise the coefficients to the power 3 and multiply the indices of each base by 3.

Remove brackets first by multiplying indices for each base.
Simplify the numerator.

Simplify the fraction by subtracting indices of the same base.

Example 17 Using the zero index

Evaluate, using the zero index.

a $4a^0$

b $2p^0 + (3p)^0$

SOLUTION

a $4a^0 = 4 \times 1$
$= 4$

b $2p^0 + (3p)^0 = 2 \times 1 + 1$
$= 3$

EXPLANATION

Any number to the power of zero is equal to 1.
Note: $(3p)^0$ is not the same as $3p^0$.

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Exercise 2F REVISION

UNDERSTANDING AND FLUENCY	1(½), 2, 3, 4–7(½)	3, 4–7(½)	4–7(½)

1 Write the following without indices.

a 2^3 **b** 3^3 **c** 4^2 **d** 5^3
e 10^2 **f** 3×2^2 **g** $(3 \times 2)^2$ **h** 5^0
i $\left(\frac{2}{3}\right)^2$ **j** $5^2 + 5^2$ **k** $(5 + 5)^2$ **l** $2^3 \times 3^2$

2 Write the following in index form.

a $2^5 \times 2^3$ **b** $2^5 \div 2^3$ **c** $(2^5)^3$
d $5^4 \times 5$ **e** $5^4 \div 5$ **f** $5 \div 5^4$

3 Find the value of x.

a $5^3 \times 25 = 5^x$ **b** $49 \times 7 = 7^x$ **c** $4 \times 8 = 2^x$
d $8^x = 1$ **e** $16 \times \frac{1}{4} = 4^x$ **f** $16 \times \frac{1}{4} = 2^x$

Example 14

4 Simplify, using the index law for multiplication.

a $a^5 \times a^4$ **b** $x^3 \times x^2$ **c** $b \times b^5$
d $7m^2 \times 2m^3$ **e** $2s^4 \times 3s^3$ **f** $t^8 \times 2t^8$
g $\frac{1}{5}p^2 \times p$ **h** $\frac{1}{4}c^4 \times \frac{2}{3}c^3$ **i** $\frac{3}{5}s \times \frac{3s}{5}$
j $2x^2y \times 3xy^2$ **k** $3a^2b \times 5ab^5$ **l** $3v^7w \times 6v^2w$
m $3x^4 \times 5xy^2 \times 10y^4$ **n** $2rs^3 \times 3r^4s \times 2r^2s^2$ **o** $4m^6n^7 \times mn \times 5mn^2$

Example 15

5 Simplify, using the index law for division.

a $x^5 \div x^2$ **b** $a^7 \div a^6$ **c** $q^9 \div q^6$
d $b^5 \div b$ **e** $\frac{y^8}{y^3}$ **f** $\frac{d^8}{d^3}$
g $\frac{j^7}{j^6}$ **h** $\frac{m^{15}}{m^9}$ **i** $2x^2y^3 \div x$
j $3r^5s^2 \div (r^3s)$ **k** $6p^4q^2 \div (3q^2p^2)$ **l** $16m^7x^5 \div (8m^3x^4)$
m $\frac{5a^2b^4}{a^2b}$ **n** $\frac{8st^4}{2t^3}$ **o** $\frac{2v^5}{8v^3}$
p $\frac{7a^2b}{14ab}$ **q** $\frac{-3x^4y}{9x^3y}$ **r** $\frac{-8x^2y^3}{16x^2y}$

Example 16

6 Use the appropriate index law to simplify the following.

a $(x^5)^2$ **b** $(t^3)^2$ **c** $4(a^2)^3$ **d** $5(y^5)^3$
e $(4t^2)^3$ **f** $(2u^2)^2$ **g** $(3r^3)^3$ **h** $(3p^4)^4$
i $\left(\frac{a^2}{b^3}\right)^2$ **j** $\left(\frac{x^3}{y^4}\right)^3$ **k** $\left(\frac{x^2y^3}{z^4}\right)^2$ **l** $\left(\frac{u^4w^2}{v^2}\right)^4$
m $\left(\frac{3f^2}{5g}\right)^3$ **n** $\left(\frac{3a^2b}{2pq^3}\right)^2$ **o** $\left(\frac{at^3}{3g^4}\right)^3$ **p** $\left(\frac{4p^2q^3}{3r}\right)^4$

7 Use the zero index to evaluate the following.

a $8x^0$ **b** $3t^0$ **c** $(5z)^0$ **d** $(10ab^2)^0$
e $5(g^3h^3)^0$ **f** $8x^0 - 5$ **g** $4b^0 - 9$ **h** $7x^0 - 4(2y)^0$

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PROBLEM-SOLVING AND REASONING	8(½), 9, 11(½)	8(½), 9, 11(½), 12	8(½), 9, 10, 12, 13

Example 17

8 Use appropriate index laws to simplify the following.

a $x^6 \times x^5 \div x^3$

b $x^2y \div (xy) \times xy^2$

c $x^4n^7 \times x^3n^2 \div (xn)$

d $\dfrac{x^2y^3 \times x^2y^4}{x^3y^5}$

e $\dfrac{m^2w \times m^3w^2}{m^4w^3}$

f $\dfrac{r^4s^7 \times r^4s^7}{r^4s^7}$

g $\dfrac{9x^2y^3 \times 6x^7y^5}{12xy^6}$

h $\dfrac{4x^2y^3 \times 12x^2y^2}{24x^4y}$

i $\dfrac{16a^8b \times 4ab^7}{32a^7b^6}$

j $(3m^2n^4)^3 \times mn^2$

k $-5(a^2b)^3 \times (3ab)^2$

l $(4f^2g)^2 \times f^2g^4 \div (3(fg^2)^3)$

m $\dfrac{4m^2n \times 3(m^2n)^3}{6m^2n}$

n $\dfrac{(7y^2z)^2 \times 3yz^2}{7(yz)^2}$

o $\dfrac{2(ab)^2 \times (2a^2b)^3}{4ab^2 \times 4a^7b^3}$

p $\dfrac{(2m^3)^2}{3(mn^4)^0} \times \dfrac{(6n^5)^2}{(-2n)^3m^4}$

9 Simplify:

a $(-3)^3$ **b** $-(3)^3$ **c** $(-3)^4$ **d** -3^4

10 Simplify:

a $\left((x^2)^3\right)^2$ **b** $\left((a^5)^3\right)^7$ **c** $\left(\left(\dfrac{a^2}{b}\right)^3\right)^5$

11 Evaluate without the use of a calculator.

a $\dfrac{13^3}{13^2}$ **b** $\dfrac{18^7}{18^6}$ **c** $\dfrac{9^8}{9^6}$ **d** $\dfrac{4^{10}}{4^7}$

e $\dfrac{25^2}{5^4}$ **f** $\dfrac{36^2}{6^4}$ **g** $\dfrac{27^2}{3^4}$ **h** $\dfrac{32^2}{2^7}$

12 When Billy uses a calculator to raise -2 to the power 4 he gets -16, however, the answer is actually 16. What has he done wrong?

13 Find the value of a in these equations in which the index is unknown.

a $2^a = 8$ **b** $3^a = 81$ **c** $2^{a+1} = 4$

d $(-3)^a = -27$ **e** $(-5)^a = 625$ **f** $4^a = \dfrac{1}{4}$

ENRICHMENT	–	–	14–16

Indices in equations

14 If $x^4 = 3$, find the value of:

a x^8 **b** $x^4 - 1$ **c** $2x^{16}$ **d** $3x^4 - 3x^8$

15 Find the value(s) of x.

a $x^4 = 16$ **b** $2^{x-1} = 16$ **c** $2^{2x} = 16$ **d** $2^{2x-3} = 16$

16 Find the possible pairs of positive integers for x and y when:

a $x^y = 16$ **b** $x^y = 64$ **c** $x^y = 81$ **d** $x^y = 1$

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2G Negative indices REVISION

Interactive
Widgets
HOTsheets
Walkthrough

Stage

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

The study of indices can be extended to include negative powers. Using the index law for division and the fact that $a^0 = 1$, we can establish rules for negative powers.

$$a^0 \div a^n = a^{0-n} \text{ (index law for division)} \quad \text{also} \quad a^0 \div a^n = 1 \div a^n \text{ (as } a^0 = 1)$$
$$= a^{-n} \qquad\qquad = \frac{1}{a^n}$$

Therefore: $a^{-n} = \dfrac{1}{a^n}$

Also: $\dfrac{1}{a^{-n}} = 1 \div a^{-n}$

$$= 1 \div \frac{1}{a^n}$$
$$= 1 \times \frac{a^n}{1}$$
$$= a^n$$

Therefore: $\dfrac{1}{a^{-n}} = a^n$

Let's start: The disappearing bank balance

Due to fees, an initial bank balance of \$64 is halved every month.

Balance (\$)	64	32	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
Positive indices only	2^6	2^5							$\frac{1}{2^2}$	
Positive and negative indices	2^6		2^4					2^{-1}		

- Copy and complete the table and continue each pattern.
- Discuss the differences in the way indices are used at the end of the rows.
- What would be a way of writing $\dfrac{1}{16}$ using positive indices?
- What would be a way of writing $\dfrac{1}{16}$ using negative indices?

Key ideas

- a^{-1} is the reciprocal of a, which is $\dfrac{1}{a}$.
  $\therefore 5^{-1} = \dfrac{1}{5}$, which is the reciprocal of 5.
- $\left(\dfrac{a}{b}\right)^{-1}$ is the reciprocal of $\dfrac{a}{b}$, which is $\dfrac{b}{a}$.
  $\therefore \left(\dfrac{2}{3}\right)^{-1} = \dfrac{3}{2}$, which is the reciprocal of $\dfrac{2}{3}$.
- $a^{-m} = \dfrac{1}{a^m}$ For example: $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$ (2^{-3} is the reciprocal of 2^3.)
- $\dfrac{1}{a^{-m}} = a^m$ For example: $\dfrac{1}{2^{-3}} = 2^3 = 8$
- $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n = \dfrac{b^n}{a^n}$ For example: $\left(\dfrac{x^2}{3}\right)^{-4} = \left(\dfrac{3}{x^2}\right)^4 = \dfrac{81}{x^8}$

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Example 18 Writing expressions using positive indices

Express each of the following using positive indices.

a b^{-4} **b** $3x^{-4}y^2$ **c** $\frac{5}{x^{-3}}$ **d** $\left(\frac{3a^2}{a^5}\right)^{-3}$

SOLUTION	EXPLANATION
a $b^{-4} = \frac{1}{b^4}$	Use $a^{-n} = \frac{1}{a^n}$.
b $3x^{-4}y^2 = \frac{3y^2}{x^4}$	x is the only base with a negative power.
c $\frac{5}{x^{-3}} = 5 \times x^3$ $= 5x^3$	Use $\frac{1}{a^{-n}} = a^n$ and note that $\frac{5}{x^{-3}} = 5 \times \frac{1}{x^{-3}}$.
d $\left(\frac{3a^2}{a^5}\right)^{-3} = \left(\frac{a^5}{3a^2}\right)^3$	Take the reciprocal of the fraction and change the power from negative to positive.
$= \left(\frac{a^3}{3}\right)^3$	Simplify the fraction.
$= \frac{a^9}{27}$	Cube the numerator and denominator.

Example 19 Simplifying more complex expressions

Simplify the following and express your answers using positive indices.

a $\frac{(p^2q)^4}{5p^{-3}q^3} \times \left(\frac{p^{-2}}{q^3}\right)^3$ **b** $\left(\frac{2m^3}{r^2n^{-4}}\right)^3 \div \left(\frac{5m^{-2}n^3}{r}\right)^2$

SOLUTION	EXPLANATION
a $\frac{(p^2q)^4}{5p^{-3}q^3} \times \left(\frac{p^{-2}}{q^3}\right)^3 = \frac{p^8q^4}{5p^{-3}q^3} \times \frac{p^{-6}}{q^9}$	Multiply the bracket power to each of the indices within the bracket.
$= \frac{p^2q^4}{5p^{-3}q^{12}}$	Combine indices of like bases: $8 + (-6) = 2$, $3 + 9 = 12$, $4 - 12 = -8$ and $2 - (-3) = 5$.
$= \frac{p^5q^{-8}}{5}$	Use $a^{-n} = \frac{1}{a^n}$ to express with a positive index.
$= \frac{p^5}{5q^8}$	
b $\left(\frac{2m^3}{r^2n^{-4}}\right)^3 \div \left(\frac{5m^{-2}n^3}{r}\right)^2 = \frac{2^3m^9}{r^6n^{-12}} \div \frac{5^2m^{-4}n^6}{r^2}$	Multiply the bracket power to each of the indices within the bracket.
$= \frac{8m^9}{r^6n^{-12}} \times \frac{r^2}{25m^{-4}n^6}$	Multiply by the reciprocal of the divisor.
$= \frac{8m^{13}r^{-4}}{25n^{-6}}$	Combine indices of like bases: $9 - (-4) = 13$, $2 - 6 = -4$ and $-12 + 6 = -6$.
$= \frac{8m^{13}n^6}{25r^4}$	Write the answer with positive powers.

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Exercise 2G REVISION

UNDERSTANDING AND FLUENCY	1–3, 4–7(½)	3, 4–8(½)	4–8(½)

1 Copy and complete this pattern.

10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}
1000						

2 Given that $3^{-1} = \frac{1}{3}$, which is the reciprocal of 3, find the following.

a 4^{-1} **b** 10^{-1} **c** x^{-1} **d** $\left(\frac{1}{2}\right)^{-1}$

e $\left(\frac{3}{2}\right)^{-1}$ **f** $\left(\frac{1}{4}\right)^{-1}$ **g** $\left(\frac{3}{4}\right)^{-1}$ **h** $\left(\frac{x}{y}\right)^{-1}$

3 Given that $2^{-3} = \frac{1}{8}$, find the following.

a 3^{-2} **b** 4^{-2} **c** 5^{-2} **d** 2^{-4}

e $\left(\frac{1}{3}\right)^{-2}$ **f** $\left(\frac{2}{3}\right)^{-2}$ **g** $\left(\frac{3}{2}\right)^{-2}$ **h** $\left(\frac{1}{2}\right)^{-5}$

Example 18a, b

4 Express the following using positive indices.

a x^{-5} **b** a^{-4} **c** $2m^{-4}$ **d** $3y^{-7}$

e $3a^2b^{-3}$ **f** $4m^3n^{-3}$ **g** $10x^{-2}y^5z$ **h** $3x^{-4}y^{-2}z^3$

i $\frac{1}{3}p^{-2}q^3r$ **j** $\frac{1}{5}d^2e^{-4}f^5$ **k** $\frac{3}{8}u^2v^{-6}w^7$ **l** $\frac{2}{5}b^3c^{-5}d^{-2}$

Example 18c

5 Express the following using positive indices.

a $\frac{1}{x^{-2}}$ **b** $\frac{2}{y^{-3}}$ **c** $\frac{4}{m^{-7}}$ **d** $\frac{3}{b^{-5}}$

e $\frac{2b^4}{d^{-3}}$ **f** $\frac{3m^2}{n^{-4}}$ **g** $\frac{4b^4}{3a^{-3}}$ **h** $\frac{5h^3}{2g^{-3}}$

6 Use the index laws to simplify the following. Write your answers using positive indices.

a $x^3 \times x^{-2}$ **b** $a^7 \times a^{-4}$ **c** $2b^5 \times b^{-9}$ **d** $3y^{-6} \times y^3$

e $3a^{-4} \times 2a^2$ **f** $4x^{-5} \times 3x^4$ **g** $5m^{-4} \times (-2m^{-2})$ **h** $-3a^{-7} \times 6a^{-3}$

i $\frac{2x^{-2}}{3x^{-3}}$ **j** $\frac{7d^{-3}}{10d^{-5}}$ **k** $\frac{6c^{-5}}{12c^{-5}}$ **l** $\frac{3b^{-2}}{4b^{-4}}$

m $\frac{5s^{-2}}{3s}$ **n** $\frac{4f^{-5}}{3f^{-3}}$ **o** $\frac{3d^{-3}}{6d^{-1}}$ **p** $\frac{15t^{-4}}{18t^{-2}}$

Example 18d

7 Express the following with positive indices.

a $\left(\frac{2x^2}{x^3}\right)^4$ **b** $\left(\frac{m^3}{4m^5}\right)^3$ **c** $2(x^{-7})^3$ **d** $4(d^{-2})^3$

e $(3t^{-4})^2$ **f** $5(x^2)^{-2}$ **g** $(3x^{-5})^4$ **h** $-8(x^5)^{-3}$

i $\left(\frac{4}{y^2}\right)^{-2}$ **j** $\left(\frac{3}{h^3}\right)^{-4}$ **k** $7(j^{-2})^{-4}$ **l** $2(t^{-3})^{-2}$

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8 Express the following in simplest form with positive indices.

a $x^2y^3 \times x^{-3}y^{-4}$
b $4a^{-6}y^4 \times a^3y^{-2}$
c $2a^{-3}b \times 3a^{-2}b^{-3}$
d $6a^4b^3 \times 3a^{-6}b$
e $a^3b^4 \div (a^2b^7)$
f $p^2q^3 \div (p^7q^2)$
g $\dfrac{a^4b^3}{a^2b^5}$
h $\dfrac{m^3n^2}{mn^3}$
i $\dfrac{p^2q^2r^4}{pq^4r^5}$
j $\dfrac{3x^2y}{6xy^2}$
k $\dfrac{4m^3n^4}{7m^2n^7}$
l $\dfrac{12r^4s^6}{9rs^{-1}}$
m $\dfrac{f^3g^{-2}}{f^{-2}g^3}$
n $\dfrac{r^{-3}s^{-4}}{r^3s^{-2}}$
o $\dfrac{3w^{-2}x^3}{6w^{-3}x^{-2}}$
p $\dfrac{15c^3d}{12c^{-2}d^{-3}}$

PROBLEM-SOLVING AND REASONING	9(½), 12	9–10(½), 12, 13	9–10(½), 11, 14, 15

Example 19

9 Simplify the following and express your answers with positive indices.

a $(a^3b^2)^3 \times (a^2b^4)^{-1}$
b $(2p^2)^4 \times (3p^2q)^{-2}$
c $2(x^2y^{-1})^2 \times (3xy^4)^3$
d $\dfrac{2a^3b^2}{a^{-3}} \times \dfrac{2a^2b^5}{b^4}$
e $\dfrac{(3rs^2)^4}{r^{-3}s^4} \times \dfrac{(2r^2s)^2}{s^7}$
f $\dfrac{4(x^{-2}y^4)^2}{x^2y^{-3}} \times \dfrac{xy^4}{2x^{-2}y}$
g $\left(\dfrac{a^2b^3}{b^{-2}}\right)^2 \div \left(\dfrac{ab^4}{a^2}\right)^{-2}$
h $\left(\dfrac{m^4n^{-2}}{r^3}\right)^2 \div \left(\dfrac{m^{-3}n^2}{r^3}\right)^2$
i $\dfrac{3(x^2y^{-4})^2}{2(xy^2)^2} \div \dfrac{(xy)^{-3}}{(3x^{-2}y^4)^2}$

10 Evaluate without the use of a calculator.

a 5^{-2}
b 4^{-3}
c 2×7^{-2}
d $5 \times (-3^{-4})$
e $3^{10} \times (3^2)^{-6}$
f $(4^2)^{-5} \times 4(4^{-3})^{-3}$
g $\dfrac{2}{7^{-2}}$
h $\dfrac{-3}{4^{-2}}$
i $\left(\dfrac{2}{3}\right)^{-2}$
j $\left(\dfrac{-5}{4}\right)^{-3}$
k $\dfrac{(4^{-2})^3}{4^{-4}}$
l $\dfrac{(10^{-4})^{-2}}{(10^{-2})^{-3}}$

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11 The width of a hair on a spider is approximately 3^{-5} cm. How many centimetres is this, correct to 4 decimal places?

12 **a** Simplify these numbers.

i $\left(\frac{2}{3}\right)^{-1}$ **ii** $\left(\frac{5}{7}\right)^{-1}$ **iii** $\left(\frac{2x}{y}\right)^{-1}$

b What is $\left(\frac{a}{b}\right)^{-1}$ when expressed in simplest form?

13 A student simplifies $2x^{-2}$ and writes $2x^{-2} = \frac{1}{2x^2}$. Explain the error made.

14 Evaluate the following by combining fractions.

a $2^{-1} + 3^{-1}$ **b** $3^{-2} + 6^{-1}$ **c** $\left(\frac{3}{4}\right)^{-1} - \left(\frac{1}{2}\right)^{0}$

d $\left(\frac{3}{2}\right)^{-1} - 5(2^{-2})$ **e** $\left(\frac{4}{5}\right)^{-2} - \left(\frac{2^{-2}}{3}\right)$ **f** $\left(\frac{3}{2^{-2}}\right) - \left(\frac{2^{-1}}{3^{-2}}\right)^{-1}$

15 Prove that $\left(\frac{1}{2}\right)^{x} = 2^{-x}$, giving reasons.

ENRICHMENT	–	–	16

Simple equations with negative indices

16 Find the value of x.

a $2^x = \frac{1}{4}$ **b** $2^x = \frac{1}{32}$ **c** $3^x = \frac{1}{27}$ **d** $\left(\frac{3}{4}\right)^x = \frac{4}{3}$

e $\left(\frac{2}{3}\right)^x = \frac{9}{4}$ **f** $\left(\frac{2}{5}\right)^x = \frac{125}{8}$ **g** $\frac{1}{2^x} = 8$ **h** $\frac{1}{3^x} = 81$

i $\frac{1}{2^x} = 1$ **j** $5^{x-2} = \frac{1}{25}$ **k** $3^{x-3} = \frac{1}{9}$ **l** $10^{x-5} = \frac{1}{1000}$

m $\left(\frac{3}{4}\right)^{2x+1} = \frac{64}{27}$ **n** $\left(\frac{2}{5}\right)^{3x-5} = \frac{25}{4}$ **o** $\left(\frac{3}{2}\right)^{3x+2} = \frac{16}{81}$ **p** $\left(\frac{7}{4}\right)^{1-x} = \frac{4}{7}$

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2H Scientific notation REVISION

Interactive Widgets HOTsheets Walkthrough

Scientific notation is useful when working with very large or very small numbers. Combined with the use of significant figures, numbers can be written down with an appropriate degree of accuracy and without the need to write all the zeros that define the position of the decimal place. The approximate distance between the Earth and the Sun is 150 million kilometres or 1.5×10^8 km when written in scientific notation using two significant figures. Negative indices can be used for very small numbers; e.g. 0.0000382 g $= 3.82 \times 10^{-5}$ g.

Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Amazing facts large and small

Think of an object, place or living thing that is associated with a very large or small number.

- Give three examples of very large numbers.
- Give three examples of very small numbers.
- Can you remember how to write these numbers using scientific notation?
- How are significant figures used when writing numbers with scientific notation?

Key ideas

- A number written using **scientific notation** is of the form $a \times 10^m$, where $1 \leqslant a < 10$ and m is an integer.
  - Large numbers: $24\,800\,000 = 2.48 \times 10^7$
    $9\,020\,000\,000 = 9.02 \times 10^9$
  - Small numbers: $0.00307 = 3.07 \times 10^{-3}$
    $0.0000012 = 1.2 \times 10^{-6}$
- **Significant figures** are counted from left to right starting at the first non-zero digit.
  - When using scientific notation the digit to the left of the decimal point is the first significant figure.
    For example: 2.019×10^7 shows four significant figures.
  - Scientific calculators have a button for scientific notation, such as $\boxed{\text{Exp}}$ or $\boxed{\times 10^x}$.

Example 20 Converting from scientific notation to a basic numeral

Write these numbers as a basic numeral.

a 5.016×10^5 **b** 3.2×10^{-7}

SOLUTION	EXPLANATION
a $5.016 \times 10^5 = 501\,600$	Move the decimal point 5 places to the right.
b $3.2 \times 10^{-7} = 0.00000032$	Move the decimal point 7 places to the left.

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Example 21 Converting to scientific notation using significant figures

Write these numbers in scientific notation, using three significant figures.

a 5218300 **b** 0.0042031

SOLUTION	EXPLANATION
a $5218300 = 5.22 \times 10^{6}$	Place the decimal point after the first non-zero digit. The digit following the third digit is at least 5, so round up.
b $0.0042031 = 4.20 \times 10^{-3}$	Round down in this case, but retain the zero to show the value of the third significant figure.

Exercise 2H REVISION

UNDERSTANDING AND FLUENCY	1–7(½)	3–8(½)	4–8(½)

1 How many significant figures are showing in these numbers?

a 2.12×10^{7} **b** 5.902×10^{4} **c** 1.81×10^{-3} **d** 1.0×10^{-7}

2 Write these numbers as powers of 10.

a 1000 **b** 10000000 **c** 0.000001 **d** $\frac{1}{1000}$

3 Convert to scientific notation.

a 43000 **b** 712000 **c** 901200 **d** 10010
e 0.00078 **f** 0.00101 **g** 0.00003 **h** 0.0300401

Example 20a
4 Write these numbers as a basic numeral.

a 3.12×10^{3} **b** 5.4293×10^{4} **c** 7.105×10^{5} **d** 8.213×10^{6}
e 5.95×10^{4} **f** 8.002×10^{5} **g** 1.012×10^{4} **h** 9.99×10^{6}
i 2.105×10^{8} **j** 5.5×10^{4} **k** 2.35×10^{9} **l** 1.237×10^{12}

Example 20b
5 Write these numbers as a basic numeral.

a 4.5×10^{-3} **b** 2.72×10^{-2} **c** 3.085×10^{-4} **d** 7.83×10^{-3}
e 9.2×10^{-5} **f** 2.65×10^{-1} **g** 1.002×10^{-4} **h** 6.235×10^{-6}
i 9.8×10^{-1} **j** 5.45×10^{-10} **k** 3.285×10^{-12} **l** 8.75×10^{-7}

Example 21a
6 Write these numbers in scientific notation, using three significant figures.

a 6241 **b** 572644 **c** 30248 **d** 423578
e 10089 **f** 34971863 **g** 72477 **h** 356088
i 110438523 **j** 909325 **k** 4555678 **l** 9826100005

Example 21b
7 Write these numbers in scientific notation, using three significant figures.

a 0.002423 **b** 0.018754 **c** 0.000125 **d** 0.0078663
e 0.0007082 **f** 0.11396 **g** 0.000006403 **h** 0.00007892
i 0.000129983 **j** 0.00000070084 **k** 0.000000009886 **l** 0.0004998

8 Write in scientific notation, using the number of significant figures given in the brackets.

a 23900 (2) **b** 5707159 (3) **c** 703780030 (2)
d 4875 (3) **e** 0.00192 (2) **f** 0.00070507 (3)
g 0.000009782 (2) **h** 0.35708 (4) **i** 0.000050034 (3)

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PROBLEM-SOLVING AND REASONING	9, 10(½), 12	9, 10(½), 12, 13(½)	10(½), 11, 13–14(½)

9 Write the following numerical facts using scientific notation.

- **a** The area of Australia is about 7 700 000 km^2.
- **b** The number of stones used to build the Pyramid of Khufu is about 2 500 000.
- **c** The greatest distance of Pluto from the Sun is about 7 400 000 000 km.
- **d** A human hair is about 0.01 cm wide.
- **e** The mass of a neutron is about 0.000000000000000000000000001675 kg.
- **f** The mass of a bacteria cell is about 0.00000000000095 g.

10 Use a calculator to evaluate the following, giving the answers in scientific notation using three significant figures.

- **a** $(2.31)^{-7}$
- **b** $(5.04)^{-4}$
- **c** $(2.83 \times 10^2)^{-3}$
- **d** $5.1 \div (8 \times 10^2)$
- **e** $9.3 \times 10^{-2} \times 8.6 \times 10^8$
- **f** $(3.27 \times 10^4) \div (9 \times 10^{-5})$
- **g** $\sqrt{3.23 \times 10^{-6}}$
- **h** $\pi(3.3 \times 10^7)^2$
- **i** $\sqrt[3]{5.73 \times 10^{-4}}$

11 The speed of light is approximately 3×10^5 km/s and the average distance between Pluto and the Sun is about 5.9×10^9 km. How long does it take for light from the Sun to reach Pluto? Give your answer correct to the nearest minute.

12 Explain why 38×10^7 is not written using scientific notation.

13 Use scientific notation to write the following.

- **a** 21×10^3
- **b** 394×10^7
- **c** 6004×10^{-2}
- **d** 179×10^{-6}
- **e** 0.2×10^4
- **f** 0.007×10^2
- **g** 0.01×10^9
- **h** 0.06×10^8
- **i** 0.4×10^{-2}
- **j** 0.0031×10^{-11}
- **k** 210.3×10^{-6}
- **l** 9164×10^{-24}

14 Combine your knowledge of index laws with scientific notation to evaluate the following and express using scientific notation.

- **a** $(3 \times 10^2)^2$
- **b** $(2 \times 10^3)^3$
- **c** $(8 \times 10^4)^2$
- **d** $(12 \times 10^{-5})^2$
- **e** $(5 \times 10^{-3})^{-2}$
- **f** $(4 \times 10^5)^{-2}$
- **g** $(1.5 \times 10^{-3})^2$
- **h** $(8 \times 10^{-8})^{-1}$
- **i** $(5 \times 10^{-2}) \times (2 \times 10^{-4})$
- **j** $(3 \times 10^{-7}) \times (4.25 \times 10^2)$
- **k** $(15 \times 10^8) \times (12 \times 10^{-11})$
- **l** $(18 \times 10^5) \div (9 \times 10^3)$
- **m** $(240 \times 10^{-4}) \div (3 \times 10^{-2})$
- **n** $(2 \times 10^{-8}) \div (50 \times 10^4)$
- **o** $(5 \times 10^2) \div (20 \times 10^{-3})$

ENRICHMENT	–	–	15

$E = mc^2$

15 $E = mc^2$ is a formula derived by Albert Einstein (1879–1955). The formula relates the energy (E joules) of an object to its mass (m kg), where c is the speed of light (approximately 3×10^8 m/s). Use $E = mc^2$ to answer these questions, using scientific notation.

- **a** Find the energy, in joules, contained inside an object with these given masses.
  - **i** 10 kg
  - **ii** 26 000 kg
  - **iii** 0.03 kg
  - **iv** 0.00001 kg
- **b** Find the mass, in kilograms, of an object that contains these given amounts of energy. Give your answer using three significant figures.
  - **i** 1×10^{25} J
  - **ii** 3.8×10^{16} J
  - **iii** 8.72×10^4 J
  - **iv** 1.7×10^{-2} J
- **c** The mass of the Earth is about 6×10^{24} kg. How much energy does this convert to?

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2I Fractional indices

Interactive Widgets HOTsheets Walkthrough

Stage: 5.3#, 5.3, 5.3§, 5.2, 5.2◊, 5.1, 4

The square and cube roots of numbers, such as $\sqrt{81} = 9$ and $\sqrt[3]{64} = 4$, can be written using fractional powers.

The following shows that $\sqrt{9} = 9^{\frac{1}{2}}$ and $\sqrt[3]{8} = 8^{\frac{1}{3}}$.

Consider:

$$\begin{aligned} \sqrt{9} \times \sqrt{9} &= 3 \times 3 \\ &= 9 \end{aligned} \quad \text{and} \quad \begin{aligned} 9^{\frac{1}{2}} \times 9^{\frac{1}{2}} &= 9^{\frac{1}{2}+\frac{1}{2}} \\ &= 9 \end{aligned}$$

$$\therefore \sqrt{9} = 9^{\frac{1}{2}}$$

Also:

$$\begin{aligned} \sqrt[3]{8} \times \sqrt[3]{8} \times \sqrt[3]{8} &= 2 \times 2 \times 2 \\ &= 8 \end{aligned} \quad \text{and} \quad \begin{aligned} 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} &= 8^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} \\ &= 8 \end{aligned}$$

$$\therefore \sqrt[3]{8} = 8^{\frac{1}{3}}$$

A rational index is an index that can be expressed as a fraction.

Let's start: Making the connection

For each part below use your knowledge of index laws and basic surds to simplify the numbers. Then discuss the connection that can be made between numbers that have a $\sqrt{\ }$ sign and numbers that have fractional powers.

- $\sqrt{5} \times \sqrt{5}$ and $5^{\frac{1}{2}} \times 5^{\frac{1}{2}}$
- $(\sqrt{5})^2$ and $\left(5^{\frac{1}{2}}\right)^2$
- $\sqrt[3]{27} \times \sqrt[3]{27} \times \sqrt[3]{27}$ and $27^{\frac{1}{3}} \times 27^{\frac{1}{3}} \times 27^{\frac{1}{3}}$
- $\sqrt[3]{64}$ and $\left(64^{\frac{1}{3}}\right)^3$

Key ideas

- $a^{\frac{1}{n}} = \sqrt[n]{a}$
  - $\sqrt[n]{a}$ is the nth root of a.

  For example: $3^{\frac{1}{2}} = \sqrt[2]{3}$ or $\sqrt{3}$, $5^{\frac{1}{3}} = \sqrt[3]{5}$, $7^{\frac{1}{10}} = \sqrt[10]{7}$

- $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m$ or $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$

  For example: $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$ or $8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}} = 64^{\frac{1}{3}} = \sqrt[3]{64} = 4$

- The index laws apply to **rational indices** (i.e. fractional indices) just as they do for indices that are integers.

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Example 22 Writing in index form

Express the following in index form.

a $\sqrt{15}$

b $\sqrt{7x^5}$

c $3\sqrt[4]{x^7}$

d $10\sqrt{10}$

SOLUTION

a $\sqrt{15} = 15^{\frac{1}{2}}$

b $\sqrt{7x^5} = (7x^5)^{\frac{1}{2}}$

$= 7^{\frac{1}{2}}x^{\frac{5}{2}}$

c $3\sqrt[4]{x^7} = 3(x^7)^{\frac{1}{4}}$

$= 3x^{\frac{7}{4}}$

d $10\sqrt{10} = 10 \times 10^{\frac{1}{2}}$

$= 10^{\frac{3}{2}}$

EXPLANATION

$\sqrt{\ }$ means the square root or $\sqrt[2]{\ }$.

Note: $\sqrt[n]{a} = a^{\frac{1}{n}}$

Rewrite $\sqrt{\ }$ as to the power of $\frac{1}{2}$, then apply index laws to simplify: $5 \times \frac{1}{2} = \frac{5}{2}$.

$\sqrt[4]{\ }$ means to the power of $\frac{1}{4}$.

Multiply the indices.

Rewrite the square root as to the power of $\frac{1}{2}$. Then add indices for the common base 10. Recall that $10 = 10^1$, so $1 + \frac{1}{2} = \frac{3}{2}$.

Example 23 Writing in surd form

Express the following in surd form.

a $3^{\frac{1}{5}}$

b $5^{\frac{2}{3}}$

SOLUTION

a $3^{\frac{1}{5}} = \sqrt[5]{3}$

b $5^{\frac{2}{3}} = \left(5^{\frac{1}{3}}\right)^2$

$= \left(\sqrt[3]{5}\right)^2$

Alternatively: $5^{\frac{2}{3}} = (5^2)^{\frac{1}{3}}$

$= \sqrt[3]{25}$

EXPLANATION

$a^{\frac{1}{n}} = \sqrt[n]{a}$

Note that $\frac{2}{3} = \frac{1}{3} \times 2$.

$5^{\frac{1}{3}} = \sqrt[3]{5}$

$\frac{1}{3} \times 2$ is the same as $2 \times \frac{1}{3}$.

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Example 24 Evaluating numbers with fractional indices

Evaluate the following without using a calculator.

a $16^{\frac{1}{2}}$ **b** $16^{\frac{1}{4}}$ **c** $27^{-\frac{1}{3}}$

SOLUTION

a $16^{\frac{1}{2}} = \sqrt{16}$
$= 4$

$16^{\frac{1}{2}}$ means $\sqrt{16}$.

b $16^{\frac{1}{4}} = \sqrt[4]{16}$
$= 2$

$16^{\frac{1}{4}}$ means $\sqrt[4]{16}$ and $2^4 = 16$.

c $27^{-\frac{1}{3}} = \dfrac{1}{27^{\frac{1}{3}}}$

Rewrite, using positive indices.

$= \dfrac{1}{\sqrt[3]{27}}$

$27^{\frac{1}{3}}$ means $\sqrt[3]{27}$ and $3^3 = 27$.

$= \dfrac{1}{3}$

Exercise 2I

UNDERSTANDING AND FLUENCY	1–6, 7(½)	3–7(½)	4–7(½)

1 Evaluate the following without using a calculator.

a $\sqrt{9}$ **b** $\sqrt{25}$ **c** $\sqrt{121}$
d $\sqrt{625}$ **e** $\sqrt[3]{8}$ **f** $\sqrt[3]{27}$
g $\sqrt[3]{125}$ **h** $\sqrt[3]{64}$ **i** $\sqrt[4]{16}$
j $\sqrt[4]{81}$ **k** $\sqrt[5]{32}$ **l** $\sqrt[5]{100000}$

2 Use a calculator to confirm the following.

a $\sqrt[3]{7} = 7^{\frac{1}{3}}$ **b** $\sqrt[5]{10} = 10^{\frac{1}{5}}$ **c** $\sqrt[13]{100} = 100^{\frac{1}{13}}$

3 Write down the value of x.

a $\sqrt[3]{5} = 5^x$ **b** $\sqrt[4]{5} = 5^x$ **c** $\sqrt{5} = 5^x$
d $\sqrt[5]{10} = 10^x$ **e** $\sqrt{y} = y^x$ **f** $\sqrt[3]{y} = y^x$

Example 22a, b

4 Express the following in index form.

a $\sqrt{29}$ **b** $\sqrt[3]{35}$
c $\sqrt[5]{x^2}$ **d** $\sqrt[4]{b^3}$
e $\sqrt{2a}$ **f** $\sqrt[3]{4t^7}$
g $\sqrt[5]{10t^2}$ **h** $\sqrt[8]{8m^4}$

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Example 22c, d

5 Express the following in index form.

a $7\sqrt{x^5}$ b $6\sqrt[3]{n^7}$ c $3\sqrt[4]{y^{12}}$
d $5\sqrt[3]{p^2r}$ e $2\sqrt[3]{a^4b^2}$ f $2\sqrt[4]{g^3h^5}$
g $5\sqrt{5}$ h $7\sqrt{7}$ i $4\sqrt[3]{4}$

Example 23

6 Express the following in surd form.

a $2^{\frac{1}{5}}$ b $8^{\frac{1}{7}}$ c $6^{\frac{1}{3}}$ d $11^{\frac{1}{10}}$
e $3^{\frac{3}{2}}$ f $7^{\frac{2}{3}}$ g $2^{\frac{3}{5}}$ h $3^{\frac{4}{7}}$

Example 24

7 Evaluate without using a calculator.

a $36^{\frac{1}{2}}$ b $27^{\frac{1}{3}}$ c $64^{\frac{1}{3}}$
d $49^{\frac{1}{2}}$ e $16^{\frac{1}{4}}$ f $125^{\frac{1}{3}}$
g $9^{-\frac{1}{2}}$ h $32^{-\frac{1}{5}}$ i $81^{-\frac{1}{4}}$
j $1000^{-\frac{1}{3}}$ k $400^{-\frac{1}{2}}$ l $10000^{-\frac{1}{4}}$

PROBLEM-SOLVING AND REASONING	8–9(½), 11	8–9(½), 11	8–10(½), 11, 12

8 Evaluate without using a calculator. Hint: $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$; e.g. $81^{\frac{3}{4}} = \left(81^{\frac{1}{4}}\right)^3 = 3^3 = 27$

a $8^{\frac{2}{3}}$ b $32^{\frac{3}{5}}$ c $36^{\frac{3}{2}}$ d $16^{\frac{5}{4}}$
e $16^{-\frac{3}{4}}$ f $27^{-\frac{2}{3}}$ g $64^{-\frac{2}{3}}$ h $25^{-\frac{3}{2}}$
i $\dfrac{1}{25^{-\frac{3}{4}}}$ j $\dfrac{2}{4^{\frac{5}{2}}}$ k $\dfrac{3}{9^{\frac{5}{2}}}$ l $\dfrac{10}{100^{\frac{3}{2}}}$

9 Use index laws to simplify the following.

a $a^{\frac{1}{2}} \times a^{\frac{3}{2}}$ b $m^{\frac{3}{2}} \times m^{\frac{3}{2}}$ c $x^{\frac{7}{3}} \div x^{\frac{4}{3}}$ d $b^{\frac{5}{4}} \div b^{\frac{3}{4}}$
e $\left(s^{\frac{3}{2}}\right)^{\frac{4}{7}}$ f $\left(y^{\frac{1}{3}}\right)^{\frac{1}{3}}$ g $\left(t^{\frac{2}{11}}\right)^0$ h $\left(\dfrac{a^{\frac{2}{3}}}{b^{\frac{4}{3}}}\right)^{\frac{3}{4}}$

10 Simplify the following.

a $\sqrt{25s^4}$ b $\sqrt[3]{27t^6}$ c $\sqrt[4]{16t^8}$ d $\sqrt[3]{125t^{12}}$
e $\left(x^3\right)^{\frac{1}{3}}$ f $\left(b^{12}\right)^{\frac{1}{3}}$ g $\left(t^{\frac{1}{4}}\right)^{12}$ h $\left(m^{\frac{1}{5}}\right)^{10}$
i $(16a^2b^8)^{\frac{1}{2}}$ j $(216m^6n^3)^{\frac{1}{3}}$ k $(32x^{10}y^{15})^{\frac{1}{5}}$ l $(343r^9t^6)^{\frac{1}{3}}$
m $\sqrt{\dfrac{25}{49}}$ n $\sqrt[3]{\dfrac{8x^3}{27}}$ o $\left(\dfrac{32}{x^{10}}\right)^{\frac{1}{5}}$ p $\left(\dfrac{10^2x^4}{0.01}\right)^{\frac{1}{4}}$

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11 $16^{\frac{5}{4}}$ can be evaluated in two ways, as shown here.

Method A

$$\begin{aligned}16^{\frac{5}{4}} &= (16^5)^{\frac{1}{4}}\\ &= (1048576)^{\frac{1}{4}}\\ &= \sqrt[4]{1048576}\\ &= 32\end{aligned}$$

Method B

$$\begin{aligned}16^{\frac{5}{4}} &= (16^{\frac{1}{4}})^5\\ &= (\sqrt[4]{16})^5\\ &= 2^5\\ &= 32\end{aligned}$$

a If $16^{\frac{5}{4}}$ is to be evaluated without using a calculator, which method above would be preferable?

b Use your preferred method to evaluate the following without a calculator.

i $8^{\frac{5}{3}}$ **ii** $36^{\frac{3}{2}}$ **iii** $16^{\frac{7}{4}}$ **iv** $27^{\frac{4}{3}}$

v $125^{\frac{4}{3}}$ **vi** $\left(\frac{1}{9}\right)^{\frac{3}{2}}$ **vii** $\left(\frac{4}{25}\right)^{\frac{5}{2}}$ **viii** $\left(\frac{27}{1000}\right)^{\frac{4}{3}}$

12 Explain why $\sqrt[6]{64}$ is not a surd.

ENRICHMENT – – 13

Does it exist?

13 We know that when $y = \sqrt{x}$, where $x < 0$, y is not a real number. This is because the square of y cannot be negative; i.e. $y^2 \neq x$ since y^2 is positive and x is negative.

But we know that $(-2)^3 = -8$ so $\sqrt[3]{-8} = -2$.

a Evaluate:

i $\sqrt[3]{-27}$

ii $\sqrt[3]{-1000}$

iii $\sqrt[5]{-32}$

iv $\sqrt[7]{-2187}$

b Decide if these are real numbers.

i $\sqrt{-5}$

ii $\sqrt[3]{-7}$

iii $\sqrt[5]{-16}$

iv $\sqrt[4]{-12}$

c When $y = \sqrt[n]{x}$ and $x < 0$, for what values of n is y a real number?

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2J Exponential equations

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Equations can take many forms. For example, $2x - 1 = 5$ and $5(a - 3) = -3(3a + 7)$ are both linear equations; $x^2 = 9$ and $3x^2 - 4x - 9 = 0$ are quadratic equations; and $2^x = 8$ and $3^{2x} - 3^x - 6 = 0$ are exponential equations. Exponential equations contain a pronumeral within the index or indices of the terms in the equation. To solve for the unknown in exponential equations we use our knowledge of indices and surds and try to equate powers where possible.

Let's start: 2 to the power of what number is 5?

We know that 2 to the power of 2 is 4 and that 2 to the power of 3 is 8. But 2 to the power of what number is 5? That is, what is x if $2^x = 5$?

- Use a calculator and trial and error to estimate the value of x when $2^x = 5$ by completing this table.

x	2	3	2.5	2.1	
2^x	4	8	5.65...		
Result	too small	too big	too big		

- Continue trying values until you find the answer, correct to 3 decimal places.

Key ideas

- A simple **exponential equation** is of the form $a^x = b$, where $a > 0$ and $a \neq 1$.
  - There is only one solution to exponential equations of this form.
- Many exponential equations can be solved by expressing both sides of the equation using the same base.
  - We use this fact: if $a^x = a^y$ then $x = y$. (Assuming $a > 0$ and $a \neq 1$.)

Example 25 Solving exponential equations

Solve for x in each of the following.

a $2^x = 16$ **b** $3^x = \frac{1}{9}$ **c** $25^x = 125$

SOLUTION

a $2^x = 16$

$2^x = 2^4$

$\therefore x = 4$

b $3^x = \frac{1}{9}$

$3^x = \frac{1}{3^2}$

$3^x = 3^{-2}$

$\therefore x = -2$

EXPLANATION

a: Rewrite 16 as a power, using the base 2.

Equate powers using the result: if $a^x = a^y$ then $x = y$.

b: Rewrite 9 as a power of 3, then write using a negative index.

Equate powers with the same base.

Example continued on next page

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c
$$25^x = 125$$
$$(5^2)^x = 5^3$$
$$5^{2x} = 5^3$$
$$2x = 3$$
$$\therefore x = \frac{3}{2}$$

Since 25 and 125 are both powers of 5, rewrite both with a base of 5.
Multiply indices, then equate powers and solve for x.

Example 26 Solving exponential equations with a variable on both sides

Solve $3^{2x-1} = 27^x$.

SOLUTION

$$3^{2x-1} = 27^x$$
$$3^{2x-1} = (3^3)^x$$
$$3^{2x-1} = 3^{3x}$$
$$\therefore 2x - 1 = 3x$$
$$-1 = x$$
$$\therefore x = -1$$

EXPLANATION

Rewrite 27 with a base of 3.
Multiply indices and then equate powers.

Subtract $2x$ from both sides and answer with x as the subject.

Exercise 2J

UNDERSTANDING AND FLUENCY	1–3, 4–5(½)	3, 4–6(½)	4–6(½)

1 **a** Evaluate the following.
i 2^2 **ii** 2^3 **iii** 2^4 **iv** 2^5
b Hence, state the value of x when:
i $2^x = 8$ **ii** $2^x = 32$ **iii** $2^x = 64$

2 Solve for x in each of the following.
a $3^x = 9$ **b** $3^x = \frac{1}{3}$ **c** $3^x = 81$
d $3^x = 243$ **e** $3^x = \frac{1}{27}$ **f** $3^x = 1$

3 Write these numbers in index form. For example, $32 = 2^5$.
a 9 **b** 125 **c** 243 **d** 128 **e** 729

Example 25a

4 Solve for x in each of the following.
a $3^x = 27$ **b** $2^x = 8$ **c** $6^x = 36$
d $9^x = 81$ **e** $5^x = 125$ **f** $4^x = 64$
g $3^x = 81$ **h** $6^x = 216$ **i** $5^x = 625$
j $2^x = 32$ **k** $10^x = 10\,000$ **l** $7^x = 343$

Example 25b

5 Solve for x in each of the following.
a $7^x = \frac{1}{49}$ **b** $9^x = \frac{1}{81}$ **c** $11^x = \frac{1}{121}$
d $4^x = \frac{1}{256}$ **e** $3^x = \frac{1}{243}$ **f** $5^{-x} = \frac{1}{125}$
g $3^{-x} = \frac{1}{9}$ **h** $2^{-x} = \frac{1}{64}$ **i** $7^{-x} = \frac{1}{343}$

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Example 25c

6 Solve for x in each of the following.

a $9^x = 27$
b $8^x = 16$
c $25^x = 125$
d $16^x = 64$
e $81^x = 9$
f $216^x = 6$
g $32^x = 2$
h $10000^x = 10$
i $7^{-x} = 49$
j $4^{-x} = 256$
k $16^{-x} = 64$
l $25^{-x} = 125$

PROBLEM-SOLVING AND REASONING	7, 8(½), 10	7, 8(½), 10, 11(½)	8(½), 9, 11–13(½)

7 The population of bacteria in a petri dish is given by the rule $P = 2^t$, where P is the bacteria population and t is the time in minutes.

a What is the initial population of bacteria; i.e. when $t = 0$?

b Determine the population of bacteria after:

i 1 minute
ii 5 minutes
iii 1 hour
iv 1 day

c Determine how long it takes for the population to reach:

i 8
ii 256
iii more than 1000

Example 26

8 Solve for x in each of the following.

a $2^{x+1} = 8^x$
b $3^{2x+1} = 27^x$
c $7^{x+9} = 49^{2x}$
d $5^{x+3} = 25^{2x}$
e $6^{2x+3} = 216^{2x}$
f $9^{x+12} = 81^{x+5}$
g $27^{x+3} = 9^{2x}$
h $25^{x+3} = 125^{3x}$
i $32^{2x+3} = 128^{2x}$
j $27^{2x+3} = 9^{2x-1}$
k $9^{x-1} = 27^{2x-6}$
l $49^{2x-3} = 343^{2x-1}$

9 Would you prefer \$1 million now or 1 cent doubled every second for 30 seconds? Give reasons for your preference.

10 Consider a^x, where $a = 1$.

a Evaluate 1^x when:

i $x = 1$
ii $x = 3$
iii $x = 10000000$

b Are there any solutions to the equation $a^x = 2$ when $a = 1$? Give a reason.

11 Recall that $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[3]{x} = x^{\frac{1}{3}}$. Now solve the following.

a $3^x = \sqrt{81}$
b $5^x = \sqrt{25}$
c $6^x = \sqrt[3]{36}$
d $4^x = \sqrt[4]{64}$
e $2^x = \sqrt[4]{32}$
f $3^x = \sqrt[9]{27}$
g $25^x = \sqrt[5]{125}$
h $9^x = \dfrac{1}{\sqrt[3]{27}}$

12 a Write these numbers as decimals.

i $\dfrac{1}{2^2}$
ii 2^{-3}
iii 10^{-3}
iv $\left(\dfrac{1}{5}\right)^4$

b Write these decimal numbers as powers of prime numbers.

i 0.04
ii 0.0625
iii 0.5
iv 0.0016

13 Show how you can use techniques from this section to solve these equations involving decimals.

a $10^x = 0.0001$
b $2^x = 0.015625$
c $5^x = 0.00032$
d $(0.25)^x = 0.5$
e $(0.04)^x = 125$
f $(0.0625)^{x+1} = \dfrac{1}{2}$

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ENRICHMENT	–	–	14

Mixing index laws with equations

14 Solve for n in the following.

a $3^n \times 9^n = 27$

b $5^{3n} \times 25^{-2n+1} = 125$

c $2^{-3n} \times 4^{2n-2} = 16$

d $3^{2n-1} = \frac{1}{81}$

e $7^{2n+3} = \frac{1}{49}$

f $5^{3n+2} = \frac{1}{625}$

g $6^{2n-6} = 1$

h $11^{3n-1} = 11$

i $8^{5n-1} = 1$

j $\frac{3^{n-2}}{9^{1-n}} = 9$

k $\frac{5^{3n-3}}{25^{n-3}} = 125$

l $\frac{36^{3+2n}}{6^n} = 1$

Many mathematicians work in scientific research and development, where index laws are often used in scientific formulas.

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2K Exponential growth and decay FRINGE

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The population of a country increasing by 5% per year and an investment increasing, on average, by 12% per year are examples of exponential growth. When an investment grows exponentially, the increase per year is not constant. The annual increase is calculated on the value of the investment at the time, and this changes from year to year because of the added investment returns. The more money you have invested, the more interest you will make in a year.

Population growth is exponential when the percentage increase remains constant each year.

In the same way, a population can grow exponentially. A growth of 5% in a large population represents many more babies born in a year than 5% of a small population.

Exponential growth and decay will be studied in more detail in Chapter 9. Here we will focus on exponential growth in relation to compound interest and other simple situations that involve growth and decay. In the financial world, it is important to understand how compound interest works and how investments can grow and decay exponentially.

Let's start: A compound rule

Imagine that you have an investment valued at $100 000 and you hope that it will return 10% per annum (p.a.). The 10% return is to be added to the investment balance each year.

- Discuss how to calculate the investment balance in the first year.
- Discuss how to calculate the investment balance in the second year.
- Complete this table.

Year	0	1	2	3
Balance ($)	100 000	100 000 × 1.1 = ______	100 000 × 1.1 × ___ = ______	______ = ______

- Discuss how indices can be used to calculate the balance after the second year.
- Discuss how indices can be used to calculate the balance after the tenth year.
- What might be the rule connecting the investment balance ($$A$) and the time n years?

Key ideas

- **Per annum** (p.a.) means 'per year'.
- Exponential growth and decay can be modelled by the rule $A = ka^t$, where A is the amount, k is the initial amount and t is the time.
  - When $a > 0$, exponential growth occurs.
  - When $0 < a < 1$, exponential decay occurs.

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Key ideas

- For a **growth** rate of $r\%$ p.a., the base 'a' is calculated using $a = 1 + \frac{r}{100}$ or $a = 1 + R$.
- For a **decay** rate of $r\%$ p.a., the base 'a' is calculated using $a = 1 - \frac{r}{100}$ or $a = 1 - R$.
- The basic **exponential formula** can be summarised as $A = A_0(1 \pm R)^n$ or $A = A_0(1 + R)^n$.
  - The subscript zero is often used to indicate the initial value of a quantity (e.g. P_0 is initial population).
- **Compounding interest** involves *adding* any interest earned to the balance at the end of each year or other period. The rule for the investment amount (\$$A$) is given by $A = P(1 + R)^n$, where:
  - P is the initial amount.
  - R is the interest rate, expressed as a decimal.
  - n is the number of periods.

Example 27 Writing exponential rules

Form exponential rules for the following situations.

a John invests his \$100 000 in savings at a rate of 14% per annum.

b A town's initial population of 50 000 is decreasing by 12% per year.

SOLUTION	EXPLANATION
a Let A = the amount of money at anytime	Define your variables.
n = the number of years the money is invested	The basic formula is $A = P(1 \pm R)^n$.
P = initial amount invested	
$R = 0.14$	
$P = 100\,000$	
$A = 100\,000(1 + 0.14)^n$	Substitute $R = 0.14$ and $P = 100\,000$ and use '+' since we have growth.
$\therefore A = 100\,000(1.14)^n$	
b Let A = the population at anytime	Define your variables.
n = the number of years the population decreases	The basic formula is $A = P(1 \pm R)^n$.
P = starting population	
$R = 0.12$	
$P = 50\,000$	
$P = 50\,000(1 - 0.12)^n$	Substitute $R = 0.12$ and $P = 50\,000$ and use '−' since we have decay.
$\therefore P = 50\,000(0.88)^n$	

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Example 28 Applying exponential rules

House prices are rising at 9% per year and Zoe's house is currently valued at \$600 000.

a Determine a rule for the value of Zoe's house (\$$V$) in n years' time.

b What will be the value of her house:

i next year? **ii** in 3 years' time?

c Use trial and error to find when Zoe's house will be valued at \$900 000, to 1 decimal place.

SOLUTION

a Let V = value of Zoe's house at anytime

V_0 = starting value \$600 000

n = number of years from now

$R = 0.9$

$V = V_0(1.09)^n$

$\therefore V = 600\,000(1.09)^n$

b **i** When $n = 1$, $V = 600\,000(1.09)^1$

$= 654\,000$

Zoe's house would be valued at \$654 000 next year.

ii When $n = 3$, $V = 600\,000(1.09)^3$

$= 777\,017.40$

In 3 years' time Zoe's house will be valued at about \$777 017.

c

n	4	5	4.5	4.8	4.7
V	846 949	923 174	891 894	907 399	899 613

Zoe's house will be valued at \$900 000 in about 4.8 years' time.

EXPLANATION

Define your variables.

The basic formula is $V = V_0(1 \pm R)^n$.

Use '+' since we have growth.

Substitute $n = 1$ for next year.

For three years, substitute $n = 3$.

Try a value of n in the rule. If V is too low, increase your n value. If V is too high, decrease your n value. Continue this process until you get close to \$900 000.

Exercise 2K FRINGE

UNDERSTANDING AND FLUENCY	1–6	3–7	5–8

1 An investment of \$1000 is to grow by 5% per year. Round your answers to the nearest cent.

a Find the interest earned in the first year.

b Find the investment balance at the end of the first year.

c Find the interest earned in the second year.

d Find the interest earned in the third year.

e Find the investment balance at the end of the fifth year.

2 The mass of a 5 kg limestone rock that is exposed to the weather is decreasing at a rate of 2% per annum.

a Find the mass of the rock at the end of the first year.

b Copy and complete the rule for the mass of the rock (M kg) after t years.

$M = 5(1 - \_\_\_)^t = 5 \times \_\_\_^t$

c Use your rule to calculate the mass of the rock after 5 years, correct to 2 decimal places.

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3 For each of the following determine if exponential *growth* or exponential *decay* is represented.

a $A = 1000 \times 1.3^t$

b $A = 200 \times 1.78^t$

c $A = 350 \times 0.9^t$

d $P = 50000 \times 0.85^t$

e $P = P_0\left(1 + \frac{3}{100}\right)^t$

f $T = T_0\left(1 - \frac{7}{100}\right)^t$

Example 27

4 Define variables and form exponential rules for the following situations.

a \$200000 is invested at 17% per annum.

b A house initially valued at \$530000 is losing value at 5% per annum.

c The value of a car, bought for \$14200, is decreasing at 3% per annum.

d A population, which is initially 172500, is increasing at 15% per year.

e A tank with 1200 litres of water is leaking at a rate of 10% of the water in the tank every hour.

f A human cell of area 0.01 cm^2 doubles its size every minute.

g An oil spill, initially covering an area of 2 square metres, is increasing at 5% per minute.

h A substance of mass 30 g is decaying at a rate of 8% per hour.

Example 28

5 The value of a house purchased for \$500000 is expected to grow by 10% per year. Let \$$A$ be the value of the house after t years.

a Copy and complete the rule connecting A and t.

$A = 500000 \times \_\_\_^t$

b Use your rule to find the expected value of the house after the following number of years. Round year answer to the nearest cent.

i 3 years

ii 10 years

iii 20 years

c Use trial and error to estimate when the house will be worth \$1 million. Round your answer to 1 decimal place.

6 A share portfolio initially worth \$300000 is reduced by 15% p.a. over a number of years. Let \$$A$ be the share portfolio value after t years.

a Copy and complete the rule connecting A and t.

$A = \_\_\_\_\_\_ \times 0.85^t$

b Use your rule to find the value of the shares after the following number of years. Round your answer to the nearest cent.

i 2 years

ii 7 years

iii 12 years

c Use trial and error to estimate when the share portfolio will be valued at \$180000. Round your answer to 1 decimal place.

7 A water tank containing 15000 L has a small hole that reduces the amount of water by 6% per hour.

a Determine a rule for the volume of water (V) left after t hours.

b Calculate (to the nearest litre) the amount of water left in the tank after:

i 3 hours

ii 7 hours

c How much water is left after two days? Round your answer to 2 decimal places.

d Using trial and error, determine when the tank holds less than 500L of water, to 1 decimal place.

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8 Megan invests \$50 000 into a superannuation scheme that has an annual return of 11%.
- **a** Determine the rule for the value of her investment (V) after n years.
- **b** How much will Megan's investment be worth in:
  - **i** 4 years?
  - **ii** 20 years?
- **c** Find the approximate time before her investment is worth \$100 000. Round your answer to 2 decimal places.

PROBLEM-SOLVING AND REASONING	9, 10, 12(a)	9, 10, 12	10–13

9 A certain type of bacteria grows according to the equation $N = 3000(2.6)^t$, where N is the number of cells present after t hours.
- **a** How many bacteria are there at the start?
- **b** Determine the number of cells (round your answer to the whole number) present after:
  - **i** 0 hours
  - **ii** 2 hours
  - **iii** 4.6 hours
- **c** If 50 000 000 bacteria are needed to make a drop of serum, determine how long you will have to wait to make a drop. (Give your answer to the nearest minute.)

10 A car tyre has 10 mm of tread when new. It is considered unroadworthy when there is only 3 mm left. The rubber wears at 12.5% every 10 000 km.
- **a** Write an equation relating the depth of tread (D) for every 10 000 km travelled.
- **b** Using trial and error, determine when the tyre becomes unroadworthy, to the nearest 10 000 km.
- **c** If a tyre lasts 80 000 km, it is considered to be a tyre of good quality. Is this a good quality tyre?

11 A cup of coffee has an initial temperature of 90°C.
- **a** If the temperature reduces by 8% every minute, determine a rule for the temperature of the coffee (T) after t minutes.
- **b** What is the temperature of the coffee (to 1 decimal place) after:
  - **i** 90 seconds?
  - **ii** 2 minutes?
- **c** When is the coffee suitable to drink if it is best consumed at a temperature of 68.8°C? Give your answer to the nearest minute.

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12 Interest on investments can be calculated using different time periods. Consider \$1000 invested at 10% p.a. over 5 years.

- If interest is compounded annually, then $R = 0.1$ and $t = 5$. So $A = 1000(1.1)^5$.
- If interest is compounded monthly, then $R = \frac{1}{121}$ and $t = 5 \times 12 = 60$.

  So $A = 1000\left(1 + \frac{0.1}{12}\right)^{60}$.

a If interest is calculated annually, find the value of the investment, to the nearest cent, after:
  i 5 years
  ii 8 years
  iii 15 years

b If interest is calculated monthly, find the value of the investment, to the nearest cent, after:
  i 5 years
  ii 8 years
  iii 15 years

13 You are given \$2000 and invest it in an account that offers 7% p.a. compound interest. Determine what the investment will be worth, to the nearest cent, after 5 years if interest is compounded:

a annually b monthly c weekly (assume 52 weeks in the year)

ENRICHMENT	–	–	14–16

Half-life

Half-life is the period of time it takes for an object to decay by half. It is often used to compare the rate of decay for radioactive materials.

14 A 100 g mass of a radioactive material decays at a rate of 10% every 10 years.

a Find the mass of the material after these time periods. Round your answer to 1 decimal place.
  i 10 years
  ii 30 years
  iii 60 years

b Estimate the half-life of the radioactive material (i.e. find how long it takes for the material to decay to 50 g). Use trial and error and round your answer to the nearest year.

15 An ice sculpture, initially containing 150 L of water, melts at a rate of 3% per minute.

a What will be the volume of the ice sculpture after half an hour? Round your answer to the nearest litre.

b Estimate the half-life of the ice sculpture. Give your answer in minutes, correct to 1 decimal place.

16 The half-life of a substance is 100 years. Find the rate of decay per annum, expressed as a percentage correct to 1 decimal place.

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True wealth

Chances are that people who become wealthy have invested money in appreciating assets, such as property, shares and other businesses, rather than depreciating assets, such as cars, televisions and other electronic devices. Appreciating assets increase in value over time and earn income. Depreciating assets lose value over time.

Appreciating or depreciating?

Imagine that you have \$100 000 to invest or spend and you have these options.

Option 1: Invest in shares and expect a return of 8% p.a.

Option 2: Buy a car that depreciates at 8% p.a.

a Find the value of the \$100 000 share investment after:
- **i** 2 years
- **ii** 5 years
- **iii** 10 years

b How long will it take for the share investment to double in value?

c Find the value of the \$100 000 car after:
- **i** 2 years
- **ii** 5 years
- **iii** 10 years

d How long will it take for the value of the car to fall to half of its original value?

e Explain why people who wish to create wealth might invest in appreciating assets.

Buying residential property

A common way to invest in Australia is to buy residential property. Imagine you have \$500 000 to buy an investment property that is expected to grow in value by 10% p.a. Stamp duty and other buying costs total \$30 000. Each year the property has costs of \$4000 (e.g. land tax, rates and insurance) and earns a rental income of \$1200 per month.

a What is the initial amount you can spend on a residential property after taking into account the stamp duty and other buying costs?

b What is the total rental income per year?

c What is the total net income from the property per year after annual expenses have been considered?

d By considering only the property's initial capital value, find the expected value of the property after:
- **i** 5 years
- **ii** 10 years
- **iii** 30 years

e By taking into account the rise in value of the property and the net income, determine the total profit after:
- **i** 1 year
- **ii** 3 years
- **iii** 5 years

Investigation

ISBN 978-1-108-46847-3

Borrowing to invest

Borrowing money to invest can increase returns but it can also increase risk. Interest must be paid on the borrowed money, but this can be offset by the income of the investment. If there is a net loss at the end of the financial year, then negative gearing (i.e. borrowing) has occurred. This net loss can be used to reduce the amount of tax paid on other income, such as salary or other business income.

Imagine that you take out a loan of \$300 000 to add to your own savings of \$200 000 so that you can spend \$500 000 on the investment property discussed on the previous page. In summary:

- The property is expected to grow in value by 10% p.a.
- Your \$300 000 loan is *interest only* at 7% p.a., meaning that only interest is paid back each year and the balance remains the same.
- Property costs are \$4000 p.a.
- Rental income is \$1200 per month.
- Your income tax rate is 30%.

a Find the amount of interest that must be paid on the loan each year.

b Find the net cash loss for the property per year. Include property costs, rent and loan interest.

c This loss reduces other income, so with a tax rate of 30% this loss is reduced by 30%. Now calculate the overall net loss, including this tax benefit.

d Now calculate the final net gain of the property investment for the following number of years. You will need to find the value of the appreciating asset (which is initially \$470 000) and subtract the net loss for each year from part **c** above.

i 1 year
ii 5 years
iii 20 years

You now know how millions of people try to become wealthy using compounding investment returns with a little help from the taxman!

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1 Write $3^{n-1} + 3^{n-1} + 3^{n-1}$ as a single term with base 3.

2 Simplify:

a $\dfrac{25^6 \times 5^4}{125^5}$

b $\dfrac{8^x \times 3^x}{6^x \times 9^x}$

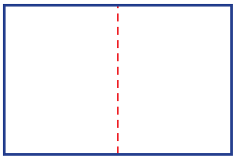
3 Solve $3^{2x} \times 27^{x+1} = 81$.

4 Simplify:

a $\dfrac{2^{n+1} - 2^{n+2}}{2^{n-1} - 2^{n-2}}$

b $\dfrac{2^{a+3} - 4 \times 2^a}{2^{2a+1} - 4^a}$

5 A rectangular piece of paper has an area of $100\sqrt{2}$ cm². The piece of paper is such that when it is folded in half along the dashed line, the new rectangle is similar (i.e. of the same shape) to the original rectangle. What are the dimensions of the piece of paper?



6 Simplify, leaving your answer with a rational denominator.

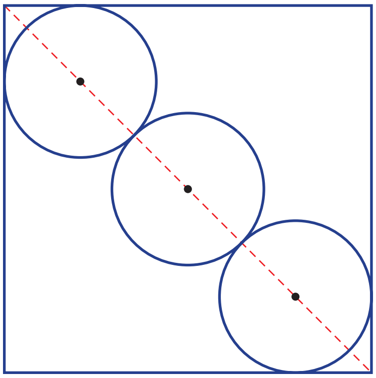
$$\frac{\sqrt{2}}{2\sqrt{2}+1} + \frac{2}{\sqrt{3}+1}$$

7 Simplify:

a $\dfrac{x^{\frac{1}{2}}y^{-\frac{1}{2}} - x^{-\frac{1}{2}}y^{\frac{1}{2}}}{\sqrt{xy}}$

b $\dfrac{x^{\frac{1}{2}}y^{-\frac{1}{2}} - x^{-\frac{1}{2}}y^{\frac{1}{2}}}{x^{-1}y^{-1}}$

8 Three circles of radius 1 unit fit inside a square such that the two outer circles touch the middle circle and the sides of the square, as shown. Given the centres of the circle lie on the diagonal of the square, find the exact area of the square.



Puzzles and challenges

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Chapter summary

Indices and surds

Negative indices

$a^{-m} = \frac{1}{a^m}$

$\frac{1}{a^{-m}} = a^m$

Scientific notation

Numbers (large and small) in scientific notation are of the form $a \times 10^m$, where $1 \le a < 10$ and m is an integer.

e.g. $3.45 \times 10^5 = 345\,000$

$2.1 \times 10^{-3} = 0.0021$

e.g. 1 475 231 in scientific notation, using three significant figures, is 1.48×10^6.

Distributive law

e.g. **1** $\sqrt{2}(2\sqrt{3}+5) = 2\sqrt{6} + 5\sqrt{2}$

2 $(\sqrt{6}+5)(2\sqrt{6}-3)$
$= 2 \times 6 - 3\sqrt{6} + 10\sqrt{6} - 15$
$= 7\sqrt{6} - 3$

3 $(\sqrt{7}-\sqrt{2})(\sqrt{7}+\sqrt{2})$
$= 7 + \sqrt{14} - \sqrt{14} - 2$
$= 5$

Index laws

$a^m \times a^n = a^{m+n}$

$a^m \div a^n = a^{m-n}$

$(a^m)^n = a^{mn}$

$(ab)^m = a^m b^m$

$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Zero index: $a^0 = 1, a \ne 0$

Rational indices

$\sqrt{a} = a^{\frac{1}{2}}$

$\sqrt[n]{a} = a^{\frac{1}{n}}$

$\sqrt[n]{a^m} = a^{\frac{m}{n}}$

e.g. $27^{\frac{1}{3}} = \sqrt[3]{27}$
$= 3$

Simplifying surds

A surd uses the symbol $\sqrt{}$ and as a decimal is infinite and non-recurring. Simplify surds: use the highest square factor.

e.g. **1** $\sqrt{20} = \sqrt{4 \times 5}$
$= \sqrt{4} \times \sqrt{5}$
$= 2\sqrt{5}$

2 $4\sqrt{27} = 4 \times \sqrt{9} \times \sqrt{3}$
$= 12\sqrt{3}$

Exponential equations

If $a^x = a^y$, then $x = y$.

e.g. **1** $2^x = 16$
$2^x = 2^4$
$\therefore x = 4$

2 $27^{x-4} = 9^x$
$(3^3)^{x-4} = (3^2)^x$
$3^{3x-12} = 3^{2x}$
$\therefore 3x - 12 = 2x$
$x = 12$

Multiply/Divide

$\sqrt{x} \times \sqrt{y} = \sqrt{xy}$

$a\sqrt{x} \times b\sqrt{y} = ab\sqrt{xy}$

$\sqrt{x} \div \sqrt{y} = \sqrt{\frac{x}{y}}$

$a\sqrt{x} \div (b\sqrt{y}) = \frac{a}{b}\sqrt{\frac{x}{y}}$

Add/Subtract

Like surds only

e.g. $3\sqrt{2} + 4\sqrt{2} = 7\sqrt{2}$

$4\sqrt{2} + \sqrt{5} - \sqrt{2} = 3\sqrt{2} + \sqrt{5}$

Simplify first

e.g. $3\sqrt{8} - 4\sqrt{2}$
$= 3 \times \sqrt{4} \times \sqrt{2} - 4\sqrt{2}$
$= 6\sqrt{2} - 4\sqrt{2}$
$= 2\sqrt{2}$

Exponential growth/decay

Rule: amount $A = P(1 \pm R)^n$

P = initial amount

R = rate of growth/decay, as a decimal

n = number of periods

Use + for growth.

Use – for decay.

Rationalise the denominator

$\frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$

$= \frac{2\sqrt{5}}{5}$

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Multiple-choice questions

1 Which of the following is a surd?

A $\sqrt{36}$ **B** π **C** $\sqrt{7}$ **D** $\sqrt[3]{8}$ **E** $1.\dot{6}$

2 A square has an area of 75 square units. Its side length, in simplified form, is:

A $3\sqrt{5}$ **B** 8.5 **C** $25\sqrt{3}$ **D** $5\sqrt{3}$ **E** $6\sqrt{15}$

3 $4\sqrt{5}$ is equivalent to:

A $\sqrt{100}$ **B** $\sqrt{80}$ **C** $2\sqrt{10}$ **D** $\sqrt{20}$ **E** $\sqrt{40}$

4 $3\sqrt{12} + 7 - 4\sqrt{3}$ simplifies to:

A $7 + \sqrt{3}$ **B** $8\sqrt{3} + 7$ **C** $6\sqrt{6} + 3\sqrt{3}$ **D** $3\sqrt{3}$ **E** $2\sqrt{3} + 7$

5 The expanded form of $2\sqrt{5}(5 - 3\sqrt{3})$ is:

A $10\sqrt{5} - 6\sqrt{15}$ **B** $7\sqrt{5} - 5\sqrt{15}$ **C** $10\sqrt{5} - 12\sqrt{2}$ **D** $10 - 5\sqrt{15}$ **E** $7\sqrt{5} - 5\sqrt{3}$

6 $\frac{2\sqrt{5}}{\sqrt{6}}$ is equivalent to:

A $\frac{2\sqrt{30}}{\sqrt{6}}$ **B** $\frac{5\sqrt{6}}{3}$ **C** $2\sqrt{5}$ **D** $\frac{\sqrt{30}}{3}$ **E** $\frac{\sqrt{30}}{10}$

7 The simplified form of $\frac{(6xy^3)^2}{3x^3y^2 \times 4x^4y^0}$ is:

A $\frac{y^4}{2x^6}$ **B** $\frac{3y^3}{x^{10}}$ **C** $\frac{y^6}{x^5}$ **D** $\frac{3y^4}{x^5}$ **E** $\frac{y^6}{2x^6}$

8 $\frac{8a^{-1}b^{-2}}{12a^3b^{-5}}$ expressed with positive indices is:

A $\frac{2a^2}{3b^3}$ **B** $\frac{a^2b^3}{96}$ **C** $\frac{2b^3}{3a^4}$ **D** $-\frac{2b^7}{3a^2}$ **E** $\frac{3}{2}a^4b^7$

9 The radius of the Earth is approximately 6 378 137 m. In scientific notation, using three significant figures, this is:

A 6.378×10^6 m **B** 6.38×10^6 m **C** 6.4×10^5 m **D** 6.37×10^6 m **E** 6.36×10^6 m

10 $\sqrt{8x^6}$ in index form is:

A $8x^3$ **B** $8x^2$ **C** $4x^3$ **D** $8^{\frac{1}{2}}x^3$ **E** $8^{\frac{1}{2}}x^4$

11 The solution to $3^{2x-1} = 9^2$ is:

A $x = \frac{3}{2}$ **B** $x = 2$ **C** $x = \frac{5}{2}$ **D** $x = 6$ **E** $x = 3$

12 A rule for the amount $\$A$ in an account after n years for an initial investment of \$5000 that is increasing at 7% per annum is:

A $A = 5000(1.7)^n$ **B** $A = 5000(0.93)^n$ **C** $A = 5000(0.3)^n$

D $A = 5000(1.07)^n$ **E** $A = 5000(0.7)^n$

Short-answer questions

1 Simplify the following surds.

a $\sqrt{24}$ **b** $\sqrt{72}$ **c** $3\sqrt{200}$ **d** $4\sqrt{54}$

e $\sqrt{\frac{4}{49}}$ **f** $\sqrt{\frac{8}{9}}$ **g** $\frac{5\sqrt{28}}{2}$ **h** $\frac{2\sqrt{45}}{15}$

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2 Simplify the following.

a $2\sqrt{3} + 4 + 5\sqrt{3}$
b $6\sqrt{5} - \sqrt{7} - 4\sqrt{5} + 3\sqrt{7}$
c $\sqrt{8} + 3\sqrt{2}$
d $4\sqrt{3} + 2\sqrt{18} - 4\sqrt{2}$
e $2\sqrt{5} \times \sqrt{6}$
f $-3\sqrt{2} \times 2\sqrt{10}$
g $\dfrac{2\sqrt{15}}{\sqrt{3}}$
h $\dfrac{5\sqrt{14}}{15\sqrt{2}}$

3 Expand and simplify.

a $\sqrt{2}(2\sqrt{3} + 4)$
b $2\sqrt{3}(2\sqrt{15} - \sqrt{3})$
c $(4\sqrt{3} + 4)(5 - 2\sqrt{3})$
d $(2\sqrt{5} + \sqrt{10})(6 - 3\sqrt{2})$
e $(\sqrt{10} - 2)(\sqrt{10} + 2)$
f $(\sqrt{11} - 2\sqrt{5})(\sqrt{11} + 2\sqrt{5})$
g $(3 + \sqrt{7})^2$
h $(4\sqrt{3} - 2\sqrt{2})^2$

4 Rationalise the denominator of each of the following.

a $\dfrac{1}{\sqrt{6}}$
b $\dfrac{10}{\sqrt{2}}$
c $\dfrac{6\sqrt{3}}{\sqrt{2}}$
d $\dfrac{4\sqrt{7}}{\sqrt{2}}$
e $\dfrac{3\sqrt{3}}{2\sqrt{6}}$
f $\dfrac{5\sqrt{5}}{4\sqrt{10}}$
g $\dfrac{\sqrt{5} + 2}{\sqrt{2}}$
h $\dfrac{4\sqrt{2} - \sqrt{3}}{\sqrt{3}}$

5 Simplify the following, expressing all answers with positive indices when required.

a $(5y^{-3})^2$
b $7m^0 - (5n)^0$
c $4x^{-2}y^3 \times 5x^5y^{-7}$
d $\left(\dfrac{3x}{y^{-3}}\right)^2 \times \dfrac{x^{-5}}{6y^2}$
e $\dfrac{(2t^2)^{-1}}{3t}$
f $\dfrac{3(a^2b^{-4})^2}{(2ab^2)^2} \div \dfrac{(ab)^{-2}}{(3a^{-2}b)^2}$

6 Express in index form.

a $\sqrt{21}$
b $\sqrt[3]{x}$
c $\sqrt[3]{m^5}$
d $\sqrt[5]{a^2}$
e $\sqrt{10x^3}$
f $\sqrt[3]{2a^9b}$
g $7\sqrt{7}$
h $4\sqrt[3]{4}$

7 Evaluate these without using a calculator.

a $25^{\frac{1}{2}}$
b $64^{\frac{1}{3}}$
c $\left(\dfrac{1}{8}\right)^{\frac{1}{3}}$
d $49^{-\frac{1}{2}}$
e $100^{-\frac{1}{2}}$
f $125^{-\frac{1}{3}}$

8 **a** Write the following numbers as a basic numeral.

i 3.21×10^3
ii 4.024×10^6
iii 7.59×10^{-3}
iv 9.81×10^{-5}

b Write the following numbers in scientific notation, using three significant figures.

i 0.0003084
ii 0.0000071753
iii 5678200
iv 119830000

9 Solve the following exponential equations for x.

a $3^x = 27$
b $7^x = 49$
c $4^{2x+1} = 64$
d $2^{x-2} = 16$
e $9^x = \dfrac{1}{81}$
f $5^x = \dfrac{1}{125}$
g $36^x = 216$
h $8^{x+1} = 32$
i $7^{3x-4} = 49^x$
j $11^{x-5} = \dfrac{1}{121}$
k $100^{x-2} = 1000^x$
l $9^{3-2x} = 27^{x+2}$

10 Form exponential rules for the following situations.

a An antique bought for \$800 is expected to grow in value by 7% per year.
b A balloon with volume 3000 cm^3 is leaking air at a rate of 18% per minute.

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Extended-response questions

1 A small rectangular jewellery box has a base with dimensions $3\sqrt{15}$ cm by $12 + \sqrt{3}$ cm and a height of $2\sqrt{5} + 4$ cm.

a Determine the exact area of the base of the box in expanded and simplified form.

b What is the exact volume of the box?

c Julie's earring boxes occupy an area of $9\sqrt{5}$ cm^2. What is the exact number that would fit across the base of the jewellery box? Give your answer with a rational denominator.

d The surface of Julie's rectangular dressing table has dimensions $\sqrt{2} - 1$ metres by $\sqrt{2} + 1$ metres.

i Find the area of the dressing table, in square centimetres.

ii What percentage of the area of the dressing table does the jewellery box occupy? Give your answer to 1 decimal place.

2 Georgia invests \$10000 in shares in a new company. She has been told that it is expected to increase at 6.5% per year.

a Write a rule for Georgia's expected amount, A dollars, in shares after n years.

b Use your rule to find how much she expects to have after:

i 2 years ii 5 years

c When her shares reach \$20000 Georgia plans to cash them in. According to this rule, how many years will it take to reach this amount? Give your answer to 1 decimal place.

d After 6 years there is a downturn in the market and the shares start to drop, losing value at 3% per year.

i How much are Georgia's shares worth prior to the downturn in the market? Give your answer to the nearest dollar.

ii Using your answer from part **d i**, write a rule for the expected value, V dollars, of Georgia's shares t years after the market downturn.

iii Ten years after Georgia initially invested in the shares the market is still falling at this rate. She decides it's time to sell her shares. How much money does she have, to the nearest dollar? How does this compare with the original amount of \$10000 she invested?

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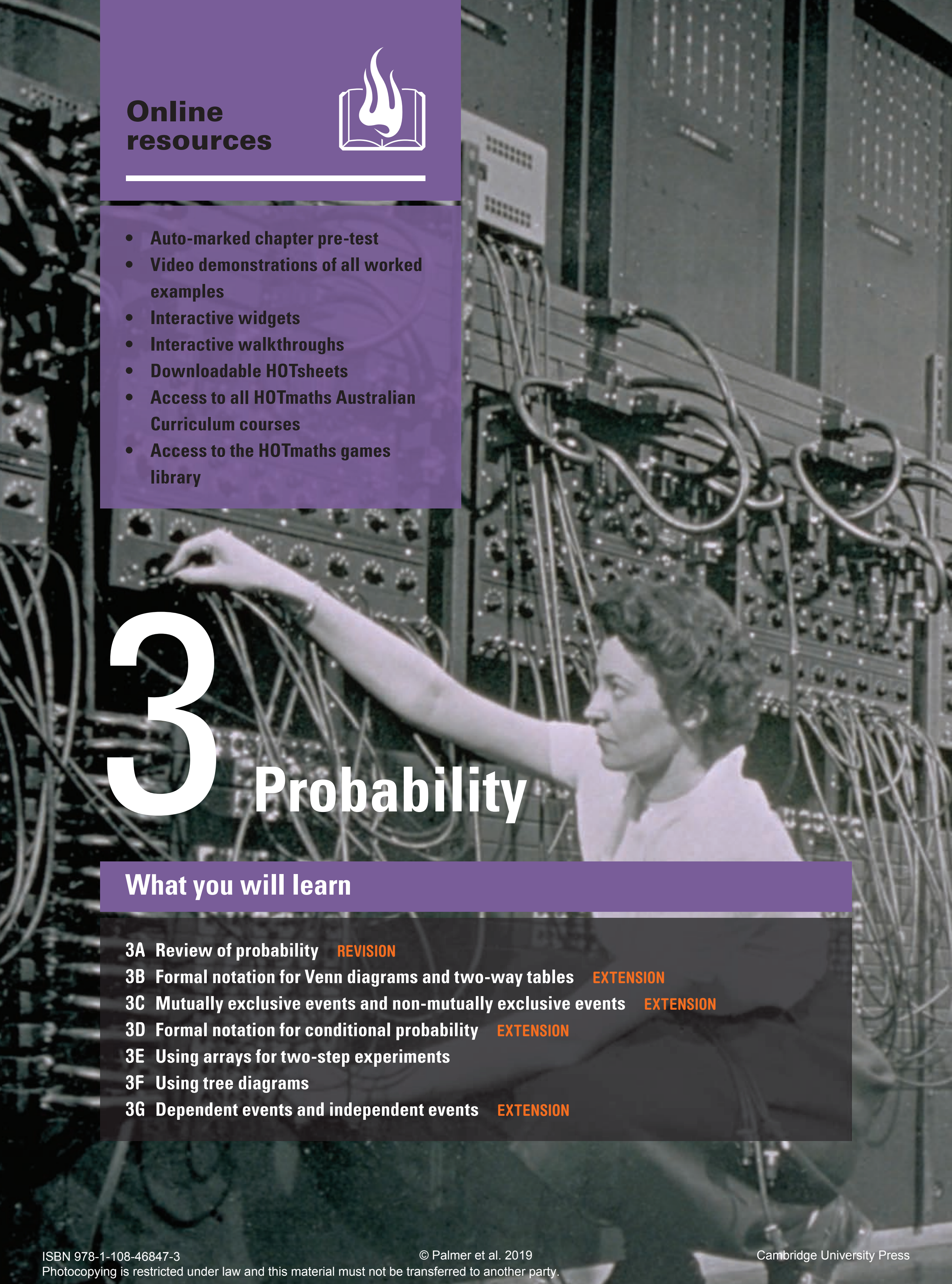

Online resources

- Auto-marked chapter pre-test
- Video demonstrations of all worked examples
- Interactive widgets
- Interactive walkthroughs
- Downloadable HOTsheets
- Access to all HOTmaths Australian Curriculum courses
- Access to the HOTmaths games library

3 Probability

What you will learn

3A Review of probability REVISION
3B Formal notation for Venn diagrams and two-way tables EXTENSION
3C Mutually exclusive events and non-mutually exclusive events EXTENSION
3D Formal notation for conditional probability EXTENSION
3E Using arrays for two-step experiments
3F Using tree diagrams
3G Dependent events and independent events EXTENSION

NSW syllabus

STRAND: STATISTICS AND PROBABILITY
SUBSTRAND: PROBABILITY

Outcomes

A student calculates relative frequencies to estimate probabilities of simple and compound events.
(MA5.1–13SP)

A student describes and calculates probabilities in multi-step chance experiments.
(MA5.2–17SP)

Chance and code breaking

A four-digit combination lock has 10 possibilities for each number, providing $10^4 = 10\,000$ combinations. So when it is locked, there is a one in ten thousand chance of guessing the correct code. To methodically test all the possibilities with, say, one second per try, it would take 10000 seconds, which is over 2.5 hours.

The probability of guessing a computer password varies with its length and random nature. The probability of cracking a password of 8 lower case letters with one try is 1 in 26^8, or 1 in 208827064576. Despite this very low chance for a single guess, a powerful computer can tryout billions of combinations per second and might crack it in minutes.

During the Second World War, English mathematicians and engineers designed various machines to break the codes of German encrypted messages. Colossus, built in 1943 to decode the settings of the German Lorenz machine, was the first electronic, digital, manually programmable computer ever built. Colossus weighed 5 tonnes and filled a room with its 8 racks of glass vacuum tubes and resistors all wired together. It processed 5000 entries per second and could decode encrypted messages in hours rather than the weeks it took for manual decoding. Ten Colossus computers were built and their success likely shortened the war by two years.

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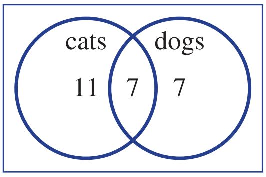
Pre-test

1 This Venn diagram shows the number of people in a group of 25 who own cats and/or dogs.



a State the number of people who own:
  i a dog **ii** a cat or a dog **iii** only a cat

b If a person is selected at random from this group, find the probability they will own:
  i a cat **ii** a cat and a dog **iii** only a dog

2 Arrange from lowest to highest: $\frac{1}{2}$, 0.4, 1 in 5, 39%, $\frac{3}{4}$, 1, 0, $\frac{9}{10}$, 0.62, 71%.

3 A spinning wheel has eight equal sectors, numbered 1 to 8. On one spin of the wheel, find the following probabilities.

a $P(5)$ **b** $P(\text{even})$ **c** $P(\text{not even})$
d $P(\text{multiple of 3})$ **e** $P(\text{factor of 12})$ **f** $P(\text{odd or a factor of 12})$
g $P(\text{both odd and a factor of 12})$

4 A letter is selected from the word PROBABILITY.

a How many letters are there in total?
b Find the chance (i.e. probability) of selecting:
  i the letter R **ii** the letter B **iii** a vowel
  iv not a vowel **v** a T or an I **vi** neither a B nor a P

5 Drew shoots from the free throw line on a basketball court. After 80 shots he counts 35 successful throws.

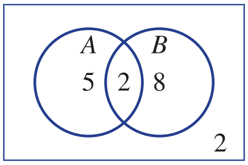
a Estimate the probability that his next throw will be successful.
b Estimate the probability that his next throw will not be successful.

6 Two fair 4-sided dice are rolled and the sum of the two numbers obtained is noted.

a Copy and complete this grid.
b What is the total number of outcomes?
c Find the probability that the total sum is:
  i 2 **ii** 4 **iii** less than 5
  iv less than or equal to 5 **v** at most 6 **vi** no more than 3

		Roll 1			
		1	2	3	4
Roll 2	1	2	3	4	
	2	3	4		
	3	4			
	4				

7 Use the information in the Venn diagram to complete the two-way table.

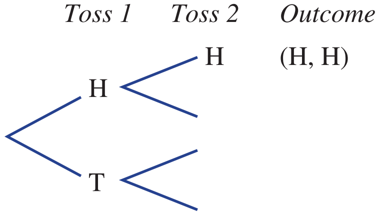


→

	A	not *A*	Total
B		8	
not *B*			
Total			

8 Two coins are tossed.

a Copy and complete this tree diagram.
b State the total number of outcomes.
c Find the probability of obtaining:
  i two heads
  ii no heads
  iii one tail
  iv at least one tail
  v one of each, a head and a tail
  vi at most two heads



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3A Review of probability REVISION

Interactive Widgets HOTsheets Walkthrough

Probability is an area of mathematics concerned with the likelihood of particular random events. In some situations, such as rolling a die, we can determine theoretical probabilities because we know the total number of outcomes and the number of favourable outcomes. In other cases, we can use statistics and experimental results to describe the chance that an event will occur. The chance that a particular soccer team will win its next match, for example, could be estimated using various results from preceding games.

A soccer team could win, lose or draw the next match it plays, but these three outcomes do not necessarily have the same probability.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

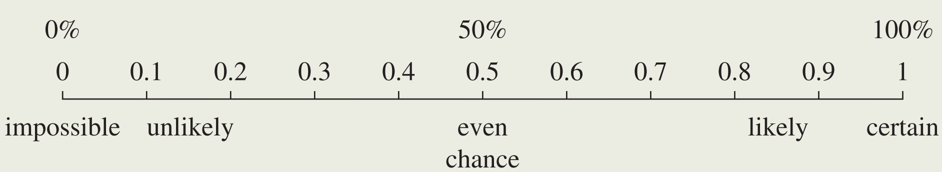
Let's start: Name the event

For each number below describe an event that has that exact or approximate probability. If you think it is exact, then give a reason.

- $\frac{1}{2}$
- 25%
- 0.2
- 0.00001
- $\frac{99}{100}$

Key ideas

- Definitions
  - A **trial** is a single occurrence of a chance experiment, such as a single roll of a die.
  - The **sample space** is the list of all possible outcomes of an experiment.
  - An **outcome** is a possible result of an experiment.
  - An **event** is the list of favourable outcomes.
  - **Equally likely outcomes** are outcomes that have the same chance of occurring.
- In the study of probability, a numerical value based on a scale from 0 to 1 is used to describe levels of **chance**.



- The probability of an event in which outcomes are **equally likely** is calculated as:

$$P(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

- **Experimental probability** is calculated in the same way as theoretical probability but uses the results of an experiment:

$$P(\text{event}) = \frac{\text{number of favourable results}}{\text{total number of trials}}$$

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Example 1 Calculating simple theoretical probabilities

A letter is chosen from the word TELEVISION. Find the probability that the letter is:

a a V **b** an E **c** not an E **d** either an E or a V

SOLUTION

a $P(\text{V}) = \frac{1}{10} (= 0.1)$

b $P(\text{E}) = \frac{2}{10}$

$= \frac{1}{5} (= 0.2)$

c $P(\text{not an } E) = \frac{8}{10}$

$= \frac{4}{5} (= 0.8)$

d $P(\text{an E or a V}) = \frac{3}{10} (= 0.3)$

EXPLANATION

$P(\text{V}) = \frac{\text{number of Vs}}{\text{total number of letters}}$

There are 2 Es in the word TELEVISION.

Simplify the fraction.

If there are 2 Es in the word TELEVISION with 10 letters, then there must be 8 letters that are not E. This is the same as 1 – P(E).

The number of letters that are either E or V is 3.

Example 2 Calculating simple experimental probabilities

An experiment involves tossing three coins and counting the number of heads. Here are the results after running the experiment 100 times.

Number of heads	0	1	2	3
Frequency	11	40	36	13

Find the experimental probability of obtaining:

a zero heads **b** two heads

c fewer than two heads **d** at least one head

SOLUTION

a $P(0 \text{ heads}) = \frac{11}{100}$

$= 0.11$

b $P(2 \text{ heads}) = \frac{36}{100}$

$= 0.36$

c $P(\text{fewer than 2 heads}) = \frac{11 + 40}{100}$

$= \frac{51}{100}$

$= 0.51$

d $P(\text{at least 1 head}) = \frac{40 + 36 + 13}{100}$

$= \frac{89}{100}$

$= 0.89$

EXPLANATION

$P(0 \text{ heads}) = \frac{\text{number of times 0 heads are observed}}{\text{total number of trials}}$

$P(2 \text{ heads}) = \frac{\text{number of times 2 heads are observed}}{\text{total number of trials}}$

Fewer than 2 heads means to observe 0 or 1 head.

At least 1 head means that 1, 2 or 3 heads can be observed. This is the same as 1 – P(0 heads).

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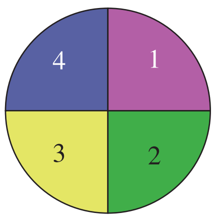
Exercise 3A REVISION

UNDERSTANDING AND FLUENCY	1–6	2–7	3–7

1 A coin is flipped once.

a How many different outcomes are possible from a single flip of the coin?
b List the sample space from a single flip of the coin.
c Are the possible outcomes equally likely?
d What is the probability of obtaining a tail?
e What is the probability of not obtaining a tail?
f What is the probability of obtaining a tail or a head?

2 For the following spinners, find the probability that the outcome will be a 4.

a


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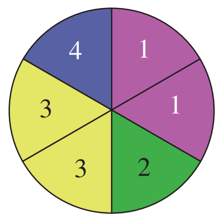
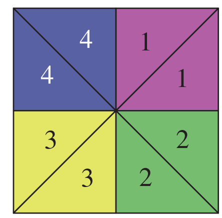
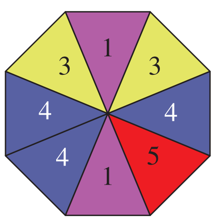
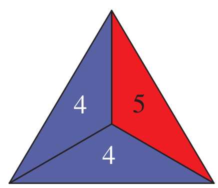
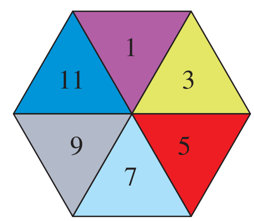


c


d


e


f


Example 1

3 A letter is chosen from the word TEACHER. Find the probability that the letter is:

a an R **b** an E **c** not an E **d** either an R or an E

4 A letter is chosen from the word EXPERIMENT. Find the probability that the letter is:

a an E **b** a vowel **c** not a vowel **d** either an X or a vowel

5 A fair 10-sided die, numbered 1 to 10, is rolled once. Find these probabilities.

a $P(8)$ **b** $P(\text{odd})$ **c** $P(\text{even})$
d $P(\text{less than } 6)$ **e** $P(\text{prime})$ (Remember that 1 is not prime.)
f $P(3 \text{ or } 8)$ **g** $P(8, 9 \text{ or } 10)$

Example 2

6 An experiment involves tossing three coins and counting the number of heads. Here are the results after running the experiment 100 times.

Number of heads	0	1	2	3
Frequency	9	38	43	10

Find the experimental probability of obtaining:

a zero heads
b two heads
c fewer than two heads
d at least one head

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7 An experiment involves rolling two dice and counting the number of sixes. Here are the results after running the experiment 100 times.

Number of sixes	0	1	2
Frequency	62	35	3

Find the experimental probability of obtaining:

a zero sixes
b two sixes
c fewer than two sixes
d at least one six

PROBLEM-SOLVING AND REASONING	8, 9, 12	9, 10, 12, 13	10, 11, 13, 14

8 Martin is a prizewinner in a competition and will be awarded a single random prize chosen from a collection of 50 prizes. The type and number of prizes to be handed out are listed below.

Prize	Car	Holiday	iPad	DVD
Number	1	4	15	30

Find the probability that Martin will be awarded the following.

a a car
b an iPad
c a prize that is not a car

9 Find the probability of choosing a red counter if a counter is chosen from a box that contains the following counters.

a 3 red and 3 yellow
b 3 red and 5 yellow
c 1 red, 1 yellow and 2 blue
d 5 red, 12 green and 7 orange
e 10 red only
f 6 blue and 4 green

10 Many of the 50 cars inspected at an assembly plant contained faults. The results of the inspection are given below.

Number of faults	0	1	2	3	4
Number of cars	30	12	4	3	1

Find the probability that a car selected from the assembly plant will have:

a one fault
b four faults
c fewer than two faults
d one or more faults
e three or four faults
f at least two faults

11 A quality control inspector examines clothing at a particular factory on a regular basis and records the number of faulty items identified each day. After 20 visits to the factory over the course of the year, the results are summarised in a table.

Number of faulty items	0	1	2	3	4	Total
Frequency	14	4	1	0	1	20

a Estimate the probability that the inspector will identify the following numbers of faulty items on any particular day.
  i 0
  ii 1
  iii 2
  iv 3
  v 4
b If the factory is fined when two or more faulty items are found, estimate the probability that the factory will be fined on the next inspection.

12 A bag contains red and yellow counters. A counter is drawn from the bag and then replaced. This happens 100 times, with 41 of the counters drawn being red.

a How many counters drawn are yellow?
b If there are 10 counters in the bag, how many do you expect are red? Give a reason.
c If there are 20 counters in the bag, how many do you expect are red? Give a reason.

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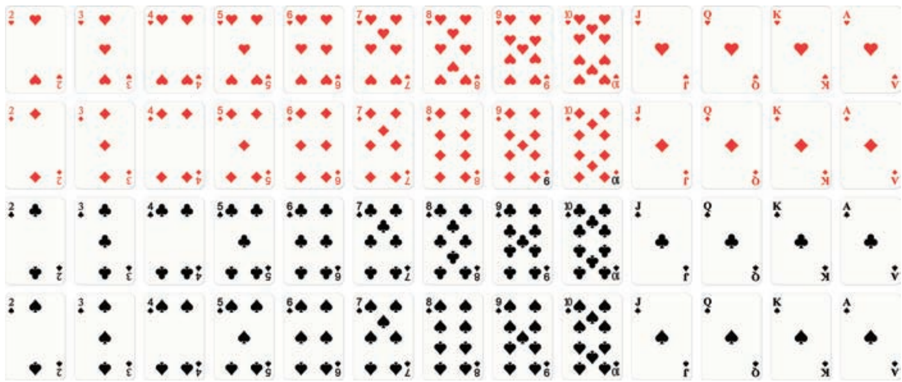
13 A card is chosen from a standard deck of 52 playing cards that includes 4 aces, 4 kings, 4 queens and 4 jacks. Find the following probabilities.

a $P(\text{heart})$ **b** $P(\text{king})$ **c** $P(\text{king of hearts})$
d $P(\text{heart or club})$ **e** $P(\text{king or jack})$ **f** $P(\text{heart or king})$
g $P(\text{not a king})$ **h** $P(\text{neither a heart nor a king})$



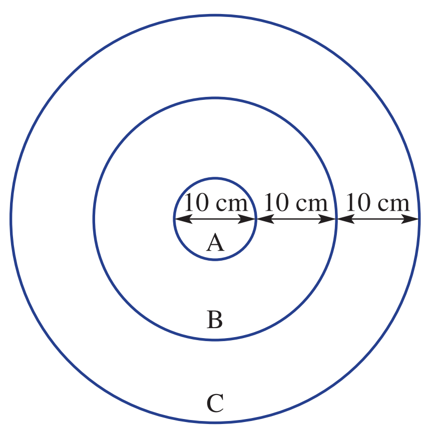
14 The probability of selecting a white chocolate from a box is $\frac{1}{5}$ and the probability of selecting a dark chocolate from the same box is $\frac{1}{3}$. The other chocolates are milk chocolates.

a Find the probability of selecting a milk chocolate.
b How many chocolates in total could be in the box? Give reasons. Is there more than one answer?

ENRICHMENT	–	–	15

Target probability

15 A target board is made up of three rings (A, B and C) that are 10 cm apart, as shown. An experienced archer shoots an arrow at the board and is guaranteed to hit it, with an equal chance of doing so at any point. Recall that the area of a circle $= \pi r^2$.



a Calculate the total area of the target and express your answer as an exact value (e.g. 10π).
b Calculate, using exact values, the area of the regions labelled:
i A **ii** B **iii** C
c Calculate the probability that the region in which the archer's arrow will hit will be:
i A **ii** B **iii** C **iv** A or B
v B or C **vi** A or C **vii** A, B or C **viii** not B
d Rachel says, "If all three rings are changed by the same amount, the answers to part **c** will remain the same." Is Rachel correct?

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3B Formal notation for Venn diagrams and two-way tables EXTENSION

Interactive

Widgets

HOTsheets

Walkthrough

When we consider two or more events it is possible that there are outcomes that are common to both events. A TV network, for example, might be collecting statistics regarding whether or not a person watches cricket and/or tennis *or* neither over the Christmas holidays. The estimated probability that a person will watch cricket or tennis will therefore depend on how many people responded yes to watching both cricket *and* tennis.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Duplication in cards

Imagine that you randomly draw one card from a standard deck of 52 playing cards.

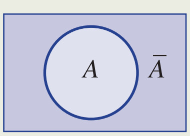
- Discuss what types of cards are included in a standard deck.
- What is the probability of selecting a heart?
- What is the probability of selecting a king?
- Now find the probability that the card is a king and a heart. Is this possible?
- Find the probability that the card is a king *or* a heart. Discuss why the probability is not just equal to $\frac{4}{52} + \frac{13}{52} = \frac{17}{52}$.

Key ideas

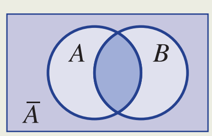
Note: The formal notation used in this section goes beyond the NSW syllabus but will prepare students for further study beyond Year 10.

■ Set notation

- A **set** is a collection or group of elements that can include numbers, letters or other objects.
- The **sample space**, denoted by S, Ω, U or ξ, is the set of all possible elements or objects considered in a particular situation. This is also called the **universal set**.
- A **Venn diagram** illustrates how all elements in the sample space are distributed among the events.



or



- A **null** or **empty set** is a set with no elements and is symbolised by { } or ∅.
- All elements that belong to both A *and* B make up the **intersection**: $A \cap B$.
- All elements that belong to either events A *or* B make up the **union**: $A \cup B$.
- Two sets A and B are **mutually exclusive** if they have no elements in common, meaning $A \cap B = \varnothing$.
- For an event A, the **complement** of A is $\overline{A}$ (or A' or A^c).

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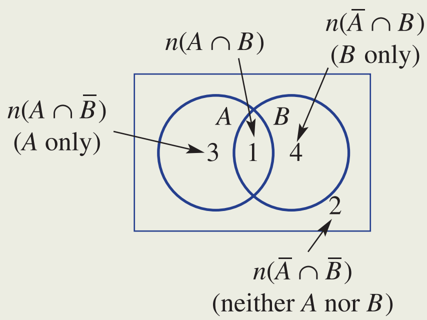
Key ideas

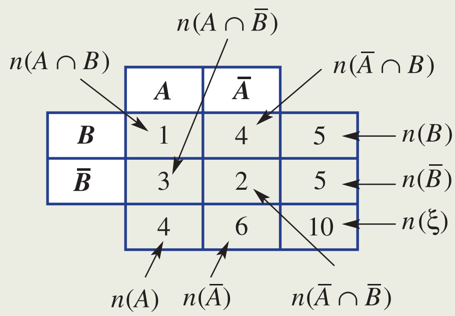
- $P(\bar{A}) = 1 - P(A)$
- A **only** (or $A \cap \bar{B}$) is defined as all the elements in A but not in any other set.
- $\boldsymbol{n(A)}$ is the number of elements in set A.

■ Venn diagrams and two-way tables are useful tools when considering two or more events.

Venn diagram



Two-way table



Example 3 Listing sets

Consider the given events A and B that involve numbers taken from the first 10 positive integers:

$A = \{1, 2, 3, 4, 5, 6\}$ $\quad$ $B = \{1, 3, 7, 8\}$

a Represent the two events A and B in a Venn diagram.

b List the following sets.

i $A \cap B$ **ii** $A \cup B$

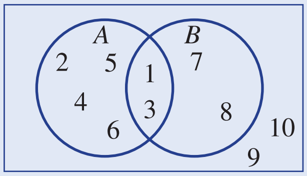
c If a number from the first 10 positive integers is selected randomly, find the probability that the following events occur.

i A **ii** $A \cap B$ **iii** $A \cup B$

d Are the events A and B mutually excusive? Why or why not?

SOLUTION

a



EXPLANATION

The elements 1 and 3 are common to both sets A and B.

The elements 9 and 10 belong to neither set A nor set B.

b **i** $A \cap B = \{1, 3\}$

$A \cap B$ is the intersection of sets A and B.

ii $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$A \cup B$ contains elements in either A or B.

c **i** $P(A) = \dfrac{6}{10} = \dfrac{3}{5}$

There are six elements in A.

ii $P(A \cap B) = \dfrac{2}{10} = \dfrac{1}{5}$

$A \cap B$ contains two elements.

iii $P(A \cup B) = \dfrac{8}{10} = \dfrac{4}{5}$

$A \cup B$ contains eight elements.

d The sets A and B are not mutually exclusive since $A \cap B \neq \varnothing$.

The set $A \cap B$ contains at least one element.

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Example 4 Using Venn diagrams

From a class of 30 students, 12 enjoy cricket (C), 14 enjoy netball (N) and 6 enjoy both cricket and netball.

a Illustrate this information in a Venn diagram.

b State the number of students who enjoy:

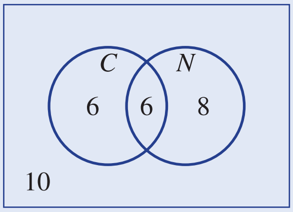
i netball only **ii** neither cricket nor netball

c Find the probability that a student chosen randomly from the class will enjoy:

i netball **ii** netball only **iii** both cricket and netball

SOLUTION

a



b **i** $n(N \text{ only}) = 8$

ii $n(\text{neither } C \text{ nor } N) = 10$

c **i** $P(N) = \frac{14}{30} = \frac{7}{15}$

ii $P(N \text{ only}) = \frac{8}{30} = \frac{4}{15}$

iii $P(C \cap N) = \frac{6}{30} = \frac{1}{5}$

EXPLANATION

First, place the 6 in the intersection (i.e. 6 enjoy cricket and netball), then determine the other values according to the given information. The total must be 30.

Includes elements in N but not in C.

These are the elements outside both C and N.

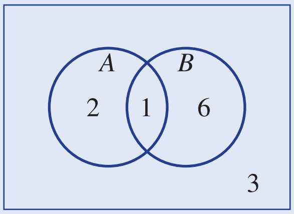
14 of the 30 students enjoy netball.

8 of the 30 students enjoy netball but not cricket.

6 students enjoy both cricket and netball.

Example 5 Venn diagrams and two-way tables

The Venn diagram shows the distribution of elements in two sets, A and B.



a Transfer the information in the Venn diagram to a two-way table.

b Find:

i $n(A \cap B)$ **ii** $n(\overline{A} \cap B)$ **iii** $n(A \cap \overline{B})$ **iv** $n(\overline{A} \cap \overline{B})$

v $n(A)$ **vi** $n(\overline{B})$ **vii** $n(A \cup B)$

c Find:

i $P(A \cap B)$ **ii** $P(\overline{A})$ **iii** $P(A \cap \overline{B})$

SOLUTION

a

	A	$\overline{A}$	
B	1	6	7
$\overline{B}$	2	3	5
	3	9	12

EXPLANATION

	A	$\overline{A}$	
B	$n(A \cap B)$	$n(\overline{A} \cap B)$	$n(B)$
$\overline{B}$	$n(A \cap \overline{B})$	$n(\overline{A} \cap \overline{B})$	$n(\overline{B})$
	$n(A)$	$n(\overline{A})$	$n(\xi)$

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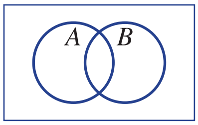
b	**i** $n(A \cap B) = 1$	$n(A \cap B)$ is the intersection of A and B.
	ii $n(\overline{A} \cap B) = 6$	$n(\overline{A} \cap B)$ is B only.
	iii $n(A \cap \overline{B}) = 2$	$n(A \cap \overline{B})$ is A only.
	iv $n(\overline{A} \cap \overline{B}) = 3$	$n(\overline{A} \cap \overline{B})$ is neither A nor B.
	v $n(A) = 3$	$n(A) = n(A \cap \overline{B}) + n(A \cap B)$
	vi $n(\overline{B}) = 5$	$n(\overline{B}) = n(A \cap \overline{B}) + n(\overline{A} \cap \overline{B})$
	vii $n(A \cup B) = 9$	$n(A \cup B) = n(A \cap B) + n(A \cap \overline{B}) + n(\overline{A} \cap B)$
c	**i** $P(A \cap B) = \frac{1}{12}$	When calculating probabilities, you will need to divide the number of elements in each set by the number of elements in the sample space, which is 12.
	ii $P(\overline{A}) = \frac{9}{12} = \frac{3}{4}$	
	iii $P(A \cap \overline{B}) = \frac{2}{12} = \frac{1}{6}$	

Exercise 3B EXTENSION

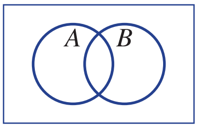
UNDERSTANDING AND FLUENCY	1–7	3–8	4, 7–9

1 Copy these Venn diagrams and shade the region described by each of the following.

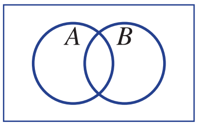
a A



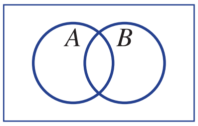
b B



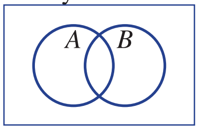
c $A \cap B$



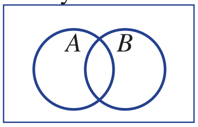
d $A \cup B$



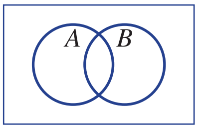
e A only



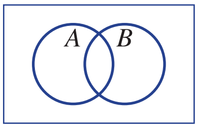
f B only



g $\overline{A}$



h neither A nor B



2 Using the symbols $\cup$, $\cap$ and $\varnothing$, rewrite the following.

a null set **b** A and B **c** A or B **d** empty set
e E and F **f** W or Z **g** A or B or C **h** A and B and C

3 Decide if the events A and B are mutually exclusive.

a $A = \{1, 3, 5, 7\}$
$B = \{5, 8, 11, 14\}$

b $A = \{-3, -2, \ldots, 4\}$
$B = \{-11, -10, \ldots, -4\}$

c $A = \{\text{prime numbers}\}$
$B = \{\text{even numbers}\}$

Example 3

4 Consider the given events A and B, which involve numbers taken from the first 10 positive integers:

$A = \{1, 2, 4, 5, 7, 8, 10\}$ $B = \{2, 3, 5, 6, 8, 9\}$

a Represent events A and B in a Venn diagram.

b List the following sets.

i $A \cap B$ **ii** $A \cup B$

c If a number from the first 10 positive integers is randomly selected, find the probability that the following events occur.

i A **ii** $A \cap B$ **iii** $A \cup B$

d Are the events A and B mutually exclusive? Why or why not?

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5 The elements of the events A and B described below are numbers taken from the first 10 prime numbers.

$A = \{2, 5, 7, 11, 13\}$ $B = \{2, 3, 13, 17, 19, 23, 29\}$

a Represent events A and B in a Venn diagram.

b List the elements belonging to the following.

i A and B ii A or B

c Find the probability that these events occur.

i A ii B iii $A \cap B$ iv $A \cup B$

Example 4

6 From a group of 50 adults, 35 enjoy reading fiction (F), 20 enjoy reading non-fiction (N) and 10 enjoy reading both fiction and non-fiction.

a Illustrate the information in a Venn diagram.

b State the number of people who enjoy:

i fiction only ii neither fiction nor non-fiction

c Find the probability that a person chosen randomly from the group will enjoy reading:

i non-fiction ii non-fiction only iii both fiction and non-fiction

7 At a show, 45 children have the choice of riding on the Ferris wheel (F) and/or the Big Dipper (B). Thirty-five of the children wish to ride on the Ferris wheel, 15 children want to ride on the Big Dipper and 10 children want to ride on both.

a Illustrate the information in a Venn diagram.

b Find:

i $n(F \text{ only})$ ii $n(\text{neither } F \text{ nor } B)$

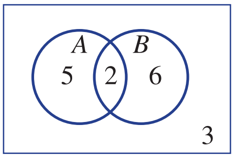
c For a child chosen at random from the group, find the following probabilities.

i $P(F)$ ii $P(F \cap B)$ iii $P(F \cup B)$

iv $P(\overline{F})$ v $P(\text{neither } F \text{ nor } B)$

Example 5

8 The Venn diagram shows the distribution of elements in two sets, A and B.



a Transfer the information in the Venn diagram to a two-way table.

b Find:

i $n(A \cap B)$ ii $n(\overline{A} \cap B)$ iii $n(A \cap \overline{B})$ iv $n(\overline{A} \cap \overline{B})$

v $n(A)$ vi $n(\overline{B})$ vii $n(A \cup B)$

c Find:

i $P(A \cap B)$ ii $P(\overline{A})$ iii $P(A \cap \overline{B})$

9 From a total of 10 people, 5 like apples (A), 6 like bananas (B) and 4 like both apples and bananas.

a Draw a Venn diagram for the 10 people.

b Draw a two-way table.

c Find:

i $n(\overline{A} \cap B)$ ii $n(\overline{A} \cap \overline{B})$ iii $P(A \cap B)$ iv $P(A \cup B)$

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PROBLEM-SOLVING AND REASONING	10, 11, 14	11, 12, 14, 15	12, 13, 15, 16

10 Decide which of the elements would need to be removed from event A if the two events A and B described below are to become mutually exclusive.

a $A = \{1, 2, 3, 4\}$
$B = \{4, 5, 6, 7\}$

b $A = \{10, 12, 14, 16, 18\}$
$B = \{9, 10, 11, 12\}$

c $A = \{\text{a, b, c, d, e}\}$
$B = \{\text{a, c, e, g}\}$

d $A = \{1, 3, 5, 8, 10, 15, 20, 22, 23\}$
$B = \{7, 9, 14, 16, 19, 21, 26\}$

11 A letter is chosen at random from the word COMPLEMENTARY and two events, C and D, are as follows.

C: choosing a letter belonging to the word COMPLETE
D: choosing a letter belonging to the word CEMENT

a Represent the events C and D in a Venn diagram. Ensure that your Venn diagram includes all the letters that make up the word COMPLEMENTARY.

b Find the probability that the randomly chosen letter will:

i belong to C **ii** belong to C and D **iii** belong to C or D
iv not belong to C **v** belong to neither C nor D

12 Complete the following two-way tables.

a

	A	$\bar{A}$	
B		3	6
$\bar{B}$			
		4	11

b

	A	$\bar{A}$	
B	2	7	
$\bar{B}$			3
	4		

13 In a group of 12 chefs, all enjoy baking cakes and/or tarts. In fact, 7 enjoy baking cakes and 8 enjoy baking tarts. Find out how many chefs enjoy baking both cakes and tarts.

14 If events A and B are mutually exclusive and $P(A) = a$ and $P(B) = b$, write expressions for:

a $P(\text{not } A)$ **b** $P(A \text{ or } B)$ **c** $P(A \text{ and } B)$

15 Use diagrams to show that $\overline{A \cup B} = \bar{A} \cap \bar{B}$.

16 Mario and Elisa are choosing a colour to paint the interior walls of their house. They have six colours to choose from: white (w), cream (c), navy (n), sky blue (s), maroon (m) and violet (v).

Mario would be happy with white or cream and Elisa would be happy with cream, navy or sky blue. As they can't decide, a colour is chosen at random for them.

Let M be the event that Mario will be happy with the colour and let E be the event that Elisa will be happy with the colour.

a Represent the events M and E in a Venn diagram.

b Find the probability that the following events occur.

i Mario will be happy with the colour choice; i.e. find $P(M)$.
ii Mario will not be happy with the colour choice.
iii Both Mario and Elisa will be happy with the colour choice.
iv Mario or Elisa will be happy with the colour choice.
v Neither Mario nor Elisa will be happy with the colour choice.

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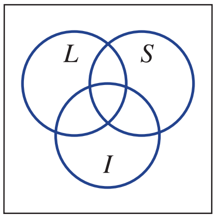
ENRICHMENT	–	–	17, 18

Triple Venn diagrams

17 Of 15 chosen courier companies, 9 offer a local service (L), 7 offer an interstate service (S) and 6 offer an international service (I). Two companies offer all three services, 3 offer both local and interstate services, 5 offer only a local service and 1 offers only an international service.



a Draw a Venn diagram displaying the given information.

b Find the number of courier companies that offer neither a local, interstate nor international service.

c If a courier is chosen at random from the 15 examined initially, find the following probabilities.

i $P(L)$ **ii** $P(L \text{ only})$ **iii** $P(L \text{ or } S)$ **iv** $P(L \text{ and } S \text{ only})$

18 Thirty-eight people were interviewed about their travelling experience in the past 12 months. Although the interviewer did not write down the details of the interviews, she remembers the following information.

In the past 12 months:

– Two people travelled overseas, interstate and within their own State.
– Two people travelled overseas and within their own State only.
– Seven people travelled interstate only.
– 22 people travelled within their own State.
– Three people did not travel at all.
– The number of people who travelled interstate and within their own State only was twice the number of people who travelled overseas and interstate only.
– The number of people who travelled overseas was equal to the number of people who travelled within their own State only.

a Use a Venn diagram to represent the information that the interviewer remembers.

b By writing down equations using the variables x (the number of people who travelled overseas and interstate only) and y (the number of people who travelled overseas only), solve simultaneously and find:

i the number of people who travelled interstate and overseas only
ii the number of people who travelled overseas

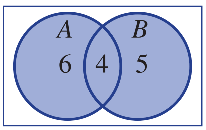
c If one person from the 38 is chosen at random, find the probability that the person will have travelled to the following places.

i within their own State only
ii overseas only
iii interstate only
iv overseas or interstate or within their own State
v interstate or overseas

3C Mutually exclusive events and non-mutually exclusive events EXTENSION

When two events are mutually exclusive we know that the probability of the union of the events can be found by simply adding the probabilities of each of the individual events. If they are not mutually exclusive then we need to take the intersection into account.

If we take 15 people who like apples (A) or bananas (B), for example, we could illustrate this with the following possible Venn diagram.



$$P(A) = \frac{10}{15}$$
$$P(B) = \frac{9}{15}$$

Clearly, the probability that a person likes apples or bananas is not $\frac{10}{15} + \frac{9}{15} = \frac{19}{15}$, as this is impossible. The intersection needs to be taken into account because, in the example above, this has been counted twice. This consideration leads to the addition rule, which will be explained in this section.

Let's start: What's the intersection?

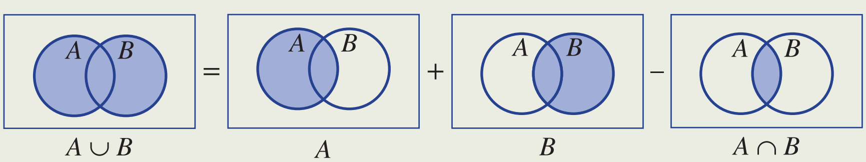
Two events, A and B, are such that $P(A) = 0.5$, $P(B) = 0.4$ and $P(A \cup B) = 0.8$.

- Are the events mutually exclusive? Why?
- Is it possible to find $P(A \cap B)$? If so, find $P(A \cap B)$.
- Can you write a rule connecting $P(A \cup B)$, $P(A)$, $P(B)$ and $P(A \cap B)$?
- Does your rule hold true for mutually exclusive events?

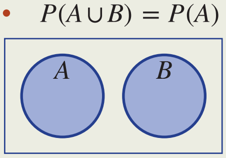
Key ideas

Note: The content and formal notation in this section goes beyond the NSW syllabus but will prepare students for further study beyond Year 10.

- The **addition rule** for two events, A and B, is:
  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



- If A and B are mutually exclusive then:
  - $P(A \cap B) = 0$
  - $P(A \cup B) = P(A) + P(B)$



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Example 6 Applying the addition rule

A card is selected from a standard deck of 52 playing cards (4 suits, no jokers). Let A be the event 'the card is a diamond' and let B be the event 'the card is a jack'.

a Find:
- **i** $n(A)$
- **ii** $n(B)$
- **iii** $n(A \cap B)$

b Find:
- **i** $P(A)$
- **ii** $P(\overline{A})$
- **iii** $P(A \cap B)$

c Use the addition rule to find $P(A \cup B)$.

d Find the probability that the card is a jack or not a diamond.

SOLUTION

a **i** $n(A) = 13$

One-quarter of the cards is the diamond suit.

ii $n(B) = 4$

There is one jack in each suit.

iii $n(A \cap B) = 1$

Only one card is both a diamond and a jack.

b **i** $P(A) = \frac{13}{52} = \frac{1}{4}$

13 out of the 52 cards are diamond.

ii $P(\overline{A}) = 1 - \frac{1}{4} = \frac{3}{4}$

The complement of A is $\overline{A}$.

iii $P(A \cap B) = \frac{1}{52}$

There is one jack of diamonds out of the 52 cards.

c
$$\begin{aligned} &P(A \cup B) \\ &= P(A) + P(B) - P(A \cap B) \\ &= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} \\ &= \frac{16}{52} \\ &= \frac{4}{13} \end{aligned}$$

Substitute $P(A)$, $P(B)$ and $P(A \cap B)$ into the addition rule to find $P(A \cup B)$.

d
$$\begin{aligned} &P(\text{jack } \textit{or} \text{ not a diamond}) \\ &= P(B \cup \overline{A}) \\ &= P(B) + P(\overline{A}) - P(B \cap \overline{A}) \\ &= \frac{4}{52} + \frac{39}{52} - \frac{3}{52} \\ &= \frac{40}{52} \\ &= \frac{10}{13} \end{aligned}$$

Use the addition rule.
There are 4 jacks and 39 cards that are not diamonds.
There are 3 cards that are both jacks and not diamonds.

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Example 7 Using the addition rule

Two events, A and B, are such that $P(A) = 0.4$, $P(B) = 0.8$ and $P(A \cup B) = 0.85$.
Find:

a $P(A \cap B)$ **b** $P(\overline{A} \cap \overline{B})$

SOLUTION

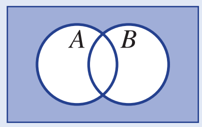
a
$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ 0.85 &= 0.4 + 0.8 - P(A \cap B) \\ 0.85 &= 1.2 - P(A \cap B) \\ \therefore P(A \cap B) &= 1.2 - 0.85 \\ &= 0.35 \end{aligned}$$

b
$$\begin{aligned} P(\overline{A} \cap \overline{B}) &= 1 - 0.85 \\ &= 0.15 \end{aligned}$$

EXPLANATION

Write the addition rule and substitute the given information.
Simplify and solve for $P(A \cap B)$.

$\overline{A} \cap \overline{B} = \overline{A \cup B}$



Exercise 3C EXTENSION

UNDERSTANDING AND FLUENCY	1–7	3, 4, 7, 8	5–8

1 A fair 6-sided die is rolled.
- Event A is tossing a number greater than 3.
- Event B is tossing an even number.

a List the sets.
i A **ii** B **iii** A or B (i.e. $A \cup B$) **iv** A and B (i.e. $A \cap B$)

b Are events A and B mutually exclusive? Give a reason.

c Find $P(A \cup B)$.

2 Substitute the given information into the addition rule to find $P(A \cup B)$.
a $P(A) = 0.7, P(B) = 0.5, P(A \cap B) = 0.4$ **b** $P(A) = 0.6, P(B) = 0.7, P(A \cap B) = 0.5$
c $P(A) = 0.65, P(B) = 0.4, P(A \cap B) = 0.35$ **d** $P(A) = 0.42, P(B) = 0.58, P(A \cap B) = 0$

3 Use the addition rule to find $P(A \cap B)$ if $P(A \cup B) = 0.9$, $P(A) = 0.5$ and $P(B) = 0.45$.

Example 6

4 A card is selected from a standard deck of 52 playing cards.
Let A be the event 'the card is a spade' and let B be the event 'the card is an ace'.

a Find:
i $n(A)$ **ii** $n(B)$ **iii** $n(A \cap B)$

b Find:
i $P(A)$ **ii** $P(\overline{A})$ **iii** $P(A \cap B)$

c Use the addition rule to find $P(A \cup B)$.

d Find the probability that the card is an ace and not a spade.

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5 A number is chosen from the set $\{1, 2, 3, \ldots, 20\}$. Let A be the event 'choosing a multiple of 3' and let B be the event 'choosing a prime number'.

a List set:

i A ii B

b Find:

i $P(A \cap B)$ ii $P(A \cup B)$

c Find the probability that the number is a prime and not a multiple of 3.

6 In a class of 24 students, 13 like Mathematics, 9 like English and 3 like both.

a Find the probability that a randomly selected student in this class likes both Mathematics and English.

b Find the probability that a randomly selected student in this class likes neither Mathematics nor English.

Example 7

7 Two events, A and B, are such that $P(A) = 0.3$, $P(B) = 0.6$ and $P(A \cup B) = 0.8$. Find:

a $P(A \cap B)$ b $P(\overline{A} \cap \overline{B})$

8 Two events, A and B, are such that $P(A) = 0.45$, $P(B) = 0.75$ and $P(A \cup B) = 0.9$.

a $P(A \cap B)$ b $P(\overline{A} \cap \overline{B})$

PROBLEM-SOLVING AND REASONING	9, 10, 12	9, 10, 12, 13	10, 11, 13, 14

9 Of 32 cars at a show, 18 cars have four-wheel drive, 21 are sports cars and 27 have four-wheel drive or are sports cars.

a Find the probability that a randomly selected car at the show is both four-wheel drive and a sports car.

b Find the probability that a randomly selected car at the show is neither four-wheel drive nor a sports car.

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10 A card is selected from a standard deck of 52 playing cards. Find the probability that the card is:

a a heart or a king
b a club or a queen
c a black card or an ace
d a red card or a jack
e a diamond or not a king
f a king or not a heart
g a 10 or not a spade
h a red card or not a 6

11 **a** Find $P(A \cap \overline{B})$ when $P(A \cup B) = 0.8$, $P(A) = 0.5$ and $P(B) = 0.4$.

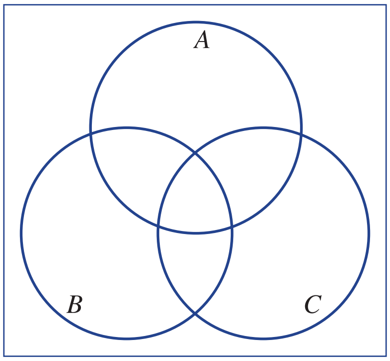
b Find $P(\overline{A} \cap B)$ when $P(A \cup B) = 0.76$, $P(A) = 0.31$ and $P(B) = 0.59$.

12 Why does the addition rule become $P(A \cup B) = P(A) + P(B)$ for mutually exclusive events?

13 Explain why the following represent impossible events.

a $P(A) = 0.3, P(B) = 0.5, P(A \cap B) = 0.4$
b $P(A \cup B) = 0.75, P(A) = 0.32, P(B) = 0.39$

14 Write down an addition rule for $P(A \cup B \cup C)$ using sets A, B and C.



ENRICHMENT	–	–	15, 16

Divisibility and the addition rule

15 A number is selected randomly from the first 20 positive integers. Find the probability that it is divisible by:

a 3
b 4
c 2 and 3
d 2 or 3
e 3 or 5
f 2 or 5

16 A number is selected randomly from the first 500 positive integers. Find the probability that it is divisible by:

a 4
b 7
c 3 and 5
d 2 and 7
e 3 and 8
f 3, 7 and 9

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3D Formal notation for conditional probability EXTENSION

Interactive
Widgets
HOTsheets
Walkthrough

The mathematics associated with the probability that an event occurs given that another event has already occurred is called conditional probability.

Consider, for example, a group of primary school students who have bicycles for a special cycling party. Some of the bicycles have gears, some have suspension and some have both gears and suspension. Consider these two questions:

- What is the probability that a randomly selected bicycle from the group has gears?
- What is the probability that a randomly selected bicycle from the group has gears given that it has suspension?

The second question is conditional in that we already know that the bicycle has suspension.

Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Gears and suspension

Suppose that in a group of 18 bicycles, 9 have gears, 11 have suspension and 5 have both gears and suspension. Discuss the solution to the following question by considering the points below.

What is the probability that a randomly selected bicycle from the group will have gears given that it has suspension?

- Illustrate the information on a Venn diagram.
- How many of the bicycles that have suspension have gears?
- Which areas in the Venn diagram are to be considered when answering the question? Give reasons.
- What would be the answer to the question in reverse; that is, what is the probability that a bicycle from the group will have suspension given that it has gears?

Key ideas

Note: The formal notation used in this section goes beyond the NSW syllabus but will prepare students for further study beyond Year 10.

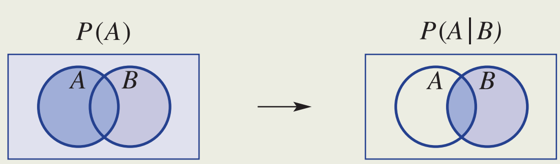
- The probability of event A occurring given that event B has occurred is denoted by $P(A|B)$, which reads 'the probability of A given B'. This is known as **conditional probability**.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \text{ and } P(B|A) = \frac{P(A \cap B)}{P(A)}$$

- For problems in this section, these rules can be simplified to:

$$P(A|B) = \frac{n(A \cap B)}{n(B)} \text{ and } P(B|A) = \frac{n(A \cap B)}{n(A)}$$

- $P(A|B)$ differs from $P(A)$ in that the sample space is reduced to the set B, as shown in these Venn diagrams.



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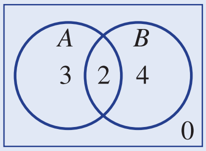
Example 8 Finding conditional probabilities using a Venn diagram

Consider this Venn diagram, displaying the number of elements belonging to the events A and B.



Find the following probabilities.

a $P(A)$
b $P(A \cap B)$
c $P(A|B)$
d $P(B|A)$

SOLUTION / EXPLANATION

a $P(A) = \frac{5}{9}$

There are 5 elements in set A and 9 in total.

b $P(A \cap B) = \frac{2}{9}$

There are 2 elements common to sets A and B.

c $P(A|B) = \frac{2}{6} = \frac{1}{3}$

2 of the 6 elements in set B are in set A.

d $P(B|A) = \frac{2}{5}$

2 of the 5 elements in set A are in set B.

Example 9 Finding conditional probabilities using a two-way table

From a group of 15 hockey players at a game of hockey, 13 played on the field, 7 sat on the bench and 5 both played and sat on the bench.

A hockey player is chosen at random from the team.

Let A be the event 'the person played on the field' and let B be the event 'the person sat on the bench'.

a Represent the information in a two-way table.
b Find the probability that the person only sat on the bench.
c Find the probability that the person sat on the bench given that they played on the field.
d Find the probability that the person played on the field given that they sat on the bench.

SOLUTION / EXPLANATION

a

	A	$\overline{A}$	
B	5	2	7
$\overline{B}$	8	0	8
	13	2	15

$n(A \cap B) - 5, n(A) - 13, n(B) - 7$. The total is 15. Insert these values and then fill in the other places to ensure the rows and columns give the required totals.

b $P(B \cap \overline{A}) = \frac{2}{15}$

Two people sat on the bench and did not play on the field.

c $P(B|A) = \frac{5}{13}$

$n(B \cap A) = 5$ and $n(A) = 13$.

d $P(A|B) = \frac{5}{7}$

$n(A \cap B) = 5$ and $n(B) = 7$.

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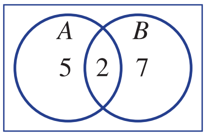
Exercise 3D EXTENSION

UNDERSTANDING AND FLUENCY | 1–3, 4–5(½) | 3, 4–5(½), 6 | 5(½), 6, 7

1 Use this Venn diagram to answer these questions.



a **i** Find $n(A \cap B)$. **ii** Find $n(B)$.

b Find $P(A|B)$ using $P(A|B) = \dfrac{n(A \cap B)}{n(B)}$.

2 Use this two-way table to answer these questions.

	A	$\overline{A}$	
B	7	5	12
$\overline{B}$	3	1	4
	10	6	16

a **i** Find $n(A \cap B)$. **ii** Find $n(A)$.

b Find $P(B|A)$ using $P(B|A) = \dfrac{n(A \cap B)}{n(A)}$.

c Find $P(A|B)$.

3 In a group of 20 people, 15 are wearing jackets and 10 are wearing hats; 5 are wearing both a jacket and a hat.

a What fraction of the people who are wearing jackets are wearing hats?

b What fraction of the people who are wearing hats are wearing jackets?

Example 8

4 The following Venn diagrams display information about the number of elements associated with the events A and B. For each Venn diagram, find:

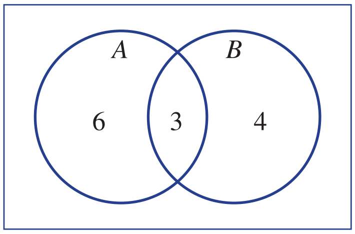
i $P(A)$

ii $P(A \cap B)$

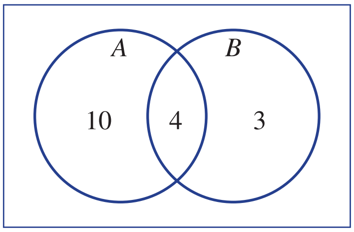
iii $P(A|B)$

iv $P(B|A)$

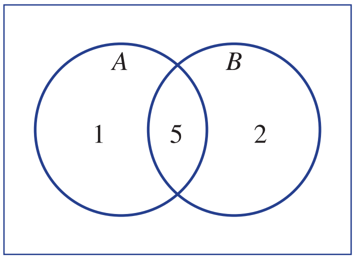
a



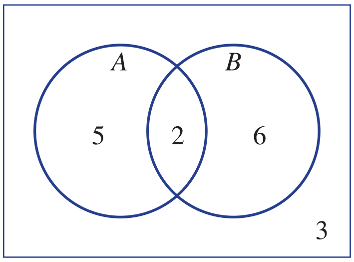
b



c



d



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5 The following two-way tables show information about the number of elements in the events A and B. For each two-way table find:

i $P(A)$ ii $P(A \cap B)$ iii $P(A|B)$ iv $P(B|A)$

a

	A	$\bar{A}$	
B	2	8	10
$\bar{B}$	5	3	8
	7	11	18

b

	A	$\bar{A}$	
B	1	4	5
$\bar{B}$	3	1	4
	4	5	9

c

	A	$\bar{A}$	
B	7	3	10
$\bar{B}$	1	6	7
	8	9	17

d

	A	$\bar{A}$	
B	4	2	6
$\bar{B}$	8	2	10
	12	4	16

Example 9

6 Of a group of 20 English cricket fans at a match, 13 purchased a pie, 15 drank beer and 9 both purchased a pie and drank beer.
Let A be the event 'the fan purchased a pie'.
Let B be the event 'the fan drank beer'.

a Represent the information in a two-way table.
b Find the probability that a fan in the group purchased only a pie (and did not drink beer).
c Find the probability that a fan in the group purchased a pie given that they drank beer.
d Find the probability that a fan in the group drank beer given that they purchased a pie.

7 Of 15 musicians surveyed to find out whether they play the violin or the piano, 5 play the violin, 8 play the piano and 2 play both instruments.

a Represent the information in a Venn diagram.
b How many of the musicians surveyed do not play either the violin or the piano?
c Find the probability that one of the 15 musicians surveyed plays piano given that they play the violin.
d Find the probability that one of the 15 musicians surveyed plays the violin given that they play the piano.

PROBLEM-SOLVING AND REASONING	8, 9, 12	9, 10, 12, 13	10, 11, 13, 14

8 On a car production line, 30 cars are due to be completed by the end of the day. Fifteen of the cars have cruise control and 20 have airbags, and 6 cars have both cruise control and airbags.

a Represent the information provided in a Venn diagram or two-way table.
b Find the probability that a car chosen at random will have:
i cruise control only ii airbags only
c Given that the car chosen has cruise control, find the probability that the car will have airbags.
d Given that the car chosen has airbags, find the probability that the car will have cruise control.

9 For each of the following, complete the given two-way tables and find:

i $n(\bar{A} \cap \bar{B})$ ii $P(B|A)$ iii $P(A|B)$

a

	A	$\bar{A}$	
B	2		4
$\bar{B}$			
	5		8

b

	A	$\bar{A}$	
B	3		16
$\bar{B}$			
	8		27

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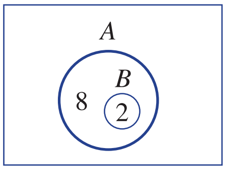
10 A card is drawn from a standard deck of 52 playing cards. Find the probability that:
- **a** the card is a king given that it is a heart
- **b** the card is a jack given that it is a red card
- **c** the card is a diamond given that it is a queen
- **d** the card is a black card given that it is an ace

11 A number is chosen from the first 24 positive integers. Find the probability that:
- **a** the number is divisible by 3 given that it is divisible by 4
- **b** the number is divisible by 6 given that it is divisible by 3

12 Two events, A and B, are mutually exclusive. What can be said about the probability of A given B (i.e. $P(A|B)$) or the probability of B given A (i.e. $P(B|A)$)? Give a reason.

13 Two events, A and B, are such that B is a subset of A, as shown in this Venn diagram.
- **a** Find $P(A \text{ given } B)$.
- **b** Find $P(B \text{ given } A)$.



14 **a** Rearrange the rule $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$ to make $P(A \cap B)$ the subject.
- **b** Hence, find $P(A \cap B)$ if $P(B|A) = 0.3$ and $P(A) = 0.6$.

ENRICHMENT	–	–	15

Investment workshops

15 People aged between 20 and 50 years attended workshops on shares, property or cash at an investment conference. The number of people attending each workshop is shown in this table.

Workshop	20–29 years	30–39 years	40–49 years
Shares	40	85	25
Property	18	57	6
Cash	5	32	61

- **a** How many people attended the conference?
- **b** Find the probability that a randomly selected person at the conference is aged between 30 and 39 years.
- **c** Find the probability that a randomly selected person at the conference attended the property workshop.
- **d** Find the probability that a randomly selected person at the conference attended the property workshop given they are not in the 30–39 age group.
- **e** Find the probability that a randomly selected person at the conference is aged between 40 and 49 years given that they did not attend the cash workshop.
- **f** Find the probability that a randomly selected person at the conference did not attend the shares workshop given they are not in the 30–39 age group.

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3E Using arrays for two-step experiments

Interactive
Widgets
HOTsheets
Walkthrough

A probability experiment might involve flipping a coin and then rolling a die. This could be called a 'two-step experiment' or a 'multi-step experiment'. In some multi-step experiments (such as choosing two chocolates from a box) the first chocolate may or may not be replaced back in the box before the second chocolate is chosen.

Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Does replacement matter?

From the digits $\{1, 2, 3, 4, 5\}$ you select two of these to form a two-digit number.

- How many numbers can be formed if selections are made with replacement?
- How many numbers can be formed if selections are made without replacement?

32 14 25

not allowed without replacement → 22 53

31 34 42

- Find the probability that the number 35 is formed if selections are made with replacement.
- Find the probability that the number 35 is formed if selections are made without replacement.

Key ideas

- A table called an **array** may be used to list the sample space of a two-step experiment.
- If **replacement** is allowed, then outcomes from each selection can be repeated.
- If selections are made **without replacement**, then outcomes from each selection cannot be repeated.

  For example: Two selections are made from the digits $\{1, 2, 3\}$.

With replacement

		1st 1	2	3
	1	(1, 1)	(2, 1)	(3, 1)
2nd	2	(1, 2)	(2, 2)	(3, 2)
	3	(1, 3)	(2, 3)	(3, 3)

Without replacement

		1st 1	2	3
	1	×	(2, 1)	(3, 1)
2nd	2	(1, 2)	×	(3, 2)
	3	(1, 3)	(2, 3)	×

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Example 10 Constructing a table with replacement

A fair 6-sided die is rolled twice.

a List the sample space, using a table.

b State the total number of outcomes.

c Find the probability of obtaining the outcome (1, 5).

d Find:

i $P(\text{double})$ **ii** $P(\text{sum of at least } 10)$ **iii** $P(\text{sum not equal to } 7)$

e Find the probability of a sum of 12, given that the sum is at least 10.

SOLUTION

a

Roll 2

		1	2	3	4	5	6
	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
Roll 1	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

EXPLANATION

Be sure to place the number from roll 1 in the first position for each outcome.

b 36 outcomes

There is a total of $6 \times 6 = 36$ outcomes.

c $P(1, 5) = \frac{1}{36}$

Only one outcome is (1, 5).

d **i** $P(\text{double}) = \frac{6}{36} = \frac{1}{6}$

Six outcomes have the same number repeated.

ii $P(\text{sum of at least } 10) = \frac{6}{36} = \frac{1}{6}$

Six outcomes have a sum of either 10, 11 or 12.

iii $P(\text{sum not equal to } 7) = 1 - \frac{6}{36} = \frac{5}{6}$

This is the complement of having a sum of 7. Six outcomes have a sum of 7.

e $P(\text{sum} = 12 \text{ given that sum} \geqslant 10) = \frac{1}{6}$

One of the 6 outcomes with a sum of at least 10 has a sum of 12.

Example 11 Constructing a table without replacement

Two letters are chosen from the word KICK without replacement.

a Construct a table to list the sample space.

b Find the probability of:

i obtaining the outcome (K, C)

ii selecting two Ks

iii selecting a K and a C

iv selecting two Ks given that at least one K is selected

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SOLUTION

a

		1st			
		K	I	C	K
2nd	K	×	(I, K)	(C, K)	(K, K)
	I	(K, I)	×	(C, I)	(K, I)
	C	(K, C)	(I, C)	×	(K, C)
	K	(K, K)	(I, K)	(C, K)	×

b **i** $P(\text{K, C}) = \frac{2}{12} = \frac{1}{6}$

ii $P(\text{K, K}) = \frac{2}{12} = \frac{1}{6}$

iii $P(K \text{ and } C) = \frac{4}{12} = \frac{1}{3}$

iv $P(\text{2 Ks, given at least 1 K}) = \frac{2}{10} = \frac{1}{5}$

EXPLANATION

Selection is without replacement, so the same letter (from the same position) cannot be chosen twice.

Two of the 12 outcomes are (K, C).

Two of the outcomes are K and K, which use different Ks from the word KICK.

Four outcomes contain a K and a C.

There are 10 outcomes with at least one K, two of which have two Ks.

Exercise 3E

UNDERSTANDING AND FLUENCY	1–5	2, 3, 5, 6	4–6

1 Two letters are chosen from the word DOG.

a Complete a table listing the sample space if selections are made:

i with replacement

		1st		
		D	O	G
2nd	D	(D, D)	(O, D)	
	O			
	G			

ii without replacement

		1st		
		D	O	G
2nd	D	×	(O, D)	
	O		×	
	G			×

b State the total number of outcomes when selection is made:

i with replacement **ii** without replacement

c If selection is made with replacement, find the probability that:

i the two letters are the same **ii** there is at least one D

iii there is not an O **iv** there is a D or a G

v there is a D and a G

d If selection is made without replacement, find the probability that:

i the two letters are the same **ii** there is at least one D

iii there is not an O **iv** there is a D or a G

v there is a D and a G

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2 Two digits are selected from the set $\{2, 3, 4\}$ to form a two-digit number. Find the number of two-digit numbers that can be formed if the digits are selected:

a with replacement **b** without replacement

Example 10a–d

3 A fair 4-sided die is rolled twice.

a List the sample space, using a table.

b State the total number of possible outcomes.

c Find the probability of obtaining the outcome $(2, 4)$.

d Find the probability of:

i a double **ii** a sum of at least 5 **iii** a sum not equal to 4

4 Two fair coins are tossed, each landing with a head (H) or tail (T).

a List the sample space, using a table.

b State the total number of possible outcomes.

c Find the probability of obtaining the outcome (H, T).

d Find the probability of obtaining:

i one tail **ii** at least one tail

e If the two coins are tossed 1000 times, how many times would you expect to get two tails?

Example 11

5 Two letters are chosen from the word SET without replacement.

a Show the sample space, using a table.

b Find the probability of:

i obtaining the outcome (E, T) **ii** selecting one T

iii selecting at least one T **iv** selecting an S and a T

v selecting an S or a T

6 A letter is chosen from the word LEVEL without replacement and then a second letter is chosen from the same word.

a Draw a table displaying the sample space for the pair of letters chosen.

b State the total number of outcomes possible.

c State the number of outcomes that contain exactly one of the following letters.

i V **ii** L **iii** E

d Find the probability that the outcome will contain exactly one of the following letters.

i V **ii** L **iii** E

e Find the probability that the two letters chosen will be the same.

PROBLEM-SOLVING AND REASONING	7, 8, 10	7, 8, 10, 11	8, 9, 11, 12

7 In a quiz, Min guesses that the probability of rolling a sum of 10 or more from two fair 6-sided dice is 10%. Complete the following to decide whether or not this guess is correct.

Die 2

Die 1	1	2	3	4	5	6
1	2	3	…			
2	3	…				
3	4					
4	⋮					
5	⋮					
6						

a Copy and complete the table representing all the outcomes for possible totals that can be obtained.

b State the total number of outcomes.

c Find the number of the outcomes that represent a sum of:

i 3 **ii** 7 **iii** less than 7

d Find the probability that the following sums are obtained.

i 7 **ii** less than 5 **iii** greater than 2 **iv** at least 11

e Find the probability that the sum is at least 10, and decide whether or not Min's guess is correct.

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8 A letter is chosen randomly from the word OLD and then a second letter is chosen from the word COLLEGE.

a Draw a table illustrating all possible pairs of letters that can be chosen.

b State the total number of outcomes.

c If a double represents selecting the same letter, find the probability of selecting a double.

9 The 10 students who completed a special flying course are waiting to see if they will be awarded the one Distinction or the one Merit award for their efforts.

a In how many ways can the two awards be given if:

i the same student can receive both awards

ii the same student cannot receive both awards

b Assuming that a student cannot receive both awards, find the probability that a particular student receives:

i the Distinction award

ii the Merit award

iii neither award

c Assuming that a student can receive both awards, find the probability that they receive at least one award.

Example 10e

10 Two fair 4-sided dice are rolled and the sum is noted.

a Find the probability of:

i a sum of 5

ii a sum of less than 6

b **i** Find the probability of a sum of 5 given that the sum is less than 6.

ii Find the probability of a sum of 2 given that the sum is less than 6.

iii Find the probability of a sum of 7 given that the sum is at least 7.

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11 Decide whether the following situations would naturally involve selections with replacement or without replacement.

- **a** selecting two people to play in a team
- **b** tossing a coin twice
- **c** rolling two dice
- **d** choosing two chocolates to eat

12 In a game of chance, six cards numbered 1 to 6 are lying face down on a table. Two cards are selected without replacement and the sum of both numbers is noted.

- **a** State the total number of outcomes.
- **b** Find the probability that the total sum is:
  - **i** equal to 3
  - **ii** equal to 4
  - **iii** at least 10
  - **iv** no more than 5
- **c** What would have been the answer to part **b i** if the experiment had been conducted with replacement?

ENRICHMENT	–	–	13

Random weights

13 In a gym, Justin considers choosing two weights to fit onto a rowing machine to make the load heavier. There are four different weights to choose from: 2.5 kg, 5 kg, 10 kg and 20 kg, and there are plenty of each weight available. After getting a friend to randomly choose both weights, Justin attempts to operate the machine.

- **a** Complete a table that displays all possible total weights that could be placed on the machine.
- **b** State the total number of outcomes.
- **c** How many of the outcomes deliver a total weight described by the following?
  - **i** equal to 10 kg
  - **ii** less than 20 kg
  - **iii** at least 20 kg
- **d** Find the probability that Justin will be attempting to lift the following weight.
  - **i** 20 kg
  - **ii** 30 kg
  - **iii** no more than 10 kg
  - **iv** less than 10 kg
- **e** If Justin is unable to lift more than 22 kg, what is the probability that he will not be able to operate the rowing machine?

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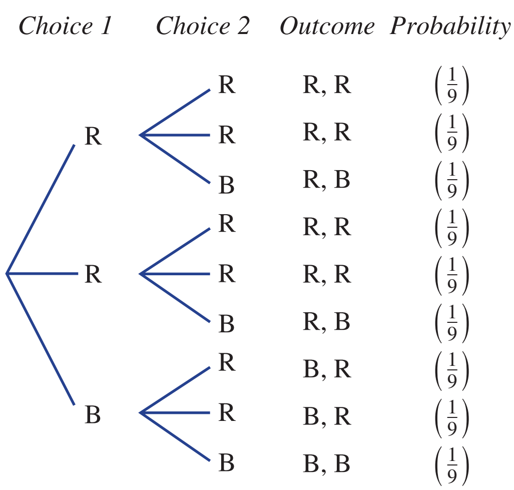
3F Using tree diagrams

Interactive
Widgets
HOTsheets
Walkthrough

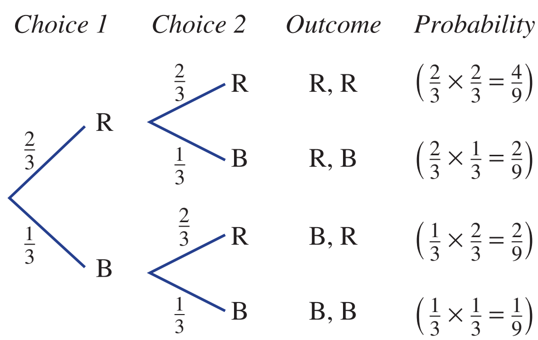
Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Suppose a bag contains two red counters and one blue counter and that two counters are selected at random with replacement. One way to display the outcomes is with a tree diagram, in which all equally likely outcomes are listed in columns, as shown below left. A more efficient way, however, is to group similar outcomes and write their corresponding probabilities on the branches, as shown below right.



$P(\mathrm{R, B}) = \frac{2}{9}$



$P(\mathrm{R, B}) = \frac{2}{9}$

You will note that in the tree diagram on the right the probability of each outcome is obtained by multiplying the branch probabilities. The reason for this relates to conditional probabilities.

Using conditional probabilities, the tree diagram on the right can be redrawn like this (below right).

We know from conditional probability that:

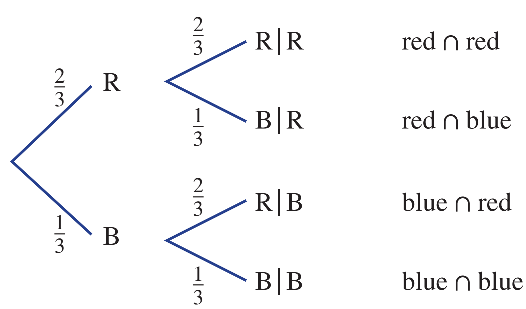
- $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

Using blue and red we could write:

- $P(\text{red} \mid \text{blue}) = \dfrac{P(\text{blue} \cap \text{red})}{P(\text{blue})}$

By rearranging we have:

$$\begin{aligned} P(\text{blue} \cap \text{red}) &= P(\text{blue}) \times P(\text{red} \mid \text{blue}) \\ &= \frac{1}{3} \times \frac{2}{3} \\ &= \frac{2}{9} \end{aligned}$$



This explains why we multiply branches on tree diagrams. This also applies when selection is made without replacement.

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Let's start: Prize probability

Two lucky door prizes are awarded randomly to a group of 7 male and 3 female partygoers.

- Use a tree diagram with branch probabilities to show how selection with replacement can be displayed.
- Use a tree diagram with branch probabilities to show how selection without replacement can be displayed.
- Which of these situations has a higher probability?
  - **a** A male and a female receive one prize each if selection is made with replacement.
  - **b** A male and a female receive one prize each if selection is made without replacement.

Key ideas

- **Tree diagrams** can be used to list the sample space for experiments involving two or more steps.
  - Branch probabilities are used to describe the chance of each outcome at each step.
  - Each outcome for the experiment is obtained by multiplying the branch probabilities.
  - Branch probabilities will depend on whether selection is made with or without replacement.

Example 12 Constructing a tree diagram for a two-step experiment

Boxes A and B contain 4 counters each. Box A contains 2 red and 2 green counters and box B contains 1 red and 3 green counters. A box is chosen at random and then a single counter is selected.

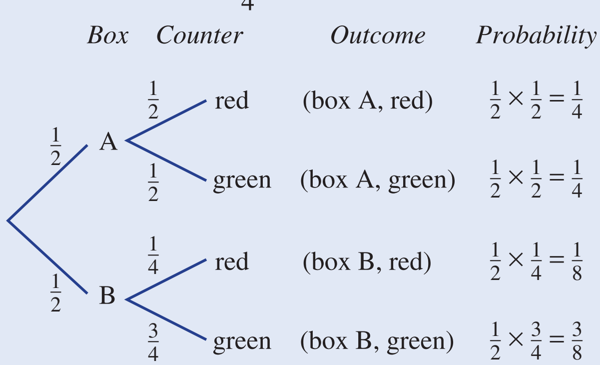
a If box A is chosen, what is the probability that a red counter is chosen from it?
b If box B is chosen, what is the probability that a red counter is chosen from it?
c Represent the options available as a tree diagram that shows all possible outcomes and related probabilities.
d What is the probability of selecting box B and a red counter?
e What is the probability of selecting a red counter?

SOLUTION

a $P(\text{red from box A}) = \frac{2}{4} = \frac{1}{2}$

b $P(\text{red from box B}) = \frac{1}{4}$

c



EXPLANATION

Two of the 4 counters in box A are red.

One of the 4 counters in box B is red.

First selection is a box followed by a counter.
Multiply each of the probabilities along the branch pathways to find the probability of each outcome.

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d $P(\text{box B, red}) = \frac{1}{2} \times \frac{1}{4}$
$= \frac{1}{8}$

The probability of choosing box B is $\frac{1}{2}$ and a red counter from box B is $\frac{1}{4}$, so multiply the probability for these two outcomes together.

e $P(1 \text{ red}) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4}$
$= \frac{1}{4} + \frac{1}{8}$
$= \frac{3}{8}$

The outcomes (A, red) and (B, red) both contain 1 red counter, so add the probabilities for these two outcomes together.

Example 13 Using a tree diagram without replacement

A bag contains 5 blue (B) and 3 white (W) marbles and two marbles are selected without replacement.

a Draw a tree diagram showing all outcomes and probabilities.

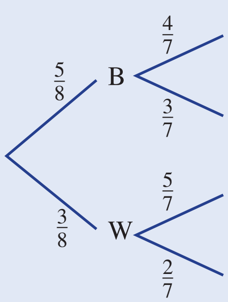
b Find the probability of selecting:

- **i** a blue marble followed by a white marble (B, W)
- **ii** 2 blue marbles
- **iii** exactly 1 blue marble

c If the experiment is repeated with replacement, find the answers to each question in part **b**.

SOLUTION

a *Selection 1* *Selection 2*



Outcome	*Probability*
(B, B)	$\frac{5}{8} \times \frac{4}{7} = \frac{5}{14}$
(B, W)	$\frac{5}{8} \times \frac{3}{7} = \frac{15}{56}$
(W, B)	$\frac{3}{8} \times \frac{5}{7} = \frac{15}{56}$
(W, W)	$\frac{3}{8} \times \frac{2}{7} = \frac{3}{28}$

b **i** $P(\text{B, W}) = \frac{5}{8} \times \frac{3}{7}$
$= \frac{15}{56}$

ii $P(\text{B, B}) = \frac{5}{8} \times \frac{4}{7}$
$= \frac{5}{14}$

iii $P(1 \text{ blue}) = \frac{5}{8} \times \frac{3}{7} + \frac{3}{8} \times \frac{5}{7}$
$= \frac{15}{28}$

EXPLANATION

After one blue marble is selected there are 7 marbles remaining: 4 blue and 3 white.
After one white marble is selected there are 7 marbles remaining: 5 blue and 2 white.

Multiply the probabilities on the (B, W) pathway.

Only 4 blue marbles remain after the first selection. Multiply the probabilities on the (B, B) pathway.

The outcomes (B, W) and (W, B) both have one blue marble. Multiply the probabilities to find individual probabilities, then sum for the final result.

Example continued on next page

c **i** $P(\text{B, W}) = \frac{5}{8} \times \frac{3}{8}$

$= \frac{15}{64}$

When selecting objects with replacement, remember that the number of marbles in the bag remains the same for each selection.

ii $P(\text{B, B}) = \frac{5}{8} \times \frac{5}{8}$

$= \frac{25}{64}$

That is, $P(\text{B}) = \frac{5}{8}$ and $P(\text{W})\ \frac{3}{8}$ throughout.

iii $P(1 \text{ blue}) = \frac{5}{8} \times \frac{3}{8} + \frac{3}{8} \times \frac{5}{8}$

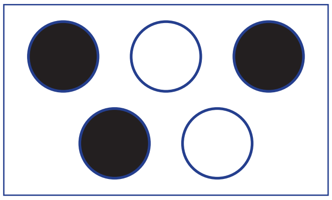
$= \frac{15}{32}$

One blue marble corresponds to the (B, W) or (W, B) outcomes.

Exercise 3F

UNDERSTANDING AND FLUENCY	1–5	2–6	4–7

1 A box contains 2 white (W) and 3 black (B) counters.



a A single counter is drawn at random. Find the probability that it is:

i white

ii black

b Two counters are now drawn at random. The first one is replaced before the second one is drawn. Find the probability that the second counter is:

i white

ii black

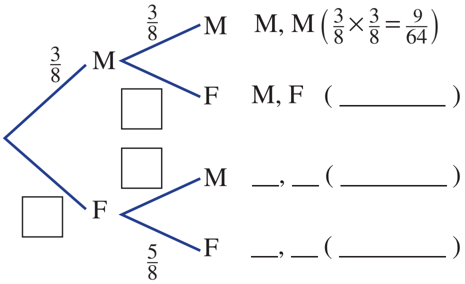
c Two counters are drawn and the first counter is not replaced before the second one is drawn. If the first counter is white, find the probability that the second counter is:

i white

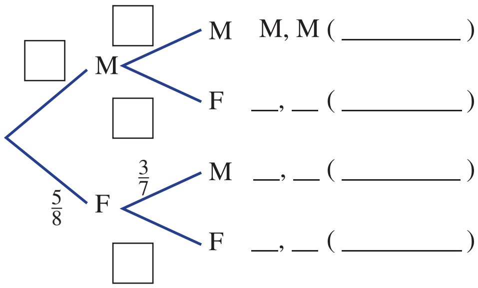
ii black

2 Two prizes are awarded to a group of 3 male (M) and 5 female (F) candidates. Copy and complete each tree diagram. Include the missing branch probabilities and outcome probabilities.

a with replacement



b without replacement



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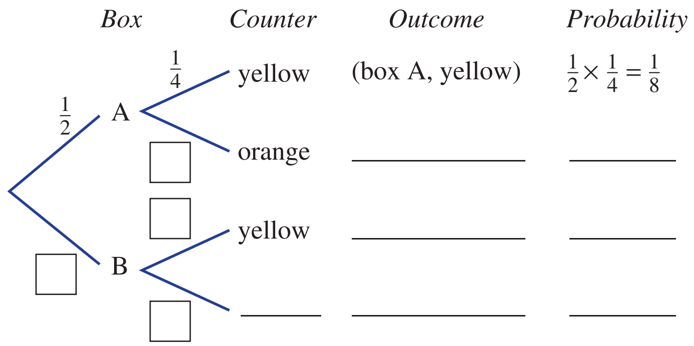
Example 12

3 Boxes A and B contain 4 counters each. Box A contains 1 yellow and 3 orange counters, whereas box B contains 3 yellow and 1 orange counter. A box is chosen at random and then a single counter is selected.

a What is the probability of selecting a yellow counter from box A?

b What is the probability of selecting a yellow counter from box B?

c Represent the options available by completing this tree diagram.



d What is the probability of selecting box B and a yellow counter?

e What is the probability of selecting 1 yellow counter?

4 A fair 4-sided die is rolled twice and the pair of numbers is recorded.

a Use a tree diagram to list the sample space.

b State the total number of outcomes.

c Find the probability of obtaining:

i a 4 then a 1; i.e. the outcome (4, 1)

ii a double

d Find the probability of obtaining a sum described by the following:

i equal to 2

ii equal to 5

iii less than or equal to 5

Example 13

5 A bag contains 4 red (R) and 2 white (W) marbles, and two marbles are selected without replacement.

a Draw a tree diagram showing all outcomes and probabilities.

b Find the probability of selecting:

i a red marble and then a white marble (R, W)

ii 2 red marbles

iii exactly 1 red marble

c The experiment is repeated with replacement. Find the answers to each question in part **b**.

6 Two students are selected from a group of 3 males (M) and 4 females (F) without replacement.

a Draw a tree diagram to find the probability of selecting:

i 2 males

ii 2 females

iii 1 male and 1 female

iv 2 people, either both male or both female

b The experiment is repeated with replacement. Find the answers to each question in part **a**.

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7 There are 6 boys and 4 girls in a team. Three people are randomly chosen without replacement.

a Draw a tree diagram showing all outcomes and probabilities.

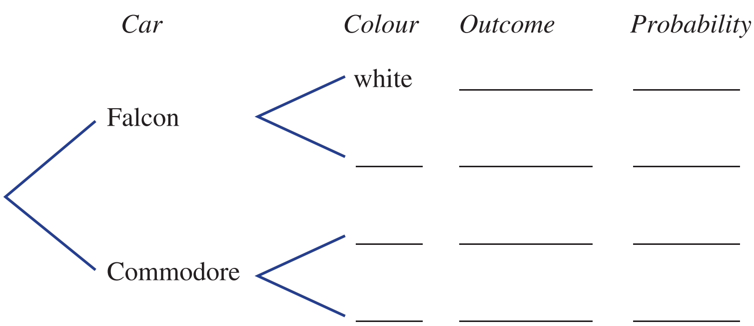
b **i** What is the probability that exactly 2 boys will be chosen?

ii What is the probability that at least 1 girl will be chosen?

PROBLEM-SOLVING AND REASONING	8, 9, 12	9, 10, 12, 13	10, 11, 13, 14

8 As part of a salary package, a person can select either a Falcon or a Commodore. There are 3 white Falcons and 1 silver Falcon and 2 white Commodores and 1 red Commodore to choose from.

a Complete a tree diagram showing a random selection of a car type then a colour.



b Find the probability that the person chooses:

i a white Falcon

ii a red Commodore

iii a white car

iv a car that is not white

v a silver car or a white car

vi a car that is neither a Falcon nor red

9 Two bottles of wine are randomly selected for tasting from a box containing 2 red and 2 white wines. Use a tree diagram to help answer the following.

a If the first bottle is replaced before the second is selected, find:

i P(2 red)

ii P(1 red)

iii P(not 2 white)

iv P(at least 1 white)

b If the first bottle is not replaced before the second is selected, find:

i P(2 red)

ii P(1 red)

iii P(not 2 white)

iv P(at least 1 white)

10 Cans of sliced apple produced by 'Just Apple' are sometimes underweight. A box of 10 cans is selected randomly from the factory and then 2 cans from the 10 are tested without replacement. This particular box of 10 cans is known to have 2 cans that are underweight.

a State the probability that the first can chosen will be:

i underweight

ii not underweight

b Use a tree diagram to find the probability that:

i both cans are underweight

ii one can is underweight

iii at most one can is underweight

c The factory passes the inspection if no cans are found to be underweight. Find the chance that this will occur and express your answer as a percentage, rounded to 1 decimal place.

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11 The probability of rain on any particular day is 0.2. However, the probability of rain on a day after a rainy day is 0.85, whereas the probability of rain on a day after a non-rainy day is 0.1.

a On two consecutive days, find the probability of having:
- **i** two rainy days
- **ii** exactly one rainy day
- **iii** at least one dry day

b On three consecutive days, find the probability of having:
- **i** three rainy days
- **ii** exactly one dry day
- **iii** at most two rainy days

12 Two socks are selected at random from a drawer containing 4 red and 4 yellow socks.

a Find the probability that the two socks will be of the same colour if the socks are drawn without replacement.

b Find the probability that the two socks will not be of the same colour if the socks are drawn without replacement.

13 A box contains 2 red (R) and 3 blue (B) counters and three counters are selected without replacement.

a Use a tree diagram to find:
- **i** P(R, R, B)
- **ii** P(2 red)
- **iii** P(3 red)
- **iv** P(at least 1 red)
- **v** P(at most 2 blue)

b If a fourth selection is made without replacement, find the probability that:
- **i** at least 1 red is selected
- **ii** 3 blue are selected

14 Containers A, B and C hold 4 marbles each, all of which are the same size. The table illustrates the marble colours in each container.

	Container *A*	Container *B*	Container *C*
Purple	1	2	3
Green	3	2	1

A container is chosen at random and then a marble is selected from the container.

a Draw a tree diagram to help determine all the possible outcomes and the associated probabilities. *Suggestion*: You will need three branches to start (representing the three different containers that can be chosen), followed by two branches for each of A, B and C (to represent the choice of either a purple or a green marble).

b State the total number of outcomes.

c Find the following probabilities.
- **i** P(container A, purple)
- **ii** P(container B, green)
- **iii** P(container C, purple)

d Find the probability of obtaining a green marble.

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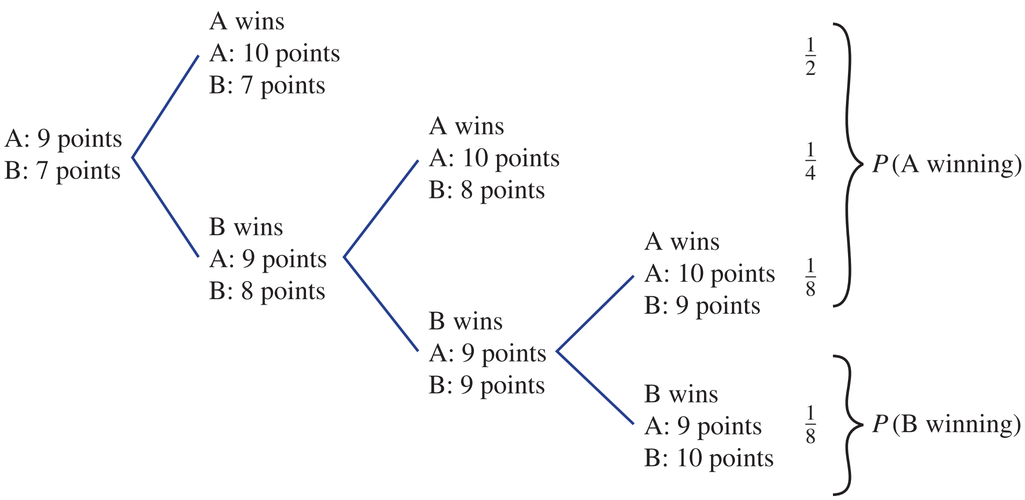
ENRICHMENT – – 15

Fermat and Pascal

15 The French mathematicians Pierre de Fermat and Blaise Pascal inspired the development of mathematical probability through their consideration of simple games. Here's one of their first problems.

Two equally skilled people play a game in which the first to earn 10 points wins \$100 and each player has an equal chance of winning a point. At some point in the game, however, one of the players has to leave and the game must be stopped. If the game score is 9 points to 7, how should the \$100 be divided between the two players?

This diagram shows the number of ways the game could have been completed.



a Use this diagram to help calculate the probability that:
- **i** player A wins the game
- **ii** player B wins the game

b Based on your answers from part **a**, describe how the \$100 should be divided between players A and B.

c Investigate how the \$100 should be divided between players A and B if the game is stopped with the following number of points. You will need to draw a new tree diagram each time.
- **i** player A: 8 points, player B: 7 points
- **ii** player A: 7 points, player B: 7 points
- **iii** player A: 8 points, player B: 6 points
- **iv** player A: 6 points, player B: 7 points
- **v** Choose your own pair of game points and investigate.

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3G Dependent events and independent events

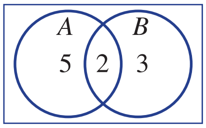
EXTENSION

Interactive
Widgets
HOTsheets
Walkthrough

Stage

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

In previous sections we have looked at problems involving conditional probability. This Venn diagram, for example, gives the following results.



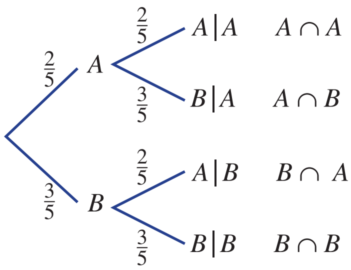
$$P(A) = \frac{7}{10} \text{ and } P(A|B) = \frac{2}{5}$$

Clearly, the condition B in $P(A|B)$ has changed the probability of A. The events A and B are therefore not independent.

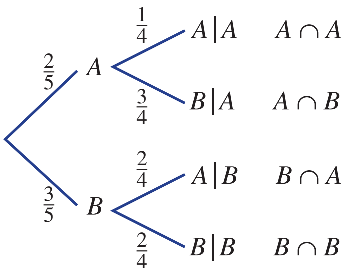
For multiple events we can consider events either with or without replacement.

These tree diagrams, for example, show two selections of marbles from a bag of 2 aqua (A) and 3 blue (B) marbles.

With replacement



Without replacement



In the first tree diagram we can see that $P(A|B) = P(A)$, so the events are independent. In the second tree diagram, we can see that $P(A|B) \neq P(A)$, so the events are not independent.

So, for independent events we have:

$$P(A|B) = P(A)(*)$$

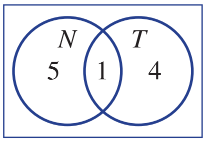
This implies that $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ becomes $P(A) = \dfrac{P(A \cap B)}{P(B)}$.

Using (*) and rearranging gives $P(A \cap B) = P(A) \times P(B)$.

Let's start: Are they independent?

Recall that two events are independent when the outcome of one event does not affect the probability of the other event. Discuss whether or not you think the following pairs of events are independent. Give reasons.

- Tossing two coins with the events:
  - getting a tail on the first coin
  - getting a tail on the second coin
- Selecting two mugs without replacement from a cupboard in which there are 3 red and 2 blue mugs, and obtaining the events:
  - first is a blue mug
  - second is a red mug
- Selecting a person from a group of 10 who enjoy playing netball (N) and/or tennis (T), as in the Venn diagram shown.
  - selecting a person from the group who enjoys netball
  - selecting a person from the group who enjoys tennis



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Key ideas

Note: The formal notation used in this section goes beyond the NSW syllabus but will prepare students for further study beyond Year 10.

- Two events are **independent** if the outcome of one event does not change the probability of obtaining the other event.
  - $P(A|B) = P(A)$ or $P(B|A) = P(B)$
  - $P(A \cap B) = P(A) \times P(B)$
- For multi-step experiments with selection made **with replacement**, successive events are independent.
- For multi-step experiments with selection made **without replacement**, successive events are not independent.

Example 14 Using Venn diagrams

A selection of 10 mobile phone offers includes four with free connection and five with a free second battery, whereas one offer has both free connection and a free second battery.
Let A be the event 'choosing a mobile phone with free connection'.
Let B be the event 'choosing a mobile phone with a free second battery'.

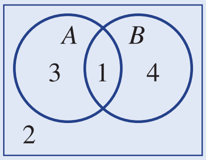
a Summarise the information about the 10 mobile phone offers, in a Venn diagram.

b Find:

  i $P(A)$

  ii $P(A|B)$

c State whether or not the events A and B are independent.

SOLUTION	EXPLANATION
a 	Start with the 1 element that belongs to both sets A and B and complete according to the given information.
b i $P(A) = \frac{4}{10} = \frac{2}{5}$	4 of the 10 elements belong to set A.
ii $P(A\|B) = \frac{1}{5}$	1 of the 5 elements in set B belongs to set A.
c The events A and B are not independent.	$P(A\|B) \neq P(A)$

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Exercise 3G EXTENSION

UNDERSTANDING AND FLUENCY	1–7	2–5, 6–7(½)	5, 6–7(½)

1 A coin is tossed twice. Let A be the event 'the first toss gives a tail' and let B be the event 'the second toss gives a tail'.

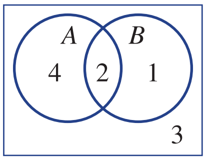
a Find:

i $P(A)$ **ii** $P(B)$

b Would you say that events A and B are independent?

c What is $P(B|A)$?

2 This Venn diagram shows the number of elements in events A and B.



a Find:

i $P(B)$ **ii** $P(B|A)$

b Is $P(B|A) = P(B)$?

c Are the events A and B independent?

3 Complete each sentence.

a For multi-step experiments, events are independent when selections are made ____________ replacement.

b For multi-step experiments, events are not independent when selections are made __________ replacement.

Example 14

4 A selection of 8 offers for computer printers includes 3 with a free printer cartridge and 4 with a free box of paper, whereas 2 have both a free printer cartridge and a free box of paper.
Let A be the event 'choosing a printer with a free printer cartridge'.
Let B be the event 'choosing a printer with a free box of paper'.

a Summarise the given information about the 8 computer printer offers, in a Venn diagram.

b Find:

i $P(A)$ **ii** $P(A|B)$

c State whether or not the events A and B are independent.

5 A selection of 6 different baby strollers includes 3 with a free rain cover, 4 with a free sun shade, and 2 offer both a free rain cover and a free sun shade.
Let A be the event 'choosing a stroller with a free sun shade'.
Let B be the event 'choosing a stroller with a free rain cover'.

a Summarise the given information about the six baby strollers, in a Venn diagram.

b Find:

i $P(A)$ **ii** $P(A|B)$

c State whether or not the events A and B are independent.

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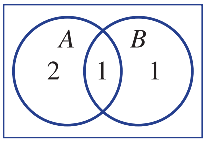
6 From events A and B in the given Venn diagrams:

- **i** Find $P(A)$ and $P(A|B)$.
- **ii** Hence, decide whether or not events A and B are independent.

a



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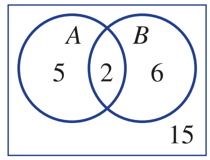

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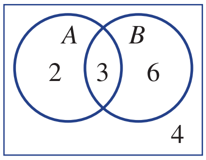

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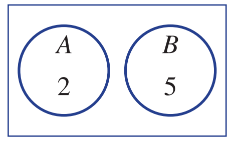

7 For the events A and B, with details provided in the given two-way tables, find $P(A)$ and $P(A|B)$ and decide whether or not the events A and B are independent.

a

	A	$\overline{A}$	
B	1	1	2
$\overline{B}$	3	3	6
	4	4	8

b

	A	$\overline{A}$	
B	1	3	4
$\overline{B}$	2	4	6
	3	7	10

c

	A	$\overline{A}$	
B	3	17	20
$\overline{B}$	12	4	16
	15	21	36

d

	A	$\overline{A}$	
B	1		9
$\overline{B}$			
	5		45

PROBLEM-SOLVING AND REASONING	8, 9, 11	8, 9, 11, 12a	9–12

8 Of 17 leading accountants, 15 offer advice on tax (T), whereas 10 offer advice on business growth (G). Eight of the accountants offer advice on both tax and business growth. One of the 17 accountants is chosen at random.

a Use a Venn diagram or two-way table to help find:

- **i** $P(T)$
- **ii** $P(T \text{ only})$
- **iii** $P(T|G)$

b Are the events T and G independent?

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9 A coin is tossed 5 times. Find the probability of obtaining:

a 5 heads

b at least 1 tail

c at least 1 head

10 A fair 6-sided die is rolled three times. Find the probability that the sum of the three throws is:

a 18

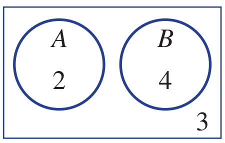
b 3

c 4

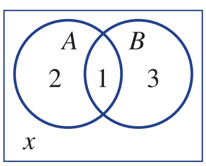
d 5

11 Use this diagram to help decide if this statement is true or false.

If two events A and B are mutually exclusive, then they are also independent.



12 Consider the events A and B, with the number of elements contained in each event given in the Venn diagrams below. In each case, find the value of x so that the events A and B are independent; i.e. $P(A) = P(A|B)$.

a


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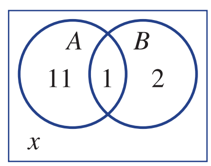

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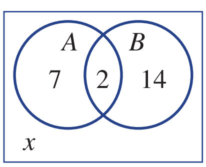

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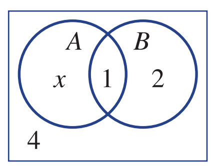

ENRICHMENT	–	–	13, 14

Using $P(A \cap B) = P(A) \neq P(B)$ and the addition rule

13 For two independent events A and B, recall that $P(A \cap B) = P(A) \times P(B)$. Two independent events A and B are such that $P(A) = 0.6$ and $P(B) = 0.4$. Find:

a $P(A \cap B)$

b $P(A \cup B)$

14 For two independent events A and B, we are given $P(A \cup B) = 0.9$ and $P(A) = 0.4$. Find $P(B)$.

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Investigation

1 From London to Paris on the Eurostar

On a special work assignment, Helena is to be paid £100 per hour for every hour she spends in Paris after travelling on the Eurostar from London.

Helena is waiting for a train at St Pancras Station in London and is placed on standby; she is not guaranteed a definite seat. If there is no seat on a given train, then she waits to see if there is a seat on the next train.

This information is provided for Helena at the station.

Train	Probability of a place	Cost
7 a.m.	$\frac{1}{2}$	£320
8 a.m.	$\frac{2}{3}$	£200
9 a.m.	$\frac{3}{4}$	£150
No further trains	–	£0

Standby

a Illustrate the information given, using a tree diagram and showing all Helena's options and the probabilities for the three trains. (Note: All branch paths might not be the same length.)

b Find the probability that Helena will catch the following trains.
i 7 a.m. **ii** 8 a.m. **iii** 9 a.m.

c What is the probability that Helena will miss all the available trains?

Maximising income

a In terms of pure financial gain, which train is the most desirable for Helena to catch? Remember that she is paid £100 for each extra hour she is in Paris.

b How much money would Helena need to earn per hour on the work assignment if the following trains were the most financially desirable to catch?
i 7 a.m. **ii** 9 a.m.

Expected cost

a Tabulate the cost of travel of each outcome and its corresponding probability using your results from part **b** in the standby section above.

Cost	£320	£200	£150	£0
Probability				

b By finding the sum of the product of each cost and its corresponding probability, find Helena's expected (average) cost for train travel from London to Paris.

c If Helena repeats this journey on 20 occasions, what would be her expected total cost?

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2 Sampling and reporting

We often see media reports in the newspaper or on television that are supported by statistics that have been collected in a variety of ways. Sometimes the ways in which these statistics are collected and interpreted can lead to bias or a misleading point of view.

Public transport success

A newspaper reports that 1000 people were surveyed to find out their preferred mode of transport. They were surveyed in the city near a Sydney train station. Of those surveyed, 855 said that they preferred public transport; that is, the chance that a person will use public transport in Sydney is 0.855 or 85.5%.

a Do you think this media story contains any bias? Why?

b How could the method of sampling be improved to better represent the people of Sydney?

Economic downturn

A television news reporter surveys four companies and finds that the profits of three of these companies has reduced over the past year. They report that this means the country is facing an economic downturn and that only one in four companies is making a profit.

a What are some of the problems in this media report?

b How could the news reporter improve their sampling methods?

c Is it correct to say that one in four companies is making a profit? Explain.

Annoying neighbours

An article on the internet reports that, on the basis of emails from 500 people, it can be said that 80% of people really dislike their neighbours and so the probability that you will move to a house with annoying neighbours is 80% or $\frac{4}{5}$.

a How would you describe the problems with the sampling method of this media report?

b Describe a sampling method that would give a better idea about the number of annoying neighbours in a town or city.

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Puzzles and challenges

1 A women's tennis match is won by the first player to win two sets. Andrea has a 0.4 chance of winning in each set against Betty.

Find the following probabilities.

a Andrea wins in two sets.

b Andrea wins in three sets.

c Betty wins after losing the first set.

2 Find $P(A)$ if $P(A \cup B) = 0.74$ and $P(B) = 0.36$, assuming that A and B are independent events.

3 A fair coin is tossed three times. Find the probability that:

a at least one head is obtained

b at least one head is obtained given that the first toss is a head

c at least two heads are obtained given that there is at least one head

4 Two digits are chosen without replacement from the set $\{1, 2, 3, 4\}$ to form a two-digit number. Find the probability that the two-digit number is:

a 32

b even

c less than 40

d at least 22

5 A fair coin is tossed six times. What is the probability that at least one tail is obtained?

6 What is the chance of choosing the correct six numbers in a 49-ball lottery game?

7 Two leadership positions are to be filled from a group of two girls and three boys. What is the probability that the positions will be filled by one girl and one boy?

8 The letters of the word DOOR are jumbled randomly. What is the probability that the final arrangement will spell DOOR?

9 True or false? In a group of 23 people, the probability that at least two people have the same birthday is more than 0.5.

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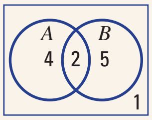
Chapter summary

Probability

Review

- Sample space is the list of all possible outcomes
- $P(\text{event}) = \dfrac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$

Venn diagram

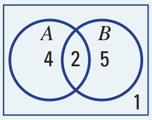


Two-way table

	A	$\bar{A}$	
B	2	5	7
$\bar{B}$	4	1	5
	6	6	12

Conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \text{ or } \frac{n(A \cap B)}{n(B)}$$



	A	$\bar{A}$	
B	2	5	7
$\bar{B}$	4	1	5
	6	6	12

$P(A \mid B) = \frac{2}{7}$

Independent events

- $P(A \mid B) = P(A)$
- $P(A \cap B) = P(A) \times P(B)$

Tables

With replacement

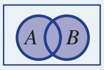
	A	B	C
A	(A, A)	(B, A)	(C, A)
B	(A, B)	(B, B)	(C, B)
C	(A, C)	(B, C)	(C, C)

Without replacement

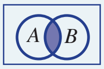
	A	B	C
A	×	(B, A)	(C, A)
B	(A, B)	×	(C, B)
C	(A, C)	(B, C)	×

Unions and intersections

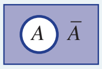
- Union $A \cup B$ (A or B)



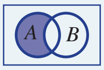
- Intersection $A \cap B$ (A and B)



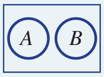
- Complement of A is $\bar{A}$ (not A)



- A only is $A \cap \bar{B}$



- Mutually exclusive events $A \cap B = \emptyset$



Addition rule

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

If A and B are mutually exclusive:
$P(A \cap B) = 0$ and
$P(A \cup B) = P(A) + P(B)$

Tree diagrams

3 white
4 black

With replacement

Choice 1	*Choice 2*	*Outcome*	*Probability*
W ($\frac{3}{7}$)	W ($\frac{3}{7}$)	(W, W)	$\frac{9}{49}$
	B ($\frac{4}{7}$)	(W, B)	$\frac{12}{49}$
B ($\frac{4}{7}$)	W ($\frac{3}{7}$)	(B, W)	$\frac{12}{49}$
	B ($\frac{4}{7}$)	(B, B)	$\frac{16}{49}$

Without replacement

Choice 1	*Choice 2*	*Outcome*	*Probability*
W ($\frac{3}{7}$)	W ($\frac{2}{6}$)	(W, W)	$\frac{1}{7}$
	B ($\frac{4}{6}$)	(W, B)	$\frac{2}{7}$
B ($\frac{4}{7}$)	W ($\frac{3}{6}$)	(B, W)	$\frac{2}{7}$
	B ($\frac{3}{6}$)	(B, B)	$\frac{2}{7}$

$P(1 \text{ white, } 1 \text{ black}) = \frac{2}{7} + \frac{2}{7}$
$= \frac{4}{7}$

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Multiple-choice questions

1 A letter is chosen from the word SUCCESS. The probability that the letter is neither C nor S is:

A $\frac{2}{7}$ **B** $\frac{3}{5}$ **C** $\frac{5}{7}$ **D** $\frac{4}{7}$ **E** $\frac{3}{7}$

2 The number of manufacturing errors detected in a car manufacturing plant on 20 randomly selected days is given by this table.

Number of errors	0	1	2	3	Total
Frequency	11	6	2	1	20

An estimate of the probability that on the next day at least one error will be observed is:

A $\frac{3}{10}$ **B** $\frac{9}{20}$ **C** $\frac{11}{20}$ **D** $\frac{17}{20}$ **E** $\frac{3}{20}$

3 A letter is chosen from each of the words CAN and TOO. The probability that the pair of letters will not have an O is:

A $\frac{2}{3}$ **B** $\frac{1}{2}$ **C** $\frac{1}{3}$ **D** $\frac{1}{9}$ **E** $\frac{5}{9}$

4 The number of times a coin must be tossed to give 16 possible outcomes is:

A 8 **B** 2 **C** 16 **D** 3 **E** 4

5 A box has 3 red and 2 blue counters. If a red counter is selected and not replaced, then the probability that a blue counter will be observed on the second selection is:

A $\frac{1}{2}$ **B** $\frac{2}{5}$ **C** $\frac{2}{3}$ **D** $\frac{1}{4}$ **E** $\frac{3}{4}$

6 The sets A and B are known to be mutually exclusive. Which of the following is therefore true?

A $n(A) = n(B)$ **B** $n(A \cap B) = 0$ **C** $A = \varnothing$ **D** $P(A \cap B) = 1$ **E** $P(A \cup B) = 0$

7 From the list of the first 10 positive integers, $A = \{1, 3, 5, 7, 9\}$ and B is the set of primes less than 10. Therefore, $P(\overline{A})$ and $P(A \text{ only})$ are, respectively:

A $\frac{1}{3}, \frac{1}{5}$ **B** $\frac{1}{2}, \frac{1}{2}$ **C** $\frac{1}{2}, \frac{3}{10}$ **D** $\frac{1}{2}, \frac{1}{5}$ **E** $\frac{1}{3}, \frac{2}{5}$

8 For this two-way table, $P(A \cap \overline{B})$ is:

A $\frac{2}{3}$ **B** $\frac{1}{4}$ **C** $\frac{1}{7}$

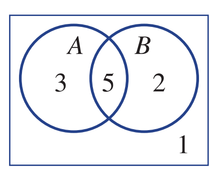
D $\frac{1}{3}$ **E** $\frac{3}{4}$

	A	$\overline{A}$	
B	2		3
$\overline{B}$			4
		4	

9 For this Venn diagram, $P(A|B)$ is:

A $\frac{5}{7}$ **B** $\frac{5}{2}$ **C** $\frac{5}{8}$

D $\frac{5}{3}$ **E** $\frac{3}{11}$



10 Two events are independent when:

A $P(A) = P(B)$ **B** $P(\overline{A}) = \varnothing$ **C** $P(A \cup B) = 0$

D $P(A|B) = P(B)$ **E** $P(A) = P(A|B)$

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Short-answer questions

1 A letter is chosen from the word INTEREST. Find the probability that the letter will be:

a I **b** E **c** a vowel
d not a vowel **e** E or T

2 A letter is chosen from the word POSITIVE. Find the probability that the letter also belongs to these words.

a NEGATIVE **b** ADDITION **c** DIVISION

3 An engineer inspects 20 houses in a street for cracks. The results are summarised in this table.

Number of cracks	0	1	2	3	4
Frequency	8	5	4	2	1

a From these results, estimate the probability that the next house inspected in the street will have the following number of cracks:
i zero **ii** one **iii** two **iv** three **v** four
b Estimate the probability that the next house will have:
i at least one crack
ii no more than two cracks

4 Of 36 people, 18 have an interest in cars, 11 have an interest in homewares and 6 have an interest in both cars and homewares.

a Represent this information using a Venn diagram.
b Represent this information using a two-way table.
c State the number of people who do not have an interest in either cars or homewares.
d If a person is chosen at random from the group, find the probability that the person will:
i have an interest in cars and homewares
ii have an interest in homewares only
iii not have any interest in cars

5 All 26 birds in an aviary have clipped wings and/or a tag. In total, 18 birds have tags and 14 have clipped wings.

a Find the number of birds that have both a tag and clipped wings.
b Find the probability that a bird chosen at random will have a tag only.

6 A card is selected from a standard deck of 52 playing cards. Let A be the event 'the card is a heart' and let B be the event 'the card is a king'.

a Find:
i $n(A)$ **ii** $n(B)$ **iii** $n(A \cap B)$
b Find:
i $P(\overline{A})$ **ii** $P(A \cap B)$
c Use the addition rule to find $P(A \cup B)$.
d Find the probability that the card is a king or not a diamond.

7 Two events, A and B, are such that $P(A) = 0.25$, $P(B) = 0.35$ and $P(A \cup B) = 0.5$. Find:

a $P(A \cap B)$ **b** $P(\overline{A} \cap \overline{B})$

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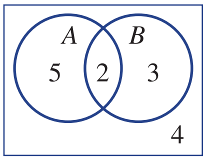
8 For these probability diagrams, find $P(A|B)$.

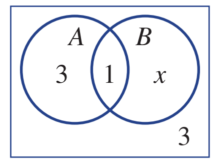
a



b

	A	$\overline{A}$	
B	1		
$\overline{B}$	2	2	
			9

9 Two events, A and B, are represented in the given Venn diagram. Also $n(B \text{ only}) = x$, where x is a positive integer.



a If $x = 4$, find:

i $P(A)$ ii $P(B)$ iii $P(A|B)$

b If $x = 4$, decide whether the events A and B are independent.

c If $x = 1$, find:

i $P(A)$ ii $P(B)$ iii $P(A|B)$

d If $x = 1$, decide if the events A and B are independent.

10 A letter is chosen at random from the word HAPPY and a second letter is chosen from the word HEY.

a List the sample space, using a table.

b State the total number of outcomes.

c Find the probability that the two letters chosen will be:

i H then E

ii the same

iii not the same

11 A fair 4-sided die is rolled twice and the total is noted.

a Use a tree diagram to list the sample space, including all possible totals.

b Find these probabilities.

i $P(2)$

ii $P(5)$

iii $P(1)$

iv $P(\text{not } 1)$

12 Two people are selected from a group of two females and three males, without replacement. Use a tree diagram to find the probability of selecting:

a a female on the first selection

b a male on the second selection after choosing a female on the first selection

c two males

d one male

e at least one female

13 Two independent events, A and B, are such that $P(A) = 0.4$ and $P(B) = 0.3$. Find:

a $P(A \cap B)$

b $P(A \cup B)$

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Extended-response questions

1 Of 15 people surveyed to find out whether they run or swim for exercise, 6 said they run, 4 said they swim and 8 said they neither run nor swim.

- **a** How many people surveyed run and swim?
- **b** One of the 15 people is selected at random. Find the probability that they:
  - **i** run or swim
  - **ii** only swim
- **c** Represent the information in a two-way table.
- **d** Find the probability that:
  - **i** a person swims given that they run
  - **ii** a person runs given that they swim

2 A bakery sells three types of bread: raisin (R) at \$2 each, sourdough (S) at \$3 each, and white (W) at \$1.50 each. Lillian is in a hurry. She randomly selects two loaves and takes them quickly to the counter.

- **a** Draw a table showing the possible combination of loaves that Lillian could have selected.
- **b** Find the probability that Lillian selects:
  - **i** two raisin loaves
  - **ii** two loaves that are the same
  - **iii** at least one white loaf
  - **iv** not a sourdough loaf

Lillian has only \$4 in her purse.

- **c** How many different combinations of bread will Lillian be able to afford?
- **d** Find the probability that Lillian will not be able to afford her two chosen loaves.

On the next day, there are only two raisin, two sourdough and three white loaves available. Lillian chooses two loaves without replacement from the limited number of loaves.

- **e** Use a tree diagram, showing branch probabilities, to find:
  - **i** P(2 raisin loaves)
  - **ii** P(1 sourdough loaf)
  - **iii** P(not more than 1 white loaf)
  - **iv** P(2 loaves that are not the same)

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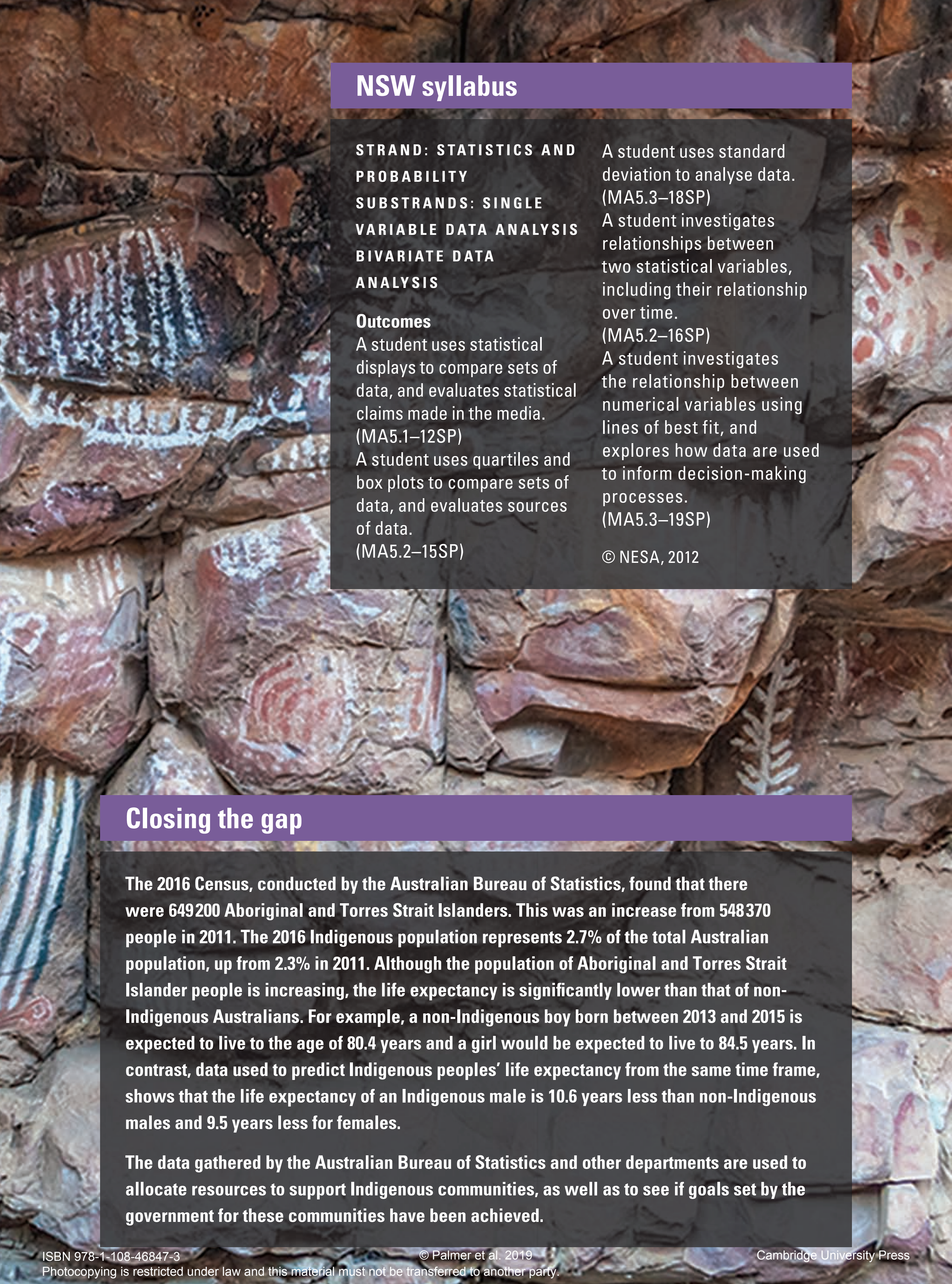
Online resources

- Auto-marked chapter pre-test
- Video demonstrations of all worked examples
- Interactive widgets
- Interactive walkthroughs
- Downloadable HOTsheets
- Access to all HOTmaths Australian Curriculum courses
- Access to the HOTmaths games library

4 Single variable and bivariate statistics

What you will learn



NSW syllabus

STRAND: STATISTICS AND PROBABILITY

SUBSTRANDS: SINGLE VARIABLE DATA ANALYSIS

BIVARIATE DATA ANALYSIS

Outcomes

A student uses statistical displays to compare sets of data, and evaluates statistical claims made in the media.
(MA5.1–12SP)

A student uses quartiles and box plots to compare sets of data, and evaluates sources of data.
(MA5.2–15SP)

A student uses standard deviation to analyse data.
(MA5.3–18SP)

A student investigates relationships between two statistical variables, including their relationship over time.
(MA5.2–16SP)

A student investigates the relationship between numerical variables using lines of best fit, and explores how data are used to inform decision-making processes.
(MA5.3–19SP)

Closing the gap

The 2016 Census, conducted by the Australian Bureau of Statistics, found that there were 649 200 Aboriginal and Torres Strait Islanders. This was an increase from 548 370 people in 2011. The 2016 Indigenous population represents 2.7% of the total Australian population, up from 2.3% in 2011. Although the population of Aboriginal and Torres Strait Islander people is increasing, the life expectancy is significantly lower than that of non-Indigenous Australians. For example, a non-Indigenous boy born between 2013 and 2015 is expected to live to the age of 80.4 years and a girl would be expected to live to 84.5 years. In contrast, data used to predict Indigenous peoples' life expectancy from the same time frame, shows that the life expectancy of an Indigenous male is 10.6 years less than non-Indigenous males and 9.5 years less for females.

The data gathered by the Australian Bureau of Statistics and other departments are used to allocate resources to support Indigenous communities, as well as to see if goals set by the government for these communities have been achieved.

Pre-test

1 For each of these datasets, find:
   i the mean (rounded to 1 decimal place)
   ii the median
   iii the mode

a 2, 2, 3, 4, 6, 6, 7, 9, 9, 11
b 47, 36, 41, 58, 41
c 0.2, 0.3, 0.3, 0.6, 0.7, 0.9, 1.2, 1.3

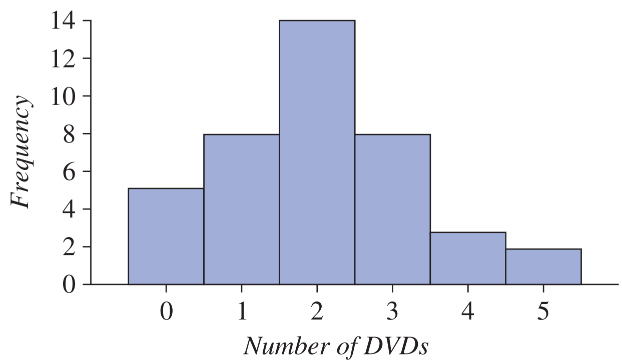
2 This table shows the frequency of scores in a test.

Score	Frequency
0–19	2
20–39	3
40–59	6
60–79	12
80–100	7

a How many scores are in the 40–59 range?
b How many scores are:
   i at least 60?
   ii less than 80?
c How many scores are there in total?
d What percentage of scores are in the 20–39 range?

3 Customers leaving a store were surveyed about the number of items they had purchased in the store.

a How many customers bought exactly three items?
b How many customers were surveyed?
c How many items were purchased by these customers?
d What percentage of customers purchased fewer than two items?



4 This stem-and-leaf plot shows the weight, in grams, of some small calculators.

a How many calculators are represented in the plot?
b Find the median weight of the calculators.
c What is the mode?
d Find the range.

Stem	Leaf
9	8
1 0	2 6
1 1	1 1 4 9
1 2	3 6
1 3	8 9 9
1 4	0 2 5

13|6 means 136 grams

5 Find the lower quartile (Q_1), the upper quartile (Q_3) and the interquartile range (IQR) for this dataset:
3, 4, 4, 5, 8, 9, 9, 12, 12, 13

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4A Collecting, using and misusing statistical data

Interactive
Widgets
HOTsheets
Walkthrough

There are many reports on television and radio that begin with the words 'A recent study has found that …'. These are usually the result of a survey or investigation that a researcher has conducted to collect information about an important issue, such as unemployment, crime or health.

Sometimes the results of these surveys are used to persuade people to change their behaviour. Sometimes they are used to pressure the government into changing the laws or the way the government spends public money.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Improving survey questions

Here is a short survey. It is not very well constructed.

Question 1: How old are you?
Question 2: How much time did you spend sitting in front of the television or a computer yesterday?
Question 3: Some people say that teenagers like you are lazy and spend way too much time sitting around when you should be outside exercising. What do you think of that comment?

Have a class discussion about the following.

- What will the answers to Question 1 look like? How could they be displayed?
- What will the answers to Question 2 look like? How could they be displayed?
- Is Question 2 going to give a realistic picture of your normal daily activity?
- Do you think Question 2 could be improved somehow?
- What will the answers to Question 3 look like? How could they be displayed?
- Do you think Question 3 could be improved somehow?

Key ideas

- **Surveys** are used to collect statistical data.
- Survey questions need to be constructed carefully so that the person knows exactly what sort of answer to give. They should use simple language and should not be ambiguous.
- Survey questions should not be worded so that they deliberately try to provoke a certain kind of response.
- Surveys should respect the privacy of the people being surveyed.
- If the question contains an option to be chosen from a list, the number of options should be an odd number, so that there is a 'neutral' choice. For example, the options could be:

strongly agree	agree	unsure	disagree	strongly disagree

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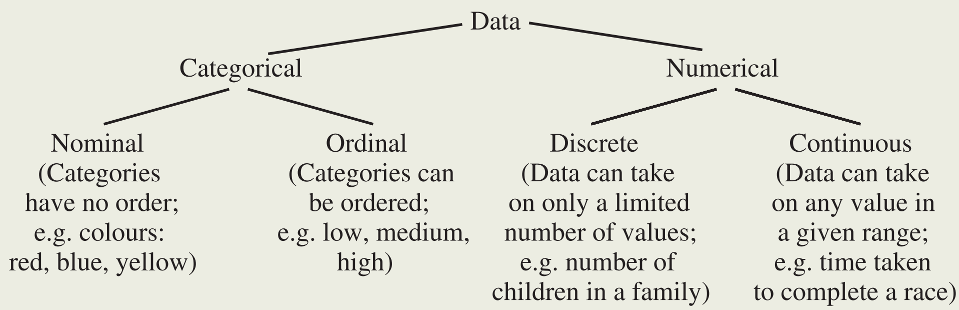
Key ideas

- A **population** is a group of people, animals or objects with something in common. Some examples of populations are:
  - all the people in Australia on Census Night
  - all the students in your school
  - all the boys in your Maths class
  - all the tigers in the wild in Sumatra
  - all the cars in Sydney
  - all the wheat farms in NSW
- A **sample** is a group that has been chosen from a population. Sometimes information from a sample is used to describe the whole population, so it is important to choose the sample carefully.
- **Statistical data** can be divided into subgroups.



Example 1 Describing types of data

What type of data would the following survey questions generate?

a How many televisions do you have in your home?

b To what type of music do you most like to listen?

SOLUTION	EXPLANATION
a numerical and discrete	The answer to the question is a number with a limited number of values; in this case, a whole number.
b categorical and nominal	The answer is a type of music and these categories have no order.

Example 2 Choosing a survey sample

A survey is carried out on the internet to determine Australia's favourite musical performer. Why will this sample not necessarily be representative of Australia's views?

SOLUTION	EXPLANATION
An internet survey is restricted to people with a computer and internet access, ruling out some sections of the community from participating in the survey.	The sample may not include some of the older members of the community or those in areas without access to the internet. Also, the survey would need to be set up so that people can do it only once so that 'fake' survey submissions are not counted.

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Exercise 4A

UNDERSTANDING AND FLUENCY	1–7	3–8	5–8

Example 2

1 A popular Australian 'current affairs' television show recently investigated the issue of spelling. They suspected that people in their twenties are not as good at spelling as people in their fifties, so they decided to conduct a statistical investigation.
They chose a sample of 12 people aged 50–59 years and 12 people aged 20–29 years.
Answer the following questions on paper, then discuss in a small group or as a whole class.

- **a** Do you think that the number of people surveyed is adequate?
- **b** How many people do you think there are in Australia aged 20–29 years?
- **c** How many people do you think there are in Australia aged 50–59 years?
- **d** Use the website of the Australian Bureau of Statistics to look up the answers to parts **b** and **c**.
- **e** Do you think it is fair and reasonable to compare the spelling ability of these two groups of people?
- **f** How would you go about comparing the spelling ability of these two groups of people?
- **g** Would you give the two groups the same set of words to spell?
- **h** How could you give the younger people an unfair advantage?
- **i** What sorts of words would you include in a spelling test for the survey?
- **j** How and where would you choose the people to do the spelling test?

2 Match each word (**a**–**h**) with its definition (**A**–**H**).

a population	**A** A group chosen from a population
b census	**B** A tool used to collect statistical data
c sample	**C** The state of being secret
d survey	**D** An element or feature that can vary
e data	**E** All the people or objects in question
f variable	**F** Statistics collected on a national scale
g statistics	**G** The practise of collecting and analysing data
h confidentiality	**H** The factual information collected from a survey or other source

3 Match each word (**a**–**f**) with its definition (**A**–**F**).

a numerical	**A** Categorical data that has no order
b continuous	**B** Data that are numbers
c discrete	**C** Numerical data that take on a limited number of values
d categorical	**D** Data that can be divided into categories
e ordinal	**E** Numerical data that take any value in a given range
f nominal	**F** Categorical data that can be ordered

Example 1

4 Which one of the following survey questions would generate numerical data?

- **A** What is your favourite colour?
- **B** What type of car does your family own?
- **C** How long does it take for you to travel to school?
- **D** What type of dog do you own?

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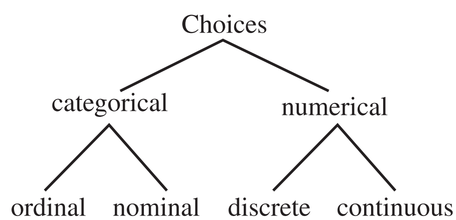
5 Which one of the following survey questions would generate categorical data?
  A How many times do you eat at your favourite fast-food place in a typical week?
  B How much do you usually spend buying your favourite fast food?
  C How many items did you buy last time you went to your favourite fast-food place?
  D Which is your favourite fast food?

6 Year 10 students were asked the following questions in a survey. Describe what type of data each question generates.
  a How many people under the age of 18 years are there in your immediate family?
  b How many letters are there in your first name?
  c Which company is the carrier of your mobile telephone calls? Optus/Telstra/Vodafone/3/Virgin/Other (Please specify.)
  d What is your height?
  e How would you describe your level of application in Maths? (Choose from very high, high, medium or low.)



7 Every student in Years 7 to 12 votes in the prefect elections. The election process is an example of:
  A a population
  B continuous data
  C a representative sample
  D a census

8 TV ‘ratings’ are used to determine the shows that are the most popular. Every week some households are chosen at random and a device is attached to their television. The device keeps track of the shows the households are watching during the week. The company that chooses the households should always attempt to find:
  A a census
  B continuous data
  C a representative sample
  D ungrouped data

PROBLEM-SOLVING AND REASONING	9, 10, 12	9–13	11–14

9 The principal decides to survey Year 10 students to determine their opinion of Mathematics. In order to increase the chance of choosing a representative sample, the principal should:
  A Give a survey form to the first 30 Year 10 students who arrive at school.
  B Give a survey form to all the students studying the most advanced maths subject.
  C Give a survey form to five students in every Maths class.
  D Give a survey form to 20% of the students in every class.

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10 Explain your choice of answer in Question 9. What is wrong with the other three options?

11 Choose a topic in which you are especially interested, such as football, cricket, movies, music, cooking, food, computer games or social media.
Make up a survey about your topic that you could give to the people in your class. It must have *four* questions.
Question 1 must produce data that are categorical and ordinal.
Question 2 must produce data that are categorical and nominal.
Question 3 must produce data that are numerical and discrete.
Question 4 must produce data that are numerical and continuous.

12 Discuss some of the problems with the selection of a survey sample for each given topic.
   a A survey at the train station of how Australians get to work.
   b An email survey on people's use of computers.
   c Phoning people on the electoral roll to determine Australia's favourite sport.

13 Search the internet for 'misleading graphs' and 'how to lie with statistics'. Describe three ways in which people use statistics to mislead or persuade others.

14 Search the internet to find three amusing/interesting quotes, jokes and cartoons about statistics and statisticians, such as this one: 'Statistics: The only science that enables different experts using the same figures to draw different conclusions.'

ENRICHMENT	–	–	15, 16

The 2016 Australian Census

15 Research the 2016 Australian Census on the website of the Australian Bureau of Statistics. Find out something interesting from the results of the 2016 Australian Census.
Write a short news report or record a 3-minute news report on your computer.

16 It is often said that Australia has an ageing population. What does this mean?
Search the internet for evidence showing that the 'average' Australian is getting older every year. By how much?

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4B Review of data displays REVISION

Interactive
Widgets
HOTsheets
Walkthrough

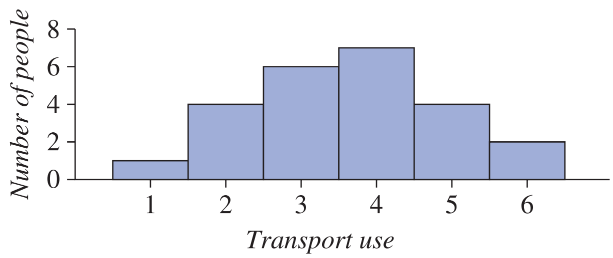
Statistical graphs are an essential element in the analysis and representation of data. Graphs can help to show the most frequent category, the range of values, the shape of the distribution and the centre of the dataset. By looking at statistical graphs the reader can quickly draw conclusions about the numbers or categories in the dataset and interpret this within the context of the data.

Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Public transport analysis

A survey is carried out to find out how many times people in the group have used public transport in the past week. The results are shown in this histogram.

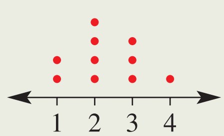


Discuss what the histogram tells you about this group of people and their use of public transport. You may wish to include these points.

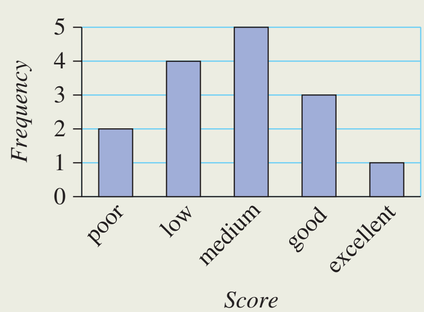
- How many people were surveyed?
- Is the data symmetrical or skewed?
- Is it possible to work out the exact mean? Why or why not?
- Do you think these people were selected from a group in your own community? Give reasons.

Key ideas

- The different types of statistical data that we saw in the previous section; i.e. categorical (nominal or ordinal) and numerical (discrete or continuous), can be displayed using different types of graphs to represent the different data.
- Graphs for a single set of categorical or discrete data
  - **Dot plot**



  - **Column graph**



  - **Stem-and-leaf plot**

0	1 3
1	2 5 9
2	1 4 6 7
3	0 4

2|4 means 24

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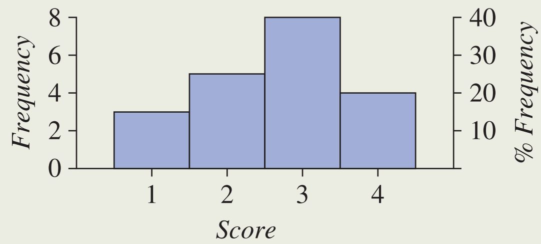
Key ideas

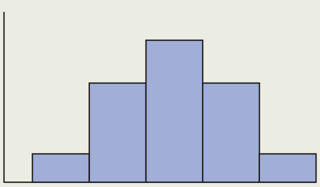
- **Histograms** can be used for grouped discrete or continuous numerical data.

Score	Frequency	Percentage frequency
1	3	15
2	5	25
3	8	40
4	4	20



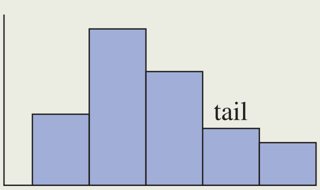
- The two most common used 'measures of centre' are:
  - **mean** ($\bar{x}$) $\bar{x} = \dfrac{\text{sum of data values}}{\text{number of data values}}$
  - **median** the middle value when data are placed in order
- The **mode** of a dataset is the data value that occurs most frequently.
- Data can be **symmetrical** or **skewed**.

Symmetrical



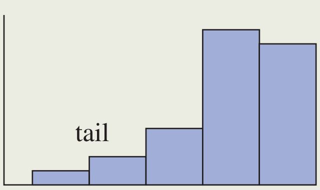
The mean and median will be equal.

Positively skewed



The median will be less than the mean.

Negatively skewed



The median will be greater than the mean.

Example 3 Presenting and analysing data

Twenty people are surveyed to find out how many times they use the internet in a week. The raw data are listed.

21, 19, 5, 10, 15, 18, 31, 40, 32, 25
11, 28, 31, 29, 16, 2, 13, 33, 14, 24

a Organise the data into a frequency table using class intervals of 10. Include a percentage frequency column.

b Construct a histogram for the data, showing both the frequency and percentage frequency on the one graph.

c Construct a stem-and-leaf plot for the data.

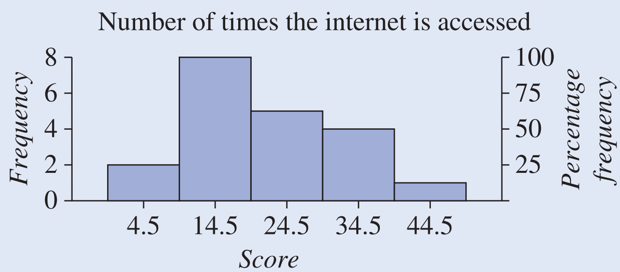
d Use your stem-and-leaf plot to find the median.

SOLUTION

a

Class interval	Frequency	Percentage frequency
0–9	2	10
10–19	8	40
20–29	5	25
30–39	4	20
40–49	1	5
Total	**20**	**100**

b



c

Stem	Leaf
0	2 5
1	0 1 3 4 5 6 8 9
2	1 4 5 8 9
3	1 1 2 3
4	0

3|1 means 31

d Median $= \dfrac{19 + 21}{2} = 20$

EXPLANATION

Calculate each percentage frequency by dividing the frequency by the total (i.e. 20) and multiplying by 100.

Transfer the data from the frequency table to the histogram. Axis scales are evenly spaced and the histogram bar is placed across the boundaries of the class interval. There is no space between the bars.

Order the data in each leaf and also show a key (e.g. 3|1 means 31).

After counting the scores from the lowest value (i.e. 2), the two middle values are 19 and 21. So the median is the mean of these two numbers.

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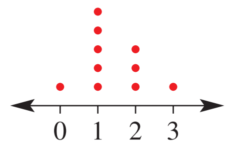
Exercise 4B REVISION

UNDERSTANDING AND FLUENCY	1–6	3–7	4–7

1 A number of families were surveyed to find the number of children in each. The results are shown in this dot plot.



a How many families were surveyed?
b Find the mean number of children in the families surveyed.
c Find the median number of children in the families surveyed.
d Find the mode for the number of children in the families surveyed.
e What percentage of the families have at most two children?

2 Classify each set of data as either categorical or numerical.
a 4.7, 3.8, 1.6, 9.2, 4.8
b red, blue, yellow, green, blue, red
c low, medium, high, low, low, medium
d 3 g, 7 g, 8 g, 7 g, 4 g, 1 g, 10 g

3 Complete these frequency tables.

a

Class interval	Frequency	Percentage frequency
0–9	2	
10–19	1	
20–29	5	
30–39	2	
Total		

b

Class interval	Frequency	Percentage frequency
80–84	8	
85–89	23	
90–94	13	
95–99		
Total	**50**	

Example 3

4 The number of wins scored this season is given for 20 hockey teams. Here are the raw data.

4, 8, 5, 12, 15, 9, 9, 7, 3, 7
10, 11, 1, 9, 13, 0, 6, 4, 12, 5

a Organise the data into a frequency table using class intervals of 5 and include a percentage frequency column.
b Construct a histogram for the data, showing both the frequency and percentage frequency on the one graph.
c Construct a stem-and-leaf plot for the data.
d Use your stem-and-leaf plot to find the median.

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5 This frequency table displays the way in which 40 people travel to and from work.

Type of transport	Frequency	Percentage frequency
Car	16	
Train	6	
Tram	8	
Walking	5	
Bicycle	2	
Bus	3	
Total	**40**	

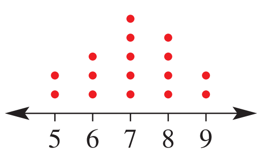
a Copy and complete the table.

b Use the table to find:

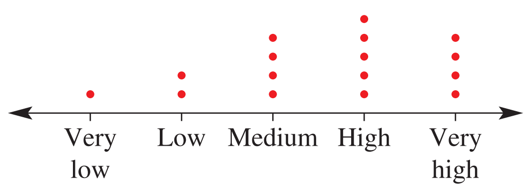
- **i** the frequency of people who travel by train
- **ii** the most popular form of transport
- **iii** the percentage of people who travel by car
- **iv** the percentage of people who walk or cycle to work
- **v** the percentage of people who travel by public transport, including trains, buses and trams

6 Describe each graph as either symmetrical, positively skewed or negatively skewed.

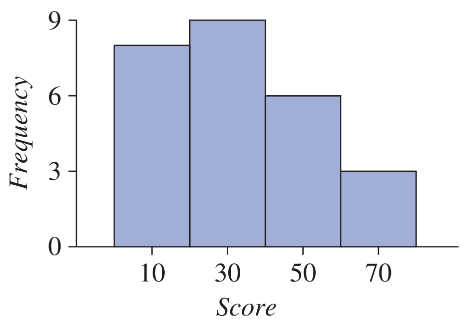
a



b



c



d

Stem	Leaf
4	1 6
5	0 5 4
6	1 8 9 9 9
7	2 7 8
8	3 8

7 For the data in these stem-and-leaf plots, find:

- **i** the mean (rounded to 1 decimal place)
- **ii** the median
- **iii** the mode

a

Stem	Leaf
2	1 3 7
3	2 8 9 9
4	4 6

3|2 means 32

b

Stem	Leaf
0	4
1	0 4 9
2	1 7 8
3	2

2|7 means 27

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PROBLEM-SOLVING AND REASONING	8, 9, 12	9, 10, 12, 13	10, 11, 13, 14

8 Two football players, Nick and Jack, compare their personal tallies of the number of goals scored for their team over a 12-match season. Their tallies are as follows.

Game	1	2	3	4	5	6	7	8	9	10	11	12
Nick	0	2	2	0	3	1	2	1	2	3	0	1
Jack	0	0	4	1	0	5	0	3	1	0	4	0

a Draw a dot plot to display Nick's goal-scoring achievement.
b Draw a dot plot to display Jack's goal-scoring achievement.
c How would you describe Nick's scoring habits?
d How would you describe Jack's scoring habits?

9 Three different electric sensors, A, B and C, are used to detect movement in Harvey's backyard over a period of 3 weeks. An in-built device counts the number of times the sensor detects movement each night. The results are as follows.

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Sensor A	0	0	1	0	0	1	1	0	0	2	0	0	0	0	0	1	1	0	0	1	0
Sensor B	0	15	1	2	18	20	2	1	3	25	0	0	1	15	8	9	0	0	2	23	2
Sensor C	4	6	8	3	5	5	5	4	8	2	3	3	1	2	2	1	5	4	0	4	9

a Using class intervals of 3 and starting at 0, draw up a frequency table for each sensor.
b Draw histograms for each sensor.
c Given that it is known that stray cats consistently wander into Harvey's backyard, how would you describe the performance of:
  i sensor A?
  ii sensor B?
  iii sensor C?

10 This tally records the number of mice that were weighed and categorised into particular mass intervals for a scientific experiment.

Mass (grams)	Tally
10–14	|||
15–19	𝍸 |
20–24	𝍸 𝍸 𝍸 |
25–29	𝍸 𝍸 𝍸 𝍸 |
30–35	||||

a Construct a table using the three column headings: Mass, Frequency and Percentage frequency.
b Find the total number of mice weighed in the experiment.
c State the percentage of mice that are in the 20–24 grams interval.
d Which is the most common weight interval?
e What percentage of mice are in the most common mass interval?
f What percentage of mice have a mass of 15 grams or more?

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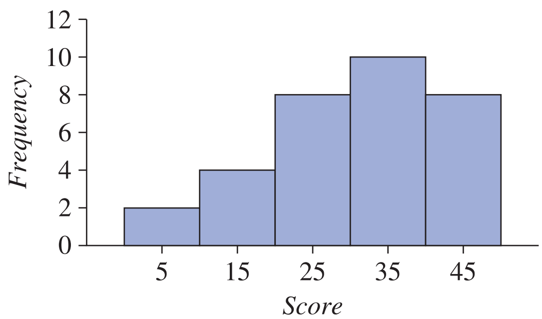
11 A school symphony orchestra contains four musical sections: strings, woodwind, brass and percussion. The number of students playing in each section is summarised in this tally.

Section	Tally
String	𝍸 𝍸 𝍸 𝍸 𝍷
Woodwind	𝍸 𝍷𝍷𝍷
Brass	𝍸 𝍷𝍷
Percussion	𝍷𝍷𝍷𝍷

a Construct and complete a percentage frequency table for the data.
b What is the total number of students in the school orchestra?
c What percentage of students play in the string section?
d What percentage of students do not play in the string section?
e If the number of students in the string section increases by 3, what will be the percentage of students who play in the percussion section? (Round your answer to 1 decimal place.)
f What will be the percentage of students in the string section of the orchestra if the entire woodwind section is away? (Round your answer to 1 decimal place.)

12 This histogram shows the distribution of test scores for a class. Explain why the percentage of scores in the 20–30 range is 25%.



13 Explain why the exact value of the mean, median and mode cannot be determined directly from a histogram that shows grouped data, like the one in Question **12**.

14 State the possible values of a, b and c in this ordered stem-and-leaf plot.

Stem	Leaf
3	2 3 a 7
4	b 4 8 9 9
5	0 1 4 9 c
6	2 6

ENRICHMENT	–	–	15

Cumulative frequency curves and percentiles

15 Cumulative frequency is obtained by adding a frequency to the total of its predecessors. It is sometimes referred to as a 'running total'.

$$\text{Percentage cumulative frequency} = \frac{\text{cumulative frequency}}{\text{total number of data elements}} \times 100$$

A cumulative frequency graph is one in which the heights of the columns are proportional to the corresponding cumulative frequencies.

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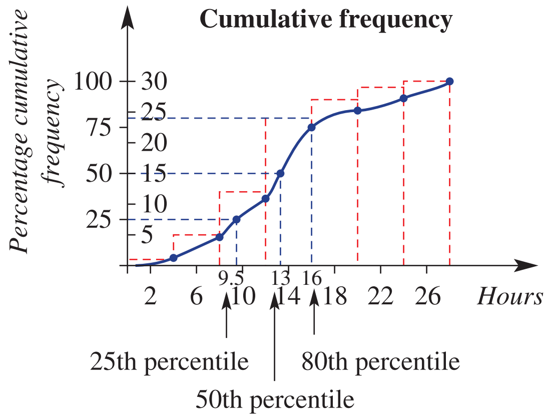
The points in the upper right-hand corners of these rectangles join to form a smooth curve called the cumulative frequency curve.

If a percentage scale is added to the vertical axis, the same graph can be used as a percentage cumulative frequency curve, which is convenient for the reading of percentiles.

Note: For the purposes of this activity, the category 0– includes all data values from 0 up to, but not including, 4.

Number of hours	Frequency	Cumulative frequency	Percentage cumulative frequency
0–	1	1	3.3
4–	4	5	16.7
8–	7	12	40.0
12–	12	24	80.0
16–	3	27	90.0
20–	2	29	96.7
24–28	1	30	100.0



The following information relates to the amount, in dollars, of winter gas bills for homes in a suburban street.

Amount ($)	Frequency	Cumulative frequency	Percentage cumulative frequency
0–	2		
40–	1		
80–	12		
120–	18		
160–	3		
200–240	1		

a Copy and complete the table. Round the percentage cumulative frequency to 1 decimal place.

b Find the number of homes that have gas bills of less than $120.

c Construct a cumulative frequency curve for the gas bills.

d Estimate the following percentiles.

  i 50th

  ii 20th

  iii 80th

e In this street, 95% of households pay less than what amount?

f What percentage of households pay less than $100?

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4C Summary statistics

Interactive
Widgets
HOTsheets
Walkthrough

In addition to the median of a single set of data, there are two related statistics called the upper and lower quartiles. When data are placed in order, then the lower quartile is central to the lower half of the data and the upper quartile is central to the upper half of the data. These quartiles are used to calculate the interquartile range, which helps to describe the spread of the data, and determine whether or not any data points are outliers.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: House prices

A real estate agent tells you that the median house price for a suburb in 2017 was \$753 000 and the mean was \$948 000.

- Is it possible for the median and the mean to differ by so much?
- Under what circumstances could this occur? Discuss.

Key ideas

- Statisticians use **summary statistics** to highlight important aspects of a dataset. These are summarised below.
- **Measures of location**
  - The two most commonly used measures of location are the **mean** and the **median**. These are also called **measures of centre** or **measures of central tendency**. They can be applied to numerical data.
  - The **mean** is sometimes called the **arithmetic mean** or the **average**. The formula used for calculating the mean is

$$\bar{x} = \frac{\text{sum of data values}}{\text{number of data values}}$$

  - The **median** divides an ordered dataset into two sets, each of which contain the same number of data values. It is often called the 'middle value'.
  - The **mode** of a dataset is the most frequently occurring data value. There can be more than one mode. When there are two modes, the data is said to be bimodal. The mode can be very useful for categorical data. It can also be used for numerical data, but it may not be an accurate measure of centre.
- **Five-figure summary** or **five-number summary**
  - **Minimum value** (min) the minimum value
  - **Lower quartile** (Q_1) the number above 25% of the ordered data
  - **Median** (Q_2) the middle value above 50% of the ordered data
  - **Upper quartile** (Q_3) the number above 75% of the ordered data
  - **Maximum value** (max) the maximum value

Odd number of data values

$$1 \quad \underbrace{2 \quad 2}_{\downarrow} \quad 3 \quad \underset{\downarrow}{5} \quad 6 \quad \underbrace{6 \quad 7}_{\downarrow} \quad 9$$

$$\underbrace{Q_1 = 2 \quad Q_2 = 5 \quad Q_3 = 6.5}_{\text{IQR} = 4.5}$$

Even number of data values

$$2 \quad 3 \quad \underset{\downarrow}{3} \quad 4 \quad \underbrace{7 \quad 8}_{\downarrow} \quad 8 \quad \underset{\downarrow}{9} \quad 9 \quad 9$$

$$\underbrace{Q_1 = 2 \quad Q_2 = 7.5 \quad Q_3 = 9}_{\text{IQR} = 6}$$

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Key ideas

- **Measures of spread**
  - **Range** = max value − min value
  - **Quantiles**, such as quartiles, deciles, percentiles
  - **Interquartile range** (IQR)

    IQR = upper quartile − lower quartile
    $= Q_3 - Q_1$
  - **Standard deviation** (see Section 4E)
- **Outliers** are data elements outside the vicinity of the rest of the data. There are many ways to define outliers. One definition states that a data point is an outlier if it is either:
  - less than $Q_1 - 1.5 \times \text{IQR}$ or
  - greater than $Q_3 + 1.5 \times \text{IQR}$
- An outlier can significantly affect the mean but usually has little impact on the median.

Example 4 Finding the range and IQR

Determine the range and IQR for these datasets by finding the five-number summary.

a 2, 2, 4, 5, 6, 8, 10, 13, 16, 20

b 1.6, 1.7, 1.9, 2.0, 2.1, 2.4, 2.4, 2.7, 2.9

SOLUTION

a Range = 20 − 2 = 18

2 2 4 5 6 | 8 10 13 16 20

Q_1 (at 4), $Q_2(7)$ (between 6 and 8), Q_3 (at 13)

$Q_2 = 7$, so $Q_1 = 4$ and $Q_3 = 13$.

IQR = 13 − 4 = 9

EXPLANATION

Range = max − min

First, split the data in half to locate the median, which is $\frac{6+8}{2} = 7$.

SOLUTION

b Range = 2.9 − 1.6 = 1.3

1.6 1.7 | 1.9 2.0 2.1 2.4 2.4 | 2.7 2.9

Q_1 (between 1.7 and 1.9), Q_2 (at 2.1), Q_3 (between 2.4 and 2.7)

$Q_1 = \frac{1.7 + 1.9}{2} = 1.8$

$Q_3 = \frac{2.4 + 2.7}{2} = 2.55$

IQR = 2.55 − 1.8 = 0.75

EXPLANATION

Max = 2.9, min = 1.6

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Example 5 Finding the five-number summary and outliers

The following dataset represents the number of flying geese spotted on each day of a 13-day tour of England.

$$5, 1, 2, 6, 3, 3, 18, 4, 4, 1, 7, 2, 4$$

a For the data, find:

- **i** the minimum and maximum number of geese spotted
- **ii** the median
- **iii** the upper and lower quartiles
- **iv** the IQR
- **v** any outliers

b Can you give a possible reason for why the outlier occurred?

SOLUTION

a **i** Min = 1, max = 18

Look for the largest and smallest numbers and order the data values:

ii 1, 1, 2, 2, 3, 3, 4, 4, 4, 5, 6, 7, 18

$\therefore$ Median = 4

1 1 2 | 2 3 3 4 4 4 5 | 6 7 18

Q_1 Q_2 Q_3

iii Lower quartile $= \dfrac{2+2}{2}$

$= 2$

Upper quartile $= \dfrac{5+6}{2}$

$= 5.5$

Since Q_2 falls on a data value, it is not included in the lower or higher halves when Q_1 and Q_3 are calculated.

iv IQR = 5.5 − 2

= 3.5

$\text{IQR} = Q_3 - Q_1$

v $Q_1 - 1.5 \times \text{IQR} = 2 - 1.5 \times 3.5$

$= -3.25$

$Q_3 + 1.5 \times \text{IQR} = 5.5 + 1.5 \times 3.5$

$= 10.75$

$\therefore$ The outlier is 18.

A data point is an outlier if it is less than $Q_1 - 1.5 \times \text{IQR}$ or greater than $Q_3 + 1.5 \times \text{IQR}$.

There are no numbers less than −3.25 but 18 is greater than 10.75.

b Perhaps a flock of geese was spotted that day.

Exercise 4C

UNDERSTANDING AND FLUENCY	1–6	3–7	4–7

1 **a** List the summary statistics used in a five-number summary.
b Explain the difference between the range and the interquartile range.
c What is an outlier?
d How do you determine if a data value in a single dataset is an outlier?

2 This dataset shows the number of cars in 13 families surveyed.

1, 4, 2, 2, 3, 8, 1, 2, 2, 0, 3, 1, 2

a Arrange the data in ascending order.
b Find the median (i.e. the middle value).
c By first removing the middle value, determine:
- **i** the lower quartile Q_1 (middle of lower half)
- **ii** the upper quartile Q_3 (middle of upper half)

d Determine the interquartile range (IQR).
e Calculate $Q_1 - 1.5 \times \text{IQR}$ and $Q_3 + 1.5 \times \text{IQR}$.
f Are there any values that are outliers (numbers below $Q - 1.5 \times \text{IQR}$ or above $Q + 1.5 \times \text{IQR}$)?

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3 The number of ducks spotted in eight different flocks are given in this dataset.

2, 7, 8, 10, 11, 11, 13, 15

a i Find the median (i.e. average of the middle two numbers).

ii Find the lower quartile (i.e. middle of the smallest four numbers).

iii Find the upper quartile (i.e. middle of the largest four numbers).

b Determine the IQR.

c Calculate $Q_1 - 1.5 \times \text{IQR}$ and $Q_3 + 1.5 \times \text{IQR}$.

d Are there any outliers (i.e. numbers below $Q_1 - 1.5 \times \text{IQR}$ or above $Q_3 + 1.5 \times \text{IQR}$)?

Example 4

4 Determine the range and IQR for these datasets by finding the five-number summary.

a 3, 4, 6, 8, 8, 10, 13

b 10, 10, 11, 14, 14, 15, 16, 18

c 1.2, 1.8, 1.9, 2.3, 2.4, 2.5, 2.9, 3.2, 3.4

d 41, 49, 53, 58, 59, 62, 62, 65, 66, 68

5 Determine the median and mean of the following sets of data. Round your answer to 1 decimal place where necessary.

a 6, 7, 8, 9, 10 b 2, 3, 4, 5, 5, 6 c 4, 6, 3, 7, 3, 2, 5

Example 5

6 The following numbers of cars were counted on each day for 15 days, travelling on a quiet suburban street.

10, 9, 15, 14, 10, 17, 15, 0, 12, 14, 8, 15, 15, 11, 13

For the given data, find:

a the minimum and maximum number of cars counted

b the median

c the lower and upper quartiles

d the IQR

e any outliers

f a possible reason for the outlier

7 Summarise the datasets below by finding the:

i minimum and maximum values

ii median (Q_2)

iii lower and upper quartiles (Q_1 and Q_3)

iv IQR

v any outliers

a 4, 5, 10, 7, 5, 14, 8, 5, 9, 9

b 24, 21, 23, 18, 25, 29, 31, 16, 26, 25, 27

PROBLEM-SOLVING AND REASONING	8, 9, 12	9, 10, 12, 13	10, 11, 13–15

8 Twelve different calculators have the following numbers of buttons.

36, 48, 52, 43, 46, 53, 25, 60, 128, 32, 52, 40

a For the given data, find:

i the minimum and maximum number of buttons on the calculators

ii the median

iii the lower and upper quartiles

iv the IQR

v any outliers

vi the mean

b Which is a better measure of the centre of the data: the mean or the median? Explain.

c Can you give a possible reason why the outlier has occurred?

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9 Using the definition of an outlier, decide whether or not any outliers exist in the following sets of data. If so, list them.

a 3, 6, 1, 4, 2, 5, 9, 8, 6, 3, 6, 2, 1
b 8, 13, 12, 16, 17, 14, 12, 2, 13, 19, 18, 12, 13
c 123, 146, 132, 136, 139, 141, 103, 143, 182, 139, 127, 140
d 2, 5, 5, 6, 5, 4, 5, 6, 7, 5, 8, 5, 5, 4

10 For the data in this stem-and-leaf plot, find:

a the IQR
b any outliers
c the median if the number 37 is added to the list
d the median if the number 22 is added to the list instead of 37

Stem	Leaf
0	1
1	6 8
2	0 4 6
3	2 3

2|4 means 24

11 Three different numbers have median 2 and range 2. Find the three numbers.

12 Explain what happens to the mean of a dataset when all the values are:

a increased by 5
b multiplied by 2
c divided by 10

13 Explain what happens to the IQR of a dataset when all values are:

a increased by 5
b multiplied by 2
c divided by 10

14 Give an example of a small dataset that satisfies the following.

a median = mean
b median = upper quartile
c range = IQR

15 Explain why in many situations the median is preferred to the mean as a measure of centre.

ENRICHMENT	–	–	16

Some research

16 Use the internet to search for data about a topic that interests you. Try to choose a single set of data that includes between 15 and 50 values.

a Organise the data using:
  i a stem-and-leaf plot
  ii a frequency table and histogram
b Find the mean and the median.
c Find the range and the interquartile range.
d Write a brief report describing the centre and spread of the data, referring to parts **a** to **c** above.
e Present your findings to your class or a partner.

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Cambridge University Press

4D Box plots

Interactive

Widgets

HOTsheets

Walkthrough

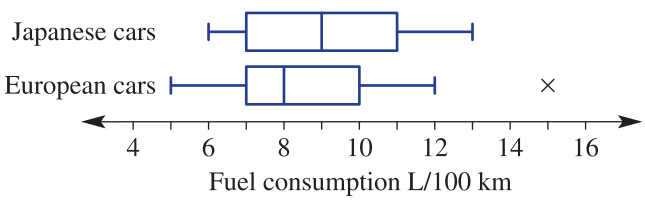
Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

The five-number summary (min, Q_1, Q_2, Q_3, max) can be represented in graphical form as a box plot. Box plots are graphs that summarise single datasets. They clearly display the minimum and maximum values, the median, the quartiles and any outliers. Box plots also give a clear indication of how data are spread, as the IQR is shown by the width of the central box.

Let's start: Fuel consumption

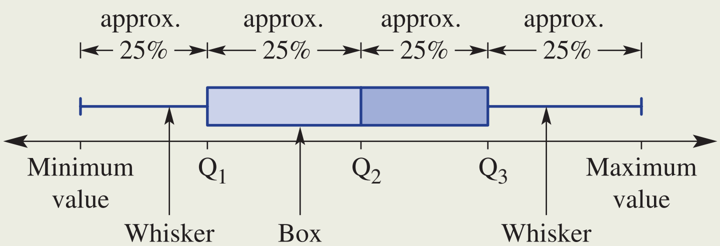
This parallel box plot summarises the average fuel consumption (litres per 100 km) for a group of Japanese-made and European-made cars.



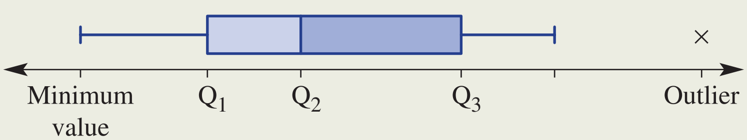
- What do the box plots say about how the fuel consumption compares between Japanese-made and European-made cars?
- What does each part of the box plot represent?
- What do you think the cross (×) represents on the European cars box plot?

Key ideas

- A **box plot** (also called a box-and-whisker plot) can be used to summarise a dataset. It divides the dataset into four groups that are approximately equal in size.



- An **outlier** is marked with a cross (×).
  - An outlier is greater than $Q_3 + 1.5 \times \text{IQR}$ or less than $Q_1 - 1.5 \times \text{IQR}$.
  - The whiskers stretch to the lowest and highest data values that are not outliers.



- **Parallel box plots** are box plots drawn on the same scale. They are used to compare datasets within the same context.

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Example 6 Constructing box plots

Consider the given dataset: 5, 9, 4, 3, 5, 6, 6, 5, 7, 12, 2, 3, 5.

a Determine whether any outliers exist by first finding Q_1 and Q_3.

b Draw a box plot to summarise the data, marking outliers if they exist.

SOLUTION

a $2\quad 3\quad \underbrace{3\quad 4}_{\uparrow\, Q_1}\quad 5\quad 5\quad \underset{\uparrow\, Q_2}{5}\quad 5\quad 6\quad \underbrace{6\quad 7}_{\uparrow\, Q_3}\quad 9\quad 12$

$$Q_1 = \frac{3+4}{2} = 3.5 \qquad Q_3 = \frac{6+7}{2} = 6.5$$

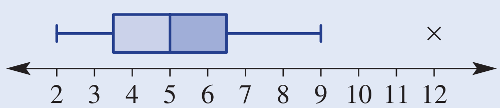
$\therefore$ IQR $= 6.5 - 3.5 = 3$

$$\begin{aligned} Q_1 - 1.5 \times \text{IQR} &= 3.5 - 1.5 \times 3 \\ &= -1 \end{aligned}$$

$$\begin{aligned} Q_3 + 1.5 \times \text{IQR} &= 6.5 + 1.5 \times 3 \\ &= 11 \end{aligned}$$

$\therefore$ 12 is an outlier.

b



EXPLANATION

Order the data values to help find the quartiles. Locate the median Q_2 and then split the data in half above and below this value.

Q_1 is the middle value of the lower half and Q_3 is the middle value of the upper half.

Determine IQR $= Q_3 - Q_1$.

Check for any outliers; that is, values below $Q_1 - 1.5 \times \text{IQR}$ or above $Q_3 + 1.5 \times \text{IQR}$.

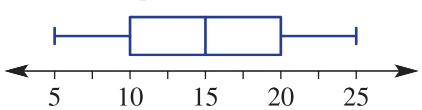
There are no data values below -1 but $12 > 11$.

Draw a line and mark in a uniform scale reaching from 2 to 12. Sketch the box plot by marking the minimum 2 and the outlier 12 and Q_1, Q_2 and Q_3. The end of the five-point summary is the nearest value below 11; i.e. 9.

Exercise 4D

UNDERSTANDING AND FLUENCY	1–5	3–5	4, 5

1 For this box plot, find:



a the median (Q_2)

b the minimum

c the maximum

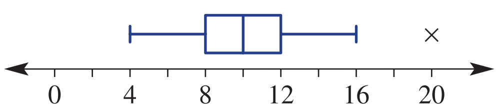
d the range

e the lower quartile (Q_1)

f the upper quartile (Q_3)

g the interquartile range (IQR)

2 Complete the following for this box plot.



a Find the IQR.

b Calculate $Q_1 - 1.5 \times \text{IQR}$.

c Calculate $Q_3 + 1.5 \times \text{IQR}$.

d State the value of the outlier.

e Check that the outlier is greater than $Q_3 + 1.5 \times \text{IQR}$.

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3 Construct a box plot that shows these features.

a min = 1, $Q_1 = 3$, $Q_2 = 4$, $Q_3 = 7$, max = 8

b outlier = 5, minimum above outlier = 10, $Q_1 = 12$, $Q_2 = 14$, $Q_3 = 15$, max = 17

Example 6

4 Consider the datasets below.

i Determine whether any outliers exist by first finding Q_1 and Q_3.

ii Draw a box plot to summarise the data, marking outliers if they exist.

a 4, 6, 5, 2, 3, 4, 4, 13, 8, 7, 6

b 1.8, 1.7, 1.8, 1.9, 1.6, 1.8, 2.0, 1.1, 1.4, 1.9, 2.2

c 21, 23, 18, 11, 16, 19, 24, 21, 23, 22, 20, 31, 26, 22

d 0.04, 0.04, 0.03, 0.03, 0.05, 0.06, 0.07, 0.03, 0.05, 0.02

5 First find Q_1, Q_2 and Q_3 and then draw box plots for the given datasets. Remember to find outliers and mark them on your box plot if they exist.

a 11, 15, 18, 17, 1, 2, 8, 12, 19, 15

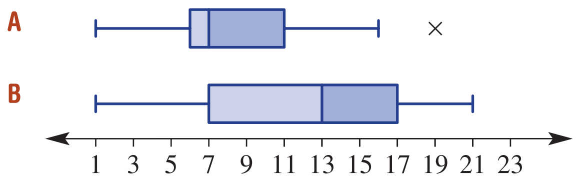
b 37, 48, 52, 51, 51, 42, 48, 47, 39, 41, 65

c 0, 1, 5, 4, 4, 4, 2, 3, 3, 1, 4, 3

d 124, 118, 73, 119, 117, 120, 120, 121, 118, 122

PROBLEM-SOLVING AND REASONING	6, 7, 9	7–10	7, 8, 10, 11

6 Consider these parallel box plots, A and B.



a What statistical measure do these box plots have in common?

b Which dataset (A or B) has a wider range of values?

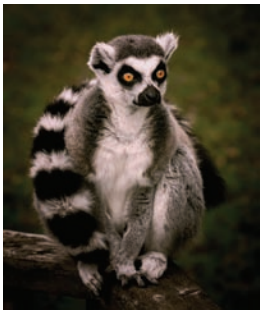
c Find the IQR for:

i dataset A **ii** dataset B

d How would you describe the main difference between the two sets of data from which the parallel box plots have been drawn?

7 The following masses, in kilograms, of 15 Madagascan lemurs were recorded as part of a conservation project.

14.4, 15.5, 17.3, 14.6, 14.7, 15.0, 15.8, 16.2, 19.7, 15.3, 13.8, 14.6, 15.4, 15.7, 14.9



a Find Q_1, Q_2 and Q_3.

b Which masses, if any, would be considered outliers?

c Draw a box plot to summarise the lemurs' masses.

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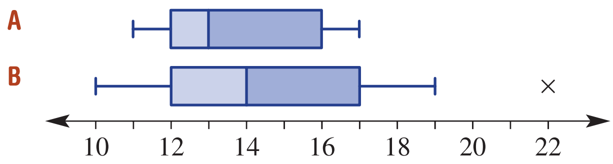
8 Two datasets can be compared using parallel box plots on the same scale, as shown below.



a What statistical measures do these box plots have in common?

b Which dataset (A or B) has a wider range of values?

c Find the IQR for:

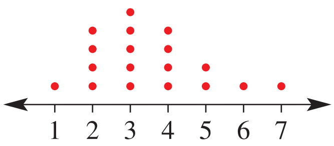
i dataset A

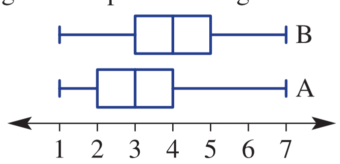
ii dataset B

d How would you describe the main difference between the two sets of data from which the parallel box plots have been drawn?

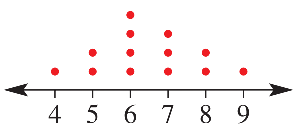
9 Select the box plot (A or B) that best matches the given dot plot or histogram.

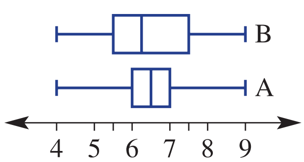
a



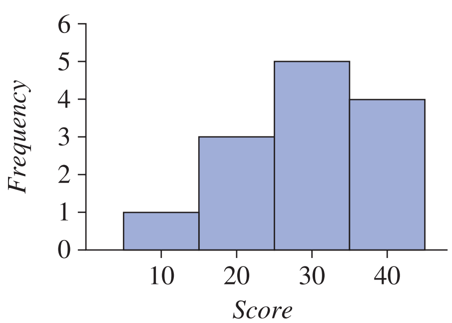


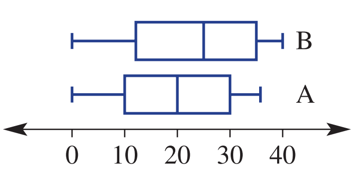
b





c





10 Fifteen essays are marked for spelling errors by a particular examiner and the following numbers of spelling errors are counted.

3, 2, 4, 6, 8, 4, 6, 7, 6, 1, 7, 12, 7, 3, 8

The same 15 essays are marked for spelling errors by a second examiner and the following numbers of spelling errors are counted.

12, 7, 9, 11, 15, 5, 14, 16, 9, 11, 8, 13, 14, 15, 13

a Draw parallel box plots for the data.

b Do you believe there is a major difference in the way the essays were marked by the two examiners? If yes, describe this difference.

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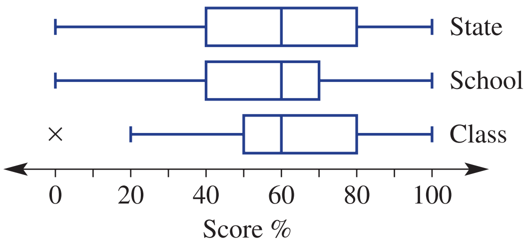
11 The results for a Year 12 class are to be compared with the Year 12 results of the school and the State, using the parallel box plots shown.



a Describe the main differences between the performance of:
- **i** the class against the school
- **ii** the class against the State
- **iii** the school against the State

b Why is an outlier shown on the class box plot but not shown on the school box plot?

ENRICHMENT	–	–	12

Creating your own parallel box plots

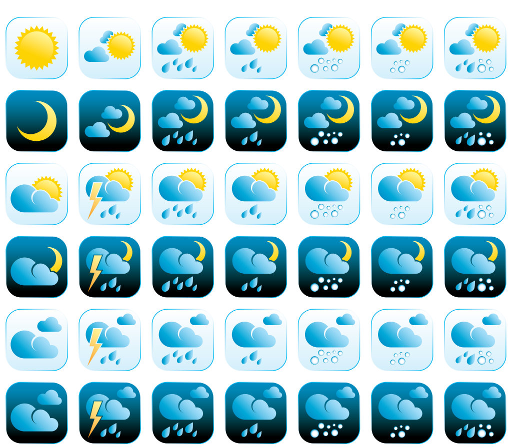
12 **a** Choose an area of study for which you can collect data easily, for example:
- the heights or weights of students
- the maximum temperatures over a weekly period
- the amount of pocket money received each week for a group of students

b Collect at least two sets of data for your chosen area of study – perhaps from two or three different sources, including the internet.

Examples:
- Measure student heights in your class and those of a second class in the same year level.
- Record maximum temperatures for one week and repeat for a second week to obtain a second dataset.
- Use the internet to obtain the football scores of two teams for each match in the previous season.

c Draw parallel box plots for your data.

d Write a report on the characteristics of each dataset and the similarities and differences between the datasets collected.



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4E Standard deviation

For a single dataset we have already used the range and interquartile range to describe the spread of the data. Another statistic commonly used to describe spread is standard deviation. The standard deviation is a number that describes how far data values are from the mean. A dataset with a relatively small standard deviation will have data values concentrated about the mean, and if a dataset has a relatively large standard deviation then the data values will be more spread out from the mean.

The standard deviation can be calculated by hand but, given the tedious nature of the calculation, technology can be used for more complex datasets. You will be able to find a function on your calculator (often denoted σ) that can be used to calculate the standard deviation.

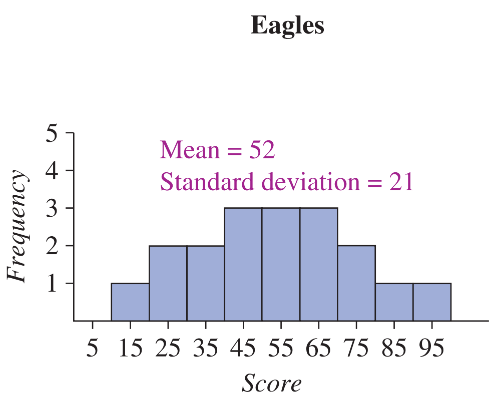
Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

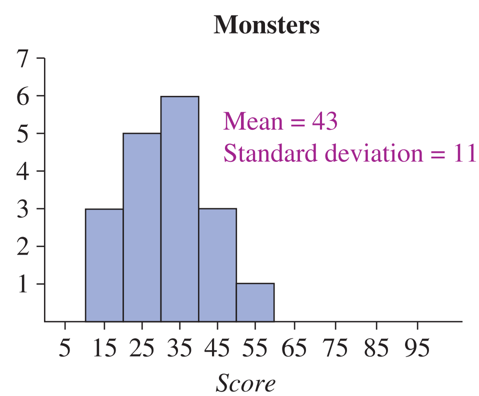
Interactive
Widgets
HOTsheets
Walkthrough

Let's start: Which is the better team?

These histograms show the number of points scored by the Eagles and Monsters basketball teams in an 18-round competition. The mean and standard deviation are given for each team.

- Which team has the higher mean? What does this say about the team's performance?
- Which team has the smaller standard deviation? What does this say about the team's performance? Discuss.





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Key ideas

- The **standard deviation** is a number that describes how far data values deviate from the mean.
  - The symbol used for standard deviation is σ (sigma).
  - If data are concentrated about the mean, then the standard deviation is relatively small.
  - If data are spread out from the mean, then the standard deviation is relatively large.
- Every scientific calculator has a button that calculates population standard deviation. Find it and learn how to use it.
  - Your calculator will also have a button for sample standard deviation. This is used when a dataset is collected from a sample.
- These are the steps that your calculator uses to calculate the standard deviation of a dataset collected from a population.
  1. Find the mean $(\bar{x})$.
  2. Find the difference between each value and the mean (called the deviation).
  3. Square each deviation.
  4. Sum the squares of each deviation.
  5. Divide by the number of data values, n.
  6. Take the square root.

  $$\sigma = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ \dots + (x_n - \bar{x})^2}{n}}$$
- In a dataset in which all the data values are equal, the standard deviation is zero.

Example 7 Calculating the standard deviation, step-by-step

Calculate the mean and standard deviation for this small dataset, correct to 1 decimal place.

2, 4, 5, 8, 9

SOLUTION

$$\bar{x} = \frac{2 + 4 + 5 + 8 + 9}{5}$$
$$= 5.6$$

EXPLANATION

Sum all the data values and divide by the number of data values (i.e. $n = 5$) to find the mean.

SOLUTION

$$\sigma = \sqrt{\frac{(-3.6)^2 + (-1.6)^2 + (-0.6)^2 + (2.4)^2 + (3.4)^2}{5}}$$
$$= 2.6 \text{ (to 1 decimal place)}$$

EXPLANATION

Sum the square of all the deviations, divide by n (i.e. 5) and then take the square root.

Deviation 1 is $2 - 5.6 = -3.6$.

Deviation 2 is $4 - 5.6 = -1.6$, etc.

Now use the σ button on your calculator to check this answer.

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Example 8 Interpreting the standard deviation

This back-to-back stem-and-leaf plot shows the distribution of distances that 17 people in Darwin and Sydney travel to work. The means and standard deviations are given.

Darwin Leaf	Stem	Sydney Leaf
8 7 4 2	0	1 5
9 9 5 5 3	1	2 3 7
8 7 4 3 0	2	0 5 5 6
5 2 2	3	2 5 9 9
	4	4 4 6
	5	2

3|5 means 35 km

Sydney
$\bar{x} = 27.9$
$\sigma = 14.7$

Darwin
$\bar{x} = 19.0$
$\sigma = 9.8$

Consider the position and spread of the data and then answer the following.

a By looking at the stem-and-leaf plot, suggest why Darwin's mean is less than that of Sydney.
b Why is Sydney's standard deviation larger than that of Darwin?
c Give a practical reason for the difference in centre and spread for the data for Darwin and Sydney.

SOLUTION

a The maximum score for Darwin is 35. Sydney's mean is affected by several values larger than 35.

EXPLANATION

The mean depends on every value in the dataset.

b The data for Sydney are more spread out from the mean. Darwin's scores are more closely clustered near its mean.

Sydney has more scores with a large distance from its mean. Darwin's scores are closer to the Darwin mean.

c Sydney is a larger city and more spread out, so people have to travel farther to get to work.

Higher populations often lead to larger cities and longer travel distances.

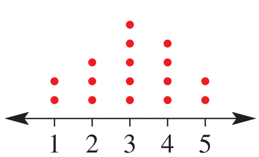
Exercise 4E

UNDERSTANDING AND FLUENCY	1–6	4, 5, 6	5, 6

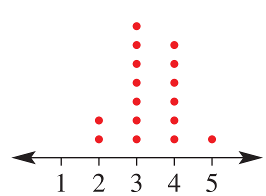
1 Insert the word *smaller* or *larger* into each sentence.
   a If data are more spread out from the mean, then the standard deviation is ___________.
   b If data are more concentrated about the mean, then the standard deviation is ___________.

2 Here are two dot plots A and B.

A



B



   a Which dataset (i.e. A or B) would have the higher mean?
   b Which dataset (i.e. A or B) would have the higher standard deviation?

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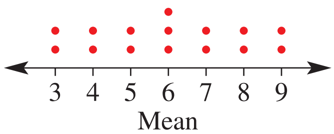
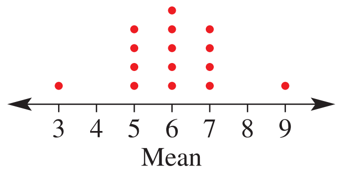
3 These dot plots show the results for a class of 15 students who sat tests A and B. Both sets of results have the same mean and range.

A



B



a Which dataset (i.e. A or B) would have the higher standard deviation?

b Give a reason for your answer to part **a**.

4 This back-to-back stem-and-leaf plot compares the number of trees or shrubs in the backyards of homes in the suburbs of Gum Heights and Oak Valley.

a Which suburb has the smaller mean number of trees or shrubs? Do not calculate the actual means.

b Which suburb has the smaller standard deviation?

Gum Heights Leaf	Stem	Oak Valley Leaf
7 3 1	0	6
9 8 6 4 0	1	0 5
9 8 7 2	2	0 2 3 6 8 8 9
6 4	3	4 6 8
	4	3

2|8 means 28

Example 7

5 Calculate the mean and standard deviation for these small datasets. Round the standard deviation to 1 decimal place where necessary.

a 3, 5, 6, 7, 9

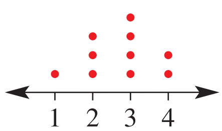
b 1, 1, 4, 5, 7

c 2, 5, 6, 9, 10, 11, 13

d 28, 29, 32, 33, 36, 37

6 Calculate the mean and standard deviation for the data in these graphs, correct to 1 decimal place.

a



b

Stem	Leaf
0	4
1	1 3 7
2	0 2

1|7 means 17

PROBLEM-SOLVING AND REASONING 7, 8, 10 | 7, 8, 10, 11 | 8, 9, 11, 12

Example 8

7 This back-to-back stem-and-leaf plot shows the distribution of distances travelled by students at an inner-city and an outer-suburb school. The means and standard deviations are given.

Inner city Leaf	Stem	Outer suburb Leaf
9 6 4 3 1 1	0	3 4 9
9 4 2 0	1	2 8 8 9
7 1	2	1 3 4
	3	4
	4	1

2|4 means 24

Inner city
$\bar{x} = 10.6$
$\sigma = 8.0$

Outer suburb
$\bar{x} = 18.8$
$\sigma = 10.7$

Consider the position and spread of the data and then answer the following.

a Why is the mean for the outer-suburb school larger than that for the inner-city school?

b Why is the standard deviation for the inner-city school smaller than that for the outer-suburb school?

c Give a practical reason for the differences in centre and spread for the two schools.

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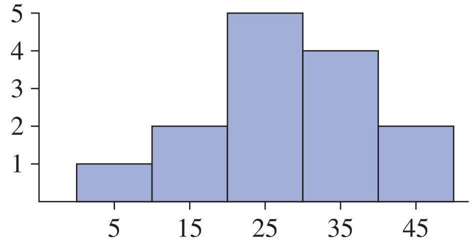
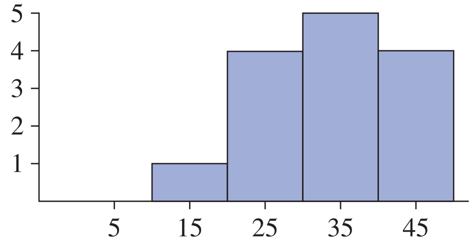
8 Consider these two histograms, and then state if the following are true or false.

A



B



- **a** The mean for set A is greater than the mean for set B.
- **b** The range for set A is greater than the range for set B.
- **c** The standard deviation for set A is greater than the standard deviation for set B.

9 Find the mean and standard deviation for the scores in these frequency tables. Round the standard deviations to 1 decimal place.

a

Score	Frequency
1	3
2	1
3	3

b

Score	Frequency
4	1
5	4
6	3

10 Two simple datasets, A and B, are identical except for the maximum value, which is an outlier for set B.

A: 4, 5, 7, 9, 10

B: 4, 5, 7, 9, 20

- **a** Is the range for set A equal to the range for set B?
- **b** Is the mean for both datasets the same?
- **c** Is the median for both datasets the same?
- **d** Would the standard deviation be affected by the outlier? Explain.

11 Datasets 1 and 2 have means $\bar{x}_1$ and $\bar{x}_2$ and standard deviations σ_1 and σ_2, respectively.

- **a** If $\bar{x}_1 > \bar{x}_2$, does this necessarily mean that $\sigma_1 > \sigma_2$? Give a reason.
- **b** If $\sigma_1 < \sigma_2$, does this necessarily mean that $\bar{x}_1 < \bar{x}_2$?

12 Datasets A and B each have 20 data values and are very similar except for an outlier in set A. Explain why the interquartile range might be a better measure of spread than the range or the standard deviation.

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ENRICHMENT	–	–	13

Study scores

13 The Mathematics study scores (out of 100) for 50 students in a school are as listed.

71, 85, 62, 54, 37, 49, 92, 85, 67, 89
96, 44, 67, 62, 75, 84, 71, 63, 69, 81
57, 43, 64, 61, 52, 59, 83, 46, 90, 32
94, 84, 66, 70, 78, 45, 50, 64, 68, 73
79, 89, 80, 62, 57, 83, 86, 94, 81, 65

The mean is 69.16 and the standard deviation is 16.0.

a Calculate:
- **i** $\bar{x} + \sigma$
- **ii** $\bar{x} - \sigma$
- **iii** $\bar{x} + 2\sigma$
- **iv** $\bar{x} - 2\sigma$
- **v** $\bar{x} + 3\sigma$
- **vi** $\bar{x} - 3\sigma$

b Use your answers from part **a** to find the percentage of students with a score within:
- **i** one standard deviation from the mean
- **ii** two standard deviations from the mean
- **iii** three standard deviations from the mean

c **i** Research what it means when we say that the data is ‘normally distributed’. Give a brief explanation.
- **ii** For data that is normally distributed, find out what percentage of data is within one, two and three standard deviations from the mean. Compare this with your results for part **b** above.

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4F Displaying and analysing time-series data

Interactive
Widgets
HOTsheets
Walkthrough

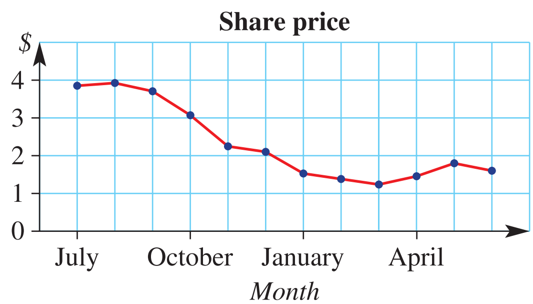
A time series is a sequence of data values that are recorded at regular time intervals. Examples include temperature recorded on the hour, speed recorded every second, population recorded every year and profit recorded every month. A line graph can be used to represent time-series data and these can help to analyse the data, describe trends and make predictions about the future.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Share price trends

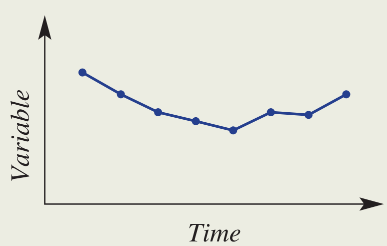
A company's share price is recorded at the end of each month of the financial year, as shown in this time-series graph.

- Describe the trend in the data at different times of the year.
- At what time of year do you think the company starts reporting poor profit results?
- Does it look like the company's share price will return to around \$4 in the next year? Why?

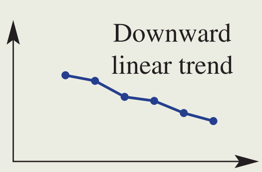


Key ideas

- **Time-series data** are recorded at regular time intervals.
- The graph or plot of a time series uses:
  - time on the horizontal axis
  - line segments connecting points on the graph



- If the time-series plot results in points being on or near a straight line, then we say that the trend is **linear**.



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Example 9 Plotting and interpreting a time-series plot

The approximate population of an outback town is recorded from 1990 to 2005.

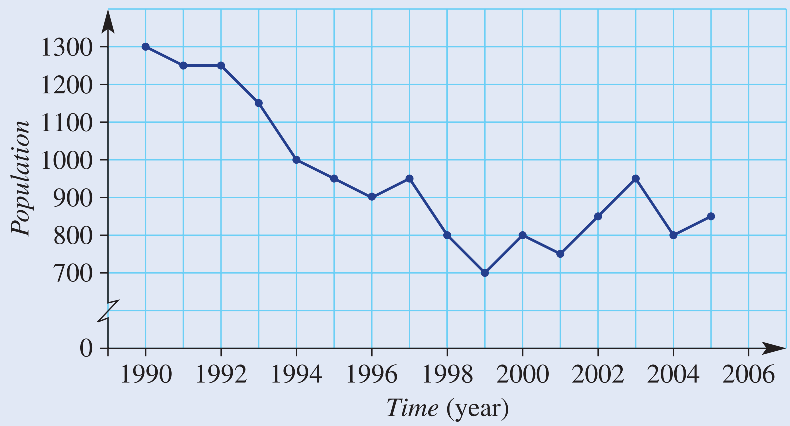
Year	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005
Population	1300	1250	1250	1150	1000	950	900	950	800	700	800	750	850	950	800	850

a Plot the time series.

b Describe the trend in the data over the 16 years.

SOLUTION

a



b The population declines steadily for the first 10 years. The population rises and falls in the last 6 years, resulting in a slight upwards trend.

EXPLANATION

Use time on the horizontal axis.

Break the y-axis so as to not include 0–700.

Join points with line segments.

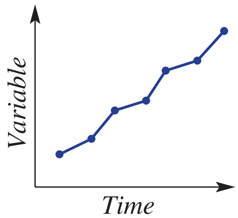
Interpret the overall rise and fall of the lines on the graph.

Exercise 4F

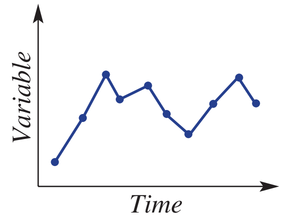
UNDERSTANDING AND FLUENCY	1–5	3–6	4–6

1 Describe the following time-series plots as having a linear (i.e. straight-line) trend, non-linear trend (i.e. a curve) or no trend.

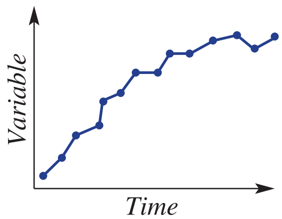
a



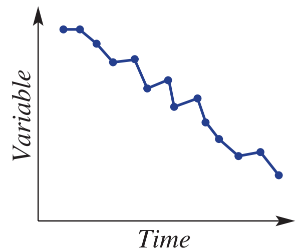
b



c



d



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2 This time-series graph shows the temperature over the course of an 8-hour school day.

a State the temperature at:

i 8 a.m. **ii** noon

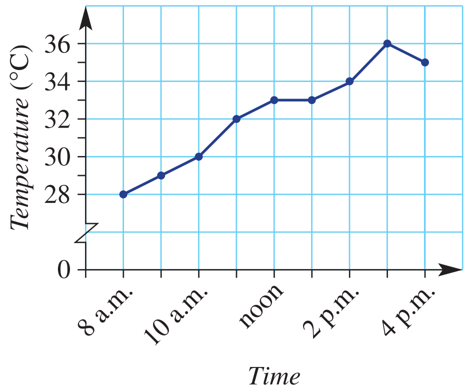
iii 1 p.m. **iv** 4 p.m.

b What was the maximum temperature?

c During what times did the temperature:

i stay the same? **ii** decrease?

d Describe the general trend in the temperature for the 8-hour school day.



Example 8

3 The approximate population of a small town is recorded from 2000 to 2010.

Year	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
Population	550	500	550	600	700	650	750	750	850	950	900

a Plot the time-series graph.

b Describe the general trend in the data over the 11 years.

c For the 11 years, what was the:

i minimum population? **ii** maximum population?

4 A company's share price over 12 months is recorded in this table.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Price ($)	1.30	1.32	1.35	1.34	1.40	1.43	1.40	1.38	1.30	1.25	1.22	1.23

a Plot the time-series graph. Break the y-axis to exclude values from \$0 to \$1.20.

b Describe the way in which the share price has changed over the 12 months.

c What is the difference between the maximum and minimum share price in the 12 months?

5 The pass rate (%) for a particular examination is given in a table over 10 years.

Year	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004
Pass rate (%)	74	71	73	79	85	84	87	81	84	83

a Plot the time-series graph for the 10 years.

b Describe the way in which the pass rate for the examination has changed in the given time period.

c In what year was the pass rate a maximum?

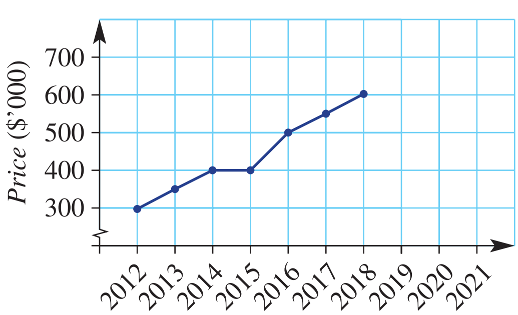
d By how much had the pass rate improved from 1995 to 1999?

6 This time-series plot shows the upwards trend of house prices in an Adelaide suburb over 7 years from 2007 to 2013.

a Would you say that the general trend in house prices is linear or non-linear?

b Assuming the trend in house prices continues for this suburb, what would you expect the house price to be in:

i 2019? **ii** 2021?



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PROBLEM-SOLVING AND REASONING	7, 8, 10	7, 8, 10, 11	8–11

7 The two top-selling book stores for a company list their sales figures for the first 6 months of the year. Sales amounts are in thousands of dollars.

	July	August	September	October	November	December
City Central ($'000)	12	13	12	10	11	13
Southbank ($'000)	17	19	16	12	13	9

a What is the difference in the sales volume for:
  i August?
  ii December?

b How many months did the City Central store sell more books than the Southbank store?

c Construct a time-series plot for both stores on the same set of axes.

d Describe the trend of sales for the 6 months for:
  i City Central
  ii Southbank

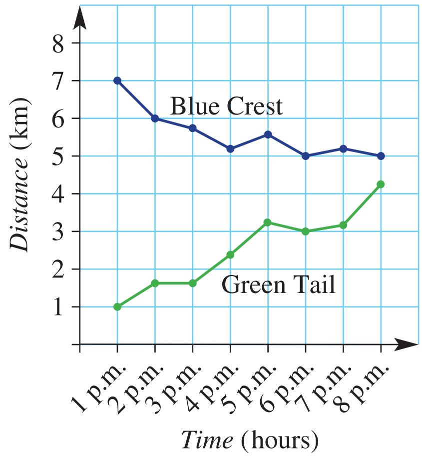
e Based on the trend for the sales by the Southbank store, what would you expect the approximate sales volume to be in January?

8 Two pigeons (Green Tail and Blue Crest) each have a beacon that communicates with a recording machine. The distance of each pigeon from the machine is recorded every hour for 8 hours.

a State the distance from the machine at 3 p.m. of:
  i Blue Crest
  ii Green Tail

b Describe the trend in the distance from the recording machine for:
  i Blue Crest
  ii Green Tail

c Assuming that the given trends continue, predict the time when the pigeons will be the same distance from the recording machine.

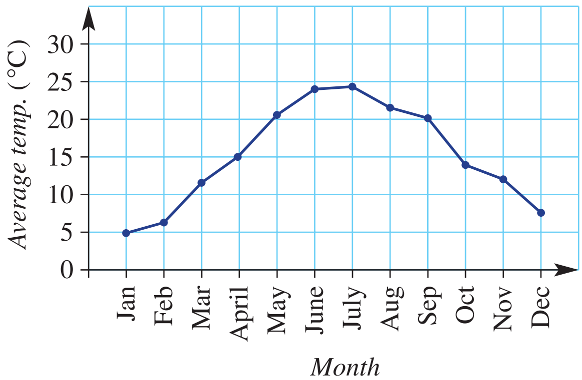


9 The average monthly maximum temperature for a city is illustrated in this graph.

a Explain why the average maximum temperature for December is close to the average maximum temperature for January.

b Do you think this graph is for an Australian city?

c Do you think the data is for a city in the Northern Hemisphere or the Southern Hemisphere? Give a reason.



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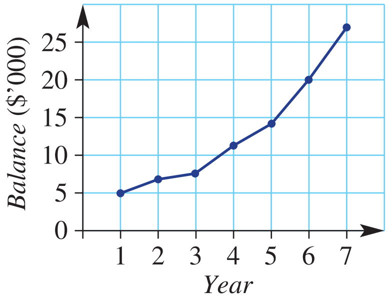
10 The balance of an investment account is shown in this time-series plot.



a Describe the trend in the account balance over the 7 years.
b Give a practical reason for the shape of the curve that models the trend in the graph.

11 A drink at room temperature is placed in a fridge with an internal temperature of 4°C.
a Sketch a time-series plot that might show the temperature of the drink after it has been placed in the fridge.
b Would the temperature of the drink ever get to 3°C? Why?

ENRICHMENT	–	–	12

Moving run average

12 A moving average is determined by calculating the average of all data values up to a particular time or place in the dataset.
Consider a batsman in cricket with the following runs scored from 10 completed innings.

Innings	1	2	3	4	5	6	7	8	9	10
Score	26	38	5	10	52	103	75	21	33	0
Moving average	26	32	23							

a Complete the table by calculating the moving average for innings 4–10. Round your answer to the nearest whole number where required.
b Plot the score and moving averages for the batter on the same set of axes.
c Describe the behaviour of the:
  i score graph
  ii moving average graph
d Describe the main difference in the behaviour of the two graphs. Give reasons.

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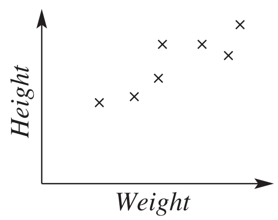
4G Bivariate data and scatter plots

Interactive
Widgets
HOTsheets
Walkthrough

When we collect information about two variables in a given context, we are collecting bivariate data. As there are two variables involved in bivariate data, we use a number plane to graph the data. These graphs are called scatter plots and are used to illustrate a relationship that may exist between the variables. Scatter plots make it very easy to see the strength of the association between the two variables.



Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: A relationship or not?

Consider the two variables in each part below.

- Would you expect there to be some relationship between the two variables in each of these cases?
- If you feel a relationship exists, would you expect the second-listed variable to increase or to decrease as the first variable increases?

a Height of person and Weight of person
b Temperature and Life of milk
c Length of hair and IQ
d Depth of topsoil and Brand of motorcycle
e Years of education and Income
f Spring rainfall and Crop yield
g Size of ship and Cargo capacity
h Fuel economy and CD track number
i Amount of traffic and Travel time
j Cost of 2 litres of milk and Ability to swim
k Background noise and Amount of work completed

Key ideas

- **Bivariate data** includes data for two variables.
  - The two variables are usually related; for example, height and weight.
- A **scatter plot** is a graph on a number plane in which the axes variables correspond to the two variables from the bivariate data.
- The words *relationship*, *correlation* and *association* are used to describe the way in which variables are related.

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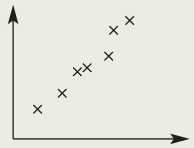
Key ideas

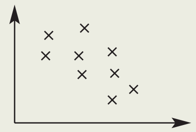
- Types of correlation

  Examples

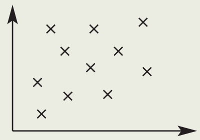
Strong, positive correlation



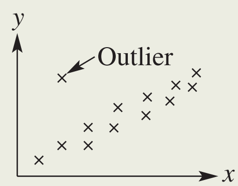
Weak, negative correlation



No correlation



- An **outlier** can clearly be identified as a data point that is isolated from the rest of the data.



Example 10 Constructing and interpreting scatter plots

Consider this simple bivariate dataset.

x	13	9	2	17	3	6	8	15
y	2.1	4.0	6.2	1.3	5.5	0.9	3.5	1.6

a Draw a scatter plot for the data.

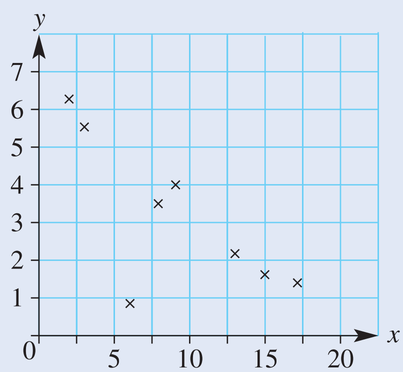
b Describe the correlation between x and y as positive or negative.

c Describe the correlation between x and y as strong or weak.

d Identify any outliers.

SOLUTION

a



EXPLANATION

On graph paper, plot each point, using a × symbol.

b negative correlation

As x increases, y decreases.

c strong correlation

The downwards trend in the data is clearly defined.

d The outlier is (6, 0.9).

This point defies the trend.

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Exercise 4G

UNDERSTANDING AND FLUENCY	1, 2a, 3, 4, 5a, 6	2b, 3, 5b, 6	3, 5c, 6

1 Decide if it is likely for there to be a strong correlation between these pairs of variables.

a Height of door and Width of door
b Weight of car and Fuel consumption
c Temperature and Length of phone calls
d Size of textbook and Quality of textbook
e Colour of flower and Strength of perfume
f Amount of rain and Size of vegetables in the vegetable garden

2 For each of the following sets of bivariate data with variables x and y:

i Draw a scatter plot by hand.
ii Decide whether y generally increases or decreases as x increases.

a

x	1	2	3	4	5	6	7	8	9	10
y	3	2	4	4	5	8	7	9	11	12

b

x	0.1	0.3	0.5	0.9	1.0	1.1	1.2	1.6	1.8	2.0	2.5
y	10	8	8	6	7	7	7	6	4	3	1

Example 10

3 Consider this simple bivariate dataset.

x	1	2	3	4	5	6	7	8
y	1.0	1.1	1.3	1.3	1.4	1.6	1.8	1.0

a Draw a scatter plot for the data.
b Describe the correlation between x and y as positive or negative.
c Describe the correlation between x and y as strong or weak.
d Identify any outliers.

4 Consider this simple bivariate dataset.

x	14	8	7	10	11	15	6	9	10
y	4	2.5	2.5	1.5	1.5	0.5	3	2	2

a Draw a scatter plot for the data.
b Describe the correlation between x and y as positive or negative.
c Describe the correlation between x and y as strong or weak.
d Identify any outliers.

5 By completing scatter plots (either by hand or using technology) for each of the following datasets, describe the correlation between x and y as positive, negative or none.

a

x	1.1	1.8	1.2	1.3	1.7	1.9	1.6	1.6	1.4	1.0	1.5
y	22	12	19	15	10	9	14	13	16	23	16

b

x	4	3	1	7	8	10	6	9	5	5
y	115	105	105	135	145	145	125	140	120	130

c

x	28	32	16	19	21	24	27	25	30	18
y	13	25	22	21	16	9	19	25	15	12

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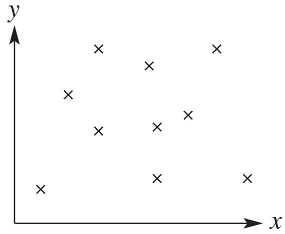
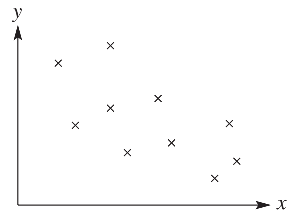
6 For the following scatter plots, describe the correlation between x and y.

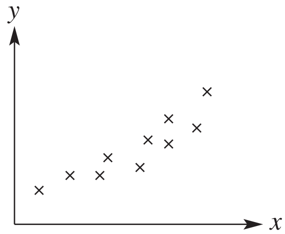
a



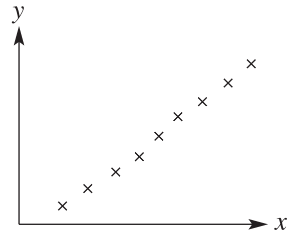
b



c



d



PROBLEM-SOLVING AND REASONING	7, 8, 11	8, 9, 11, 12	8, 10, 12, 13

7 For common motor vehicles, consider the two variables *Engine size* (cylinder volume) and *Fuel economy* (number of kilometres travelled for every litre of petrol).

a Do you expect there to be some relationship between these two variables?

b As the engine size increases, would you expect the fuel economy to increase or decrease?

c The following data were collected for 10 vehicles.

Car	A	B	C	D	E	F	G	H	I	J
Engine size	1.1	1.2	1.2	1.5	1.5	1.8	2.4	3.3	4.2	5.0
Fuel economy	21	18	19	18	17	16	15	20	14	11

i Do the data generally support your answers to parts **a** and **b**?

ii Which car gives a fuel economy reading that does not support the general trend?

8 A tomato grower experiments with a new organic fertiliser and sets up five separate garden beds: A, B, C, D and E. The grower applies different amounts of fertiliser to each bed and records the diameter of each ripe tomato picked.
The average diameter of a tomato from each garden bed and the corresponding amount of fertiliser are recorded below.

Bed	A	B	C	D	E
Fertiliser (grams per week)	20	25	30	35	40
Average diameter (cm)	6.8	7.4	7.6	6.2	8.5

a Draw a scatter plot for the data with *Diameter* on the vertical axis and *Fertiliser* on the horizontal axis. Label the points A, B, C, D and E.

b Which garden bed appears to go against the trend?

c According to the given results, would you be confident in saying that the amount of fertiliser fed to tomato plants does affect the size of the tomato produced?

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9 In a newspaper, the number of photos and number of words were counted for 15 different pages. Here are the results.

Page	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of photos	3	2	1	2	6	4	5	7	4	5	2	3	1	0	1
Number of words	852	1432	1897	1621	912	1023	817	436	1132	1201	1936	1628	1403	2174	1829

a Sketch a scatter plot using *Number of photos* on the horizontal axis and *Number of words* on the vertical axis.

b From your scatter plot, describe the general relationship between the number of photos and the number of words per page. Use the words positive, negative, strong correlation or weak correlation.

10 On 14 consecutive days, a local council measures the volume of sound heard from a freeway at various points in a local suburb. The volume of sound, in decibels, is recorded against the distance, in metres, between the freeway and the point in the suburb.

Distance (m)	200	350	500	150	1000	850	200	450	750	250	300	1500	700	1250
Volume (dB)	4.3	3.7	2.9	4.5	2.1	2.3	4.4	3.3	2.8	4.1	3.6	1.7	3.0	2.2

a Draw a scatter plot of *Volume* against *Distance*, plotting *Volume* on the vertical axis and *Distance* on the horizontal axis.

b Describe the correlation between *Distance* and *Volume* as positive, negative or none.

c Generally, as *Distance* increases, does *Volume* increase or decrease?

11 A government department is interested in convincing the electorate that a larger number of police on patrol leads to lower crime rates. Two separate surveys are completed over a one-week period and the results are listed in this table.

	Area	A	B	C	D	E	F	G
Survey 1	Number of police	15	21	8	14	19	31	17
	Incidence of crime	28	16	36	24	24	19	21
Survey 2	Number of police	12	18	9	12	14	26	21
	Incidence of crime	26	25	20	24	22	23	19

a Using scatter plots, determine whether or not there is a relationship between the number of police on patrol and the incidence of crime, using the data in:

i survey 1 **ii** survey 2

b Which survey results do you think the government will use to make its point? Why?

12 A student collects some data and finds that there is a positive correlation between height and the ability to play tennis. Does that mean that if you are tall you will be better at tennis? Explain.

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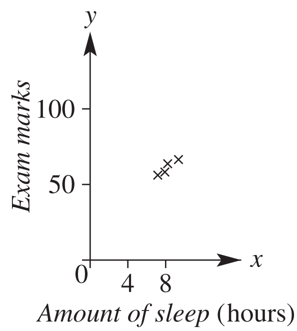
13 A person presents you with this scatter plot and suggests a strong correlation between the amount of sleep and exam marks. What do you suggest is the problem with the person's graph and conclusions?



ENRICHMENT	–	–	14

Does television provide a good general knowledge?

14 A university graduate is conducting a test to see whether a student's general knowledge is in some way linked to the number of hours of television watched.
Twenty Year 10 students sit a written general knowledge test marked out of 50. Each student also provides the graduate with details about the number of hours of television watched per week. The results are given in the table below.

Student	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
Hours of TV	11	15	8	9	9	12	20	6	0	15	9	13	15	17	8	11	10	15	21	3
Test score	30	4	13	35	26	31	48	11	50	33	31	28	27	6	39	40	36	21	45	48

a Which two students performed best on the general knowledge test, having watched TV for the following numbers of hours?
  i less than 10
  ii more than 4
b Which two students performed worst on the general knowledge test, having watched TV for the following numbers of hours?
  i less than 10
  ii more than 4
c Which two students best support the argument that the more hours of TV watched, the better your general knowledge will be?
d Which two students best support the argument that the more hours of TV watched, the worse your general knowledge will be?
e From the given data, would you say that the graduate should conclude that a student's general knowledge is definitely linked to the number of hours of TV watched per week?

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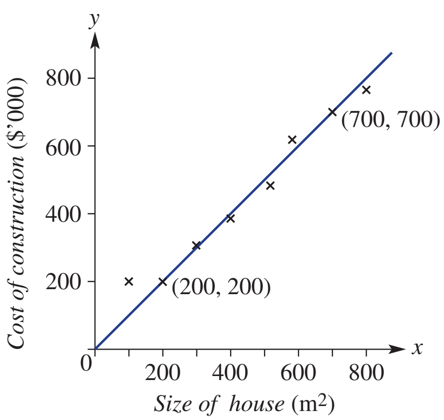
4H Line of best fit by eye

When bivariate data have a strong linear correlation, we can model the data with a straight line. This line is called a trend line or line of best fit. When we fit the line 'by eye', we try to balance the number of data points above the line with the number of points below the line. This trend line and its equation can then be used to construct other data points within and outside the existing data points.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Size versus cost

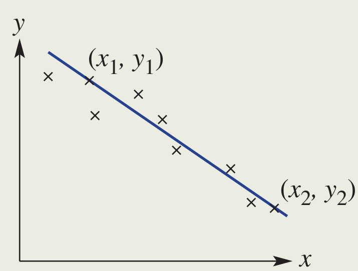
This scatter plot shows the estimated cost of building a house of a given size, as quoted by a building company. The given trend line passes through the points (200, 200) and (700, 700).



- Do you think the trend line is a good fit to the points on the scatter plot? Why?
- How can you find the equation of the trend line?
- How can you predict the cost of a house of 1000 m^2 with this building company?

Key ideas

- A **line of best fit** or **trend line** is positioned by eye by balancing the number of points above the line with the number of points below the line.
  - The distance of each point from the trend line also must be taken into account.
- The equation of the line of best fit can be found using two points that are on the line of best fit.



- For $y = mx + b$:

  $$m = \frac{y_2 - y_1}{x_2 - x_1}$$ and substitute a point to find the value of b.
  - Alternatively, use $y - y_1 = m(x - x_1)$.
- The line of best fit and its equation can be used for:
  - **interpolation**: constructing points within the given data range
  - **extrapolation**: constructing points outside the given data range

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Example 11 Fitting a line of best fit

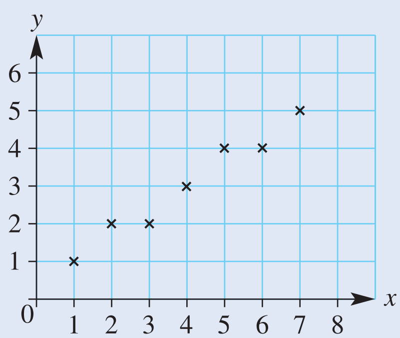
Consider the variables x and y and the corresponding bivariate data.

x	1	2	3	4	5	6	7
y	1	2	2	3	4	4	5

a Draw a scatter plot for the data.
b Is there positive, negative or no correlation between x and y?
c Fit a line of best fit by eye to the data on the scatter plot.
d Use your line of best fit to estimate:
- **i** y when $x = 3.5$
- **ii** y when $x = 0$
- **iii** x when $y = 1.5$
- **iv** x when $y = 5.5$

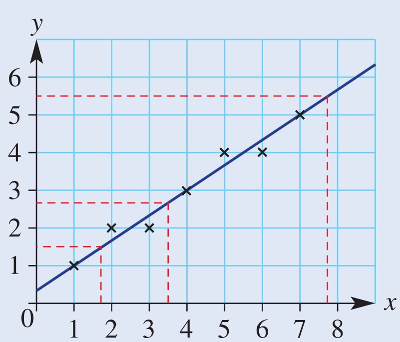
SOLUTION

a



b positive correlation

c



d **i** $y \approx 2.7$
ii $y \approx 0.4$
iii $x \approx 1.7$
iv $x \approx 7.8$

EXPLANATION

a Plot the points on graph paper.

b As x increases, y increases.

c Since a relationship exists, draw a line on the plot, keeping as many points above the line as below the line. (There are no outliers in this case.)

d Extend the vertical and horizontal lines from the values given and read off your solution. As they are approximations, we use the $\approx$ sign and not the $=$ sign.

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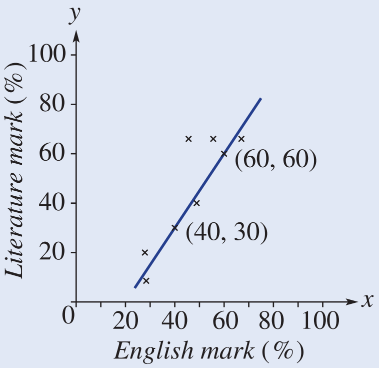
Example 12 Finding the equation of a line of best fit

This scatter plot shows a linear relationship between English marks and Literature marks in a small class of students. A trend line passes through (40, 30) and (60, 60).



a Find the equation of the trend line.

b Use your equation to estimate a Literature score if the English score is:

i 50

ii 86

c Use your equation to estimate the English score if the Literature score is:

i 42

ii 87

SOLUTION

a $y = mx + b$

$m = \frac{60 - 30}{60 - 40} = \frac{30}{20} = \frac{3}{2}$

$\therefore\ y = \frac{3}{2}x + b$

Use $m = \frac{y_2 - y_1}{x_2 - x_1}$ for the two given points.

$(40, 30)$: $30 = \frac{3}{2}(40) + b$

$30 = 60 + b$

$b = -30$

$\therefore\ y = \frac{3}{2}x - 30$

Substitute either (40, 30) or (60, 60) to find b.

b **i** $y = \frac{3}{2}(50) - 30 = 45$

$\therefore$ Literature score is 45.

Substitute $x = 50$ and find the value of y.

ii $y = \frac{3}{2}(86) - 30 = 99$

$\therefore$ Literature score is 99.

Repeat for $x = 86$.

c **i** $42 = \frac{3}{2}x - 30$

$72 = \frac{3}{2}x$

$x = 48$

$\therefore$ English score is 48.

Substitute $y = 42$ and solve for x.

ii $87 = \frac{3}{2}x - 30$

$117 = \frac{3}{2}x$

$x = 78$

$\therefore$ English score is 78.

Repeat for $y = 87$.

EXPLANATION

(Explanations are given beside each step above.)

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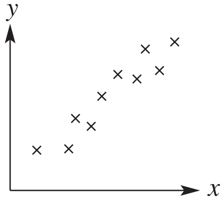
Exercise 4H

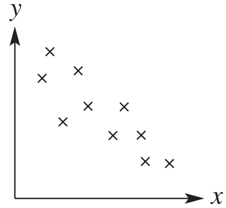
UNDERSTANDING AND FLUENCY	1–6	3–6	4–6

1 Practise fitting a line of best fit on these scatter plots by trying to balance the number of points above the line with the number of points below the line. (Using a pencil might help.)

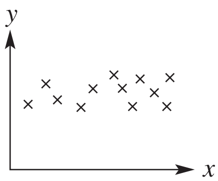
a



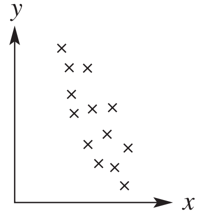
b



c

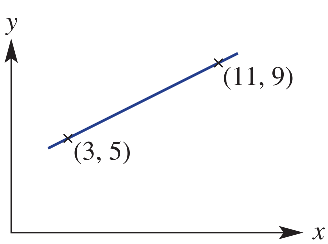


d

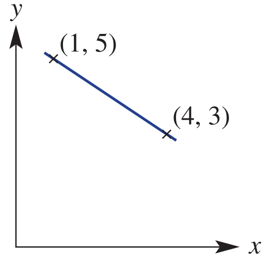


2 For each graph find the equation of the line in the form $y = mx + b$. First, find the gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and then subsititute a point.

a



b



3 Using $y = \dfrac{5}{4}x - 3$, find:

a y when:

i $x = 16$ **ii** $x = 7$

b x when:

i $y = 4$ **ii** $y = \dfrac{1}{2}$

Example 11

4 Consider the variables x and y and the corresponding bivariate data.

x	1	2	3	4	5	6	7
y	2	2	3	4	4	5	5

a Draw a scatter plot for the data.

b Is there positive, negative or no correlation between x and y?

c Fit a line of best fit by eye to the data on the scatter plot.

d Use your line of best fit to estimate:

i y when $x = 3.5$ **ii** y when $x = 0$ **iii** x when $y = 2$ **iv** x when $y = 5.5$

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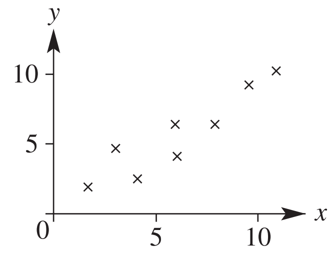
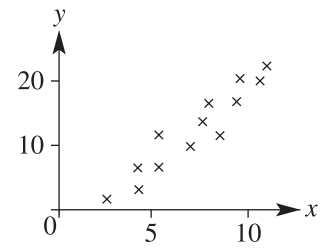
5 For the following scatter plots, pencil in a line of best fit by eye, and then use your line to estimate the value of y when $x = 5$.

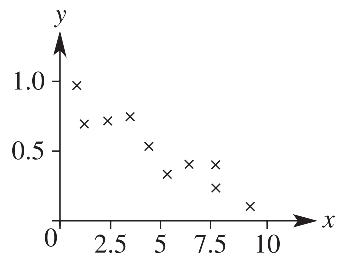
a



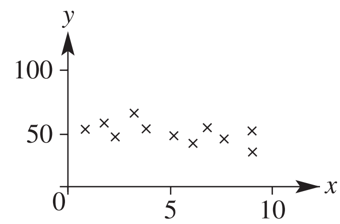
b



c



d



Example 12

6 This scatter plot shows a linear relationship between Mathematics marks and Physics marks in a small class of students. A trend line passes through (20, 30) and (70, 60).

a Find the equation of the trend line.

b Use your equation to find the Physics score if the Mathematics score is:

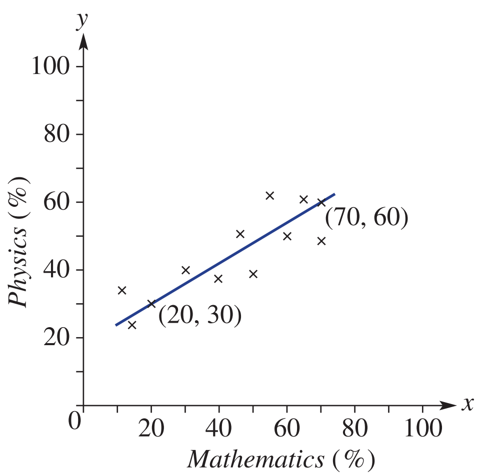
i 40

ii 90

c Use your equation to find the Mathematics score if the Physics score is:

i 36

ii 78



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PROBLEM-SOLVING AND REASONING	7, 9	7–10	7–10

7 Over eight consecutive years, a city nursery has measured the growth of an outdoor bamboo species for that year. The annual rainfall in the area where the bamboo was growing was also recorded. The data are listed in the table.

Rainfall (mm)	450	620	560	830	680	650	720	540
Growth (cm)	25	45	25	85	50	55	50	20

a Draw a scatter plot for the data, showing growth on the vertical axis.

b Fit a line of best fit by eye.

c Use your line of best fit to estimate the growth expected for the following rainfall readings. You do not need to find the equation of the line.

i 500 mm **ii** 900 mm

d Use your line of best fit to estimate the rainfall for a given year if the growth of the bamboo was:

i 30 cm **ii** 60 cm

8 A line of best fit for a scatter plot relating the weight (kg) and length (cm) of a group of dogs passes through the points (15, 70) and (25, 120). Assume weight is on the x-axis.

a Find the equation of the trend line.

b Use your equation to estimate the length of an 18 kg dog.

c Use your equation to estimate the weight of a dog that has a length of 100 cm.

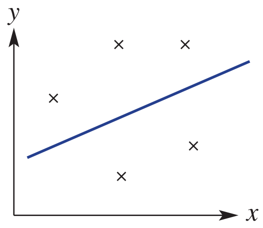
9 Choose one or more of the following reasons as to why these trend lines should not be used.

A The data points indicate the relationship is not linear.

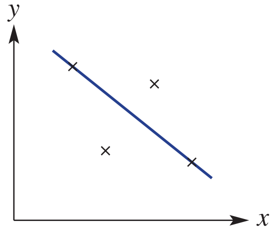
B There are too few data points.

C The data is not strongly correlated.

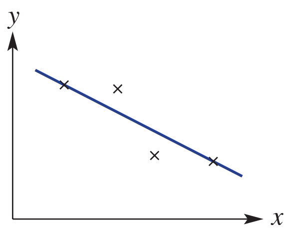
a



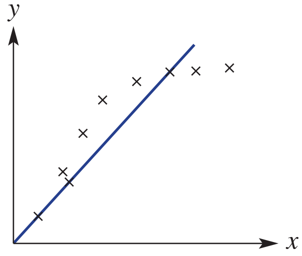
b



c



d



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10 A trend line relating the percentage scores for Music performance (y) and Music theory (x) is given by $y = \frac{4}{5}x + 10$.

a Find the value of x when:

i $y = 50$ **ii** $y = 98$

b What problem occurs in predicting Music theory scores when using high Music performance scores?

ENRICHMENT	–	–	11

Heart rate and age

11 Two independent scientific experiments confirmed a correlation between *Maximum heart rate* (in beats per minutes or b.p.m.) and *Age* (in years). The data for the two experiments are as follows.

Experiment 1													
Age (years)	15	18	22	25	30	34	35	40	40	52	60	65	71
Max. heart rate (b.p.m.)	190	200	195	195	180	185	170	165	165	150	125	128	105
Experiment 2													
Age (years)	20	20	21	26	27	32	35	41	43	49	50	58	82
Max. heart rate (b.p.m.)	205	195	180	185	175	160	160	145	150	150	135	140	90

a Sketch separate scatter plots for experiment 1 and experiment 2.

b By fitting a line of best fit by eye to your scatter plots, estimate the maximum heart rate for a person aged 55 years, using the results from:

i experiment 1 **ii** experiment 2

c Estimate the age of a person who has a maximum heart rate of 190 b.p.m., using the results from:

i experiment 1 **ii** experiment 2

d For a person aged 25 years, which experiment estimates a lower maximum heart rate?

e Research the average maximum heart rate of people according to age and compare with the results given above.

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4I Linear regression with technology

Interactive

Widgets

HOTsheets

Walkthrough

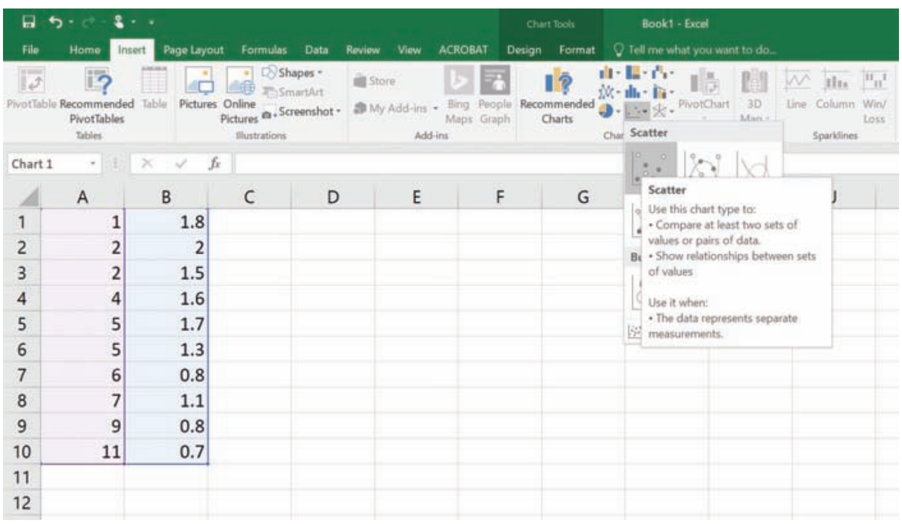
Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

In Section 4H we used a line of best fit by eye to describe a general linear (i.e. straight line) trend for bivariate data. In this section we look at the more formal methods for fitting straight lines to bivariate data. This is called linear regression. There are many different methods used by statisticians to model bivariate data. Two common methods are least squares regression and median–median regression. These methods are best handled with the use of technology, including Excel spreadsheets.

Let's start: Linear regression on computers and calculators

Explore the menus of your chosen technology to see what kind of regression tools are available. Your options include Excel spreadsheets, graphics calculators and CAS calculators.

- Can you find the least squares regression and median–median regression tools?
- Use your technology to try Example 13 on the following page.



Key ideas

- **Linear regression** involves using a method to fit a straight line to bivariate data.
  - The result is a straight-line equation that can be used for interpolation and extrapolation.
- The **least squares regression line** minimises the sum of the square of the deviations of each point from the line.
  - Outliers have an impact on the least squares regression line because all deviations are included in the calculation of the equation of the line.

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Example 13 Using an Excel spreadsheet

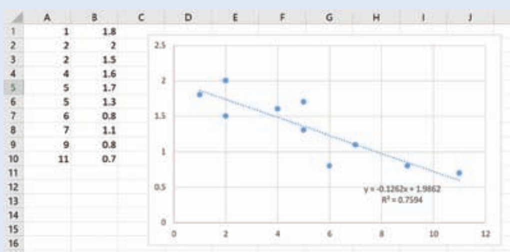
Search the internet for instructions on how to insert a trend line on an Excel spreadsheet. Use the instructions to create a scatter plot and answer the questions below.

x	1	2	2	4	5	5	6	7	9	11
y	1.8	2	1.5	1.6	1.7	1.3	0.8	1.1	0.8	0.7

a Construct a scatter plot for the data.
b Find the equation of the least squares regression line.
c Sketch the graph of the regression line onto the scatter plot.
d Using the least squares regression line, estimate the value of y when x is:
 i 4.5 **ii** 15

SOLUTION

a–c



d **i** Let $x = 4.5$
$y \approx -0.1262x + 1.9862$
$y \approx -0.1262 \times 4.5 + 1.9862$
$y \approx 1.42$

ii Let $x = 15$
$y \approx -0.1262x + 1.9862$
$y \approx -0.1262 \times 15 + 1.9862$
$y \approx 0.09$

EXPLANATION

In this version of Excel:
- Enter the data in two columns.
- Highlight the data.
- From the insert tab, choose 'Scatter'.
- Right click on one of the points, then choose 'Add trend line'.
- Tick the box that says 'Display Equation on chart'.

Substitute the given value into the equation of the trend line.
This is called *interpolation* because 4.5 is between the known data values given in the table.

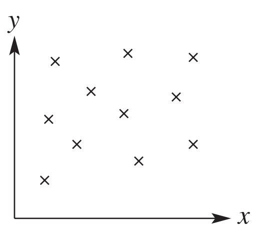
Substitute the given value into the equation of the trend line.
This is called *extrapolation* because 15 is beyond the known data values given in the table.

Exercise 4I

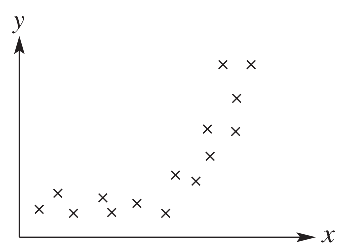
UNDERSTANDING AND FLUENCY 1–4 2–4 3–4

1 A regression line for a bivariate dataset is given by $y = 2.3x - 4.1$. Use this equation to find:
 a the value of y when x is:
  i 7 **ii** 3.2
 b the value of x when y is:
  i 12 **ii** 0.5

2 Give a brief reason why a linear regression line is not very useful in the following scatter plots.
 a



 b



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Example 13

3 Consider the data in tables **A–C** and use a graphics or CAS calculator or an Excel spreadsheet to help answer the following questions.

A

x	1	2	3	4	5	6	7	8
y	3.2	5	5.6	5.4	6.8	6.9	7.1	7.6

B

x	3	6	7	10	14	17	21	26
y	3.8	3.7	3.9	3.6	3.1	2.5	2.9	2.1

C

x	0.1	0.2	0.5	0.8	0.9	1.2	1.6	1.7
y	8.2	5.9	6.1	4.3	4.2	1.9	2.5	2.1

a Construct a scatter plot for the data.
b Find the equation of the least squares regression line.
c Sketch the graph of the regression line onto the scatter plot.
d Using the least squares regression line, estimate the value of y when x is:
i 7 **ii** 12

4 The values and ages of 14 cars are summarised in these tables.

Age (years)	5	2	4	9	10	8	7
Price ($'000)	20	35	28	14	11	12	15

Age (years)	11	2	1	4	7	6	9
Price ($'000)	5	39	46	26	19	17	14

a Find the equation of the least squares regression line. (Use *Age* for the x-axis.)
b Use your least squares regression line to estimate the value of a 3-year-old car.
c Use your least squares regression line to estimate the age of a $15 000 car.

PROBLEM-SOLVING AND REASONING 5, 6, 8 5, 6, 8, 9 6–9

5 A factory that produces denim jackets does not have air-conditioning. It was suggested that the high temperatures inside the factory were having an effect on the number of jackets able to be produced, so a study was completed and data collected on 14 consecutive days.

Max. daily temp. inside factory (°C)	28	32	36	27	24	25	29	31	34	38	41	40	38	31
Number of jackets produced	155	136	120	135	142	148	147	141	136	118	112	127	136	132

Use a graphics or CAS calculator or an Excel spreadsheet to complete the following.
a Draw a scatter plot for the data.
b Find the equation of the least squares regression line.
c Graph the line onto the scatter plot.
d Use the regression line to estimate how many jackets would be able to be produced if the maximum daily temperature in the factory is:
i 30°C **ii** 35°C **iii** 45°C

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6 A particular brand of electronic photocopier is considered suitable for scrap once it has broken down more than 50 times or if it has produced more than 200000 copies. A study of one particular copier gave the following results.

Number of copies ($\times$ 1000)	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150
Total number of breakdowns	0	0	1	2	2	5	7	9	12	14	16	21	26	28	33

a Sketch a scatter plot for the data.
b Graph the least squares regression line onto the scatter plot.
c Using your regression line, estimate the number of copies the photocopier will have produced at the point when you would expect 50 breakdowns.
d Would you expect this photocopier to be considered suitable for scrap because of the number of breakdowns or the number of copies made?

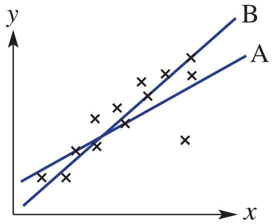
7 At a suburban sports club, the distance record for the hammer throw has increased over time. The first recorded value was 72.3 m in 1967 and the most recent record was 118.2 m in 1996. Further details are as follows.

Year	1967	1968	1969	1976	1978	1983	1987	1996
New record (m)	72.3	73.4	82.7	94.2	99.1	101.2	111.6	118.2

a Draw a scatter plot for the data.
b Use your regression equation to estimate the distance record for the hammer throw for the year:
i 2000 **ii** 2015
c Would you say that it is realistic to use your regression equation to estimate distance records beyond 2015? Why?

8 Briefly explain why the least squares regression line is affected by outliers.

9 This scatter plot shows both the least squares regression line and another line. Which line (i.e. A or B) do you think is the least squares line? Give a reason.



ENRICHMENT | – | – | 10

Correlation coefficient

10 Use the internet to find out about the Pearson correlation coefficient and then answer these questions.
a What is the coefficient used for?
b Can you use Excel to calculate the correlation coefficient?
c What does a relatively large or small correlation coefficient mean?

ISBN 978-1-108-46847-3

Investigation

1 Indigenous population comparison

The following data were collected by the Australian Bureau of Statistics during the 2006 National Census. It shows the population of Indigenous and non-Indigenous people in Australia and uses class intervals of 5 years.

Cat. No. 2068.0 – 2006 Census Tables
2006 Census of Population and Housing
Australia (Australia)
IND1GENOUS STATUS BY AGE
Count of persons
Based on place of usual residence
Commonwealth of Australia 2007

Age (years)	Indigenous	Non-Indigenous	Indigenous status not stated	Total
0–4	55 568	1 130 011	74 826	1 260 405
5–9	57 954	1 175 829	75 083	1 308 866
10–14	57 595	1 235 753	74 595	1 367 943
15–19	43 507	1 236 569	71 831	1 356 907
20–24	37 511	1 222 101	87 750	1 347 362
25–29	30 835	1 164 021	82 067	1 276 924
30–34	31 477	1 288 479	79 513	1 399 469
35–39	30 872	1 357 704	77 608	1 466 184
40–44	25 889	1 368 909	75 861	1 471 659
45–49	22 535	1 351 118	73 077	1 446 730
50–54	17 981	1 231 407	66 403	1 315 791
55–59	13 221	1 158 490	62 890	1 234 601
60–64	8978	897 984	51 118	958 080
65–69	6027	707 565	43 793	757 385
70–74	4089	572 688	39 273	616 050
75–79	2517	502 833	38 252	543 602
80–84	1343	371 173	31 964	404 480
85–89	696	195 242	18 376	214 314
90–94	259	78 163	7308	85 730
95–99	92	17 886	1672	19 650
100 and over	80	2887	188	3155
Total	**455 027**	**18 266 812**	**1 133 448**	**19 855 237**

Indigenous histogram

a Use the given data to construct a histogram for the population of Indigenous people in Australia in 2006.
b Which age group contains the most Indigenous people?
c Describe the shape of the histogram. Is it symmetrical or skewed?

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Non-Indigenous histogram

a Use the given data to construct a histogram for the population of non-Indigenous people in Australia in 2006. Try to construct this histogram so it is roughly the same width and height as the histogram for the Indigenous population. You will need to rescale the y-axis.
b Which age group contains the most non-Indigenous people?
c Describe the shape of the histogram. Is it symmetrical or skewed?

Comparisons

a Explain the main differences in the shapes of the two histograms.
b What do the histograms tell you about the age of Indigenous and non-Indigenous people in Australia in 2006?
c What do the graphs tell you about the difference in life expectancy for Indigenous and non-Indigenous people?

2 Antarctic ice

According to many different studies, the Antarctic ice mass is decreasing over time. The following data show the approximate change in ice mass in gigatonnes (Gt; 10^9 tonnes) from 2002 to 2009.

Year	2002	2003	2004	2005	2006	2007	2008	2009
Change (Gt)	450	300	400	200	−100	−400	−200	−700

A change of 300, for example, means that the ice mass has increased by 300 Gt in that year.

Data interpretation

a By how much did the Antarctic ice mass increase in:
  i 2002? **ii** 2005?
b By how much did the Antarctic ice mass decrease in
  i 2006? **ii** 2009?
c What was the overall change in ice mass from the beginning of 2005 to the end of 2007?

Time-series plot

a Construct a time-series plot for the given data.
b Describe the general trend in the change in ice mass over the 8 years.

Line of best fit

a Fit a line of best fit by eye to your time-series plot.
b Find an equation for your line of best fit.
c Use your equation to estimate the change in ice mass for:
  i 2010 **ii** 2015

Regression

a Use technology to find the least squares regression line for your time-series plot.
b How does the equation of this line compare to your line of best fit found above?

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Puzzles and challenges

1 The mean mass of six boys is 71 kg, and the mean mass of five girls is 60 kg. Find the average mass of all 11 people put together.

2 Sean has a current four-topic average of 78% for Mathematics. What score does he need in the fifth topic to have an overall average of 80%?

3 A single-ordered dataset includes the following data.

2, 4, 5, 6, 8, 10, x

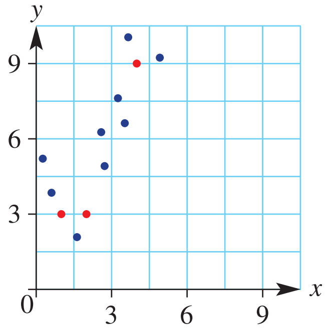
What is the largest possible value of x if it is not an outlier?

4 Find the interquartile range for a set of data if 75% of the data are above 2.6 and 25% of the data are above 3.7.

5 A single dataset has 3 added to every value. Describe the change in:

a the mean
b the median
c the range
d the interquartile range
e the standard deviation

6 Three key points on a scatter plot have coordinates (1, 3), (2, 3) and (4, 9). Find a quadratic equation that fits these three points exactly.



7 Six numbers are written in ascending order: 1.4, 3, 4.7, 5.8, a, 11.
Find all possible values of a if the number 11 is considered to be an outlier.

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Single variable and bivariate statistics

Collection data
A survey can be used to collect data from a population. The sample of the population chosen to be surveyed should be selected without bias and should be representative of the population.

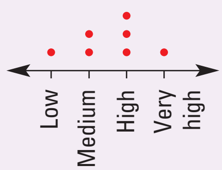
Data

Categorical
- Nominal (red, blue, ...)
- Ordinal (low, medium, ...)

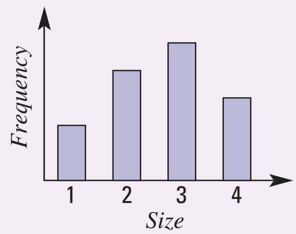
Numerical
- Discrete (1, 2, 3, ...)
- Continuous (0.31, 0.481, ...)

Graphs for single set of categorical or discrete data

Dot plot



Column graphs



Stem-and-leaf plot

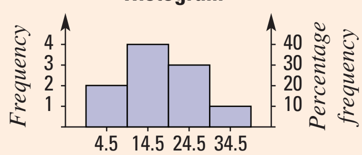
Stem	Leaf
0	1 6
1	2 7 9
2	3 8
3	4

2|3 means 23

Grouped data

Class interval	Frequency	Percentage frequency
0–9	2	20
10–19	4	40
20–29	3	30
30–39	1	10
Total	10	100

Histogram



Measures of centre
- Mean $(\bar{x}) = \dfrac{\text{sum of all values}}{\text{number of scores}}$
- Median (Q_2) = middle value
- Mode = most common value

Measures of spread
- Range = max – min
- Interquartile range (IQR) = $Q_3 - Q_1$
- Standard deviation (σ) for n data values

$$= \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n}}$$

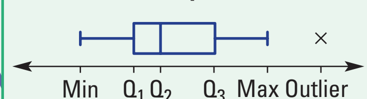
– If σ is relatively small, the data are concentrated about the mean.
– If σ is relatively large, the data are spread out from the mean.

Quartiles

Q_1: above 25% of the data
Q_3: above 75% of the data

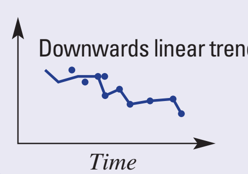
$$2 \quad 3 \quad \underset{\uparrow \atop Q_1}{5} \quad 7 \quad \underbrace{8 \,|\, 9}_{\uparrow \atop Q_2 = 8.5} \quad 11 \quad \underset{\uparrow \atop Q_3}{12} \quad 14 \quad 15$$

Box plots

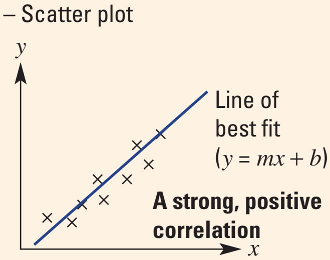


Outliers
- Single dataset
  – less than $Q_1 - 1.5 \times \text{IQR}$ or more than $Q_3 + 1.5 \times \text{IQR}$
- Bivariate
  – not in the vicinity of the rest of the data

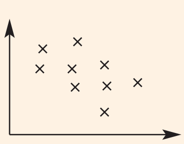
Time-series data



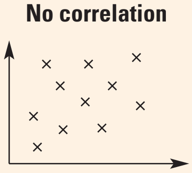
Bivariate data
– Two related variables
– Scatter plot



Weak, negative correlation



No correlation



Linear regression with technology
– Least squares line

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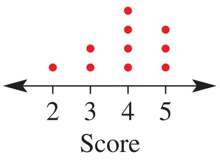
Chapter review

Multiple-choice questions

Questions 1–4 refer to this dot plot.



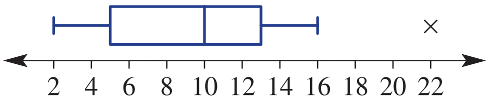
1 The mean of the scores in the data is:
A 3.5 B 3.9 C 3
D 4 E 5

2 The median for the data is:
A 3.5 B 5 C 3 D 2 E 4

3 The mode for the data is:
A 3.5 B 2 C 3 D 4 E 5

4 The dot plot is:
A symmetrical B positively skewed C negatively skewed
D bimodal E correlated

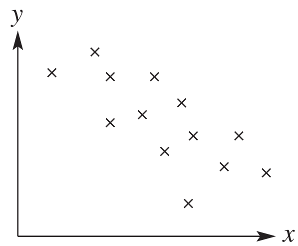
Questions 5 and 6 refer to this box plot.



5 The interquartile range is:
A 8 B 5 C 3 D 20 E 14

6 The range is:
A 5 B 3 C 20 D 14 E 8

7 The variables x and y in this scatter plot could be described as having:
A no correlation
B a strong, positive correlation
C a strong, negative correlation
D a weak, negative correlation
E a weak, positive correlation



8 The equation of the line of best fit for a set of bivariate data is given by $y = 2.5x–3$. An estimate for the value of x when $y = 7$ is:
A -1.4 B 1.2 C 1.6 D 7 E 4

9 The standard deviation for the small dataset 1, 1, 2, 3, 3 is closest to:
A 0.8 B 2 C 0.9 D 1 E 2.5

10 The equation of the line of best fit connecting the points (1, 1) and (4, 6) is:
A $y = 5x + 3$ B $y = \frac{5}{3}x - \frac{2}{3}$ C $y = \frac{5}{3}x + \frac{8}{3}$
D $y = \frac{5}{3}x - \frac{8}{3}$ E $y = \frac{3}{5}x - \frac{2}{3}$

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Short-answer questions

1 A group of 16 people was surveyed to find the number of hours of television they watch in a week. The raw data are: 6, 5, 11, 13, 24, 8, 1, 12, 7, 6, 14, 10, 9, 16, 8, 3

a Organise the data into a table with class intervals of 5 and include a percentage frequency column.

b Construct a histogram for the data, showing both the frequency and percentage frequency on the graph.

c Would you describe the data as symmetrical, positively skewed or negatively skewed?

d Construct a stem-and-leaf plot for the data, using 10s as the stem.

e Use your stem-and-leaf plot to find the median.

2 For each set of data below, complete the following tasks.

i Find the range.

ii Find the lower quartile (Q_1) and the upper quartile (Q_3).

iii Find the interquartile range.

iv Locate any outliers.

v Draw a box plot.

a 2, 2, 3, 3, 3, 4, 5, 6, 12

b 11, 12, 15, 15, 17, 18, 20, 21, 24, 27, 28

c 2.4, 0.7, 2.1, 2.8, 2.3, 2.6, 2.6, 1.9, 3.1, 2.2

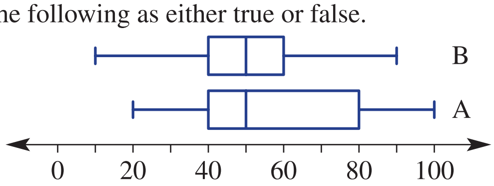
3 Compare these parallel box plots, A and B, and answer the following as either true or false.

a The range for A is greater than the range for B.

b The median for A is equal to the median for B.

c The interquartile range is smaller for B.

d 75% of the data for A sit below 80.



4 Consider the simple bivariate dataset.

x	1	4	3	2	1	4	3	2	5	5
y	24	15	16	20	22	11	5	17	6	8

a Draw a scatter plot for the data.

b Describe the correlation between x and y as positive or negative.

c Describe the correlation between x and y as strong or weak.

d Identify any outliers.

5 The line of best fit passes through the two points labelled on this graph.

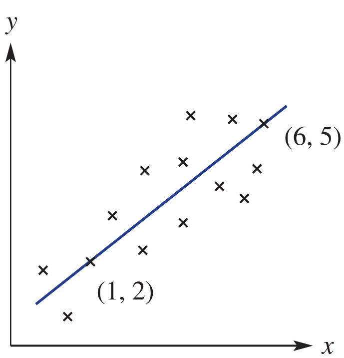
a Find the equation of the line of best fit.

b Use your equation to estimate the value of y when:

i $x = 4$ **ii** $x = 10$

c Use your equation to estimate the value of x when:

i $y = 3$ **ii** $y = 12$



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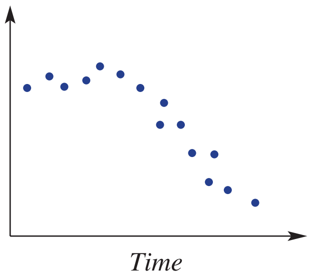
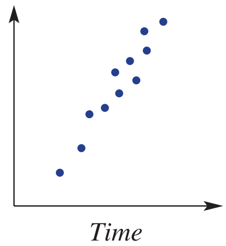
6 Describe the trend in these time-series plots as linear, non-linear or no trend.

a



b



7 Calculate the mean and standard deviation for these small datasets. Round the standard deviation to 1 decimal place.

a 4, 5, 7, 9, 10

b 1, 1, 3, 5, 5, 9

8 The Eagles and The Vipers basketball teams compare their number of points per match for a season. The data are presented in this back-to-back stem-and-leaf plot. State which team has:

a the higher range

b the higher mean

c the higher median

d the higher standard deviation

The Eagles Leaf	Stem	The Vipers Leaf
	0	9
2	1	9
83	2	0489
74	3	24789
97410	5	28
762	6	0

9 For the simple bivariate dataset in Question **4**, use technology to find the equation of the least squares regression line.

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Extended-response questions

1 The number of flying foxes taking refuge in two different fig trees was recorded over a period of 14 days. The data collected are presented here.

Tree 1	56	38	47	59	63	43	49	51	60	77	71	48	50	62
Tree 2	73	50	36	82	15	24	73	57	65	86	51	32	21	39

a Find the IQR for:
  i tree 1 **ii** tree 2
b Identify any outliers for:
  i tree 1 **ii** tree 2
c Draw parallel box plots for the data.
d By comparing your box plots, describe the difference in the ways the flying foxes use the two fig trees for taking refuge.

2 The approximate number of shoppers in an air-conditioned shopping plaza was recorded for 14 days, along with the corresponding maximum daily outside temperatures for those days.

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Max. daily temp. (T) (°C)	27	26	28	33	38	36	28	30	32	25	25	27	29	33
No. of shoppers (N)	1050	950	1200	1550	1750	1800	1200	1450	1350	900	850	700	950	1250

a Draw a scatter plot for the number of shoppers versus the maximum daily temperatures, with the number of shoppers on the vertical axis, and describe the correlation between the variables as either positive, negative or none.
b Find the equation of the least squares regression line for the data.
c Use your least squares regression equation to estimate:
  i the number of shoppers on a day with a maximum daily temperature of 24°C
  ii the maximum daily temperature if the number of shoppers at the plaza is 1500

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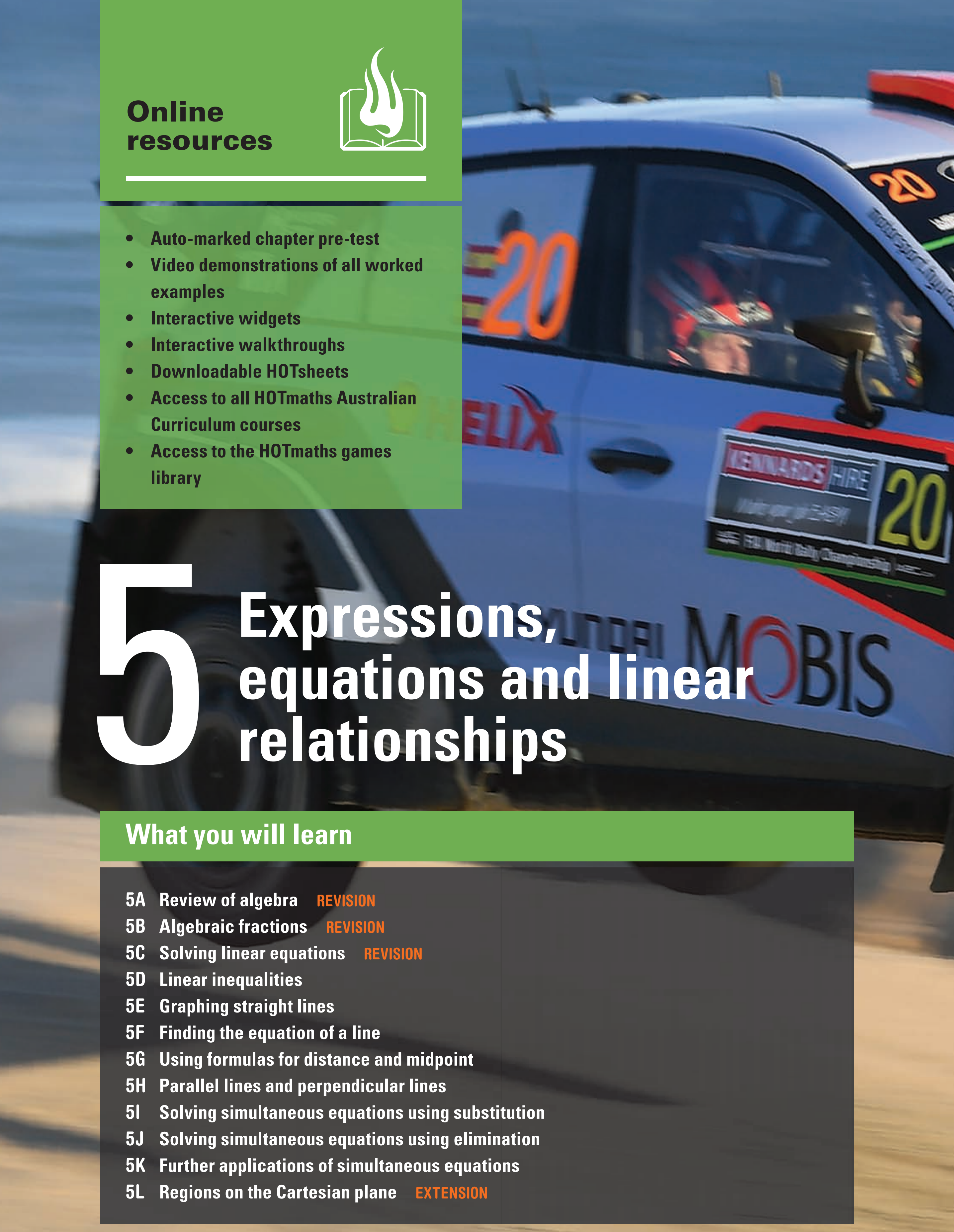

Online resources

- Auto-marked chapter pre-test
- Video demonstrations of all worked examples
- Interactive widgets
- Interactive walkthroughs
- Downloadable HOTsheets
- Access to all HOTmaths Australian Curriculum courses
- Access to the HOTmaths games library

5 Expressions, equations and linear relationships

What you will learn

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NSW syllabus

STRAND: NUMBER AND ALGEBRA
SUBSTRANDS: ALGEBRAIC TECHNIQUES
LINEAR RELATIONSHIPS
EQUATIONS

Outcomes

A student simplifies algebraic fractions, and expands and factorises quadratic expressions.
(MA5.2–6NA)

A student selects and applies appropriate algebraic techniques to operate with algebraic expressions.
(MA5.3–5NA)

A student solves linear and simple quadratic equations, linear inequalities and linear simultaneous equations, using analytical and graphical techniques.
(MA5.2–8NA)

A student solves complex linear, quadratic, simple cubic and simultaneous equations, and rearranges literal equations.
(MA5.3–7NA)

A student determines the midpoint, gradient and length of an interval, and graphs linear relationships.
(MA5.1–6NA)

A student uses the gradient–intercept form to interpret and graph linear relationships.
(MA5.2–9NA)

A student uses formulas to find midpoint, gradient and distance on the Cartesian plane, and applies standard forms of the equation of a straight line.
(MA5.3–8NA)

Rally driving and suspension

Rally drivers rely on their car's suspension to keep them safe on difficult and intense tracks around the world. Drivers must understand how their suspension system works in order to ensure that it performs as it should under extreme conditions.

The springs used in a suspension system obey the principles of a linear relationship. The force acting on the springs versus the extension of the springs forms a linear relation. Mechanics investigate the performance of various suspension springs that are suitable for the make and model of the rally car and decide which is the best for the car and driver.

Linear relationships in cars are not just restricted to suspension systems; running costs and speed efficiency can also be modelled using linear relationships. Fixed costs are the y-intercept of the equation and then the variable costs are set as the gradient, as this gives the rate of increase in the overall costs per kilometre.

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Pre-test

1 Write expressions for the following.

a one more than twice x
b the product of one less than x and 5
c the sum of x and half of x
d 3 less than twice x
e one-third of the sum of x and 4
f the difference between $3x$ and 7 (assume $3x$ is the larger number)

2 Simplify by collecting like terms.

a $4x + 3x$ **b** $-8ab + 3ab$ **c** $-3x^2 - 2x^2$

3 Simplify these expressions.

a $4y \times 2$ **b** $-3 \times 7x$ **c** $-5a \times (-3a)$

4 Expand the following.

a $3(x + 1)$ **b** $-4(x - 1)$ **c** $-2x(5 - x)$

5 Simplify by cancelling.

a $\frac{8m}{2}$ **b** $\frac{9x}{3x}$ **c** $\frac{5a^2}{5a}$ **d** $\frac{7x^2}{5} \times \frac{10}{21x}$

6 Simplify without the use of a calculator.

a $\frac{1}{2} + \frac{2}{3}$ **b** $1\frac{1}{3} - \frac{3}{4}$ **c** $\frac{3}{7} + \frac{2}{14}$ **d** $1\frac{8}{9} - \frac{13}{8}$

e $\frac{4}{5} \times \frac{5}{7}$ **f** $1\frac{2}{5} \times \frac{3}{4}$ **g** $\frac{3}{8} \div \frac{1}{2}$ **h** $2\frac{1}{3} \div \frac{3}{7}$

7 Solve these equations for x.

a $x - 6 = -2$ **b** $2x - 5 = 9$ **c** $\frac{x + 2}{3} = 4$

8 For the following relations:

i Find the x- and y-intercepts (if possible).
ii State the gradient.
iii Sketch a graph.

a $y = x + 4$ **b** $2x - 3y = 6$ **c** $y = 4$

9 Find the gradient between the following pairs of points.

a (0, 3) and (2, 9) **b** (−1, 4) and (3, −4) **c** (−2, −5) and (4, 1)

10 Write an equation connecting x and y.

a

x	−2	−1	0	1	2
y	−3	−1	1	3	5

b

x	−2	−1	0	1	2
y	7	6	5	4	3

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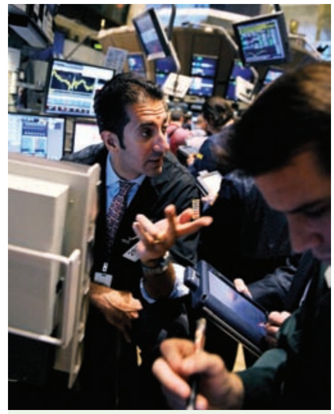
5A Review of algebra REVISION

Interactive
Widgets
HOTsheets
Walkthrough

Algebra involves the use of pronumerals, which are letters representing numbers. Combinations of numbers and pronumerals form terms (i.e. numbers and pronumerals connected by multiplication and division), expressions (i.e. a term or terms connected by addition and subtraction) and equations (i.e. mathematical statements that include an equals sign). Skills in algebra are important when dealing with the precise and concise nature of mathematics. The complex nature of many problems in finance and engineering usually result in algebraic expressions and equations that need to be simplified and solved.



Stockmarket traders rely on finacial modelling based on complex algebraic expressions.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Mystery problem

Between one school day and the next, the following problem appeared on a student noticeboard.

Prove that $8 - x^2 + \frac{3x - 9}{3} + 5(x - 1) - x(6 - x) = 0$.

- By working with the left-hand side of the equation, show that this equation is true for any value of x.
- At each step of your working, discuss what algebraic processes you have used.

Key ideas

- Algebra uses the following words.
  - **term**: $5x$, $7x^2y$, $\frac{2a}{3}$, 7 (a constant term)
  - **coefficient**: In $-2x + y$, the coefficient of x is -2 and the coefficient of y is 1.
  - **expression**: $7x$, $3x + 2xy$, $\frac{x + 3}{2}$
  - **equation**: $x = 5$, $7x - 1 = 2$, $x^2 + 2x = -4$
- Expressions can be evaluated by substituting a value for each pronumeral.
  - Order of operations are followed:
    First brackets, then indices, then multiplication and division (from left to right), then addition and subtraction (from left to right).
    If $x = -2$ and $y = 4$:

$$\begin{aligned}\frac{3x^2 - y}{2} &= \frac{3(-2)^2 - 4}{2}\\ &= \frac{3 \times 4 - 4}{2}\\ &= 4\end{aligned}$$

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Key ideas

- **Like terms** contain the same pronumerals. Some expressions containing like terms can be simplified.
  For example: $3x - 7x + x = -3x$
  $6a^2b - ba^2 = 5a^2b$

- The symbols for multiplication ($\times$) and division ($\div$) are usually not shown in simplified expressions.
  For example: $7 \times x \div y = \frac{7x}{y}$
  $$-6a^2b \div (ab) = \frac{-6a^2b}{ab} = -6a$$

- The **distributive law** is used to expand expressions containing grouping symbols.
  - $a(b + c) = ab + ac$ e.g. $2(x + 7) = 2x + 14$
  - $a(b - c) = ab - ac$ e.g. $-x(3 - x) = -3x + x^2$

- **Factorisation** involves writing expressions as a product of factors.
  - Many expressions can be factorised by taking out the highest common factor (HCF).
    For example: $3x - 12 = 3(x - 4)$
    $9x^2y - 6xy + 3x = 3x(3xy - 2y + 1)$

- Other general properties are:
  - **associative** $a \times (b \times c) = (a \times b) \times c$ or $a + (b + c) = (a + b) + c$
  - **commutative** $ab = ba$ or $a + b = b + a$ (Note: $\frac{a}{b} \neq \frac{b}{a}$ and $a - b \neq b - a$)
  - **identity** $a \times 1 = a$ or $a + 0 = a$
  - **inverse** $a \times \frac{1}{a} = 1$ or $a + (-a) = 0$

Example 1 Collecting like terms

Simplify by collecting like terms.

a $3a^2b - 2a^2b$

b $5xy + 2xy^2 - 2xy + 3y^2x$

SOLUTION	EXPLANATION
a $3a^2b - 2a^2b = a^2b$	$3a^2b$ and $2a^2b$ have the same pronumeral part, so they are like terms. Subtract coefficients. Recall: $1a^2b = a^2b$.
b $5xy + 2xy^2 - 2xy + 3y^2x = 3xy + 5xy^2$	Collect like terms, noting that $3y^2x = 3xy^2$. The + or – sign belongs to the term that follows it.

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Example 2 Multiplying and dividing expressions

Simplify the following.

a $2h \times 7l$ **b** $-3p^2r \times 2pr$ **c** $-\frac{7xy}{14y}$

SOLUTION

a $2h \times 7l = 14hl$

b $-3p^2r \times 2pr = -6p^3r^2$

c $-\frac{7xy}{14y} = -\frac{x}{2}$

EXPLANATION

Multiply the numbers and remove the × sign.

Remember the basic index law: When you multiply terms with the same base, you add the powers.

Cancel the highest common factor of 7 and 14, and cancel the y.

Example 3 Expanding the brackets

Expand the following, using the distributive law. Simplify where possible.

a $2(x + 4)$ **b** $-3x(x - y)$ **c** $3(x + 2) - 4(2x - 4)$

SOLUTION

a $2(x + 4) = 2x + 8$

b $-3x(x - y) = -3x^2 + 3xy$

c $3(x + 2) - 4(2x - 4) = 3x + 6 - 8x + 16$
$= -5x + 22$

EXPLANATION

$2(x + 4) = 2 \times x + 2 \times 4$

Note that $x \times x = x^2$ and $-3 \times (-1) = 3$.

Expand each pair of brackets and simplify by collecting like terms.

Example 4 Factorising simple algebraic expressions

Factorise:

a $3x - 9$ **b** $2x^2 + 4x$

SOLUTION

a $3x - 9 = 3(x - 3)$

b $2x^2 + 4x = 2x(x + 2)$

EXPLANATION

HCF of $3x$ and 9 is 3.

Check that $3(x - 3) = 3x - 9$.

HCF of $2x^2$ and $4x$ is $2x$.

Check that $2x(x + 2) = 2x^2 + 4x$.

Example 5 Evaluating expressions

Evaluate $a^2 - 2bc$ when $a = -3$, $b = 5$ and $c = -1$.

SOLUTION

$$\begin{aligned} a^2 - 2bc &= (-3)^2 - 2(5)(-1) \\ &= 9 - (-10) \\ &= 19 \end{aligned}$$

EXPLANATION

Substitute for each pronumeral:

$(-3)^2 = -3 \times (-3)$ and $2 \times 5 \times (-1) = -10$

To subtract a negative number, add its opposite.

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Exercise 5A REVISION

UNDERSTANDING AND FLUENCY	1–5, 6–11(½)	4, 5, 6–11(½)	6–11(½)

1 Which of the following is an equation?
A $3x - 1$ **B** $\frac{x+1}{4}$ **C** $7x + 2 = 5$ **D** $3x^2y$

2 Which expression contains a term with a coefficient of −9?
A $8 + 9x$ **B** $2x + 9x^2y$ **C** $9a - 2ab$ **D** $b - 9a^2$

3 State the coefficient of a^2 in these expressions.
a $4 + 7a^2$ **b** $a + a^2$ **c** $\frac{3}{2} - 4a^2$ **d** $-9a^2 + 2a^3$
e $\frac{a^2}{2}$ **f** $1 - \frac{a^2}{5}$ **g** $\frac{2}{7}a^2 + a$ **h** $-\frac{7a^2}{3} - 1$

4 Decide if the following pairs of terms are like terms.
a xy and $2yx$
b $7a^2b$ and $-7ba^2$
c $-4abc^2$ and $8ab^2c$

5 Evaluate:
a $(-3)^2$ **b** $(-2)^3$ **c** -2^3 **d** -3^2

Example 1

6 Simplify by collecting like terms.
a $6a + 4a$ **b** $8d + 7d$ **c** $5y - 5y$
d $2xy + 3xy$ **e** $9ab - 5ab$ **f** $4t + 3t + 2t$
g $7b - b + 3b$ **h** $3st^2 - 4st^2$ **i** $4m^2n - 7nm^2$
j $0.3a^2b - ba^2$ **k** $4gh + 5 - 2gh$ **l** $7xy + 5xy - 3y$
m $4a + 5b - a + 2b$ **n** $3jk - 4j + 5jk - 3j$ **o** $2ab^2 + 5a^2b - ab^2 + 5ba^2$
p $3mn - 7m^2n + 6nm^2 - mn$ **q** $4st + 3ts^2 + st - 4s^2t$ **r** $7x^3y^4 - 3xy^2 - 4y^4x^3 + 5y^2x$

Example 2

7 Simplify the following.
a $4a \times 3b$ **b** $5a \times 5b$ **c** $-2a \times 3d$ **d** $5h \times (-2m)$
e $-6h \times (-5t)$ **f** $-5b \times (-6l)$ **g** $2s^2 \times 6t$ **h** $-3b^2 \times 7d^5$
i $4ab \times 2ab^3$ **j** $-6p^2 \times (-4pq)$ **k** $6hi^4 \times (-3h^4i)$ **l** $7mp \times 9mr$
m $\frac{7x}{7}$ **n** $\frac{6ab}{2}$ **o** $-\frac{3a}{9}$ **p** $-\frac{2ab}{8}$
q $\frac{4ab}{2a}$ **r** $-\frac{15xy}{5y}$ **s** $-\frac{4xy}{8x}$ **t** $-\frac{28ab}{56b}$

Example 3a, b

8 Expand the following, using the distributive law.
a $5(x + 1)$ **b** $2(x + 4)$ **c** $3(x - 5)$
d $-5(4 + b)$ **e** $-2(y - 3)$ **f** $-7(a + c)$
g $-6(-m - 3)$ **h** $4(m - 3n + 5)$ **i** $-2(p - 3q - 2)$
j $2x(x + 5)$ **k** $6a(a - 4)$ **l** $-4x(3x - 4y)$
m $3y(5y + z - 8)$ **n** $9g(4 - 2g - 5h)$ **o** $-2a(4b - 7a + 10)$
p $7y(2y - 2y^2 - 4)$ **q** $-3a(2a^2 - a - 1)$ **r** $-t(5t^3 + 6t^2 + 2)$

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Example 3c

9 Expand and simplify the following, using the distributive law.

a $2(x+4)+3(x+5)$
b $4(a+2)+6(a+3)$
c $6(3y+2)+3(y-3)$
d $3(2m+3)+3(3m-1)$
e $2(2+6b)-3(4b-2)$
f $3(2t+3)-5(2-t)$
g $2x(x+4)+x(x+7)$
h $4(6z-4)-3(3z-3)$
i $3d^2(2d^3-d)-2d(3d^4+4d^2)$
j $q^3(2q-5)+q^2(7q^2-4q)$

Example 4

10 Factorise:

a $3x-9$
b $4x-8$
c $10y+20$
d $6y+30$
e x^2+7x
f $2a^2+8a$
g $5x^2-5x$
h $9y^2-63y$
i $xy-xy^2$
j $x^2y-4x^2y^2$
k $8a^2b+40a^2$
l $7a^2b+ab$
m $-5t^2-5t$
n $-6mn-18mn^2$
o $-y^2-8yz$
p $-3a^2b-6ab-3a$

Example 5

11 Evaluate these expressions when $a=-4$, $b=3$ and $c=-5$.

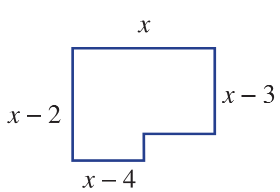
a $-2a^2$
b $b-a$
c $abc+1$
d $-ab$
e $\dfrac{a+b}{2}$
f $\dfrac{3b-a}{5}$
g $\dfrac{a^2-b^2}{c}$
h $\dfrac{\sqrt{a^2+b^2}}{\sqrt{c^2}}$

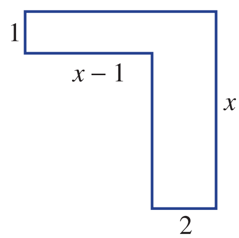
PROBLEM-SOLVING AND REASONING	12, 14, 15	12–15	12, 13, 15, 16

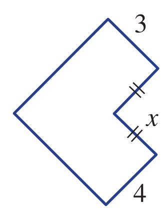
12 Find an expression for the area of a floor of a rectangular room with the following side lengths. Expand and simplify your answer.

a $x+3$ and $2x$
b x and $x-5$

13 Find expressions, in simplest form, for the perimeter (P) and area (A) of these shapes.

a


b


c


14 When $a=-2$, give reasons why:

a $a^2>0$
b $-a^2<0$
c $a^3<0$

15 Decide if the following are true or false for all values of a and b. If false, give an example to show that it is false.

a $a+b=b+a$
b $a-b=b-a$
c $ab=ba$
d $\dfrac{a}{b}=\dfrac{b}{a}$
e $a+(b+c)=(a+b)+c$
f $a-(b-c)=(a-b)-c$
g $a\times(b\times c)=(a\times b)\times c$
h $a\div(b\div c)=(a\div b)\div c$

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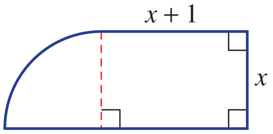
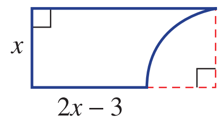
16 a Write an expression for the statement 'the sum of x and y divided by 2'.
b Explain why the statement above is ambiguous.
c Write an unambiguous statement describing $\frac{a+b}{2}$.

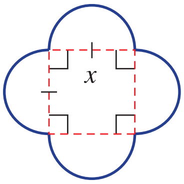
ENRICHMENT	–	–	17

Algebraic circular spaces

17 Find expressions, in simplest form, for the perimeter (P) and area (A) of these shapes. Your answers may contain π; for example, 4π. Do not use decimals.

a


b


c


Architects, builders, carpenters and landscapers are among the many occupations that use algebraic formulas to calculate areas and perimeters in their daily work.

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5B Algebraic fractions REVISION

Interactive
Widgets
HOTsheets
Walkthrough

Since pronumerals represent numbers, the rules for algebraic fractions are the same as those for simple numerical fractions. This includes processes such as cancelling common factors; adding or subtracting with a lowest common denominator (LCD); and dividing by multiplying by the reciprocal of the fraction that follows the division sign.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Describe the error

Here are three problems involving algebraic fractions. Each simplification contains one critical error. Find and describe the errors, then give the correct answer.

a $\dfrac{6x - \cancel{8}^{2}}{\cancel{4}_{1}} = \dfrac{6x-2}{1}$
$= 6x - 2$

b $\dfrac{x}{2} + \dfrac{x}{3} = \dfrac{x}{5}$

c $\dfrac{2a}{9} \div \dfrac{2}{3} = \dfrac{2a}{9} \times \dfrac{2}{3}$
$= \dfrac{4a}{27}$

Key ideas

- Simplify **algebraic fractions** by factorising expressions where possible and cancelling common factors.
- Add and subtract algebraic fractions by first finding the lowest common denominator (LCD) and then combining the numerators.
- For multiplication, cancel common factors and then multiply the numerators and denominators.
- For division, multiply by the **reciprocal** of the fraction that follows the division sign. For example, the reciprocal of a is $\frac{1}{a}$ and the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

Example 6 Cancelling common factors

Simplify by cancelling common factors.

a $\dfrac{8a^2b}{2a}$ **b** $\dfrac{3-9x}{3}$

SOLUTION

a $\dfrac{8a^2b}{2a} = \dfrac{\cancel{8}^{4} \times \cancel{a}^{1} \times a \times b}{\cancel{2}_{1} \times \cancel{a}_{1}}$
$= 4ab$

b $\dfrac{3-9x}{3} = \dfrac{\cancel{3}^{1}(1-3x)}{\cancel{3}^{1}}$
$= 1 - 3x$

EXPLANATION

a Cancel the common factors 2 and a.

b Factorise the numerator, then cancel the common factor of 3.

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Example 7 Multiplying and dividing algebraic fractions

Simplify the following.

a $\frac{2}{a} \times \frac{a+2}{4}$

b $\frac{2x-4}{3} \div \frac{x-2}{6}$

SOLUTION

a $\frac{\cancel{2}^1}{a} \times \frac{a+2}{\cancel{4}_2} = \frac{a+2}{2a}$

b $\frac{2x-4}{3} \div \frac{x-2}{6} = \frac{2x-4}{3} \times \frac{6}{x-2}$

$= \frac{2\cancel{(x-2)}^1}{\cancel{3}_1} \times \frac{\cancel{6}^2}{\cancel{x-2}^1}$

$= 4$

EXPLANATION

Cancel the common factor of 2 and then multiply the numerators and the denominators.

Multiply by the reciprocal of the second fraction.
Factorise $2x - 4$ and cancel the common factors.

Example 8 Adding and subtracting algebraic fractions

Simplify the following.

a $\frac{3}{4} - \frac{a}{2}$

b $\frac{2}{5} + \frac{3}{a}$

SOLUTION

a $\frac{3}{4} - \frac{a}{2} = \frac{3}{4} - \frac{2a}{4}$

$= \frac{3-2a}{4}$

b $\frac{2}{5} + \frac{3}{a} = \frac{2a}{5a} + \frac{15}{5a}$

$= \frac{2a+15}{5a}$

EXPLANATION

The LCD of 2 and 4 is 4. Express each fraction as an equivalent fraction with a denominator of 4.
Subtract the numerators.

The LCD of 5 and a is $5a$.
Add the numerators.

Example 9 Adding and subtracting more complex algebraic fractions

Simplify the following algebraic expressions.

a $\frac{x+3}{2} + \frac{x-2}{5}$

b $\frac{3}{x-6} - \frac{2}{x+2}$

SOLUTION

a $\frac{x+3}{2} + \frac{x-2}{5} = \frac{5(x+3)}{10} + \frac{2(x-2)}{10}$

$= \frac{5(x+3) + 2(x-2)}{10}$

$= \frac{5x + 15 + 2x - 4}{10}$

$= \frac{7x + 11}{10}$

EXPLANATION

LCD is 10.
Use brackets to ensure that you retain equivalent fractions.
Combine the numerators, then expand the brackets and simplify.

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b $\dfrac{3}{x-6}-\dfrac{2}{x+2}=\dfrac{3(x+2)}{(x-6)(x+2)}-\dfrac{2(x-6)}{(x-6)(x+2)}$ — $(x-6)(x+2)$ is the lowest common multiple of $(x-6)$ and $(x+2)$.

$=\dfrac{3(x+2)-2(x-6)}{(x-6)(x+2)}$ — Combine the numerators and then expand the brackets.

$=\dfrac{3x+6-2x+12}{(x-6)(x+2)}$ — Recall that $-2\times(-6)=12$.

$=\dfrac{x+18}{(x-6)(x+2)}$ — Collect like terms to simplify.

Exercise 5B REVISION

UNDERSTANDING AND FLUENCY	1–8	4, 5–9(½)	5–9(½)

1 Simplify and write the answer in simplest form.

a $\dfrac{5}{8}+\dfrac{3}{4}$ **b** $\dfrac{12}{10}\div 3\dfrac{1}{5}$ **c** $\dfrac{5}{12}-\dfrac{3}{8}$ **d** $\left(\dfrac{3}{7}\right)^2\times 4\dfrac{1}{5}\times 3\dfrac{8}{9}$

2 Write down the reciprocal of these fractions.

a $\dfrac{3}{2}$ **b** $\dfrac{7a}{3}$ **c** $\dfrac{-4xy}{7t}$ **d** $\dfrac{-8x^2a}{b^2c}$

3 Write down the lowest common denominator for these pairs of fractions.

a $\dfrac{a}{3},\dfrac{7a}{4}$ **b** $\dfrac{x}{2},\dfrac{4xy}{6}$ **c** $\dfrac{3xy}{7},\dfrac{-3x}{14}$ **d** $\dfrac{2}{x},\dfrac{3}{2x}$

Example 6a

4 Simplify by cancelling common factors.

a $\dfrac{10x}{2}$ **b** $\dfrac{24x}{6}$ **c** $\dfrac{5a}{20}$ **d** $\dfrac{7}{21a}$

e $\dfrac{35x^2}{7x}$ **f** $\dfrac{-14x^2y}{7xy}$ **g** $\dfrac{-36ab^2}{4ab}$ **h** $\dfrac{8xy^3}{-4xy^2}$

i $\dfrac{-15pq^2}{30p^2q^2}$ **j** $\dfrac{-20s}{45s^2t}$ **k** $\dfrac{-48x^2}{16xy}$ **l** $\dfrac{120ab^2}{140ab}$

Example 6b

5 Simplify by cancelling common factors.

a $\dfrac{4x+8}{4}$ **b** $\dfrac{6a-30}{6}$ **c** $\dfrac{6x-18}{2}$ **d** $\dfrac{5-15y}{5}$

e $\dfrac{-2-12b}{-2}$ **f** $\dfrac{21x-7}{-7}$ **g** $\dfrac{9t-27}{-9}$ **h** $\dfrac{44-11x}{-11}$

i $\dfrac{x^2+2x}{x}$ **j** $\dfrac{6x-4x^2}{2x}$ **k** $\dfrac{a^2-a}{a}$ **l** $\dfrac{7a+14a^2}{21a}$

Example 7a

6 Simplify the following.

a $\dfrac{3}{x}\times\dfrac{x-1}{6}$ **b** $\dfrac{x+4}{10}\times\dfrac{2}{x}$ **c** $\dfrac{-8a}{7}\times\dfrac{7}{2a}$

d $\dfrac{x+3}{9}\times\dfrac{4}{x+3}$ **e** $\dfrac{y-7}{y}\times\dfrac{5y}{y-7}$ **f** $\dfrac{10a^2}{a+6}\times\dfrac{a+6}{4a}$

g $\dfrac{2m+4}{m}\times\dfrac{m}{m+2}$ **h** $\dfrac{6-18x}{2}\times\dfrac{5}{1-3x}$ **i** $\dfrac{b-1}{10}\times\dfrac{5}{1-b}$

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Example 7b

7 Simplify the following.

a $\frac{x}{5} \div \frac{x}{15}$ **b** $\frac{x+4}{2} \div \frac{x+4}{6}$ **c** $\frac{6x-12}{5} \div \frac{x-2}{3}$

d $\frac{3-6y}{8} \div \frac{1-2y}{2}$ **e** $\frac{2}{a-1} \div \frac{3}{2a-2}$ **f** $\frac{2}{10x-5} \div \frac{10}{2x-1}$

g $\frac{5}{3a+4} \div \frac{15}{-15a-20}$ **h** $\frac{2x-6}{5x-20} \div \frac{x-3}{x-4}$ **i** $\frac{t-1}{9} \div \frac{1-t}{3}$

Example 8

8 Simplify the following.

a $\frac{2}{3} + \frac{a}{7}$ **b** $\frac{3}{8} + \frac{a}{2}$ **c** $\frac{3}{10} - \frac{3b}{2}$ **d** $\frac{2}{5} + \frac{4x}{15}$

e $\frac{1}{9} - \frac{2a}{3}$ **f** $\frac{3}{4} + \frac{2}{a}$ **g** $\frac{7}{9} - \frac{3}{a}$ **h** $\frac{4}{b} - \frac{3}{4}$

i $\frac{2}{7} - \frac{3}{2b}$ **j** $\frac{3}{2y} - \frac{7}{9}$ **k** $\frac{-4}{x} - \frac{2}{3}$ **l** $\frac{-9}{2x} - \frac{1}{3}$

Example 9a

9 Simplify the following algebraic expressions.

a $\frac{x+3}{4} + \frac{x+2}{5}$ **b** $\frac{x+2}{3} + \frac{x+1}{4}$ **c** $\frac{x-3}{4} - \frac{x+2}{2}$

d $\frac{x+4}{3} - \frac{x-3}{9}$ **e** $\frac{2x+1}{2} - \frac{x-2}{3}$ **f** $\frac{3x+1}{5} + \frac{2x+1}{10}$

g $\frac{x-2}{8} + \frac{2x+4}{12}$ **h** $\frac{5x+3}{10} + \frac{2x-2}{4}$ **i** $\frac{3-x}{14} - \frac{x-1}{7}$

PROBLEM-SOLVING AND REASONING	10–12	10–13	10, 11, 13

Example 9b

10 Simplify the following algebraic expressions.

a $\frac{5}{x+1} + \frac{2}{x+4}$ **b** $\frac{4}{x-7} + \frac{3}{x+2}$ **c** $\frac{1}{x-3} + \frac{2}{x+5}$

d $\frac{3}{x+3} - \frac{2}{x-4}$ **e** $\frac{6}{2x-1} - \frac{3}{x-4}$ **f** $\frac{4}{x-5} + \frac{2}{3x-4}$

g $\frac{5}{2x-1} - \frac{6}{x+7}$ **h** $\frac{2}{x-3} - \frac{3}{3x+4}$ **i** $\frac{8}{3x-2} - \frac{3}{1-x}$

11 **a** Write the LCD for these pairs of fractions.

i $\frac{3}{a}, \frac{2}{a^2}$

ii $\frac{7}{x^2}, \frac{3+x}{x}$

b Now, simplify these expressions.

i $\frac{2}{a} - \frac{3}{a^2}$

ii $\frac{a+1}{a} - \frac{4}{a^2}$

iii $\frac{7}{2x^2} + \frac{3}{4x}$

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12 Describe the error in this working, then fix the solution.

$$\begin{aligned}\frac{x}{2} - \frac{x+1}{3} &= \frac{3x}{6} - \frac{2(x+1)}{6}\\ &= \frac{3x}{6} - \frac{2x+2}{6}\\ &= \frac{x+2}{6}\end{aligned}$$

13 **a** Explain why $2x - 3 = -(3 - 2x)$.

b Use this idea to help simplify these expressions.

- **i** $\dfrac{1}{x-1} - \dfrac{1}{1-x}$
- **ii** $\dfrac{3x}{3-x} + \dfrac{x}{x-3}$
- **iii** $\dfrac{x+1}{7-x} - \dfrac{2}{x-7}$
- **iv** $\dfrac{2-a}{3} \times \dfrac{7}{a-2}$
- **v** $\dfrac{6x-3}{x} \div \dfrac{1-2x}{4}$
- **vi** $\dfrac{18-x}{3x-1} \div \dfrac{2x-36}{7-21x}$

ENRICHMENT	–	–	14, 15

Fraction challenges

14 Simplify these expressions.

- **a** $\dfrac{a-b}{b-a}$
- **b** $\dfrac{5}{a} + \dfrac{2}{a^2}$
- **c** $\dfrac{3}{x+1} + \dfrac{2}{(x+1)^2}$
- **d** $\dfrac{x}{(x-2)^2} - \dfrac{x}{x-2}$
- **e** $\dfrac{x}{2(3-x)} - \dfrac{x^2}{7(x-3)^2}$
- **f** $\dfrac{1}{x} - \dfrac{1}{y} - \dfrac{1}{z}$

15 By first simplifying the left-hand side of these equations, find the value of a.

- **a** $\dfrac{a}{x-1} - \dfrac{2}{x+1} = \dfrac{4}{(x-1)(x+1)}$
- **b** $\dfrac{3}{2x-1} + \dfrac{a}{x+1} = \dfrac{5x+2}{(2x-1)(x+1)}$

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5C Solving linear equations REVISION

Interactive
Widgets
HOTsheets
Walkthrough

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

A linear equation is a statement that contains an equals sign and includes a pronumeral that is raised to the power of 1. Here are some examples.

$2x - 5 = 7$ $\quad\quad$ $\frac{x + 1}{3} = x + 4$ $\quad\quad$ $-3(x + 2) = \frac{1}{2}$

We solve linear equations by operating on both sides of the equation until the value of the pronumeral is found.

Let's start: What's the best method?

Here are four linear equations.

- Discuss what you think is the best method to solve them, using 'by hand' techniques.
- Discuss how it might be possible to check if a solution is correct.

a $\frac{7x - 2}{3} = 4$ $\quad$ **b** $3(x - 1) = 6$ $\quad$ **c** $4x + 1 = x - 2$ $\quad$ **d** $\frac{2x + 1}{5} = \frac{x - 1}{3}$

Key ideas

- An equation is true for the given values of the pronumerals, when the left-hand side equals the right-hand side.
  For example: $2x - 4 = 6$ is true when $x = 5$ but false when $x \neq 5$.
- A **linear equation** contains a pronumeral with a power of 1 and no other powers.
- Useful steps in solving linear equations are:
  - using inverse operations (backtracking)
  - collecting like terms
  - expanding brackets
  - multiplying by the lowest common denominator
- Sometimes it is helpful to swap the left-hand side and right-hand side.
  For example: $11 = 3x + 2$ is equivalent to $3x + 2 = 11$.

Example 10 Solving linear equations

Solve the following equations and check your solution by substitution.

a $4x + 5 = 17$ $\quad$ **b** $3(2x + 5) = 4x$

SOLUTION

a
$$\begin{aligned} 4x + 5 &= 17 \\ 4x &= 12 \\ x &= 3 \end{aligned}$$
Check:
LHS $= 4 \times 3 + 5 = 17$ $\quad$ RHS $= 17$

EXPLANATION

Subtract 5 from both sides and then divide both sides by 4.

Check by seeing if $x = 3$ makes the equation true.

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b $3(2x + 5) = 4x$
$6x + 15 = 4x$
$2x + 15 = 0$
$2x = -15$
$x = -\frac{15}{2}$

Expand the brackets.
Gather like terms by subtracting $4x$ from both sides.

Subtract 15 from both sides and then divide both sides by 2.

Check:
$\text{LHS} = 3\left(2 \times \left(-\frac{15}{2}\right) + 5\right)$
$= 3(-15 + 5)$
$= -30$

$\text{RHS} = 4 \times \left(-\frac{15}{2}\right)$
$= -30$

Check by seeing if $x = -\frac{15}{2}$ makes the equation true.

Example 11 Solving equations involving algebraic fractions

Solve the following equations and check your solution by substitution.

a $\frac{x+3}{4} = 2$ **b** $\frac{4x-2}{3} = \frac{3x-1}{2}$ **c** $\frac{x+2}{3} - \frac{2x-1}{2} = 4$

SOLUTION	EXPLANATION
a $\frac{x+3}{4} = 2$	Multiply both sides by 4.
$x + 3 = 8$	Subtract 3 from both sides.
$x = 5$	
Check:	Check by seeing if $x = 5$ makes the equation true.
$\text{LHS} = \frac{5+3}{4} = 2$ $\quad$ $\text{RHS} = 2$	
b $\frac{4x-2}{3} = \frac{3x-1}{2}$	Multiply both sides by the LCD of 2 and 3, which is 6.
$\frac{{}^{2}\cancel{6}(4x-2)}{\cancel{3}_{1}} = \frac{{}^{3}\cancel{6}(3x-1)}{\cancel{2}_{1}}$	Cancel common factors.
$2(4x - 2) = 3(3x - 1)$	
$8x - 4 = 9x - 3$	Expand the brackets.
$9x - 3 = 8x - 4$	Swap LHS and RHS.
$x = -1$	Check by seeing if $x = -1$ makes the equation true.
c $\frac{x+2}{3} - \frac{2x-1}{2} = 4$	Multiply both sides by the LCD of 2 and 3, which is 6. Cancel common factors.
$\frac{{}^{2}\cancel{6}(x+2)}{\cancel{3}_{1}} - \frac{{}^{3}\cancel{6}(2x-1)}{\cancel{2}_{1}} = 4 \times 6$	
$2(x + 2) - 3(2x - 1) = 24$	Expand, noting that $-3 \times (-1) = 3$.
$2x + 4 - 6x + 3 = 24$	Simplify and solve for x.
$-4x + 7 = 24$	
$-4x = 17$	
$x = -\frac{17}{4}$	Check your solution using substitution.

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Exercise 5C REVISION

UNDERSTANDING AND FLUENCY	1–3, 4–7(½)	4–6(½), 7	5–6(½), 7

1 Decide if the following are linear equations.

a $x^2 - 1 = 0$ **b** $\sqrt{x} + x = 3$ **c** $\frac{x-1}{2} = 5$ **d** $\frac{3x}{4} = 2x - 1$

2 Decide if these equations are true when $x = 2$.

a $3x - 1 = 5$ **b** $4 - x = 1$ **c** $\frac{2x+1}{5} = x + 4$

3 Decide if these equations are true when $x = -6$.

a $-3x + 17 = x$ **b** $2(4 - x) = 20$ **c** $\frac{2-3x}{10} = \frac{-12}{x}$

Example 10a

4 Solve the following equations and check your solution by substitution.

a $x + 8 = 13$ **b** $x - 5 = 3$ **c** $-x + 4 = 7$ **d** $-x - 5 = -9$
e $2x + 9 = 14$ **f** $4x + 3 = 14$ **g** $3x - 3 = -4$ **h** $6x + 5 = -6$
i $-3x + 5 = 17$ **j** $-2x + 7 = 4$ **k** $-4x - 9 = 9$ **l** $-3x - 7 = -3$
m $8 - x = 10$ **n** $5 - x = -2$ **o** $6 - 5x = 16$ **p** $4 - 9x = -7$

Example 10b

5 Solve the following equations and check your solution by substitution.

a $4(x + 3) = 16$ **b** $2(x - 3) = 12$
c $2(x - 4) = 15$ **d** $3(1 - 2x) = 8$
e $3(2x + 3) = -5x$ **f** $2(4x - 5) = -7x$
g $3(2x + 3) + 2(x + 4) = 25$ **h** $2(2x - 3) + 3(4x - 1) = 23$
i $2(3x - 2) - 3(x + 1) = 5$ **j** $5(2x + 1) - 3(x - 3) = 63$
k $5(x - 3) = 4(x - 6)$ **l** $4(2x + 5) = 3(x + 15)$
m $5(x + 2) = 3(2x - 3)$ **n** $3(4x - 1) = 7(2x - 7)$

Example 11a

6 Solve the following equations and check your solution by substitution.

a $\frac{x-4}{2} = 3$ **b** $\frac{x+2}{3} = 5$ **c** $\frac{x+4}{3} = -6$
d $\frac{2x+7}{3} = 5$ **e** $\frac{2x+1}{3} = -3$ **f** $\frac{3x-2}{4} = 4$
g $\frac{x}{2} - 5 = 3$ **h** $\frac{3x}{2} + 2 = 8$ **i** $\frac{2x}{3} - 2 = -8$
j $5 - \frac{x}{2} = 1$ **k** $4 - \frac{2x}{3} = 0$ **l** $5 - \frac{4x}{7} = 9$
m $\frac{x+1}{3} + 2 = 9$ **n** $\frac{x-3}{2} - 4 = 2$ **o** $4 + \frac{x-5}{2} = -3$

7 For each of the following statements, write an equation and solve it to find x.

a When 3 is added to x, the result is 7.
b When x is added to 8, the result is 5.
c When 4 is subtracted from x, the result is 5.
d When x is subtracted from 15, the result is 22.
e Twice the value of x is added to 5 and the result is 13.
f 5 less than x doubled is -15.
g 8 added to 3 times x is 23.
h 5 less than twice x is 3 less than x.

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PROBLEM-SOLVING AND REASONING	8–10, 15	10–12, 15, 16	12–16

Example 11b

8 Solve the following equations that involve algebraic fractions.

a $\frac{2x + 12}{7} = \frac{3x + 5}{4}$ **b** $\frac{5x - 4}{4} = \frac{x - 5}{5}$ **c** $\frac{3x - 5}{4} = \frac{2x - 8}{3}$

d $\frac{1 - x}{5} = \frac{2 - x}{3}$ **e** $\frac{6 - 2x}{3} = \frac{5x - 1}{4}$ **f** $\frac{10 - x}{2} = \frac{x + 1}{3}$

g $\frac{2(x + 1)}{3} = \frac{3(2x - 1)}{2}$ **h** $\frac{-2(x - 1)}{3} = \frac{2 - x}{4}$ **i** $\frac{3(6 - x)}{2} = \frac{-2(x + 1)}{5}$

9 Substitute the given values and then solve for the unknown in each of the following common formulas.

a $v = u + at$ Solve for a, given that $v = 6$, $u = 2$ and $t = 4$.

b $s = ut + \frac{1}{2}at^2$ Solve for u, given that $s = 20$, $t = 2$ and $a = 4$.

c $A = h\left(\frac{a + b}{2}\right)$ Solve for b, given that $A = 10$, $h = 4$ and $a = 3$.

d $A = P\left(1 + \frac{r}{100}\right)$ Solve for r, given that $A = 1000$ and $P = 800$.

10 The perimeter of a square is 68 cm. Determine its side length.

11 The sum of two consecutive numbers is 35. What are the numbers?

12 I ride four times faster than I walk. If a trip took me 45 minutes and I spent 15 of these minutes walking 3 km, how far did I ride?

13 A service technician charges \$30 up front and \$46 for each hour that he works.

a What will a 4-hour job cost?

b If the technician works on a job for 2 days and averages 6 hours per day, what will be the overall cost?

c Find how many hours he worked if the cost is:

i \$76

ii \$513

iii \$1000 (round to the nearest half hour)

14 The capacity of a petrol tank is 80 litres. If it initially contains 5 litres and a petrol pump fills it at 3 litres per 10 seconds, find:

a the amount of fuel in the tank after 2 minutes

b how long it will take to fill the tank to 32 litres

c how long it will take to fill the tank

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15 Solve $2(x - 5) = 8$ using the following two methods and then decide which method you prefer. Give a reason.

a Method 1: First expand the brackets.
b Method 2: First divide both sides by 2.

16 A family of equations can be represented by other pronumerals (sometimes called parameters). For example, the solution to the family of equations $2x - a = 4$ is $x = \frac{4 + a}{2}$.

Find the solution for x in these equation families.

a $x + a = 5$
b $6x + 2a = 3a$
c $ax + 2 = 7$
d $ax - 1 = 2a$
e $\frac{ax - 1}{3} = a$
f $ax + b = c$

ENRICHMENT	–	–	17, 18

More complex equations

17 Solve the following equations by multiplying both sides by the LCD.

a $\frac{x}{2} + \frac{2x}{3} = 7$
b $\frac{x}{4} + \frac{3x}{3} = 5$
c $\frac{3x}{5} - \frac{2x}{3} = 1$
d $\frac{2x}{5} - \frac{x}{4} = 3$
e $\frac{x - 1}{2} + \frac{x + 2}{5} = 2$
f $\frac{x + 3}{3} + \frac{x - 4}{2} = 4$
g $\frac{x + 2}{3} - \frac{x - 1}{2} = 1$
h $\frac{x - 4}{5} - \frac{x + 2}{3} = 2$
i $\frac{7 - 2x}{3} - \frac{6 - x}{2} = 1$

18 Make a the subject in these equations.

a $a(b + 1) = c$
b $ab + a = b$
c $\frac{1}{a} + b = c$
d $\frac{a + b}{a - b} = 2$
e $\frac{1}{a} + \frac{1}{b} = 0$
f $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$

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5D Linear inequalities

Interactive
Widgets
HOTsheets
Walkthrough

There are many situations in which a solution to the problem is best described using one of the symbols $<, \leqslant, >$ or $\geqslant$. For example, a pharmaceutical company might determine the smallest number of packets of a particular drug that need to be sold so that the product is financially viable.

The safe dosage range of a drug can be expressed as an inequality.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

An inequality is a mathematical statement that uses *is less than* ($<$), *is less than or equal to* ($\leqslant$), *is greater than* ($>$) or *is greater than or equal to* ($\geqslant$) symbol. Inequalities may result in an infinite number of solutions and these can be illustrated using a number line.

Let's start: What does it mean for x?

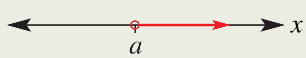
The following inequalities provide some information about the number x.

a $2 \geqslant x$ **b** $-2x < 4$ **c** $3 - x \leqslant -1$

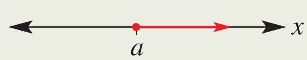
- Can you describe the possible values for x that satisfy each inequality?
- Test some values to check.
- How would you write the solution for x? Illustrate this on a number line.

Key ideas

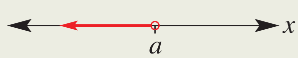
- The four **inequality signs** are $<, \leqslant, >$ and $\geqslant$.
  - An open circle is used for $<$ and $>$ where the end point is not included.
  - A closed circle is used for $\leqslant$ and $\geqslant$ where the end point is included.
  - $x > a$ means x is greater than a.



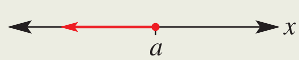
  - $x \geqslant a$ means x is greater than or equal to a.



  - $x < a$ means x is less than a.



  - $x \leqslant a$ means x is less than or equal to a.



  - Also $a < x \leqslant b$ could be illustrated as shown.

- Solving **linear inequalities** follows the same rules as solving linear equations, except:
  - We reverse the inequality sign if we multiply or divide by a negative number.
    For example: If $-5 < -3$ then $5 > 3$ and if $-2x < 4$ then $x > -2$.
  - We reverse the inequality sign if the sides are switched.
    For example: If $2 \geqslant x$, then $x \leqslant 2$.

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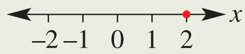
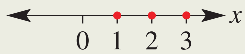
Key ideas

- A finite set of discrete numbers can be shown on a number line as follows:

$x = 2$

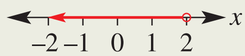


$x = 1, 2, 3$



- An infinite and continuous set of numbers can be shown on a number line as follows:

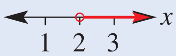
$x < 2$



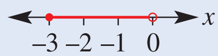
Example 12 Writing inequalities from number lines

Write as an inequality.

a



b



SOLUTION

a $x > 2$

b $-3 \leqslant x < 0$

EXPLANATION

An open circle means 2 is not included.

-3 is included but 0 is not.

Example 13 Solving linear inequalities

Solve the following inequalities and graph their solutions on a number line.

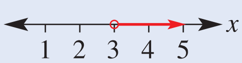
a $3x + 4 > 13$ **b** $4 - \dfrac{x}{3} \leqslant 6$ **c** $3x + 2 > 6x - 4$

SOLUTION

a $3x + 4 > 13$

$3x > 9$

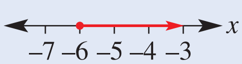
$\therefore x > 3$



b $4 - \dfrac{x}{3} \leqslant 6$

$-\dfrac{x}{3} \leqslant 2$

$\therefore x \geqslant -6$



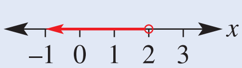
c $3x + 2 > 6x - 4$

$2 > 3x - 4$

$6 > 3x$

$2 > x$

$\therefore x < 2$



EXPLANATION

a Subtract 4 from both sides and then divide both sides by 3.

Use an open circle since x does not include 3.

b Subtract 4 from both sides.

Multiply both sides by -3 and reverse the inequality sign.

Use a closed circle since x includes the number -6.

c Subtract $3x$ from both sides to gather the terms containing x.

Add 4 to both sides and then divide both sides by 3.

Make x the subject. Switching sides means the inequality sign must be reversed.

Use an open circle since x does not include 2.

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Exercise 5D

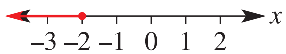
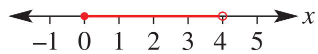
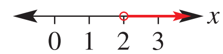
UNDERSTANDING AND FLUENCY 1–6 4–7 5–7

1 Write down three numbers which satisfy each of these inequalities.

a $x \geqslant 3$ **b** $x < -1.5$ **c** $0 < x \leqslant 7$ **d** $-8.7 \leqslant x < -8.1$

2 Match the graph with the inequality.

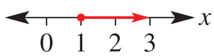
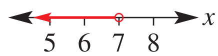
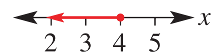
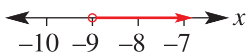
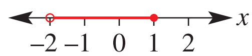
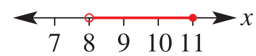
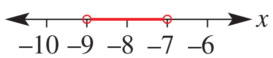
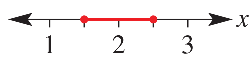
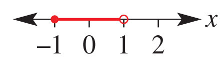
A $x > 2$ **B** $x \leqslant -2$ **C** $0 \leqslant x < 4$

a 
b 
c 

3 Phil owns x rabbits. If $10 < x \leqslant 13$, how many rabbits could Phil own?

Example 12

4 Write each of the following as an inequality.

a 
b 
c 
d 
e 
f 
g 
h 
i 

Example 13a

5 Solve the following inequalities and graph their solutions on a number line.

a $2x + 6 < 14$ **b** $3x + 5 \geqslant 20$ **c** $4x - 7 \geqslant 9$

d $\frac{x}{5} \leqslant 2$ **e** $\frac{x+4}{3} \leqslant 2$ **f** $\frac{5x-3}{2} > 6$

g $\frac{2x+3}{5} > 3$ **h** $\frac{x}{3} + 4 \leqslant 6$ **i** $\frac{x}{9} + 6 < 4$

j $-3 + \frac{x}{4} > 5$ **k** $3(3x - 1) \leqslant 7$ **l** $2(4x + 4) < 5$

Example 13b

6 Solve the following inequalities. Remember: If you multiply or divide by a negative number, you must reverse the inequality sign.

a $-5x + 7 \leqslant 9$ **b** $4 - 3x > -2$ **c** $-5x - 7 \geqslant 18$ **d** $\frac{3-x}{2} \geqslant 5$

e $\frac{5-2x}{3} > 7$ **f** $\frac{4-6x}{5} \leqslant -4$ **g** $3 - \frac{x}{2} \leqslant 8$ **h** $-\frac{x}{3} - 5 > 2$

Example 13c

7 Solve the following inequalities.

a $x + 1 < 2x - 5$ **b** $5x + 2 \geqslant 8x - 4$ **c** $7 - x > 2 + x$

d $3(x + 2) \leqslant 4(x - 1)$ **e** $7(1 - x) \geqslant 3(2 + 3x)$ **f** $-(2 - 3x) < 5(4 - x)$

PROBLEM-SOLVING AND REASONING 8, 9, 11 9–12 9–13

8 For the following situations, write an inequality and solve it to find the possible values for x.

a 7 more than twice a number x is less than 12.

b Half of a number x subtracted from 4 is greater than or equal to -2.

c The product of 3 and one more than a number x is at least 2.

d The sum of two consecutive even integers, of which the smaller is x, is no more than 24.

e The sum of four consecutive even integers, of which x is the largest, is at most 148.

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9 The cost of a mobile phone call is 30 cents plus 20 cents per minute.

a Find the possible cost of a call if it is:

i shorter than 5 minutes

ii longer than 10 minutes

b For how many minutes can the phone be used if the cost per call is:

i less than \$2.10?

ii greater than or equal to \$3.50?

10 Solve these inequalities by first multiplying by the LCD.

a $\frac{1+x}{2} < \frac{x-1}{3}$

b $\frac{2x-3}{2} \geqslant \frac{x+1}{3}$

c $\frac{3-2x}{5} \leqslant \frac{5x-1}{2}$

d $\frac{x}{2} \leqslant \frac{7-x}{3}$

e $\frac{5x}{3} \geqslant \frac{3(3-x)}{4}$

f $\frac{2(4-3x)}{5} > \frac{2(1+x)}{3}$

11 How many whole numbers satisfy these inequalities? Give a reason.

a $x > 8$

b $2 < x \leqslant 3$

12 Solve these families of inequalities by writing x in terms of a.

a $10x - 1 \geqslant a + 2$

b $\frac{2-x}{a} > 4$

c $a(1-x) > 7$

13 Describe the sets (i.e. in a form like $2 < x \leqslant 3$ or $-1 \leqslant x < 5$) that simultaneously satisfy these pairs of inequalities.

a $x < 5$
$x \geqslant -4$

b $x \leqslant -7$
$x > -9.5$

c $x \leqslant 10$
$x \geqslant 10$

ENRICHMENT	–	–	14, 15

Mixed inequalities

14 Solve the inequalities and graph their solutions on a number line. Consider this example first.

Solve $-2 \leqslant x - 3 \leqslant 6$.

$1 \leqslant x \leqslant 9$ (Add 3 to both sides.)

a $1 \leqslant x - 2 \leqslant 7$

b $-4 \leqslant x + 3 \leqslant 6$

c $-2 \leqslant x + 7 < 0$

d $0 \leqslant 2x + 3 \leqslant 7$

e $-5 \leqslant 3x + 4 \leqslant 11$

f $-16 \leqslant 3x - 4 \leqslant -10$

g $7 \leqslant 7x - 70 \leqslant 14$

h $-2.5 < 5 - 2x \leqslant 3$

15 Solve these inequalities for x.

a $\frac{3-x}{5} + \frac{1+x}{4} \geqslant 2$

b $\frac{2x-1}{3} - \frac{x+1}{4} < 1$

c $\frac{7x-1}{6} - \frac{2x-1}{2} \leqslant \frac{1}{2}$

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5E Graphing straight lines

In two dimensions, a straight-line graph can be described by a linear equation. Common forms of such equations are $y = mx + b$ and $ax + by = d$, where a, b, d and m are constants. From a linear equation a graph can be drawn by considering such features as x- and y-intercepts and the gradient.

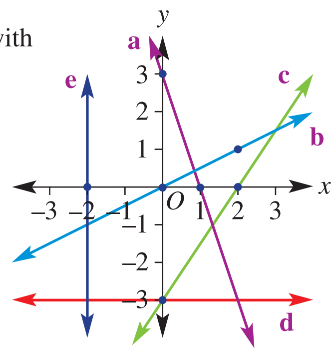
For any given straight-line graph the y value changes by a fixed amount for each 1 unit change in the x value. This change in y tells us the gradient of the line.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Five graphs, five equations

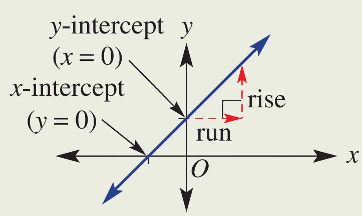
Here are five equations and five graphs. Match each equation with its graph. Give reasons for each of your choices.

- $y = -3$
- $x = -2$
- $y = \frac{1}{2}x$
- $y = -3x + 3$
- $3x - 2y = 6$



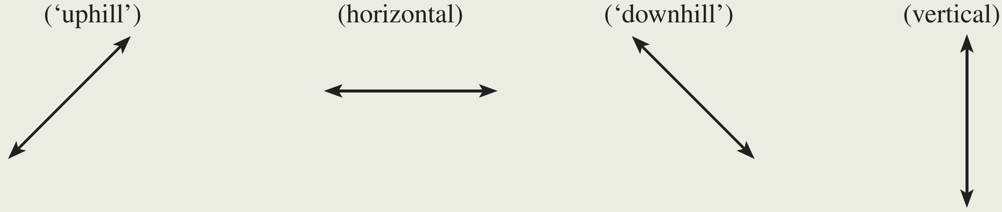
Key ideas

- The **gradient**, m, is a number that describes the slope of a line.
  - Gradient $= \frac{\text{rise}}{\text{run}}$



  - The gradient is the change in y per 1 unit change in x. Gradient is also referred to as the 'rate of change of y'.
- Gradient can be positive, zero, negative or undefined.

positive gradient ('uphill') | zero gradient (horizontal) | negative gradient ('downhill') | undefined (vertical)



- The **intercepts** are the points where the line crosses the x- and the y-axis.
  - The y-intercept is where $x = 0$.
  - The x-intercept is where $y = 0$.

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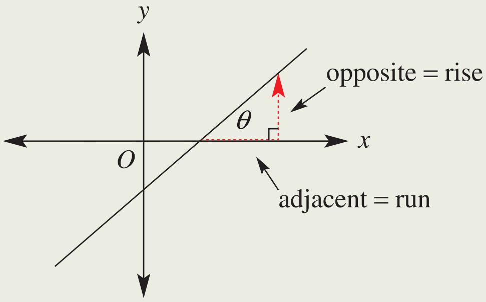
Key ideas

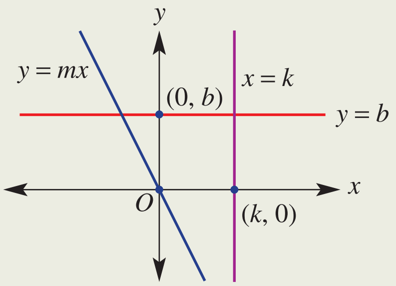
- Gradient can also be calculated using the formula $m = \tan\ \theta$, where θ is the angle between the line and the positive x-axis. This is explained in the Investigation at the end of this chapter and also in Section 7B.



$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{rise}}{\text{run}} = \text{gradient} = m$$

$$\therefore m = \tan\theta$$

- The gradient–intercept form of a straight line is $y = mx + b$, where m is the gradient and b is the y-coordinate of the y-intercept.
  - For example: $y = 3x + 1$ is a straight line with a gradient of 3 and y-intercept at 1. Sometimes this is written using function rotation: $f(x) = 3x + 1$ (This is explained later in Chapter 9.)
- Special lines include those with only one intercept:
  - horizontal lines $y = b$
  - vertical lines $x = k$
  - lines passing through the origin $y = mx$



Example 14 Deciding if a point is on a line

Does the point $(-2, 7)$ lie on these lines?

a $y = -3x + 1$ **b** $2x + 2y = 1$

SOLUTION

a $y = -3x + 1$

$$\begin{aligned} \text{LHS} &= y & \text{RHS} &= -3x + 1 \\ &= 7 & &= -3 \times (-2) + 1 \\ & & &= 7 = \text{LHS} \end{aligned}$$

$\therefore (-2, 7)$ is on the line.

EXPLANATION

Substitute $x = -2$ and $y = 7$ into the equation of the line.
If the equation is true (i.e. LHS = RHS), then the point is on the line.

SOLUTION

b $2x + 2y = 1$

$$\begin{aligned} \text{LHS} &= 2x + 2y & \text{RHS} &= 1 \\ &= 2 \times (-2) + 2 \times 7 & &\neq \text{LHS} \\ &= 10 \end{aligned}$$

$\therefore (-2, 7)$ is not on the line.

EXPLANATION

By substituting the point we find that the equation is false, so the point is not on the line.

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Example 15 Sketching linear graphs using the gradient–intercept method

Find the gradient and y-intercept for these linear relations and sketch each graph.

a $y = 2x - 1$

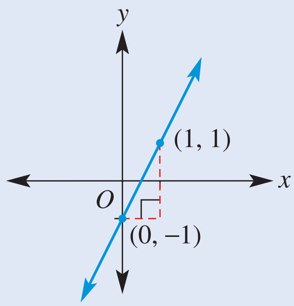
b $2x + 3y = 3$

SOLUTION

a $y = 2x - 1$

Gradient $= 2$

y-intercept $= -1$



EXPLANATION

In the form $y = mx + b$, the gradient is m (the coefficient of x) and b is the y-intercept.

Start by plotting the y-intercept at $(0, -1)$ on the graph.

Gradient $= 2 = \frac{2}{1}$, thus rise $= 2$ and run $= 1$.

From the y-intercept, move 1 unit right (run) and 2 units up (rise) to the point $(1, 1)$. Join the two points with a line.

SOLUTION

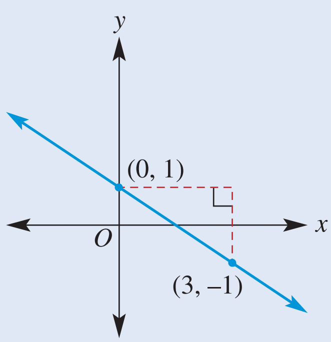
b $2x + 3y = 3$

$$3y = -2x + 3$$

$$y = -\frac{2}{3}x + 1$$

Gradient $= -\frac{2}{3}$

y-intercept $= 1$



EXPLANATION

Rewrite in the form $y = mx + b$ by subtracting $2x$ from both sides and then dividing both sides by 3. The gradient is the coefficient of x and the y-intercept is the constant term.

Start the graph by plotting the y-intercept at $(0, 1)$.

Gradient $= -\frac{2}{3}$ (run $= 3$ and fall $= 2$). From the point $(0, 1)$, move 3 units right (run) and 2 units down (fall) to $(3, -1)$.

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Example 16 Finding x- and y-intercepts of linear graphs

Sketch the following by finding the x- and y-intercepts.

a $y = 2x - 8$

b $-3x - 2y = 6$

SOLUTION

a $y = 2x - 8$

y-intercept (when $x = 0$):

$y = 2(0) - 8$

$y = -8$

The y-intercept is -8.

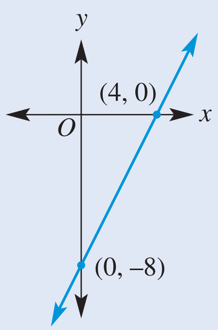
x-intercept (when $y = 0$):

$0 = 2x - 8$

$8 = 2x$

$x = 4$

The x-intercept is 4.



b $-3x - 2y = 6$

y-intercept (when $x = 0$):

$-3(0) - 2y = 6$

$-2y = 6$

$y = -3$

The y-intercept is -3.

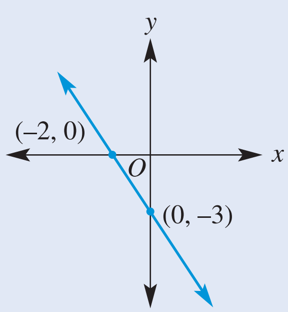
x-intercept (when $y = 0$):

$-3x - 2(0) = 6$

$-3x = 6$

$x = -2$

The x-intercept is -2.



EXPLANATION

a The y-intercept is at $x = 0$. For $y = mx + b$, b is the y-intercept.

The x-intercept is on the x-axis, so $y = 0$. Solve the equation for x.

Plot and label the intercepts and join with a straight line.

b The y-intercept is on the y-axis, so substitute $x = 0$. Simplify and solve for y.

The x-intercept is on the x-axis, so substitute $y = 0$. Simplify and solve for x.

Sketch by drawing a line passing through the two axes intercepts. Label the intercepts with their coordinates.

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Example 17 Sketching lines with one intercept

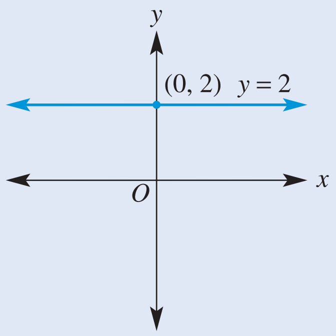
Sketch these special lines.

a $y = 2$ **b** $x = -3$ **c** $y = -\frac{1}{2}x$

SOLUTION

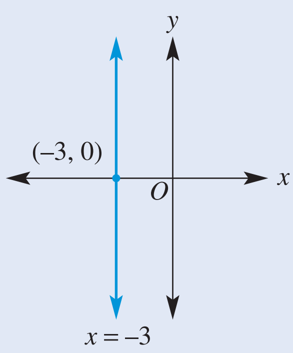
EXPLANATION

a



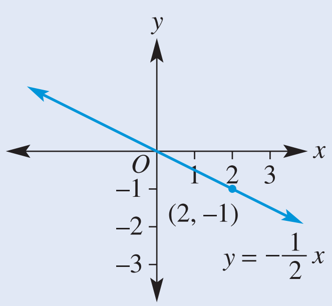
The y-coordinate of every point must be 2; hence, $y = 2$ is a horizontal line passing through $(0, 2)$.

b



The x-coordinate of every point must be -3; hence, $x = -3$ is a vertical line passing through $(-3, 0)$.

c



Both the x- and y-intercepts are $(0, 0)$, so the gradient can be used to find a second point.

The gradient $= -\frac{1}{2}$; hence, use run $= 2$ and fall $= 1$.

Alternatively, substitute $x = 1$ to find a second point.

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Exercise 5E

UNDERSTANDING AND FLUENCY	1–4, 5–8(½)	3, 4–8(½)	5–8(½)

1 Rearrange these equations and write them in the form $y = mx + b$. Then state the gradient (m) and y-intercept (b).

a $y + 2x = 5$ **b** $2y = 4x - 6$ **c** $x - y = 7$ **d** $-2x - 5y = 3$

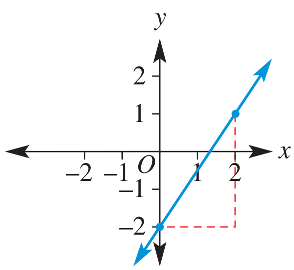
2 The graph of $y = \frac{3}{2}x - 2$ is shown.

a State the rise of the line if the run is:

i 2 **ii** 4 **iii** 7

b State the run of the line if the rise is:

i 3 **ii** 9 **iii** 4

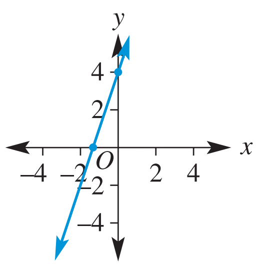


3 Match each of the following equations to one of the graphs shown.

a $y = 3x + 4$ **b** $y = -2x - 4$ **c** $y = 4$

d $y = x$ **e** $y = -2x + 4$ **f** $x = -4$

i



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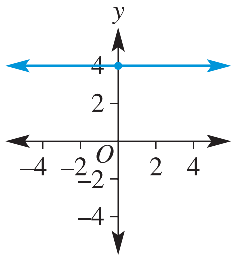

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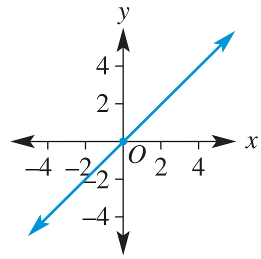

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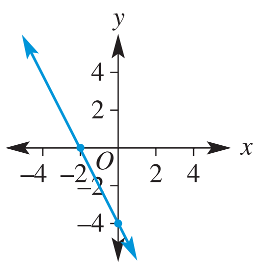

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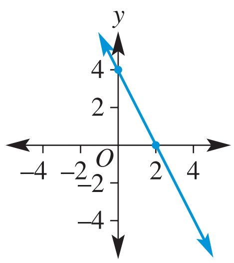

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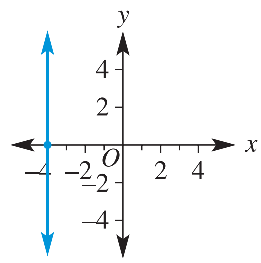

Example 14

4 Does the point $(3, -1)$ lie on these lines?

a $y = x - 4$ **b** $y = -x + 2$ **c** $y = -3x + 9$

d $x + 2y = 6$ **e** $-2x - y = -5$ **f** $3y - 4x = -9$

Example 15a

5 Find the gradient and y-intercept for these lines and sketch a graph.

a $y = 5x - 3$ **b** $y = 2x + 3$ **c** $y = -2x - 1$

d $y = -x + 2$ **e** $y = x - 4$ **f** $y = -\frac{3}{2}x + 1$

g $y = \frac{4}{3}x - 2$ **h** $y = -\frac{7}{2}x + 6$ **i** $y = 0.5x - 0.5$

j $y = 1 - x$ **k** $y = 3 + \frac{2}{3}x$ **l** $y = 0.4 - 0.2x$

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Example 15b

6 Find the gradient and y-intercept for these lines and sketch each graph.

a $3x + y = 12$ **b** $10x + 2y = 5$ **c** $x - y = 7$ **d** $3x - 3y = 6$

e $4x - 3y = 9$ **f** $-x - y = \frac{1}{3}$ **g** $-y - 4x = 8$ **h** $x - 2y = \frac{1}{2}$

Example 16

7 Sketch the following by finding x- and y-intercepts.

a $y = 3x - 6$ **b** $y = 2x + 4$ **c** $y = 4x + 10$ **d** $y = 3x - 4$

e $y = 7 - 2x$ **f** $y = 4 - \frac{x}{2}$ **g** $3x + 2y = 12$ **h** $2x + 5y = 10$

i $4y - 3x = 24$ **j** $x + 2y = 5$ **k** $3x + 4y = 7$ **l** $5y - 2x = 12$

Example 17

8 Sketch these special lines.

a $y = -4$ **b** $y = 1$ **c** $x = 2$ **d** $x = -\frac{5}{2}$

e $y = 0$ **f** $x = 0$ **g** $y = 4x$ **h** $y = -3x$

i $y = -\frac{1}{3}x$ **j** $y = \frac{5x}{2}$ **k** $x + y = 0$ **l** $4 - y = 0$

PROBLEM-SOLVING AND REASONING	9, 10, 14	9–12, 14, 15	10–13, 15, 16

9 Sam is earning some money picking apples. She gets paid \$10 plus \$2 per kilogram of apples that she picks. If Sam earns \$$C$ for n kg of apples picked, complete the following.

a Write a rule for C in terms of n.

b Sketch a graph for $0 \leqslant n \leqslant 10$, labelling the end points.

c Use your rule to find:

i the amount Sam earned after picking 9 kg of apples

ii the number of kilograms of apples Sam picked if she earned \$57

10 A 90 litre tank full of water begins to leak at a rate of 1.5 litres per hour. If V litres is the volume of water in the tank after t hours, complete the following.

a Write a rule for V in terms of t.

b Sketch a graph for $0 \leqslant t \leqslant 60$, labelling the end points.

c Use your rule to find:

i the volume of water after 5 hours

ii the time taken to empty the tank completely

11 Alex earns \$84 for 12 hours of work.

a Write her rate of pay per hour.

b Write the equation for Alex's total pay, P, after t hours of work.

12 It costs Jack \$1600 to maintain and drive his car for 32 000 km.

a Write the rate of cost in \$ per km.

b Write a formula for the cost, \$$C$, of driving Jack's car for k kilometres.

c If Jack also pays a total of \$1200 for registration and insurance, write the new formula for the cost to Jack of owning and driving his car for k kilometres.

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13 $D = 25t + 30$ is an equation for calculating the distance, D km, from home that a cyclist has travelled after t hours.

- **a** What is the gradient of the graph of the given equation?
- **b** What could the 30 represent?
- **c** If a graph of D against t was drawn, what would be the intercept on the D-axis?

14 A student with a poor understanding of straight-line graphs writes down some incorrect information next to each equation. Decide how the error might have been made and then correct the information.

- **a** $y = \dfrac{2x + 1}{2}$ (gradient = 2)
- **b** $y = 0.5(x + 3)$ (y-intercept = 3)
- **c** $3x + y = 7$ (gradient = 3)
- **d** $x - 2y = 4$ (gradient = 1)

15 Write expressions for the gradient and y-intercept of these equations.

- **a** $ay = 3x + 7$
- **b** $ax - y = b$
- **c** $by = 3 - ax$

16 A straight line is written in the form $ax + by = d$. In terms of a, b and d, find:

- **a** the x-intercept
- **b** the y-intercept
- **c** the gradient

ENRICHMENT	–	–	17

Graphical areas

17 Find the area enclosed by these lines.

- **a** $x = 2, x = -1, y = 0, y = 4$
- **b** $x = 3, y = 2x, y = 0$
- **c** $x = -3, y = -\dfrac{1}{2}x + 2, y = -2$
- **d** $2x - 5y = -10, y = -2, x = 1$
- **e** $y = 3x - 2, y = -3, y = 2 - x$

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5F Finding the equation of a line

It is a common procedure in mathematics to find the equation (or rule) of a straight line. Once the equation of a line is determined, it can be used to find the exact coordinates of other points on the line. Mathematics such as this can be used, for example, to predict a future company share price or the water level in a dam after a period of time.

The level of the water in the dam can be modelled with a linear equation.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Fancy formula

Here is a proof of a rule for the equation of a straight line between any two given points. Some of the steps are missing. See if you can fill them in.

$$y = mx + b$$

$$y_1 = mx_1 + b \qquad \text{(i.e. substitute } (x, y) = (x_1, y_1))$$

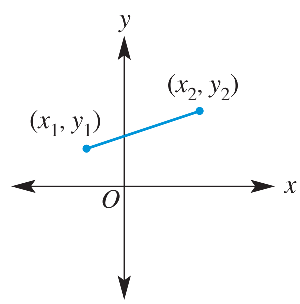
$$\therefore b = \_\_\_\_\_\_\_\_$$

$$\therefore y = mx + \_\_\_\_\_\_\_\_$$

$$\therefore y - y_1 = m(\_\_\_\_\_\_)$$

$$\text{Now } m = \frac{\boxed{}}{x_2 - x_1}$$

$$\therefore y - y_1 = \_\_\_\_\_\_\_\_$$



Key ideas

- Horizontal lines have the equation $y = b$, where b is the y-intercept.
- Vertical lines have the equation $x = k$, where k is the x-intercept.
- Given the gradient (m) and the y-intercept (b), use $y = mx + b$ to state the equation of the line.
- To find the equation of a line when given any two points, find the gradient (m) and then:
  - Substitute a point to find b in $y = mx + b$, or
  - Use $y - y_1 = m(x - x_1)$, where $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and (x_1, y_1), (x_2, y_2) are points on the line.
- If A, B and C are points on the Cartesian plane and the gradient of AB is equal to the gradient of BC, then the points A, B and C lie on a single straight line; i.e. If $m_{AB} = m_{BC}$, then A, B and C are **collinear**.

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Example 18 Finding the gradient of a line joining two points

Determine the gradient of the line joining the pair of points (−3, 8) and (5, −2).

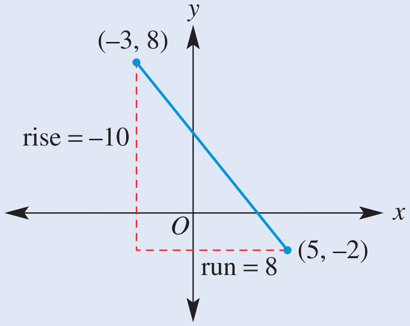
SOLUTION

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-2 - 8}{5 - (-3)}$$
$$= -\frac{10}{8}$$
$$= -\frac{5}{4}$$

EXPLANATION

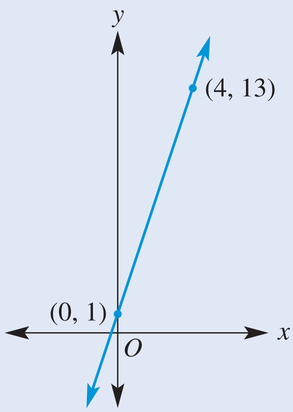
Use $(x_1, y_1) = (-3, 8)$ and $(x_2, y_2) = (5, -2)$.
Remember that $5 - (-3) = 5 + 3$.
Alternatively, draw a graph to help find the rise and run between the two points.

$$m = \frac{\text{rise}}{\text{run}}$$



Example 19 Finding the equation of a line when given the *y*-intercept and a point

Find the equation of the straight line shown.



SOLUTION

$$y = mx + b$$
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{13 - 1}{4 - 0}$$
$$= \frac{12}{4}$$
$$= 3$$

and $b = 1$

$\therefore y = 3x + 1$

EXPLANATION

The equation of a straight line is of the form $y = mx + b$.
Find m, using $(x_1, y_1) = (0, 1)$ and $(x_2, y_2) = (4, 13)$.

The y-intercept is given.
Substitute $m = 3$ and $b = 1$ into $y = mx + b$.

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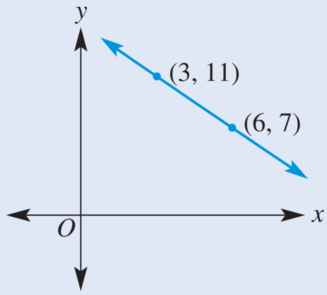
Example 20 Finding the equation of a line when given two points

Find the equation of the straight line shown.



SOLUTION

Method 1

$y = mx + b$

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{7 - 11}{6 - 3}$

$= -\dfrac{4}{3}$

$y = -\dfrac{4}{3}x + b$

$7 = -\dfrac{4}{3} \times (6) + b$

$7 = -8 + b$

$15 = b$

$\therefore\ y = -\dfrac{4}{3}x + 15$

Method 2

$y - y_1 = m(x - x_1)$

$y - 11 = -\dfrac{4}{3}(x - 3)$

$y = -\dfrac{4}{3}x + 4 + 11$

$= -\dfrac{4}{3}x + 15$

EXPLANATION

Substitute $(x_1, y_1) = (3, 11)$ and $(x_2, y_2) = (6, 7)$ into the gradient formula.

Substitute $m = -\dfrac{4}{3}$ into $y = mx + b$.

Substitute the point (6, 7) or (3, 11) to find the value of b.

Write the rule with both m and b.

Choose $(x_1, y_1) = (3, 11)$ or, alternatively, choose (6, 7).

$m = -\dfrac{4}{3}$ was found using method 1.

Expand brackets and make y the subject.

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Exercise 5F

UNDERSTANDING AND FLUENCY	1–6	3–6	4–7

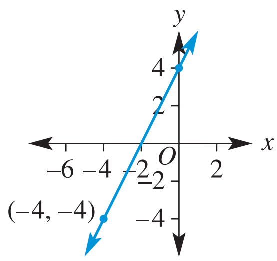
1 Use the formula $y - y_1 = m(x - x_1)$ to find the equation of each line, given one point and the gradient.

a $(x_1, y_1) = (3, 2)$ and $m = 4$ **b** $(x_1, y_1) = (4, -1)$ and $m = -1$

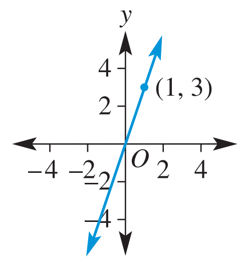
c $(x_1, y_1) = (-3, -4)$ and $m = -1$ **d** $(x_1, y_1) = (-7, 2)$ and $m = \frac{1}{2}$

2 Calculate the gradient of the following lines.

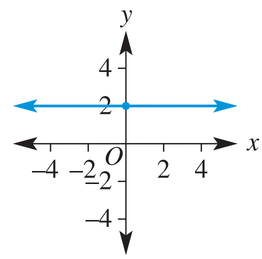
a



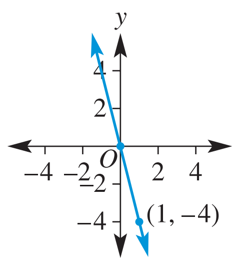
b



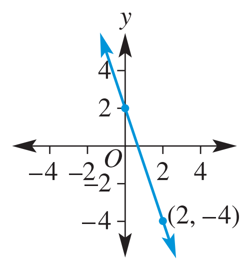
c



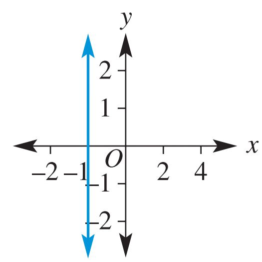
d



e



f



Example 18 **3** Determine the gradient of the line joining the following pairs of points.

a (4, 2), (12, 4) **b** (1, 4), (3, 8) **c** (0, 2), (2, 7)

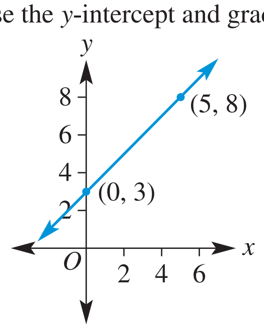
d (3, 4), (6, 13) **e** (8, 4), (5, 4) **f** (2, 7), (4, 7)

g (−1, 3), (2, 0) **h** (−3, 2), (−1, 7) **i** (−3, 4), (4, −3)

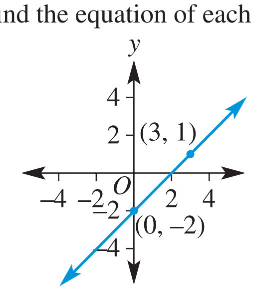
j (2, −3), (2, −5) **k** (2, −3), (−4, −12) **l** (−2, −5), (−4, −2)

Example 19 **4** Use the y-intercept and gradient to find the equation of each line.

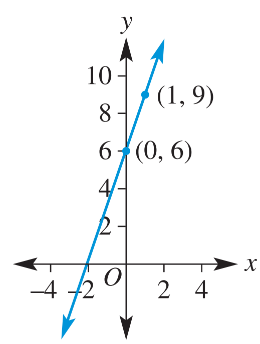
a



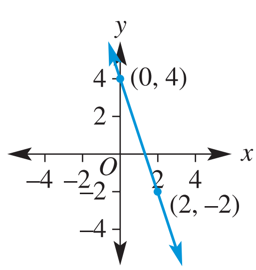
b



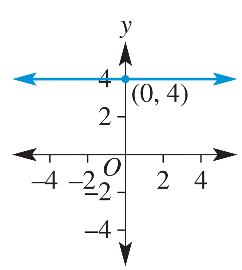
c



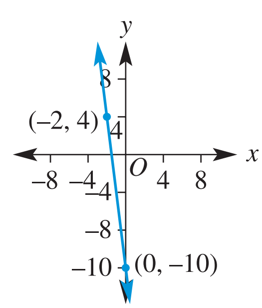
d



e



f



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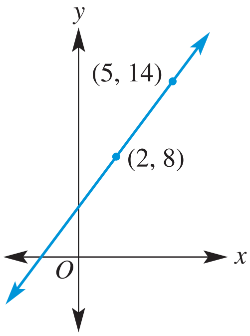
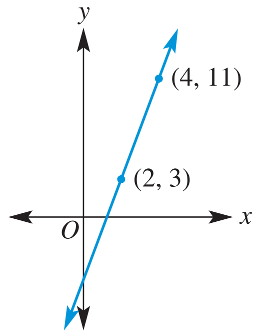
Example 20

5 Find the equation of the straight lines with the given points.

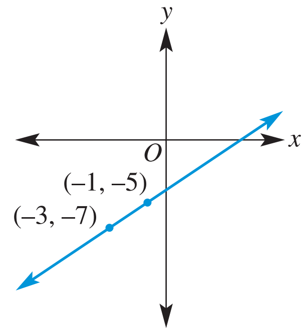
a



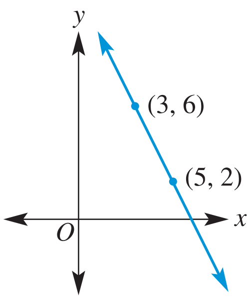
b



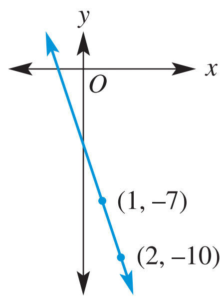
c



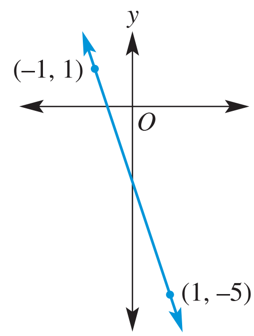
d



e



f



6 Given the following tables of values, determine the linear equation relating x and y in each case.

a

x	0	3
y	5	14

b

x	4	6
y	–4	–8

c

x	–1	3
y	–2	0

d

x	–2	1
y	2	–4

7 Consider the points $A(-3, -1)$, $B(0, 1)$ and $C(6, 5)$.

a Calculate the gradient of AB.

b Calculate the gradient of BC.

c Is $m_{AB} = m_{BC}$? What does that tell you about A, B and C?

PROBLEM-SOLVING AND REASONING	8, 9, 11	8–11	8–12

8 Randy invests some money in a simple savings fund and the amount increases at a constant rate over time. He hopes to buy a boat when the investment amount reaches \$20 000. After 3 years the amount is \$16 500 and after 6 years the amount is \$18 000.

a Find a rule linking the investment amount (\$$A$) and time ($t$ years).

b How much did Randy invest initially (i.e. when $t = 0$)?

c How long does Randy have to wait before he buys his boat?

d What would be the value of the investment after $12\frac{1}{2}$ years?

9 The cost of hiring a surfboard involves an up-front fee plus an hourly rate. Three hours of hire costs \$50 and 7 hours costs \$90.

a Sketch a graph of cost, \$$C$, for t hours of hire using the information given above.

b Find a rule linking the cost, \$$C$, in terms of t hours.

c **i** State the cost per hour. **ii** State the up-front fee.

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10 **a** The following information applies to the filling of a container with water at a constant rate. In each case, find a rule for the volume, V litres, in terms of t minutes.

i Initially (i.e. at $t = 0$), the container is empty ($V = 0$) and after 1 minute it contains 4 litres of water.

ii Initially (i.e. at $t = 0$), the container is empty ($V = 0$) and after 3 minutes it contains 9 litres of water.

iii After 1 and 2 minutes the container has 2 and 3 litres of water, respectively.

iv After 1 and 2 minutes the container has 3.5 and 5 litres of water, respectively.

b For parts **iii** and **iv** above, find how much water was in the container initially.

c Write your own information that would give the rule $V = -t + b$.

11 A line joins the two points $(-1, 3)$ and $(4, -2)$.

a Calculate the gradient of the line, using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and $(x_1, y_1) = (-1, 3)$ and $(x_2, y_2) = (4, -2)$.

b Calculate the gradient of the line, using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and $(x_1, y_1) = (4, -2)$ and $(x_2, y_2) = (-1, 3)$.

c What conclusions can you draw from your results from parts **a** and **b** above?

d Explain why the formula $m = \dfrac{y_1 - y_2}{x_1 - x_2}$ gives the same value for gradient as $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

12 A line passes through the points $(1, 3)$ and $(4, -1)$.

a Calculate the gradient.

b Using $y - y_1 = m(x - x_1)$ and $(x_1, y_1) = (1, 3)$, find the rule for the line.

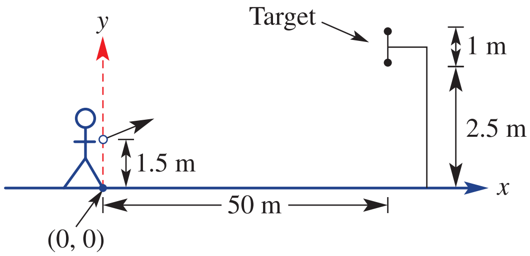
c Using $y - y_1 = m(x - x_1)$ and $(x_1, y_1) = (4, -1)$, find the rule for the line.

d What do you notice about your results from parts **b** and **c**? Can you explain why this is the case?

ENRICHMENT	–	–	13

Linear archery

13 An archer's target is 50 m away and is 2.5 m off the ground, as shown. Arrows are fired from a height of 1.5 m and the circular target has a diameter of 1 m.



a Find the gradient of the straight trajectory from the arrow (in firing position) to:

i the bottom of the target

ii the top of the target

b If the position of the ground directly below the firing arrow is the point $(0, 0)$ on a Cartesian plane, find the equation of the straight trajectory to:

i the bottom of the target

ii the top of the target

c If $y = mx + b$ is the equation of the arrow's trajectory, what are the possible values of m if the arrow is to hit the target?

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5G Using formulas for distance and midpoint

Interactive
Widgets
HOTsheets
Walkthrough

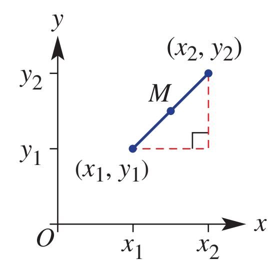
A line segment (also known as an interval) is defined using two end points. The length of a line segment is the distance between those two points. This can be found using Pythagoras' theorem. The midpoint is the point exactly halfway between the two points. This can be determined by finding the average of the x values and the average of the y values.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

There are many applications for finding the distance between two coordinates, especially in mapping and navigation.

Let's start: Developing the rules

The line segment shown has end points (x_1, y_1) and (x_2, y_2).



- Length: Use your knowledge of Pythagoras' theorem to find the rule for the length of the segment.
- Midpoint: State the coordinates of M (i.e. the midpoint) in terms of x_1, y_1, x_2 and y_2. Give reasons for your answer.

Key ideas

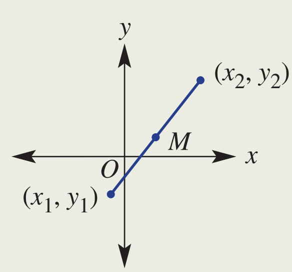
- The **length of a line segment** (or distance between two points (x_1, y_1) and (x_2, y_2)) is given by the formula:

  $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

  This formula is derived in Question 14 of Exercise 5G.

- The **midpoint**, M, of a line segment between (x_1, y_1) and (x_2, y_2) is given by:

  $$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$



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Example 21 Finding the distance between two points

Find the exact distance between each pair of points.

a (0, 2) and (1, 7) **b** (−3, 8) and (4, −1)

SOLUTION

a $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(1 - 0)^2 + (7 - 2)^2}$

$= \sqrt{1^2 + 5^2}$

$= \sqrt{26}$ units

b $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(4 - (-3))^2 + (-1 - 8)^2}$

$= \sqrt{7^2 + (-9)^2}$

$= \sqrt{49 + 81}$

$= \sqrt{130}$ units

EXPLANATION

Let $(x_1, y_1) = (0, 2)$ and $(x_2, y_2) = (1, 7)$.

Simplify and express your answer exactly, using a surd.

Let $(x_1, y_1) = (-3, 8)$ and $(x_2, y_2) = (4, -1)$.

Alternatively, let $(x_1, y_1) = (4, -1)$ and $(x_2, y_2) = (-3, 8)$.

Either way, the answers will be the same.

Example 22 Finding the midpoint of a line segment joining two points

Find the midpoint of the line segment that joins (−3, −5) and (2, 8).

SOLUTION

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$= \left(\frac{-3 + 2}{2}, \frac{-5 + 8}{2}\right)$$

$$= \left(-\frac{1}{2}, \frac{3}{2}\right)$$

EXPLANATION

Let $(x_1, y_1) = (-3, -5)$ and $(x_2, y_2) = (2, 8)$.

This is equivalent to finding the average of the x-coordinates and the average of the y-coordinates of the two points.

Example 23 Using a given distance to find coordinates

Find the values of a if the distance between $(2, a)$ and $(4, 9)$ is $\sqrt{5}$.

SOLUTION

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\sqrt{5} = \sqrt{(4 - 2)^2 + (9 - a)^2}$

$\sqrt{5} = \sqrt{2^2 + (9 - a)^2}$

$5 = 4 + (9 - a)^2$

$1 = (9 - a)^2$

$\pm 1 = 9 - a$

So $9 - a = 1$ or $9 - a = -1$

$\therefore a = 8$ or $a = 10$

EXPLANATION

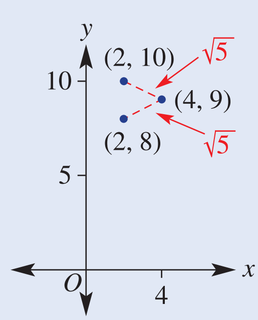
Substitute all the given information into the rule for the distance between two points.

Simplify and then square both sides to eliminate the square roots.

Subtract 4 from both sides and take the square root of each side.

Remember, if $x^2 = 1$ then $x = \pm 1$.

Solve for a. You can see there are two solutions.



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Exercise 5G

UNDERSTANDING AND FLUENCY	1–6	3–6	4–6

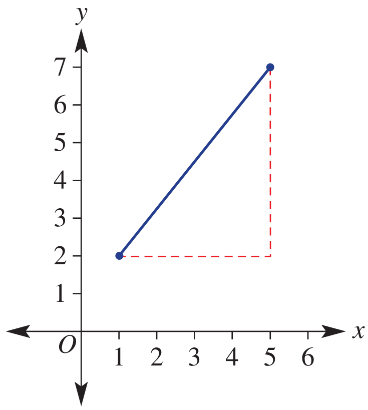
1 The end points for the given line segment are (1, 2) and (5, 7).

a What is the horizontal distance between the two end points?

b What is the vertical distance between the two end points?

c Use Pythagoras' theorem ($c^2 = a^2 + b^2$) to find the exact length of the segment.

d State the midpoint of the segment.



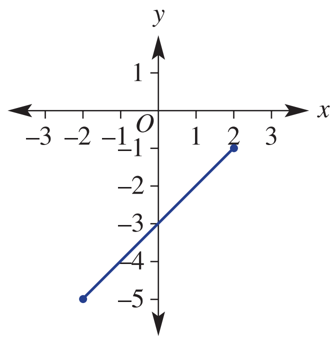
2 The end points for the given line segment are (−2, −5) and (2, −1).

a What is the horizontal distance between the two end points?

b What is the vertical distance between the two end points?

c Use Pythagoras' theorem ($c^2 = a^2 + b^2$) to find the exact length of the segment.

d State the midpoint of the segment.



3 Substitute the given values of x_1, x_2, y_1 and y_2 into the formulas below to find the exact distance between the two points, d, and the coordinates of the midpoint, M.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

a $(x_1, y_1) = (1, 3)$ and $(x_2, y_2) = (3, 7)$

b $(x_1, y_1) = (0, 8)$ and $(x_2, y_2) = (4, -1)$

c $(x_1, y_1) = (-3, 4)$ and $(x_2, y_2) = (1, -1)$

d $(x_1, y_1) = (-4, -1)$ and $(x_2, y_2) = (2, -2)$

Example 21

4 Find the exact distance between these pairs of points.

a (0, 4) and (2, 9) **b** (0, −1) and (3, 6) **c** (−1, 4) and (0, −2)

d (−3, 8) and (1, 1) **e** (−2, −1) and (4, −2) **f** (−8, 9) and (1, −3)

g (−8, −1) and (2, 0) **h** (−4, 6) and (8, −1) **i** (−10, 11) and (−4, 10)

Example 22

5 Find the midpoint of the line segment that joins the points given in Question **4**.

6 Consider the points $P(4, 7)$ and $Q(6, -1)$.

a Use the formula to calculate the distance PQ.

b Find the midpoint of the internal PQ.

c What is the gradient of PQ?

d Find the equation of the line passing through P and Q.

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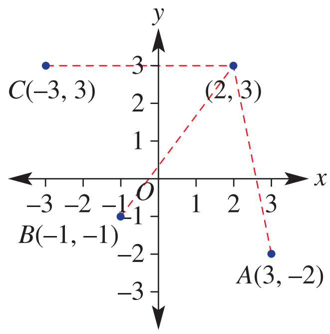
PROBLEM-SOLVING AND REASONING	7, 8, 11	8, 9, 11, 12	9, 10, 12–14

7 Which of the points A, B or C shown on these axes is closest to the point $(2, 3)$?



8 Find the values of a and b when:

a The midpoint of $(a, 3)$ and $(7, b)$ is $(5, 4)$.

b The midpoint of $(a, -1)$ and $(2, b)$ is $(-1, 2)$.

c The midpoint of $(-3, a)$ and $(b, 2)$ is $\left(-\frac{1}{2}, 0\right)$.

d The midpoint of $(-5, a)$ and $(b, -4)$ is $\left(-\frac{3}{2}, \frac{7}{2}\right)$.

Example 23

9 Find the values of a when:

a The distance between $(1, a)$ and $(3, 5)$ is $\sqrt{8}$.

b The distance between $(2, a)$ and $(5, 1)$ is $\sqrt{13}$.

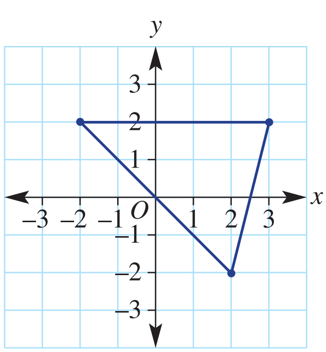
c The distance between $(a, -1)$ and $(4, -3)$ is $\sqrt{29}$.

d The distance between $(-3, -5)$ and $(a, -9)$ is 5.

10 A block of land is illustrated on this simple map, which uses the ratio 1 : 100, meaning that 1 unit represents 100 m.

a Find the perimeter of the block, correct to the nearest metre.

b The block is to be split up into four triangular areas by building three fences that join the three midpoints of the sides of the block. Find the perimeter of the inside triangular area to the nearest metre.



11 A line segment has end points $(-2, 3)$ and $(1, -1)$.

a Find the midpoint, using $(x_1, y_1) = (-2, 3)$ and $(x_2, y_2) = (1, -1)$.

b Find the midpoint, using $(x_1, y_1) = (1, -1)$ and $(x_2, y_2) = (-2, 3)$.

c Give a reason why the answers to parts **a** and **b** are the same.

d Find the length of the segment, using $(x_1, y_1) = (-2, 3)$ and $(x_2, y_2) = (1, -1)$.

e Find the length of the segment, using $(x_1, y_1) = (1, -1)$ and $(x_2, y_2) = (-2, 3)$.

f What do you notice about your answers to parts **d** and **e**? Give an explanation for this.

12 The distance between the points $(-2, -1)$ and $(a, 3)$ is $\sqrt{20}$. Find the values of a and use a Cartesian plane to illustrate why there are two solutions for a.

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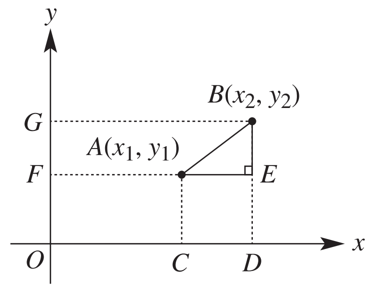
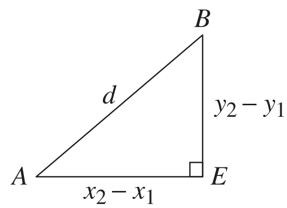
13 Find the coordinates of the point that divides the segment joining $(-2, 0)$ and $(3, 4)$ in the given ratio. Ratios are to be considered from left to right.

a $1:1$ **b** $1:2$ **c** $2:1$ **d** $4:1$ **e** $1:3$ **f** $2:3$

14 Copy into your workbook the diagram shown at right. Consider the points $A(x_1, y_1)$ and $B(x_2, y_2)$.



a Write down the coordinates of C.
b Write down the coordinates of D.
c Write down an expression for the horizontal distance CD.
d Is CD equal in length to AE?
e Write down the coordinates of F.
f Write down the coordinates of G.
g Write down an expression for the vertical distance FG.
h Is BE equal in length to FG?
i Consider $\triangle ABE$, shown at right. Copy and complete the following.



Using Pythagoras' theorem:

$AB^2 = AE^2 + \_\_^2$

$d^2 = (x_2 - x_1)^2 + (\_\_\_\_)^2$

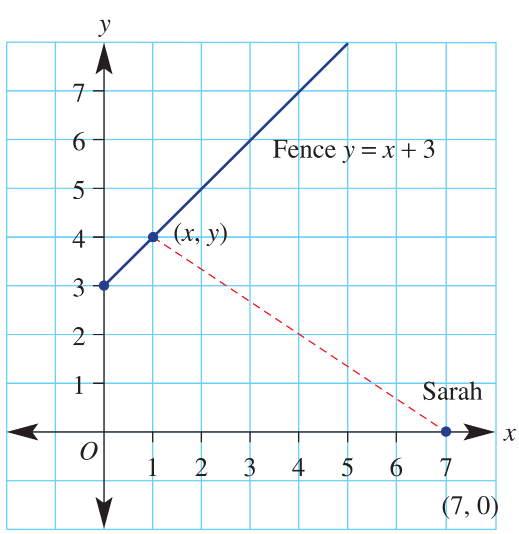
$d = \sqrt{(\_\_\_\_)^2 + (\_\_\_\_)^2}$

j The formula is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Explain why this formula gives the same value as $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

ENRICHMENT	–	–	15

Shortest distance

15 Sarah pinpoints her position on a map as $(7, 0)$ and wishes to hike towards a fence line that follows the path $y = x + 3$, as shown. Note: 1 unit = 100 m.



a Using the points $(7, 0)$ and (x, y), write a rule in terms of x and y for the distance between Sarah and the fence.
b Use the equation of the fence line to write the rule in part **a** in terms of x only.
c Use your rule from part **b** to find the distance between Sarah and the fence line, to the nearest metre, when:

i $x = 1$ **ii** $x = 2$
iii $x = 3$ **iv** $x = 4$

d Which x value from part **c** gives the shortest distance?
e Consider any point on the fence line and find the coordinates of the point such that the distance will be a minimum. Give reasons.

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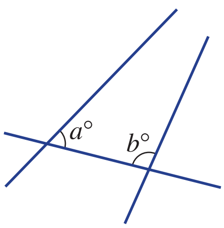
5H Parallel lines and perpendicular lines

Stage

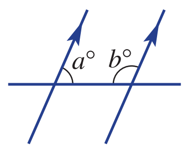
- 5.3#
- 5.3
- 5.3§
- 5.2
- 5.2◊
- 5.1
- 4

Euclid of Alexandria (300 BC) was a Greek mathematician and is known as the 'Father of geometry'. In his texts, known as *Euclid's Elements*, his work is based on five simple axioms. The fifth axiom, called the 'Parallel Postulate', states: 'It is true that if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, intersect on that side on which are the angles less than the two right angles'.

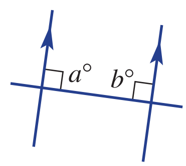
In simple terms, the Parallel Postulate says that if cointerior angles do not sum to 180°, then the two lines are not parallel. Furthermore, if the two interior angles are equal and also sum to 180°, then the third line must be perpendicular to the other two.



$a + b \neq 180$



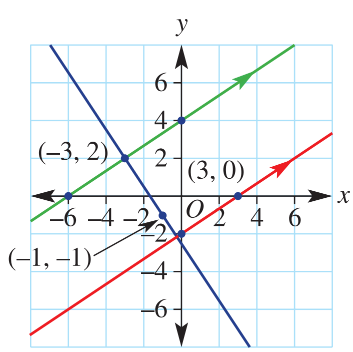
$a + b = 180$



$a = b = 90$

Let's start: Gradient connection

Shown here is a pair of parallel lines and a third line that is perpendicular to the other two lines.



- Find the gradient of each line using the coordinates shown on the graph.
- What is common about the gradients for the two parallel lines?
- Is there any connection between the gradients of the parallel lines and the perpendicular line? Can you write down this connection as a formula?

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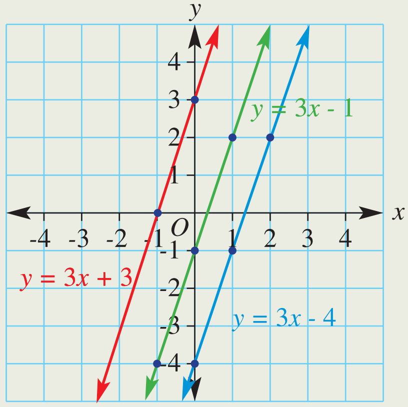
Key ideas

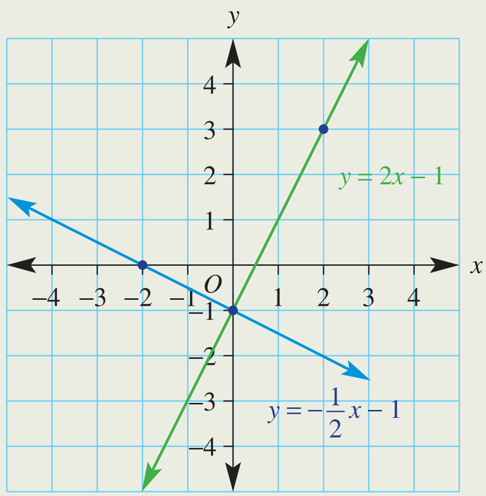
- All **parallel lines** have equal gradient.
  For example: $y = 3x - 1$, $y = 3x + 3$ and $y = 3x - 4$ have the same gradient of 3.



- Two **perpendicular lines** with gradients m_1 and m_2 satisfy the rule:

  $m_1 \times m_2 = -1$ or $m_2 = -\frac{1}{m_1}$ (i.e. m_2 is the negative reciprocal of m_1).

  In the diagram $m_1 \times m_2 = 2 \times \left(-\frac{1}{2}\right) = -1$.



- Equations of parallel or perpendicular lines can be found by:
  - first finding the gradient (m)
  - then substituting a point to find b in $y = mx + b$

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Example 24 Deciding if lines are parallel or perpendicular

Decide if the graph of each pair of rules will be parallel, perpendicular or neither.

a $y = \frac{1}{2}x + 2$ and $2y - x = 5$ **b** $y = -3x - 8$ and $y = \frac{1}{3}x + 1$ **c** $3x + 2y = -5$ and $x - y = 2$

SOLUTION

a $y = \frac{1}{2}x + 2, m = \frac{1}{2}$ (1)

$2y - x = 5$

$2y = x + 5$

$y = \frac{1}{2}x + \frac{5}{2}, m = \frac{1}{2}$ (2)

So the lines are parallel.

b $y = -3x - 8, m = -3$ (1)

$y = \frac{1}{3}x + 1, m = \frac{1}{3}$ (2)

$-3 \times \frac{1}{3} = -1$

So the lines are perpendicular.

c $3x + 2y = -5$

$2y = -3x - 5$

$y = -\frac{3}{2}x - \frac{5}{2}, m = -\frac{3}{2}$ (1)

$x - y = 2$

$-y = -x + 2$

$y = x - 2, m = 1$ (2)

$-\frac{3}{2} \times 1 \neq -1$

So the lines are neither parallel nor perpendicular.

EXPLANATION

a Write both equations in the form $y = mx + b$.

Both lines have a gradient of $\frac{1}{2}$, so the lines are parallel.

b Both equations are in the form $y = mx + b$.

Test: $m_1 \times m_2 = -1$

c Write both equations in the form $y = mx + b$.

The gradients are not equal and $m_1 \times m_2 \neq -1$.

Example 25 Finding the equation of a parallel or perpendicular line

Find the equation of the line that is:

a parallel to $y = -2x - 7$ and passes through $(1, 9)$

b perpendicular to $y = \frac{3}{4}x - 1$ and passes through $(3, -2)$

SOLUTION

a $y = mx + b$

$m = -2$

$y = -2x + b$

Substitute $(1, 9)$: $9 = -2(1) + b$

$11 = b$

$\therefore y = -2x + 11$

EXPLANATION

Since the line is parallel to $y = -2x - 7$, $m = -2$.

Substitute the given point $(1, 9)$ and solve for b.

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b $y = mx + b$

$m = \dfrac{-1}{\left(\frac{3}{4}\right)}$

$= -\dfrac{4}{3}$

$y = -\dfrac{4}{3}x + b$

The gradient is the negative reciprocal of $\dfrac{3}{4}$.

Substitute (3, −2): $-2 = -\dfrac{4}{3}(3) + b$

$-2 = -4 + b$

$b = 2$

$\therefore y = -\dfrac{4}{3}x + 2$

Substitute (3, −2) and solve for b.

Exercise 5H

UNDERSTANDING AND FLUENCY	1–3, 4–5(½)	3, 4–5(½)	4–5(½)

1 Write down the gradient of all the lines that are parallel to each line.

a $y = 4x - 6$
b $y = -7x - 1$
c $y = -\dfrac{3}{4}x + 2$
d $y = \dfrac{8}{7}x - \dfrac{1}{2}$

2 Use $m_2 = -\dfrac{1}{m_1}$ to find the gradient of all the lines that are perpendicular to each line.

a $y = 3x - 1$
b $y = -2x + 6$
c $y = \dfrac{7}{8}x - \dfrac{2}{3}$
d $y = -\dfrac{4}{9}x - \dfrac{4}{7}$

3 A line is parallel to the graph of the rule $y = 5x - 2$ and its y-intercept is 4. The equation of the line is of the form $y = mx + b$.

a State the value of m.
b State the value of b.
c Write the rule.

Example 24

4 Decide if each pair of lines is parallel, perpendicular or neither.

a $y = 3x - 1$ and $y = 3x + 7$
b $y = \dfrac{1}{2}x - 6$ and $y = \dfrac{1}{2}x - 4$
c $y = -\dfrac{2}{3}x + 1$ and $y = \dfrac{2}{3}x - 3$
d $y = -4x - 2$ and $y = x - 7$
e $y = -\dfrac{3}{7}x - \dfrac{1}{2}$ and $y = \dfrac{7}{3}x + 2$
f $y = -8x + 4$ and $y = \dfrac{1}{8}x - 2$
g $2y + x = 2$ and $y = -\dfrac{1}{2}x - 3$
h $x - y = 4$ and $y = x + \dfrac{1}{2}$
i $8y + 2x = 3$ and $y = 4x + 1$
j $3x - y = 2$ and $x + 3y = 5$

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Example 25

5 Find the equation of the line that is:

a parallel to $y = x + 3$ and passes through $(1, 5)$
b parallel to $y = -x - 5$ and passes through $(1, -7)$
c parallel to $y = -4x - 1$ and passes through $(-1, 3)$
d parallel to $y = \frac{2}{3}x + 1$ and passes through $(3, -4)$
e parallel to $y = -\frac{4}{5}x + \frac{1}{2}$ and passes through $(5, 3)$
f perpendicular to $y = 2x + 3$ and passes through $(2, 5)$
g perpendicular to $y = -4x + 1$ and passes through $(-4, -3)$
h perpendicular to $y = \frac{2}{3}x - 4$ and passes through $(4, -1)$
i perpendicular to $y = \frac{4}{3}x + \frac{1}{2}$ and passes through $(-4, -2)$
j perpendicular to $y = -\frac{2}{7}x - \frac{3}{4}$ and passes through $(-8, 3)$

PROBLEM-SOLVING AND REASONING	6, 7, 10	7, 8, 10, 11	8–12

6 This question involves vertical and horizontal lines. Find the equation of the line that is:

a parallel to $x = 3$ and passes through $(6, 1)$
b parallel to $x = -1$ and passes through $(0, 0)$
c parallel to $y = -3$ and passes through $(8, 11)$
d parallel to $y = 7.2$ and passes through $(1.5, 8.4)$
e perpendicular to $x = 7$ and passes through $(0, 3)$
f perpendicular to $x = -4.8$ and passes through $(2.7, -3)$
g perpendicular to $y = -\frac{3}{7}$ and passes through $\left(\frac{2}{3}, \frac{1}{2}\right)$
h perpendicular to $y = \frac{8}{13}$ and passes through $\left(-\frac{4}{11}, \frac{3}{7}\right)$

7 Find the equation of the line that is parallel to these equations and passes through the given points.

a $y = \frac{2x - 1}{3}$, $(0, 5)$
b $y = \frac{3 - 5x}{7}$, $(1, 7)$
c $3y - 2x = 3$, $(-2, 4)$
d $7x - y = -1$, $(-3, -1)$

8 Find the equation of the line that is perpendicular to the equations given in Question **7** and passes through the same given points.

9 A line with equation $3x - 2y = 12$ intersects a second line at the point where $x = 2$. If the second line is perpendicular to the first line, find where the second line cuts the x-axis.

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10 **a** Find the value of a if $y = \frac{a}{7}x + b$ is parallel to $y = 2x - 4$.

b Find the value of a if $y = \frac{2a + 1}{3}x + b$ is parallel to $y = -x - 3$.

c Find the value of a if $y = \left(\frac{1 - a}{2}\right)x + b$ is perpendicular to $y = \frac{1}{2}x - \frac{3}{5}$.

d Find the value of a if $ay = 3x + b$ is perpendicular to $y = -\frac{3}{7}x - 1$.

11 Find an expression for the gradient of a line if it is:

a parallel to $y = mx + 8$
b parallel to $ax + by = 4$
c perpendicular to $y = mx - 1$
d perpendicular to $ax + by = -3$

12 Find the equation of a line that is:

a parallel to $y = 2x + b$ and passes through (c, d)
b parallel to $y = mx + b$ and passes through (c, d)
c perpendicular to $y = -x + b$ and passes through (c, d)
d perpendicular to $y = mx + b$ and passes through (c, d)

ENRICHMENT	–	–	13–15

Perpendicular and parallel geometry

13 A quadrilateral, $ABCD$, has vertex coordinates $A(2, 7)$, $B(4, 9)$, $C(6, 3)$ and $D(8, 5)$.

a Find the gradient of the line segments:
 i AB
 ii BC
 iii CD
 iv DA

b What do you notice about the gradient of the opposite sides?
c What type of quadrilateral is $ABCD$?

14 The vertices of triangle ABC are $A(0, 0)$, $B(3, 4)$ and $C\left(\frac{25}{3}, 0\right)$.

a Find the gradient of the line segments:
 i AB
 ii BC
 iii CA

b What type of triangle is $\triangle ABC$?
c Find the perimeter of $\triangle ABC$.

15 Find the equation of the perpendicular bisector of the line segment that joins $(1, 1)$ with $(3, 5)$ and find where this bisector cuts the x-axis.

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5I Solving simultaneous equations using substitution

Interactive
Widgets
HOTsheets
Walkthrough

When we try to find a solution to a set of equations rather than just a single equation, we say that we are solving simultaneous equations. For two linear simultaneous equations, we are interested in finding the point where the graphs of the two equations meet. The point, for example, at the intersection of a company's cost equation and revenue equation is the 'break-even' point. This determines the point at which a company will start making a profit.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Give up and do the algebra

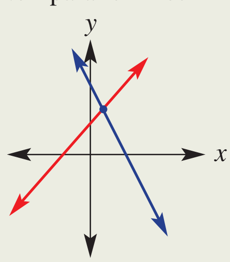
The two simultaneous equations $y = 2x - 3$ and $4x - y = 5\frac{1}{2}$ have a single solution.

- Use a guess and check (i.e. trial and error) technique to try to find the solution.
- Try a graphical technique to try and find the solution. Is this helpful?
- Now find the exact solution using the method of substitution.
- Which method is better? Discuss.

Key ideas

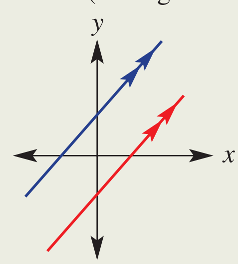
■ Solving two **simultaneous equations** involves finding a solution that satisfies both equations.
- When two straight lines are not parallel then there will be a single (**unique**) solution.

Non-parallel lines



One point of intersection

Parallel lines (same gradient)



No points of intersection

■ The **substitution** method is usually used when at least one of the equations has one pronumeral as the subject. For example: $y = 3x + 2$ or $x = 3y - 1$.
- By substituting one equation into the other, a single equation in terms of one pronumeral is formed and can then be solved.

  For example:

  $$x + y = 8 \quad (1)$$
  $$y = 3x + 4 \quad (2)$$

  Substitute equation (2) into equation (1): $x + y = 8$ becomes

  $$x + (3x + 4) = 8$$
  $$4x + 4 = 8$$
  $$\therefore x = 1$$

  Substituting $x = 1$ into equation (1) gives

  $y = 3(1) + 4 = 7$

  The solution is $x = 1$, $y = 7$.

  Therefore, the point of intersection is $(1, 7)$.

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Example 26 Solving simultaneous equations using substitution

Solve these pairs of simultaneous equations, using the method of substitution. Write down the point of intersection.

a $2x - 3y = -8$ and $y = x + 3$ **b** $y = -3x + 2$ and $y = 7x - 8$

SOLUTION	EXPLANATION
a $2x - 3y = -8$ (1) $y = x + 3$ (2)	Label your equations.
Substitute equation (2) into equation (1): $2x - 3(x + 3) = -8$ $2x - 3x - 9 = -8$ $-x - 9 = -8$ $-x = 1$ $x = -1$	Substitute equation (2) into equation (1) since equation (2) has a pronumeral as the subject. Expand and simplify, then solve the equation for x.
Substitute $x = -1$ into equation (2): $y = -1 + 3$ $= 2$	Substitute the solution for x into one of the equations to find y.
Point of intersection is $(-1, 2)$.	State the solution and check by substituting the solution into both equations.
b $y = -3x + 2$ (1) $y = 7x - 8$ (2)	Write down and label each equation.
Substitute equation (2) into equation (1): $7x - 8 = -3x + 2$ $10x = 10$ $x = 1$	Alternatively, substitute equation (1) into equation (2). Solve the equation for x.
Substitute $x = 1$ into equation (1): $y = -3(1) + 2$ $= -1$ Point of intersection is $(1, -1)$.	Substitute the solution for x into one of the equations. Mentally check the solution to both equations: $-1 = 7(1) - 8$

Exercise 5I

UNDERSTANDING AND FLUENCY	1–4	3, 4–5(½)	4–5(½)

1 By substituting the given point into both equations, decide whether it is the point of intersection for these straight lines.

- **a** $x + y = 5$ and $x - y = -1$, point $(2, 3)$
- **b** $3x - y = 2$ and $x + 2y = 10$, point $(2, 4)$
- **c** $3x + y = -1$ and $x - y = 0$, point $(-1, 2)$
- **d** $5x - y = 7$ and $2x + 3y = 9$, point $(0, 3)$
- **e** $2y = x + 2$ and $x - y = 4$, point $(-2, -6)$
- **f** $x = y + 7$ and $3y - x = 20$, point $(-3, 10)$
- **g** $2(x + y) = -20$ and $3x - 2y = -20$, point $(-8, -2)$
- **h** $7(x - y) = 6$ and $y = 2x - 9$, point $(1, 6)$

2 Determine the point of intersection of each pair of equations by plotting accurate graphs.

a $y = 3$
$y = 2x - 4$

b $y = -2$
$y = 2x - 3$

c $x = 2$
$y = 4$

d $x = -1$
$y = 0$

e $y = x + 3$
$2x + 3y = 14$

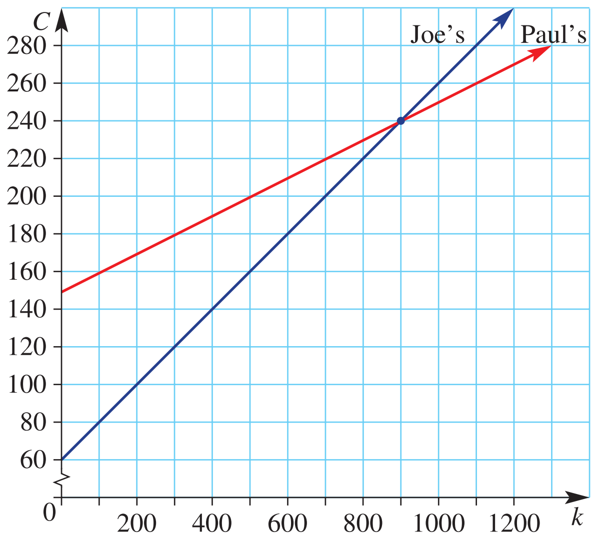
f $y = 2x - 6$
$3x - y = 7$

g $y = 3x + 9$
$3x - y = 3$

h $y = x$
$y = x + 4$

i $y = -2x + 3$
$y = 3x + 4$

3 This graph represents the rental cost $\$C$ after k kilometres of a new car from two car rental firms called Paul's Motor Mart and Joe's Car Rental.



a **i** Determine the initial rental cost from each company.
ii Find the cost per kilometre when renting from each company.
iii Find the linear equations for the total rental cost from each company.
iv Determine the number of kilometres for which the cost is the same from both rental firms.

b If you had to travel 300 km, which company would you choose?

c If you had \$260 to spend on travel, which firm would give you the most kilometres?

Example 26a

4 Solve the following pairs of simultaneous equations, using the method of substitution.

a $y = x + 5$ and $3x + y = 13$
b $y = x + 3$ and $6x + y = 17$
c $y = x - 2$ and $3x - 2y = 7$
d $y = x - 1$ and $3x + 2y = 8$
e $y = x$ and $4x + 3y = 7$
f $y = x$ and $7x + 3y = 10$
g $x = 2y + 3$ and $11y - 5x = -14$
h $x = 3y - 2$ and $7y - 2x = 8$
i $x = 3y - 5$ and $3y + 5x = 11$
j $x = 4y + 1$ and $2y - 3x = -23$

Example 26b

5 Solve the following pairs of simultaneous equations, using the method of substitution.

a $y = 4x + 2$ and $y = x + 8$
b $y = -2x - 3$ and $y = -x - 4$
c $x = y - 6$ and $x = -2y + 3$
d $x = -7y - 1$ and $x = -y + 11$
e $y = 4 - x$ and $y = x - 2$
f $y = 5 - 2x$ and $y = \frac{3}{2}x - 2$
g $y = 5x - 1$ and $y = \frac{11 - 3x}{2}$
h $y = 8x - 5$ and $y = \frac{5x + 13}{6}$

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PROBLEM-SOLVING AND REASONING	6, 7, 10	6, 8, 10(½), 11	8, 9, 11, 12

6 The salary structures for companies A and B are given by:
Company A: \$20 per hour
Company B: \$45 plus \$15 per hour

- **a** Find a rule for $\$E$ earned for t hours for:
  - **i** company A
  - **ii** company B
- **b** Solve your two simultaneous equations from part **a**.
- **c** **i** State the number of hours worked for which the earnings are the same for the two companies.
  - **ii** State the amount earned when the earnings are the same for the two companies.

7 The value of two cars is depreciating (i.e. decreasing) at a constant rate according to the information in this table.

Car	Initial value	Annual depreciation
Luxury sports coupe	\$62 000	\$5000
Family sedan	\$40 000	\$3000

- **a** Write rules for the value $\$V$ after t years for:
  - **i** the luxury sports coupe
  - **ii** the family sedan
- **b** Solve your two simultaneous equations from part **a**.
- **c** **i** State the time taken for the cars to have the same value.
  - **ii** State the value of the cars when they have the same value.

8 The sum of the ages of a boy and his mother is 48. If the mother is three more than twice the boy's age, find the difference in the ages of the boy and his mother.

9 The perimeter of a rectangular farm is 1 800 m and its length is 140 m longer than its breadth. Find the area of the farm.

10 When two lines have the same gradient, there will be no intersection point. Use this idea to decide whether these pairs of simultaneous equations will have a solution.

- **a** $y = 3x - 1$ and $y = 3x + 2$
- **b** $y = 7x + 2$ and $y = 7x + 6$
- **c** $y = 2x - 6$ and $y = -2x + 1$
- **d** $y = -x + 7$ and $y = 2x - 1$
- **e** $2x - y = 4$ and $y = 2x + 1$
- **f** $7x - y = 1$ and $y = -7x + 2$
- **g** $2y - 3x = 0$ and $y = -\frac{3x}{2} - 1$
- **h** $\frac{x - 2y}{4} = 0$ and $y = \frac{x}{2} + 3$

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11 For what value of k will these pairs of simultaneous equations have no solution?

a $y = -4x - 7$ and $y = kx + 2$

b $y = kx + 4$ and $3x - 2y = 5$

c $kx - 3y = k$ and $y = 4x + 1$

12 Solve these simple equations for x and y. Your solution should contain the pronumeral k.

a $x + y = k$ and $y = 2x$

b $x - y = k$ and $y = -x$

c $2x - y = -k$ and $y = x - 1$

d $y - 4x = 2k$ and $x = y + 1$

ENRICHMENT	–	–	13, 14

Factorise to solve

13 Factorisation can be used to help solve harder literal equations (i.e. equations that include other pronumerals).

For example: $ax + y = b$ and $y = bx$

Substituting $y = bx$ into $ax + y = b$ gives:

$$\begin{aligned} ax + bx &= b \\ x(a + b) &= b \qquad \text{(Factor out } x.) \\ x &= \frac{b}{a+b} \\ \therefore y &= b \times \frac{b}{a+b} = \frac{b^2}{a+b} \end{aligned}$$

Now solve these literal equations.

a $ax - y = b$ and $y = bx$

b $ax + by = 0$ and $y = x + 1$

c $x - by = a$ and $y = -x$

d $y = ax + b$ and $y = bx$

e $y = (a - b)x$ and $y = bx + 1$

f $ax + by = c$ and $y = ax + c$

g $\frac{x}{a} + \frac{y}{b} = 1$ and $y = ax$

14 Make up your own literal equation. Solve it and then test it on a classmate.

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5J Solving simultaneous equations using elimination

Interactive
Widgets
HOTsheets
Walkthrough

The elimination method for the solution of simultaneous equations is commonly used when the equations are written in the same form. One pronumeral is eliminated using addition or subtraction and the value of the other pronumeral can then be found using one of the original equations.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Weather forecasting uses techniques such as those in this chapter to solve the equations that model atmospheric conditions. Similiar techniques are used in all kinds of forecasting, especially in science, technology and economics.

Let's start: Which operation?

Below are four sets of simultaneous equations.

- For each set discuss whether addition or subtraction would be used to eliminate one pronumeral.
- State which pronumeral might be eliminated first in each case.
- Describe how you would first deal with parts **c** and **d** so that elimination can be used.

a $x + y = 5$
$2x + y = 7$

b $-4x - 2y = -8$
$4x + 3y = 10$

c $5x - y = 1$
$3x - 2y = -5$

d $3x + 2y = -5$
$4x - 3y = -1$

Key ideas

- The method of **elimination** is generally used when both equations are in the form $ax + by = d$.

  For example: $2x - y = 6$, $3x + y = 10$ *or* $-5x + y = -2$, $6x + 3y = 5$

  - When there is no matching pair (as in the second example above) one or both of the equations can be multiplied by a chosen factor. This is shown in **Example 27b** and **c**.

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Example 27 Using the elimination method to solve simultaneous equations

Solve the following pairs of simultaneous equations using the elimination method. Write down the point of intersection.

a $x + y = 6$ and $3x - y = 10$
b $y - 3x = 1$ and $2y + 5x = 13$
c $3x + 2y = 6$ and $5x + 3y = 11$

SOLUTION	EXPLANATION
a $x + y = 6 \quad (1)$ $3x - y = 10 \quad (2)$	Label your equations to help you refer to them in your working.
$(1) + (2)$ gives: $4x = 16$ $x = 4$	Add the two equations to eliminate y. Then solve for the remaining pronumeral, x.
Substitute $x = 4$ into equation (1): $(4) + y = 6$	Substitute $x = 4$ into one of the equations to find y.
$\therefore y = 2$ Point of intersection is $(4, 2)$.	State the solution and check by substituting the solution into the original equations.
b $y - 3x = 1 \quad (1)$ $2y + 5x = 13 \quad (2)$	Label your equations.
$(1) \times 2$: $\quad 2y - 6x = 2 \quad (3)$ $(2) - (3)$: $\quad 11x = 11$ $x = 1$	Multiply equation (1) by 2 so that there is a matching pair $(2y)$. $2y - 2y = 0$ and $5x - (-6x) = 11x$.
Substitute $x = 1$ into equation (1): $y - 3(1) = 1$ $\therefore y = 4$	Solve for x. Substitute into one of the equations to find y.
Point of intersection is $(1, 4)$.	State and check the solution.
c $3x + 2y = 6 \quad (1)$ $5x + 3y = 11 \quad (2)$ $(1) \times 3$: $\quad 9x + 6y = 18 \quad (3)$ $(2) \times 2$: $\quad 10x + 6y = 22 \quad (4)$	Multiply equation (1) by 3 and equation (2) by 2 to generate $6y$ in each equation. (Alternatively, multiply (1) by 5 and (2) by 3 to obtain matching x coefficients.)
$(4) - (3)$: $\quad x = 4$	Subtract to eliminate y.
Substitute $x = 4$ into equation (1): $3(4) + 2y = 6$ $2y = -6$ $\therefore y = -3$	Substitute $x = 4$ into one of the equations to find y.
Point of intersection is $(4, -3)$.	State and check the solution.

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Exercise 5J

UNDERSTANDING AND FLUENCY	1–5	3, 4–5(½)	4–5(½)

1 **a** Add these equations:

$$\begin{array}{l} 2x + y = 11 \\ \underline{2x - y = 5} \\ \underline{} \end{array}$$

b Subtract these equations:

$$\begin{array}{l} 2x + y = 11 \\ \underline{2x - y = 5} \\ \underline{} \end{array}$$

2 Decide if you would add or subtract the two given terms to give a result of zero.

a $7x, 7x$ **b** $2y, 2y$ **c** $-x, x$ **d** $-5y, 5y$
e $2y, -2y$ **f** $10x, -10x$ **g** $-3x, -3x$ **h** $-7y, -7y$

3 Write the resulting equation when $2x - 3y = 4$ is multiplied by:

a 2 **b** 3 **c** 4 **d** 10

Example 27a

4 Solve the following pairs of simultaneous equations using the elimination method.

a $x + y = 7$ and $5x - y = 5$
b $x + y = 5$ and $3x - y = 3$
c $x - y = 2$ and $2x + y = 10$
d $x - y = 0$ and $4x + y = 10$
e $3x + 4y = 7$ and $2x + 4y = 6$
f $x + 3y = 5$ and $4x + 3y = 11$
g $2x + 3y = 1$ and $2x + 5y = -1$
h $4x + y = 10$ and $4x + 4y = 16$
i $2x + 3y = 8$ and $2x - 4y = -6$
j $3x + 2y = 8$ and $3x - y = 5$
k $-3x + 2y = -4$ and $5x - 2y = 8$
l $-2x + 3y = 8$ and $-4x - 3y = -2$

Example 27b

5 Solve the following pairs of simultaneous equations using the elimination method.

a $3x + 5y = 8$ and $x - 2y = -1$
b $2x + y = 10$ and $3x - 2y = 8$
c $x + 2y = 4$ and $3x - y = 5$
d $3x - 4y = 24$ and $x - 2y = 10$
e $y - 3x = -\frac{1}{2}$ and $x + 2y = \frac{5}{2}$
f $7x - 2y = -\frac{5}{2}$ and $3x + y = -2$

PROBLEM-SOLVING AND REASONING	6(½), 7, 10	6(½), 7, 8, 10, 11	6(½), 8, 9, 11, 12

Example 27c

6 Solve the following pairs of simultaneous equations using the elimination method.

a $3x + 2y = 6$ and $5x + 3y = 11$
b $3x + 2y = 5$ and $2x + 3y = 5$
c $4x - 3y = 0$ and $3x + 4y = 25$
d $2x + 3y = 10$ and $3x - 4y = -2$
e $-2y - 4x = 0$ and $3y + 2x = -2$
f $-7x + 3y = 22$ and $3x - 6y = -11$

7 The sum of two numbers is 1633 and their difference is 35. Find the two numbers.

8 The cost of one apple and one banana at the school canteen is \$1 and the cost of 3 apples and 2 bananas is \$2.40. Find the cost of a single banana.

9 A group of 5 adults and 3 children paid a total of \$108 for their concert tickets. Another group of 3 adults and 10 children paid \$155. Find the cost of an adult ticket and the cost of a child's ticket.

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10 Describe the error made in this working and then correct the error to find the correct solution.

$$\begin{aligned} 3x - 2y &= 5 \quad (1)\\ -4x - 2y &= -2 \quad (2)\\ (1) + (2)\text{: } -x &= 3\\ \therefore x &= -3\\ 3(-3) - 2y &= 5\\ -9 - 2y &= 5\\ -2y &= 14\\ y &= -7 \end{aligned}$$

Point of intersection is $(-3, -7)$.

11 Solve these literal simultaneous equations for x and y.

a $ax + y = 0$ and $ax - y = 2$

b $x - by = 4$ and $2x + by = 9$

c $ax + by = 0$ and $ax - by = -4$

d $ax + by = a$ and $ax - by = b$

e $ax + by = c$ and $bx + ay = c$

12 Explain why there is no solution to the set of equations $3x - 7y = 5$ and $3x - 7y = -4$.

ENRICHMENT	–	–	13

Partial fractions

13 Writing $\dfrac{6}{(x-1)(x+1)}$ as a sum of two 'smaller' fractions $\dfrac{a}{x-1} + \dfrac{b}{x+1}$, known as partial fractions, involves a process of finding the values of a and b for which the two expressions are equal. Here is the process.

$$\begin{aligned} \frac{6}{(x-1)(x+1)} &= \frac{a}{x-1} + \frac{b}{x+}\\ &= \frac{a(x+1) + b(x-1)}{(x-1)(x+1)} \end{aligned}$$

$$\begin{aligned} \therefore\ a(x+1) + b(x-1) &= 6\\ ax + a + bx - b &= 6\\ ax + bx + a - b &= 6\\ x(a+b) + (a-b) &= 0x = 6 \end{aligned}$$

By equating coefficients $\quad a + b = 0 \quad (1)$
$\qquad a - b = 6 \quad (2)$

$(1) + (2)$ gives $2a = 6$
$\therefore a = 3$

and so $b = -3$.

$$\therefore \frac{6}{(x-1)(x+1)} = \frac{3}{x-1} - \frac{3}{x+1}$$

Use this technique to write the following as the sum of two fractions.

a $\dfrac{4}{(x-1)(x+1)}$

b $\dfrac{7}{(x+2)(2x-3)}$

c $\dfrac{-5}{(2x-1)(3x+1)}$

d $\dfrac{9x+4}{(3x-1)(x+2)}$

e $\dfrac{2x-1}{(x+3)(x-4)}$

f $\dfrac{1-x}{(2x-1)(4-x)}$

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5K Further applications of simultaneous equations

Interactive
Widgets
HOTsheets
Walkthrough

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

When a problem involves two unknown pronumerals, simultaneous equations can be used to find the solution to the problem, provided that the two pronumerals can be identified and two equations can be written from the problem description.

Let's start: 19 scores but how many goals?

Nathan heard on the news that his AFL team scored 19 times during a game and the total score was 79 points. He wondered how many goals (which are worth 6 points each) and how many behinds (which are worth 1 point each) were scored in the game. Nathan looked up simultaneous equations in his Maths book and it said to follow these steps.

1 Define two pronumerals.
2 Write two equations.
3 Solve the equations.
4 Answer the question in words.

Can you help Nathan with the four steps to find out what he wants to know?

Simultaneous equations enable you to calculate from the score in an AFL game the exact combinations of goals and behinds that could have formed that score, if the number of scoring shots is known.

Key ideas

- When solving problems with two unknowns:
  - Define a pronumeral for each unknown.
  - Write down two equations from the information given.
  - Solve the equations simultaneously to find the solution.
  - Check that the solution is reasonable.
  - Interpret the solution and answer the question in words.

Example 28 Setting up and solving simultaneous equations

The sum of the ages of two children is 17 and the difference in their ages is 5. If Kara is the older sister of Ben, determine their ages.

SOLUTION

Let k be Kara's age and let b be Ben's age.

$$k + b = 17 \qquad (1)$$
$$k - b = 5 \qquad (2)$$

(1) + (2): $2k = 22$

$\therefore k = 11$

Substitute $k = 11$ into equation (1):

$11 + b = 17$
$b = 6$

$\therefore$ Kara is 11 years old and Ben is 6 years old.

EXPLANATION

Define the unknowns and use these to write two equations from the information in the question.

- The sum of their ages is 17.
- The difference in their ages is 5.

Add the equations to eliminate b, and then solve to find k.

Substitute $k = 11$ into one of the equations to find the value of b.

Answer the question in words.

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Example 29 Solving further applications with two variables

John buys three daffodils and five petunias from the nursery and pays \$25. Julia buys four daffodils and three petunias for \$26. Determine the cost of each type of flower.

SOLUTION

Let \$$d$ be the cost of a daffodil and let \$$p$ be the cost of a petunia.

$3d + 5p = 25 \quad (1)$

$4d + 3p = 26 \quad (2)$

$(1) \times 4: \quad 12d + 20p = 100 \quad (3)$

$(2) \times 3: \quad 12d + 9p = 78 \quad (4)$

$(3) - (4): \quad 11p = 22$

$\therefore p = 2$

Substitute $p = 2$ into equation (1):

$3d + 5(2) = 25$

$3d + 10 = 25$

$3d = 15$

$\therefore d = 5$

A petunia costs \$2 and a daffodil costs \$5.

EXPLANATION

Define the unknowns and set up two equations from the question.

If 1 daffodil costs d dollars, then 3 will cost $3 \times d = 3d$.

- 3 daffodils and 5 petunias cost \$25.
- 4 daffodils and 3 petunias cost \$26.

Multiply equation (1) by 4 and multiply equation (2) by 3 to generate $12d$ in each equation.

Subtract the equations to eliminate d and then solve for p.

Substitute $p = 2$ into one of the equations to find the value of d.

Answer the question in words.

Exercise 5K

UNDERSTANDING AND FLUENCY	1–7	3–8	5, 7, 8

1 Let x and y be two numbers that satisfy the following statements. Set up two linear equations according to the information and then solve them simultaneously to determine the numbers in each case.

- **a** They sum to 16 but their difference is 2.
- **b** They sum to 30 but their difference is 10.
- **c** They sum to 7 and twice the larger number plus the smaller number is 12.
- **d** The sum of twice the first plus three times the second is 11 and the difference between four times the first and three times the second is 13.

2 The perimeter of a rectangle is 56 cm. If the length of the rectangle is three times its breadth, determine the dimensions by setting up and solving two simultaneous equations.

Example 28

3 The sum of the ages of two children is 24 and the difference between their ages is 8. If Nikki is the older sister of Travis, determine their ages by setting up and solving a pair of simultaneous equations.

4 Cam is 3 years older than Lara. If their combined age is 63, determine their ages by solving an appropriate pair of equations.

Example 29

5 Alex buys 4 bolts and 6 washers for \$2.20 and Holly spends \$1.80 on 3 bolts and 5 washers at the same local hardware store. Determine the costs of each bolt and each washer.

6 It costs \$3 for children and \$7 for adults to attend a basketball game. If 5000 people attended the game and the total takings at the door was \$25 000, determine the number of children and the number of adults that attended the game.

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7 A vanilla thickshake is \$2 more than a fruit juice. If 3 vanilla thickshakes and 5 fruit juices cost \$30, determine their individual prices.

8 A paddock contains both ducks and sheep. There are a total of 42 heads and 96 feet in the paddock. How many ducks and how many sheep are in the paddock?

PROBLEM-SOLVING AND REASONING	9, 10, 12	9, 10, 12, 13	10, 11, 13, 14

9 Connie the fruiterer sells two fruit packs.
Pack 1 : 10 apples and 5 mangoes (\$12)
Pack 2 : 15 apples and 4 mangoes (\$14.15)
Determine the cost of 1 apple and 5 mangoes.

10 James has \$10 in 5-cent and 10-cent coins in his change jar and counts 157 coins in total. How many 10-cent coins does he have?

11 Five years ago I was 5 times older than my son. In 8 years' time I will be 3 times older than my son. How old am I today?

12 Erin goes off on a long bike ride, averaging 10 km/h. One hour later her brother Alistair starts chasing after her at 16 km/h. How long will it take Alistair to catch up to Erin?

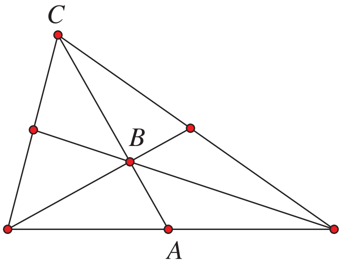
13 Two ancient armies are 1 km apart and begin walking towards each other. The Vikons walk at a pace of 3 km/h and the Mohicas walk at a pace of 4 km/h. For how long will they walk before the battle begins?

14 A river is flowing downstream at a rate of 2 metres per second. Brendan, who has an average swimming speed of 3 metres per second, decides to go for a swim in the river. He dives into the river and swims downstream to a certain point, then swims back upstream to the starting point. The total time taken is 4 minutes. How far did Brendan swim downstream?

ENRICHMENT	–	–	15

Incentre, circumcentre or centroid?

15 Consider the given diagram.



a Use the internet to:
  i Find out whether point B is the incentre, circumcentre or centroid of this triangle.
  ii Find the name given to line segments, such as AC, that join the midpoint of one side to the opposite vertex.
  iii Investigate the location of point B on the line segment AC.
  iv Use dynamic geometry software to construct this diagram.

b Plot the following points on a Cartesian plane: $O(0, 0)$, $D(4, 0)$ and $C(2, 6)$.

c Find the coordinates of point A, which is the midpoint of OD.

d Find the equation of the line AC.

e Find the coordinates of point E, which is the midpoint of DC.

f Find the equation of the line OE.

g Use simultaneous equations to find the coordinates of the point B, which is where the line AC and OE intersect.

h Calculate the lengths of AB and BC.

i What is the ratio of $AB : BC$?

j Is this ratio the same in every triangle?

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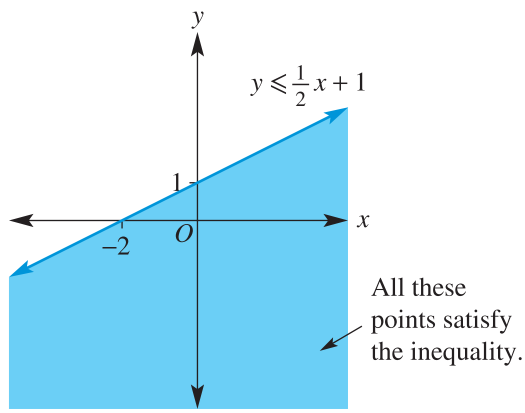
5L Regions on the Cartesian plane EXTENSION

Interactive
Widgets
HOTsheets
Walkthrough

You will remember that an inequality is a mathematical statement that contains one of these symbols: $<, \leqslant, >$ or $\geqslant$. The linear inequality with one pronumeral such as $2x - 5 > -10$ has the solution $x > -2.5$.

Linear inequalities can also have two pronumerals: $2x - 3y \geqslant 5$ and $y < 3 - x$ are two examples. The solutions to such inequalities will be an infinite set of points on a plane sitting on one side of a line. The shaded region on the diagram below is sometimes called a half plane.



Let's start: Which side do I shade?

You are asked to shade all the points on a graph that satisfy the inequality $4x - 3y \geqslant 12$.

- First, graph the equation $4x - 3y = 12$.
- Substitute the point $(3, -3)$ into the inequality $4x - 3y \geqslant 12$. Does the point satisfy the inequality? Plot the point on your graph.
- Now test these points:

  a $(3, -2)$ **b** $(3, -1)$ **c** $(3, 0)$ **d** $(3, 1)$
- Can you now decide which side of the line is to be shaded to represent all the solutions to the inequality? Should the line itself be included in the solution?

Key ideas

- The solution to a linear inequality with two pronumerals is illustrated by shading a region called a **half plane**.
- When y is the subject of the inequality, follow these simple rules:
  - $y \geqslant mx + b$ Draw a solid line (as it is included in the region) and shade above.
  - $y > mx + b$ Draw a broken line (as it is not included in the region) and shade above.
  - $y \leqslant mx + b$ Draw a solid line (as it is included in the region) and shade below.
  - $y < mx + b$ Draw a broken line (as it is not included in the region) and shade below.

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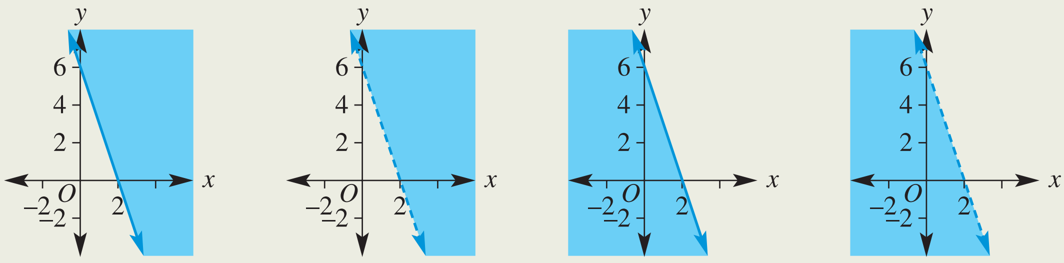
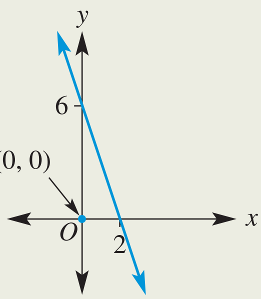
Key ideas

Here are examples of each.



$y \geqslant -3x + 6$ $y > -3x + 6$ $y \leqslant -3x + 6$ $y < -3x + 6$

- If the equation is of the form $ax + by = d$, it is usually simpler to test a point; for example, $(0, 0)$, to see which side of the line is to be included in the region.



Equation of line: $3x + y = 6$

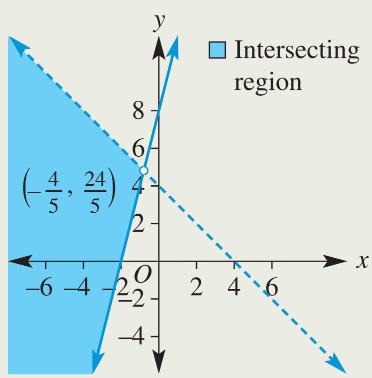
Test $(0, 0)$: $3 \times 0 + 0 < 6$

Therefore $(0, 0)$ is in the region $3x + y \leqslant 6$.

The other region is $3x + y \geqslant 6$.

- When two or more half planes are sketched on the same set of axes, the half planes overlap and form an **intersecting region**. The set of points inside the intersecting region will be the solution to the simultaneous inequalities.

  For example: $y \geqslant 4x + 8$
  $y < -x + 4$



- To help define the intersecting region correctly, you should determine and label the point of intersection. In the example above, the point of intersetion is on a dashed line, so the point is shown as ○, not ●. This point is not part of the intersecting region.

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Example 30 Sketching half planes

Shade the region for the following linear inequalities.

a $y > 1.5x - 3$ **b** $y + 2x \leqslant 4$

SOLUTION

a $y = 1.5x - 3$

y-intercept (let $x = 0$):

$y = -3$

x-intercept (let $y = 0$):

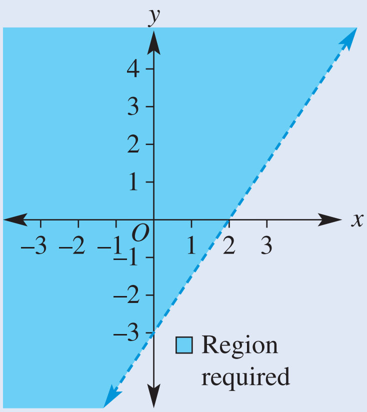
$0 = 1.5x - 3$

$1.5x = 3$

$\therefore x = 2$

EXPLANATION

First, sketch $y = 1.5x - 3$ by finding the x- and y-intercepts.



Sketch a dotted line (since the sign is $>$ not $\geqslant$) that joins the intercepts and shade above the line, since y is greater than $1.5x - 3$.

b $y + 2x = 4$

y-intercept (let $x = 0$):

$y = 4$

x-intercept (let $y = 0$):

$2x = 4$

$x = 2$

First, sketch $y + 2x = 4$ by finding the x- and y-intercepts.

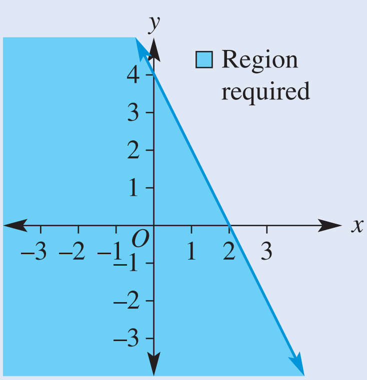
Shading: Test $(0, 0)$.

$0 + 2(0) \leqslant 4$

$0 \leqslant 4$ (true)

$\therefore (0, 0)$ is included.

Decide which side to shade by testing the point $(0, 0)$; i.e. substitute $x = 0$ and $y = 0$. Since $0 \leqslant 4$, the point $(0, 0)$ should sit inside the shaded region.



Sketch a solid line since the inequality is $\leqslant$ (not $<$). Shade the region that includes $(0, 0)$.

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Example 31 Finding the intersecting region

Sketch both the inequalities $4x + y \leqslant 12$ and $3x - 2y < -2$ on the same set of axes, show the region of intersection and find the point of intersection of the two lines.

SOLUTION

$4x + y = 12$
y-intercept (let $x = 0$):
$y = 12$
x-intercept (let $y = 0$):
$4x = 12$
$x = 3$

Shading: Test (0, 0).
$4(0) + 0 \leqslant 12$
$0 \leqslant 12$ (true)

So (0, 0) is included.

$3 - 2y = -2$
y-intercept (let $x = 0$):
$-2y = -2$
$y = 1$
x-intercept (let $y = 0$):
$3x = -2$
$x = -\frac{2}{3}$

Shading: Test (0, 0).
$3(0) + 2(0) < -2$
$0 < -2$ (false)

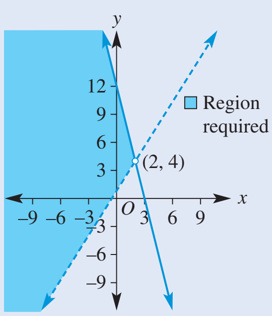
So (0, 0) is not included.

Point of intersection:
$4x + y = 12$ (1)
$3x - 2y = -2$ (2)

(1) × 2: $8x + 2y = 24$ (3)
(2) + (3): $11x = 22$
$x = 2$

Substitute $x = 2$ into equation (1):
$4(2) + y = 12$
$y = 4$

The point of intersection is (2, 4).



EXPLANATION

First, sketch $4x + y = 12$ by finding the x- and y-intercepts.

Test (0, 0) to see if it is in the included region.

Sketch $3x - 2y = -2$ by finding x- and y-intercepts.

Test (0, 0) to see if it is in the included region.

Find the point of intersection by solving the equations simultaneously, using the method of elimination.

Sketch both regions and label the intersection point. Also label the intersecting region.

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Exercise 5L EXTENSION

UNDERSTANDING AND FLUENCY	1–6	3, 4–5(½), 6, 7	4–5(½), 6, 7

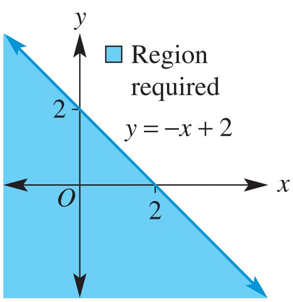
1 Substitute the point (0, 0) into these inequalities to decide if the point satisfies the inequality; i.e. is the inequality true for $x = 0$ and $y = 0$?

- **a** $y < 3x - 1$
- **b** $y \leqslant -4x + 6$
- **c** $y > -\frac{x}{2} - 3$
- **d** $y \geqslant 1 - 7x$
- **e** $3x - 2y < -1$
- **f** $y - 4x \geqslant \frac{1}{2}$
- **g** $x - y > 0$
- **h** $2x - 3y \leqslant 0$
- **i** $y \geqslant 4x$

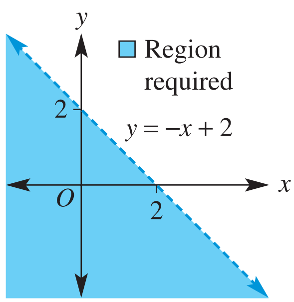
2 Match the rules (**a**–**c**) with the graphs (**A**–**C**) below.

a $x + y < 2$ **b** $y \geqslant -x + 2$ **c** $y \leqslant -x + 2$

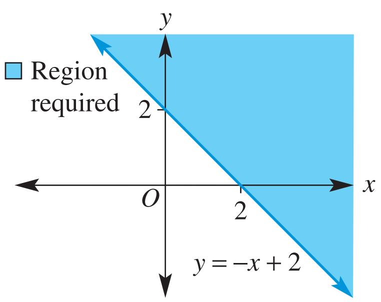
A



B



C



3 **a** Sketch the vertical line $x = -1$ and the horizontal line $y = 4$ on the same set of axes.
b Shade the region $x \geqslant -1$ (i.e. all points with an x-coordinate greater than or equal to -1).
c Shade the region $y \leqslant 4$ (i.e. all points with a y-coordinate less than or equal to 4).
d Now use a different colour to shade all the points that satisfy both $x \geqslant -1$ and $y \leqslant 4$ simultaneously.

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Example 30a

4 Sketch the half planes for the following linear inequalities.

a $y \geqslant x + 4$ **b** $y < 3x - 6$ **c** $y > 2x - 8$ **d** $y \leqslant 3x - 5$

e $y < -4x + 2$ **f** $y < 2x + 7$ **g** $y < 4x$ **h** $y > 6 - 3x$

i $y < -x$ **j** $x > 3$ **k** $x < -2$ **l** $y > 2$

Example 30b

5 Shade the region for the following linear inequalities.

a $x + 3y < 9$ **b** $3x - y \geqslant 3$ **c** $4x + 2y \geqslant 8$ **d** $2x - 3y > 18$

e $-2x + y \leqslant 5$ **f** $-2x + 4y \leqslant 6$ **g** $2x + 5y > -10$ **h** $4x + 9y < -36$

6 Decide whether the following points are in the region defined by $2x - 3y > 8$.

a $(5, 0)$ **b** $(2.5, -1)$ **c** $(0, -1)$ **d** $(2, -5)$

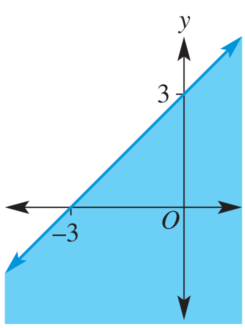
7 Decide whether the following points are in the region defined by $4x + 3y \leqslant -2$.

a $\left(-\frac{1}{2}, \frac{1}{2}\right)$ **b** $(-5, 6)$ **c** $(2, -3)$ **d** $(-3, 4)$

PROBLEM-SOLVING AND REASONING	8, 10	8, 9–10(½)	8, 9–10(½), 11

8 Write down the inequalities that give these half planes.

a



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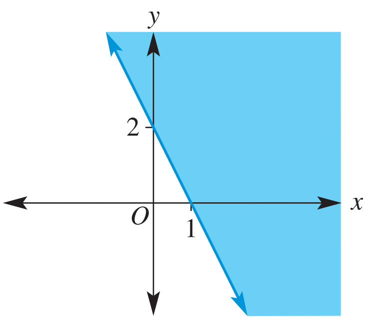

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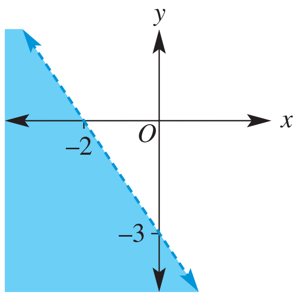

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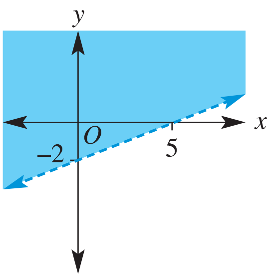

Example 31

9 Sketch both inequalities on the same set of axes, shade the region of intersection and find the point of intersection of the two lines.

a $x + 2y \geqslant 4$, $2x + 2y < 8$ **b** $3x + 4y \leqslant 12$, $3x + y > 3$ **c** $2x - 3y > 6$, $y < x - 2$ **d** $3x - 5y \leqslant 15$, $y - 3x > -3$

e $y \geqslant -x + 4$, $2x + 3y \geqslant 6$ **f** $2y - x \leqslant 5$, $y < 10 - x$ **g** $3x + 2y \leqslant 18$, $4y - x < 8$ **h** $2y \geqslant 5 + x$, $y < 6 - 3x$

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10 Sketch the following systems of inequalities on the same axes. Show the intersecting region and label the points of intersection. In each case, the result should be a triangle.

a $x \geqslant 0$
$y \geqslant 0$
$3x + 6y \leqslant 6$

b $x \geqslant 0$
$y \leqslant 0$
$2x - y \leqslant 4$

c $x \geqslant 0$
$5x + 2y \leqslant 30$
$4y - x \geqslant 16$

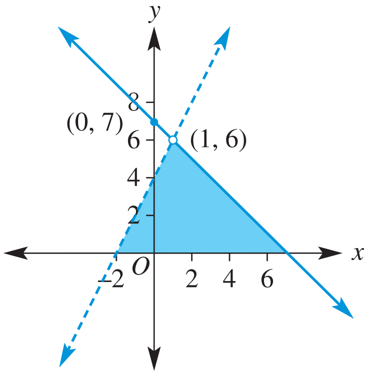
d $x < 2$
$y < 3$
$2x + 5y > 10$

e $x \leqslant 0$
$y < x + 7$
$2x + 3y \geqslant -6$

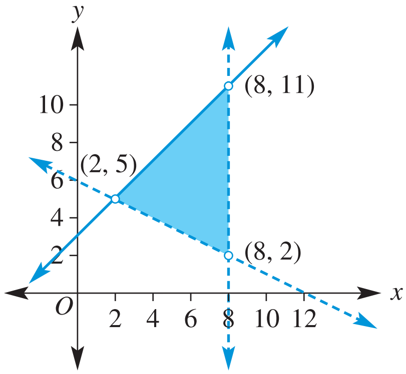
f $x + y \leqslant 9$
$2y - x \geqslant 6$
$3x + y \geqslant -2$

11 Determine the original inequalities that would give the following regions of intersection.

a



b



ENRICHMENT	–	–	12, 13

Areas of regions

12 Find the area of the triangles formed in Question **10** parts **a** to **d**.

13 **a** Find the exact area bound by:

i $x < 0$
$y > 0$
$x + 2y < 6$
$x - y > -7$

ii $y < 7$
$x + y > 5$
$3x - 2y < 14$

b Make up your own set of inequalities that gives an area of 6 square units.

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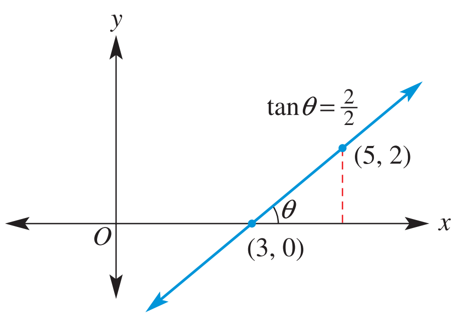
1 Angles between lines

Consider what happens if we pick up one end of a classroom desk and tilt the desk so that its surface has a non-zero gradient. The more we raise the desk, the greater the slope of the surface of the desk – the gradient of the desk is changing as we raise it higher above the ground. We are also changing the angle that the desk makes with the horizontal as we do this. Here we will investigate the relationship between this angle and the gradient.

Angle and gradient

a Find the gradient of the line joining these points:
  i (3, 0) and (5, 2) **ii** (2, 4) and (5, 8) **iii** (−1, 2) and (4, 4)

b Using your knowledge of trigonometry from Year 9, create a right-angled triangle using each pair of points in part **a**. Write a trigonometric ratio for the angle θ as shown. The first one is shown here.



c What do you notice about your trigonometric ratio and the gradient of each line?

d Hence, write a statement defining the gradient (m) of a line in terms of the angle (θ) the line makes with the positive x-axis:

i.e. Complete the rule: $m =$ _______

e Using your rule from part **d**, determine the gradient of a line that makes the following angles with the x-axis. Give your answer to 1 decimal place.
  i 35° **ii** 54° **iii** 83°

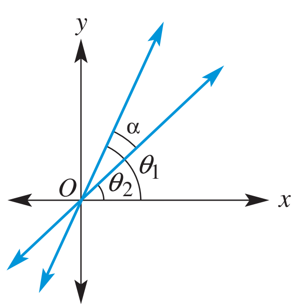
f Find the equation of a line that passes through the point (−2, 5) and makes an angle of 45° with the positive part of the x-axis.

g By reviewing parts **a** and **b**, consider what happens in the case where the line has a negative gradient.

Acute angle between lines

Here we will use the result $m = \tan\theta$ to find the acute angle between two intersecting lines.

a Sketch the lines $y = 4x$ and $y = x$ on the same set of axes. Label the acute angle between the two lines α.
  i Use the gradient of each line to find the angles (θ_1 and θ_2) that each line makes with the x-axis. Round your answer to 1 decimal place where necessary.
  ii How can θ_1 and θ_2 be used to find the acute angle α between the two lines?



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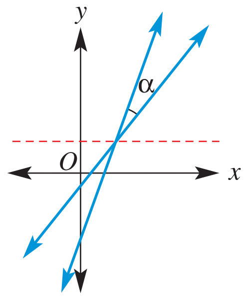
b Sketch the lines $y = 5x - 12$ and $y = \sqrt{3}x - 2$ on the same set of axes.

  i Use the gradient of each line to find the angles (θ_1 and θ_2) that each line makes with the x-axis. Round your answer to 1 decimal place where necessary.

  ii Insert a dashed line parallel to the x-axis and passing through the point of intersection of the two lines. Using this line, label the angles θ_1 and θ_2.



  iii Hence, use θ_1 and θ_2 to find the acute angle α between the two lines.

c Apply the method from part **b** to find the acute angle between the lines $y = 4x + 4$ and $y = -5x + 19$.

d By considering the gradients of $y = 3x + 6$ and $y = -\frac{x}{3} + 20$, what can you say about the two lines? Verify this using the process of part **b**.

2 Linear programming

Linear programming is a technique that can be applied in industry to maximise profit and productivity while keeping costs to a minimum. Although these problems can involve a number of variables, we will limit this investigation to problems containing two variables. The following parts will step you through a simple linear programming problem in which the goal is to maximise a company's profit.

The problem

An ice-cream vendor produces and sells ice-cream from his van. He specialises in two flavours: strawberry and mint choc chip.

Let x be the number of litres of mint choc chip made and let y be the number of litres of strawberry made.

Constraints

The limit on storage space for the ingredients means that the vendor cannot make any more than 210 litres of ice-cream in a day. His ice-cream making machine takes 2 minutes to produce 1 litre of strawberry ice-cream and 5 minutes to produce 1 litre of mint choc chip ice-cream. The machine cannot be used for more than 9 hours in the day.

Write inequalities involving x and y for:

a the amount of ice-cream that can be produced

b the amount of time the ice-cream machine can be used

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Sketching the feasible region

a On the same set of axes sketch the previous two inequalities, labelling all intercepts and points of intersection.

b Given two more inequalities $x \geqslant 0$ and $y \geqslant 0$, shade the intersecting region for all four inequalities. This is called the feasible region.

c Given the inequalities:
 i What is the maximum number of litres of mint choc chip that can be produced in a day?
 ii What is the maximum number of litres of strawberry that can be produced in a day?

The profit equation

The vendor makes a \$2 profit on each litre of mint choc chip sold and a \$1 profit on each litre of strawberry ice-cream sold.

a Write a rule for the profit \$$P$ in terms of x and y.

b Sketch a graph of your profit equation on the same graph as above for the following P values. Use a dashed line for each graph.
 i \$150 **ii** \$230 **iii** \$300

c Describe the differences and similarities in the graphs of the profit equation as P increases.

d Is a profit of \$300 possible for this vendor?

e For most profit equations the maximum profit occurs at one of the vertices of the intersecting region. Substitute the three sets of coordinates from each vertex to determine:
 i the amount of mint choc chip (x) and the amount of strawberry (y) that should be produced to give the maximum profit
 ii the maximum profit for the ice-cream vendor

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Puzzles and challenges

1 Tom walks at 4 km/h and runs at 6 km/h. He can save $3\frac{3}{4}$ minutes by running from his house to the train station instead of walking. How many kilometres is it from his house to the station?

2 A fraction is such that when its numerator is increased by 1 and its denominator is decreased by 1 it equals 1, and when its numerator is doubled and its denominator increased by 4 it is also equal to 1. What is the fraction?

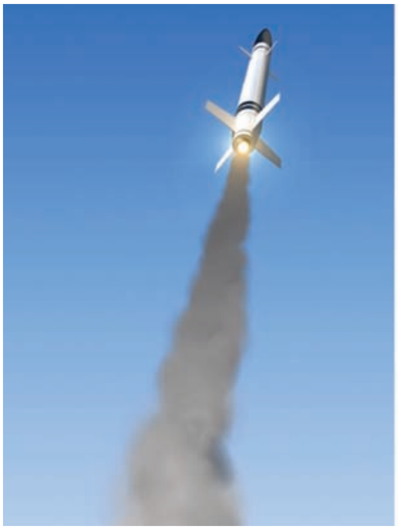
3 Show that the following sets of points are collinear (i.e. lie in a straight line).

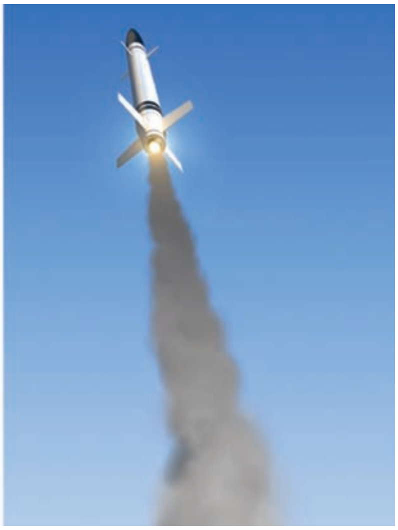
a $(2, 12)$, $(-2, 0)$ and $(-5, -9)$

b $(a, 2b)$, $(2a, b)$ and $(-a, 4b)$

4 Use two different methods from this chapter to prove that triangle ABC with vertices $A(1, 6)$, $B(4, 1)$ and $C(-4, 3)$ is a right-angled triangle.

5 Two missiles, 2420 km apart, are launched at the same time and are headed towards each other. They pass after 1.5 hours. The average speed of one missile is twice that of the other. What is the average speed of each missile?





6 Show that the points $(7, 5)$ and $(-1, 9)$ lie on a circle centred at $(2, 5)$ with radius 5 units.

7 One property of a rhombus is that its diagonals bisect each other at right angles. Prove that the points $A(0, 0)$, $B(4, 3)$, $C(0, 6)$ and $D(-4, 3)$ are the vertices of a rhombus. Is it also a square?

8 Solve these simultaneous equations.

$$x - 2y - z = 9$$
$$2x - 3y + 3z = 10$$
$$3x + y - z = 4$$

9 A triangle, PQR, has $P(8, 0)$, $Q(0, -8)$ and point R on the line $y = x - 2$. Find the area of the triangle PQR.

10 The average age of players at a ten pin bowling alley increases by 1 when either four 10-year olds leave or, alternatively, if four 22-year olds arrive. How many players were there originally and what was their average age?

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Expressions, equations and linear relationships

Midpoint and length of a line segment

Line segment joining (x_1, y_1) and (x_2, y_2) has:

Midpoint, $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Length or distance

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Parallel and perpendicular lines

Parallel lines have the same gradient; e.g. $y = 2x + 7$ and $y = 2x - 4$

For perpendicular lines with gradients m_1 and m_2:

$m_1 m_2 = -1$ or $m_2 = -\frac{1}{m_1}$

e.g. $y = 3x + 4$ and $y = -\frac{1}{3}x + 2$

Finding a rule for a graph

For points (x_1, y_1) and (x_2, y_2):

$m = \frac{y_2 - y_1}{x_2 - x_1}$

To find line equation $y = mx + b$:
- Find gradient m.
- Substitute a point if b is unknown, using $y = mx + b$ or $y - y_1 = m(x - x_1)$

Algebra review

Add/subtract like terms only.

e.g. $2x + 4x + y = 6x + y$

Multiply: $3x \times 2y = 6xy$

Divide: $4ab \div (2b) = \frac{\cancel{4}^{2}a\cancel{b}}{\cancel{2}_{1}\cancel{b}}$

$= 2a$

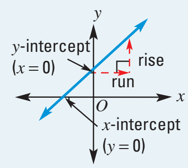
Expand: $a(b + c) = ab + ac$

$a(b - c) = ab - ac$

Graphs of linear relations

In the form $y = mx + b$, m is the gradient and b is the y-intercept.

$m = \frac{\text{rise}}{\text{run}}$



To graph:

a Use gradient and y-intercept *or*

b Locate x-intercept $(y = 0)$ and y-intercept $(x = 0)$.

Algebraic fractions

Add/subtract: Find the LCD.

e.g. $\frac{2}{a} + \frac{3}{5} = \frac{10}{5a} + \frac{3a}{5a} = \frac{10 + 3a}{5a}$

Multiply: Cancel common factors first.

e.g. $\frac{\cancel{2}^{1}}{x + 2} \times \frac{3x + 6}{\cancel{8}_{4}} = \frac{1}{\cancel{x+2}_{1}} \times \frac{3\cancel{(x+2)}^{1}}{4} = \frac{3}{4}$

Divide: Multiply by the reciprocal of the fraction following $\div$.

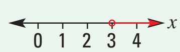
Linear inequalities

Equation contains '=' sign.

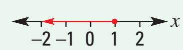
Inequalities have $>, \geqslant, <$ or $\leqslant$.

On a number line:

$x > 3$



$x \leqslant 1$



Solve as per linear equations except when multiplying or dividing by a negative, reverse the inequality sign.

Solving linear equations

Some steps include:
- Using inverse operations.
- Collecting like terms on one side.
- Multiplying by the LCD.

Simultaneous equations

Solve two equations in two variables by:
- Substitution when one variable is the subject; e.g. $y = x + 4$
- Elimination: add or subtract multiples of equations to eliminate one variable.

Applications of simultaneous equations

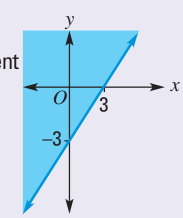
- Define unknowns using pronumerals.
- Form two equations from the information in the problem.
- Solve simultaneously.
- Answer the question in words.

Regions

Inequalities with two pronumerals represent the region above or below the line.

For $y \geqslant x - 3$, shade above the line.

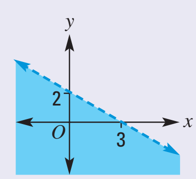
For $y > x - 3$, use a dashed line.



For $2x + 3y < 6$, test $(0, 0)$ to decide which region to shade.

$2(0) + 3(0) < 6$ (true)

Hence, include $(0, 0)$.



Chapter summary

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Multiple-choice questions

1 The simplified form of $3x + 2x \times 7y - 6x^2y \div (2x) + 5x$ is:

A $10x + 4y$ **B** $35xy - 6x^2y$ **C** $\frac{35y - 6xy}{7}$

D $5xy - 6x^2$ **E** $8x + 11xy$

2 $\frac{3x - 6}{2} \times \frac{8}{x - 2}$ simplifies to:

A 9 **B** -6 **C** $-4(x - 3)$ **D** 12 **E** $4(x - 2)$

3 $\frac{4}{x - 1} - \frac{5}{2x - 3}$ simplifies to:

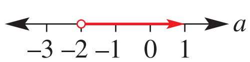
A $\frac{-1}{(x - 1)(2x - 3)}$ **B** $\frac{3x - 7}{(x - 1)(2x - 3)}$ **C** $\frac{3x - 17}{(2x - 3)(x - 1)}$

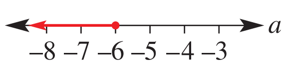
D $\frac{11 - 6x}{(x - 1)(2x - 3)}$ **E** $\frac{x - 7}{(2x - 3)(x - 1)}$

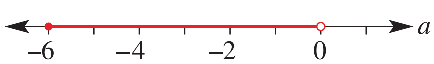
4 The solution to $-4(2x - 6) = 10x$ is:

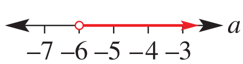
A $x = \frac{3}{2}$ **B** $x = 12$ **C** $x = \frac{4}{3}$ **D** $x = -12$ **E** $x = -\frac{4}{3}$

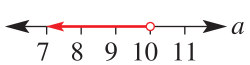
5 The number line that represents the solution to the inequality $2 - \frac{a}{3} < 4$ is:

A 

B 

C 

D 

E 

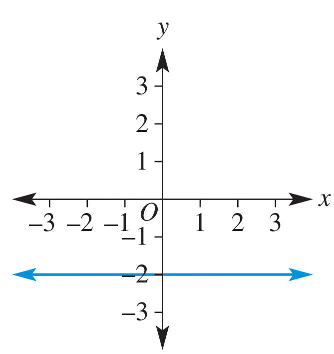
6 If $(-1, 2)$ is a point on the line $ax - 4y + 11 = 0$, the value of a is:

A -19 **B** 3 **C** $-\frac{15}{2}$ **D** 5 **E** -1

7 The graph shown has equation:

A $x = -2$ **B** $y = -2x$ **C** $y = -2$

D $x + y = -2$ **E** $y = x - 2$



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8 The gradient and the y-intercept of the graph of $3x + 8y = 2$ are, respectively:

A $-\frac{3}{8}, \frac{1}{4}$ **B** $3, 2$ **C** $\frac{2}{3}, \frac{1}{4}$

D $-3, 2$ **E** $\frac{3}{8}, 2$

9 The equation of the line joining the points $(-1, 3)$ and $(1, -1)$ could be written in the form:

A $y + 1 = -2(x - 3)$ **B** $y - 1 = \frac{1}{2}(x + 1)$ **C** $y - 3 = -2(x + 1)$

D $y - 3 = 2(x - 1)$ **E** $y - 1 = -(x - 1)$

10 The midpoint of the line segment that joins the points $(a, -6)$ and $(7, b)$ is $(4.5, -1)$. The values of the pronumerals are:

A $a = 2, b = 8$ **B** $a = 3, b = -11$ **C** $a = 9, b = 5$

D $a = 2, b = 4$ **E** $a = 2.5, b = 5$

11 The line that is perpendicular to the line with equation $y = -3x + 7$ is:

A $y = -3x + 2$ **B** $3x + y = -1$ **C** $y = 3x - 3$

D $3y = 4 - x$ **E** $3y - x = 4$

12 The line that is parallel to the line with equation $y = 2x + 3$ and passes through the point $(-3, 2)$ has the equation:

A $2x + y = 5$ **B** $y = 2x + 8$ **C** $y = -\frac{1}{2}x + \frac{1}{2}$

D $y = 2x - 4$ **E** $y - 2x = -7$

13 The solution to the simultaneous equations $2x - 3y = -1$ and $y = 2x + 3$ is:

A $x = -2, y = -1$ **B** $x = \frac{5}{2}, y = 8$ **C** $x = 2, y = 7$

D $x = -\frac{2}{3}, y = -\frac{1}{9}$ **E** $x = -3, y = 3$

14 A community fundraising concert raises \$3540 from ticket sales to 250 people. Children's tickets were sold for \$12 and adult tickets for \$18. If x adults and y children attended the concert, the two equations that represent this problem are:

A $x + y = 250$
$18x + 12y = 3540$

B $x + y = 3540$
$216xy = 3540$

C $x + y = 250$
$12x + 18y = 3540$

D $x + y = 3540$
$18x + 12y = 250$

E $3x + 2y = 3540$
$x + y = 250$

15 The point that is *not* in the region defined by $2x - 3y \leqslant 5$ is:

A $(0, 0)$ **B** $(1, -1)$ **C** $(-3, 2)$

D $(2, -1)$ **E** $\left(\frac{5}{2}, 3\right)$

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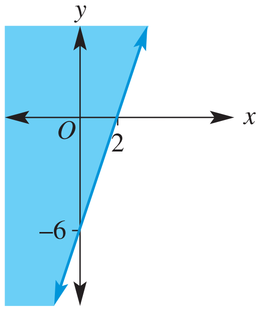
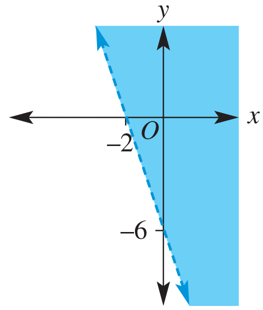
16 The region that represents the inequality $y < 3x - 6$ is:

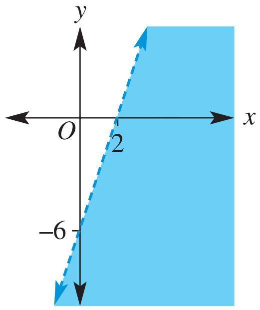
A



B



C



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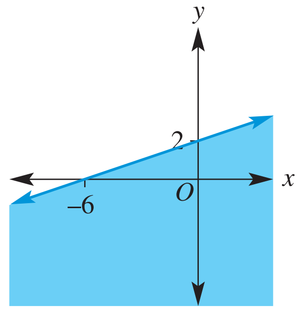

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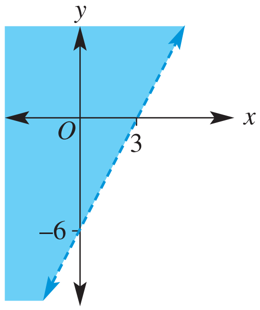

Short-answer questions

1 Simplify the following. You may need to expand the brackets first.

a $8xy + 5x - 3xy + x$ **b** $3a \times 4ab$

c $18xy \div (12y)$ **d** $3(b + 5) + 6$

e $-3m(2m - 4) + 4m^2$ **f** $3(2x + 4) - 5(x + 2)$

2 Simplify the following algebraic fractions.

a $\frac{3}{7} - \frac{a}{2}$ **b** $\frac{5}{6} + \frac{3}{a}$ **c** $\frac{x + 4}{6} + \frac{x + 3}{15}$ **d** $\frac{2}{x - 3} - \frac{3}{x + 1}$

3 Simplify these algebraic fractions by first looking to cancel common factors.

a $\frac{12x - 4}{4}$ **b** $\frac{5}{3x} \times \frac{6x}{5x + 10}$ **c** $\frac{3x - 3}{28} \div \frac{x - 1}{7}$

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4 Solve these linear equations.

a $3 - 2x = 9$

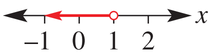
b $3(2x + 1) = 7 - 2(x + 5)$

c $\dfrac{5x - 9}{4} = -2$

d $\dfrac{2x - 1}{3} = \dfrac{x + 2}{4}$

5 Write as an inequality.

a



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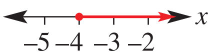

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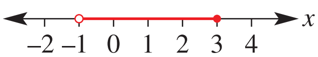


6 Solve the following inequalities.

a $4x - 3 > 17$

b $3x + 2 \leqslant 4(x - 2)$

c $1 - \dfrac{x}{3} < 2$

d $-2x \geqslant -4(1 - 3x)$

7 Marie's watering can is initially filled with 2 litres of water. However, the watering can has a small hole in the base and is leaking at a rate of 0.4 litres per minute.

a Write a rule for the volume of water, V litres, in the can after t minutes.

b What volume of water remains after 90 seconds?

c How long will it take for all the water to leak out?

d If Marie fills the can with 2 litres of water at her kitchen sink, what is the maximum amount of time she can take to get to her garden if she needs at least 600 mL to water her roses?

8 Sketch graphs of the following linear relations, labelling the points where the graph cuts the axes.

a $y = 3x - 9$

b $y = 5 - 2x$

c $y = 3$

d $x = 5$

e $y = 2x$

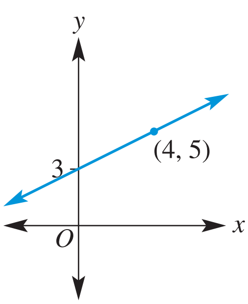
f $y = -5x$

g $x + 2y = 8$

h $3x + 8y = 24$

9 Find the equation of these straight lines.

a



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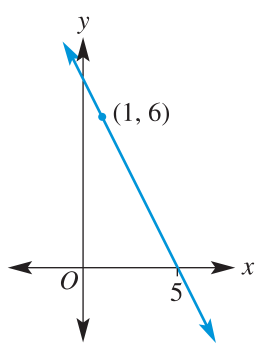

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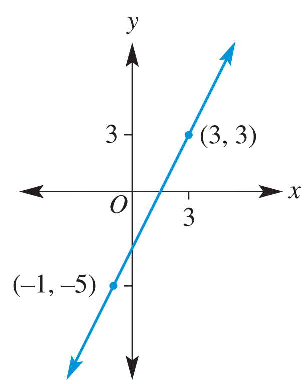

10 For the line that passes through the points $(-2, 8)$ and $(3, 5)$, determine:

a the gradient of the line

b the equation of the line

11 Find the midpoint and the exact length of the line segment joining these points.

a $(2, 5)$ and $(6, 11)$

b $(3, -2)$ and $(8, 4)$

c $(-1, -4)$ and $(2, -1)$

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12 Determine the equation of the line that is:

a parallel to the line $y = 3x + 8$ and passes through the point $(2, 4)$

b parallel to the line with equation $y = 4$ and passes through the point $(3, -1)$

c perpendicular to the line $y = 2x - 4$ and has a y-intercept of $(0, 5)$

d perpendicular to the line with equation $x + 3y = 5$ and passes through the point $(2, 5)$

13 Find the value(s) of the pronumeral in each situation below.

a The gradient of the line joining the points $(2, -5)$ and $(6, a)$ is 3.

b The line $bx + 2y = 7$ is parallel to the line $y = 4x + 3$.

c The distance between $(c, -1)$ and $(2, 2)$ is $\sqrt{13}$.

14 Solve the following simultaneous equations, using the substitution method.

a $y = 5x + 14$
$y = 2x + 5$

b $3x - 2y = 18$
$y = 2x - 5$

15 Solve these simultaneous equations by elimination.

a $3x + 2y = -11$
$2x - y = -5$

b $2y - 5x = 4$
$3y - 2x = 6$

16 At the movies Jodie buys three regular popcorns and five small drinks for her friends at a cost of \$24.50. Her friend Renee buys four regular popcorns and three small drinks for her friends at a cost of \$23.50. Find the individual costs of a regular popcorn and a small drink.

17 Sketch these regions.

a $y \geqslant 3x - 4$

b $2x - 3y > -8$

18 Shade the intersecting region of the inequalities $x + 2y \geqslant 4$ and $3x - 2y < 12$ by sketching their regions on the same axes and finding their point of intersection.

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Extended-response questions

1 There are two shrubs in Chen's backyard that grow at a constant rate. Shrub A had an initial height of 25 cm and has grown to 33 cm after 2 months. Shrub B was 28 cm high after 2 months and 46 cm high after 5 months.

a Write a rule for the height, h cm, after t months for:

i shrub A

ii shrub B

b What was the initial height ($t = 0$) of shrub B?

c Refer to your rules in part **a** to explain which shrub is growing at a faster rate.

d Graph each of your rules from part **a** on the same set of axes for $0 \leqslant t \leqslant 12$.

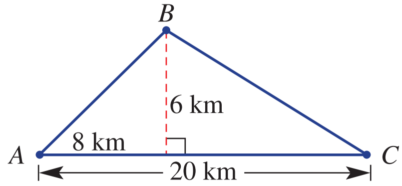
e Determine graphically after how many months the height of shrub B will overtake the height of shrub A.

f **i** Shrub B reaches its maximum height after 18 months. What is this height?

ii Shrub A has a maximum expected height of 1.3 m. After how many months will it reach this height?

iii Chen will prune shrub A when it is between 60 cm and 70 cm. Within what range of months after it is planted will Chen need to prune the shrub?

2 A triangular course has been roped off for a cross-country run. The run starts and ends at point A and goes via checkpoints B and C, as shown.



a Draw the area of land onto a set of axes, taking point A to be the origin (0, 0). Label the coordinates of B and C.

b Find the length of the course, to 1 decimal place, by calculating the distance of legs AB, BC and CA.

c A drink station is located at the midpoint of BC. Label the coordinates of the drink station on your axes.

d Find the equation of each leg of the course:

i AB

ii BC

iii CA

e Write a set of three inequalities that would overlap to form an intersecting region equal to the area occupied by the course.

f A fence line runs beyond the course. The fence line passes through point C and would intersect AB at right angles if AB was extended to the fence line. Find the equation of the fence line.

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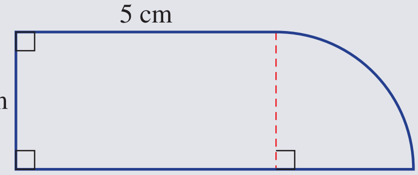
Semester review 1

Chapter 1: Measurement

Multiple-choice questions

1 The perimeter and area for this shape, correct to 2 decimal places, are:

A $P = 12.71$ cm, $A = 16.77$ cm^2
B $P = 20.71$ cm, $A = 22.07$ cm^2
C $P = 25.42$ cm, $A = 29.14$ cm^2
D $P = 18.36$ cm, $A = 43.27$ cm^2
E $P = 17.71$ cm, $A = 17.25$ cm^2



2 0.04 m^2 is equivalent to:

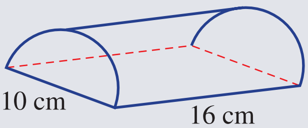
A 4 cm^2 B 40 mm^2 C 0.0004 cm^2 D 400 cm^2 E 0.00004 km^2

3 A square-based pyramid has a base area of 30 m^2 and a height of 7 m. Therefore, its volume is:

A 105 m^3 B 70 m^3 C 210π m^3 D 210 m^3 E 140 m^3

4 The curved surface area of this half cylinder, in exact form, is:

A 80π cm^2 B 105π cm^2 C 92.5π cm^2
D 120π cm^2 E 160π cm^2



5 The volume of a sphere of diameter 30 cm is closest to:

A 113 097 cm^3 B 2827 cm^3 C 11 310 cm^3 D 14 137 cm^3 E 7069 cm^3

Short-answer questions

1 Convert each of the following units to those given in the brackets.

a 23 m (mm)
b 8 s (ms)
c 7.8 s (ns)
d 8000 t (Mt)
e 2.3×10^{12} MB (TB)

2 Give the limits of accuracy for each of the following measurements.

a 7 s
b 8.99 g
c 700 km (given to three significant figures)
d 700 km (given to two significant figures)

3 A rectangle has the dimensions 4.3 m by 6.8 m.

a Between what two values does the true length lie?
b What are the limits of accuracy for the breadth of this rectangle?
c What are the limits of accuracy for the perimeter and area of this rectangle?

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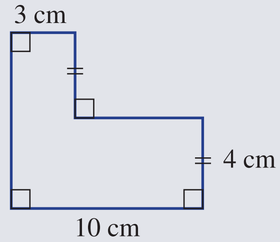
4 Find the perimeter and area of these shapes. Give your answers correct to 1 decimal place where necessary. You will need to use Pythagoras' theorem for part **c**.

a



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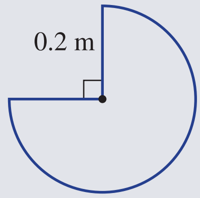

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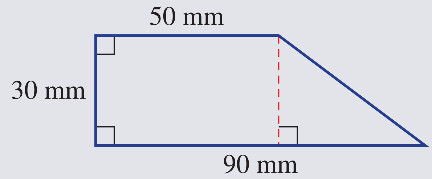
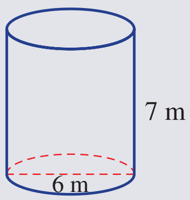


5 Find the surface area and volume for these solids. Give your answers correct to 1 decimal place.

a



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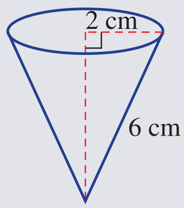

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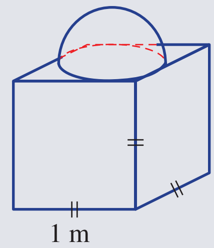

6 A rectangular prism has length 5 cm, breadth 3 cm and volume 27 cm^3.

a Find the height of the prism.

b Find the surface area of the prism.

7 A cone has volume 90 cm^3 and height 10 cm. Find the exact radius of the cone.

Extended-response question

1 A cylindrical glass vase is packaged inside a box that is a rectangular prism, so that the vase touches the box on all four sides and is the same height as the box. The vase has a diameter of 8 cm and a height of 15 cm. Complete the following, rounding your answers to 2 decimal places where necessary.

a Find the volume of the vase.

b Find the volume of space inside the box but outside the vase.

c A glass stirring rod is included in the vase. Find the length of the longest rod that can be packaged inside the vase.

d Find the difference in the length of rod in part **c** and the longest rod that can fit inside the empty box.

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Chapter 2: Indices and surds

Multiple-choice questions

1 The simplified form of $2\sqrt{45}$ is:

A $\sqrt{90}$ **B** $6\sqrt{5}$ **C** $10\sqrt{3}$ **D** $18\sqrt{5}$ **E** $6\sqrt{15}$

2 $7\sqrt{3} - 4\sqrt{2} + \sqrt{12} + \sqrt{2}$ simplifies to:

A $7\sqrt{3} - 3\sqrt{2}$ **B** $5\sqrt{3} + 3\sqrt{2}$ **C** $\sqrt{3} - 3\sqrt{2}$ **D** $9\sqrt{3} - 3\sqrt{2}$ **E** $3\sqrt{3} + 5\sqrt{2}$

3 The expanded form of $(2\sqrt{6} + 1)(3 - 4\sqrt{6})$ is:

A 3 **B** $\sqrt{6} - 33$ **C** $8\sqrt{6} + 3$ **D** $2\sqrt{6}$ **E** $2\sqrt{6} - 45$

4 The simplified form of $\dfrac{12(a^3)^{-2}}{(2ab)^2 \times a^2b^{-1}}$, when written using positive indices, is:

A $\dfrac{6}{a^2b}$ **B** $3a^2$ **C** $\dfrac{6a}{b}$ **D** $\dfrac{3}{a^2b^3}$ **E** $\dfrac{3}{a^{10}b}$

5 0.00032379 in scientific notation, using three significant figures, is:

A 3.23×10^{-4} **B** 3.24×10^{4} **C** 3.24×10^{-4}

D 32.4×10^{3} **E** 0.324×10^{-5}

Short-answer questions

1 Simplify:

a $\sqrt{54}$ **b** $4\sqrt{75}$ **c** $\dfrac{3\sqrt{24}}{2}$ **d** $\sqrt{5} \times \sqrt{2}$ **e** $3\sqrt{7} \times \sqrt{7}$

f $3\sqrt{6} \times 4\sqrt{8}$ **g** $\sqrt{15} \div \sqrt{5}$ **h** $\dfrac{3\sqrt{30}}{9\sqrt{6}}$ **i** $\sqrt{\dfrac{200}{49}}$

2 Simplify fully.

a $2\sqrt{5} + 3\sqrt{7} + 5\sqrt{5} - 4\sqrt{7}$ **b** $\sqrt{20} - 2\sqrt{5}$ **c** $\sqrt{18} - 4 + 6\sqrt{2} - 2\sqrt{50}$

3 Expand and simplify these expressions.

a $2\sqrt{3}(\sqrt{5} - 2)$ **b** $(3\sqrt{5} - 2)(1 - 4\sqrt{5})$

c $(\sqrt{7} - \sqrt{3})(\sqrt{7} + \sqrt{3})$ **d** $(3\sqrt{3} + 4\sqrt{2})^2$

4 Rationalise the denominator.

a $\dfrac{3}{\sqrt{2}}$ **b** $\dfrac{2\sqrt{3}}{5\sqrt{6}}$ **c** $\dfrac{2 - \sqrt{5}}{\sqrt{5}}$

5 Use the index laws to simplify the following. Express all answers with positive indices.

a $(2x^2)^3 \times 3x^4y^2$ **b** $\left(\dfrac{3a}{b^4}\right)^2 \times \dfrac{2b^{10}}{6(2a^5)^0}$ **c** $3a^{-5}b^2$ **d** $\dfrac{4x^{-2}y^3}{10x^{-4}y^6}$

6 Convert:

a to a basic numeral

  i 3.72×10^4 **ii** 4.9×10^{-6}

b to scientific notation, using three significant figures

  i 0.000072973 **ii** 4 725 400 000

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7 **a** Express in index form.

i $\sqrt{10}$ **ii** $\sqrt{7x^6}$ **iii** $4^5\sqrt{x^3}$ **iv** $15\sqrt{15}$

b Express in surd form.

i $6^{\frac{1}{2}}$ **ii** $20^{\frac{1}{5}}$ **iii** $7^{\frac{3}{4}}$

8 Evaluate without using a calculator.

a 5^{-1} **b** 2^{-4} **c** $81^{\frac{1}{4}}$ **d** $8^{-\frac{1}{3}}$

9 Solve these exponential equations for x.

a $4^x = 64$ **b** $7^{-x} = \frac{1}{49}$ **c** $9^x = 27$ **d** $5^{5x+1} = 125^x$

Extended-response question

1 Lachlan's share portfolio is rising at 8% per year and is currently valued at \$80000.

a Determine a rule for the value of Lachlan's share portfolio (V dollars) in n years' time.

b To the nearest dollar, what will be the value of the portfolio:

i next year? **ii** in 4 years' time?

c Use trial and error to find when Lachlan's share portfolio will be worth \$200000. Give your answer to 2 decimal places.

d After 4 years, however, the market takes a downwards turn and the share portfolio begins losing value. Two years after the downturn, Lachlan sells his shares for \$96170. If the market was declining in value at a constant percentage per year, what was this rate of decline, to the nearest percentage?

Chapter 3: Probability

Multiple-choice questions

1 The number of tails obtained from 100 tosses of two coins is shown in the table.

Number of tails	0	1	2
Frequency	23	57	20

From this table, the experimental probability of obtaining two tails is:

A 0.23 **B** 0.25 **C** 0.2 **D** 0.5 **E** 0.77

2 From the given two-way table $P(A \cap \overline{B})$ is:

	A	$\overline{A}$	
B	2		
$\overline{B}$			5
	6		12

A $\frac{1}{2}$ **B** $\frac{2}{3}$ **C** $\frac{1}{4}$ **D** $\frac{4}{5}$ **E** $\frac{1}{3}$

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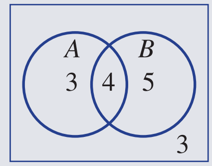
3 Two events, A and B, are such that $P(A) = 0.7$, $P(B) = 0.4$, and $P(A \cap B) = 0.3$. $P(A \cup B)$ is equal to:

A 1.4 **B** 0.8 **C** 0.6 **D** 0 **E** 0.58

4 From the information in the Venn diagram, $P(A|B)$ is:



A $\frac{5}{12}$ **B** $\frac{4}{5}$ **C** $\frac{4}{7}$

D $\frac{4}{9}$ **E** $\frac{1}{3}$

5 A bag of 5 marbles contains 2 green ones. Two marbles are selected without replacement. The probability of selecting the 2 green marbles is:

A $\frac{9}{20}$ **B** $\frac{2}{25}$ **C** $\frac{1}{10}$ **D** $\frac{2}{5}$ **E** $\frac{4}{25}$

Short-answer questions

1 Consider events A and B. Event A is the set of letters in the word 'grape' and event B is the set of letters in the word 'apricot':

$A = \{g, r, a, p, e\}$ $B = \{a, p, r, i, c, o, t\}$

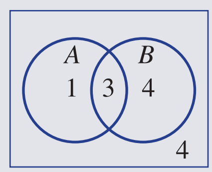
a Represent the two events A and B in a Venn diagram.

b If a letter is selected randomly from the alphabet, find:

i $P(A)$ **ii** $P(A \cap B)$ **iii** $P(A \cup B)$ **iv** $P(\overline{B})$

c Are the events A and B mutually exclusive? Why or why not?

2 The Venn diagram shows the distribution of elements in two sets, A and B.



a Transfer the information in the Venn diagram to a two-way table.

b Find:

i $n(A \cap B)$ **ii** $n(\overline{A} \cap B)$ **iii** $n(\overline{B})$ **iv** $n(A \cup B)$

c Find:

i $P(A \cap B)$ **ii** $P(A \cap \overline{B})$ **iii** $P(B)$ **iv** $P(B|A)$

3 Two events, A and B, are such that $P(A) = 0.24$, $P(B) = 0.57$ and $P(A \cup B) = 0.63$. Find:

a $P(A \cap B)$ **b** $P(\overline{A} \cap \overline{B})$

4 Two fair 4-sided dice, numbered 1 to 4, are rolled and the total is noted.

a List the sample space as a table.

b State the total number of outcomes.

c Find the probability of obtaining:

i a sum of 4

ii a sum of at least 5

iii a sum of 7, given the sum is at least 5

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5 In a group of 12 friends, 8 study German, 4 study German only and 2 study neither German nor Mandarin.
Let A be the event 'studies German'.
Let B be the event 'studies Mandarin'.
- a Summarise the information in a Venn diagram.
- b Find:
  - i $P(A)$
  - ii $P(A|B)$
- c State whether or not the events A and B are independent.

Extended-response question

1 Lindiana Jones selects two weights from her pocket to sit on a weight-sensitive trigger device after removing the goblet of fire. Her pocket contains three weights, each weighing 200 g, and five weights, each weighing 250 g. The two weights are selected without replacement. Use a tree diagram to help answer the following.
- a Find the probability that Lindiana selects two weights totalling:
  - i 400 g
  - ii 450 g
  - iii 500 g
- b If the total weight selected is less than 480 g, a poison dart will shoot from the wall. Find the probability that Lindiana is at risk from the poison dart.
- c By feeling the weight of her selection, Lindiana knows that the total weight is more than 420 g. Given this information, what is the probability that the poison dart will be fired from the wall?

Chapter 4: Single variable and bivariate statistics

Multiple-choice questions

1 For the given stem-and-leaf plot, the range and median, respectively, of the data are:

Stem	Leaf
0	2 2 6 7
1	0 1 2 3 5 8
2	3 3 5 7 9

1|5 means 15

A 20, 12.5 B 7, 12 C 27, 12.5 D 29, 3 E 27, 13

2 The interquartile range (IQR) for the dataset 2, 3, 3, 7, 8, 8, 10, 13, 15 is:
A 5 B 8.5 C 7 D 13 E 8

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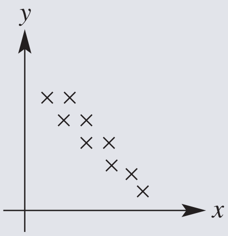
3 The best description of the correlation between the variables for the scatter plot shown is:



A weak, negative
B strong, positive
C strong, negative
D weak, positive
E no correlation

4 A line of best fit passes through the points (10, 8) and (15, 18). The equation of the line is:

A $y = \frac{2}{3}x + 8$
B $y = 2x - 12$
C $y = -\frac{1}{2}x + 13$
D $y = 2x + 6$
E $y = \frac{1}{2}x + 3$

5 The mean and standard deviation of the small dataset 2, 6, 7, 10, 12, correct to 1 decimal place, are:

A $\bar{x} = 7.4$ and $\sigma = 3.4$
B $\bar{x} = 7$ and $\sigma = 3.7$
C $\bar{x} = 7.4$ and $\sigma = 3.8$
D $\bar{x} = 7$ and $\sigma = 7.7$
E $\bar{x} = 27.1$ and $\sigma = 9.9$

Short-answer questions

1 Twenty people were surveyed to find out how many days in the past completed month they had used public transport. The results are as follows.

7, 16, 22, 23, 28, 12, 18, 4, 0, 5
8, 19, 20, 22, 14, 9, 21, 24, 11, 10

a Organise the data into a frequency table with class intervals of 5 and include a percentage frequency column.
b Construct a histogram for the data, showing both the frequency and the percentage frequency on the same graph.
c **i** State the frequency of people who used public transport on 10 or more days.
ii State the percentage of people who used public transport on fewer than 15 days.
iii State the most common interval of days for which public transport was used. Can you think of a reason for this?

2 By first finding quartiles and checking for outliers, draw box plots for the following datasets.

a 8, 10, 2, 17, 6, 25, 12, 7, 12, 15, 4
b 5.7, 4.8, 5.3, 5.6, 6.2, 5.7, 5.8, 5.1, 2.6, 4.8, 5.7, 8.3, 7.1, 6.8

3 Farsan's bank balance over 12 months is recorded below.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Balance ($)	1500	2100	2300	2500	2200	1500	1200	1600	2000	2200	1700	2000

a Plot the time series for the 12 months.
b Describe the way in which the bank balance has changed over the 12 months.
c Between which consecutive months did the biggest change in the bank balance occur?
d What is the overall change in the bank balance over the year?

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4 Consider the variables x and y and the corresponding bivariate data below.

x	4	5	6	7	8	9	10
y	2.1	2.5	3.1	2.8	4	3.6	4.9

a Draw a scatter plot for the data.
b Describe the correlation between x and y as positive, negative or none.
c Fit a line of good fit by eye to the data on the scatter plot.
d Use your line of good fit to estimate:
i y when $x = 7.5$ **ii** x when $y = 5.5$

5 The back-to-back stem-and-leaf plot shows the number of DVDs owned by people in two different age groups.

Over 40 years Leaf	Stem	Under 40 years Leaf
9 7 6 6 4 3 2	0	7 8
6 4 3 2 2 0	1	2 5 5 7 8
8 3	2	4 4 6
	3	2 6 9
	4	1 8

2|4 means 24

a By considering the centre and spread of the data, state with reasons:
i Which dataset will have the higher mean?
ii Which dataset will have the smaller standard deviation?
b Calculate the mean and standard deviation for each dataset. Round your answer to 1 decimal place where necessary.

Extended-response question

1 The height of a group of the same species of plant after a month of watering with a set number of millilitres of water per day is recorded below.

Water (mL)	8	5	10	14	12	15	18
Height (cm)	25	27	34	40	35	38	45

a Using *Water* for the x-axis, find the equation of the least squares regression line.
b Use your least squares regression line to estimate the height (to the nearest cm) of a plant watered with 16 mL of water per day.

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Chapter 5: Expressions, equations and linear relationships

Multiple-choice questions

1 The simplified form of $2x(4 - 3y) - 3(3x - 4xy)$ is:

A $6xy - x$ **B** $2xy$ **C** $x - 18xy$
D $12xy - 7x$ **E** $2x - 3y - 4xy$

2 The point that is not on the line $y = 3x - 2$ is:

A $(-1, -5)$ **B** $(1, 1)$ **C** $(-2, -4)$
D $(4, 10)$ **E** $(0, -2)$

3 The length, d, and midpoint, M, of the line segment that joins the points $(-2, 4)$ and $(3, -2)$ are:

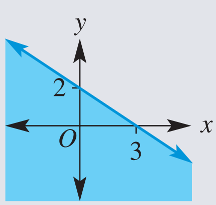
A $d = \sqrt{5}, M = (0.5, 1)$ **B** $d = \sqrt{61}, M = (2.5, 3)$ **C** $d = \sqrt{29}, M = (1, 1)$
D $d = \sqrt{61}, M = (0.5, 1)$ **E** $d = \sqrt{11}, M = (1, 2)$

4 The equation of the line that is perpendicular to the line with equation $y = -2x - 1$ and passes through the point $(1, -2)$ is:

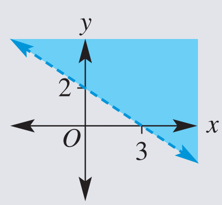
A $y = -\frac{1}{2}x + \frac{3}{2}$ **B** $y = 2x - 2$ **C** $y = -2x - 4$
D $y = x - 2$ **E** $y = \frac{1}{2}x - \frac{5}{2}$

5 The graph of $3x + 2y < 6$ is:

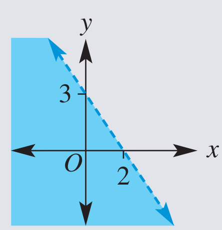
A



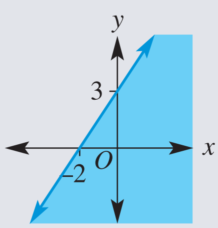
B



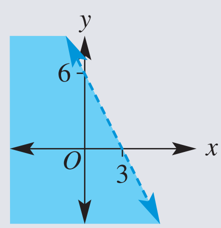
C



D



E



Short-answer questions

1 Simplify:

a $\frac{12 - 8x}{4}$ **b** $\frac{5x - 10}{3} \times \frac{12}{x - 2}$ **c** $\frac{3}{4} - \frac{2}{a}$ **d** $\frac{4}{x + 2} + \frac{5}{x - 3}$

2 **a** Solve these equations for x.

i $2 - 3x = 14$ **ii** $2(2x + 3) = 7x$ **iii** $\frac{x - 3}{2} = 5$ **iv** $\frac{3x - 2}{4} = \frac{2x + 1}{5}$

b Solve these inequalities for x and graph their solutions on a number line.

i $3x + 2 \leqslant 20$ **ii** $2 - \frac{x}{3} > 1$

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3 **a** Find the gradient and y-intercept for these linear relations and sketch each graph.

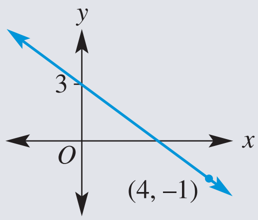
i $y = 3x - 2$ **ii** $4x + 3y = 6$

b Sketch by finding the x- and y-intercepts where applicable.

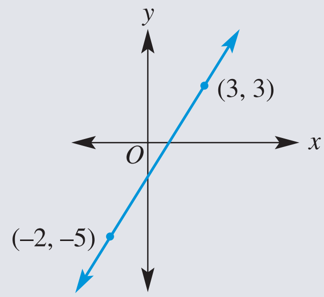
i $y = 2x - 6$ **ii** $3x + 5y = 15$ **iii** $x = 3$ **iv** $y = -2x$

4 Find the equation of the straight lines shown.

a



b



5 Find the value(s) of a in each of the following when:

a The lines $y = ax - 3$ and $y = -3x + 2$ are parallel.

b The gradient of the line joining the points $(3, 2)$ and $(5, a)$ is -3.

c The distance between $(3, a)$ and $(5, 4)$ is $\sqrt{13}$.

d The lines $y = ax + 4$ and $y = \frac{1}{4}x - 3$ are perpendicular.

6 Solve these pairs of simultaneous equations.

a $y = 2x - 1$, $y = 5x + 8$ **b** $2x - 3y = 8$, $y = x - 2$ **c** $2x + y = 2$, $5x + 3y = 7$ **d** $3x - 2y = 19$, $4x + 3y = -3$

7 At a fundraising event, two hot dogs and three cans of soft drink cost \$13, and four hot dogs and two cans of soft drink cost \$18. What are the individual costs of a hot dog and a can of soft drink?

8 Shade the region for these linear inequalities.

a $y \geqslant 3 - 2x$ **b** $3x - 2y < 9$ **c** $y > -3$

Extended-response question

1 A block of land is marked on a map with coordinate axes and with boundaries given by the equations $y = 4x - 8$ and $3x + 2y = 17$.

a Solve the two equations simultaneously to find their point of intersection.

b Sketch each equation on the same set of axes, labelling axis intercepts and the point of intersection.

The block of land is determined by the intersecting region $x \geqslant 0$, $y \geqslant 0$, $y \geqslant 4x - 8$ and $3x + 2y \leqslant 17$.

c Shade the area of the block of land (i.e. the intersecting region on the graph in part **b**).

d Find the area of the block of land if 1 unit represents 100 m.

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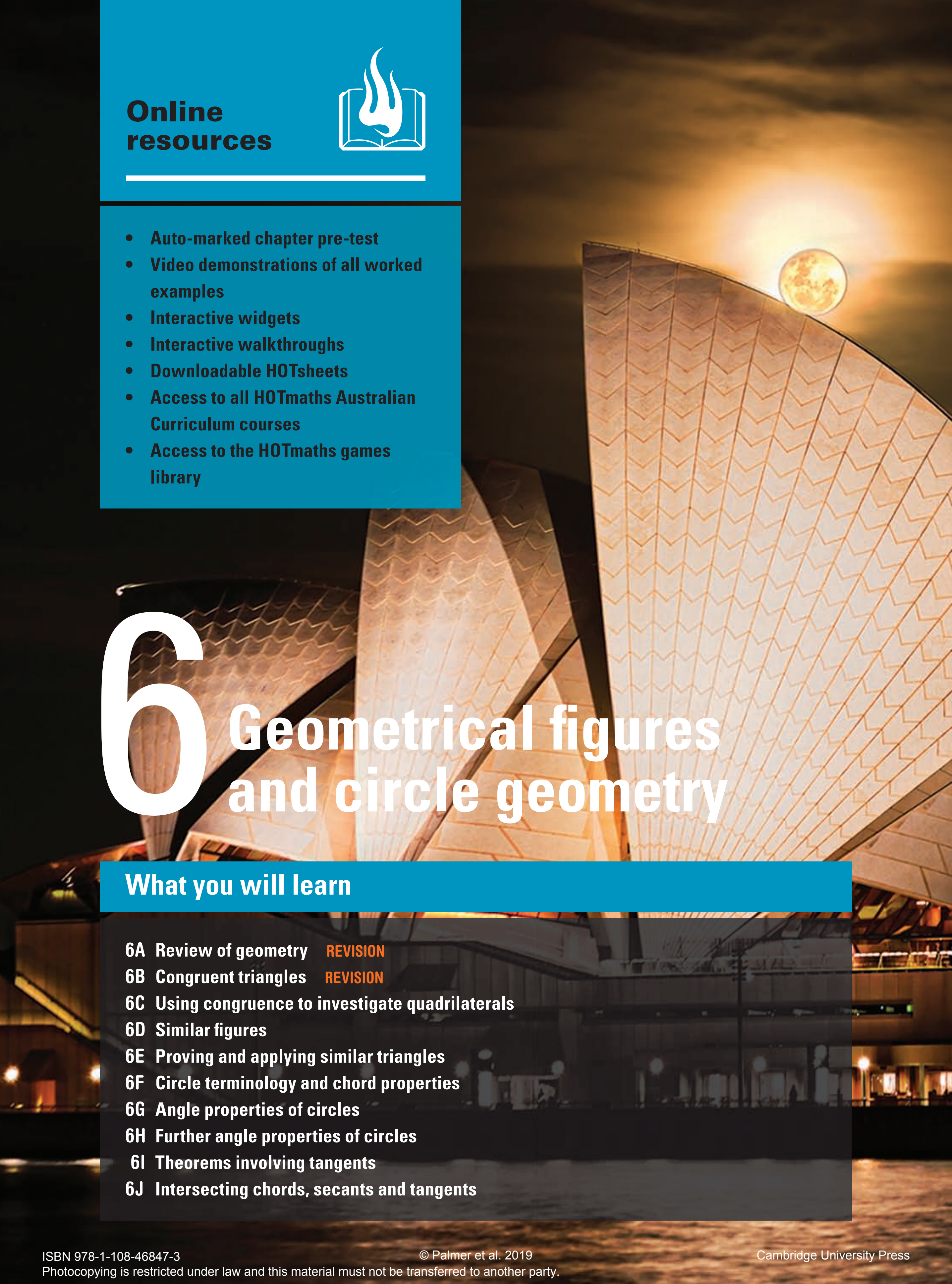

Online resources

- Auto-marked chapter pre-test
- Video demonstrations of all worked examples
- Interactive widgets
- Interactive walkthroughs
- Downloadable HOTsheets
- Access to all HOTmaths Australian Curriculum courses
- Access to the HOTmaths games library

6 Geometrical figures and circle geometry

What you will learn

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NSW syllabus

STRAND: MEASUREMENT AND GEOMETRY

SUBSTRANDS: PROPERTIES OF GEOMETRICAL FIGURES
CIRCLE GEOMETRY

Outcomes

A student describes and applies the properties of similar figures and scale drawings.
(MA5.1–11MG)

A student calculates the angle sum of any polygon and uses minimum conditions to prove triangles are congruent or similar.
(MA5.2–14MG)

A student proves triangles are similar, and uses formal geometrical reasoning to establish properties of triangles and quadrilaterals.
(MA5.3–16MG)

A student applies deductive reasoning to prove circle theorems and to solve related problems.
(MA5.3–17MG)

Opera House geometry

The geometry of the Sydney Opera House is based on triangles drawn on a sphere. To understand how this was achieved, imagine an orange sliced into wedges and then each wedge cut into two pieces in a slanting line across the wedge. The orange skin of each wedge portion is a 3D triangle and illustrates the shape of one side of an Opera House sail. One full sail has two congruent curved triangular sides joined, each a reflection of the other. The edges of the sails form arcs of circles. All the curved sides of the 14 Opera House sails could be joined to form one very large, whole sphere.

The Danish architect Jørn Utzon and engineer Ove Arup chose a radius of 75 m for the virtual sphere from which to design the curved triangular sails. The calculations required high-level mathematical modelling, including the applications of circle and spherical geometry. Designed in the 1950s and 1960s, it was one of the first ever projects to use CAD (computer-assisted drawing).

Overall, the Opera House is 120 m wide, 85 m long and 67 m high (20 storeys above the water level). The sails are covered with about 1 million tiles and it is visited by over 8 million people each year. This photo shows the Opera House lit up during a 'Vivid Sydney' festival.

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Pre-test

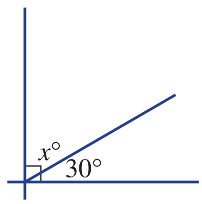
1 Write down as many properties of these shapes as you can remember.

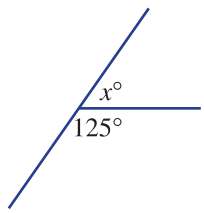
a equilateral triangle **b** isosceles triangle
c scalene triangle **d** right-angled triangle
e parallelogram **f** rectangle
g square **h** trapezium
i rhombus **j** kite

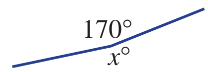
2 Which shapes from Question **1** have exactly:

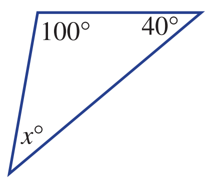
a one line of symmetry?
b two lines of symmetry?

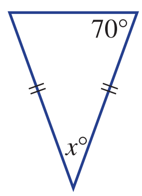
3 Find the value of the pronumeral in each diagram.

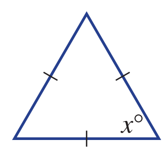
a


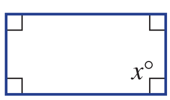
b


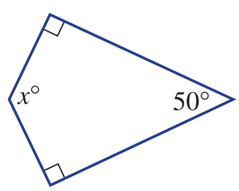
c


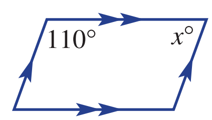
d


e


f


g


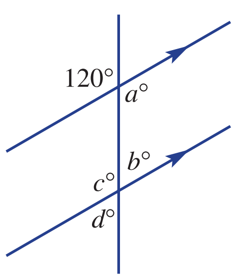
h


i


4 State whether the given pairs of angles are complementary, supplementary or neither.

a 36°, 144° **b** 18°, 72° **c** 110°, 40° **d** 94°, 86°

5 Find the values of a, b, c and d in the diagram shown.



6 Write down the names of all the polygons that have 3, 4, 5, 6, 7, 8, 9 and 10 sides.

7 Solve these simple equations for x.

a $x + 75 = 180$ **b** $2x + 42 = 180$ **c** $3x = 8$

d $\frac{x}{4} = 2.2$ **e** $\frac{x + 1}{3} = \frac{9}{2}$ **f** $x^2 = 36$ if $x > 0$

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6A Review of geometry REVISION

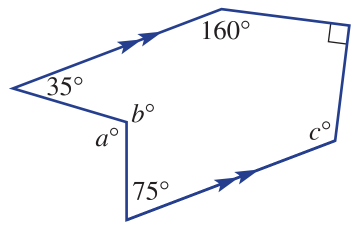
Interactive
Widgets
HOTsheets
Walkthrough

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Based on just five simple axioms (i.e. known or self-evident facts) the famous Greek mathematician Euclid (about 300 BC) was able to deduce many hundreds of propositions (theorems and constructions), which he systematically presented in the 13-volume book collection called the *Elements*. All the basic components of school geometry are outlined in these books, including the topics *angle sums of polygons* and *angles in parallel lines*, which are studied in this section.

Let's start: Three unknown angles

This hexagon contains a pair of parallel sides and three unknown angles: a, b and c.

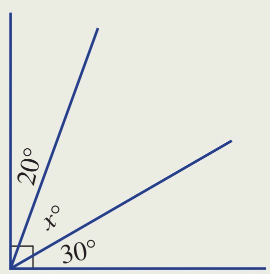


- Find the value of a using the given angles in the hexagon. Hint: Add a construction line parallel to the two parallel sides so that $a°$ is divided into two smaller angles. Give reasons throughout your solution.
- Find the value of b, giving a reason.
- What is the angle sum of a hexagon? Use this to find the value of c.
- Can you find a different method to find the value of c, using parallel lines?

Key ideas

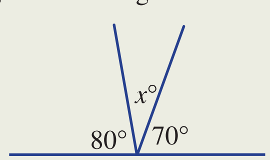
Terminology

- Two angles are **complementary** if their sum is 90°.
  For example: 70° is the **complement** of 20°.
- Two angles are **supplementary** if their sum is 180°.
  For example: 30° is the **supplement** of 150°.
- Angles in a right angle add to 90°.



$x + 20 + 30 = 90$ (angles in a right angle)

- Angles on a straight line add to 180°.



$x + 80 + 70 = 180$ (angles on a straight line)

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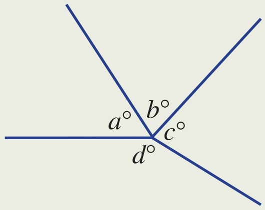
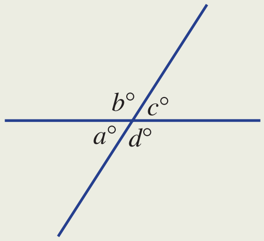
Key ideas

- Angles about a point add to 360° and form a revolution.



$a + b + c + d = 360$ (angles at a point)

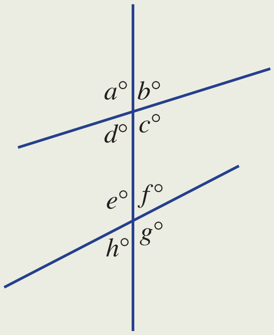
- Vertically opposite angles are equal.



$a = c$ (vertically opposite angles)

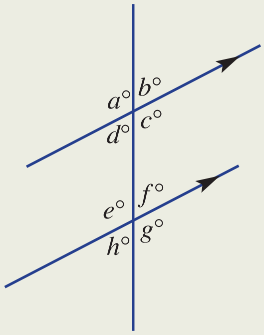
$b = d$ (vertically opposite angles)

- When a transversal crosses two lines, eight angles are formed.



Pairs of corresponding angles: $a°$ and $e°$
$b°$ and $f°$
$c°$ and $g°$
$d°$ and $h°$
Pairs of alternate angles: $c°$ and $e°$
$d°$ and $f°$
Pairs of cointerior angles: $c°$ and $f°$
$d°$ and $e°$

- When a transversal crosses parallel lines:



$a = e$ (corresponding angles on parallel lines)
$b = f$ (corresponding angles on parallel lines)
$c = g$ (corresponding angles on parallel lines)
$d = h$ (corresponding angles on parallel lines)

$c = e$ (alternate angles on parallel lines)
$d = f$ (alternate angles on parallel lines)

$c + f = 180$ (cointerior angles on parallel lines)
$d + e = 180$ (cointerior angles on parallel lines)

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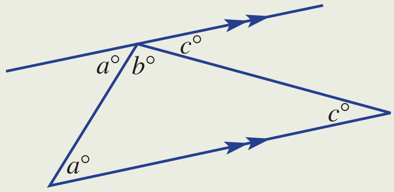
Key ideas

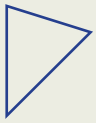
- Triangles
  - The angle sum of a triangle is 180°.
    To prove this, draw a line parallel to one side, then mark the alternate angles in parallel lines. Note that angles on a straight line add to 180°.



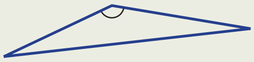
$a + b + c = 180$ (angle sum of a triangle)

  - Triangles classified by angles

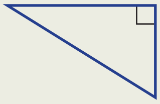
Acute: all angles acute



Obtuse: one angle obtuse

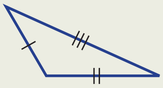


Right: one right angle

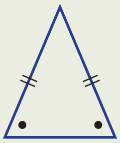


- In the following examples, the words in the brackets are 'reasons'. When you are asked to 'give reasons', take care not to abbreviate the reasons to such an extent that the meaning is lost.
  - Triangles classified by side lengths

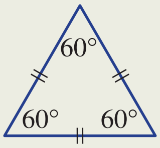
Scalene
(3 different sides)



Isosceles
(2 different sides oppposite
2 equal angles)



Equilateral
(3 equal sides)



- Formal definitions of the special quadrilaterals
  - A **kite** is a quadrilateral with two pairs of adjacent sides that are equal.
  - A **trapezium** is a quadrilateral with at least one pair of opposite sides that are parallel.
  - A **parallelogram** is a quadrilateral with both pairs of opposite sides parallel.
  - A **rhombus** is a parallelogram with two adjacent sides that are equal in length.
  - A **rectangle** is a parallelogram with one angle as a right angle.
  - A **square** is a rectangle with two adjacent sides that are equal in length.

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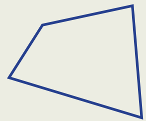
Key ideas

Quadrilateral

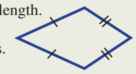


Two pairs of adjacent sides of equal length

One pair of parallel sides

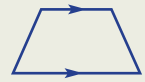
Kite properties

- Two pairs of adjacent sides of a kite are equal in length.
- One diagonal of a kite bisects the other diagonal.
- One diagonal of a kite bisects the opposite angles.
- The diagonals of a kite are perpendicular.



Trapezium properties

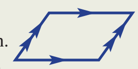
- One pair of sides of a trapezium are parallel.



Another pair of parallel sides

Parallelogram properties

- The opposite sides of a parallelogram are parallel.
- The opposite sides of a parallelogram are equal in length.
- The opposite angles of a parallelogram are equal in size.
- The diagonals of a parallelogram bisect each other.



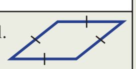
All sides are of equal length

All sides are of equal length

One right angle
∴ Four right angles

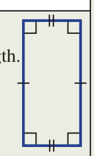
Rhombus properties

- The opposite sides of a rhombus are parallel.
- All sides of a rhombus are equal in length.
- The opposite angles of a rhombus are equal in size.
- The diagonals of a rhombus bisect the vertex angles.
- The diagonals of a rhombus bisect each other.
- The diagonals of a rhombus are perpendicular.



Rectangle properties

- The opposite sides of a rectangle are parallel.
- The opposite sides of a rectangle are equal in length.
- All angles at the vertices of a rectangle are 90°.
- The diagonals of a rectangle are equal in length.
- The diagonals of a rectangle bisect each other.

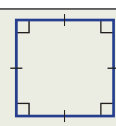


One right angle
∴ Four right angles

All sides are equal in length

Square properties

- Opposite sides of a square are parallel.
- All sides of a square are equal in length.
- All angles at the vertices of a square are 90°.
- The diagonals of a square are equal in length.
- The diagonals of a square bisect the vertex angles.
- The diagonals of a square bisect each other.
- The diagonals of a square are perpendicular.



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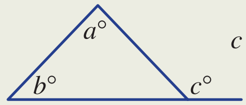
Key ideas

- **Polygons** have an angle sum given by $S = 180(n - 2)$, where n is the number of sides.
  - Regular polygons have equal angles and sides of equal length.

    A single interior angle $= \dfrac{180(n - 2)}{n}$

n	Name
3	triangle
4	quadrilateral
5	pentagon
6	hexagon
7	heptagon
8	octagon
9	nonagon
10	decagon

- An **exterior angle** is formed by extending a side.
  - For a triangle, the exterior angle theorem states that the exterior angle is equal to the sum of the two opposite interior angles.



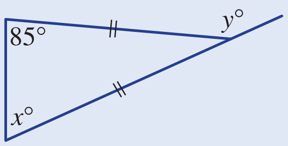
$c = a + b$ (exterior angle of a triangle)

- The sum of the exterior angles of any polygon is 360°.

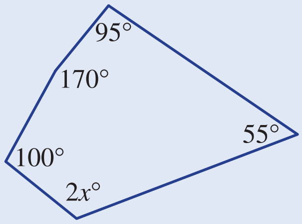
Example 1 Using exterior angles and angle sums

Find the value of x in the following, giving reasons.

a



b



SOLUTION	EXPLANATION
a $x = 85$ (equal angles are opposite sides of equal length)	Triangle is isosceles.
$y = 85 + 85$ $= 170$ (exterior angle of triangle)	Use the exterior angle theorem for a triangle.
b $S = 180(n - 2)$ $= 180 \times (5 - 2)$ $= 540$	Use the rule for the angle sum of a polygon (5 sides, so $n = 5$).
$\therefore 2x + 100 + 170 + 95 + 55 = 540$ $2x + 420 = 540$ $2x = 120$ $\therefore x = 60$	The sum of all the angles is 540°.

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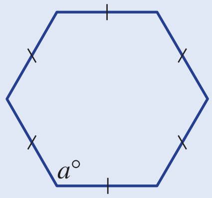
Example 2 Working with parallel lines and regular polygons

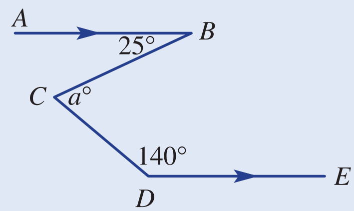
Find the value of the pronumeral, giving reasons.

a



Regular Hexagon

b



SOLUTION

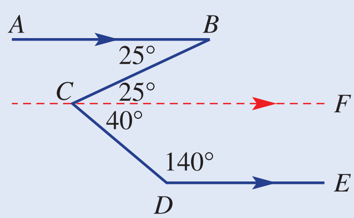
a $S = 180(n - 2)$
$= 180 \times (6 - 2)$
$= 720$
$a = 720 \div 6$
$= 120$

b Construct a third parallel line, CF.
$\angle BCF = 25°$ (alternate angles, $AB \parallel CF$)
$\angle FCD = 180° - 140°$
$= 40°$ (cointerior angles, $CF \parallel DE$)
$\therefore a = 25 + 40$ (adjacent angles)
$= 65$

EXPLANATION

Use the angle sum rule for a polygon with $n = 6$.

In a regular hexagon there are 6 equal angles.



$\angle ABC$ and $\angle FCB$ are alternate angles in parallel lines. $\angle FCD$ and $\angle EDC$ are cointerior angles in parallel lines.

Exercise 6A REVISION

UNDERSTANDING AND FLUENCY	1–7	3, 4, 5–6(½), 7	4, 5–6(½), 7

1 Copy and complete the table of polygons.

	Number of sides	Name	Interior angle sum	Exterior angle sum	If regular, size of interior angles
a	3				
b	4				
c	5				
d	6				
e	8				
f	10				
g	12				

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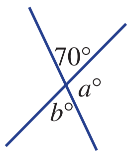
2 Decide if each of the following is true or false.

a The angle sum of a quadrilateral is 300°.
b A square has four lines of symmetry.
c An isosceles triangle has two sides of equal length.
d An exterior angle on an equilateral triangle is 120°.
e A kite has two pairs of equal opposite angles.
f A parallelogram is a rhombus.
g A square is a rectangle.
h Vertically opposite angles are supplementary.
i Cointerior angles in parallel lines are supplementary.

3 Find the values of the pronumerals, giving reasons.

a



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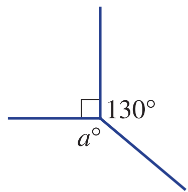
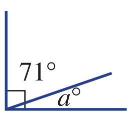


c



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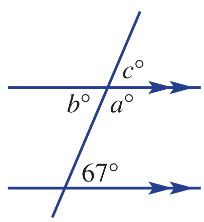

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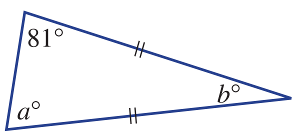

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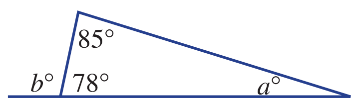
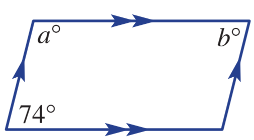


g



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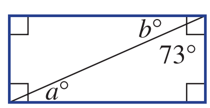

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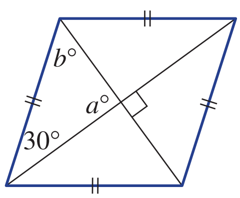
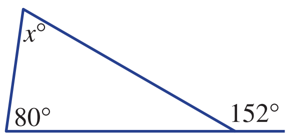


Example 1a

4 Using the exterior angle theorem, find the value of the pronumeral.

a



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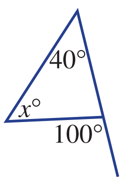

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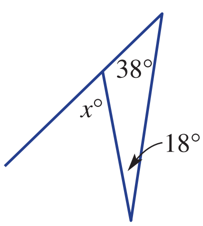

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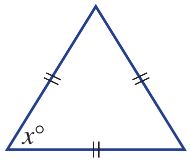
5 Find the value of the pronumeral.

a


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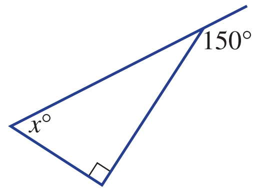

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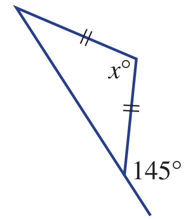

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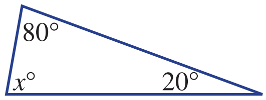

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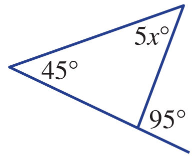

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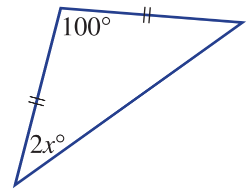

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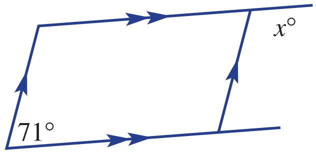

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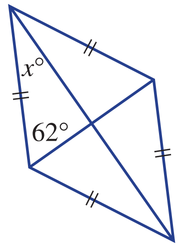

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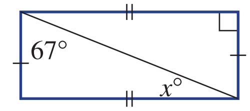

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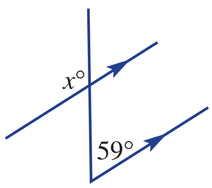

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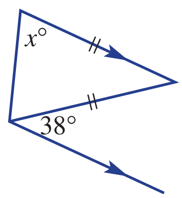

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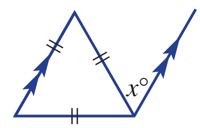
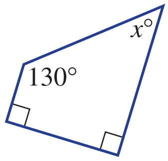


Example 1b

6 Find the value of x in the following, giving reasons.

a


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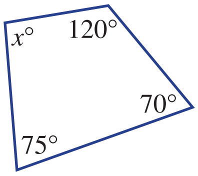

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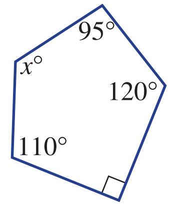

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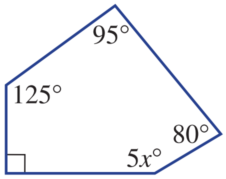

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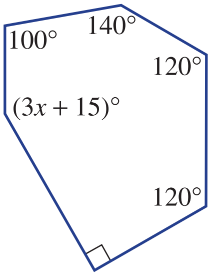

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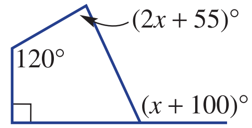

Example 2a

7 Find the size of an interior angle of these polygons if they are regular.

a pentagon **b** octagon **c** decagon

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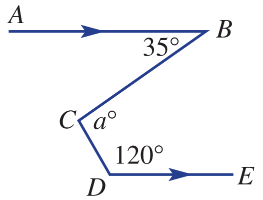
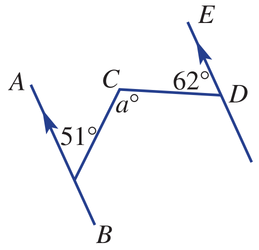
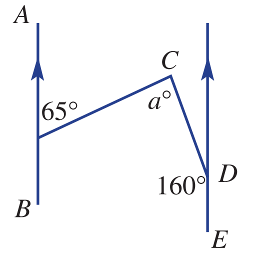
PROBLEM-SOLVING AND REASONING | 8(½), 9, 10, 13, 14 | 8(½), 9–11, 13–15 | 8(½), 10–12, 14–17

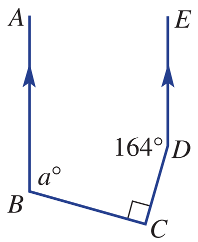
Example 2b

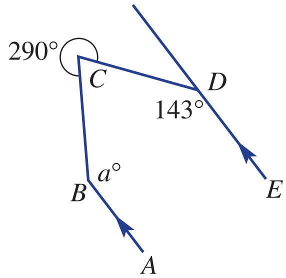
8 Find the value of the pronumeral a.

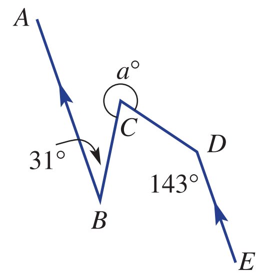
a 

b 

c 

d 

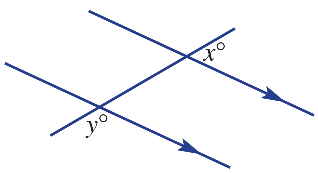
e 

f 

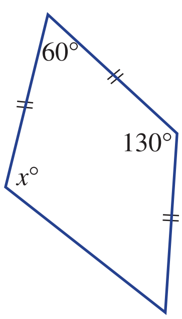
9 **a** Find the size of an interior angle of a regular polygon with 100 sides.
b What is the size of an exterior angle of a 100-sided regular polygon?

10 Find the number of sides of a polygon that has the following interior angles.
a 150° **b** 162° **c** 172.5°

11 In this diagram $y = 4x$. Find the values of x and y.



12 Find the value of x in this diagram. Hint: Form isosceles and/or equilateral triangles.



13 The rule for the sum of the interior angles of a polygon is given by $S = 180(n - 2)$.
a Show that $S = 180n - 360$.
b Find a rule for the number of sides n of a polygon with an angle sum S; i.e. write n in terms of S.
c Write the rule for the size of an interior angle I of a regular polygon with n sides.
d Write the rule for the size of an exterior angle E of a regular polygon with n sides.

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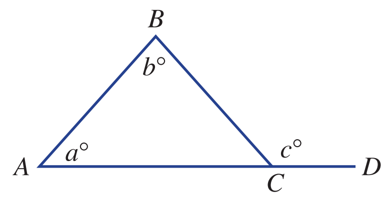
14 Prove that the exterior angle of a triangle is equal to the sum of the two opposite interior angles by following these steps.

a Write $\angle BCA$ in terms of a and b and give a reason.

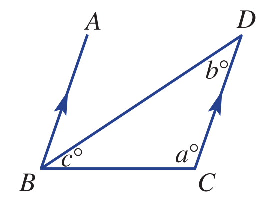
b Find c in terms of a and b using $\angle BCA$ and give a reason.



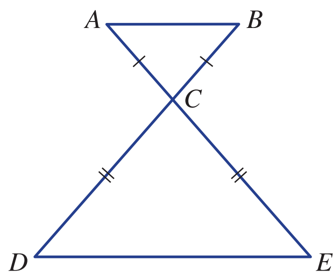
15 **a** Explain why $\angle ABD$ is equal to $b°$ in this diagram.

b Using $\angle ABC$ and $\angle BCD$, what can be said about a, b and c?

c What does your answer to part **b** show?



16 Give reasons why AB and DE in this diagram are parallel; i.e. $AB \parallel DE$.



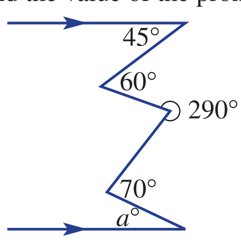
17 Each point on the Earth's surface can be described by a line of longitude (i.e. degrees east or west from Greenwich, England) and a line of latitude (i.e. degrees north or south from the equator). Investigate and write a brief report (providing examples) describing how places on Earth can be located with the use of longitude and latitude.

ENRICHMENT	–	–	18–20

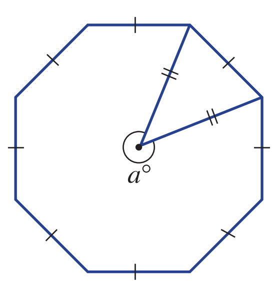
Multilayered reasoning

18 Find the value of the pronumerals, giving reasons.

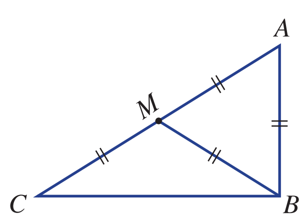
a



b



19 Give reasons why $\angle ABC = 90°$.



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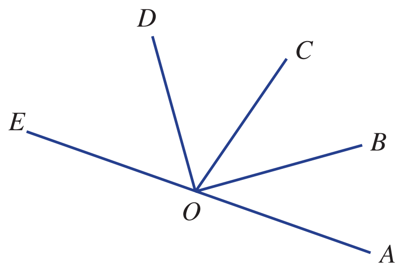
20 In this diagram $\angle AOB = \angle BOC$ and $\angle COD = \angle DOE$. Give reasons why $\angle BOD = 90°$.

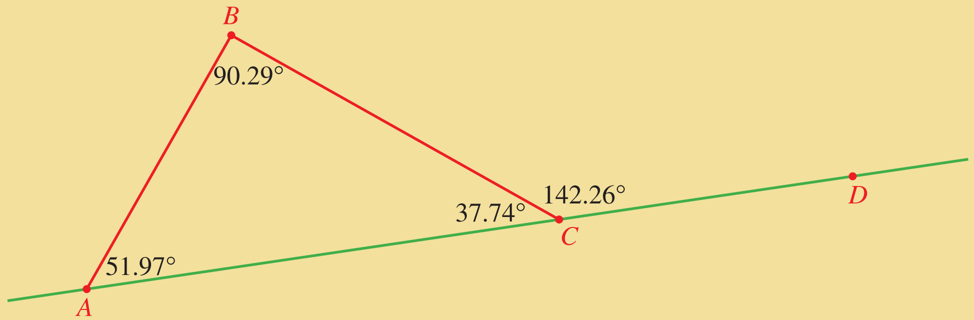


Exploring triangles with dynamic geometry software

1 Construct a triangle and illustrate the angle sum and the exterior angle theorem.

Dynamic geometry instructions and screens

a Construct a line AD and triangle ABC, as shown below.
b Measure all the interior angles and the exterior angle $\angle BCD$.
c Check that the angle sum is 180° and that $\angle BCD = \angle BAC + \angle ABC$.
d Drag one of the points to demonstrate that these properties are retained for all triangles.



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6B Congruent triangles REVISION

Interactive
Widgets
HOTsheets
Walkthrough

In geometry it is important to know whether or not two objects are, in fact, identical in shape and size. If two objects are identical, then we say they are congruent. Two shapes that are congruent will have corresponding (i.e. matching) sides equal in length and corresponding angles that are also equal. For two triangles it is not necessary to know every side and angle to determine if they are congruent. Instead, a set of minimum conditions is enough. There are four sets of minimum conditions for triangles and these are known as the tests for congruence of triangles.

Are the Deria Twin Towers in Dubai congruent?

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Which are congruent?

Consider these four triangles.

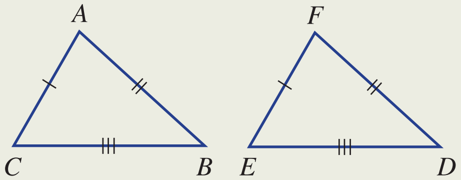
1. $\triangle ABC$ with $\angle A = 37°$, $\angle B = 112°$ and $AC = 5\,\text{cm}$
2. $\triangle DEF$ with $\angle D = 37°$, $DF = 5\,\text{cm}$ and $\angle E = 112°$
3. $\triangle GHI$ with $\angle G = 45°$, $GH = 7\,\text{cm}$ and $HI = 5\,\text{cm}$
4. $\triangle JKL$ with $\angle J = 45°$, $JK = 7\,\text{cm}$ and $KL = 5\,\text{cm}$

Shamila says that only $\triangle ABC$ and $\triangle DEF$ are congruent. George says that only $\triangle GHI$ and $\triangle JKL$ are congruent, whereas Tristan says that both pairs ($\triangle ABC$, $\triangle EDF$) and ($\triangle GHI$, $\triangle JKL$) are congruent.

- Discuss which pairs of triangles might be congruent, giving reasons.
- What drawings can be made to support your argument?
- Who is correct: Shamila, George or Tristan? Explain why.

Key ideas

- Two objects are said to be **congruent** if they are exactly the same size and shape. For two congruent triangles $\triangle ABC$ and $\triangle DEF$, we write $\triangle ABC \equiv \triangle DEF$.
  - When comparing two triangles, corresponding sides are equal in length and corresponding angles are equal in size.
  - When we prove congruence in triangles, we usually write vertices in matching order.
- Two triangles can be proved **congruent** using one of the following tests.
  - Three sides of a triangle are equal to three sides of another triangle (SSS test).



$\triangle ABC \equiv \triangle FDE$

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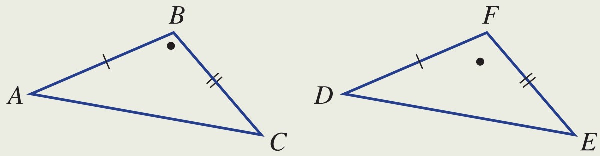
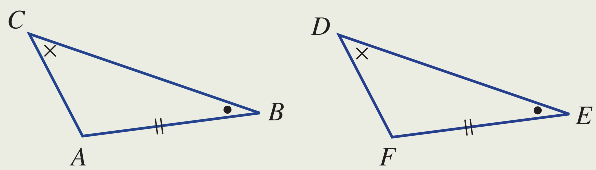
Key ideas

- Two sides and the included angle of a triangle are equal to two sides and the included angle of another triangle (SAS test).



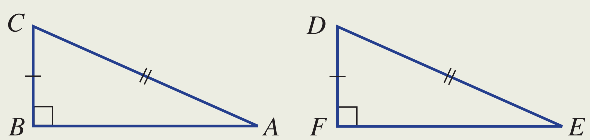
$\triangle ABC \equiv \triangle DFE$

- Two angles and one side of a triangle are respectively equal to two angles and one side of another triangle (AAS test).



$\triangle ABC \equiv \triangle FED$

- The hypotenuse and a second side of one triangle are respectively equal to the hypotenuse and a second side of another triangle (RHS test).



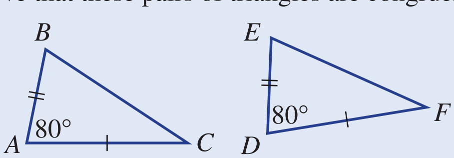
$\triangle ABC \equiv \triangle EFD$

- $AB \parallel CD$ means AB is parallel to CD.
- $AB \perp CD$ means AB is perpendicular to CD.

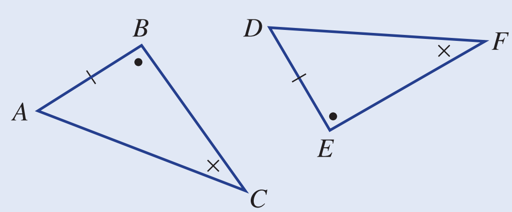
Example 3 Proving congruence in triangles

Prove that these pairs of triangles are congruent.

a



b



SOLUTION

a In $\triangle ABC$ and $\triangle DEF$:
$AB = DE$ (given)
$\angle BAC = \angle EDF = 80°$ (given)
$AC = DF$ (given)
$\therefore \triangle ABC \equiv \triangle DEF$ (SAS)

EXPLANATION

List all pairs of corresponding sides and angles.
The two triangles are therefore congruent, with two pairs of corresponding sides and the included angle being equal.

b In $\triangle ABC$ and $\triangle DEF$:
$\angle ABC = \angle DEF$ (given)
$\angle ACB = \angle DFE$ (given)
$AB = DE$ (given)
$\therefore \triangle ABC \equiv \triangle DEF$ (AAS)

List all pairs of corresponding sides and angles.
The two triangles are therefore congruent with two pairs of corresponding angles and a corresponding side being equal.

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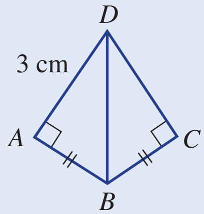
Example 4 Using congruence in proof

In this diagram $\angle A = \angle C = 90°$ and $AB = BC$.

a Prove $\triangle ABD \equiv \triangle CBD$.

b Prove $AD = CD$.

c State the length of CD.



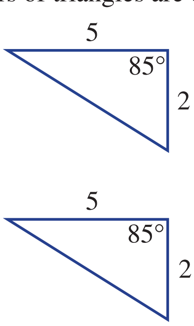
SOLUTION	EXPLANTION
a In $\triangle ABD$ and $\triangle CBD$: $\angle DAB = \angle DCB = 90°$ (given) BD is common, $AB = BC$ (given) $\therefore \triangle ABD \equiv \triangle CBD$ (RHS)	Systematically list corresponding pairs of equal angles and lengths.
b $\triangle ABD \equiv \triangle CBD$, so $AD = CD$ (corresponding sides in congruent triangles)	Since $\triangle ABD$ and $\triangle CBD$ are congruent, then the matching sides AD and CD are equal. In this context, the words 'matching' and 'corresponding' are interchangeable.
c $CD = 3$ cm	$AD = CD$ from part **b** above.

Exercise 6B REVISION

UNDERSTANDING AND FLUENCY	1–4	2–5	3–5(½)

1 Which of the tests (i.e. SSS, SAS, AAS or RHS) would be used to decide whether the following pairs of triangles are congruent?

a



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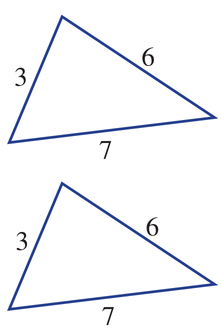
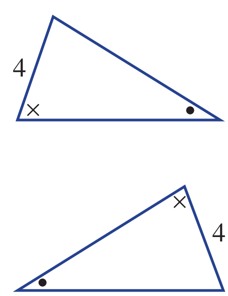
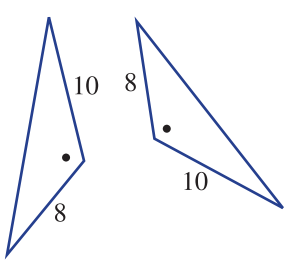


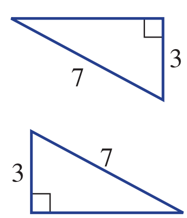
c



d



e



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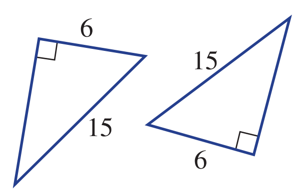

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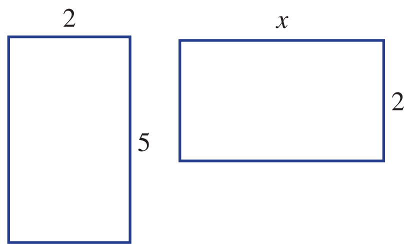
2 Assume these pairs of figures are congruent to find the value of the pronumeral in each case.

a



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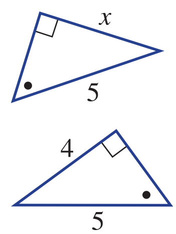

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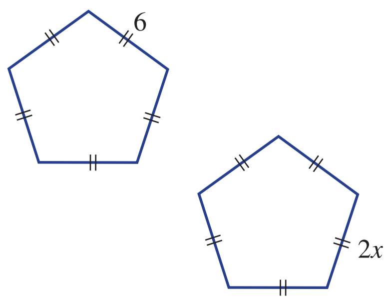

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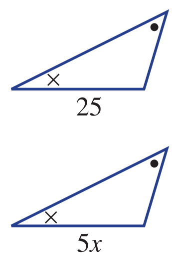

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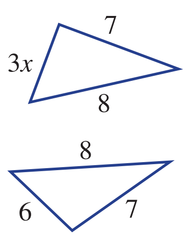

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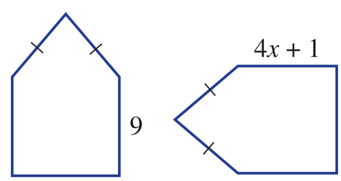
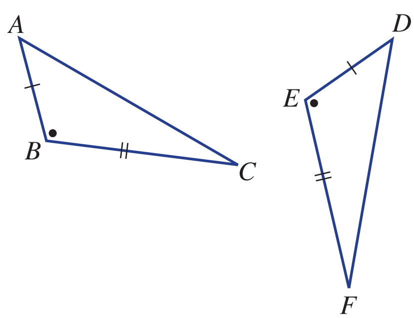


Example 3

3 Prove that these pairs of triangles are congruent.

a



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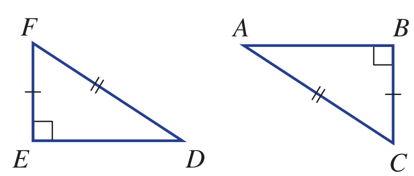

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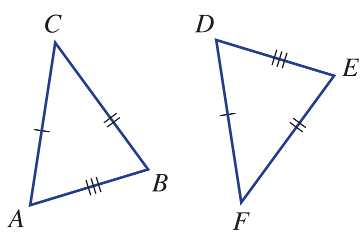

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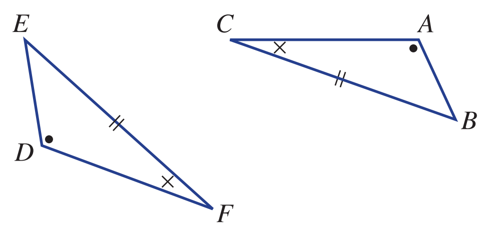

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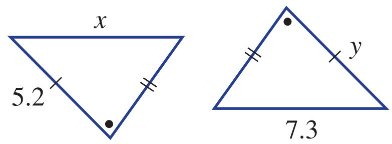
4 Find the value of the pronumerals in these diagrams that include congruent triangles.

a



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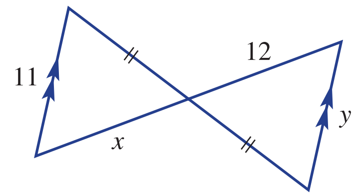

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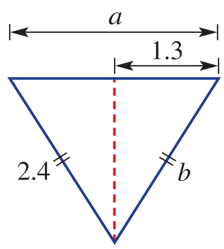

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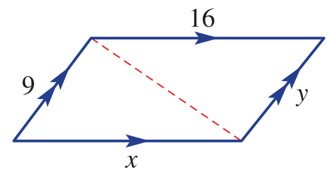
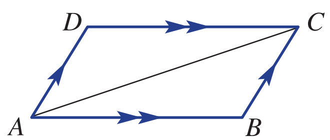


5 Prove that each pair of triangles is congruent, giving reasons. Write the vertices in matching order.

a



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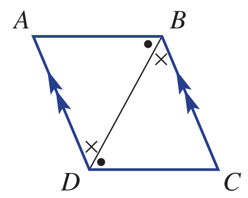

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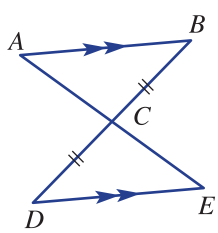

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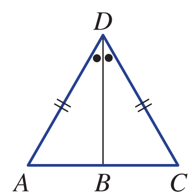

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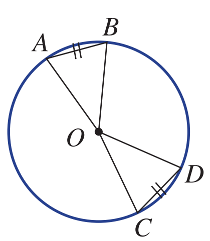

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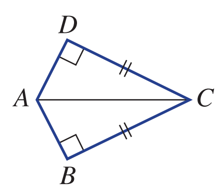


PROBLEM-SOLVING AND REASONING	6, 7, 9	7–10	8–11

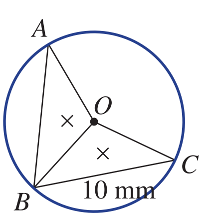
Example 4

6 In this diagram O is the centre of the circle and $\angle AOB = \angle COB$.

a Prove $\triangle AOB \equiv \triangle COB$.

b Prove $AB = BC$.

c State the length of AB.



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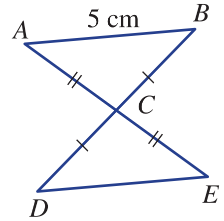
7 In this diagram $BC = DC$ and $AC = EC$.



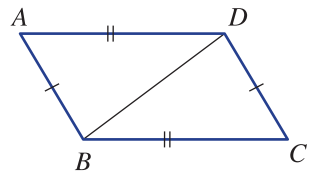
a Prove $\triangle ABC \equiv \triangle EDC$.

b Prove $AB = DE$.

c Prove $AB \parallel DE$.

d State the length of DE.

8 In this diagram $AB = CD$ and $AD = CB$.



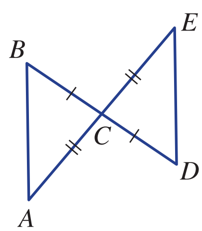
a Prove $\triangle ABD \equiv \triangle CDB$.

b Prove $\angle DBC = \angle BDA$.

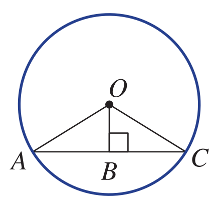
c Prove $AD \parallel BC$.

9 Prove the following for the given diagrams. Give reasons.

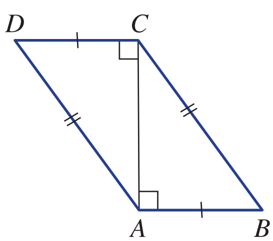
a $AB \parallel DE$



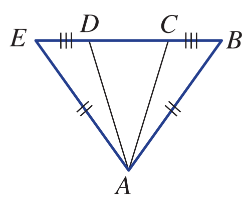
b OB bisects AC; i.e. $AB = BC$



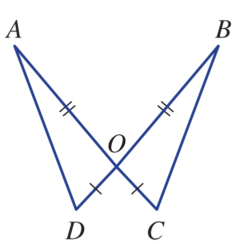
c $AD \parallel BC$



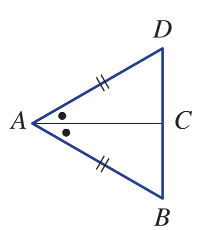
d $AD = AC$



e $\angle OAD = \angle OBC$



f $AC \perp BD$; i.e. $\angle ACD = \angle ACB$



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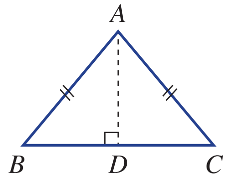
10 Consider the following triangle, which has two equal sides.

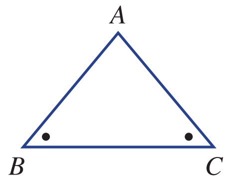


a A line segment is drawn from A perpendicular to BC. It meets BC at D. Prove that $\triangle ABD \equiv \triangle ACD$.

b Explain why $\angle ABC = \angle ACB$.

c Hence, what have you proven about the angles in a triangle with two equal sides?

11 In Question **10** we used two equal sides to prove that the angles opposite those sides *must* be equal. We will now try to show that the converse is true; i.e. if two angles of a triangle are equal, then the sides oppposite those angles are also equal.



Find three errors in the following proof.

Construct AD perpendicular to BC.	(1)
In $\triangle ABD$ and $\triangle ACD$:	
$\angle ABD = \angle ABC$ (given)	(2)
$\angle BDA = \angle CDA = 90°$ ($AD \perp BC$)	(3)
AB is common.	(4)
$\therefore \triangle ABD \equiv \triangle ADC$ (RHS test)	(5)

$\therefore AB = AC$ (matching sides in congruent triangles)

$\therefore$ If two angles in a triangle are equal, then the sides opposite those angles are also equal.

ENRICHMENT	–	–	12

Draw your own diagram

12 a A circle with centre O has a chord AB. M is the midpoint of the chord AB. Prove $OM \perp AB$.

b Two overlapping circles with centres O and C intersect at A and B. Prove $\angle AOC = \angle BOC$.

c $\triangle ABC$ is isosceles with $AC = BC$, D is a point on AC such that $\angle ABD = \angle CBD$, and E is a point on BC such that $\angle BAE = \angle CAE$. AE and BD intersect at F. Prove $AF = BF$.

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6C Using congruence to investigate quadrilaterals

Interactive
Widgets
HOTsheets
Walkthrough

Recall that parallelograms are quadrilaterals with two pairs of parallel sides. We therefore classify rectangles, squares and rhombuses as special types of parallelograms, the properties of which can be explored using congruence. By dividing a parallelogram into 'smaller' triangles, we can prove congruence for pairs of these triangles and use this to deduce the properties of the shape.

Stage
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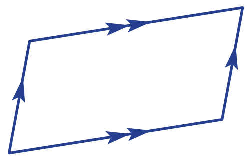
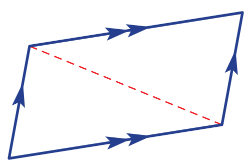


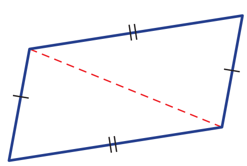
Let's start: Aren't they the same proof?

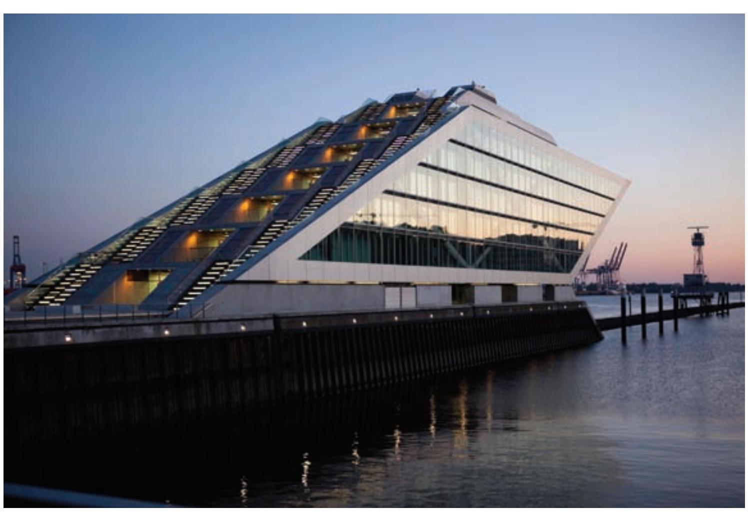
Here are two statements involving the properties of a parallelogram.

1 A parallelogram (with parallel opposite sides) has opposite sides of equal length.



2 A quadrilateral with opposite sides of equal length is a parallelogram.
- Are the two statements saying the same thing?
- Discuss how congruence can be used to help prove each statement.
- Formulate a proof for each statement.





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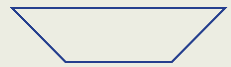
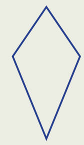
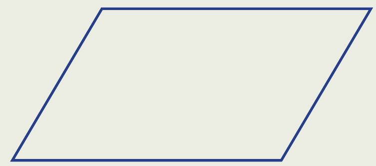
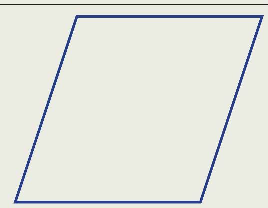
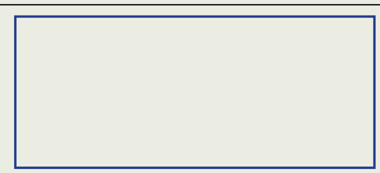
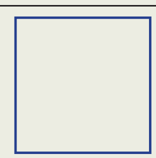
Key ideas

- Some vocabulary and symbols:
  - If AB is parallel to BC, then we write $AB \parallel CD$.
  - If AB is perpendicular to CD , then we write $AB \perp CD$.
  - To bisect means to cut in half.
- Minimum conditions needed to prove that a quadrilateral is ‘special’.
- Note: This list is not exhaustive.

A quadrilateral is a **trapezium** if: • It has one pair of parallel sides	
A quadrilateral is a **kite** if: • Two pairs of adjacent sides are equal or • The diagonals meet at right angles and one of them is bisected by the other	
A quadrilateral is a **parallelogram** if: • both pairs of opposite sides are parallel or • both pairs of opposite sides are equal or • both pairs of opposite angles are equal or • the diagonals bisect each other (i.e. the diagonals have the same midpoint) or • two sides are equal *and* parallel	
A quadrilateral is a **rhombus** if: • all sides are equal or • diagonals bisect each other at right angles or • the diagonals bisect the angles at the vertices or • a pair of adjacent sides are equal and opposite angles are equal or • the diagonals create four congruent triangles	
A quadrilateral is a **rectangle** if: • the diagonals are equal and they bisect each other or • it has three right angles or • it has two pairs of parallel sides and one right angle or • it has two pairs of opposite sides equal and one right angle	
A quadrilateral is a **square** if: • it has four equal sides and one right angle or • the diagonals are equal, bisect each other and meet at right angles	

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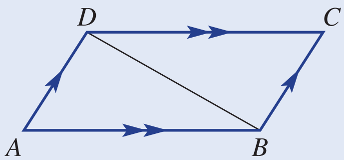
Example 5 Proving properties of quadrilaterals

a Opposite sides in a parallelogram are parallel. Prove that opposite angles are equal.

b Use the property that opposite sides of a parallelogram are equal to prove that a rectangle (with all angles 90°) has diagonals of equal length.

SOLUTION

a



$\angle ABD = \angle CDB$ (alternate angles $AB \parallel DC$)
$\angle ADB = \angle CBD$ (alternate angles $AD \parallel BC$)
BD is common.
$\therefore \triangle ABD \equiv \triangle CDB$ (AAS)
$\therefore \angle DAB = \angle BCD$ (corresponding angles in congruent triangles)
Similarly, $\triangle ADC \equiv \triangle ABC$
$\therefore \angle ADC = \angle ABC$ (corresponding angle in congruent triangles)
$\therefore$ Opposite angles of a parallelogram are equal.

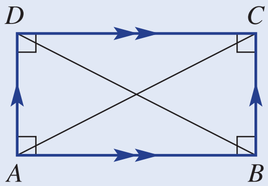
EXPLANATION

Draw a parallelogram with parallel sides and the segment BD.

Prove congruence of $\triangle ABD$ and $\triangle CDB$, noting alternate angles in parallel lines. Note also that BD is common to both triangles.

SOLUTION

b



Consider $\triangle ABC$ and $\triangle BAD$.
AB is common.
$\angle ABC = \angle BAD = 90°$
$BC = AD$ (opposite sides of a parallelogram are equal)
$\therefore \triangle ABC \equiv \triangle BAD$ (SAS)
$\therefore AC = BD$ (corresponding sides in congruent triangles)
$\therefore$ Diagonals are of equal length.

EXPLANATION

First, draw a rectangle with the given properties.

Choose $\triangle ABC$ and $\triangle BAD$, which each contain one diagonal.

Prove congruent triangles using SAS.

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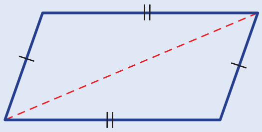
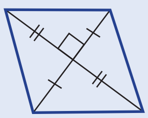
Example 6 Testing for a type of quadrilateral

a Prove that if opposite sides of a quadrilateral are equal, then it is a parallelogram.

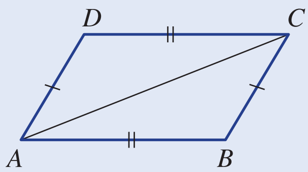


b Prove that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.



SOLUTION

a



First, label your quadrilateral and choose two triangles, in this case $\triangle ABC$ and $\triangle CDA$.

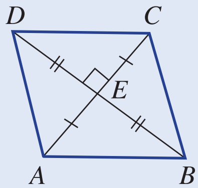
$AB = CD$ (given)
$BC = DA$ (given)
AC is common.
$\therefore \triangle ABC \equiv \triangle CDA$ (SSS)

Prove that they are congruent using SSS.

$\therefore \angle BAC = \angle DCA$
So $AB \parallel DC$ (since alternate angles are equal)
Also $\angle ACB = \angle CAD$.
$\therefore AD \parallel BC$ (since alternate angles are equal)
$\therefore ABCD$ is a parallelogram.

Choose corresponding angles in the congruent triangles to show that opposite sides are parallel. If alternate angles between lines are equal, then the lines must be parallel.

b



All angles at the point E are 90°, so it is easy to prove congruency of all four triangles using SAS.

$\triangle ABE \equiv \triangle CBE \equiv \triangle ADE \equiv \triangle CDE$ by SAS
$\therefore AB = CB = CD = DA$
$\therefore ABCD$ is a rhombus.

Corresponding sides in congruent triangles. Every quadrilateral with four equal sides is a rhombus.

Exercise 6C

UNDERSTANDING AND FLUENCY	1–5	2–6	4–6

1 Name the special quadrilateral given by these descriptions.
- **a** a parallelogram with all angles 90°
- **b** a quadrilateral with opposite sides parallel
- **c** a parallelogram that is a rhombus and a rectangle
- **d** a parallelogram with all sides of equal length

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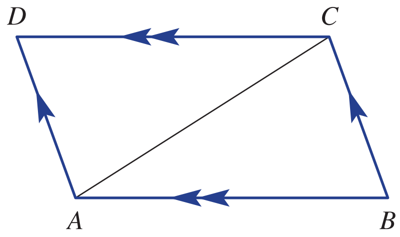
2 List all the special quadrilaterals that have these properties.
- a all angles 90°
- b diagonals are of equal length
- c diagonals bisect each other
- d diagonals bisect each other at 90°
- e diagonals bisect the interior angles

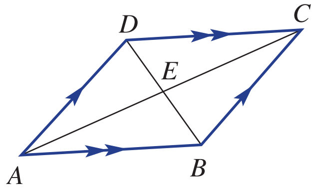
3 Give a reason why:
- a a trapezium is not a parallelogram
- b a kite is not a parallelogram

Example 5

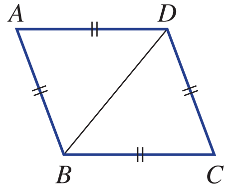
4 Complete these steps to prove that a parallelogram (with opposite sides parallel) has opposite sides equal in length.
- a Prove $\triangle ABC \equiv \triangle CDA$.
- b Hence, prove opposite sides are equal in length.



5 Complete these steps to prove that a parallelogram (with opposite sides parallel) has diagonals that bisect each other.
- a Prove $\triangle ABE \equiv \triangle CDE$.
- b Hence, prove $AE = CE$ and $BE = DE$.



6 Complete these steps to prove that a rhombus (with sides of equal length) has diagonals that bisect the vertex angles.
- a Prove $\triangle ABD \equiv \triangle CDB$.
- b Hence, prove BD bisects both $\angle ABC$ and $\angle CDA$.



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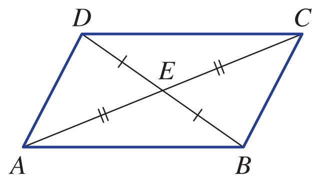
PROBLEM-SOLVING AND REASONING	7, 10	7, 8, 10, 11	8, 9, 11, 12

Example 6

7 Complete these steps to prove that if the diagonals in a quadrilateral bisect each other, then it is a parallelogram.

a Prove $\triangle ABE \equiv \triangle CDE$.

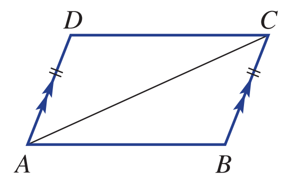
b Hence, prove $AB \parallel DC$ and $AD \parallel BC$.



8 Complete these steps to prove that if one pair of opposite sides is equal in length and parallel in a quadrilateral, then it is a parallelogram.

a Prove $\triangle ABC \equiv \triangle CDA$.

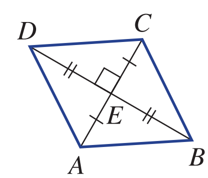
b Hence, prove $AB \parallel DC$.



9 Complete these steps to prove that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.

a Give a brief reason why $\triangle ABE \equiv \triangle CBE \equiv \triangle ADE \equiv \triangle CDE$.

b Hence, prove $ABCD$ is a rhombus.



10 Prove that the diagonals of a rhombus (a parallelogram with sides of equal length):

a intersect at right angles

b bisect the interior angles

11 Prove that a parallelogram with one right angle is a rectangle.

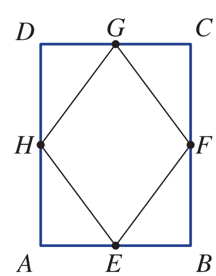
12 Prove that if the diagonals of a quadrilateral bisect each other and are of equal length, then it is a rectangle.

ENRICHMENT	–	–	13, 14

Join the midpoints

13 In this diagram, E, F, G and H are the midpoints of AB, BC, CD and DA, respectively, and $ABCD$ is a rectangle.

Prove that $EFGH$ is a rhombus.



14 Choose four random points and plot them on a Cartesian plane. Join the four points to make a quadrilateral. Use the midpoints of the sides to form a quadrilateral. Prove that the quadrilateral is a parallelogram.

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6D Similar figures

Interactive

Widgets

HOTsheets

Walkthrough

You will recall that a scale factor can be used to reduce or enlarge the size of an object. The image will be of the same shape but different in size. This means that matching pairs of angles will be equal and matching sides will be in the same ratio, just as in an accurate scale model.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

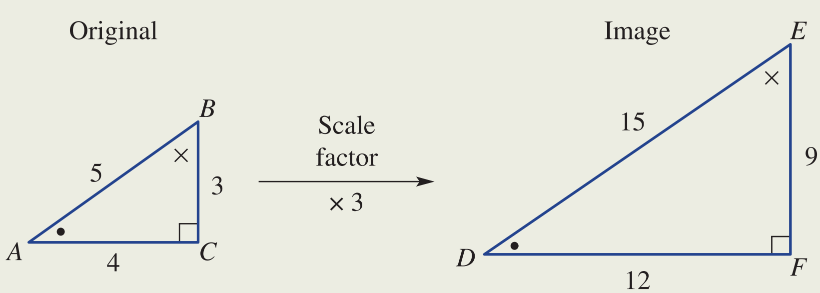
Let's start: The Q1 tower

The Q1 tower, pictured here, is located on the Gold Coast. Including its spire, the tower measures 322.5 metres tall.

- Measure the height and width of the Q1 tower in this photograph.
- Can a scale factor for the photograph and the actual Q1 tower be calculated? How?
- How can you calculate the actual width of the Q1 tower using this photograph? Discuss.

Key ideas

- **Similar figures** have the same shape but are of different size.
  - Corresponding angles are equal.
  - Corresponding sides are in the same proportion (or ratio).



$$\frac{9}{3} = \frac{12}{4} = \frac{15}{5} = 3$$

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Key ideas

- Scale factor $= \dfrac{\text{image length}}{\text{original length}}$
  - If scale factor > 1, image is larger than original.
  - If scale factor $= 1$, image is congruent to original.
  - If $0 <$ scale factor < 1, image is smaller than original.
- The symbol $|||$ is used to describe similarity and to write similarity statements.
  For example: $\triangle ABC \ ||| \ \triangle DEF$.

Example 7 Finding and using scale factors

These two shapes are similar.

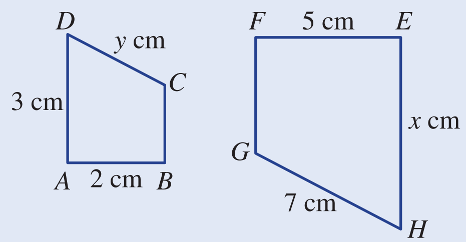
a Write a similarity statement for the two shapes.

b Complete the following: $\dfrac{EH}{\square} = \dfrac{FG}{\square}$

c Find the scale factor.

d Find the value of x.

e Find the value of y.

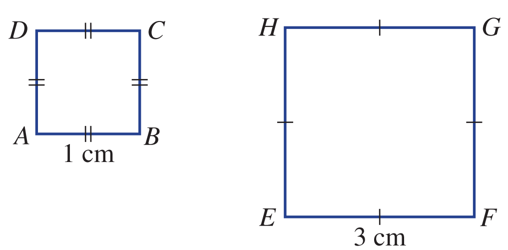


SOLUTION	EXPLANATION
a $ABCD \ ||| \ EFGH$	Use the symbol $|||$ in similarity statements.
b $\dfrac{EH}{AD} = \dfrac{FG}{BC}$	Ensure you match corresponding vertices.
c $\dfrac{EF}{AB} = \dfrac{5}{2}$ or 2.5	EF and AB are matching sides and both lengths are given.
d $x = 3 \times 2.5$ $= 7.5$	EH corresponds to AD, which is 3 cm in length. Multiply by the scale factor.
e $y = 7 \div 2.5$ $= 2.8$	DC corresponds to HG, which is 7 cm in length.

Exercise 6D

UNDERSTANDING AND FLUENCY	1–5	3–6	5, 6

1 These two figures are squares.



a Would you say that the two squares are similar? Why?

b What is the scale factor when the smaller square is enlarged to the larger square?

c If the larger square is enlarged by a factor of 5, what would be the side length of the image?

d What is the ratio of perimeter $ABCD$: perimeter $EFGH$?

e What is the ratio of area $ABCD$: area $EFGH$?

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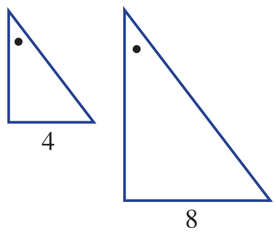
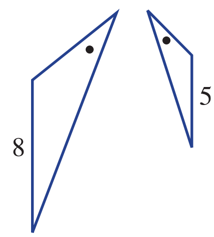
2 Find the scale factor for each shape and its larger similar image.

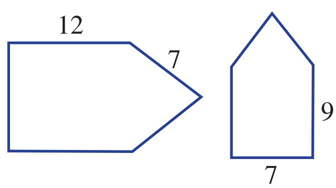
a



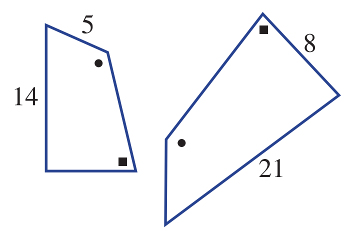
b



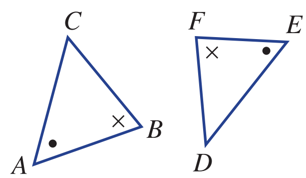
c



d



3 These two triangles are similar.



a In $\triangle ABC$, which vertex corresponds to (i.e. matches) vertex E?

b In $\triangle ABC$, which angle corresponds to $\angle D$?

c In $\triangle DEF$, which side corresponds to BC?

d Write a similarity statement for the two triangles. Write matching vertices in the same order.

Example 7

4 These two shapes are similar.

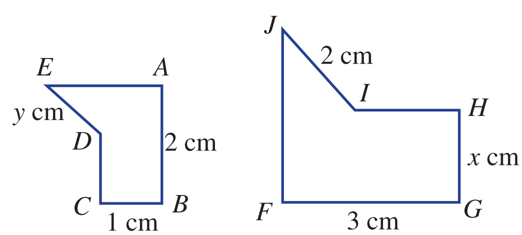
a Write a similarity statement for the two shapes.

b Complete the following: $\dfrac{AB}{\square} = \dfrac{DE}{\square}$

c Find the scale factor.

d Find the value of x.

e Find the value of y.



5 These two shapes are similar.

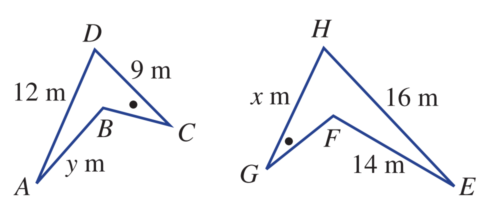
a Write a similarity statement for the two shapes.

b Complete the following: $\dfrac{EF}{\square} = \dfrac{\square}{CD}$

c Find the scale factor.

d Find the value of x.

e Find the value of y.



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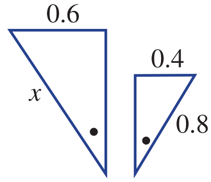
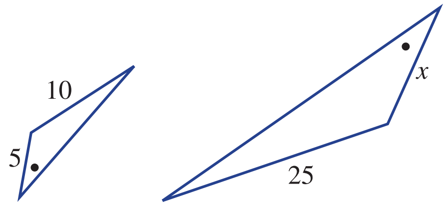
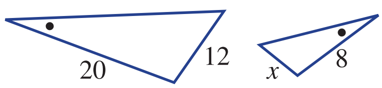
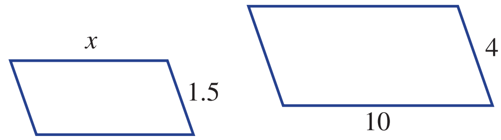
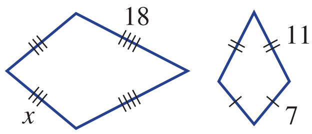
6 Find the value of the pronumeral in each pair of similar figures. Round your answer to 1 decimal place where necessary.

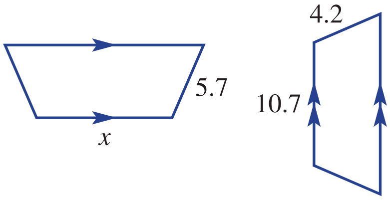
a


b


c


d


e


f


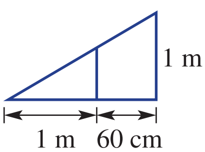
PROBLEM-SOLVING AND REASONING	7, 8, 10	7, 8, 10, 11	8, 9, 11, 12

7 A 50 m tall structure casts a shadow that is 30 m in length. At the same time, a person casts a shadow of 1.02 m. Estimate the height of the person. Hint: Draw a diagram of two triangles.

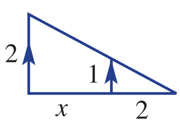
8 A BMX ramp has two vertical supports, as shown.

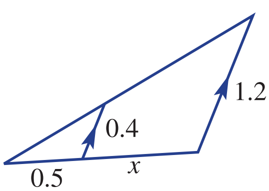
a Find the scale factor for the two triangles in the diagram.

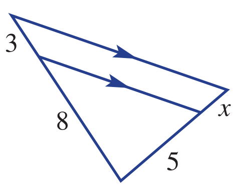
b Find the length of the inner support.

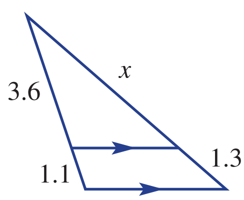


9 Find the value of the pronumeral if the pairs of triangles are similar. Round your answer to 1 decimal place in part **d**.

a


b


c


d


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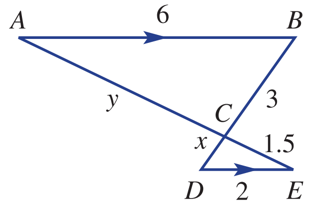
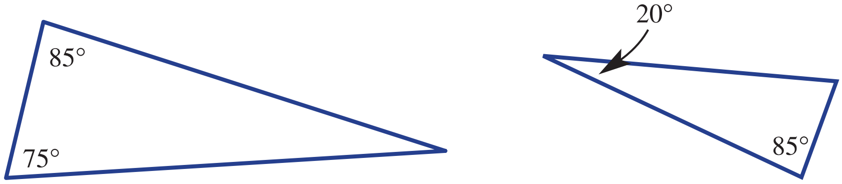
10 In this diagram the two triangles are similar and $AB \parallel DE$.



a Which side in $\triangle ABC$ corresponds to DC? Give a reason.
b Write a similarity statement by matching the vertices.
c Find the value of x.
d Find the value of y.

11 Are these statements true or false?
a All circles are similar.
b All squares are similar.
c All rectangles are similar.
d All rhombuses are similar.
e All parallelograms are similar.
f All trapeziums are similar.
g All kites are similar.
h All isosceles triangles are similar.
i All equilateral triangles are similar.
j All regular hexagons are similar.

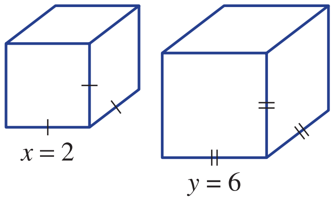
12 Both of these two triangles have two given angles. Decide whether they are similar and give reasons.



ENRICHMENT	–	–	13

Length, area, volume in similar figures

13 Shown here is a cube of side length 2 and its image after enlargement.



a Copy and complete:
Ratio of $x : y$
$= 2 : \square$
$= 1 : \square$
b Find the area of one face of:
i the smaller cube
ii the larger cube
c Find the volume of:
i the smaller cube
ii the larger cube
d Complete this table.

Cube	Length	Area	Volume
small	2		
large	6		
ratio of small : large	$1 : \square$	$1 : \square^2$	$1 : \square^3$

e A centicube has edges measuring 1 cm long. A total of 64 centicubes are used to build a cube. Give the:
i ratio of lengths $= 1 : \square$
ii ratio of surface areas $= 1 : \square$
iii ratio of volumes $= 1 : \square$
f If the ratio of the side lengths is $1 : k$, then:
i What is the ratio of the surface areas?
ii What is the area of the volumes?

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6E Proving and applying similar triangles

Interactive
Widgets
HOTsheets
Walkthrough

As with congruent triangles, there are four tests for proving that two triangles are similar. When solving problems involving similar triangles it is important to recognise and be able to prove that they are in fact similar.

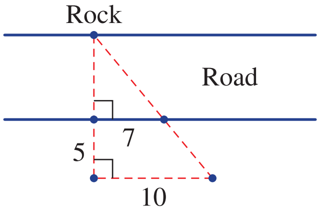
Similar triangles are used in this bridge structure.

Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: How far did the chicken travel?

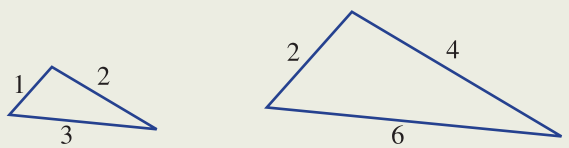
A chicken is considering how far it is across a road, so it places four pebbles in certain positions on one side of the road. Two of the pebbles are directly opposite a rock on the other side of the road. The number of chicken paces between three pairs of the pebbles is shown in the diagram.



- Has the chicken constructed any similar triangles? If so, discuss why they are similar.
- What scale factor is associated with the two triangles?
- Is it possible to find how many paces the chicken needs to take to get across the road? If so, show a solution.
- Why did the chicken cross the road?
  Answer: To explore similar triangles.

Key ideas

- Two objects are said to be **similar** if they are of the same shape but different in size.
  - For two similar triangles $\triangle ABC$ and $\triangle DEF$, we write $\triangle ABC \mathbin{|||} \triangle DEF$.
  - When comparing two triangles, corresponding sides are opposite equal angles.
- There are four tests that can be used to prove that two triangles are similar.
  - Three sides of a triangle are proportional to three sides of another triangle.



$$\frac{6}{3} = \frac{4}{2} = \frac{2}{1} = 2$$

This may be abbreviated to 'three pairs of sides in proportion'.

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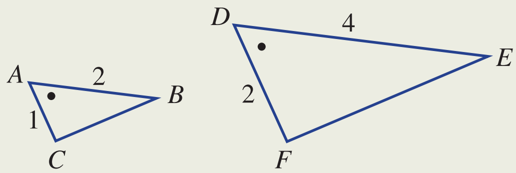
Key ideas

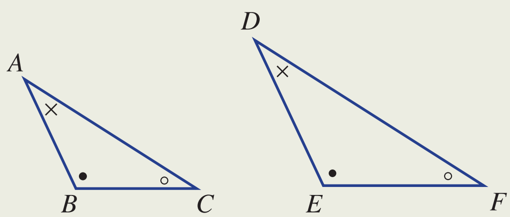
- Two sides of a triangle are proportional to two sides of another triangle and the included angles are equal.



$\frac{4}{2} = \frac{2}{1} = 2$ and $\angle A = \angle D$

This may be abbreviated to 'sides adjacent to equal angles are in proportion.'

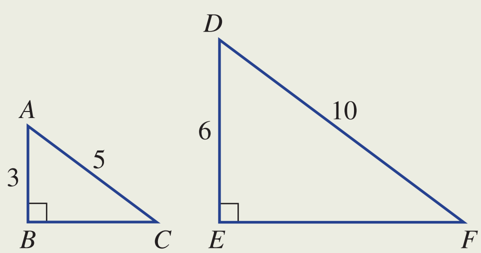
- Two angles of a triangle are equal to two angles of another triangle. (Two angles are sufficient because two pairs of corresponding equal angles implies that all three pairs of corresponding angles are equal because the angle sum of both triangles is 180°.)



$\angle A = \angle D$
$\angle B = \angle E$
$\angle C = \angle F$

This may be abbreviated to 'equiangular'.

- The hypotenuse and a second side of a right-angled triangle are proportional to the hypotenuse and second side of another right-angled triangle.



$\angle B = \angle E = 90°$

$\frac{10}{5} = \frac{6}{3} = 2$

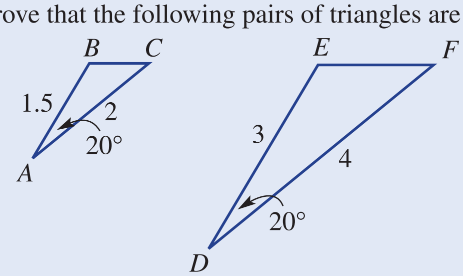
This may be abbreviated to 'hypotenuse and second side of a right-angled triangle in proportion'.

Note: Abbreviations such as SSS and SAS are generally not used for similarity in the NSW syllabus.

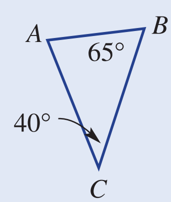
Example 8 Using similarity tests to prove similar triangles

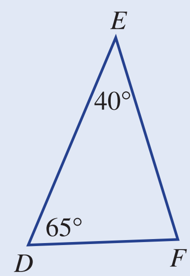
Prove that the following pairs of triangles are similar.

a



b





Example continued over page

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SOLUTION

a $\frac{DF}{AC} = \frac{4}{2} = 2$

$\frac{DE}{AB} = \frac{3}{1.5} = 2$

(ratio of corresponding sides)

$\angle BAC = \angle EDF = 20°$ (given)

$\therefore \triangle ABC \parallel\!| \triangle DEF$ (sides about equal angles are in proportion)

b $\angle ABC = \angle FDE = 65°$ (given)

$\angle ACB = \angle FED = 40°$ (given)

$\therefore \triangle ABC \parallel\!| \triangle FDE$ (matching angles are equal)

EXPLANATION

DF and *AC* are corresponding sides and *DE* and *AB* are corresponding sides, and both pairs are in the same ratio.

The corresponding angle between the pair of corresponding sides in the same ratio is also equal. The two triangles are therefore similar.

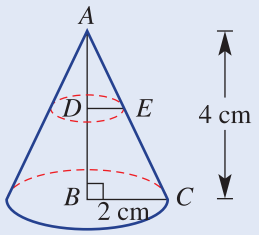
There are two pairs of given corresponding angles. If two pairs of corresponding angles are equal, then the third pair must also be equal. The two triangles are therefore similar.

Example 9 Establishing and using similarity

A cone has radius 2 cm and height 4 cm. The top of the cone is cut horizontally through *D*.

a Prove $\triangle ADE \parallel\!| \triangle ABC$.

b If $AD = 1$ cm, find the radius *DE*.



SOLUTION

a $\angle BAC$ is common.

$\angle ABC = \angle ADE$ (corresponding angles in parallel lines)

$\therefore \triangle ADE \parallel\!| \triangle ABC$ (matching angles are equal)

b $\frac{DE}{BC} = \frac{AD}{AB}$ (corresponding sides of similar triangles)

$\frac{DE}{2} = \frac{1}{4}$

$\therefore DE = \frac{2}{4}$

$= 0.5$ cm

EXPLANATION

All three pairs of corresponding angles are equal.

Therefore the two triangles are similar.

Given the triangles are similar, the ratio of corresponding sides must be equal.

Solve for *DE*.

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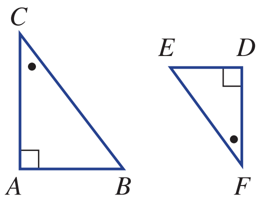
Exercise 6E

UNDERSTANDING AND FLUENCY	1–5	3–6	4–6(½)

1 This diagram includes two similar triangles.



a Which vertex in $\triangle DEF$ corresponds to vertex B ?

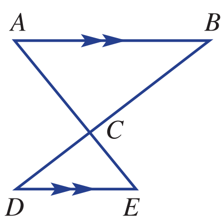
b Which angle in $\triangle ABC$ corresponds to $\angle F$?

c Which side in $\triangle ABC$ corresponds to DE ?

d Write a similarity statement for the two triangles.

Architecture and engineering are just two fields in which the skills covered in this chapter can be used.

2 This diagram includes two similar triangles.



a Which angle in $\triangle CDE$ corresponds to $\angle B$ in $\triangle ABC$ and why?

b Which angle in $\triangle ABC$ corresponds to $\angle E$ in $\triangle CDE$ and why?

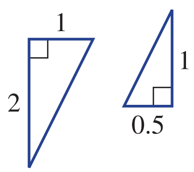
c Which angle is vertically opposite $\angle ACB$?

d Which side on $\triangle ABC$ corresponds to side CE on $\triangle CDE$?

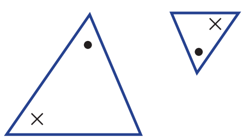
e Write a similarity statement, making sure to write the matching vertices in the same order.

3 Which similarity test would be used to prove that these pairs of triangles are similar?

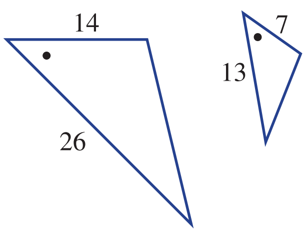
a



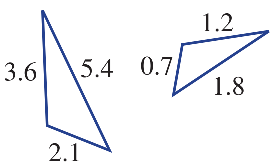
b



c



d



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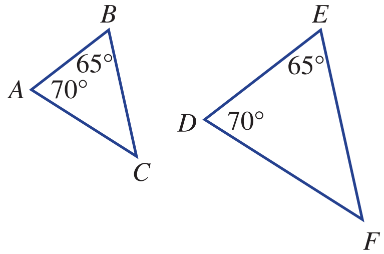
Example 8

4 Prove that the following pairs of triangles are similar.

a



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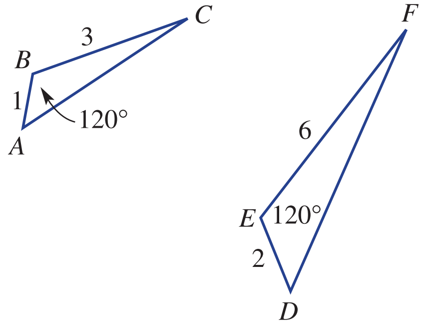
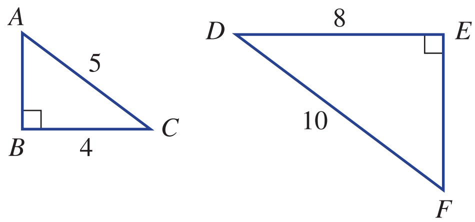
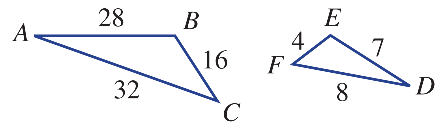


c

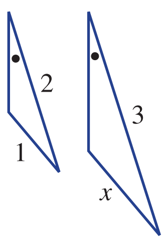


d

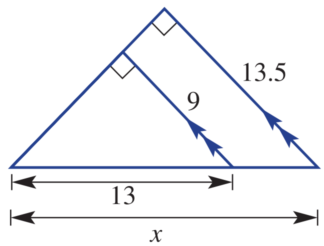


5 Find the value of the pronumerals in these pairs of similar triangles.

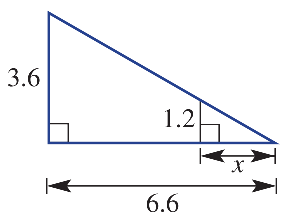
a



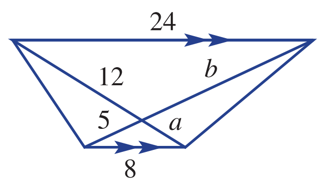
b



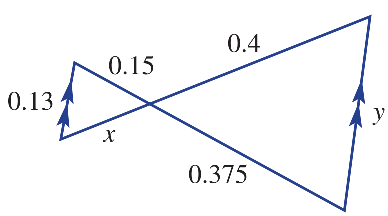
c



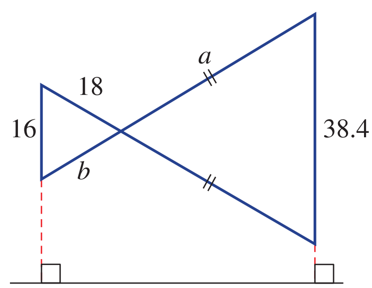
d



e



f



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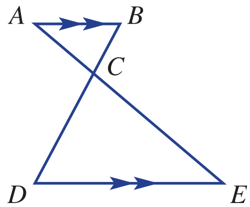
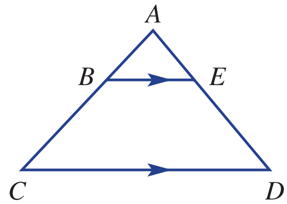
6 For the following proofs, give reasons at each step.

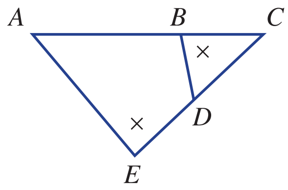
a Prove $\triangle ABC \ ||| \ \triangle EDC$.



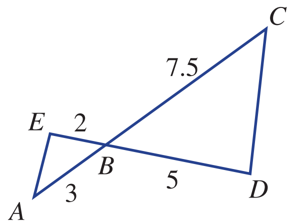
b Prove $\triangle ABE \ ||| \ \triangle ACD$.



c Prove $\Delta BCD \ ||| \ \Delta ECA$.



d Prove $\triangle AEB \ ||| \ \triangle CDB$.



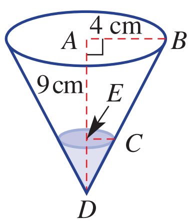
PROBLEM-SOLVING AND REASONING	7, 8, 11	8, 9, 11, 12	9, 10, 11–13

Example 9

7 A right cone with radius 4 cm has a total height of 9 cm. It contains an amount of water, as shown.

a Prove $\triangle EDC \ ||| \ \triangle ADB$.

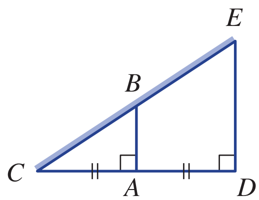
b If the depth of water in the cone is 3 cm, find the radius of the water surface in the cone.



8 A ramp is supported by a vertical stud, AB, where A is the midpoint of CD. It is known that $CD = 4$ m and that the ramp is 2.5 m high; i.e. $DE = 2.5$ m.

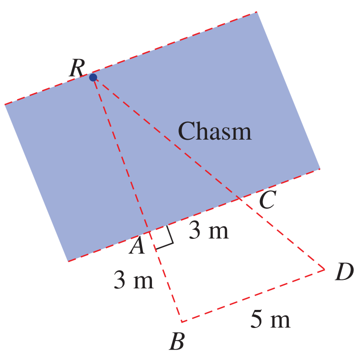
a Prove $\triangle BAC \ ||| \ \triangle EDC$.

b Find the length of the stud AB.



9 At a particular time in the day, Aaron casts a shadow 1.3 m long and Jack, who is 1.75 m tall, casts a shadow 1.2 m long. Find Aaron's height, correct to 2 decimal places.

10 To determine the width of a chasm, a marker (A) is placed directly opposite a rock (R) on the other side. Point B is placed 3 m away from point A, as shown. Marker C is placed 3 m along the edge of the chasm, and marker D is placed so that BD is parallel to AC. Markers C and D and the rock are collinear (i.e. lie in a straight line). If BD measures 5 m, find the width of the chasm (AR).



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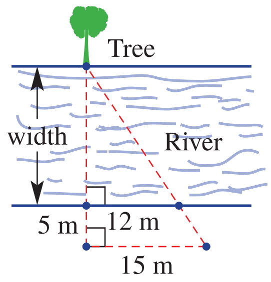
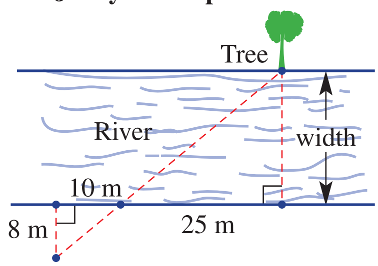
11 Aiden and Jenny come to a river and notice a tree on the opposite bank. Separately they decide to place rocks (indicated by dots) on their side of the river to try to calculate the river's width. They then measure the distances between some pairs of rocks, as shown.

Aiden's rock placement

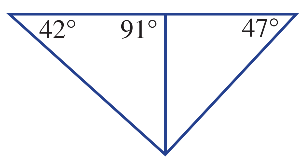


Jenny's rock placement



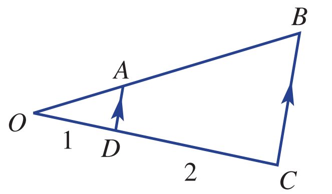
a Have both Aiden and Jenny constructed a pair of similar triangles? Give reasons.
b Use Jenny's triangles to calculate the width of the river.
c Use Aiden's triangles to calculate the width of the river.
d Which pair of triangles did you prefer to use? Give reasons.

12 There are two triangles in this diagram, each showing two given angles. Explain why they are similar.

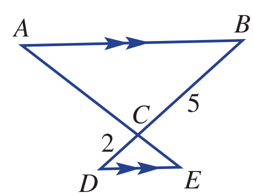


13 Prove the following, giving reasons.

a $OB = 3OA$



b $AE = \frac{7}{5}AC$

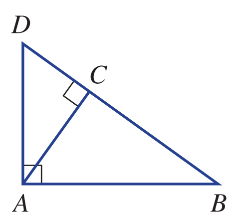


ENRICHMENT – – 14

Proving Pythagoras' theorem

14 In this figure $\triangle ABD$, $\triangle CBA$ and $\triangle CAD$ are right angled.

a Prove $\triangle ABD \ ||| \ \triangle CBA$. Hence, prove $AB^2 = CB \times BD$.
b Prove $\triangle ABD \ ||| \ \triangle CAD$. Hence, prove $AD^2 = CD \times BD$.
c Hence, prove Pythagoras' theorem $AB^2 + AD^2 = BD^2$.



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6F Circle terminology and chord properties

Interactive
Widgets
HOTsheets
Walkthrough

Although a circle appears to be a very simple object, it has many interesting geometrical properties. In this section we look at radii and chords in circles, and then explore and apply the properties of these objects. We use congruence to prove many of these properties.

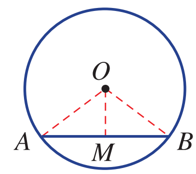
Stage
- 5.3#
- 5.3
- 5.3§
- 5.2
- 5.2◊
- 5.1
- 4

Let's start: Dynamic chords

This activity would be enhanced with the use of dynamic geometry software.

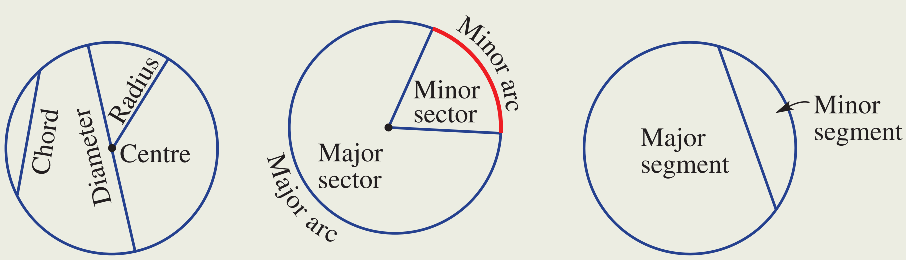
Chord AB sits on a circle with centre O. M is the midpoint of chord AB. Explore with dynamic geometry software or discuss the following.

- Is $\triangle OAB$ isosceles and if so why?
- Is $\triangle OAM \equiv \triangle OBM$ and if so why?
- Is $AB \perp OM$ and if so why?
- Is $\angle AOM = \angle BOM$ and if so why?

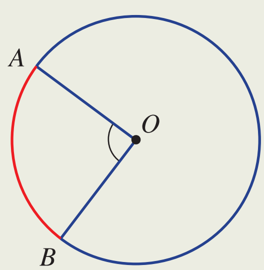


Key ideas

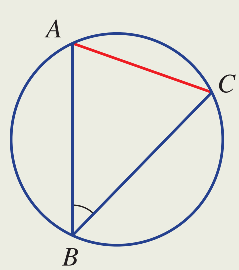
- Circle language



- An angle is **subtended** by an arc/chord if the arms of the angle meet the end points of the arc/chord.



$\angle AOB$ is subtended at the centre by the minor arc AB.



$\angle ABC$ is subtended at the circumference by the chord AC.

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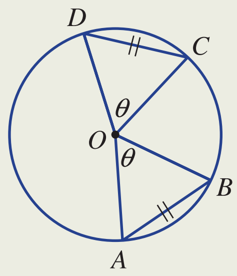
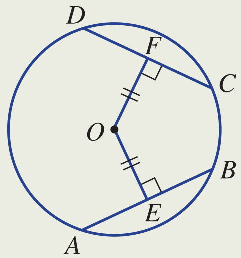
Key ideas

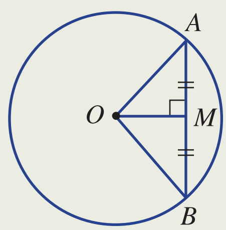
- Chords of equal length in a circle subtend equal angles at the centre.
  - If $AB = CD$, then $\angle AOB = \angle COD$.
  - Conversely, if chords subtend equal angles at the centre of the circle, then the chords are of equal length.



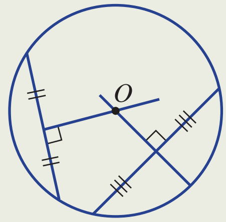
- Chords of equal length in a circle are equidistant (i.e. of equal distance) from the centre.
  - If $AB = CD$, then $OE = OF$.
  - Conversely, if chords are equidistant from the centre of the circle, then the chords are of equal length.



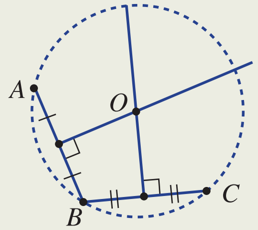
- The perpendicular from the centre of a circle bisects the chord.
  - If $OM \perp AB$, then $AM = BM$.
  - Conversely, a line through the centre of a circle that bisects a chord is perpendicular to the chord.



- The perpendicular bisectors of every chord of a circle intersect at the centre of the circle.
  - Constructing perpendicular bisectors of two chords will therefore locate the centre of a circle.



- When given any three non-collinear points (e.g. A, B, C shown here), the point of intersection of the perpendicular bisectors of any two sides of the triangle formed by the three points is the centre of the circle that passes through all three points.



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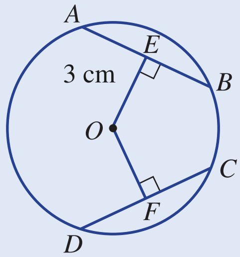
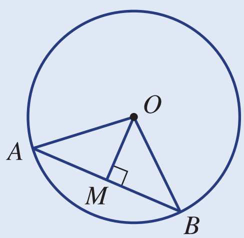
Example 10 Using theorems that involve chords

For each part, use the given information and state which chord theorem is used.

a Given $AB = CD$ and $OE = 3$ cm, find OF.



b Given $OM \perp AB$, $AB = 10$ cm and $\angle AOB = 92°$, find AM and $\angle AOM$.



SOLUTION

a $OF = 3$ cm (Chords of equal length are equidistant from the centre.)

b $AM = 5$ cm (The perpendicular from the centre to the chord bisects the chord.)
$\angle AOM = 92° \div 2 = 46°$

EXPLANATION

AB and CD are chords of equal length, so the distances EO and FO must be equal.

AB is a chord. MO is perpendicular to the chord and passes through the centre, so it bisects AB and $\angle AOB$.

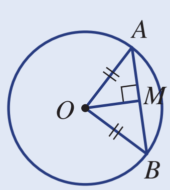
Example 11 Proving theorems that involve chords

Prove that the perpendicular from the centre of the circle to the chord, bisects the chord and the angle at the centre subtended by the chord.

SOLUTION

$OA = OB$ (both radii)
$\angle OMA = \angle OMB = 90°$ (given)
OM is common.
$\therefore \triangle OMA \equiv \triangle OMB$ (RHS)
$\therefore AM = BM$ and $\angle AOM = \angle BOM$

EXPLANATION



The perpendicular forms a pair of congruent triangles.

Corresponding sides and angles in congruent triangles are equal.

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Exercise 6F

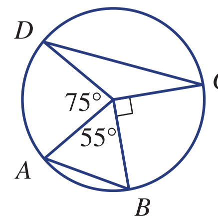
UNDERSTANDING AND FLUENCY	1–6	4–7	5–7

1 Draw a large circle, then draw and label these features.

a a chord **b** radius **c** centre
d minor sector **e** major sector **f** arc

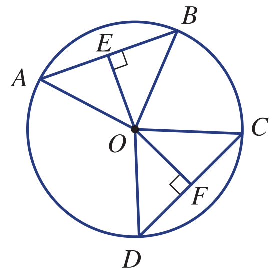
2 Give the size of the angle in this circle subtended by the following.

a chord AB **b** minor arc BC
c minor arc AD **d** chord DC



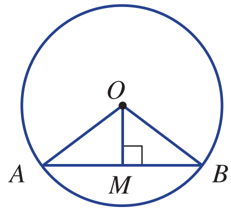
3 In this circle chords AB and CD are of equal length.

a Measure $\angle AOB$ and $\angle COD$ with a protractor.
b What do you notice about your answers from part **a**? Which chord theorem does this relate to?
c Measure OE and OF.
d What do you notice about your answers from part **c**? Which chord theorem does this relate to?



4 In this circle $OM \perp AB$.

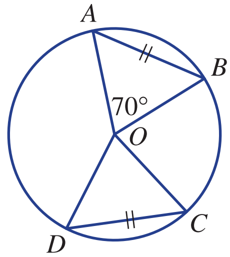
a Measure AM and BM.
b Measure $\angle AOM$ and $\angle BOM$ with a protractor.
c What do you notice about your answers from parts **a** and **b**?



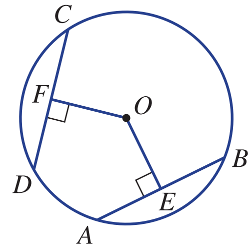
Example 10

5 For each part, use the information given and state which chord theorem is used.

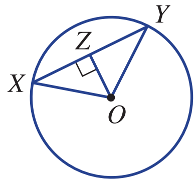
a Given $AB = CD$ and $\angle AOB = 70°$, find $\angle DOC$.



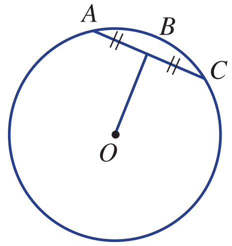
b Given $AB = CD$ and $OF = 7.2$ cm, find OE.



c Given $OZ \perp XY$, $XY = 8$ cm and $\angle XOY = 102°$, find XZ and $\angle XOZ$.



d Given $AB = BC$, find $\angle OBC$.



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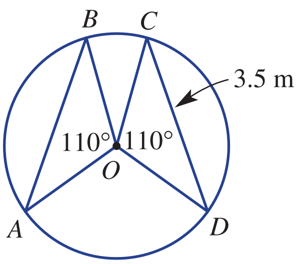
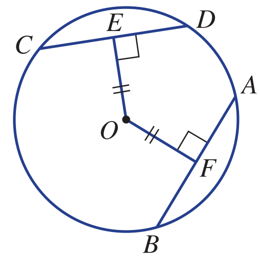
6 The perpendicular bisectors of two different chords of a circle are constructed. Describe where they intersect.

7 Use the information given to answer the following.

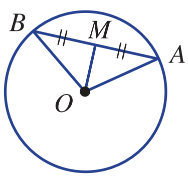
a Given $\angle AOB = \angle COD$ and $CD = 3.5$ m, find AB.



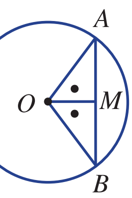
b Given $OE = OF$ and $AB = 9$ m, find CD.



c Given M is the midpoint of AB, find $\angle OMB$.



d Given $\angle AOM = \angle BOM$, find $\angle OMB$.

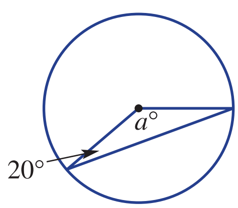


PROBLEM-SOLVING AND REASONING

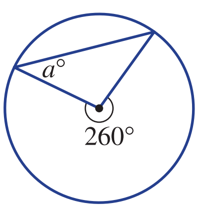
PROBLEM-SOLVING AND REASONING	8, 11	8(½), 9, 11, 12	9, 10, 12, 13

8 Find the size of each unknown angle a.

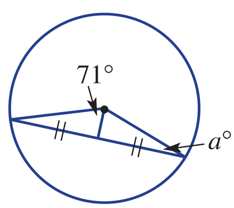
a



b



c



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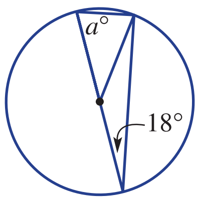

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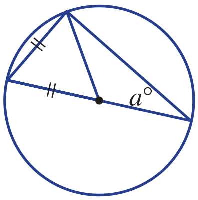

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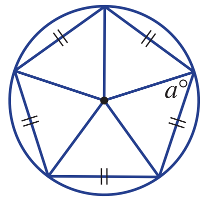
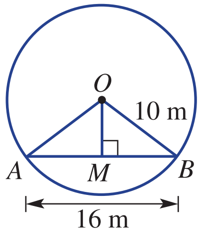


9 Find the length OM. Hint: Use Pythagoras' theorem.



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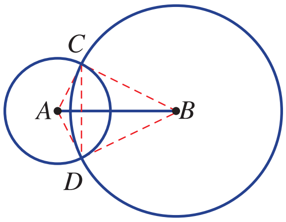
10 In this diagram, radius $AD = 5$ mm, radius $BD = 12$ mm and chord $CD = 8$ mm. Find the exact length of AB, in surd form.

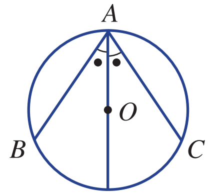


Example 11

11 **a** Prove that chords of equal length subtend equal angles at the centre of the circle.
b Prove that if chords subtend equal angles at the centre of the circle, then the chords are of equal length.

12 **a** Prove that if a radius bisects a chord of a circle, then the radius is perpendicular to the chord.
b Prove that if a radius bisects the angle at the centre subtended by the chord, then the radius is perpendicular to the chord.

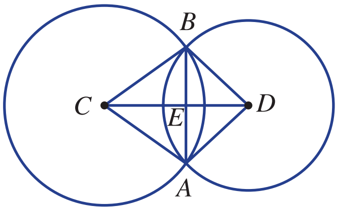
13 In this circle $\angle BAO = \angle CAO$. Prove $AB = AC$. Suggestion: Draw BC.



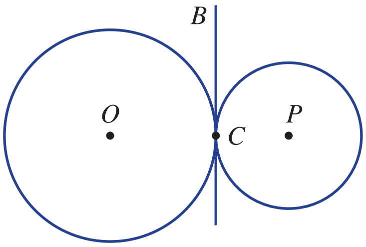
ENRICHMENT	–	–	14, 15

Common chord proof

14 In this question we will prove the following theorem. When two circles intersect, the line joining their centres bisects that common chord at right angles.
a Prove $\triangle ACD \equiv \triangle BCD$.
b Hence, prove $\triangle ACE \equiv \triangle BCE$.
c Hence, prove $CD \perp AB$ and $BE = EA$.



15 Three points are collinear if it is possible to draw a single straight line through them. Consider the two circles at right, which are touching at a single point, C. The points O and P are the centres of the circles.
a Explain why $\angle OCB = 90°$.
b Can the same be said of $\angle PCB$?
c $\angle OCB + \angle PCB = $ ________°
d What does this tell you about $\angle OCP$?
e Is OCP a straight line?
f When two circles touch, their centres and their point of contact are ________.



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Exploring chord theorems with dynamic geometry software

1 Construct a circle with centre O and any given radius.

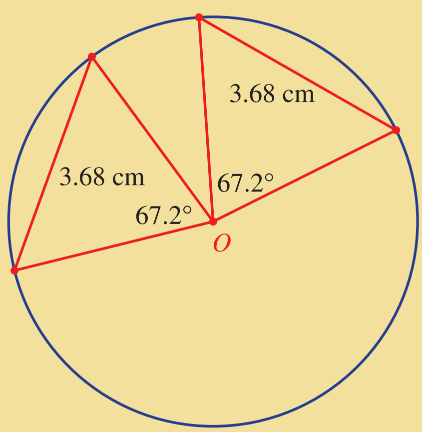
2 Construct chords of equal length by rotating a chord about the centre by a given angle.

3 Illustrate the four chord properties by constructing line segments, as shown below. Measure corresponding angles and lengths to illustrate the chord properties.
- Chord theorem 1: Chords of equal length subtend equal angles at the centre of the circle.
- Chord theorem 2: Chords of equal length are equidistant from the centre of the circle.
- Chord theorem 3: The perpendicular from the centre of the circle to the chord bisects the chord and the angle at the centre subtended by the chord.
- Chord theorem 4: The perpendiculars of every chord of a circle intersect at the centre of the circle.

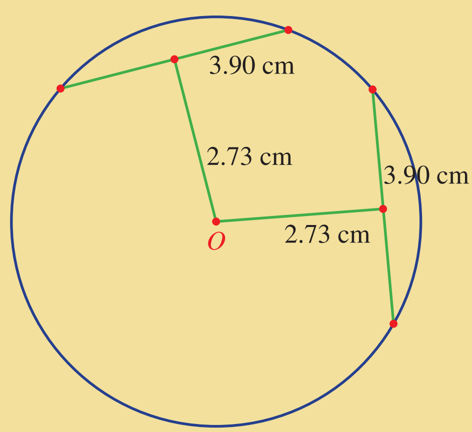
4 Drag the circle or one of the points on the circle to check that the properties are retained.

Chord theorem illustrations

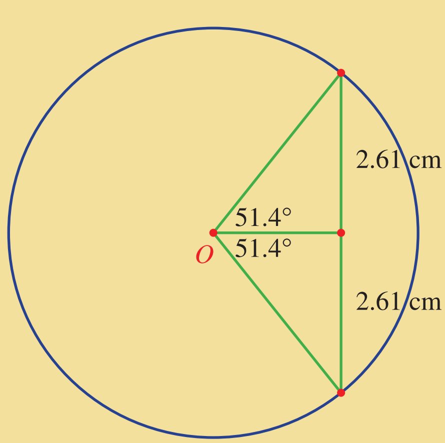
Chord theorem 1



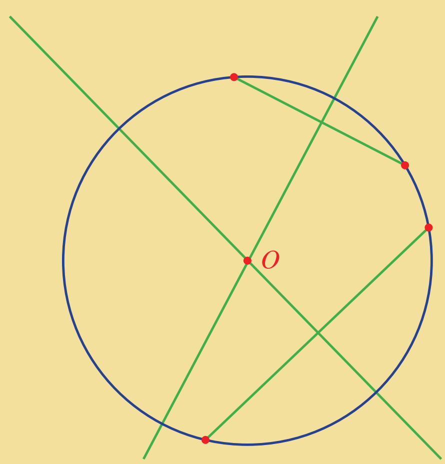
Chord theorem 2



Chord theorem 3



Chord theorem 4



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6G Angle properties of circles

Interactive
Widgets
HOTsheets
Walkthrough

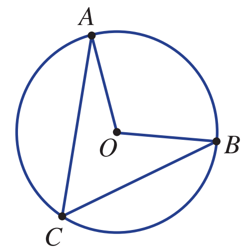
Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

The special properties of circles extend to the pairs of angles formed by radii and chords intersecting at the circumference. In this section we explore the relationship between angles at the centre and at the circumference subtended by the same arc.

Let's start: Discover angle properties

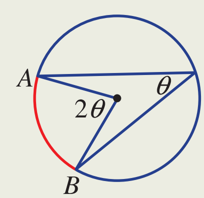
This activity can be completed with the use of a protractor and pair of compasses, but would be enhanced by dynamic geometry software.

- First, construct a circle and include two radii and two chords, as shown. The size of the circle and position of points A, B and C on the circumference can vary.
- Measure $\angle ACB$ and $\angle AOB$. What do you notice?
- Now construct a new circle with points A, B and C at different points on the circumference. (If dynamic software is used simply drag the points.) Measure $\angle ACB$ and $\angle AOB$ once again. What do you notice?
- Construct a new circle with $\angle AOB = 180°$ so that AB is a diameter. What do you notice about $\angle ACB$?



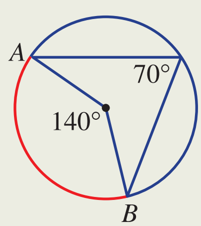
Key ideas

- Angles at the centre and circumference
  - The angle at the centre of a circle is twice the angle at the circumference when standing on the same arc.



For example:

1



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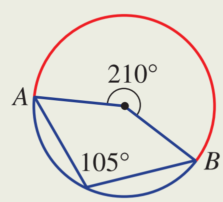

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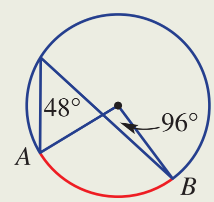
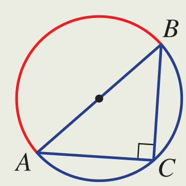


- Angle in a semicircle
  - The angle in a semicircle is a right angle.
    Note: $\angle ACB$ is known as the angle in a semicircle.



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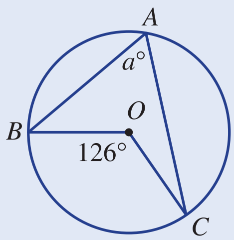
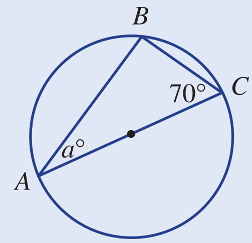
Example 12 Applying circle theorems

Find the value of the pronumerals in these circles.

a



b



SOLUTION

a $2a = 126$ (angles at centre and circumference on arc BC)

$\therefore a = 63$

b $\angle ABC$ is 90° (angle in a semicircle)

$\therefore a + 90 + 70 = 180$

$\therefore a = 20$

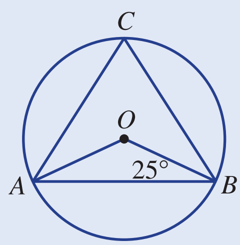
EXPLANATION

The angle at the circumference is half the angle at the centre.

The $\triangle ABC$ is inscribed inside a semicircle with diameter AC, so $\angle ABC$ must be a right angle.

Example 13 Combining circle theorems with other circle properties

Find the size of $\angle ACB$.



SOLUTION

$OA = OB$ (equal radii)

$\therefore \angle OAB = 25°$ (equal angles are opposite sides of equal length)

$\angle AOB = 180° - 2 \times 25°$ (angle sum of $\triangle AOB$)

$= 130°$

$\therefore \angle ACB = 65°$ (angle at centre and circumference on arc AB)

EXPLANATION

$\triangle AOB$ is isosceles.

Angle sum of a triangle is 180°.

The angle at the circumference is half the angle at the centre subtended by the same arc.

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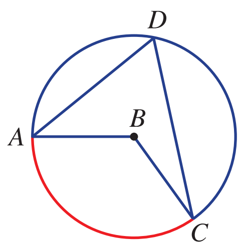
Exercise 6G

UNDERSTANDING AND FLUENCY	1–6	3, 4(½), 5–7	4(½), 5–7

1 Name another angle that is subtended by the same arc as $\angle ABC$ in these circles.

a



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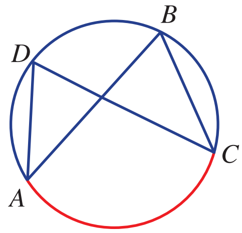
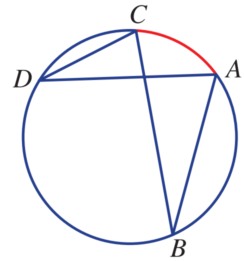


c



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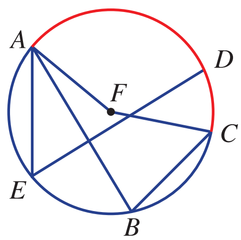

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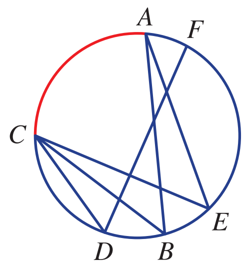

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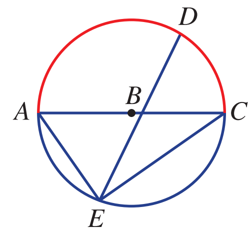


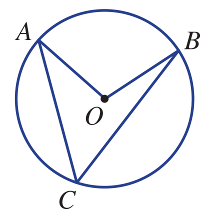
2 For this circle, O is the centre.

a Name the angle at the centre of the circle.

b Name the angle at the circumference of the circle.

c If $\angle ACB = 40°$, find $\angle AOB$.

d If $\angle AOB = 122°$, find $\angle ACB$.



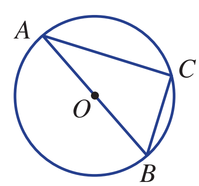
3 For this circle, AB is a diameter.

a What is the size of $\angle AOB$?

b What is the size of $\angle ACB$?

c If $\angle CAB = 30°$, find $\angle ABC$.

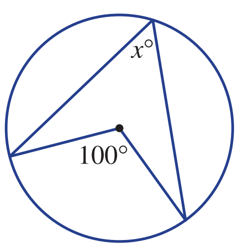
d If $\angle ABC = 83°$, find $\angle CAB$.



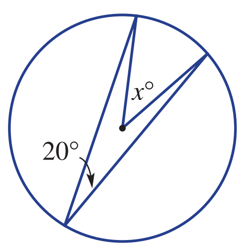
Example 12a

4 Find the value of x in these circles.

a



b



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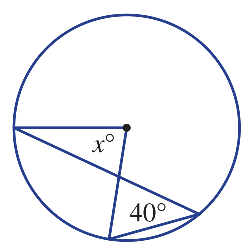

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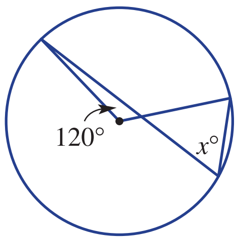
d



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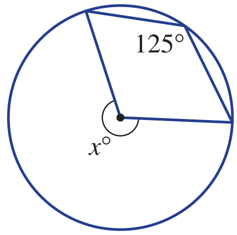

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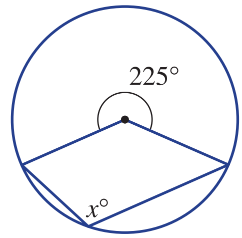
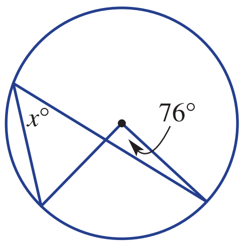


g



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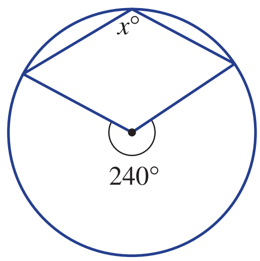
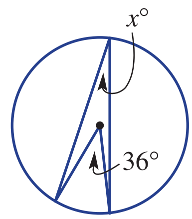


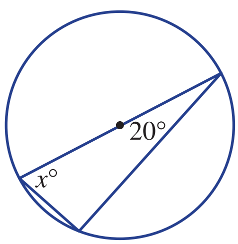
i



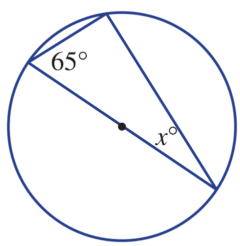
Example 12b

5 Find the value of x in these circles.

a



b



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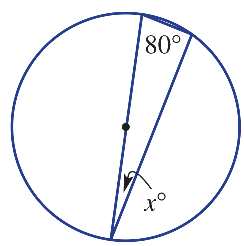
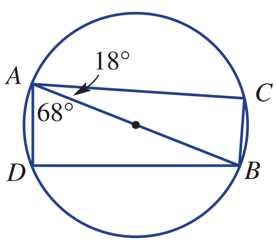


6 Find the size of both $\angle ABC$ and $\angle ABD$.

a



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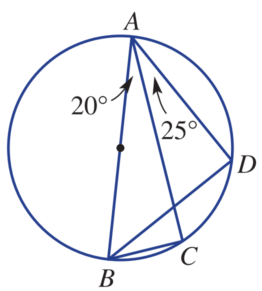

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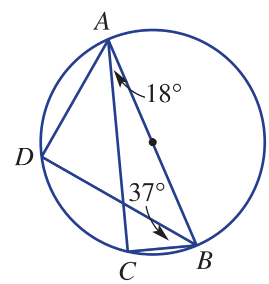
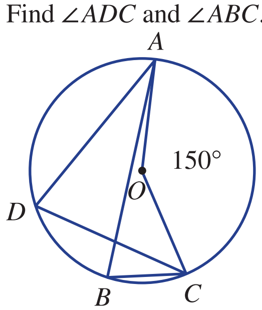
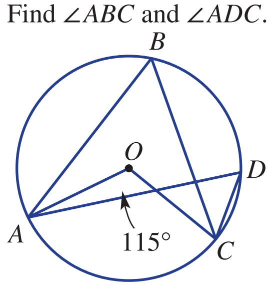


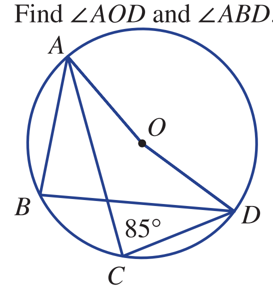
7 **a** Find $\angle ADC$ and $\angle ABC$.



a Find $\angle ABC$ and $\angle ADC$.



b Find $\angle AOD$ and $\angle ABD$.



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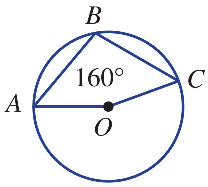
PROBLEM-SOLVING AND REASONING	8, 10	8–9(½), 10–12	8–9(½), 11–13

Example 13

8 Find $\angle ABC$.

a



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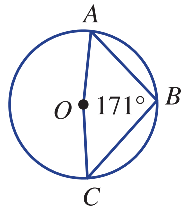

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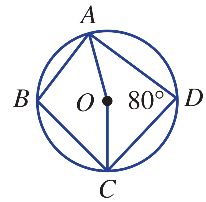

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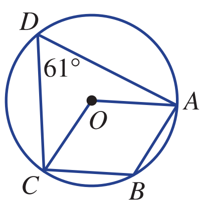

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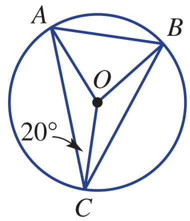

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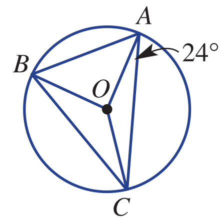
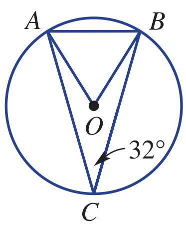
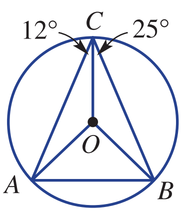


9 Find $\angle OAB$.

a



b



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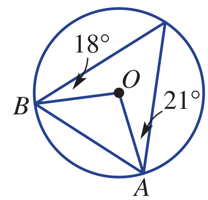

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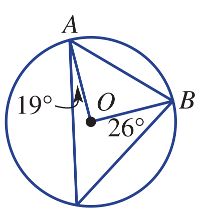

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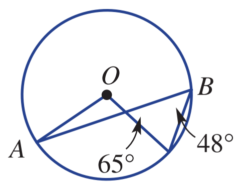

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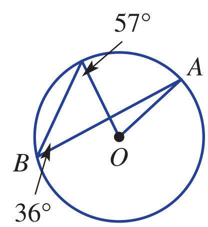
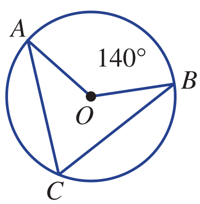


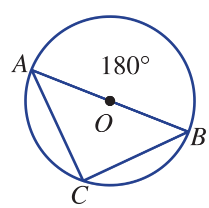
10 **a** For the first circle shown, find $\angle ACB$.



b For the second circle shown, find $\angle ACB$.

c For the second circle what can you say about $\angle ACB$?

d Explain why the second circle can be thought of as a special case of the first circle.



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11 Consider the two circles.

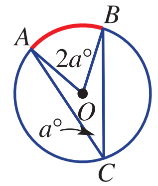
a If a minor arc is used, answer true or false.

- **i** $\angle AOB$ is always acute.
- **ii** $\angle AOB$ can be acute or obtuse.
- **iii** $\angle ACB$ is always acute.
- **iv** $\angle ACB$ can be acute or obtuse.

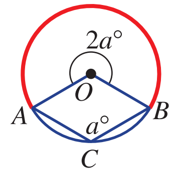
b If a major arc is used, answer true or false.

- **i** $\angle ACB$ can be acute.
- **ii** $\angle ACB$ is always obtuse.
- **iii** The angle at the centre $(2a^\circ)$ is a reflex angle.
- **iv** The angle at the centre $(2a^\circ)$ can be obtuse.

Minor arc AB



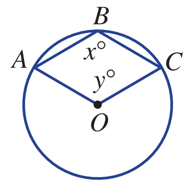
Major arc AB



12 Consider this circle.

a Write reflex $\angle AOC$ in terms of x.

b Write y in terms of x.



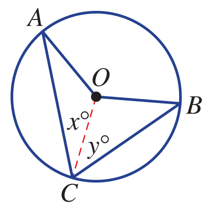
13 Consider this circle and let $\angle OCA = x^\circ$ and $\angle OCB = y^\circ$.

a Find $\angle AOC$ in terms of x, giving reasons.

b Find $\angle BOC$ in terms of y, giving reasons.

c Find $\angle AOB$ in terms of x and y.

d Explain why $\angle AOB = 2\angle ACB$.

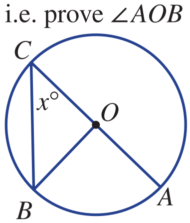


ENRICHMENT	–	–	14, 15

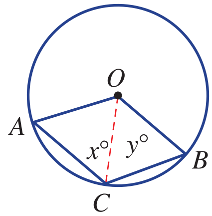
Proving all cases

14 Question **13** sets out a proof using a given illustration. Now use a similar technique for these cases.

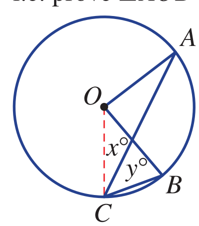
a Prove $\angle AOB = 2\angle ACB$; i.e. prove $\angle AOB = 2x^\circ$.



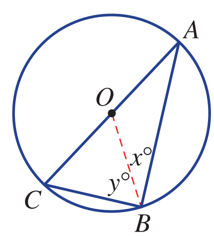
b Prove reflex $\angle AOB = 2\angle ACB$; i.e. prove reflex $\angle AOB = 2(x + y)^\circ$.



c Prove $\angle AOB = 2\angle ACB$; i.e. prove $\angle AOB = 2y^\circ$.



15 Prove that the angle in a semicircle is a right angle by showing that $x + y = 90$.



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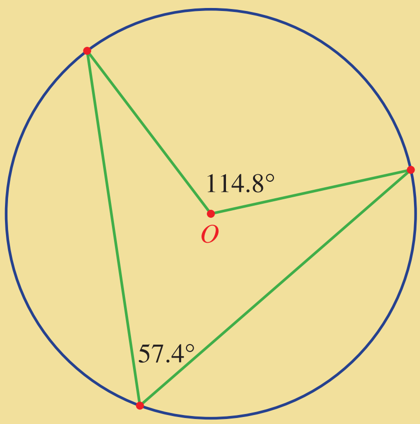
Exploring circle theorems with dynamic geometry software

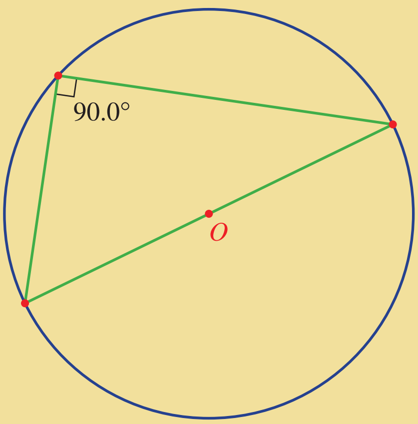
1 Construct a circle with centre O and any given radius.

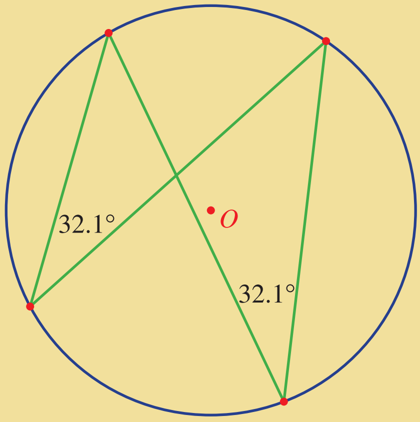
2 Illustrate the four angle properties by constructing line segments, as shown. Measure corresponding angles to illustrate the angle properties.
- The angle at the centre of a circle is twice the angle at a point on the circle subtended by the same arc.
- The angle in a semicircle is 90°.
- Angles at the circumference of a circle subtended by the same arc are equal.
- Opposite angles in a cyclic quadrilateral are supplementary.

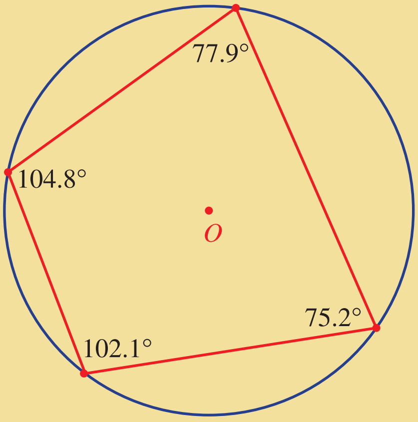
3 Drag the circle or one of the points on the circle to check that the properties are retained.

Angle theorem illustrations









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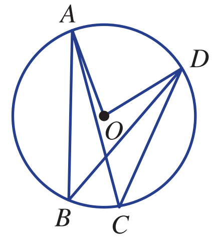
6H Further angle properties of circles

Interactive
Widgets
HOTsheets
Walkthrough

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

There are two more important properties of pairs of angles in a circle if both angles are at the circumference.

For the circle shown, you will recall that $\angle AOD = 2\angle ABD$ and also $\angle AOD = 2\angle ACD$. This implies that $\angle ABD = \angle ACD$, which proves that angles at the circumference subtended by the same arc are equal.

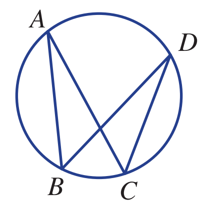


Another theorem relates to cyclic quadrilaterals, which have all four vertices sitting on the same circle. This will also be explored in this section.

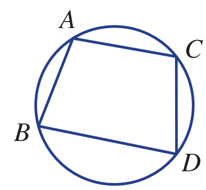
Let's start: Discover angle properties

Once again, use a protractor and a pair of compasses for this exercise or use dynamic geometry software.

- Construct a circle with four points at the circumference, as shown.
- Measure $\angle ABD$ and $\angle ACD$. What do you notice? Drag A, B, C or D and compare the angles.



- Now construct this cyclic quadrilateral (or drag point C if using dynamic geometry software).
- Measure $\angle ABD$, $\angle BDC$, $\angle DCA$ and $\angle CAB$. What do you notice? Drag A, B, C or D and compare angles.



The study of angle properties of circles by ancient Greek mathematicians laid the foundations of trigonometry. Much of the work was done by Ptolemy, who lived in the city of Alexandria, Egypt. The photo shows ancient monuments that existed in Ptolemy's time and are still standing in Alexandria today.

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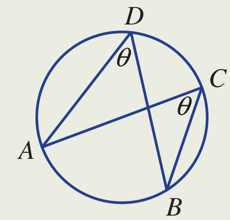
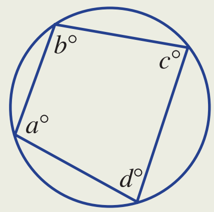
Key ideas

- Angles at the circumference
  - Angles at the circumference, subtended by the same arc, are equal. In the diagram, arc AB has subtended two equal angles.
    $\angle C = \angle D$
  - Similarly arc CD has subtended.
    $\angle A = \angle B$



- Opposite angles in cyclic quadrilaterals
  - A cyclic quadrilateral has all four vertices sitting on the same circle.
  - The opposite angles of a cyclic quadrilateral are supplementary.

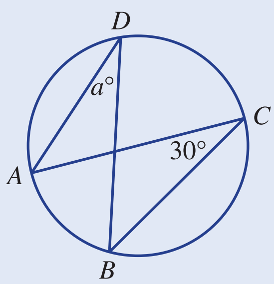
$$a + c = 180$$
$$b + d = 180$$



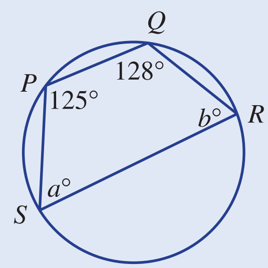
Example 14 Applying circle theorems

Find the value of the pronumerals in these circles.

a



b



SOLUTION

a $a = 30$ (angles in the same segment on arc AB)

b $a + 128 = 180$ (opposite angles of cyclic quadrilateral $PQRS$)
$\therefore a = 52$

$b + 125 = 180$ (opposite angles of cyclic quadrilateral $PQRS$)
$\therefore b = 55$

EXPLANATION

$a°$ and $30°$ are subtended by the same arc and are in the same segment.

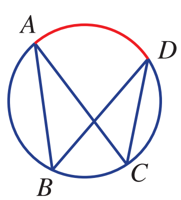
The quadrilateral is cyclic, so opposite angles sum to $180°$.

Exercise 6H

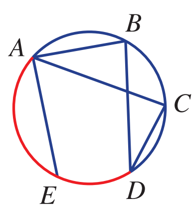
UNDERSTANDING AND FLUENCY	1–5	2–5	4–5(½)

1 Name another angle that is subtended by the same arc as $\angle ABD$.

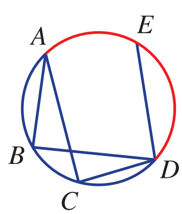
a



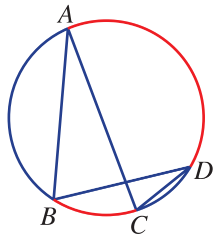
b



c

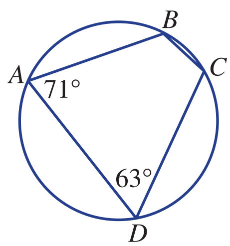


2 For this circle, answer the following.



a Name two angles subtended by arc AD.
b State the size of $\angle ACD$ if $\angle ABD = 85°$.
c Name two angles subtended by arc BC.
d State the size of $\angle BAC$ if $\angle BDC = 17°$.

3 Opposite angles in a cyclic quadrilateral are supplementary.

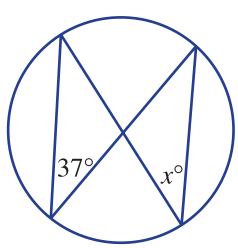


a What does it mean when we say two angles are supplementary?
b Find $\angle ABC$.
c Find $\angle BCD$.
d Check that $\angle ABC + \angle BCD + \angle CDA + \angle DAB = 360°$.

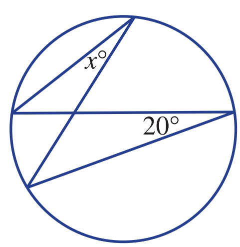
Example 14a

4 Find the value of x in these circles.

a



b



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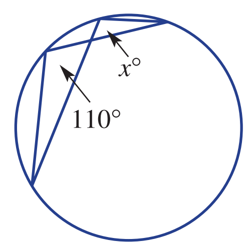

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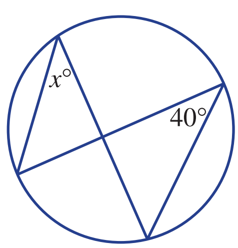

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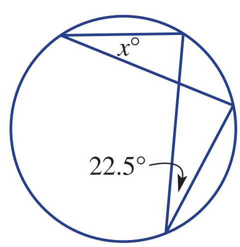
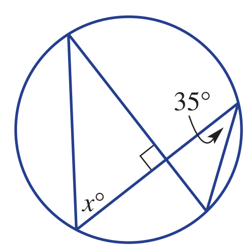


f



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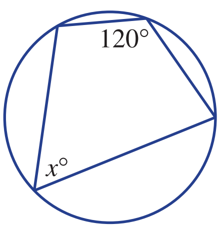
Example 14b

5 Find the value of the pronumerals in these circles.

a


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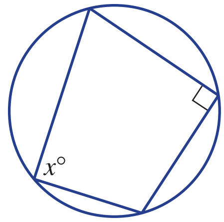

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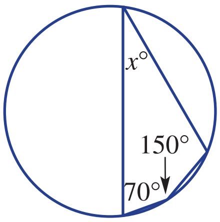

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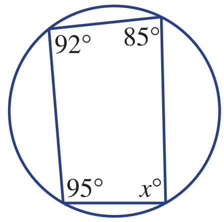

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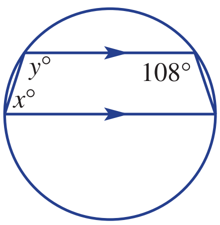

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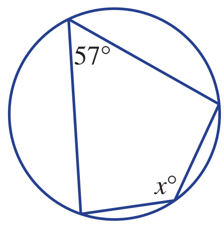
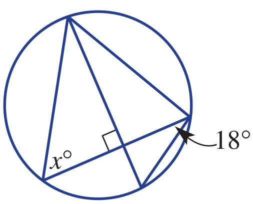


PROBLEM-SOLVING AND REASONING	6, 8	6–7(½), 8, 9	6–7(½), 9, 10

6 Find the value of x.

a


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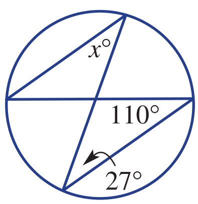

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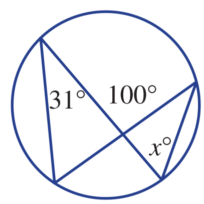

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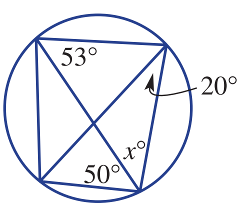

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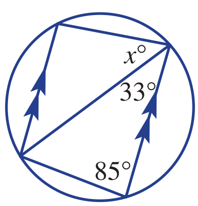

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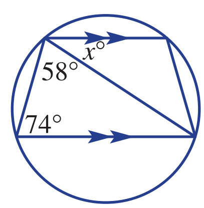

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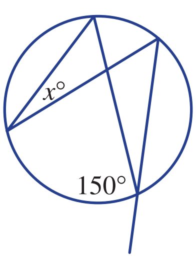

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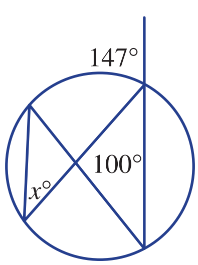

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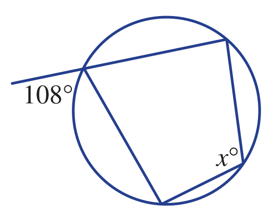

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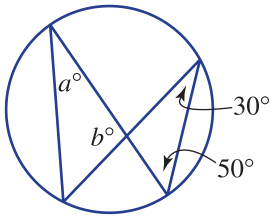
7 Find the values of the pronumerals in these circles.

a



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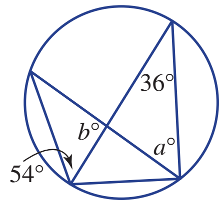

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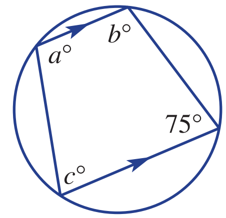

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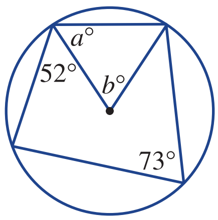

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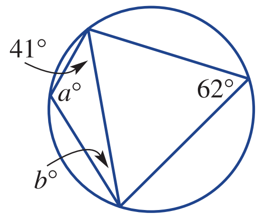

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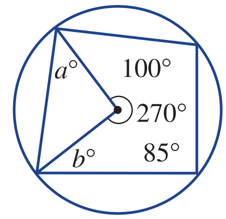
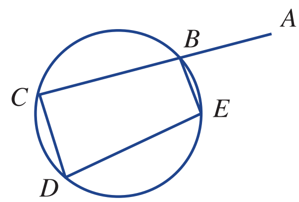


8 $\angle ABE$ is an exterior angle to the cyclic quadrilateral $BCDE$.

a If $\angle ABE = 80°$, find $\angle CDE$.

b If $\angle ABE = 71°$, find $\angle CDE$.

c Prove that $\angle ABE = \angle CDE$.

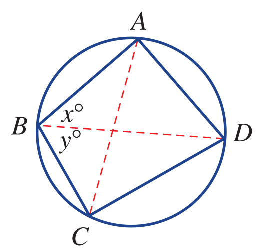


9 Prove that opposite angles in a cyclic quadrilateral are supplementary by following these steps.

a Explain why $\angle ACD = x°$ and $\angle DAC = y°$.

b Prove that $\angle ADC = 180° - (x + y)°$.

c What does this say about $\angle ABC$ and $\angle ADC$?

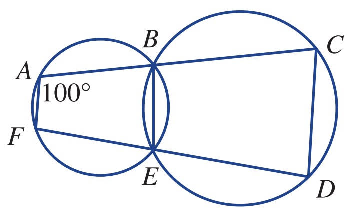


10 If $\angle BAF = 100°$, complete the following.

a Find:

i $\angle FEB$ **ii** $\angle BED$ **iii** $\angle DCB$

b Explain why $AF \parallel CD$.



ENRICHMENT	–	–	11

The sine rule

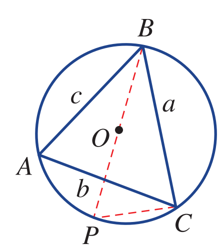
11 Consider a triangle ABC inscribed in a circle. The construction line BP is a diameter and PC is a chord. If r is the radius, then $BP = 2r$.

a What can be said about $\angle PCB$? Give a reason.

b What can be said about $\angle A$ and $\angle P$? Give a reason.

c If $BP = 2r$, use trigonometry with $\angle P$ to write an equation linking r and a.

d Prove that $2r = \dfrac{a}{\sin A}$, giving reasons.



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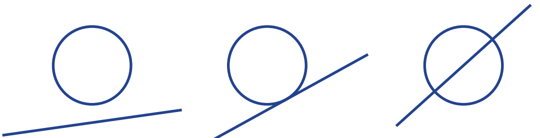
6I Theorems involving tangents

Interactive
Widgets
HOTsheets
Walkthrough

When a line and a circle are drawn, three possibilities arise – they could intersect once, twice or not at all.



0 points 1 point 2 points

If the line intersects the circle once, then it is called a tangent; if it intersects twice, it is called a secant.

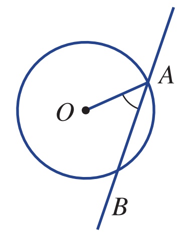
The surface of the road is a tangent to the tyre.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let’s start: From secant to tangent

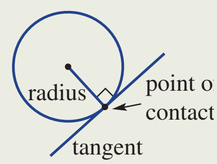
This activity is best completed using dynamic geometry software.

- Construct a circle with centre O and a secant line that intersects at A and B. Then measure $\angle BAO$.
- Drag B to alter $\angle BAO$. Can you place B so that line AB is a tangent? In this case, what is $\angle BAO$?

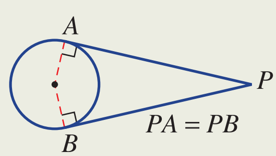


Key ideas

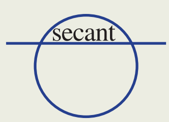
- A **tangent** is a line that touches a circle at a point called the **point of contact**.
  - A tangent intersects the circle exactly once.
  - A tangent is perpendicular to the radius at the point of contact.



- The two tangents drawn to a circle from an external point are equal in length.

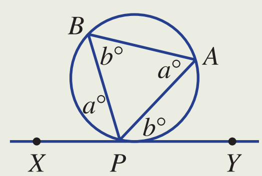


- A **secant** is a line that cuts a circle twice.



- The angle between a tangent and a chord drawn to the point of contact is equal to the angle at the circumference in the alternate segment. This is called the **alternate segment theorem**.

$\angle APY = \angle ABP$ and
$\angle BPX = \angle BAP$



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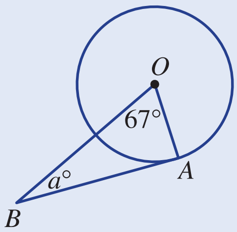
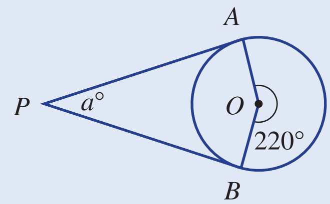
Example 15 Finding angles with tangents

Find the value of a in these diagrams that include tangents.

a



b



SOLUTION

a $\angle BAO = 90°$ (angle between radius and tangent)
$a + 90 + 67 = 180$ (angle sum of $\triangle OAB$)
$\therefore a = 23$

b $\angle PAO = \angle PBO = 90°$ (angle between radius and tangent)
Obtuse $\angle AOB = 360° - 220° = 140°$ (revolution)
$a + 90 + 90 + 140 = 360$ (angle sum of a quadrilateral)
$\therefore a = 40$

EXPLANATION

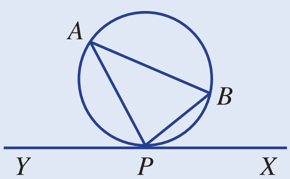
BA is a tangent, so $OA \perp BA$.
The sum of the angles in a triangle is 180°.

$PA \perp OA$ and $PB \perp OB$.
Angles in a revolution sum to 360°.
Angles in a quadrilateral sum to 360°.

Example 16 Using the alternate segment theorem

In this diagram, XY is a tangent to the circle.

a Find $\angle BPX$ if $\angle BAP = 38°$.
b Find $\angle ABP$ if $\angle APY = 71°$.



SOLUTION

a $\angle BPX = 38°$ (angle in alternate segment)
b $\angle ABP = 71°$ (angle in alternate segment)

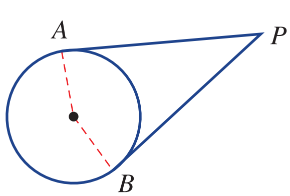
EXPLANATION

The angle between a tangent and a chord is equal to the angle in the alternate segment.

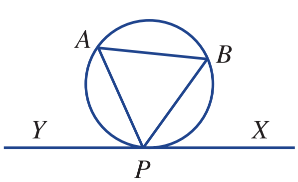
Exercise 6I

UNDERSTANDING AND FLUENCY	1–4, 5(½), 6, 7	2–4, 5(½), 6–8	4, 5(½), 6–8

1 **a** How many times does a tangent intersect a circle?
b At the point of contact, what angle does the tangent make with the radius?
c If AP is 5 cm, what is the length of BP in this diagram?



2 For this diagram name the angle that is:
a equal to $\angle BPX$
b equal to $\angle BAP$
c equal to $\angle APY$
d equal to $\angle ABP$



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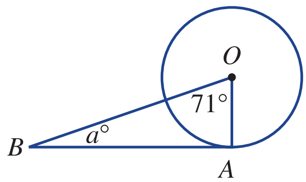
3 In Question **2**, if $\angle BPX = 70°$ and $\angle ABP = 40°$ find the size of:

a $\angle BAP$ **b** $\angle APY$ **c** $\angle APB$

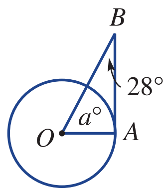
Example 15a

4 Find the value of a in these diagrams that include tangents.

a



b



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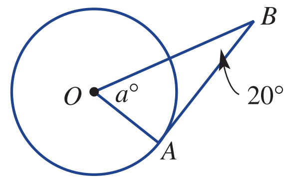
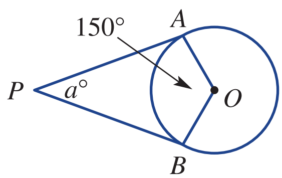


Example 15b

5 Find the value of a in these diagrams that include two tangents.

a



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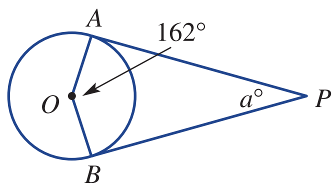

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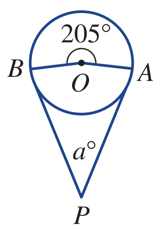

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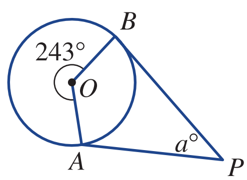
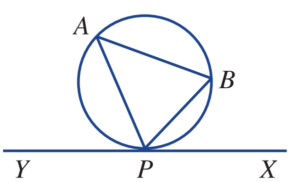


Example 16

6 In this diagram, XY is a tangent to the circle. Find:

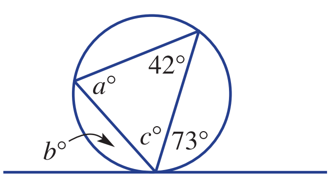
a $\angle PAB$ if $\angle BPX = 50°$

b $\angle APY$ if $\angle ABP = 59°$



7 Find the value of a, b and c in these diagrams.

a



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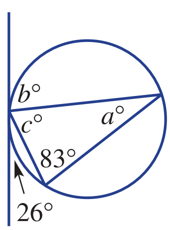

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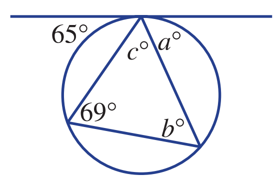
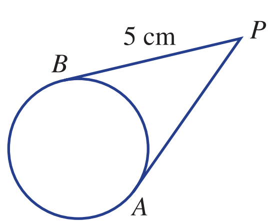


8 Find the length of AP.

a



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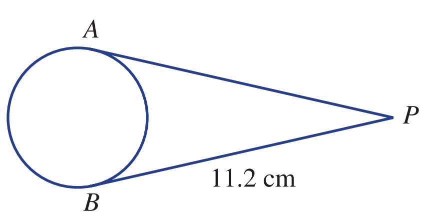

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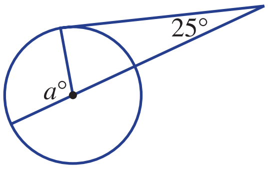
PROBLEM-SOLVING AND REASONING	9–10(½), 12, 13	9–10(½), 12, 14	10(½), 11, 13–15

9 Find the value of a.

a


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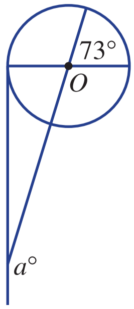

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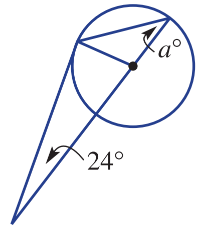

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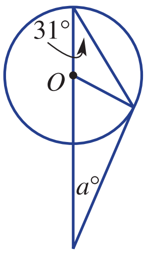

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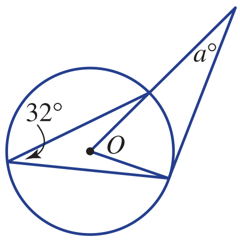

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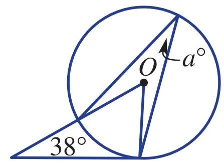

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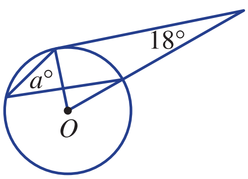

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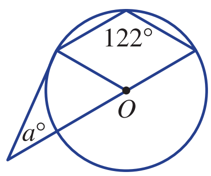

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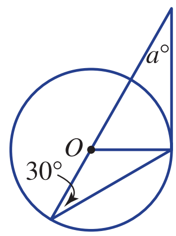
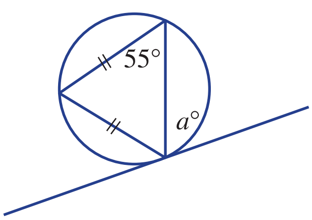


10 Find the value of a.

a


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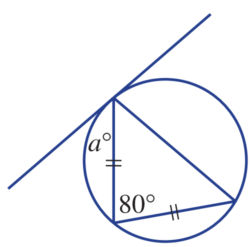

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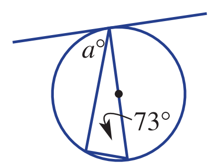

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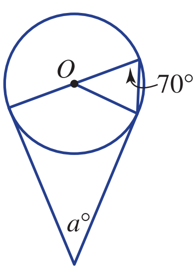

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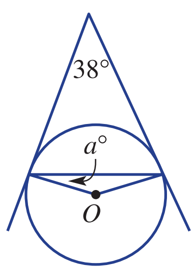

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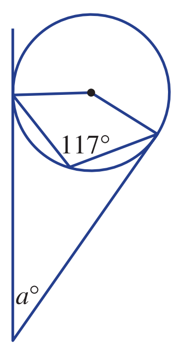

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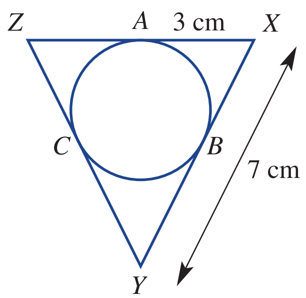
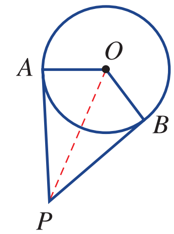
11 Find the length of CY in this diagram.

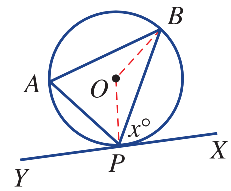


12 Prove that $AP = BP$ by following these steps.



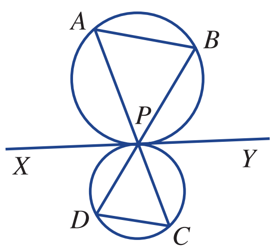
a Explain why $OA = OB$.
b What is the size of $\angle OAP$ and $\angle OBP$?
c Hence, prove that $\triangle OAP \equiv \triangle OBP$.
d Explain why $AP = BP$.

13 Prove the alternate segment theorem using these steps. First, let $\angle BPX = x°$, then give reasons at each step.

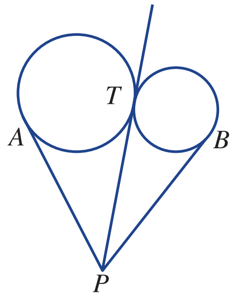


a Write $\angle OPB$ in terms of x.
b Write obtuse $\angle BOP$ in terms of x.
c Write $\angle BAP$ in terms of x.

14 These two circles touch with a common tangent XY. Prove that $AB \parallel DC$.



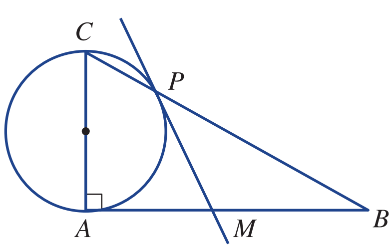
15 PT is a common tangent. Explain why $AP = BP$.



ENRICHMENT	–	–	16

Bisecting tangent

16 In this diagram, $\triangle ABC$ is right angled, AC is a diameter and PM is a tangent at P, where P is the point at which the circle intersects the hypotenuse.



a Prove that PM bisects AB; i.e. that $AM = MB$.
b Construct this figure using dynamic geometry software and check the result. Drag A, B or C to check different cases.

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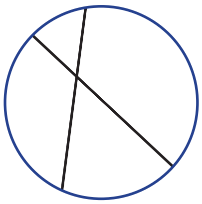
6J Intersecting chords, secants and tangents

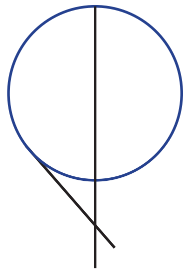
Interactive
Widgets
HOTsheets
Walkthrough

In circle geometry, the lengths of the line segments (or intervals) formed by intersecting chords, secants or tangents are connected by special rules. There are three situations in which this occurs:

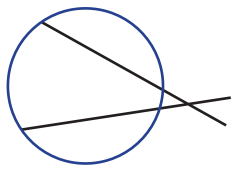
1 intersecting chords

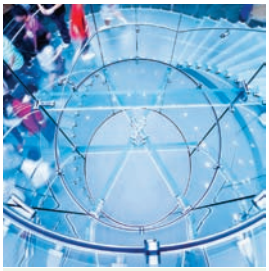


2 intersecting secant and tangent



3 intersecting secants





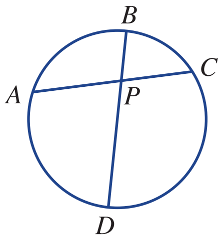
Architects use circle and chord geometry to calculate the dimensions of constructions, such as the glass structure.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Equal products

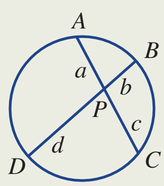
Use dynamic geometry software to construct this figure and then measure AP, BP, CP and DP.

- Calculate $AP \times CP$ and $BP \times DP$. What do you notice?
- Drag A, B, C or D. What can be said about $AP \times CP$ and $BP \times DP$ for any pair of intersecting chords?

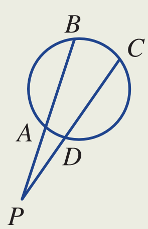


Key ideas

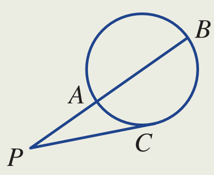
- When two chords intersect as shown, then AP, PC, BP and PD are called intercepts.
  - The products of the intercepts of two intersecting chords are equal.
    $AP \times CP = BP \times DP$ or $ac = bd$



- When two secants intersect at an external point P, as shown, then the products of the intercepts of two intersecting secants to a circle from an external point are equal. $AP \times BP = DP \times CP$



- When a secant intersects a tangent at an external point, as shown, then the square of a tangent to a circle from an external point equals the product of the intercepts of any secants from the point.
  $CP^2 = AP \times BP$



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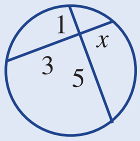
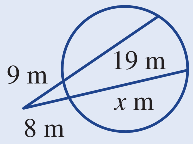
Example 17 Finding lengths using intersecting chords, secants and tangents

Find the length of x in each figure.

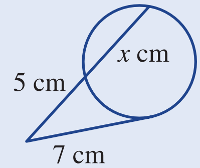
a



b



c



SOLUTION

a $x \times 3 = 1 \times 5$ (intercepts of intersecting chords)

$$3x = 5$$
$$x = \frac{5}{3}$$

b $8 \times (x + 8) = 9 \times 28$ (intercepts of intersecting secants)

$$x + 8 = \frac{252}{8}$$
$$x = \frac{188}{8}$$
$$= \frac{47}{2}$$

c $5 \times (x + 5) = 7^2$ (intersecting tangent and chord)

$$x + 5 = \frac{49}{5}$$
$$x = \frac{24}{5}$$

EXPLANATION

a Equate the products of each pair of line segments on each chord.

b Multiply the entire length of the secant by the length from the external point to the first intersection point with the circle. Then equate both products.

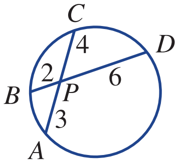
c Square the length of the tangent and then equate with the product from the other secant.

Exercise 6J

UNDERSTANDING AND FLUENCY	1–3, 4–6(a, b)	3, 4–6(a, c)	4–7(b)

1 State these lengths for the given diagram.

a AP **b** DP **c** AC **d** BD



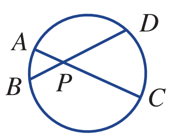
2 Solve these equations for x.

a $x \times 2 = 7 \times 3$ **b** $4x = 5 \times 2$

c $(x + 3) \times 7 = 6 \times 9$ **d** $7(x + 4) = 5 \times 11$

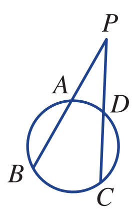
3 Complete the rules for each diagram.

a



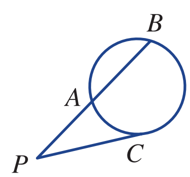
$AP \times CP = \_\_ \times \_\_$

b



$AP \times \_\_ = \_\_ \times \_\_$

c



$AP \times \_\_ = \_\_^2$

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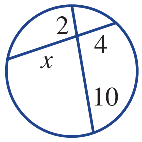
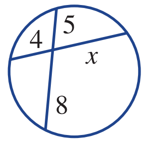
Example 17a

4 Find the length of x in each figure.

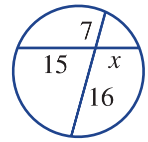
a



b



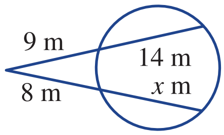
c



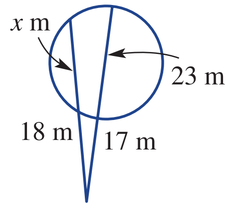
Example 17b

5 Find the value of x in each figure.

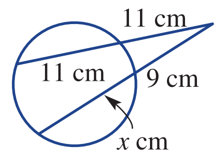
a



b



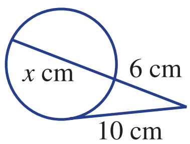
c



Example 17c

6 Find the value of x in each figure.

a



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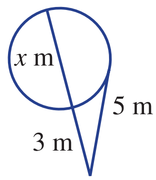
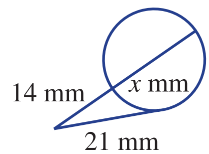
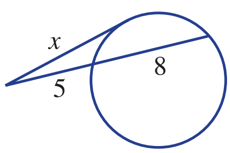


c



7 Find the exact value of x, in surd form.

a



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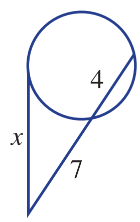
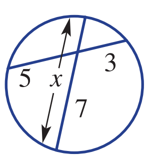


PROBLEM-SOLVING AND REASONING	8(½), 10, 11	8(½), 10–12	8(½), 9, 12–14

8 Find the value of x.

a



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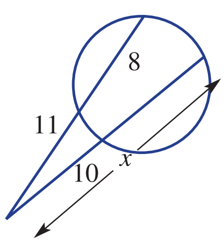

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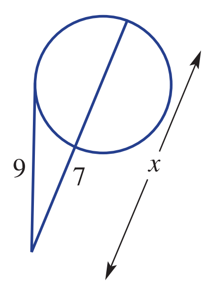

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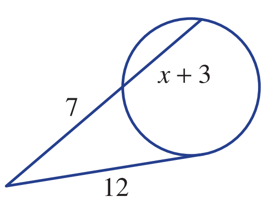

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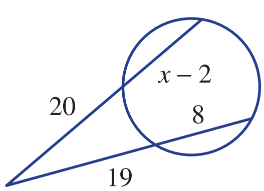

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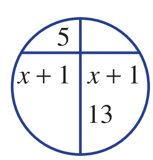

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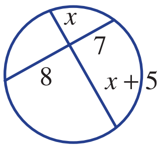
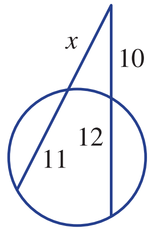
9 For each diagram, derive the given equations.

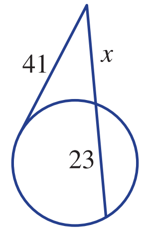
a $x^2 + 5x - 56 = 0$



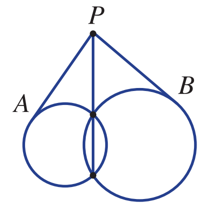
b $x^2 + 11x - 220 = 0$



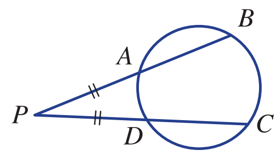
c $x^2 + 23x - 1681 = 0$



10 Explain why $AP = BP$ in this diagram, using your knowledge from this section.



11 In this diagram $AP = DP$. Explain why $AB = DC$.



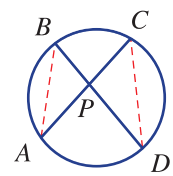
12 Prove that $AP \times CP = BP \times DP$ by following these steps.

a What can be said about the pair of angles $\angle A$ and $\angle D$ and also about the pair of angles $\angle B$ and $\angle C$? Give a reason.

b Prove $\triangle ABP ||| \triangle DCP$.

c Complete:
$$\frac{AP}{\square} = \frac{\square}{CP}$$

d Prove $AP \times CP = BP \times DP$.

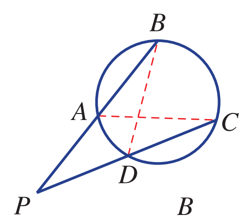


13 Prove that $AP \times BP = DP \times CP$ by following these steps.

a Consider $\triangle PBD$ and $\triangle PCA$. What can be said about $\angle B$ and $\angle C$? Give a reason.

b Prove $\triangle PBD ||| \triangle PCA$.

c Prove $AP \times BP = DP \times CP$.



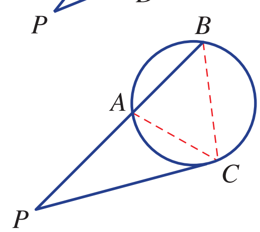
14 Prove that $AP \times BP = CP^2$ by following these steps.

a Consider $\triangle BPC$ and $\triangle CPA$. Is $\angle P$ common to both triangles?

b Explain why $\angle ACP = \angle ABC$.

c Prove $\triangle BPC ||| \triangle CPA$.

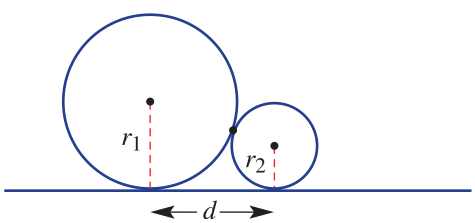
d Prove $AP \times BP = CP^2$.



ENRICHMENT	–	–	15

Horizontal wheel distance

15 Two touching circles have radii r_1 and r_2. The horizontal distance between their centres is d. Find a rule for d in terms of r_1 and r_2.



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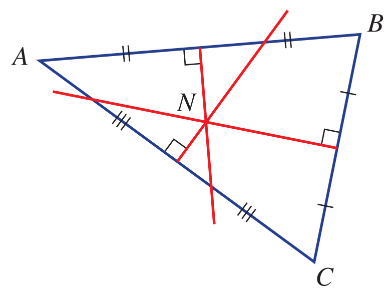
Investigation

1 Some special points of triangles and Euler's line

From Year 9, you may recall finding the circumcentre and the centroid of a triangle. Here we will review the construction of these two points and consider in further detail some of their properties, as well as their relationship with a third point – the orthocentre. This is best done using a dynamic geometry software package.

The circumcentre of a triangle

The perpendicular bisectors of each of the three sides of a triangle meet at a common point called the **circumcentre**.



a Construct and label a triangle ABC and measure each of its angles.

b Construct the perpendicular bisector of each side of the triangle.

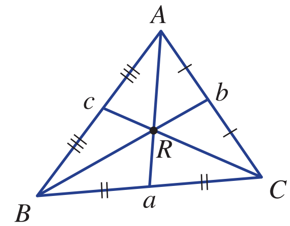
c Label the point of intersection of the lines as N. This is the circumcentre.

d By dragging the points of your triangle, observe what happens to the location of the circumcentre. Can you draw any conclusions about the location of the circumcentre for some of the different types of triangles; for example, equilateral, isosceles, right-angled or obtuse?

e Construct a circle centred at N with radius NA. This is the **circumcircle**. Drag the vertex A to different locations. What do you notice about vertices B and C in relation to this circle?

The centroid of a triangle

The three medians of a triangle intersect at a common point called the **centroid**. A **median** is the line drawn from a vertex to the midpoint of the opposite side.



a Construct and label a new triangle ABC and measure each of its angles.

b Mark the midpoint of each side of the triangle and construct a segment from each vertex to the midpoint of the opposite side.

c Observe the common point of intersection of these three medians and label this point R – the centroid. Point R is the centre of gravity of the triangle.

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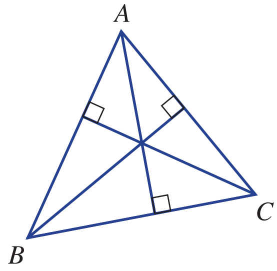
d Label the points of your triangle as shown and complete the table below by measuring the required lengths. Drag the vertices of your triangle to obtain different triangles to use for the table.

Lengths	AR	Ra	BR	Rb	CR	Rc
Triangle 1						
Triangle 2						
Triangle 3						
Triangle 4						

e From your table, what can you observe about how far along each median the centroid lies?

f Explain why each median divides the area of triangle ABC in half. What can be said about the area of the six smaller triangles formed by the three medians?

The orthocentre of a triangle



The three altitudes of a triangle intersect at a common point called the **orthocentre**. An **altitude** of a triangle is a line drawn from a vertex to the opposite side of the triangle, meeting it at right angles.

a Construct a triangle ABC and measure each angle.

b For each side, construct a line that is perpendicular to it and that passes through its opposite vertex.

c Label the point where these three lines intersect as O. This is the orthocentre.

d By dragging the vertices of the triangle, comment on what happens to the location of the orthocentre for different types of triangles.

e Can you create a triangle for which the orthocentre is outside the triangle? Under what circumstances does this occur?

All in one

a Construct a large triangle, ABC, and measure the angles. On this one triangle use the previous instructions to locate the circumcentre (N), the centroid (R) and the orthocentre (O).

b By dragging the vertices, can you make these three points coincide? What type of triangle achieves this?

c Construct a line joining the points N and R. Drag the vertices of the triangle. What do you notice about the point O in relation to this line?

d The line in part **c** is called **Euler's line**. Use this line to determine the ratio of the length NR to the length RO.

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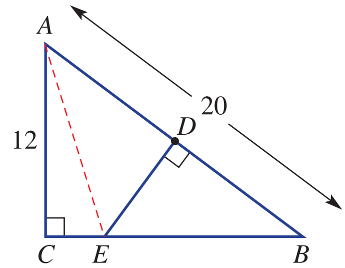
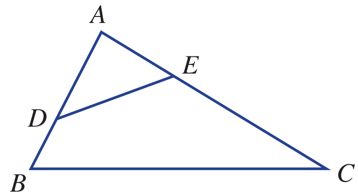
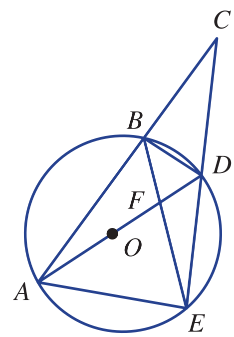
1 In a triangle ABC, angle C is a right angle, D is the midpoint of AB and DE is perpendicular to AB. The length of AB is 20 and the length of AC is 12. What is the area of triangle ACE?



2 In this diagram, $AB = 15$ cm, $AC = 25$ cm, $BC = 30$ cm and $\angle AED = \angle ABC$. If the perimeter of $\triangle ADE$ is 28 cm, find the lengths of BD and CE.



3 Name all the pairs of equal angles in the diagram shown.



4 A person stands in front of a cylindrical water tank and has a viewing angle of 27° to the sides of the tank. What percentage of the circumference of the tank can they see?

5 An isosceles triangle, ABC, is such that its vertices lie on the circumference of a circle. $AB = AC$ and the chord from A to the point D on the circle bisects BC at E. Prove that $AB^2 - AE^2 = BE \times CE$.

6 Points D, E and F are the midpoints of the three sides of $\triangle ABC$. The straight line formed by joining two midpoints is parallel to the third side and half its length.

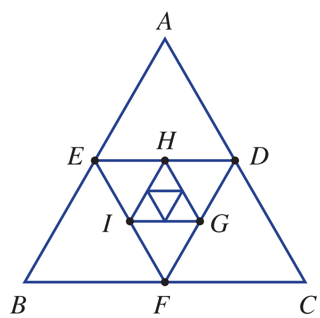
a Prove $\triangle ABC \,|||\, \triangle FDE$.

$\triangle GHI$ is drawn in the same way such that G, H and I are the midpoints of the sides of $\triangle DEF$.

b Find the ratio of the area of:

i $\triangle ABC$ to $\triangle FDE$ **ii** $\triangle ABC$ to $\triangle HGI$

c Hence, if $\triangle ABC$ is the first triangle drawn, what is the ratio of the area of $\triangle ABC$ to the area of the nth triangle drawn in this way?



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Chapter summary

Geometrical figures and circle geometry

Similar figures

All corresponding angles are equal and corresponding sides are in the same ratio; i.e. same shape but different in size.
There are four tests for similar triangles. $\triangle ABC \,|||\, \triangle DEF$ is a similarity statement.

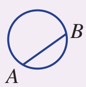
Polygons

Angle sum of polygon $S = 180(n - 2)$, where n is the number of sides.

Congruent triangles

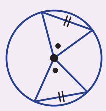
These triangles are identical, written $\triangle ABC \equiv \triangle DEF$.
Tests for congruence are SSS, SAS, AAS and RHS.

Circles and chords

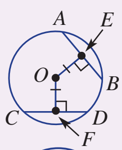


AB is a chord.

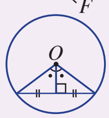
Chords of equal length subtend equal angles.



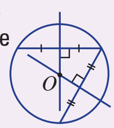
If $AB = CD$, then $OE = OF$.



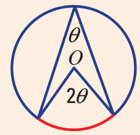
The perpendicular from centre to chord bisects the chord.



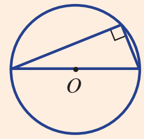
Perpendicular bisectors of every chord of a circle intersect at the centre.



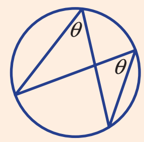
Angle properties of circles



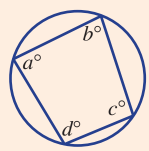
Angle at centre is twice that of angle at the circumference subtended by the same arc.



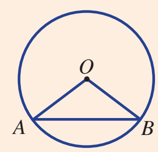
The angle in a semicircle is 90°.



Angles at circumference subtended by the same arc are equal.



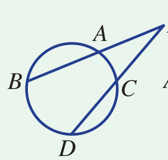
Opposite angles in cyclic quadrilaterals are supplementary.
$a + c = 180$
$b + d = 180$



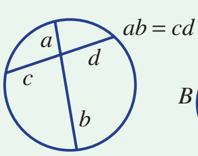
$\triangle OAB$ is isosceles given that OA and OB are radii.

Intersecting chords, secants, tangents

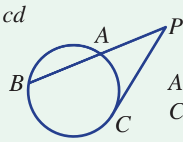
secant



$AP \times BP = CP \times DP$

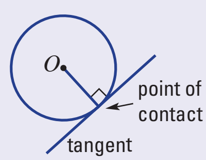


$ab = cd$

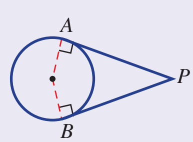


$AP \times BP = CP^2$
CP is a tangent.

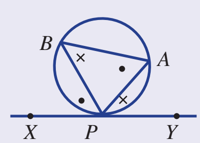
Tangents



A tangent touches a circle once and is perpendicular to the radius at point of contact.



Tangents PA and PB have equal length; i.e. $PA = PB$.



Angle between tangent and chord is equal to angle in alternate segment.

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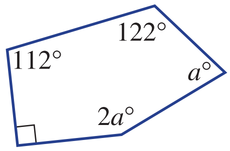
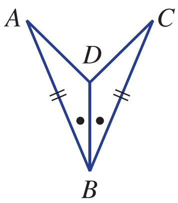
Multiple-choice questions

1 The value of a in the polygon shown is:



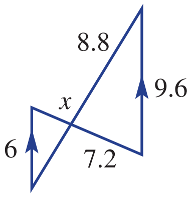
A 46 B 64 C 72
D 85 E 102

2 The test that proves $\triangle ABD \equiv \triangle CBD$ is:



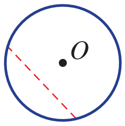
A RHS B SAS C SSS D AAA E AAS

3 The value of x in the diagram shown is:



A 4.32 B 4.5 C 3.6 D 5.5 E 5.2

4 The name given to the dashed line in the circle with centre O is:

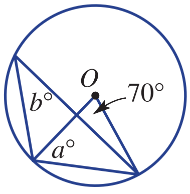


A a diameter B a minor arc C a chord
D a tangent E a secant

5 A circle of radius 5 cm has a chord 4 cm from the centre of the circle. The length of the chord is:

A 4.5 cm B 6 cm C 3 cm
D 8 cm E 7.2 cm

6 In the diagram shown, the values of the pronumerals are:



A $a = 55, b = 35$
B $a = 30, b = 70$
C $a = 70, b = 35$
D $a = 55, b = 70$
E $a = 40, b = 55$

7 A cyclic quadrilateral has one angle measuring 63° and another angle measuring 108°. Another angle in the cyclic quadrilateral is:

A 63° B 108° C 122°
D 75° E 117°

Chapter review

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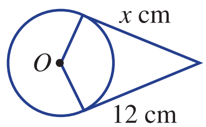
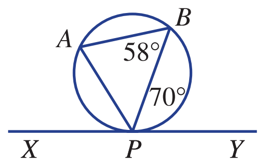
8 For the circle shown with radius 5 cm, the value of x is:



A 13 **B** 10.9 **C** 12 **D** 17 **E** 15.6

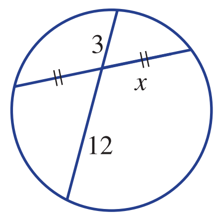
9 For the circle shown, the value of $\angle APB$ is:



A 50° **B** 45° **C** 10° **D** 52° **E** 25°

10 The value of x in the diagram shown is:

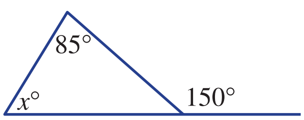
A 7.5 **B** 6 **C** 3.8
D 4 **E** 5



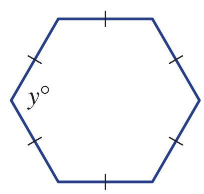
Short-answer questions

1 Determine the value of each pronumeral.

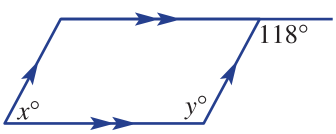
a



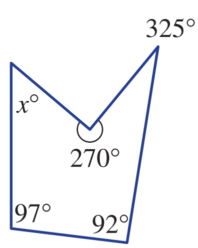
b



c

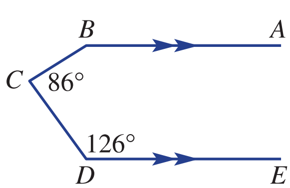


d

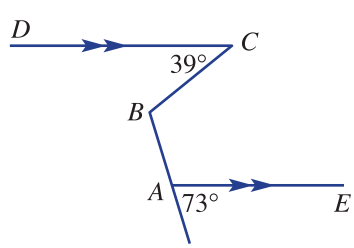


2 Find the value of $\angle ABC$ by adding a third parallel line.

a



b



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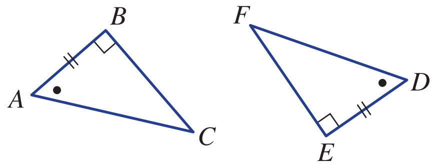
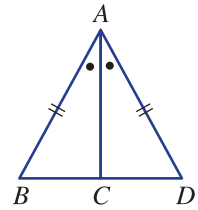
3 Prove that each pair of triangles is congruent, giving reasons.

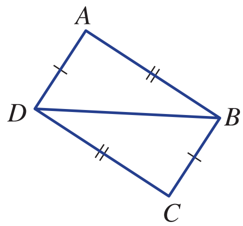
a



b



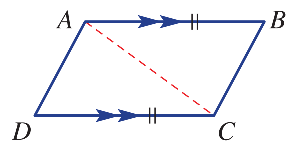
c



4 Complete these steps to prove that if one pair of opposite sides is equal and parallel in a quadrilateral, then it is a parallelogram.

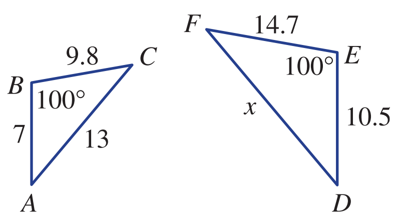
a Prove $\triangle ABC \equiv \triangle CDA$, giving reasons.

b Hence, prove $AB \parallel DC$.

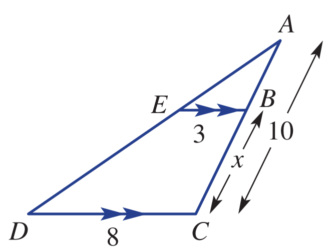


5 In each of the following, identify pairs of similar triangles by proving similarity, giving reasons, and then use this to find the value of x.

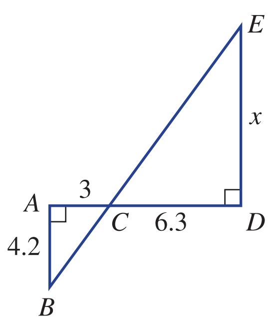
a



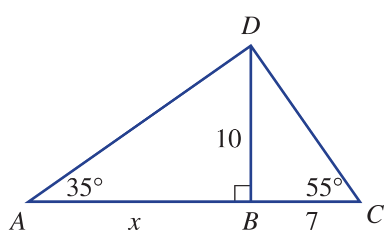
b



c

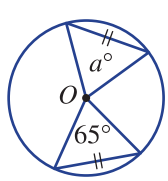


d

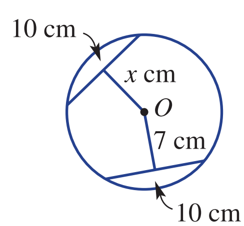


6 Find the value of each pronumeral.

a



b



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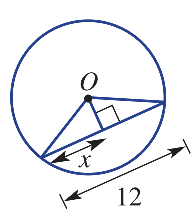

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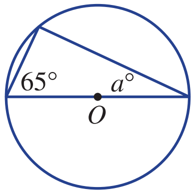
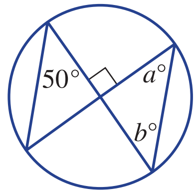
7 Use the circle theorems to help find the values of the pronumerals.

a


b


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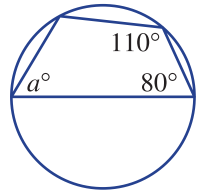
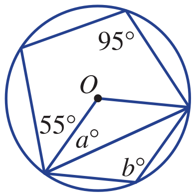
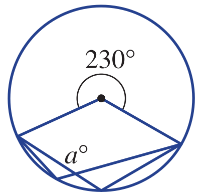
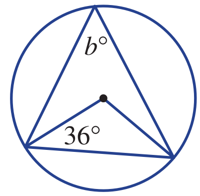
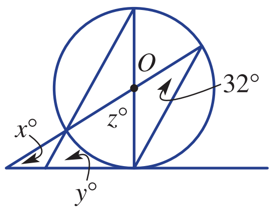


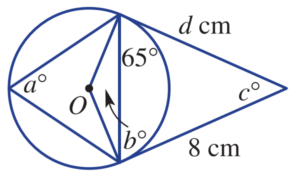
d


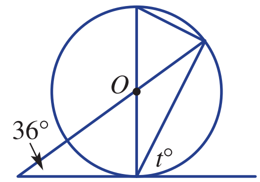
e


f


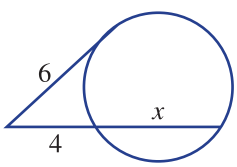
8 Find the value of the pronumerals in these diagrams involving tangents and circles.

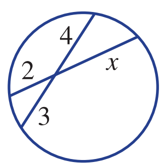
a


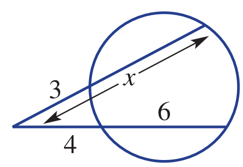
b


c


9 Find the value of x in each figure.

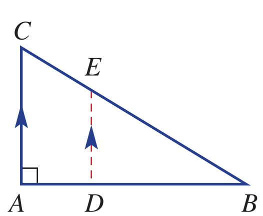
a


b


c


Extended-response questions

1 The triangular area of land shown is to be divided into two areas such that $AC \parallel DE$. The land is to be divided so that $AC:DE = 3:2$.



a Prove that $\triangle ABC \,|||\, \triangle DBE$.

b If $AC = 1.8$ km, find DE.

c If $AD = 1$ km and $DB = x$ km:

  i Show that $2(x + 1) = 3x$. **ii** Solve for x.

d For the given ratio, what percentage of the land area does $\triangle DBE$ occupy? Give your answer to 1 decimal place.

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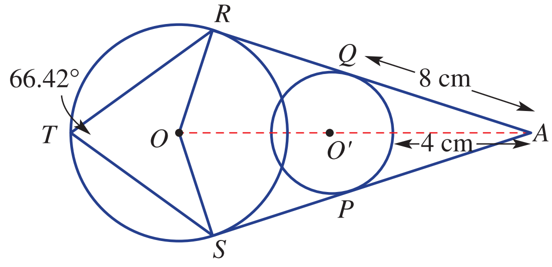
2 The diagram shows two intersecting circles sharing common tangents AR and AS. The large circle has a radius of 10 cm and the distance between the centres O and O' of the two circles is 15 cm. Other measurements are as shown. Given that the two centres O and O' and the point A are in a straight line, complete the following.



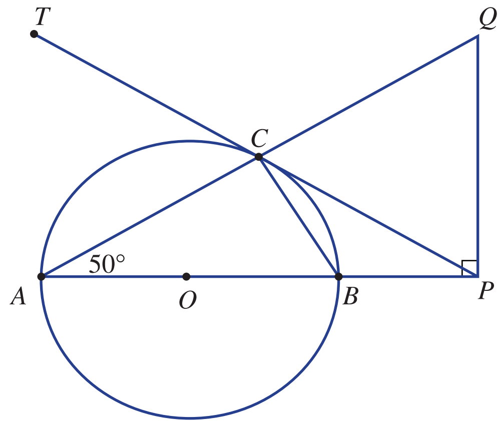
a Find the values of these angles.
  i $\angle ROS$
  ii $\angle RAS$

b Use the rule for intersecting secants and tangents to help find the diameter of the smaller circle.

c Hence, what is the distance from A to O?

d By first finding AR, determine the perimeter of $AROS$.

3 AB is a diameter of circle centre O and AB is produced to P, as shown. PQ meets AP at 90° and TP is a tangent to the circle at C.



a Explain why $\angle ACB = 90°$.

b Prove that $BCQP$ is a cyclic quadrilateral.

c Find the size of angles ABC, CBP and CQP.

d Find the size of angle TCA, giving a reason.

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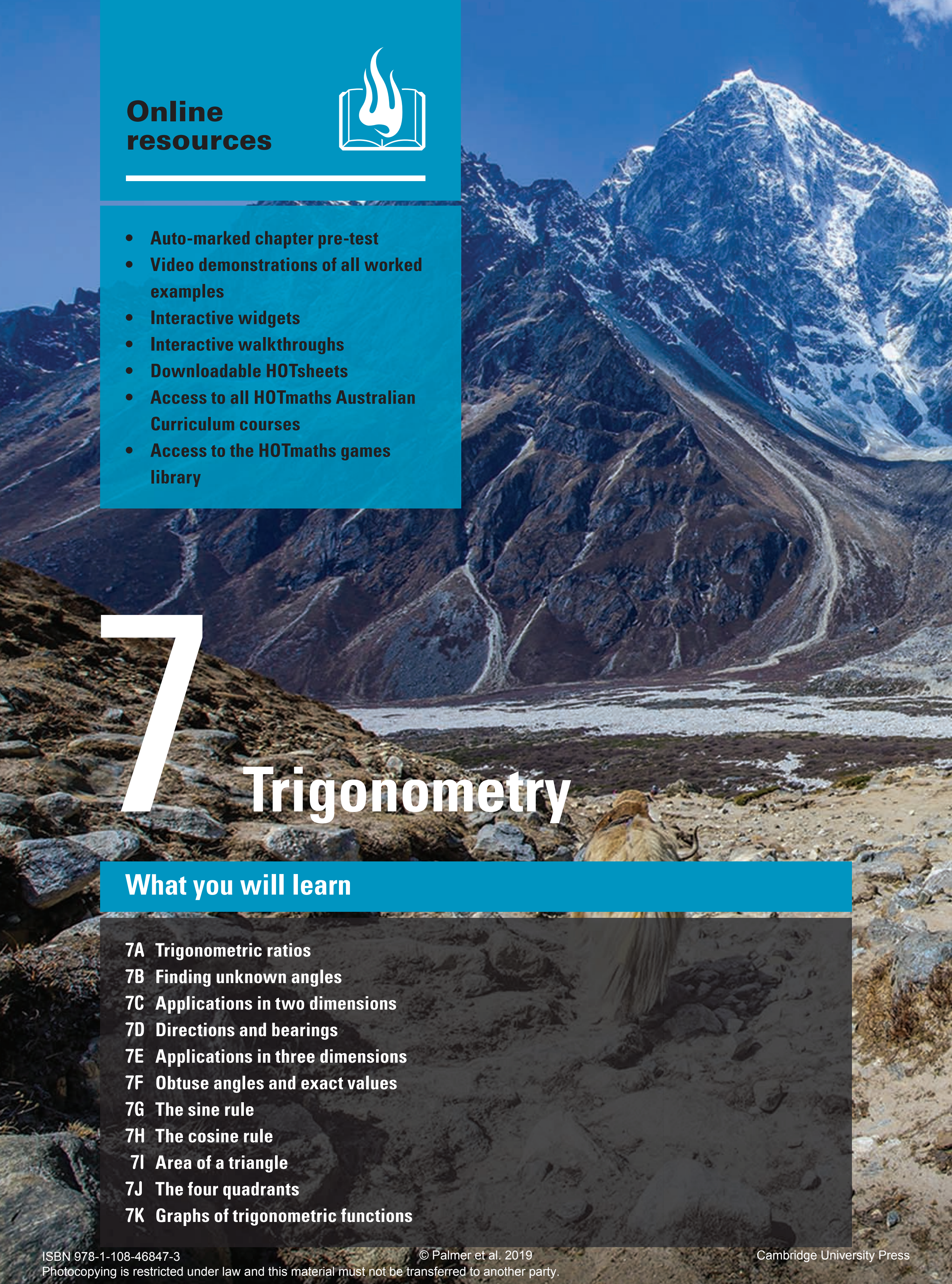

Online resources

- Auto-marked chapter pre-test
- Video demonstrations of all worked examples
- Interactive widgets
- Interactive walkthroughs
- Downloadable HOTsheets
- Access to all HOTmaths Australian Curriculum courses
- Access to the HOTmaths games library

7 Trigonometry

What you will learn

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NSW syllabus

STRAND: MEASUREMENT AND GEOMETRY

SUBSTRAND: RIGHT-ANGLED TRIANGLES (TRIGONOMETRY) TRIGONOMETRY AND PYTHAGORAS' THEOREM

Outcomes

A student applies trigonometry, given diagrams, to solve problems, including problems involving angles of elevation and depression. (MA5.1–10MG)

A student applies trigonometry to solve problems, including problems involving bearings. (MA5.2–13MG)

A student applies Pythagoras' theorem, trigonometric relationships, the sine rule, the cosine rule and the area rule to solve problems, including problems involving three dimensions. (MA5.3–15MG)

Trigonometry on a vast scale

The Great Trigonometrical Survey of India by the British began in 1802 and took over 60 years to complete. It was a huge mapping project requiring exacting mathematical calculations and physical endurance. Marching through jungles and across rugged country, the party included surveyors riding on elephants, about 30 soldiers on horses, oxen, over 40 camels carrying supplies and up to 700 walking labourers.

Surveyors used the triangulation method to map a network of large, linked triangles across India. Triangulation starts with a triangle formed by joining each end of a horizontal baseline, of known length, to the visible top of a hill. Its base angles are measured and, using high school trigonometry, the side lengths calculated. Each side of this triangle can be used as a baseline for a new triangle. Also, by measuring the angle of elevation of a hilltop, its height above the baseline altitude can be calculated.

The surveying team gradually travelled from South India 2400 km north to the Himalayas in Nepal. After measuring Mt Everest from six different locations, Radhanath Sikdar declared it the highest mountain on Earth at 8840 m (29 002 feet). It was an accurate measurement: using GPS (global positioning system) Mt Everest has since been found to have a height of 8850 m (29 035 feet). GPS also involves a process of triangulation using radio waves sent between satellites and Earth.

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1 Use a calculator to calculate the following, correct to 2 decimal places.

a $\cos 27°$ **b** $\tan 84°$ **c** $\sin 15.7°$

d $8 \sin 24°$ **e** $\dfrac{2}{\cos 32.5°}$ **f** $\dfrac{7}{\tan 42.1°}$

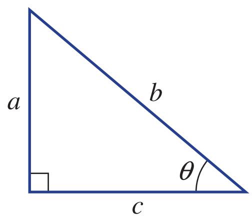
2 For these equations, solve for x, correct to 1 decimal place.

a $x = 3 \times \cos 68°$ **b** $\dfrac{x}{4} = \sin 65°$ **c** $x \tan 22° = 9$

d $x \sin 73° = 5.2$ **e** $\dfrac{5}{x} = \tan 33°$ **f** $\dfrac{\sqrt{3}}{x} = \cos 52°$

3 State which of the sides labelled a, b and c are the hypotenuse (H), opposite side (O) and adjacent side (A) to the angle θ in these right-angled triangles.

a



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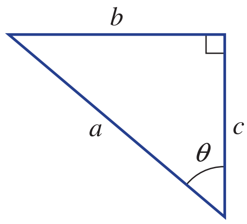

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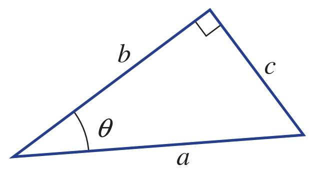
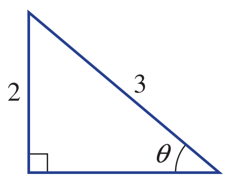


4 Complete these trigonometric ratios.

a $\sin \theta = \dfrac{\text{opposite}}{\boxed{}}$ **b** $\cos \theta = \dfrac{\boxed{}}{\text{hypotenuse}}$ **c** $\dfrac{\boxed{}}{} = \dfrac{\text{opposite}}{\text{adjacent}}$

5 Using the three trigonometric ratios in Question **4**, write the appropriate ratio for the following right-angled triangles; for example, $\sin \theta = \dfrac{3}{4}$.

a



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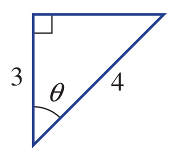

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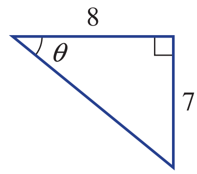


6 Find the value of θ in the following, correct to 1 decimal place.

a $\sin \theta = 0.7$ **b** $\tan \theta = \dfrac{1}{8}$ **c** $\cos \theta = \dfrac{3}{5}$

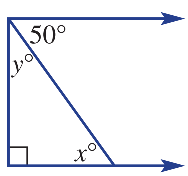
d $\cos \theta = \dfrac{1}{4}$ **e** $\tan \theta = 1.87$ **f** $\sin \theta = 0.42$

7 Write these as bearings (degrees clockwise from north).

a south **b** west **c** north-east **d** south-west

8 Find the value of x and y.

a



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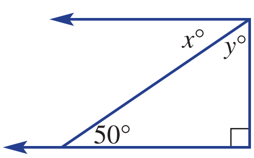

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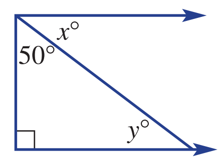

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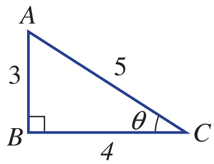
7A Trigonometric ratios

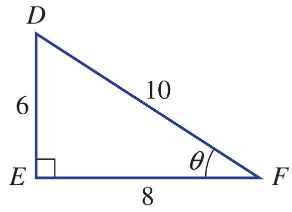
Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Interactive
Widgets
HOTsheets
Walkthrough

The study of trigonometry explores the relationship between the angles and side lengths of triangles. Trigonometry can be applied to simple problems, such as finding the angle of elevation of a kite, to solving complex problems in surveying and design.

Trigonometry is built upon the three ratios sine, cosine and tangent. These ratios do not change for triangles that are similar, such as $\triangle ABC$ and $\triangle DEF$.





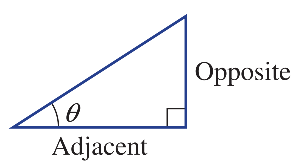
$$\sin\theta = \frac{3}{5} \qquad \sin\theta = \frac{6}{10} = \frac{3}{5}$$

$$\cos\theta = \frac{4}{5} \qquad \cos\theta = \frac{8}{10} = \frac{4}{5}$$

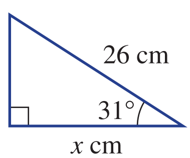
$$\tan\theta = \frac{3}{4} \qquad \tan\theta = \frac{6}{8} = \frac{3}{4}$$

Let's start: Which ratio?

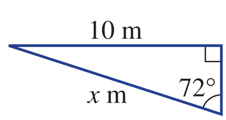
In a group or with a partner, see if you can recall some facts from Year 9 trigonometry to answer the following questions.



- What is the name given to the longest side of a right-angled triangle?
- $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$ is one trigonometric ratio. What are the other two?
- Which ratio would be used to find the value of x in this triangle? Can you also find the value of x?



- Which ratio would be used to find the value of x in this triangle? Can you find also the value of x?



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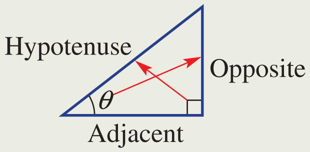
Key ideas

- The hypotenuse is the longest side in a right-angled triangle. It is opposite the right-angle.
- Given a right-angled triangle containing an angle θ, the three trigonometric ratios are:

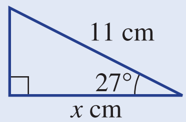


  - The sine ratio: $\sin\theta = \dfrac{\text{length of the opposite side}}{\text{length of the hypotenuse}}$
  - The cosine ratio: $\cos\theta = \dfrac{\text{length of the adjacent side}}{\text{length of the hypotenuse}}$
  - The tangent ratio: $\tan\theta = \dfrac{\text{length of the opposite side}}{\text{length of the adjacent side}}$
- Many people like to use SOHCAHTOA to help remember the three ratios.
- To find an unknown length on a right-angled triangle:
  - Choose a trigonometric ratio that links one known angle and a known side length with the unknown side length.
  - Solve for the unknown side length.
- There are some relationships between the ratios, such as $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. See the Enrichment section of **Exercise 7A**.
- Angles are measured in degrees (°), minutes (′) and seconds (″).
  - 1 degree = 60 minutes
  - 1 minute = 60 seconds
  - For example: 24°21′54″ is 24 degrees, 21 minutes and 54 seconds.
  - On most calculators you can enter angle values in this way.

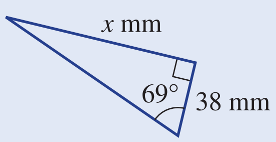
Example 1 Solving for an unknown in the numerator

Find the value of x in these right-angled triangles, correct to 2 decimal places.

a



b



SOLUTION	EXPLANATION
a $\cos\theta = \dfrac{A}{H}$	Choose the ratio $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$.
$\cos 27° = \dfrac{x}{11}$	Multiply both sides by 11, then use a calculator.
$\therefore x = 11 \times \cos 27°$	
$= 9.80$ (to 2 decimal places)	Round your answer as required.

b $\tan\theta = \frac{O}{A}$

$\tan 69° = \frac{x}{38}$

$\therefore x = 38 \times \tan 69°$

$= 98.99$ (to 2 decimal places)

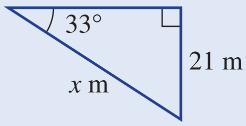
The tangent ratio uses the opposite and the adjacent sides.

Multiply both sides by 38.

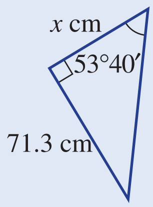
Example 2 Solving for an unknown in the denominator

Find the value of x in these right-angled triangles, rounding your answer to 2 decimal places.

a



b



SOLUTION

a $\sin\theta = \frac{O}{H}$

$\sin 33° = \frac{21}{x}$

$x \times \sin 33° = 21$

$x = \frac{21}{\sin 33°}$

$= 38.56$ (to 2 decimal places)

b $\tan\theta = \frac{O}{A}$

$\tan 53°40' = \frac{71.3}{x}$

$x \times \tan 53°40' = 71.3$

$x = \frac{71.3}{\tan 53°40'}$

$= 52.44$ (to 2 decimal places)

EXPLANATION

a Choose the sine ratio since the adjacent side is not marked.

Multiply both sides by x to remove the fraction, then divide both sides by sin 33°.

b The hypotenuse is unmarked, so use the tangent ratio.

Multiply both sides by x, then solve by dividing both sides by tan 53°40′.

Exercise 7A

UNDERSTANDING AND FLUENCY	1–5	3, 4–5(½), 6	4–5(½), 6

1 Use a calculator to evaluate the following, correct to 3 decimal places.

a cos 37° **b** sin 72° **c** tan 50° **d** cos 21.4°
e sin 15.9° **f** tan 85.1° **g** cos 78°43′ **h** sin 88°10′

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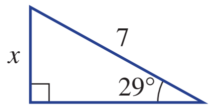
2 Decide which ratio (i.e. $\sin\theta = \dfrac{O}{H}$, $\cos\theta = \dfrac{A}{H}$ or $\tan\theta = \dfrac{O}{A}$) would be best to help find the value of x in these triangles. Do not find the value of x.

a



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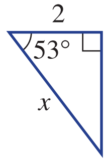

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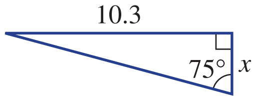


3 Solve for x in these equations, correct to 2 decimal places.

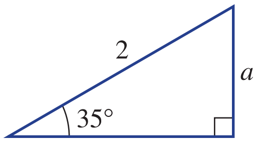
a $\dfrac{x}{3} = \tan 31°$ **b** $\dfrac{x}{5} = \cos 54°$ **c** $\dfrac{x}{12.7} = \sin 15.6°$

d $\sin 57° = \dfrac{2}{x}$ **e** $\cos 63.4° = \dfrac{10}{x}$ **f** $\tan 71.6° = \dfrac{37.5}{x}$

Example 1

4 Use trigonometric ratios to find the values of the pronumerals, to 2 decimal places.

a



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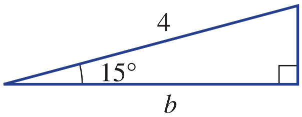

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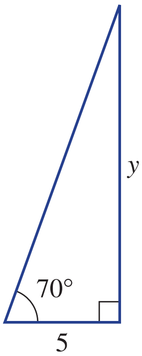

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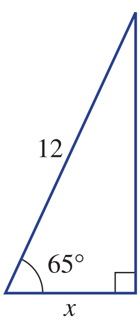

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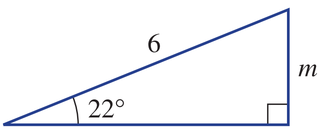

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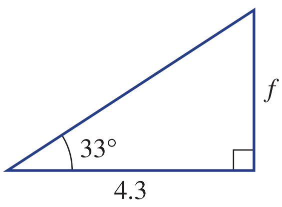

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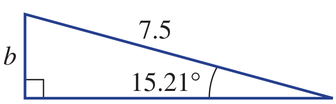

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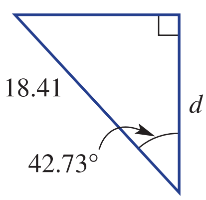

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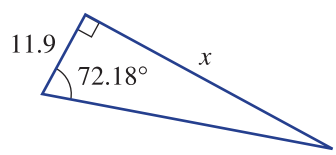

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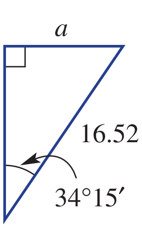

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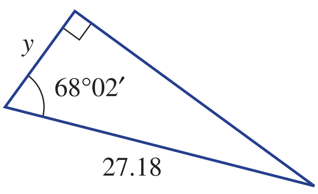

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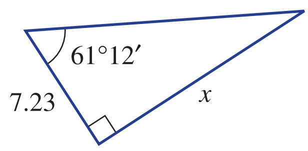

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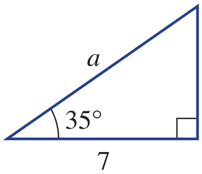
Example 2

5 Use trigonometric ratios to find the values of the pronumerals, to 2 decimal places, for these right-angled triangles.

a



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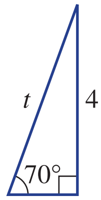

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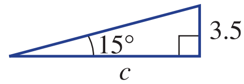

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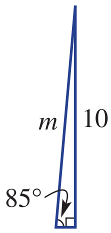

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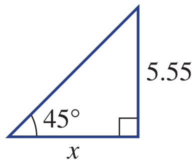

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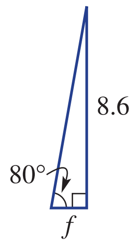

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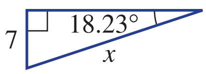

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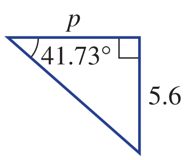

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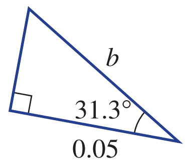

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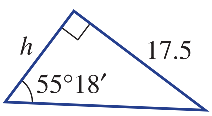

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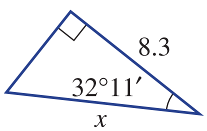

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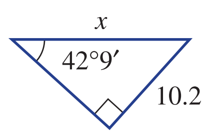
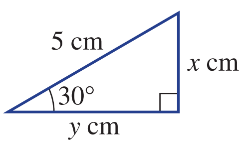


6 Find the unknown side lengths for these right-angled triangles, correct to 2 decimal places where necessary.

a



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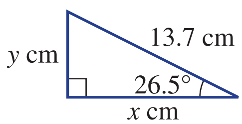

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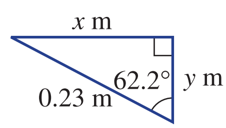
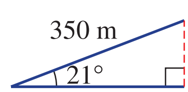


PROBLEM-SOLVING AND REASONING	7, 8, 14	9–11, 14	10–15

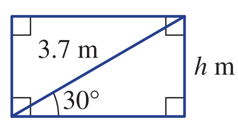
7 A 4WD climbs a 350 m straight slope at an angle of 21° to the horizontal.

a Find the vertical distance travelled, correct to the nearest metre.

b Find the horizontal distance travelled, correct to the nearest metre.



8 A diagonal wall brace of length 3.7 metres is at an angle of 30° to the horizontal. Find the height (h m) of the face of the wall, to the nearest centimetre.



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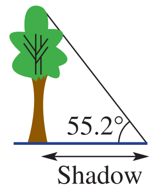
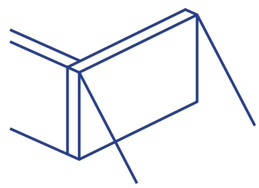
9 The angle from the horizontal of the line of sight, from the end of a tree's shadow to the top of the tree, is 55.2°. The length of the shadow is 15.5 m. Find the height of the tree, correct to 1 decimal place.



10 On a construction site, large concrete slabs of height 5.6 metres are supported at the top by steel beams positioned at an angle of 42° from the vertical. Find the length of the steel beams, to 2 decimal places.

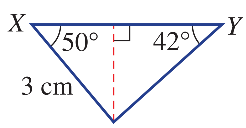


11 By measuring the diagonals, a surveyor checks the dimensions of a rectangular revegetation area of length 25 metres. If the angle of the diagonal to the side length is 28.6°, find the length of the diagonals, correct to 1 decimal place.

12 A right-angled triangular flag is made for the premiers of a school competition. The top edge of the flag is 25 cm and the second-largest angle on the flag is 71°. Find the length of the longest edge of the flag, to the nearest centimetre.

13 Find the length XY in these diagrams, correct to 1 decimal place.

a



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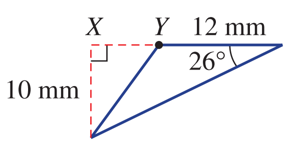
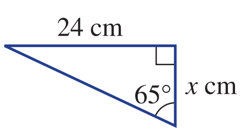


14 A student solves for x, to 2 decimal places, in the given triangle and gets 11.21, as shown. But the answer is 11.19. Explain the student's error.



$$\tan 65° = \frac{24}{x}$$
$$x = \frac{24}{\tan 65°}$$
$$= \frac{24}{2.14}$$
$$= 11.21$$

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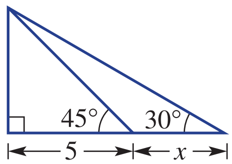
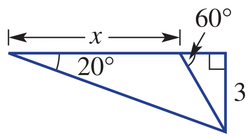
15 Find the value of x, correct to 1 decimal place, in these triangles.

a


b


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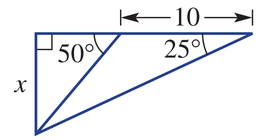
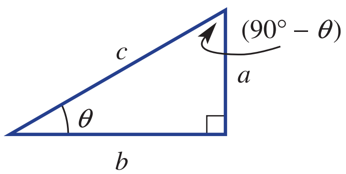


ENRICHMENT	–	–	16

Relationships between the ratios

16 For these proofs, consider the right-angled triangle shown below.



a Show that $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$ by completing these steps.
   - i Write a in terms of c and θ.
   - ii Write b in terms of c and θ.
   - iii Write $\tan \theta$ in terms of a and b.
   - iv Substitute your expressions from parts i and ii into your expression for $\tan \theta$ in part iii. Simplify to prove $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$.
   - v Can you find a different way of proving the rule described above?

b Show that $(\sin \theta)^2 + (\cos \theta)^2 = 1$ by completing these steps.
   - i Write a in terms of c and θ.
   - ii Write b in terms of c and θ.
   - iii State Pythagoras' theorem using a, b and c.
   - iv Use your results from parts i, ii and iii to show that $(\sin \theta)^2 + (\cos \theta)^2 = 1$.

c Show that $\sin \theta = \cos (90° - \theta)$ by completing these steps.
   - i $\sin \theta = \dfrac{a}{c}$, $\cos \theta = \dfrac{\square}{\square}$, $\sin (90° - \theta) = $ ______, $\cos (90° - \theta) = $ ______
   - ii Is $\sin \theta = \cos (90° - \theta)$?
   - iii Can you see another relationship?

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7B Finding unknown angles

Interactive
Widgets
HOTsheets
Walkthrough

The three trigonometric ratios discussed earlier can also be used to find unknown angles in right-angled triangles if at least two side lengths are known. For example, if $\cos\theta = \frac{1}{2}$, we can use a calculator to find the size of the unknown angle.

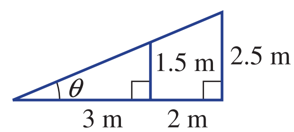
ANZAC Bridge in Sydney is a cable-stayed bridge in which each cable forms a triangle with the pylons and the bridge deck.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: The ramp

A ski ramp is 2.5 m high and 5 m long (horizontally) with a vertical strut of 1.5 m placed as shown.

- Discuss which triangle could be used to find the angle of incline, θ. Does it matter which triangle is used?
- Which trigonometric ratio is to be used and why?
- How does $\boxed{\tan^{-1}}$ on a calculator help to calculate the value of θ?
- Discuss how you can check if your calculator is in degree mode.

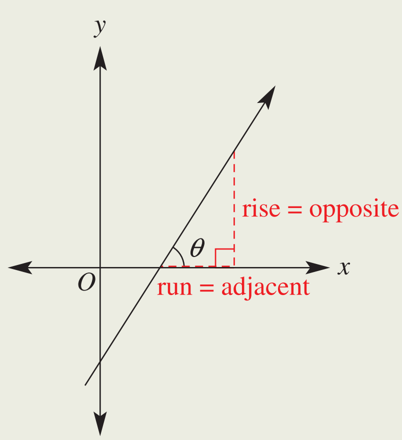


Key ideas

- The $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ buttons on calculators are used to find angles when the trigonometric ratio is known.
  - If $\sin\theta = 0.5$ then $\theta = 30°$.
  - If $\cos\theta = 0.5$ then $\theta = 60°$.
  - If $\tan\theta = 0.5$ then $\theta = 26°34'$ (to the nearest minute).
- On the Cartesian plane, gradient (m) can be calculated using the formula $m = \tan\theta$, where θ is the angle between a line and the positive direction of the x-axis.

  $$\text{Gradient} = m = \frac{\text{rise}}{\text{run}} = \frac{\text{opposite}}{\text{adjacent}} = \tan\theta$$

  $$\therefore m = \tan\theta$$



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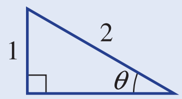
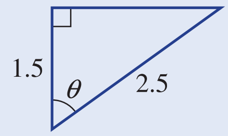
Example 3 Finding angles

Find the value of θ in the following right-angled triangles, rounding to the nearest minute in part **b**.

a



b



SOLUTION

a $\sin\theta = \dfrac{1}{2}$

$\theta = 30°$

b $\cos\theta = \dfrac{1.5}{2.5}$

$\theta = 53.1301\ldots$

$\theta = 53°8'$ (nearest minute)

EXPLANATION

Use $\sin\theta$, as the opposite side and the hypotenuse are given. Use inverse sine on a calculator to find the angle (e.g. $\sin^{-1}\left(\dfrac{1}{2}\right)$).

The adjacent side and the hypotenuse are given, so use $\cos\theta$.

Use inverse cosine on a calculator to find the angle and round your answer to the nearest minute. Note: $53°7'48.3''$ rounds to $8'$ since $48.3\ldots > 30$.

Example 4 Working with simple applications

A long, straight mine tunnel is sunk into the ground. Its final depth is 120 m and the end of the tunnel is 100 m horizontally from the ground entrance. Find the angle that the tunnel makes with the horizontal (θ), correct to the nearest minute.

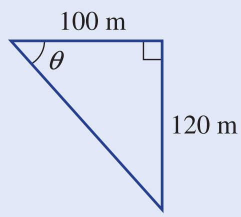
SOLUTION

$\tan\theta = \dfrac{120}{100}$

$\theta = 50.194\ldots$

$= 50°12'$ (nearest minute)

EXPLANATION



Draw a diagram, using the information given.

Use $\tan\theta$ since the opposite and adjacent are known sides

Exercise 7B

UNDERSTANDING AND FLUENCY	1–5	3, 4–6	4–6

1. Use your calculator to find the missing part in each sentence.
   - **a** If $\cos\theta = 0.5$, then $\theta =$ _____.
   - **b** If $\sin\theta = \dfrac{1}{2}$, then $\theta =$ _____.
   - **c** If $\tan\theta = 1$, then $\theta =$ _____.

2. Find θ in the following, rounding your answer to 2 decimal places where necessary.
   - **a** $\sin\theta = 0.4$
   - **b** $\cos\theta = 0.5$
   - **c** $\tan\theta = 0.2$
   - **d** $\sin\theta = 0.1$
   - **e** $\cos\theta = 0.9$
   - **f** $\tan\theta = 1$
   - **g** $\sin\theta = 0.25$
   - **h** $\cos\theta = 0.85$

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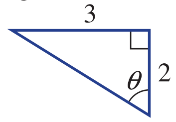
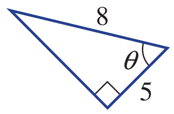
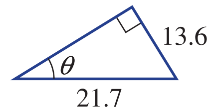
3 Decide which trigonometric ratio (i.e. sine, cosine or tangent) would be used to find θ in these triangles.

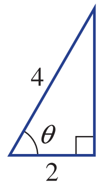
a


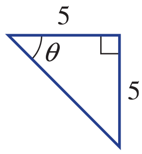
b


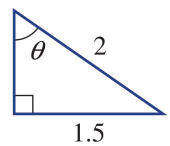
c


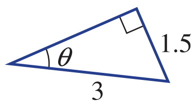
Examples 3

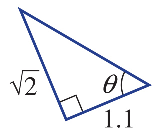
4 Find the value of θ in the following right-angled triangles, rounding your answer to the nearest minute where necessary.

a


b


c


d


e


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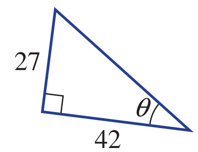
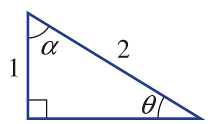
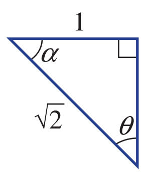


5 Find the value of α and θ, to 1 decimal place where necessary, for these special triangles.

a


b


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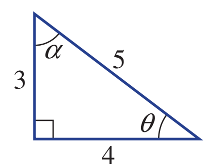

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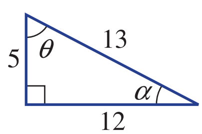

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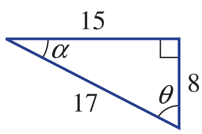
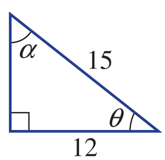


f


6 The lengths of two sides of a right-angled triangle are provided. Use this information to find the size of the two interior acute angles, and round each answer to 1 decimal place.

a hypotenuse 5 cm, opposite 3.5 cm
b hypotenuse 7.2 m, adjacent 1.9 m
c hypotenuse 0.4 mm, adjacent 0.21 mm
d opposite 2.3 km, adjacent 5.2 km
e opposite 0.32 cm, adjacent 0.04 cm
f opposite $\sqrt{5}$ cm, hypotenuse $\sqrt{11}$ cm

PROBLEM-SOLVING AND REASONING	7, 8, 11	8, 9, 11–13	9, 10, 12–14

Examples 4

7 A ladder reaches 5.5 m up a wall and sits 2 m from the base of the wall. Find the angle the ladder makes with the horizontal, correct to 2 decimal places.

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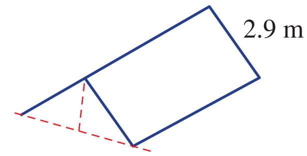
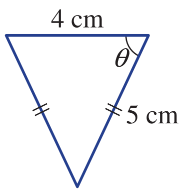
8 A tarpaulin with a simple A-frame design is set up as a shelter. The breadth of half of the tarpaulin is 2.9 metres, as shown. Find the angle to the ground that the sides of the tarpaulin make if the height at the middle of the shelter is 1.5 metres. Round your answer to the nearest 0.1 of a degree.



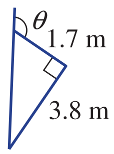
9 A diagonal cut of length 2.85 metres is to be made on a rectangular wooden slab from one corner to the other. The front of the slab measures 1.94 metres. Calculate the angle with the front edge at which the carpenter needs to begin the cut. Round your answer to 1 decimal place.

10 Find the value of θ in these diagrams, correct to 1 decimal place.

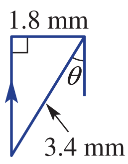
a



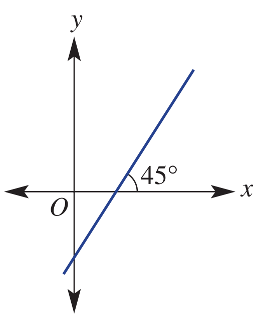
b



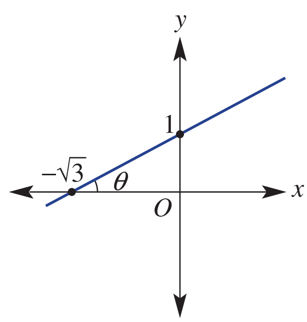
c



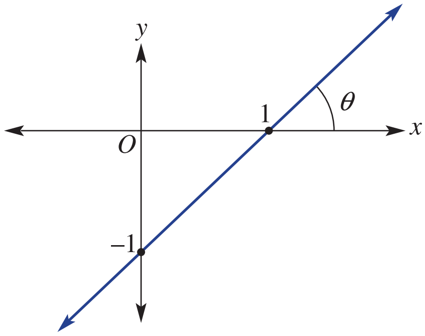
11 a What is the gradient of this line?
Hint: $m = \tan\theta$.



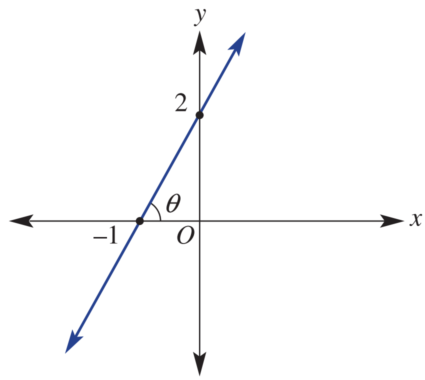
b Find the value of θ, using the formula $m = \tan\theta$.



c The diagram shows the line $y = x - 1$.
What is the angle of inclination, θ?



d The diagram shows the line $y = 2x + 2$.
What is the angle of inclination, θ, to the nearest minute?



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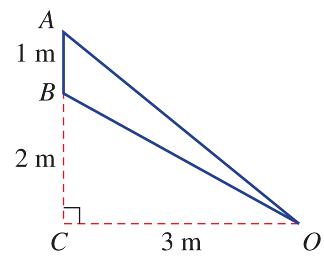
12 Consider $\triangle OAC$ and $\triangle OBC$.

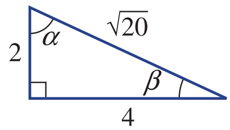


a Find, correct to 1 decimal place where necessary:

i $\angle AOC$ **ii** $\angle BOC$

b Hence, find $\angle AOB$.

13 This triangle includes the unknown angles α and β.



a Explain why only one inverse trigonometric ratio needs to be used to find the values of both α and β.

b Find α and β, correct to 1 decimal place, using your method from part **a**.

14 **a** Draw a right-angled isosceles triangle and show all the internal angles.

b If one of the shorter sides is of length x, show that $\tan 45° = 1$.

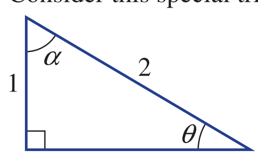
c Find the exact length of the hypotenuse in terms of x.

d Show that $\sin 45° = \cos 45°$.

ENRICHMENT	–	–	15

A special triangle

15 Consider this special triangle.



a Find θ.

b Find α.

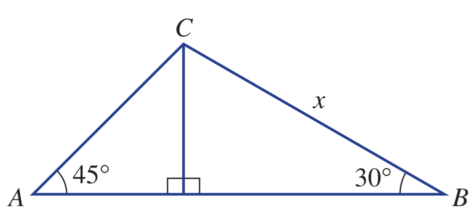
c Use Pythagoras' theorem to find the exact length of the unknown side, in surd form.

d Hence, write down the exact value for the following, in surd form.

i $\sin 30°$ **ii** $\cos 60°$ **iii** $\sin 60°$

iv $\cos 30°$ **v** $\tan 30°$ **vi** $\tan 60°$

e For the diagram below, show that $AB = \left(\dfrac{\sqrt{3}+1}{2}\right)x$.



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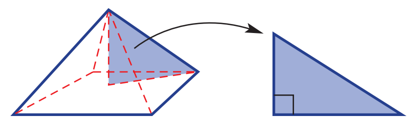
7C Applications in two dimensions

A key problem-solving strategy for many types of problems in mathematics is to draw a diagram. This strategy is particularly important when using trigonometry to solve worded problems that include right-angled triangles.



Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

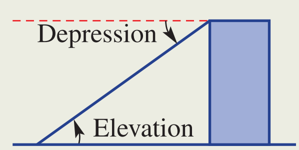
Let's start: Mountain peaks

Two mountain peaks in Victoria are Mt Stirling (1749 m) and Mt Buller (1805 m). A map shows a horizontal distance between them of 6.8 km.

- Discuss if you think there is enough information to find the angle of elevation of Mt Buller from Mt Stirling.
- What diagram could be used to summarise the information?
- Show how trigonometry can be used to find this angle of elevation.
- Discuss what is meant by the words *elevation* and *depression* in this context.

Key ideas

- The **angle of elevation** is measured *up* from the horizontal.
- The **angle of depression** is measured *down* from the horizontal.
  - On the same diagram, the angle of elevation and the angle of depression are equal. They are alternate angles in parallel lines.



- To solve more complex problems involving trigonometry:
  - Visualise and draw a right-angled triangle with the relevant information.
  - Use a trigonometric ratio to find the unknown.
  - Check that your answer is reasonable.
  - Answer the question in words.

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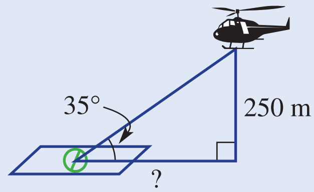
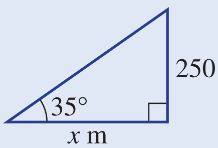
Example 5 Applying trigonometry in word problems

A helicopter is hovering at an altitude of 250 metres, and the angle of elevation from the helipad to the helicopter is 35°. Find the horizontal distance of the helicopter from the helipad, to the nearest centimetre.



SOLUTION

Let x metres be the horizontal distance from the helicopter to the helipad.



$$\begin{aligned} \tan 35^\circ &= \frac{250}{x} \\ \therefore x \times \tan 35^\circ &= 250 \\ x &= \frac{250}{\tan 35^\circ} \\ &= 357.037\ldots \end{aligned}$$

The horizontal distance from the helicopter to the helipad is 357.04 m, to the nearest centimetre.

EXPLANATION

Use $\tan\theta = \frac{\text{O}}{\text{A}}$ since the opposite and adjacent sides are being used. Solve for x.

There are 100 cm in 1 m, so round your answer to 2 decimal places.

Answer the question in words.

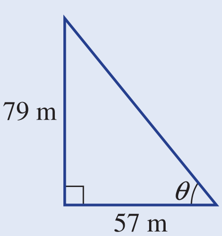
Example 6 Combining trigonometry with problem-solving

Two vertical buildings positioned 57 metres apart are 158 metres and 237 metres tall, respectively. Find the angle of elevation from the top of the shorter building to the top of the taller building, correct to the nearest minute.

SOLUTION

Let θ be the angle of elevation from the top of the shorter building to the top of the taller building.

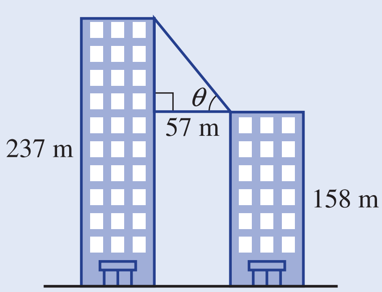
$$\begin{aligned} \text{Height difference} &= 237 - 158 \\ &= 79 \text{ m} \end{aligned}$$



$$\begin{aligned} \tan\theta &= \frac{79}{57} \\ \theta &= 54.188\ldots \end{aligned}$$

The angle of elevation from the top of the shorter building to the top of the taller building is 54°11′ (to the nearest minute).

EXPLANATION



Draw the relevant right-angled triangle separately.

We are given the opposite (O) and the adjacent (A) sides; hence, use tan.

Use the inverse tan function to find θ. Round your answer to the nearest minute.

Answer the question in words.

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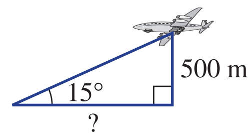
Exercise 7C

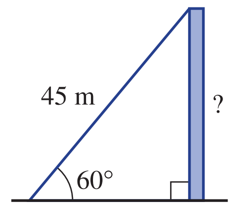
UNDERSTANDING AND FLUENCY	1–7	4–8	6–8

Example 5

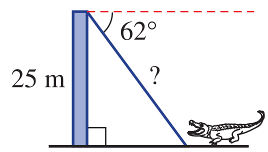
1 The altitude of an aeroplane is 500 metres, and the angle of elevation from the runway to the aeroplane is 15°. Find the horizontal distance from the aeroplane to the runway, to the nearest centimetre.



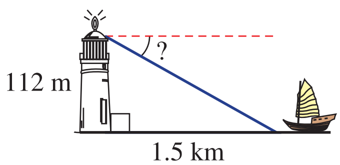
2 A cable of length 45 metres is anchored from the ground to the top of a communications mast. The angle of elevation of the cable to the top of the mast is 60°. Find the height of the communications mast, to the nearest metre.



3 The angle of depression from the top of a 25 metre tall viewing tower to a crocodile on the ground is 62°. Find the direct distance from the top of the tower to the crocodile, to the nearest centimetre.



4 Find the angle of depression from the top of a lighthouse beacon that is 112 m above sea level to a boat that is at a horizontal distance of 1.5 km from the lighthouse. Round your answer to the nearest minute.



5 The distance between two buildings is 24.5 metres. Find the height of the taller building, to the nearest metre, if the angle of elevation from the top of the shorter building to the top of the taller building is 85° and the height of the shorter building is 40 metres.

6 The angle of depression from one mountain summit to another is 15.9°. If the two mountains differ in height by 430 metres, find the horizontal distance between the two summits, to the nearest centimetre.

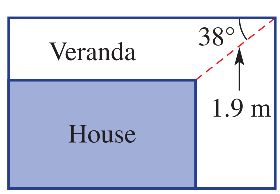
Example 6

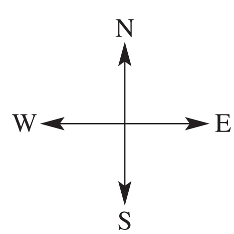
7 Two vertical buildings positioned 91 metres apart are 136 metres and 192 metres tall, respectively. Find the angle of elevation from the top of the shorter building, to the top of the taller building, to the nearest degree.

8 An L-shaped veranda has dimensions as shown. Find the width, to the nearest centimetre, of the veranda for the following sides of the house.

a north side

b east side





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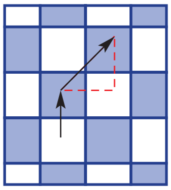
PROBLEM-SOLVING AND REASONING	9, 10, 12	9, 10, 12, 13	10, 11, 14, 15

9 A knight on a chessboard is moved forward 3.6 cm from the centre of one square to another, then diagonally across to the centre of the destination square. How far did the knight move in total? Give your answer to 2 decimal places.



10 Two unidentified flying discs are detected by a receiver. The angle of elevation from the receiver to each disc is 39.48°. The discs are hovering at a direct distance of 826 m and 1.296 km from the receiver. Find the difference in height between the two unidentified flying discs, to the nearest metre.

11 Initially a ship and a submarine are stationary at sea level, positioned 1.78 km apart. The submarine then manoeuvres to position A, 45 metres directly below its starting point. In a second manoeuvre, the submarine dives a further 62 metres to position B. (Give all answers to 2 decimal places.)

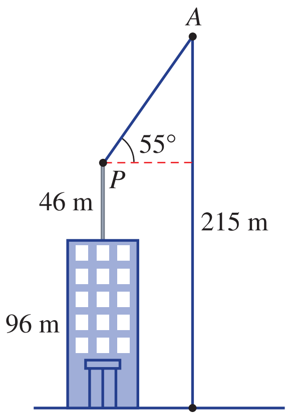
a Find the angle of elevation of the ship from the submarine when the submarine is at position A.

b Find the angle of elevation of the ship from the submarine when the submarine is at position B.

c Find the difference in the angles of elevation from the submarine to the ship when the submarine is at positions A and B.

12 A communications technician claims that when the horizontal distance between two television antennas is less than 12 metres, an interference problem will occur. The heights of two antennas above ground level are 7.5 metres and 13.9 metres, respectively, and the angle of elevation from the top of the shorter antenna to the top of the taller antenna is 29.5°. According to the technician's claim, will there be an interference problem for these two antennas?

13 The pivot point (P) of the main supporting arm (AP) of a construction crane is 46 metres above the top of a 96 metre tall office building. When the supporting arm is at an angle of 55° to the horizontal, the length of cable dropping from the point A to the ground is 215 metres. Find the length of the main supporting arm (AP), to the nearest centimetre.



14 Consider a regular hexagon with internal angles of 120° and side lengths of 10 cm.

a For the given diagram find, to the nearest millimetre, the lengths:

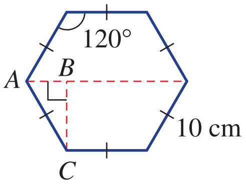
i BC **ii** AB

b Find the distance, to the nearest millimetre, between:

i two parallel sides

ii two opposite vertices

c Explore and describe how changing the side lengths of the hexagon changes the answers to part **b**.



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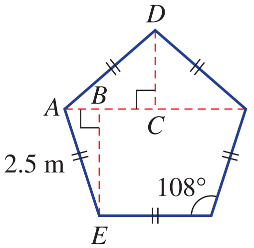
15 An aeroplane is flying horizontally, directly towards the city of Melbourne at an altitude of 400 metres. At a given time the pilot views the city lights of Melbourne at an angle of depression of 1.5°. Two minutes later the angle of depression of the city lights is 5°. Find the speed of the aeroplane in km/h, correct to 1 decimal place.

ENRICHMENT	–	–	16

Vegetable garden design

16 A vegetable garden is to be built in the shape of a regular pentagon using red gum sleepers of length 2.5 metres, as shown. It is known that the internal angles of a regular pentagon are 108°.



a Find the following angles.
- **i** $\angle AEB$
- **ii** $\angle EAB$
- **iii** $\angle CAD$
- **iv** $\angle ADC$

b Find these lengths, to 2 decimal places.
- **i** AB
- **ii** BE
- **iii** AC
- **iv** CD

c Find the distance between a vertex on the border of the vegetable garden and the centre of its opposite side, to 2 decimal places.

d Find the distance between any two non-adjacent vertices on the border of the vegetable garden, to 2 decimal places.

e Show that when the length of the red gum sleepers is x metres, the distance between a vertex and the centre of its opposite side of the vegetable garden will be $1.54x$ metres, using 2 decimal places.

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7D Directions and bearings

Bearings are used to communicate direction, and therefore are important in navigation. Ship and aeroplane pilots, bushwalkers and military personnel all use bearings to navigate and communicate direction.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

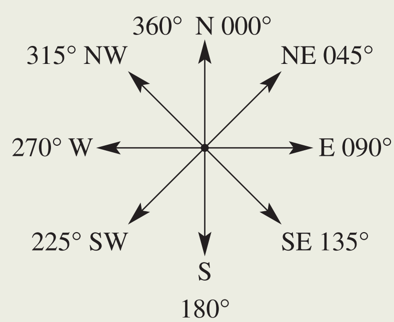
Let's start: Navigating a square

A mining surveyor starts walking from base camp to map out an area for soil testing. She starts by walking 2 km on a bearing of 020° and wishes to map out an area that is approximately square.

- Draw a diagram showing the first leg of the walk and the direction of north.
- If the surveyor turns right for the next leg, what will be the bearing for this section?
- List the direction (as a bearing) and the distance for all four legs of the walk. Remember that the mapped area must be a square.

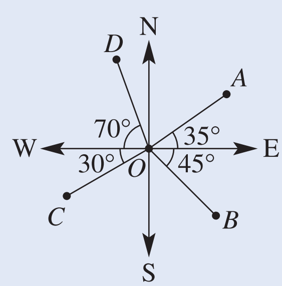
Key ideas

- **Bearings** are measured clockwise from due north. Some angles and directions are shown in this diagram; for example, NE means north-east.
  - Bearings are usually written using three digits.
  - Opposite directions differ by 180°.



Example 7 Stating a direction

Give the bearing for each point from the origin, O, in this diagram.



SOLUTION	EXPLANATION
The bearing of A is $90° - 35° = 055°$.	East is 090°, so subtract 35° from 90°.
The bearing of B is $90° + 45° = 135°$.	B is 90° plus the additional 45° in a clockwise direction.
The bearing of C is $270° - 30° = 240°$.	West is 270°, so subtract 30° from 270°.
The bearing of D is $270° + 70° = 340°$.	Alternatively, subtract 20° from 360°.

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Example 8 Using bearings with trigonometry

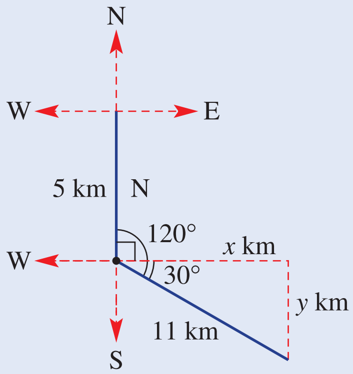
A ship travels south for 5 km, then on a bearing of 120° for 11 km.

a Find how far east the ship is from its starting point, correct to 2 decimal places.

b Find how far south the ship is from its starting point.

SOLUTION

a



Explanation: Draw a clear diagram labelling all relevant angles and lengths. Draw a compass at each change of direction. Clearly show a right-angled triangle, which will help to solve the problem.

$$\cos 30° = \frac{x}{11}$$
$$x = 11 \times \cos 30°$$
$$= 9.53 \text{ (to 2 decimal places)}$$

Explanation: As x is adjacent to 30° and the hypotenuse has length 11 km, use cos.

The ship is 9.53 km east of its initial position.

Explanation: Answer in words.

b

$$\sin 30° = \frac{y}{11}$$

Explanation: Use sine for opposite and hypotenuse.

$$y = 11 \times \sin 30°$$
$$= 5.5$$

Explanation: Multiply both sides by 11.

Distance south = 5.5 + 5 = 10.5 km

Explanation: Find total distance south by adding the initial 5 km.

The ship is 10.5 km south of its initial position.

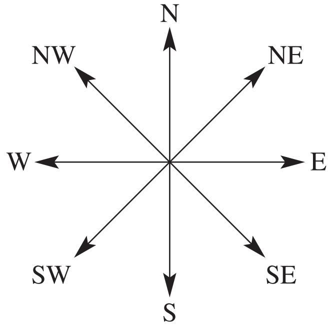
Explanation: Answer in words.

Exercise 7D

UNDERSTANDING AND FLUENCY	1–3(½), 4–7	2(½), 4–8	4, 6, 8, 9

1 Give the bearing for each of these directions.

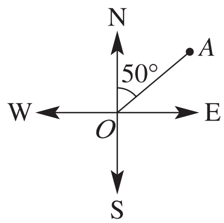
a N	**b** NE
c E	**d** SE
e S	**f** SW
g W	**h** NW



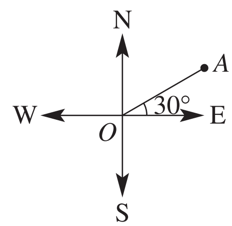
Example 7

2 For each diagram, give the bearing from O to A.

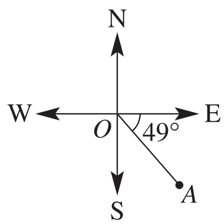
a



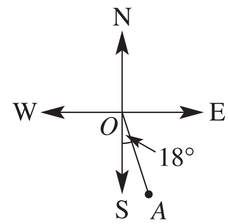
b



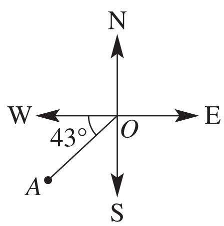
c



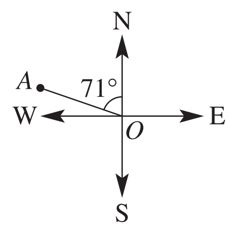
d



e



f



3 Write down the bearing that is the opposite direction to the following.

a 020° **b** 262° **c** 155° **d** 344°

4 **a** Using the internet, find a diagram that shows the sixteen points of a mariner's compass.

b Write down the bearing for these directions.

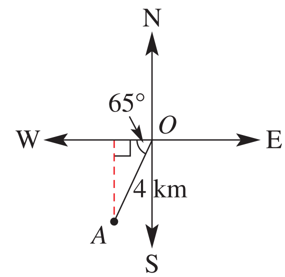
i NNE **ii** NNW **iii** SSE **iv** WSW

5 For this simple map, find the following, correct to 1 decimal place where necessary.

a How far west is point A from O?

b How far south is point A from O?

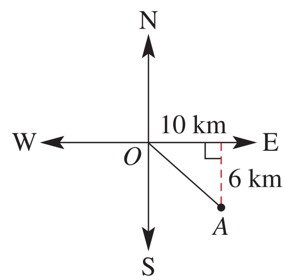
c The bearing of O from A.



6 Find the bearing, correct to the nearest degree, of:

a point A from O

b point O from A



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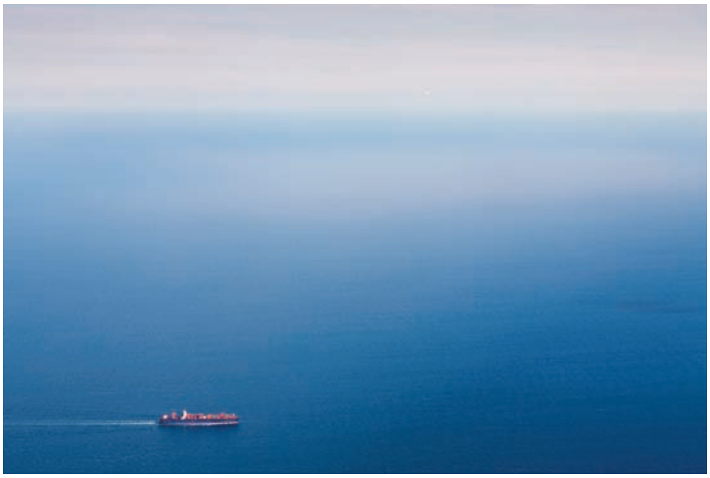
Example 8a

7 A ship travels due south for 3 km, then on a bearing of 130° for 5 km.

a Find how far east the ship is from its starting point, correct to 2 decimal places.

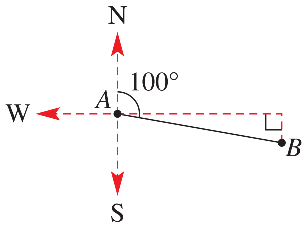
b Find how far south the ship is from its starting point, correct to 2 decimal places.



8 Two points, A and B, positioned 15 cm apart, are such that B is on a bearing of 100° from A.

a Find how far east point B is from A, correct to 2 decimal places.

b Find how far south point B is from A, correct to the nearest millimetre.



9 An aeroplane flies 138 km in a southerly direction from a military air base to a drop-off point. The drop-off point is 83 km west of the air base. Find the bearing, correct to the nearest degree, of:

a the drop-off point from the air base

b the air base from the drop-off point

PROBLEM-SOLVING AND REASONING	10, 11, 14	10–12, 14, 15	12–16

Example 8b

10 From a resting place, a bushwalker hikes due north for 1.5 km to a waterhole and then on a bearing of 315° for 2 km to base camp.

a Find how far west the base camp is from the waterhole, to the nearest metre.

b Find how far north the base camp is from the waterhole, to the nearest metre.

c Find how far north the base camp is from the initial resting place, to the nearest metre.

11 On a map, point C is 4.3 km due east of point B, and point B is 2.7 km on a bearing of 143° from point A. Give your answer to 2 decimal places for the following.

a Find how far east point B is from A.

b Find how far east point C is from A.

c Find how far south point C is from A.

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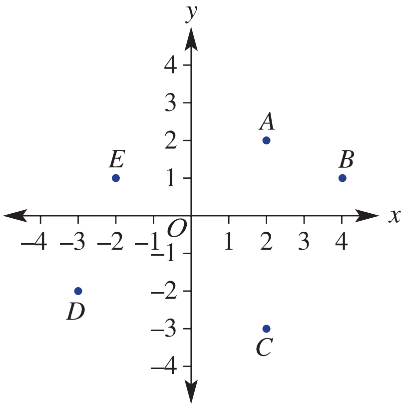
12 A military desert tank manoeuvres 13.5 km from point A on a bearing of 042° to point B. From point B, how far due south must the tank travel to be at a point due east of point A? Give the answer correct to the nearest metre.

13 Consider the points O, A, B, C, D and E on this Cartesian plane. Round the answers to 1 decimal place.



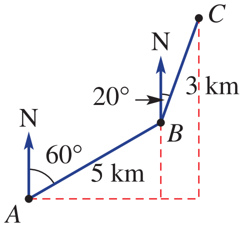
a Find the bearing of:

i A from O **ii** D from O **iii** B from C **iv** E from C

b Find the bearing from:

i O to E **ii** A to B **iii** D to C **iv** B to D

14 The overall direction and distance of a journey can be calculated by considering two (or more) smaller parts (or legs). Find the bearing of C from A and the length AC in this journey by answering these parts.



a Find, correct to 2 decimal places where necessary, how far north:

i point B is from A

ii point C is from B

iii point C is from A

b Find, correct to 2 decimal places, how far east:

i point B is from A

ii point C is from B

iii point C is from A

c Now use your answers above to find the following, correct to 1 decimal place:

i the bearing of C from A

ii the distance from A to C (Hint: Use Pythagoras' theorem.)

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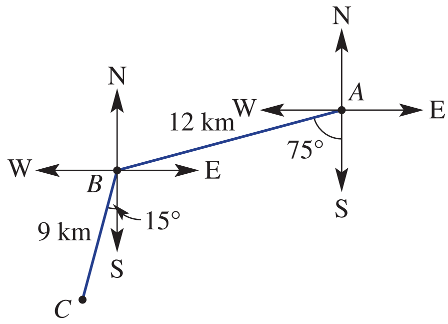
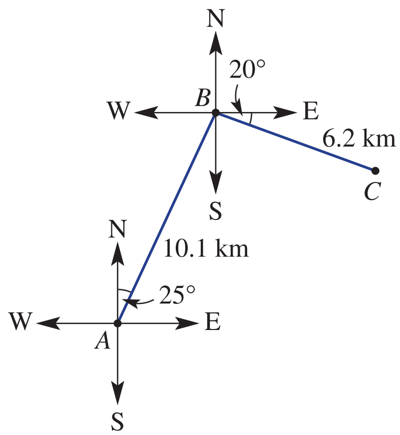
15 Use the technique outlined in Question **14** to find the distance AC and the bearing of C from A in these diagrams. Give your answers correct to 1 decimal place.

a



b



16 Tour groups A and B view a rock feature from different positions on a road heading east–west. Group A views the rock at a distance of 235 m on a bearing of 155° while group B views the rock feature on a bearing of 162° at a different point on the road. Find the following, rounding all answers to 2 decimal places.

a Find how far south the rock feature is from the road.

b Find how far east the rock feature is from:

  i group A

  ii group B

c Find the distance between group A and group B.

ENRICHMENT	–	–	17, 18

Navigation challenges

17 A light aeroplane is flown from a farm airstrip to a city runway 135 km away. The city runway is due north from the farm airstrip. To avoid a storm, the pilot flies the aeroplane on a bearing of 310° for 50 km, and then due north for 45 km. The pilot then heads directly to the city runway. Round your answers to 2 decimal places in the following.

a Find how far west the aeroplane diverged from the direct line between the farm airstrip and the city runway.

b Find how far south the aeroplane was from the city runway before heading directly to the city runway on the final leg of the flight.

c Find the bearing the aeroplane was flying on when it flew on the final leg of the flight.

18 A racing yacht sails from the start position to a floating marker on a bearing of 205.2° for 2.82 km, then to a finishing line on a bearing of 205.9° for 1.99 km. For each of the following, round your answers to 2 decimal places.

a Find how far south the finishing line is from the start position.

b Find how far west the finishing line is from the start position.

c Use Pythagoras' theorem to find the distance between the finishing line and the start position.

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7E Applications in three dimensions

Interactive
Widgets
HOTsheets
Walkthrough

Although a right-angled triangle is a two-dimensional shape, it can also be used to solve problems in three dimensions. Being able to visualise right-angled triangles included in three-dimensional diagrams is an important part of the process of finding angles and lengths associated with three-dimensional objects.

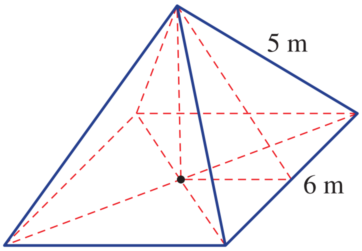
Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: How many right-angled triangles?

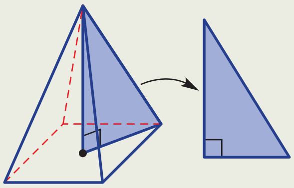
A right square pyramid has the apex above the centre of the base. In this example, the base length is 6 m and slant height is 5 m. Other important lines are dashed.

- Using the given dashed lines and the edges of the pyramid, how many different right-angled triangles can you draw?
- Is it possible to determine the exact side lengths of all your right-angled triangles?
- Is it possible to determine all the angles inside all your right-angled triangles?



Key ideas

- Using trigonometry to solve problems in three dimensions involves:
  - visualising and drawing any relevant two-dimensional triangles
  - using trigonometric ratios to find unknowns
  - relating answers from two-dimensional diagrams to the original three-dimensional object.

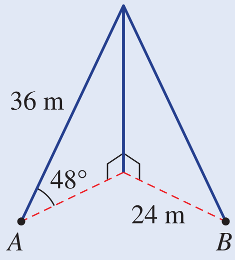


Example 9 Applying trigonometry in 3D

A vertical mast is supported at the top by two cables reaching from two points, A and B. The cable reaching from point A is 36 metres long and is at an angle of 48° to the horizontal. Point B is 24 metres from the base of the mast.

a Find the height of the mast, correct to 3 decimal places.

b Find the angle to the horizontal of the cable reaching from point B, to 2 decimal places.



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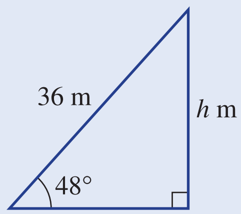
SOLUTION

EXPLANATION

a Let h be the height of the mast, in metres.

First, draw the right-angled triangle, showing the information given.



$\sin 48° = \dfrac{h}{36}$

The opposite (O) and hypotenuse (H) are given, so use sine.

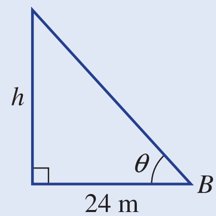
$h = 36 \times \sin 48°$
$= 26.7532\ldots$

Multiply both sides by 36.

The height of the mast is 26.753 m (to 3 decimal places).

Answer the question in words.

b



Draw the second triangle, including the answer from part **a**.

$\tan\theta = \dfrac{26.7532\ldots}{24}$
$\theta = 48.1050\ldots$

Use the inverse tan function to find the angle. Use the more precise answer from part **a** (i.e. 26.7532...) that is stored in your calculator.

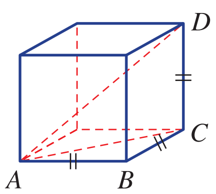
The cable reaching from point B is at an angle of 48.11° to the horizontal (to 2 decimal places).

Answer the question in words, rounding your answer appropriately.

Exercise 7E

UNDERSTANDING AND FLUENCY	1–5	2–6	3–6

1 The cube shown here has a side length of 2 m.



a Draw the right-angled triangle ABC and find and label all the side lengths. Pythagoras' theorem can be used. Answer using exact values (e.g. $\sqrt{5}$).

b Draw the right-angled triangle ACD and find and label all the side lengths. Pythagoras' theorem can be used. Answer using exact values.

c Use trigonometry to find $\angle DAC$, correct to 1 decimal place.

d Find $\angle CAB$.

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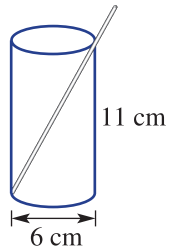
2 Find the angle of elevation that this red drinking straw makes with the base of the can, which has diameter 6 cm and height 11 cm. Round your answer to 1 decimal place.



Example 9

3 A vertical mast is supported at the top by two cables reaching from two points, A and B. The cable reaching from point A is 43 metres long and is at an angle of 61° to the horizontal. Point B is 37 metres from the base of the mast.

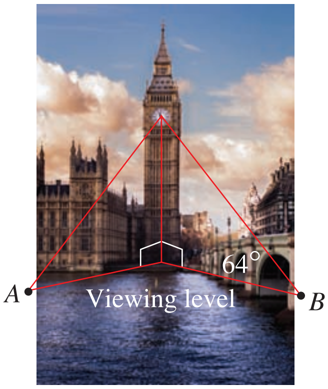
a Find the height of the mast, correct to 3 decimal places.

b Find the angle to the horizontal of the cable reaching from point B, to 2 decimal places.

4 Viewing points A and B are at a horizontal distance of 36 metres and 28 metres, respectively, from a clock tower. The viewing angle to the clockface at point B is 64°.

a Find the height of the clockface above the viewing level, to 3 decimal places.

b Find the viewing angle to the clockface at point A, to 2 decimal places.

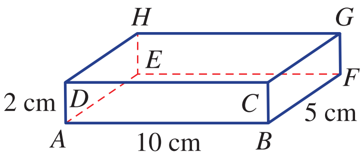


5 A rectangular prism, $ABCDEFGH$, is 5 cm wide, 10 cm long and 2 cm high.

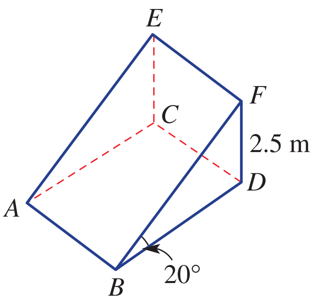
a By drawing the triangle ABF find, to 2 decimal places:

i $\angle BAF$ **ii** AF

b By drawing the triangle AGF, find $\angle GAF$, to 2 decimal places.



6 A ramp, $ABCDEF$, rests at an angle of 20° to the horizontal and the highest point on the ramp is 2.5 metres above the ground, as shown.



Give your answers to 2 decimal places in the following questions.

a Find the length of the ramp BF.

b Find the length of the horizontal BD.

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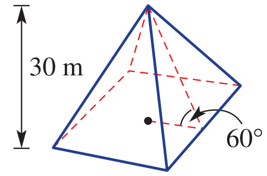
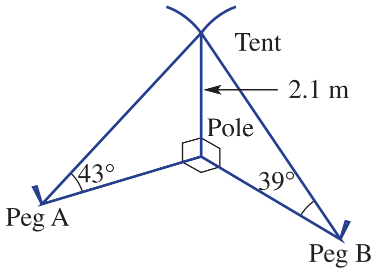
PROBLEM-SOLVING AND REASONING	7, 8, 11	8, 9, 11	9–12

7 The triangular faces of a right square pyramid are at an angle of 60° to the base. The height of the pyramid is 30 m. Find the perimeter of the base of the pyramid, correct to 2 decimal places.



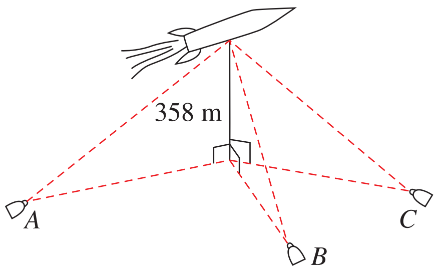
8 A tent pole measuring 2.1 metres tall is secured by ropes in two directions. The ropes are held by pegs A and B at angles of 43° and 39°, respectively, from the horizontal. The line from the base of the pole to peg A is at right angles to the line from the base of the pole to peg B. Round your answers to 2 decimal places in these questions.



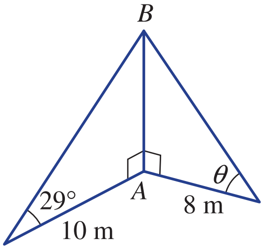
a Find the distance from the base of the tent pole to:
i peg A **ii** peg B
b Find the angle at peg B formed by peg A, peg B and the base of the pole.
c Find the distance between peg A and peg B.

9 The communities of Wood Town and Green Village live in a valley. Communication between the two communities is enhanced by a repeater station on the summit of a nearby mountain. It is known that the angles of depression from the repeater station to Wood Town and Green Village are 44.6° and 58.2°, respectively. Also, the horizontal distances from the repeater to Wood Town and Green Village are 1.35 km and 1.04 km, respectively.
a Find the vertical height, to the nearest metre, between the repeater station and:
i Wood Town **ii** Green Village
b Find the difference in height between the two communities, to the nearest metre.

10 Three cameras operated at ground level view a rocket being launched into space.
At 5 seconds immediately after launch, the rocket is 358 m above ground level and the three cameras, A, B and C, are positioned at an angle of 28°, 32° and 36°, respectively, to the horizontal.
At the 5 second mark, find:
a which camera is closest to the rocket
b the distance between the rocket and the closest camera, to the nearest centimetre



11 It is important to use a high degree of accuracy for calculations that involve multiple parts.
For this 3D diagram, complete these steps.
a Find AB, correct to 1 decimal place.
b Use your answer from part **a** to find θ, correct to 1 decimal place.
c Now recalculate θ using a more accurate value for AB. Round θ to 1 decimal place.
d What is the difference between the answers for parts **b** and **c**?



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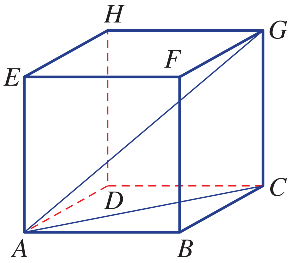
12 For a cube $ABCDEFGH$ with side length 1 unit, as shown, use trigonometry to find the following, correct to 2 decimal places where necessary. Be careful that errors do not accumulate.

a $\angle BAC$
b AC
c $\angle CAG$
d AG

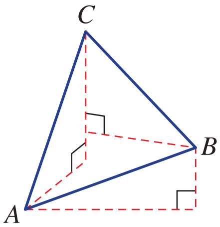


ENRICHMENT	–	–	13–15

Three points in 3D

13 Three points, A, B and C, in three-dimensional space are such that $AB = 6$, $BC = 3$ and $AC = 5$. The angles of elevation from A to B and from B to C are 15° and 25°, respectively. Round your answer to 2 decimal places in the following.

a Find the vertical difference in height between:
   i A and B
   ii B and C
   iii A and C
b Find the angle of elevation from A to C.



14 The points A, B and C in 3D space are such that:

- $AB = 10$ mm, $AC = 17$ mm and $BC = 28$ mm.
- The angle of elevation from A to B is 20°.
- The angle of elevation from A to C is 55°.

Find the angle of elevation from B to C, to the nearest degree.

15 From a point, A, due east of the foot of a tower, O, the angle of elevation is 40°. From another point, B, due south of the tower and 100 m from point A, the angle of elevation is 42°. Let the top of the tower be T, and the height of the tower be h metres.

a Use triangle TOA to show that $\tan 50° = \dfrac{OA}{h}$.
b Find an expression using triangle TOB for the length of OB, in terms of tan.
c Use Pythagoras' theorem and triangle AOB to find the height of the tower, correct to 1 decimal place.

Triangulation points or 'trig stations' such as this are used in geodetic surveying to mark points at which measurements are made to calculate local altitude. The calculations involved are similar to those in the enrichment questions above.

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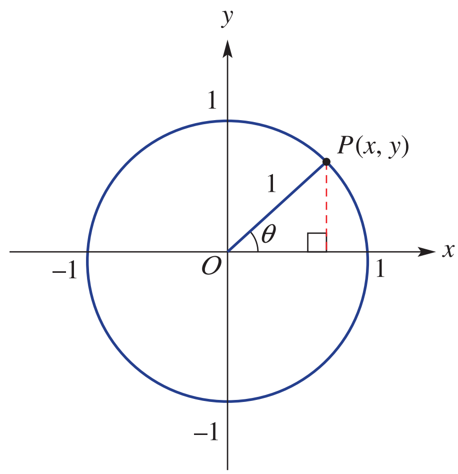
7F Obtuse angles and exact values

Interactive
Widgets
HOTsheets
Walkthrough

Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

To explain how obtuse angles are used in trigonometry, we use a circle with radius 1 unit on a number plane. This is called the unit circle, in which angles are defined anticlockwise from the positive x-axis. The unit circle can be used to consider any angle, but for now we will consider angles between 0° and 180°.



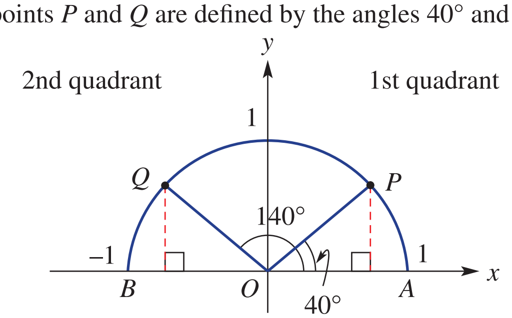
Using a point $P(x, y)$ on the unit circle, we define the three trigonometric ratios:

- $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{y}{1} = y$
- $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{x}{1} = x$
- $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{y}{x} = \dfrac{\sin\theta}{\cos\theta}$

So $\sin\theta$ is the y-coordinate of point P and $\cos\theta$ is the x-coordinate of point P. The ratio $\tan\theta$ is the y-coordinate divided by the x-coordinate, which leads to the identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. This is true for any point on the unit circle defined by any angle θ.

Let's start: The first and second quadrants

On the unit circle shown, the points P and Q are defined by the angles 40° and 140°.



- If $\cos\theta$ is the x-coordinate of a point on the unit circle, find, correct to 2 decimal places:
  - the x-coordinate of P
  - the x-coordinate of Q
- What do you notice about the x-coordinates of P and Q? Discuss.
- If $\sin\theta$ is the y-coordinate of a point on the unit circle, find, correct to 2 decimal places:
  - the y-coordinate of P
  - the y-coordinate of Q
- What do you notice about the y-coordinates of P and Q? Discuss.
- Use $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ to compare the values of tan 40° and tan 140°. What do you notice?

v

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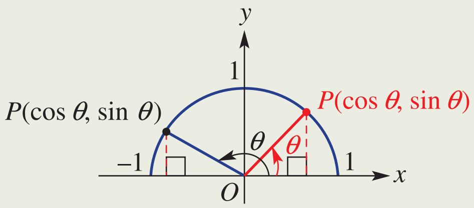
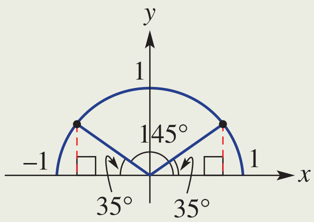
Key ideas

- The **unit circle** is a circle with radius 1 unit.
- The unit circle is used to define $\cos\theta$, $\sin\theta$ and $\tan\theta$ for all angles θ.
  - A point $P(x, y)$ is a point on the unit circle that is defined by an angle θ, which is measured anticlockwise from the positive x-axis.
  - **$\cos\theta$ is the x-coordinate of P.**
  - **$\sin\theta$ is the y-coordinate of P.**
  - **$\tan\theta = \dfrac{y}{x}$**

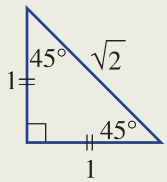


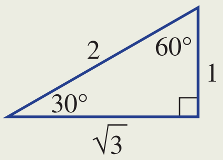
- For supplementary angles θ and $180° - \theta$:
  - $\cos(180° - \theta) = -\cos\theta$
  - $\sin(180° - \theta) = \sin\theta$
  - $\tan(180° - \theta) = -\tan\theta$

  For example:
  - $\cos 145° = -\cos 35°$
  - $\sin 145° = \sin 35°$
  - $\tan 145° = -\tan 35°$



- For acute angles ($0° < \theta < 90°$) all three trigonometric ratios are positive.
- For obtuse angles ($90° < \theta < 180°$), $\sin\theta$, is positive, $\cos\theta$ is negative and $\tan\theta$ is negative.
- Exact values for $\sin\theta$, $\cos\theta$ and $\tan\theta$ can be obtained using two special triangles. Pythagoras' theorem can be used to confirm the length of each side.





$\sin 45° = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ $\quad$ $\sin 30° = \dfrac{1}{2}$ $\quad$ $\sin 60° = \dfrac{\sqrt{3}}{2}$

$\cos 45° = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ $\quad$ $\cos 30° = \dfrac{\sqrt{3}}{2}$ $\quad$ $\cos 60° = \dfrac{1}{2}$

$\tan 45° = \dfrac{1}{1} = 1$ $\quad$ $\tan 30° = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$ $\quad$ $\tan 60° = \dfrac{\sqrt{3}}{1} = \sqrt{3}$

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Key ideas

- Exact values for $\sin\theta$, $\cos\theta$ and $\tan\theta$ for angles of 30°, 45°, 60° and 90° are given in this table.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	undefined

Example 10 Choosing supplementary angles

Choose an obtuse angle to complete each statement.

a $\sin 30° = \sin$ _____

b $\cos 57° = -\cos$ _____

c $\tan 81° = -\tan$ _____

SOLUTION

a $\sin 30° = \sin 150°$

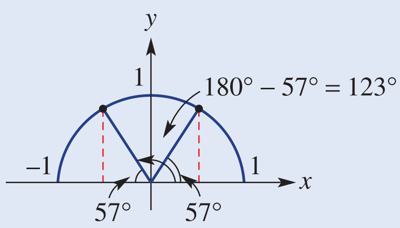
EXPLANATION

Choose the supplement of 30°, which is $180° - 30° = 150°$.

SOLUTION

b $\cos 57° = -\cos 123°$

EXPLANATION



SOLUTION

c $\tan 81° = -\tan 99°$

EXPLANATION

The supplement of 81° is 99°.

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Example 11 Using exact values

Find the exact value of each of the following.

a $\cos 60°$

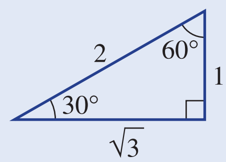
b $\sin 150°$

c $\tan 135°$

SOLUTION

a $\cos 60° = \frac{1}{2}$

EXPLANATION

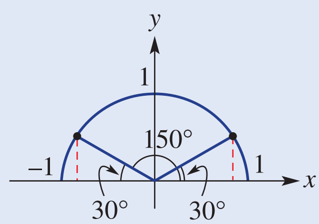


$\cos 60° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{2}$

SOLUTION

b $\sin 150° = \sin 30°$
$= \frac{1}{2}$

EXPLANATION



The sine of supplementary angles are equal and the exact value of $\sin 30°$ is $\frac{1}{2}$.

SOLUTION

c $\tan 135° = -\tan 45°$
$= -1$

EXPLANATION

45° and 135° are supplementary angles.
Also, $\tan(180° - \theta) = -\tan\theta$ and $\tan 45° = 1$.

Exercise 7F

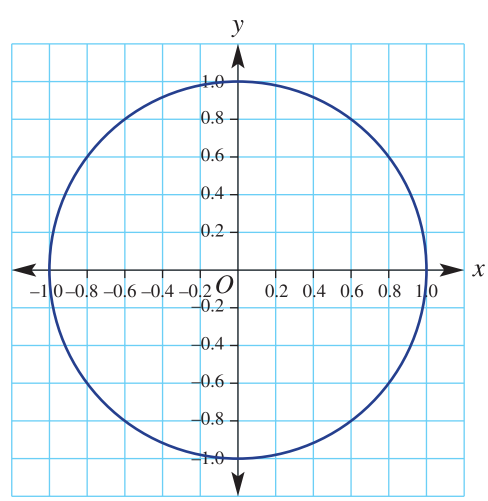
UNDERSTANDING AND FLUENCY	1–7	4, 5–8(½)	5–8(½)

1 **a** Use a pair of compasses to draw a unit circle on graph paper and label the axes in increments of 0.2.

b Use a protractor to draw an angle of 135° and then use the axes scales to estimate the values of sin 135° and cos 135°.

c Use a protractor to draw an angle of 160° and then use the axes scales to estimate the values of sin 160° and cos 160°.

d Check your answers to parts **b** and **c** with a calculator.



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2 What is the supplement of each of the following angles?

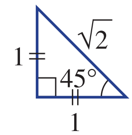
a 31° b 52° c 45° d 87°
e 139° f 124° g 151° h 111°

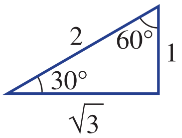
3 Use a calculator to evaluate each pair, correct to 2 decimal places.

a $\sin 20^\circ$, $\sin 160^\circ$ b $\sin 80^\circ$, $\sin 100^\circ$ c $\sin 39^\circ$, $\sin 141^\circ$
d $\cos 40^\circ$, $\cos 140^\circ$ e $\cos 25^\circ$, $\cos 155^\circ$ f $\cos 65^\circ$, $\cos 115^\circ$
g $\tan 50^\circ$, $\tan 130^\circ$ h $\tan 70^\circ$, $\tan 110^\circ$ i $\tan 12^\circ$, $\tan 168^\circ$

4 Use trigonometric ratios with these triangles to write down an exact value for:

a $\sin 45^\circ$ b $\cos 45^\circ$ c $\tan 45^\circ$
d $\cos 30^\circ$ e $\sin 30^\circ$ f $\tan 30^\circ$
g $\tan 60^\circ$ h $\cos 60^\circ$ i $\sin 60^\circ$





Example 10

5 Choose an obtuse angle to complete each statement.

a $\sin 40^\circ = \sin$ _____
b $\sin 20^\circ = \sin$ _____
c $\sin 65^\circ = \sin$ _____
d $\cos 25^\circ = -\cos$ _____
e $\cos 42^\circ = -\cos$ _____
f $\cos 81^\circ = -\cos$ _____
g $\tan 37^\circ = -\tan$ _____
h $\tan 56^\circ = -\tan$ _____
i $\tan 8^\circ = -\tan$ _____

6 Choose an acute angle to complete each statement.

a $\sin 150^\circ = \sin$ _____
b $\sin 125^\circ = \sin$ _____
c $\sin 94^\circ = \sin$ _____
d $-\cos 110^\circ = \cos$ _____
e $-\cos 135^\circ = \cos$ _____
f $-\cos 171^\circ = \cos$ _____
g $-\tan 159^\circ = \tan$ _____
h $-\tan 102^\circ = \tan$ _____
i $-\tan 143^\circ = \tan$ _____

7 Decide whether the following will be positive or negative.

a $\sin 153^\circ$ b $\tan 37^\circ$ c $\cos 84^\circ$ d $\cos 171^\circ$
e $\tan 136^\circ$ f $\sin 18^\circ$ g $\tan 91^\circ$ h $\cos 124^\circ$

Example 11

8 Find an exact value for each.

a $\cos 30^\circ$ b $\sin 45^\circ$ c $\tan 60^\circ$ d $\cos 45^\circ$
e $\cos 150^\circ$ f $\tan 120^\circ$ g $\sin 135^\circ$ h $\cos 135^\circ$
i $\sin 120^\circ$ j $\tan 150^\circ$ k $\cos 120^\circ$ l $\sin 150^\circ$
m $\tan 135^\circ$ n $\sin 90^\circ$ o $\cos 90^\circ$ p $\tan 90^\circ$

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PROBLEM-SOLVING AND REASONING	9, 10, 13	10–14	10–12, 15, 16

9 State the two values of θ if $0° < \theta < 180°$.

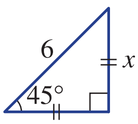
a $\sin\theta = \frac{1}{2}$ **b** $\sin\theta = \frac{\sqrt{2}}{2}$ **c** $\sin\theta = \frac{\sqrt{3}}{2}$

10 Find θ if $90° < \theta < 180°$.

a $\cos\theta = -\frac{1}{2}$ **b** $\cos\theta = -\frac{\sqrt{2}}{2}$ **c** $\cos\theta = -\frac{\sqrt{3}}{2}$

d $\tan\theta = -\sqrt{3}$ **e** $\tan\theta = -1$ **f** $\tan\theta = -\frac{\sqrt{3}}{3}$

11 Use trigonometric ratios to find the exact value of x. Calculators are not required.

a


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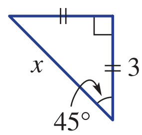

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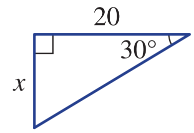

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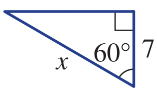

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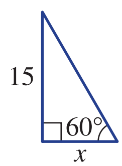

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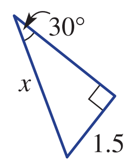
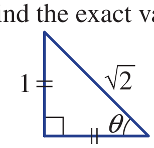


12 Find the exact value of θ without the use of a calculator.

a


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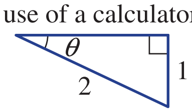

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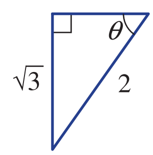
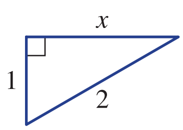


13 Use Pythagoras' theorem to find the exact value of x in these special triangles.

a


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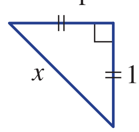
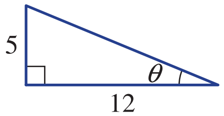


14 This right-angled triangle has its two shorter sides of length 5 and 12.



a Use Pythagoras' theorem to find the length of the hypotenuse.

b Find:

i $\sin\theta$

ii $\cos\theta$

iii $\tan\theta$

c Use $\tan\theta = \frac{\sin\theta}{\cos\theta}$ to verify your result from part **b iii**.

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15 If $0° < \theta < 90°$, find $\tan\theta$ when:

a $\sin\theta = \dfrac{1}{\sqrt{10}}$ and $\cos\theta = \dfrac{3}{\sqrt{10}}$

b $\sin\theta = \dfrac{2}{\sqrt{11}}$ and $\cos\theta = \dfrac{\sqrt{7}}{\sqrt{11}}$

c $\sin\theta = \dfrac{2}{5}$ and $\cos\theta = \dfrac{\sqrt{21}}{5}$

16 If $90° < \theta < 180°$, find $\tan\theta$ when:

a $\sin\theta = \dfrac{5}{\sqrt{34}}$ and $\cos\theta = -\dfrac{3}{\sqrt{34}}$

b $\sin\theta = \dfrac{\sqrt{20}}{6}$ and $\cos\theta = -\dfrac{2}{3}$

c $\sin\theta = \dfrac{1}{5\sqrt{2}}$ and $\cos\theta = -\dfrac{7}{5\sqrt{2}}$

ENRICHMENT	–	–	17

Complementary ratios

17 You will recall that complementary angles sum to 90°. Answer these questions to explore the relationship between sine and cosine ratios of complementary angles.

a Evaluate the following, correct to 2 decimal places.

i $\sin 10°$
ii $\cos 80°$
iii $\sin 36°$
iv $\cos 54°$
v $\cos 7°$
vi $\sin 83°$
vii $\cos 68°$
viii $\sin 22°$

b Describe what you notice from part **a**.

c Complete the following.

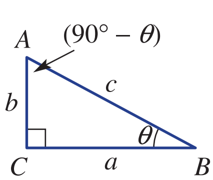
i $\cos\theta = \sin$ (____)

ii $\sin\theta = \cos$ (____)

d State the value of θ if θ is acute.

i $\sin 20° = \cos\theta$
ii $\sin 85° = \cos\theta$
iii $\cos 71° = \sin\theta$
iv $\cos 52° = \sin\theta$

e For this triangle, $\angle B = \theta$.



i Write $\angle A$ in terms of θ.

ii Write a ratio for $\sin\theta$ in terms of b and c.

iii Write a ratio for $\cos(90° - \theta)$ in terms of b and c.

f If $\cos(90° - \theta) = \dfrac{2}{3}$, find $\tan\theta$.

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7G The sine rule

Interactive
Widgets
HOTsheets
Walkthrough

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

The use of sine, cosine and tangent functions can be extended to non right-angled triangles. First, consider this triangle with sides a, b and c and with opposite angles $\angle A$, $\angle B$ and $\angle C$. Height h is also shown.

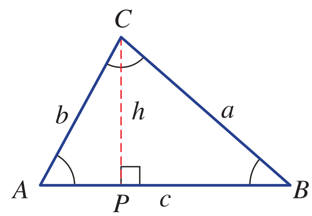
From $\triangle CPB$, $\dfrac{h}{a} = \sin B$

so $\quad h = a \sin B$

From $\triangle CPA$, $\dfrac{h}{b} = \sin A$

so $\quad h = b \sin A$

$\therefore\ a \sin B = b \sin A$ or $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$



Similarly, it can be shown that $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$ and $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

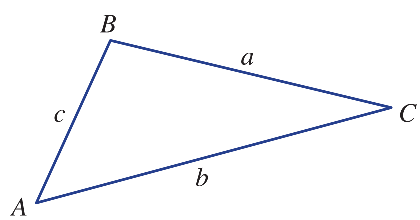
Let's start: Explore the sine rule

Use a ruler and a protractor or dynamic geometry software to measure the side lengths (a, b and c) in centimetres, correct to 1 decimal place, and the angles (A, B and C), correct to the nearest degree, for this triangle.

- Calculate the following.

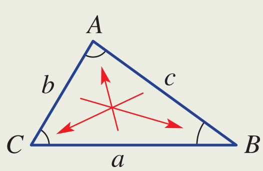
  a $\dfrac{a}{\sin A}$ **b** $\dfrac{b}{\sin B}$ **c** $\dfrac{c}{\sin C}$

- What do you notice about the three answers above?
- Draw your own triangle and check to see if your observations are consistent for any triangle.



Key ideas

- When using the sine rule, label triangles with capital letters for vertices and the corresponding lower-case letter for the side opposite the angle.



- The **sine rule** states that the ratios of each side of a triangle to the sine of the opposite angle are equal.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

  - The sine rule holds true for both acute- and obtuse-angled triangles.

- The sine rule is useful when you know:
  - the size of one angle
  - the length of the side opposite that angle, and
  - another side or angle.

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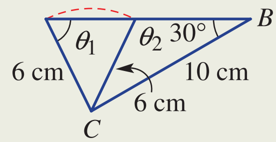
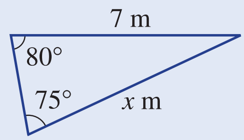
Key ideas

- The **ambiguous case** may arise when we are given two sides and an angle that is not the included angle.
  - This example shows a diagram with two given side lengths and one angle. Two different triangles contain this information.
  - Using $\dfrac{6}{\sin 30^\circ} = \dfrac{10}{\sin \theta}$ could give two results for θ (i.e. θ_1 or θ_2).
    Sometimes there are two valid angles (i.e. one acute and one obtuse). The obtuse angle is the supplement of the acute angle. Sometimes the obtuse angle is not a valid solution.



Example 12 Finding a side length using the sine rule

Find the value of x in this triangle, correct to 1 decimal place.



SOLUTION

$$\frac{x}{\sin 80^\circ} = \frac{7}{\sin 75^\circ}$$

$$x = \frac{7}{\sin 75^\circ} \times \sin 80^\circ$$

$$= 7.1368\ldots$$

$$= 7.1 \text{ (to 1 decimal place)}$$

EXPLANATION

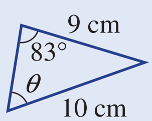
Use the sine rule $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$.

Multiply both sides by $\sin 80^\circ$.

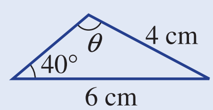
Example 13 Finding angles using the sine rule

Find the value(s) of θ in these triangles, correct to 1 decimal place.

a θ is acute



b θ is obtuse



SOLUTION

a
$$\frac{10}{\sin 83^\circ} = \frac{9}{\sin \theta}$$

$$10 \times \sin \theta = 9 \times \sin 83^\circ$$

$$\sin \theta = \frac{9 \times \sin 83^\circ}{10}$$

$$\theta = 63.289\ldots$$

$$= 63.3^\circ \text{ (to 1 decimal place)}$$

b
$$\frac{4}{\sin 40^\circ} = \frac{6}{\sin \theta}$$

$$4 \times \sin \theta = 6 \times \sin 40^\circ$$

$$\sin \theta = \frac{6 \times \sin 40^\circ}{4}$$

$\theta = 74.6^\circ$ or $180^\circ - 74.6^\circ = 105.4^\circ$

θ is obtuse, so $\theta = 105.4^\circ$.

EXPLANATION

a Substitute known values into the sine rule and rearrange to make $\sin \theta$ the subject.

Use $\sin^{-1}$ on your calculator to find the value of θ.
Note: The obtuse angle would be $180^\circ - 63.3^\circ = 116.7^\circ$ but this is not possible.

b Solve for θ using the sine rule.

The obtuse solution is the supplement of the acute solution.

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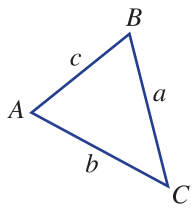
Exercise 7G

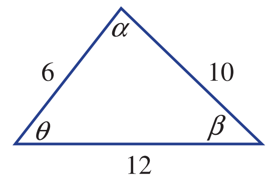
UNDERSTANDING AND FLUENCY	1–3, 4–5(½)	3, 4–5(½)	4–5(½)

1 Write the sine rule for each triangle

a



b



2 Solve each equation for a, b or c, correct to 1 decimal place.

a $\dfrac{a}{\sin 47^\circ} = \dfrac{2}{\sin 51^\circ}$ **b** $\dfrac{b}{\sin 31^\circ} = \dfrac{7}{\sin 84^\circ}$ **c** $\dfrac{5}{\sin 63^\circ} = \dfrac{c}{\sin 27^\circ}$

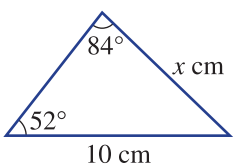
3 Find θ, correct to 1 decimal place, if θ is acute.

a $\dfrac{4}{\sin 38^\circ} = \dfrac{5}{\sin \theta}$ **b** $\dfrac{11}{\sin 51^\circ} = \dfrac{9}{\sin \theta}$ **c** $\dfrac{1.2}{\sin \theta} = \dfrac{1.8}{\sin 47^\circ}$

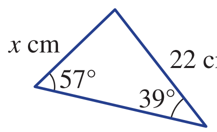
Example 12

4 Find the value of x in these triangles, correct to 1 decimal place.

a



b



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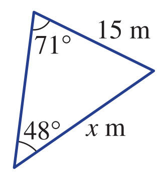

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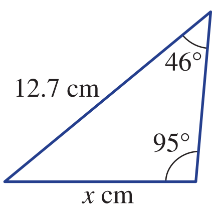

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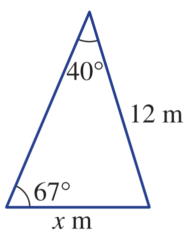

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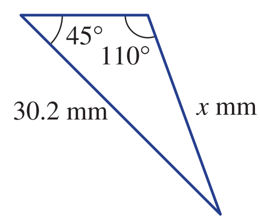
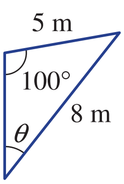


Example 13a

5 Find the value of θ, correct to 1 decimal place, if θ is acute.

a



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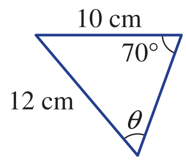

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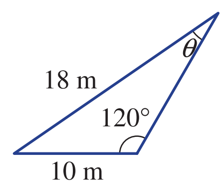

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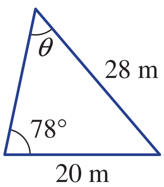

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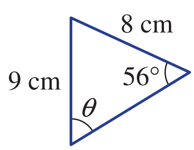

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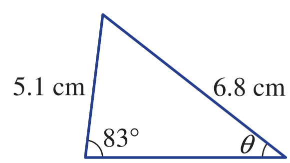

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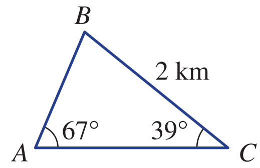
PROBLEM-SOLVING AND REASONING	6, 7, 11	7–9, 11, 12	8–10, 12–14

6 Three markers, A, B and C, map out the course for a cross-country race. The angles at A and C are 67° and 39°, respectively, and BC is 2 km.

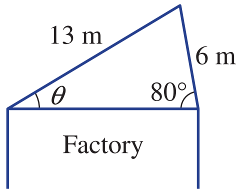
a Find the length AB, correct to 3 decimal places.

b Find the angle at B.

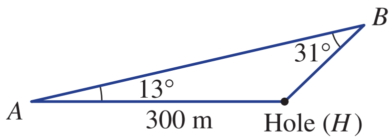
c Find the length AC, correct to 3 decimal places.



7 A factory roof has a steep 6 m section at 80° to the horizontal and another 13 m section. What is the angle of elevation of the 13 m section of roof? Give your answer to 1 decimal place.



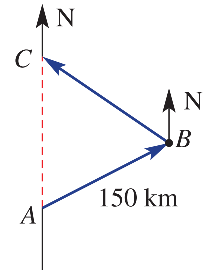
8 A golf ball is hit off-course by 13° to point B. The shortest distance to the hole is 300 m and the angle formed by the new ball position is 31°, as shown. Find the new distance to the hole (BH), correct to 1 decimal place.



9 An aeroplane is flying due north but, to avoid a storm, it flies 150 km on a bearing of 060° and then on a bearing of 320° until it reaches its original course.

a Find the angles $\angle ABC$ and $\angle ACB$.

b As a result of the diversion, how much farther did the aeroplane have to fly? Round your answer to the nearest km.

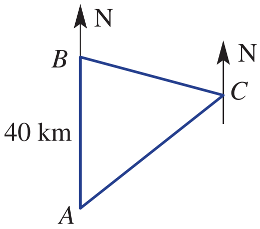


10 A ship heads due north from point A for 40 km to point B, and then heads on a bearing of 100° to point C. The bearing from C to A is 240°.

a Find $\angle ABC$.

b Find the distance from A to C, correct to 1 decimal place.

c Find the distance from B to C, correct to 1 decimal place.



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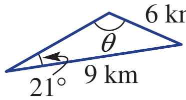
Example 13b

11 Find the value of θ, correct to 1 decimal place, if θ is obtuse.

a



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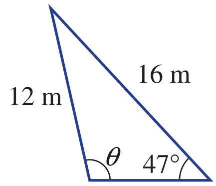

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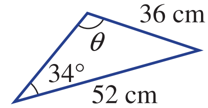

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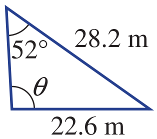

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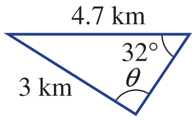

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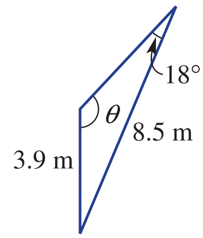
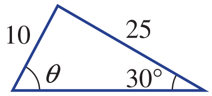
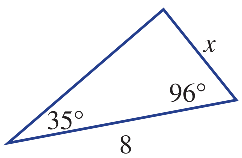


12 Try to find the angle θ in this triangle. What do you notice? Can you explain this result?

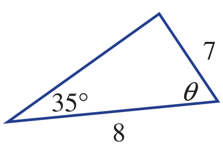


13 A triangle ABC has $\angle C = 25°$, $AC = 13$ cm and $AB = 9$ cm. Find all possible values of $\angle B$, correct to 1 decimal place.

14 **a** Find the value of x. Hint: There are 180° in a triangle.



b Find the value of θ. Hint: Find the other angle.



ENRICHMENT	–	–	15

More on the ambiguous case

15 When finding a missing angle θ in a triangle, the number of possible solutions for θ can be one or two, depending on the given information.

Two solutions: A triangle ABC has $AB = 3$ cm, $AC = 2$ cm and $\angle B = 35°$.

a Find the possible values of $\angle C$, correct to 1 decimal place.

b Draw a triangle for each angle for $\angle C$ in part **a**.

One solution: A triangle ABC has $AB = 6$ m, $AC = 10$ m and $\angle B = 120°$.

c Find the possible values of $\angle C$, correct to 1 decimal place.

d Explain why there is only one solution for $\angle C$ and not the extra supplementary angle, as in parts **a** and **b** above.

e Draw a triangle for your solution to part **c**.

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7H The cosine rule

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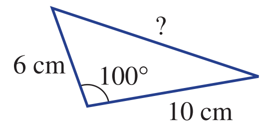
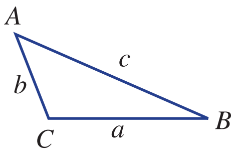
HOTsheets
Walkthrough

When a triangle is defined by two sides and the included angle, the sine rule is unhelpful in finding the length of the third side because at least one of the other two angles is needed.



In such situations, a new rule called the cosine rule can be used. It relates all three side lengths and the cosine of one angle. This means that the cosine rule can also be used to find an angle inside a triangle when given all three sides.



The proof of the cosine rule will be considered in the Enrichment question of this section.

Stage

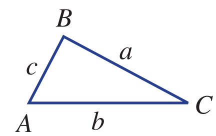
- 5.3#
- 5.3
- 5.3§
- 5.2
- 5.2◊
- 5.1
- 4

Let's start: Cosine rule in three ways

One way to write the cosine rule is like this:

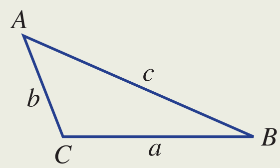
$c^2 = a^2 + b^2 - 2ab\cos C$, where c^2 is the subject of the rule.

- Rewrite the cosine rule by replacing c with a, a with c and C with A.
- Rewrite the cosine rule by replacing c with b, b with c and C with B.



Key ideas

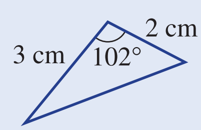
- The **cosine rule** relates one angle and three sides of any triangle.
- The cosine rule is used to find:
  - the third side of a triangle when given two sides and the included angle $\quad a^2 = b^2 + c^2 - 2bc\cos A$
  - an angle when given three sides $\quad \cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$



- When checking that your answer is reasonable, keep in mind that:
  - the longest side is opposite the largest angle
  - the shortest side is opposite the smallest angle

Example 14 Finding a side length using the cosine rule

Find the length of the third side in this triangle, correct to 2 decimal places.



SOLUTION

$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 3^2 + 2^2 - 2(3)(2)\cos 102°$

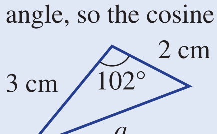
$ = 15.49494\ldots$

$\therefore a = 3.9363\ldots$

The length of the third side is 3.94 cm (to 2 decimal places).

EXPLANATION

We have been given two sides and the included angle, so the cosine rule applies.



a is the length of the unknown side.

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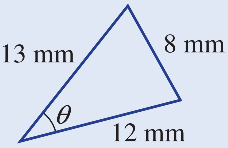
Example 15 Finding an angle using the cosine rule

Find the angle θ in this triangle, correct to the nearest minute.



SOLUTION

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

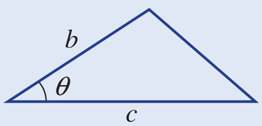
$$\cos \theta = \frac{13^2 + 12^2 - 8^2}{2 \times 13 \times 12}$$

$$\cos \theta = \frac{249}{312}$$

$$\theta = 37.053\ldots$$

$$\theta = 37°3' \text{ (to the nearest minute)}$$

EXPLANATION



b and c are the two sides nearest to the angle θ.

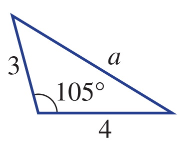
Use $\cos^{-1}$ to solve for θ.

Exercise 7H

UNDERSTANDING AND FLUENCY	1, 2, 3–4(½)	2–4(½)	3–4(½)

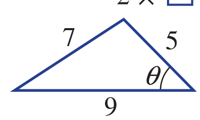
1 Copy and complete the cosine rule for each triangle below.

a $a^2 = b^2 + c^2 - 2bc \cos A$
$a^2 = 3^2 + \_\_\_ - 2 \times \_\_ \times \_\_ \times \cos \_\_$



b $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$$\cos \theta = \frac{\square + \square - \square}{2 \times \square \times \square}$$



2 Simplify and solve for the unknown (a or θ) in these equations, correct to 1 decimal place.

a $a^2 = 4^2 + 7^2 - 2 \times 4 \times 7 \times \cos 120°$

b $a^2 = 1.5^2 + 1.1^2 - 2 \times 1.5 \times 1.1 \times \cos 70°$

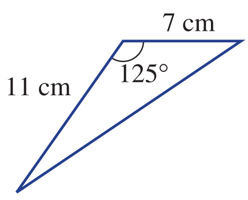
c $\cos \theta = \dfrac{7^2 + 6^2 - 9^2}{2 \times 7 \times 6}$

d $\cos \theta = \dfrac{21^2 + 30^2 - 18^2}{2 \times 21 \times 30}$

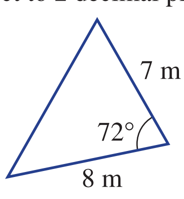
Example 14

3 Find the length of the third side, correct to 2 decimal places.

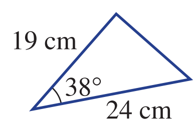
a



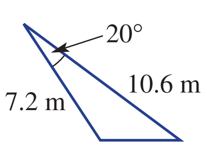
b



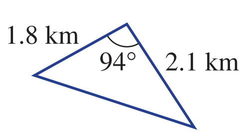
c



d



e



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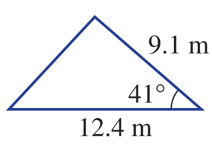

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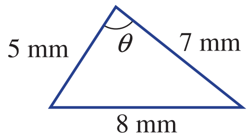
Example 15

4 Find the angle θ, correct to 2 decimal places.

a



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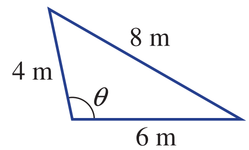

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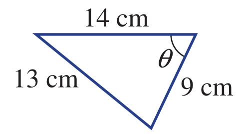

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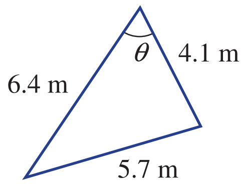
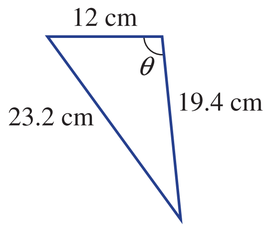


e



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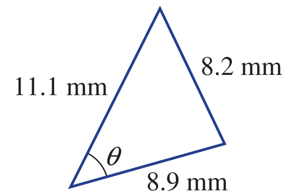
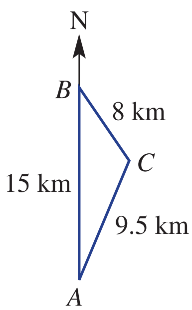


PROBLEM-SOLVING AND REASONING	5–7, 10(½)	6–8, 10(½), 11	7–9, 10(½), 11, 12

5 A triangular goat paddock has two sides of lengths 320 m and 170 m, and a 71° angle between them. Find the length of the third side, correct to the nearest metre.

6 Find the size of all three angles in a triangle that has side lengths 10 m, 7 m and 13 m. Round each angle to 1 decimal place.

7 Three camp sites, A, B and C, are planned for a hike and the distances between the camp sites are 8 km, 15 km and 9.5 km, as shown. If campsite B is due north of camp site A, find the following, correct to 1 decimal place.

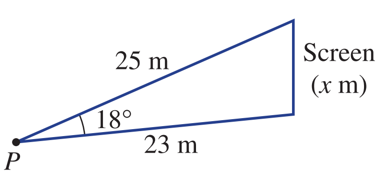


a the bearing from campsite B to campsite C

b the bearing from campsite C to campsite A

8 A helicopter on a joy flight over Kakadu National Park travels due east for 125 km, then on a bearing of 215° for 137 km before returning to its starting point. Find the total length of the journey, correct to the nearest kilometre.

9 The viewing angle to a vertical screen is 18° and the distances between the viewing point P and the top and bottom of the screen are 25 m and 23 m, respectively. Find the height of the screen (x m), correct to the nearest centimetre.



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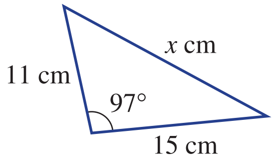
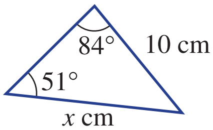
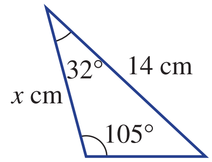
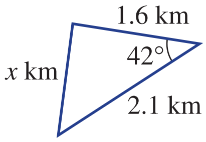
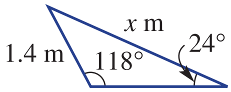
10 Decide whether the cosine rule or sine rule would be used to calculate the value of x in these triangles. Then find the value of x, correct to 1 decimal place.

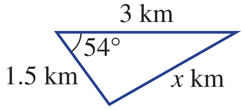
a 

b 

c 

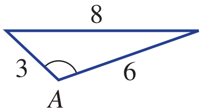
d 

e 

f 

11 A student uses the cosine rule to find an angle in a triangle and simplifies the equation to $\cos\theta = -0.17$. Is the triangle acute or obtuse? Give a reason.

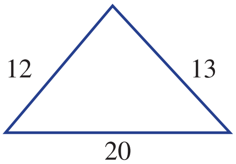
12 **a** Rearrange $a^2 = b^2 + c^2 - 2bc\cos A$ to make $\cos A$ the subject.
b Use your rule to find angle A in this triangle, correct to 1 decimal place.



ENRICHMENT	–	–	13, 14

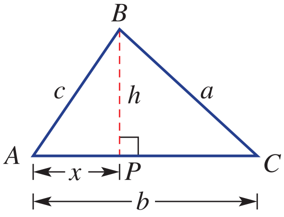
Proof of the cosine rule

13 Consider the triangle shown.



a Use the cosine rule to find the size of the largest angle.
b Use the sine rule to find the size of the smallest angle.

14 Triangle ABC shown here includes point P such that $PB \perp CA$, $BP = h$ and $AP = x$.



a Write an expression for length CP.
b Use Pythagoras' theorem and $\triangle ABP$ to write an equation in c, x and h.
c Use Pythagoras' theorem and $\triangle CPB$ to write an equation in b, a, x and h.
d Combine your equations from parts **b** and **c** to eliminate h. Simplify your result.
e Use $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ to write an expression for $\cos A$.
f Combine your equations from parts **d** and **e** to prove $a^2 = b^2 + c^2 - 2bc\cos A$.

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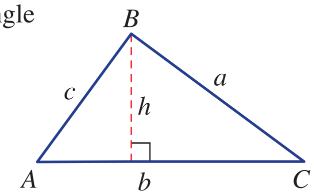
7I Area of a triangle

Interactive
Widgets
HOTsheets
Walkthrough

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

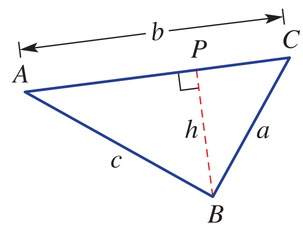
We can use trigonometry to establish a rule for the area of a triangle using two sides and the included angle.



We can see in this triangle that $\frac{h}{a} = \sin C$, so $h = a \sin C$.

$\therefore A = \frac{1}{2}bh$ becomes $A = \frac{1}{2}ba \sin C$.

Let's start: Calculating area in two ways



Draw any triangle ABC and construct the height PB. Measure the following as accurately as possible.

i AC ii BC iii BP iv $\angle C$

Now calculate the area using:

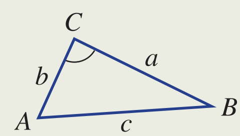
- Area $= \frac{1}{2}bh$
- Area $= \frac{1}{2}ba \sin C$

How close are your answers? They should be equal!

Key ideas

- The area of a triangle is equal to half the product of two sides and the sine of the included angle.

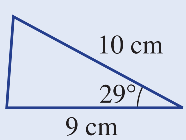
  Area $= \frac{1}{2}ab \sin C$



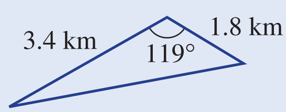
Example 16 Finding the area of a triangle

Find the area of these triangles, correct to 1 decimal place.

a



b



SOLUTION

a Area $= \frac{1}{2}ab \sin C$

$= \frac{1}{2} \times 9 \times 10 \times \sin 29°$

$= 21.8\text{ cm}^2$ (to 1 decimal place)

EXPLANATION

Substitute the two known sides (a and b) and the included angle into the rule.

b Area $= \frac{1}{2}ab \sin C$

$= \frac{1}{2} \times 1.8 \times 3.4 \times \sin 119°$

$= 2.7\text{ km}^2$ (to 1 decimal place)

If C is obtuse, then $\sin C$ is positive and the rule can still be used for obtuse-angled triangles.

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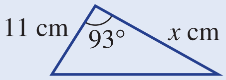
Example 17 Finding a side length when given the area

Find the value of x, correct to 2 decimal places, given that the area of this triangle is 70 cm^2.



SOLUTION

$$\text{Area} = \frac{1}{2}ab\sin C$$

$$70 = \frac{1}{2} \times 11 \times x \times \sin 93°$$

$$140 = 11x\sin 93°$$

$$x = \frac{140}{11 \times \sin 93°}$$

$$= 12.74 \text{ (to 2 decimal places)}$$

EXPLANATION

Substitute all the given information into the rule, letting $a = 11$ and $b = x$. Use $\angle C = 93°$ as the included angle.

Multiply both sides by 2 and then divide by $11 \times \sin 93°$.

Solve for x.

Exercise 7I

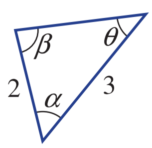
UNDERSTANDING AND FLUENCY	1–5	3, 4(½), 5, 6(½)	4(½), 5, 6(½)

1 Evaluate $\frac{1}{2}ab\sin C$, correct to 1 decimal place, for the given values of a, b and C.

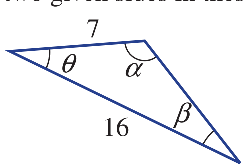
a $a = 3$, $b = 4$, $C = 38°$ **b** $a = 6$, $b = 10$, $C = 74°$ **c** $a = 15$, $b = 7$, $C = 114°$

2 Name the included angle between the two given sides in these triangles.

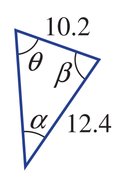
a



b



c



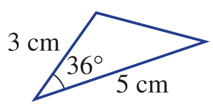
3 Solve these equations for C. Round your answer to 2 decimal places.

a $10 = \frac{1}{2} \times 4 \times 6 \times \sin C$ **b** $25 = \frac{1}{2} \times 7 \times 10 \times \sin C$ **c** $42 = \frac{1}{2} \times 11 \times 9 \times \sin C$

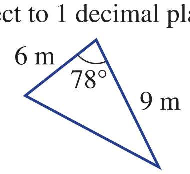
Example 16

4 Find the area of these triangles, correct to 1 decimal place.

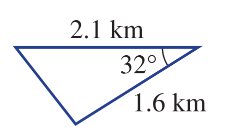
a



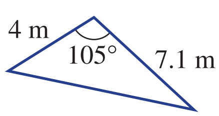
b



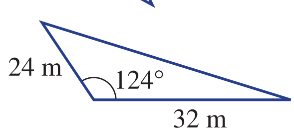
c



d



e



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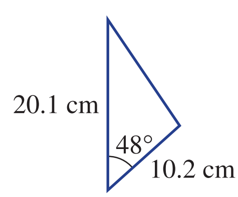

5 Find the area of these triangles, correct to 1 decimal place.

a $\triangle XYZ$ if $XY = 5$ cm, $XZ = 7$ cm and $\angle X = 43°$

b $\triangle STU$ if $ST = 12$ m, $SU = 18$ m and $\angle S = 78°$

c $\triangle EFG$ if $EF = 1.6$ km, $FG = 2.1$ km and $\angle F = 112°$

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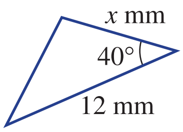
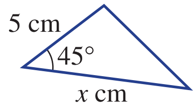
Example 17

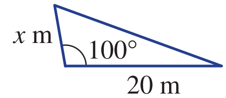
6 Find the value of x, correct to 1 decimal place, for these triangles with given areas.

a


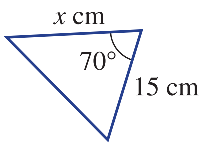
Area = 22 mm²

b


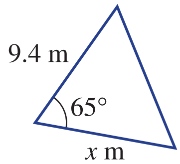
Area = 14 cm²

c


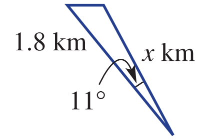
Area = 90 m²

d


Area = 128 cm²

e


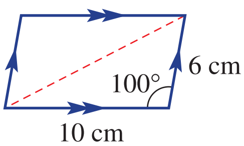
Area = 45 m²

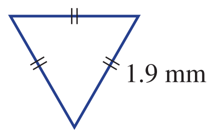
f


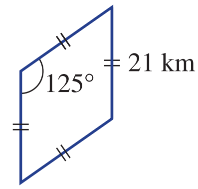
Area = 0.23 km²

PROBLEM-SOLVING AND REASONING	7, 8, 11	8, 9, 11, 12	8–10, 11–13

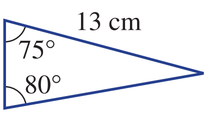
7 Find the area of these shapes, correct to 2 decimal places.

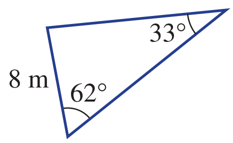
a


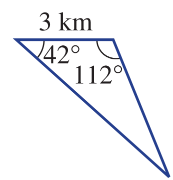
b


c


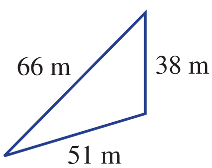
8 First use the sine rule to find another side length, and then find the area of these triangles, correct to 2 decimal places.

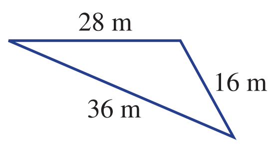
a


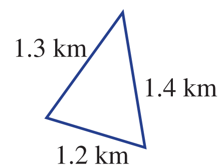
b


c


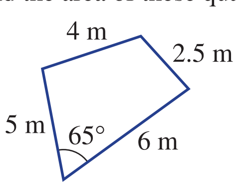
9 First use the cosine rule to find an angle, and then calculate the area of these triangles, correct to 2 decimal places.

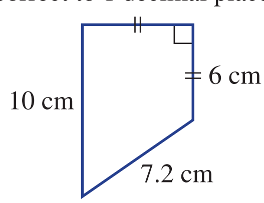
a


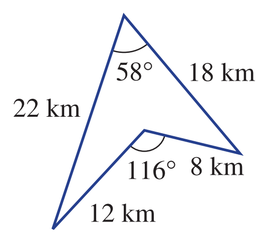
b


c


10 Find the area of these quadrilaterals, correct to 1 decimal place.

a


b


c


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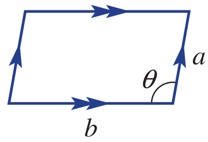
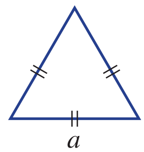
11 Write a rule for the area of these shapes, using the given pronumerals.

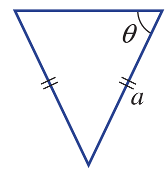
a



b

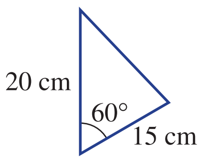


c

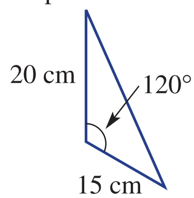


12 **a** Find the area of these two triangles, correct to 1 decimal place.

i

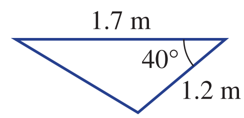


ii



b What do you notice about your answers in part **a**? How can you explain this?

c Draw another triangle that has the same two given lengths and area as the triangle on the right.



13 **a** Use the rule Area $= \frac{1}{2}ab \sin C$ to find the two possible values of θ in the triangle described below. Round your answer to 1 decimal place.

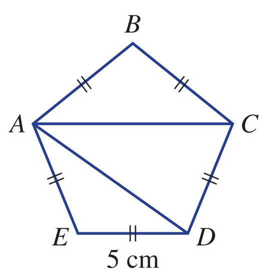
$\triangle ABC$ with $AB = 11$ m, $AC = 8$ m, included angle θ and area $= 40$ m^2.

b Draw the two triangles for the two sets of results found in part **a**.

ENRICHMENT	–	–	14

Polygon areas

14 You will recall that the sum (S) of the interior angles of a polygon with n sides is given by $S = 180(n - 2)$.



a This regular pentagon has each side measuring 5 cm.

i Calculate the angle sum of a pentagon.

ii Calculate the size of one interior angle of a regular pentagon.

iii Find the area of $\triangle AED$, correct to 2 decimal places.

iv Find the length AD, correct to 2 decimal places.

v Find $\angle ADC$ and $\angle DAC$.

vi Find the area of $\triangle ADC$, correct to 2 decimal places.

vii Find the total area of the pentagon, correct to 1 decimal place.

b Use a similar approach to find the area of a regular hexagon of side length 5 cm, correct to 1 decimal place.

c Can this method be used for other regular polygons? Explore and give examples.

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7J The four quadrants

Interactive
Widgets
HOTsheets
Walkthrough

Stage

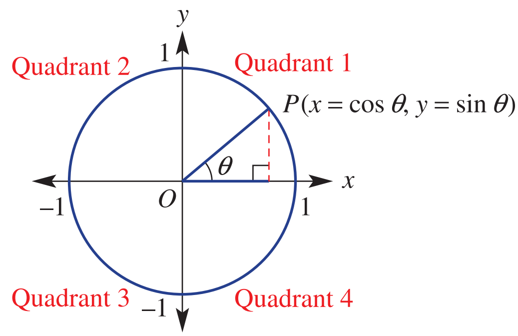
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

From Section 7F we calculated $\sin\theta$, $\cos\theta$ and $\tan\theta$ $\left(=\dfrac{\sin\theta}{\cos\theta}\right)$ using obtuse angles.

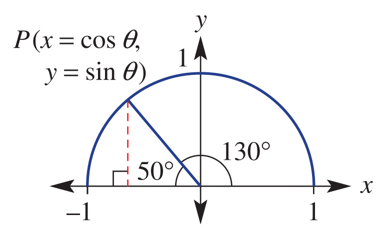
We will now extend this to include the four quadrants of the unit circle, using $0° < \theta < 360°$.

Recall that:

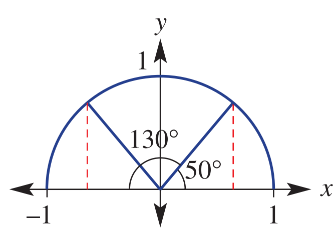
- The unit circle has a radius of 1 unit and has centre (0, 0) on a number plane.
- θ is defined anticlockwise from the positive x-axis.
- There are four quadrants, as shown.
- The coordinates of P, a point on the unit circle, are $(x, y) = (\cos\theta, \sin\theta)$.
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$



- In the second quadrant, $\sin\theta$ is positive, $\cos\theta$ is negative and $\tan\theta$ is negative. In this diagram we can see $P(\cos 130°, \sin 130°)$, where $\cos 130°$ is negative, $\sin 130°$ is positive and so $\tan 130°$ will be negative.



In the diagram at right showing 130°, a 50° angle (180° − 130°) drawn in the first quadrant can help relate trigonometric values from the second quadrant to the first quadrant. By symmetry we can see that $\sin 130° = \sin 50°$ and $\cos 130° = -\cos 50°$. This 50° angle is called the **reference angle** (or related angle).



In this section we explore these symmetries and reference angles in the third and fourth quadrants.

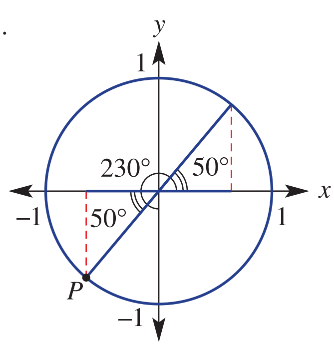
Let's start: Positive or negative

For the angle 230°, the reference angle is 50° and $P = (\cos 230°, \sin 230°)$.
Since P is in the third quadrant we can see that:

- $\cos 230° = -\cos 50°$, which is negative.
- $\sin 230° = -\sin 50°$, which is negative.

Now determine the following for each point P given below.
You should draw a unit circle for each.

a $\theta = 240°$ **b** $\theta = 210°$
c $\theta = 335°$ **d** $\theta = 290°$
e $\theta = 162°$



- What is the reference angle?
- Is $\cos\theta$ positive or negative?
- Is $\sin\theta$ positive or negative?
- Is $\tan\theta$ positive or negative?

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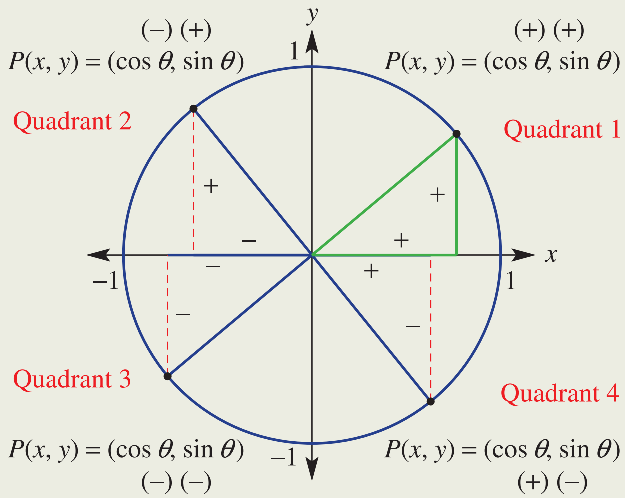
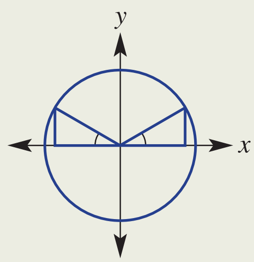
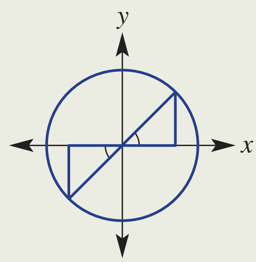
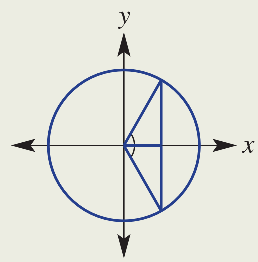
Key ideas

- Every point $P(x, y)$ on the unit circle can be described in terms of the angle θ such that: $x = \cos\theta$ and $y = \sin\theta$, where $-1 \leqslant \sin\theta \leqslant 1$ and $-1 \leqslant \cos\theta \leqslant 1$.



  - For different quadrants, $\cos\theta$ and $\sin\theta$ can be positive or negative.
  - Recall that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.
  - ASTC means:
    - **a** Quadrant 1: All $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive.
    - **b** Quadrant 2: only Sin θ is positive.
    - **c** Quadrant 3: only Tan θ is positive.
    - **d** Quadrant 4: only Cos θ is positive.

- A **reference angle** (sometimes called a related angle) is an acute angle that helps to relate $\cos\theta$ and $\sin\theta$ to the first quadrant.

Angle θ	90° to 180°	180° to 270°	270° to 360°
Reference angle	$180^\circ - \theta$	$\theta - 180^\circ$	$360^\circ - \theta$
			

- **Exact values** can be used when the reference angles are 30°, 45° or 60°.

- Multiples of 90°.

θ	0°	90°	180°	270°	360°
$\sin\theta$	0	1	0	-1	0
$\cos\theta$	1	0	-1	0	1
$\tan\theta$	0	undefined	0	undefined	0

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Example 18 Positioning a point on the unit circle

Decide in which quadrant θ lies and state whether $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive or negative.

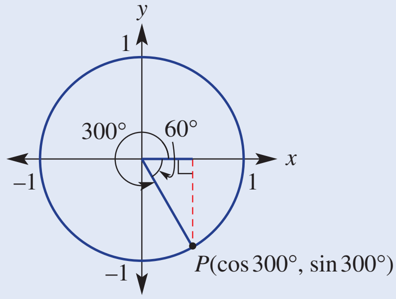
a $\theta = 300°$ **b** $\theta = 237°$

SOLUTION

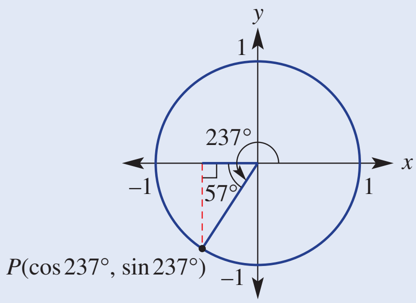
a $\theta = 300°$ is in quadrant 4.
$\sin\theta$ is negative
$\cos\theta$ is positive
$\tan\theta$ is negative

EXPLANATION

Note: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$



b $\theta = 237°$ is in quadrant 3.
$\sin\theta$ is negative
$\cos\theta$ is negative
$\tan\theta$ is positive



Example 19 Using a reference angle

Write the following using their reference angle.

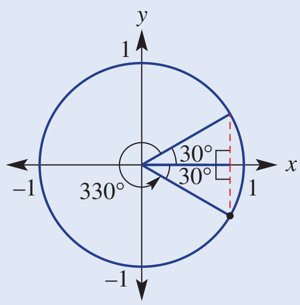
a $\sin 330°$ **b** $\cos 162°$ **c** $\tan 230°$

SOLUTION

a $\sin 330° = -\sin 30°$

EXPLANATION

$\sin 330°$ is negative and the reference angle is $360° - 330° = 30°$.



Example continued over page

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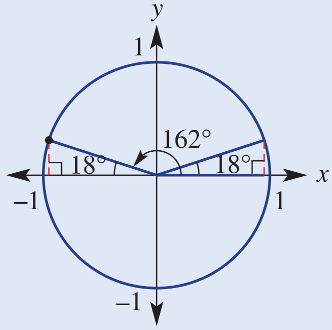
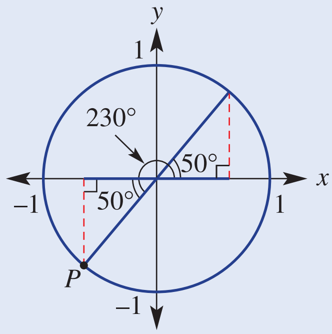
b $\cos 162^\circ = -\cos 18^\circ$

$\cos 162^\circ$ is negative and the reference angle is $180^\circ - 162^\circ = 18^\circ$.



c $\tan 230^\circ = \tan 50^\circ$

$\tan 230^\circ$ is positive (negative ÷ negative) and the reference angle is $230^\circ - 180^\circ = 50^\circ$.



Exercise 7J

UNDERSTANDING AND FLUENCY	1–6	2, 3, 5–7(½)	5–7(½)

1 Which quadrant in the unit circle corresponds to these values of θ?

a $0^\circ < \theta < 90^\circ$

b $180^\circ < \theta < 270^\circ$

c $270^\circ < \theta < 360^\circ$

d $90^\circ < \theta < 180^\circ$

2 Decide which quadrants make the following true.

a $\sin\theta$ is positive

b $\tan\theta$ is negative

c $\cos\theta$ is negative

d $\cos\theta$ is positive

e $\tan\theta$ is positive

f $\sin\theta$ is negative

3 Complete this table.

θ	0°	90°	180°	270°	360°
$\sin\theta$					0
$\cos\theta$			–1		
$\tan\theta$		undefined			

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4 Use a calculator to evaluate the following, correct to 3 decimal places.

a $\sin 172°$ b $\sin 84°$ c $\sin 212°$ d $\sin 325°$
e $\cos 143°$ f $\cos 255°$ g $\cos 321°$ h $\cos 95°$
i $\tan 222°$ j $\tan 134°$ k $\tan 42°$ l $\tan 337°$

Example 18

5 Decide in which quadrant θ lies and state whether $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive or negative.

a $\theta = 172°$ b $\theta = 295°$ c $\theta = 252°$ d $\theta = 73°$
e $\theta = 318°$ f $\theta = 154°$ g $\theta = 197°$ h $\theta = 221°$
i $\theta = 210°$ j $\theta = 53°$ k $\theta = 346°$ l $\theta = 147°$

Example 19

6 Write each of the following using its reference angle.

a $\sin 280°$ b $\cos 300°$ c $\tan 220°$ d $\sin 140°$
e $\cos 125°$ f $\tan 315°$ g $\sin 345°$ h $\cos 238°$
i $\tan 227°$ j $\sin 112°$ k $\cos 294°$ l $\tan 123°$

7 If θ is acute, find the value of θ.

a $\sin 150° = \sin\theta$
b $\sin 240° = -\sin\theta$
c $\sin 336° = -\sin\theta$
d $\cos 220° = -\cos\theta$
e $\cos 109° = -\cos\theta$
f $\cos 284° = \cos\theta$
g $\tan 310° = -\tan\theta$
h $\tan 155° = -\tan\theta$
i $\tan 278° = -\tan\theta$

PROBLEM-SOLVING AND REASONING	8, 9, 12	9, 10, 12, 13	10–12, 13(½), 14, 15

8 Write the reference angle (i.e. related angle) in the first quadrant for these angles.

a 138° b 227° c 326° d 189°
e 213° f 298° g 194° h 302°

9 For what values of θ, in degrees, are the following true?

a All of $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive.
b Only $\sin\theta$ is positive.
c Only $\cos\theta$ is positive.
d Only $\tan\theta$ is positive.

10 Complete the table by finding a second angle, θ_2, that gives the same value for the trigonometric function as θ_1. Use the unit circle to help in each case and assume $0 \leqslant \theta_2 \leqslant 360°$.

Trigonometric function	$\sin\theta$	$\cos\theta$	$\sin\theta$	$\tan\theta$	$\cos\theta$	$\tan\theta$	$\cos\theta$	$\sin\theta$	$\tan\theta$
θ_1	30°	45°	190°	15°	125°	320°	260°	145°	235°
θ_2									

11 Decide which quadrant suits the given information.

a $\sin\theta < 0$ and $\cos\theta > 0$ b $\tan\theta > 0$ and $\cos\theta > 0$
c $\tan\theta < 0$ and $\cos\theta < 0$ d $\sin\theta > 0$ and $\tan\theta < 0$
e $\sin\theta > 0$ and $\tan\theta > 0$ f $\sin\theta < 0$ and $\cos\theta < 0$

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12 Recall the exact sine, cosine and tangent values for 30°, 45° and 60°.

θ	30°	45°	60°
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

- **a** State the reference angle for 225°.
- **b** Hence, give the exact value of the following.
  - **i** sin 225° **ii** cos 225° **iii** tan 225°
- **c** State the reference angle for 330°.
- **d** Hence, give the exact value of the following.
  - **i** sin 330° **ii** cos 330° **iii** tan 330°
- **e** State the reference angle for 120°.
- **f** Hence, give the exact value of the following.
  - **i** sin 120° **ii** cos 120° **iii** tan 120°

13 Give the exact value of the following.

- **a** sin 135° **b** tan 180° **c** cos 150° **d** sin 240°
- **e** tan 315° **f** cos 210° **g** sin 330° **h** tan 120°
- **i** cos 225° **j** sin 270° **k** tan 330° **l** cos 300°
- **m** sin 180° **n** tan 270° **o** tan 225° **p** cos 180°

14 Explain why $\tan\theta > 0$ if $180° < \theta < 270°$.

15 Explain why tan 90° and tan 270° are undefined.

ENRICHMENT	–	–	16, 17

Exploring truth

16 By considering a unit circle, state whether the following are true or false.

- **a** $\sin 10° < \cos 10°$
- **b** $\sin 50° < \tan 50°$
- **c** $\cos 80° > \sin 80°$
- **d** $\cos 90° = \sin 0°$
- **e** $\tan 180° = \sin 180°$
- **f** $\cos 170° > \sin 170°$
- **g** $\sin 120° > \tan 120°$
- **h** $\sin 90° = \cos 180°$
- **i** $\tan 230° < \cos 230°$
- **j** $\cos 350° < \sin 85°$
- **k** $\sin 260° < \cos 110°$
- **l** $\tan 270° = \cos 180°$

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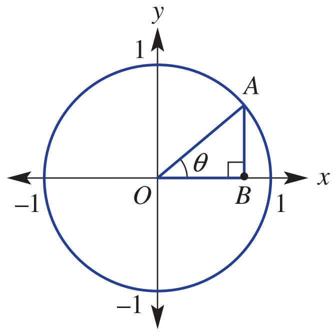
17 Trigonometric identities are mathematical statements that may involve $\sin\theta$, and/or $\cos\theta$ and/or $\tan\theta$ and hold true for all values of θ. In previous exercises you will have already considered the trigonometric identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.



a Consider the triangle OAB in the unit circle shown.

i Given that $OA = 1$, $OB = \cos\theta$ and $AB = \sin\theta$, use Pythagoras' theorem to prove the trigonometric identity: $\sin^2\theta + \cos^2\theta = 1$. Note: $\sin^2\theta = (\sin\theta)^2$.

ii Check your identity using a calculator to see if it holds true for $\theta = 30°, 145°, 262°$ and $313°$.

b **i** Evaluate the given pairs of numbers using a calculator:
(sin 60°, cos 30°), (sin 80°, cos 10°), (sin 110°, cos − 20°), (sin 195°, cos − 105°)

ii What do you notice about the value of each number in the pairs above? Drawing a unit circle illustrating each pair of values may help.

iii What is the relationship between θ in $\sin\theta$ and θ in $\cos\theta$ that is true for all pairs in part **b i**?

iv In terms of θ, complete this trigonometric identity: $\sin\theta = \cos(\_\_\_\_)$.

v Check this identity for $\theta = 40°, 155°, 210°$ and $236°$.

c Explore further trigonometric identities by drawing diagrams and checking different angles, such as:

i $\sin\theta = \sin(180 - \theta)$
ii $\cos\theta = \cos(360 - \theta)$
iii $\tan\theta = (180 + \theta)$
iv $\sin 2\theta = 2\sin\theta\cos\theta$
v $\cos^2\theta = \cos^2\theta - \sin^2\theta$
vi $\cos 2\theta = 2\cos^2\theta - 1$
vii $\cos 2\theta = 1 - 2\sin^2\theta$

Medieval Arab astronomers invented this device, a sine quadrant, for measuring the angles of stars overhead.

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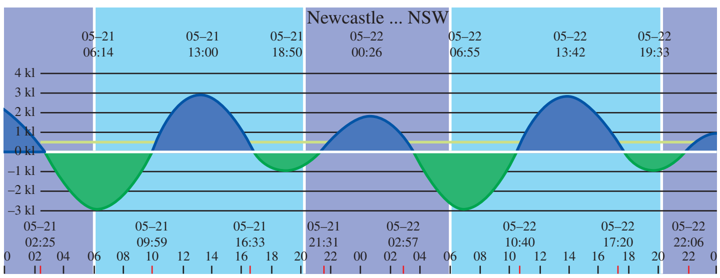
7K Graphs of trigonometric functions

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

As the angle θ increases from $0°$ to $360°$, the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ increase or decrease depending on the value of θ. Graphing the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ against θ gives a clear picture of this.

These wave-like graphs based on trigonometric functions are used to model many variables, from the height of the tide on a beach to the width of a soundwave giving high or low pitch sound.

The chart below shows the rise and fall of the tide. It strongly resembles a sine curve.

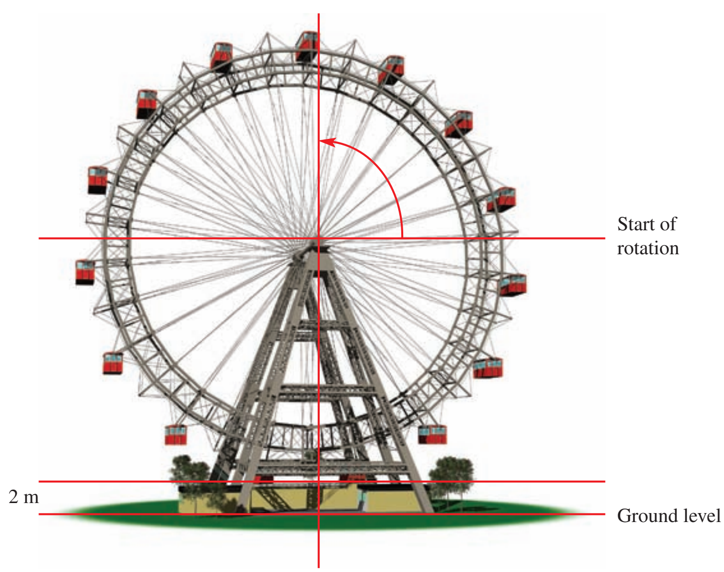


Let's start: Ferris wheel ride

Have you ever had a ride on a Ferris wheel? Imagine yourself riding a Ferris wheel again. The wheel rotates at a constant rate, but on which part of the ride will your vertical upwards movement be fastest? On which part of the ride will you move quite slowly upwards or downwards?

Work in groups to discuss these questions and help each other to complete the table and graph below.

For this example, assume that the bottom of the Ferris wheel is 2 m above the ground and the diameter of the wheel is 18 m. Count the start of a rotation from halfway up on the right, as shown below. The wheel rotates in an anticlockwise direction.



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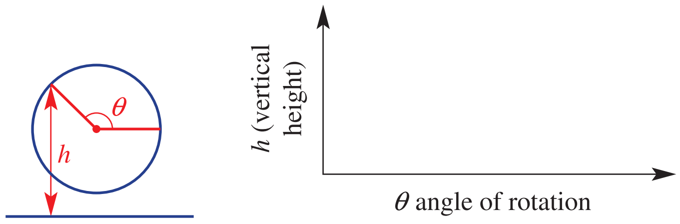
Position	Angle of rotation, θ, from halfway up	Vertical height, h, above ground level (m)
Halfway up	0°	
Top	90°	
Halfway down		
Bottom		2
Halfway up		
Top		
Halfway down		

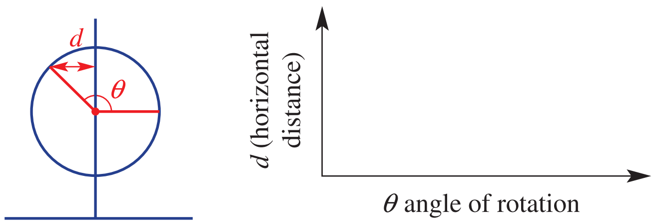
Now draw a graph of the **vertical height** (h) above the ground (vertical axis) versus the angle (θ) of anticlockwise rotation for two complete turns of the Ferris wheel.



As a group, discuss some of the key features of the graph.

- What are the maximum and minimum values for the height?
- Discuss any symmetry you see in your graph. How many values of θ (rotation angle) have the same value for height? Give some examples.

Discuss how the shape would change for a graph of the **horizontal distance** (d) from the circumference (where you sit) to the central **vertical** axis of the Ferris wheel versus the angle (θ) of rotation. Sketch this graph.



The shapes of the Ferris wheel graphs you have drawn are examples of **periodic functions** because the graph shape continuously repeats one cycle (for each period of 360°) as the wheel rotates. The graph of height above the ground illustrates a sine function ($\sin\theta$). The graph of the distance from a point on the circumference to the central vertical axis of the Ferris wheel illustrates a cosine function ($\cos\theta$).

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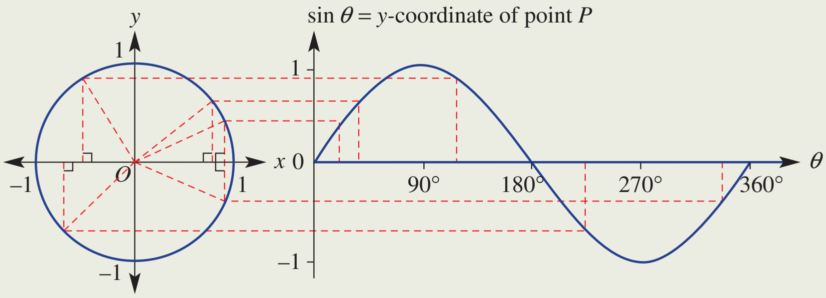
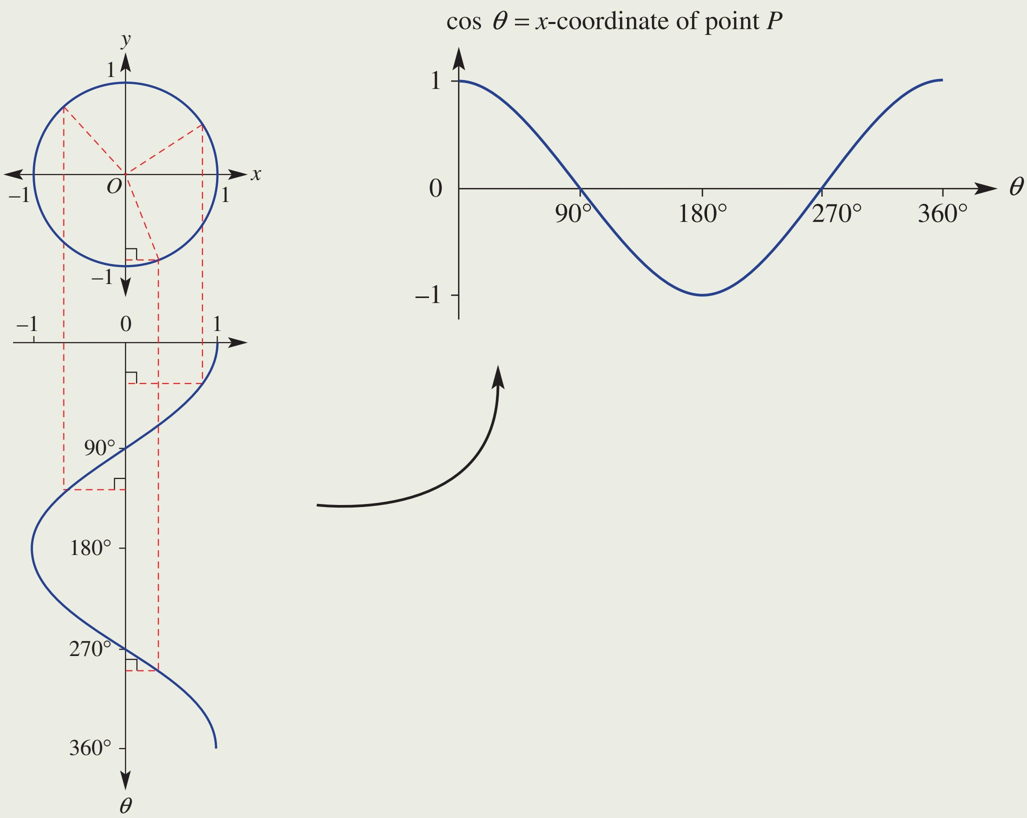
Key ideas

- **Amplitude** is the maximum displacement of the graph from a reference level (here it is the x-axis).
- The **period** of a graph is the number of degrees taken to make one complete cycle.
- By plotting θ on the x-axis and $\sin\theta$ on the y-axis, we form the graph of $\sin\theta$.
  $\sin\theta = y$-coordinate of point P on the unit circle.
  - $y = \sin\theta$ Amplitude $= 1$ Period $= 360°$



- By plotting θ on the x-axis and $\cos\theta$ on the y-axis, we form the graph of $\cos\theta$.
  $\cos\theta = x$-coordinate of point P on the unit circle.
  - When we write $y = \cos\theta$, the y variable is not to be confused with the y-coordinate of the point P on the unit circle.
  - $y = \cos\theta$ Amplitude $= 1$ Period $= 360°$



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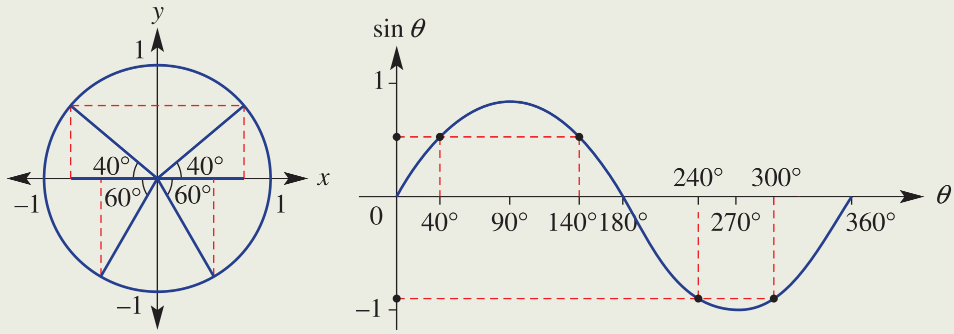
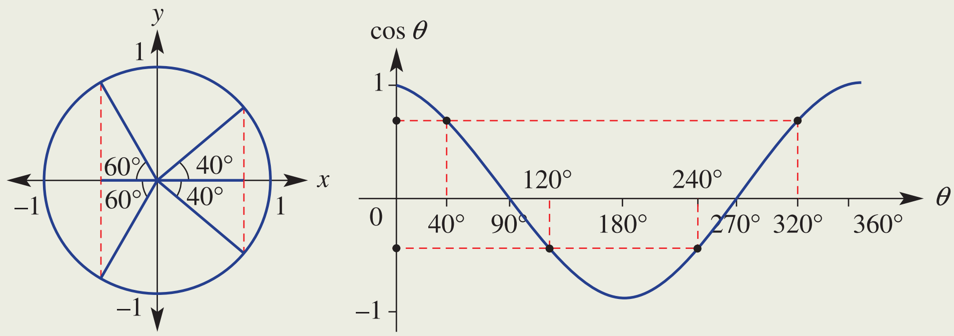
Key ideas

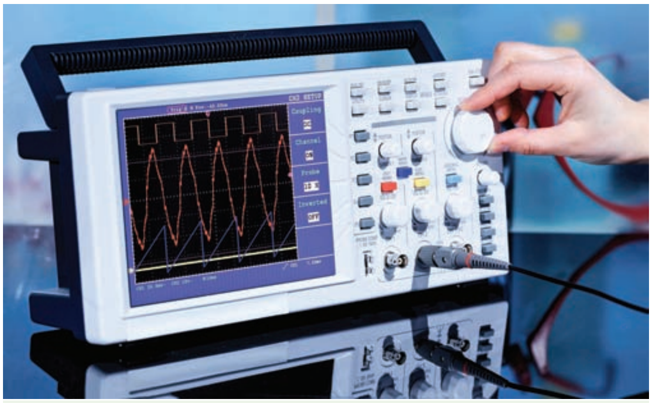
- **Symmetry** within the unit circle using reference angles can be illustrated using graphs of trigonometric functions.
  - This shows sin 40° = sin 140° (reference angle 40°) and sin 240° = sin 300° (reference angle 60°).



  - This shows cos 40° = cos 320° (reference angle 40°) and cos 120° = cos 240° (reference angle 60°).



Note: The Investigation section at the end of this chapter explains how to make the graphs discussed in the Key ideas, using dynamic geometry software.



There are many applications of trigonometric functions in physics and engineering. The sine wave displayed on this oscilloscope, for example, could represent an electromagnetic wave or an alternating power source.

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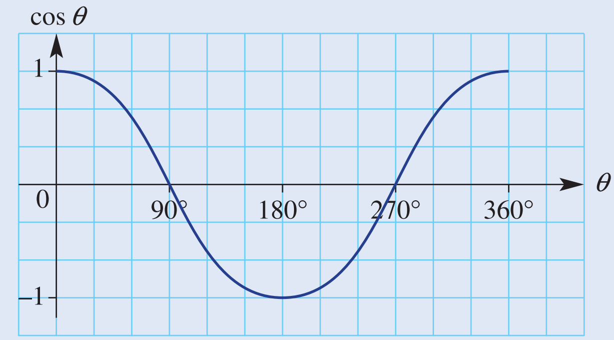
Example 20 Using a trigonometric graph

Use this graph of $\cos\theta$ to estimate:

a the value of $\cos\theta$ for $\theta = 200°$

b the two values of θ for which $\cos\theta = 0.7$

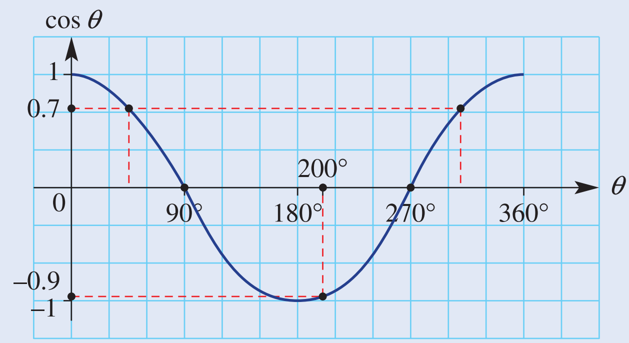


SOLUTION

a $\cos 200° \approx -0.9$

b $\cos\theta = 0.7$
$\theta \approx 46°$ or $314°$

EXPLANATION



Example 21 Comparing the size of the sine of angles

Use the graph of $y = \sin\theta$ to state whether or not the following are true or false.

a $\sin 70° < \sin 130°$

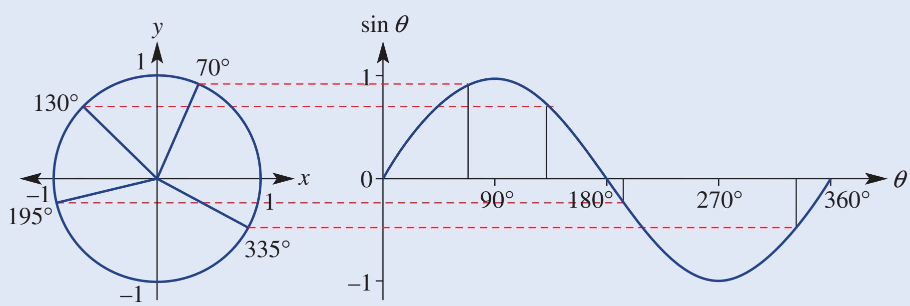
b $\sin 195° > \sin 335°$

SOLUTION

a false

b true

EXPLANATION



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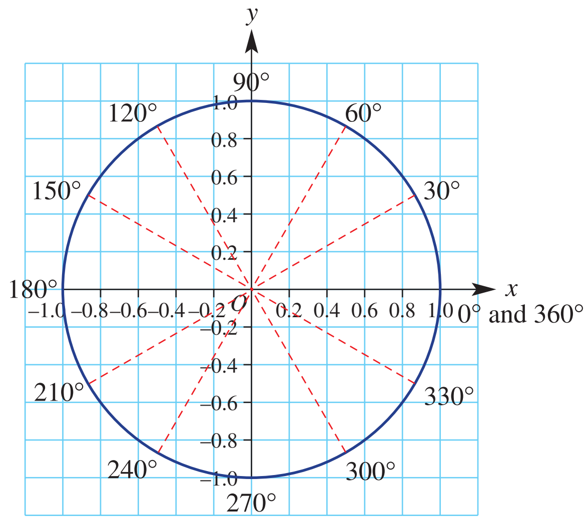
Exercise 7K

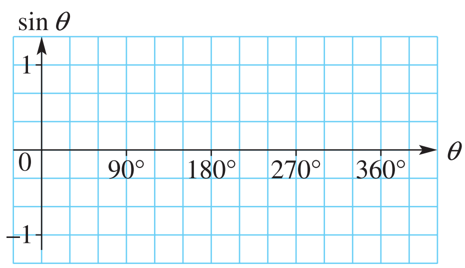
UNDERSTANDING AND FLUENCY	1–5	3–6	4–6

1 **a** Complete the table below for $\sin\theta$, writing the y-coordinate of each point at which the angle intersects the unit circle.

θ	0°	30°	60°	90°	120°	150°	180°
$\sin\theta$	0	0.5			0.87		
θ	210°	240°	270°	300°	330°	360°	
$\sin\theta$	−0.5						



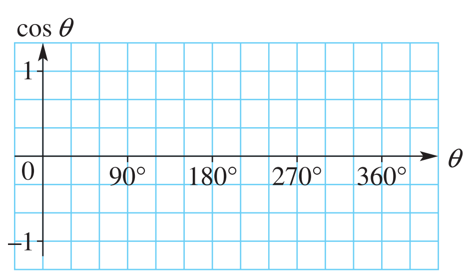
b Graph the points above and join them to make a smooth curve for $\sin\theta$.



2 **a** Using the diagram in Question **1**, complete the table below for $\cos\theta$, writing the x-coordinate of each point at which the angle intersects the unit circle.

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\cos\theta$	0	0.87			−0.5			−0.87					

b Graph the points above and join them to make a smooth curve for $\cos\theta$.



3 **a** For the graph of $\sin\theta$ and using $0° \leqslant \theta \leqslant 360°$, state:

i the maximum and minimum values of $\sin\theta$

ii the values of θ for which $\sin\theta = 0$

b For the graph of $\cos\theta$ and using $0° \leqslant \theta \leqslant 360°$, state:

i the maximum and minimum values of $\cos\theta$

ii the values of θ for which $\cos\theta = 0$

c State the values of θ for which:

i $\cos\theta < 0$ **ii** $\sin\theta < 0$

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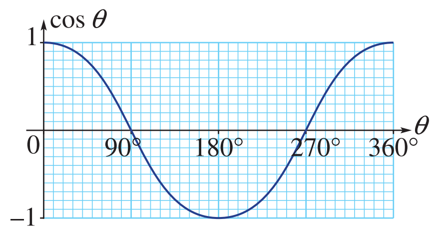
Example 20

4 This graph shows $\cos\theta$ for $0° \leqslant \theta \leqslant 360°$.



a Use this graph to estimate the value of $\cos\theta$ for the following.

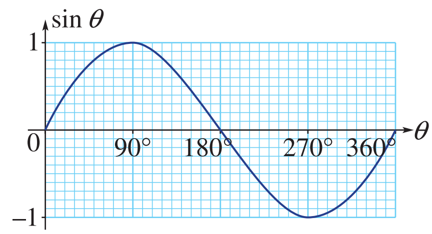
i $\theta = 35°$	**ii** $\theta = 190°$
iii $\theta = 330°$	**iv** $\theta = 140°$
v $\theta = 260°$	**vi** $\theta = 75°$
vii $\theta = 115°$	**viii** $\theta = 305°$

b Use the same graph to estimate the two values of θ for each of the following.

i $\cos\theta = 0.8$	**ii** $\cos\theta = 0.6$
iii $\cos\theta = 0.3$	**iv** $\cos\theta = 0.1$
v $\cos\theta = -0.4$	**vi** $\cos\theta = -0.2$
vii $\cos\theta = -0.8$	**viii** $\cos\theta = -0.6$

c Use your calculator to extend the graph for values of θ from $-720°$ to $720°$.

5 This graph shows $\sin\theta$ for $0° \leqslant \theta \leqslant 360°$.



a Use this graph to estimate the value of $\sin\theta$ for the following.

i $\theta = 25°$	**ii** $\theta = 115°$
iii $\theta = 220°$	**iv** $\theta = 310°$
v $\theta = 160°$	**vi** $\theta = 235°$
vii $\theta = 320°$	**viii** $\theta = 70°$

b Use the same graph to estimate the two values of θ for each of the following.

i $\sin\theta = 0.6$	**ii** $\sin\theta = 0.2$	**iii** $\sin\theta = 0.3$	**iv** $\sin\theta = 0.9$
v $\sin\theta = -0.4$	**vi** $\sin\theta = -0.8$	**vii** $\sin\theta = -0.7$	**viii** $\sin\theta = -0.1$

c Use your calculator to extend the graph for values of θ from $-720°$ to $720°$.

Example 21

6 By considering the graphs of $y = \sin\ \theta$ and $y = \cos\ \theta$, state whether the following are true or false.

a $\sin 60° > \sin 200°$	**b** $\sin 100° < \sin 300°$	**c** $\sin 135° < \sin 10°$	**d** $\sin 200° = \sin 340°$
e $\cos 70° < \cos 125°$	**f** $\cos 315° > \cos 135°$	**g** $\cos 310° = \cos 50°$	**h** $\cos 95° > \cos 260°$
i $\sin 90° = \cos 360°$	**j** $\cos 180° = \sin 180°$	**k** $\sin 210° > \sin 285°$	**l** $\cos 15° > \cos 115°$

PROBLEM-SOLVING AND REASONING	7–8(½), 12	7–9(½), 11, 12	9–10(½), 12, 13

7 For each of the following angles, state the second angle between 0° and 360° that gives the same value for $\sin\theta$.

a 70°	**b** 120°	**c** 190°	**d** 280°
e 153°	**f** 214°	**g** 307°	**h** 183°

8 For each of the following angles, state the second angle between 0° and 360° that gives the same value for $\cos\theta$.

a 80°	**b** 10°	**c** 165°	**d** 285°
e 224°	**f** 147°	**g** 336°	**h** 199°

9 Give the reference angle in the first quadrant that matches these angles.

a 150°	**b** 120°	**c** 195°	**d** 290°
e 235°	**f** 260°	**g** 125°	**h** 205°
i 324°	**j** 252°	**k** 117°	**l** 346°

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10 Recall the exact values for $\sin\theta$ and $\cos\theta$ for 30°, 45° and 60° in the first quadrant.

a Complete this table.

θ	0°	30°	45°	60°	90°
$\sin\theta$		$\frac{1}{2}$			
$\cos\theta$					0

b Using the reference angles in the table above, state the exact value of each of the following.

i sin 150° **ii** cos 120° **iii** cos 225° **iv** sin 180°
v cos 300° **vi** sin 240° **vii** cos 270° **viii** sin 135°
ix cos 210° **x** sin 330° **xi** sin 315° **xii** cos 240°
xiii sin 225° **xiv** sin 120° **xv** cos 150° **xvi** cos 330°

11 For θ between 0° and 360°, find the two values of θ that satisfy the following.

a $\cos\theta = \frac{\sqrt{2}}{2}$ **b** $\sin\theta = \frac{\sqrt{3}}{2}$ **c** $\sin\theta = \frac{1}{2}$
d $\sin\theta = -\frac{1}{2}$ **e** $\cos\theta = -\frac{1}{2}$ **f** $\cos\theta = -\frac{\sqrt{3}}{2}$

12 Use a calculator to find the two values of θ for $0° \leqslant \theta \leqslant 360°$, correct to 1 decimal place, for these simple equations.

a $\sin\theta = 0.3$ **b** $\sin\theta = 0.7$ **c** $\cos\theta = 0.6$
d $\cos\theta = 0.8$ **e** $\sin\theta = -0.2$ **f** $\sin\theta = -0.8$
g $\cos\theta = -0.4$ **h** $\cos\theta = 0.65$ **i** $\sin\theta = 0.48$

13 **a** How many values of θ satisfy $\sin\theta = 2$? Give a reason.
b How many values of θ satisfy $\cos\theta = -4$? Give a reason.

ENRICHMENT	–	–	14

Trigonometric functions with technology

14 Use technology to sketch the graph of the following families of curves on the same axes, and then write a sentence describing the effect of the changing constant. Use values of x from −360° to 360°.

a **i** $y = \sin x$ **ii** $y = -\sin x$
b **i** $y = \cos x$ **ii** $y = -\cos x$
c **i** $y = \sin x$ **ii** $y = 3\sin x$ **iii** $y = \frac{1}{2}\sin x$
d **i** $y = \cos x$ **ii** $y = \cos(2x)$ **iii** $y = \cos\left(\frac{x}{3}\right)$
e **i** $y = \sin x$ **ii** $y = \sin(x) + 2$ **iii** $y = \sin(x) - 1$
f **i** $y = \cos x$ **ii** $y = \cos(x - 45°)$ **iii** $y = \cos(x + 60°)$

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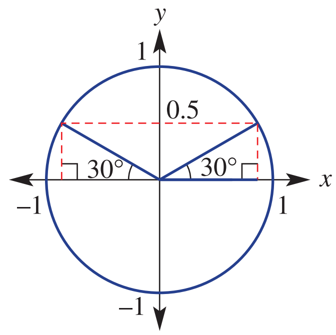
Investigation

1 Solving trigonometric equations

Trigonometric relations are not necessarily restricted to angles of less than 90°, and this is illustrated by drawing a graph of a trigonometric relation for angles up to 360°. Solving problems using trigonometric relations will therefore result in an equation that can have more than one solution.

For example, consider the equation $\sin\theta = 0.5$ for $0° \leqslant \theta \leqslant 360°$. Since $\sin\theta$ is the y-coordinate on the unit circle, there are two angles that satisfy $\sin\theta = 0.5$.



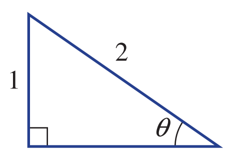
Solution 1: $\sin\theta = 0.5$
$\theta = \sin^{-1}(0.5)$
$= 30°$

Solution 2: $\theta = 180° - 30°$
$= 150°$

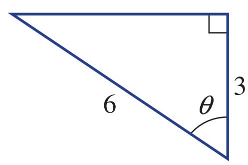
Single solutions ($0° \leqslant \theta \leqslant 90°$)

For these right-angled triangles, write an equation in terms of θ and then solve the equation to find θ.

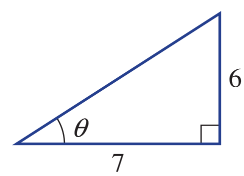
a



b

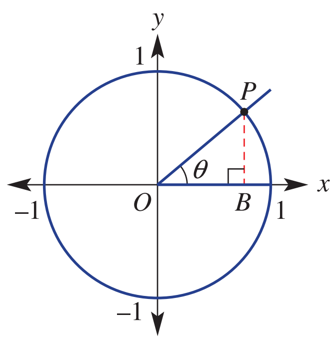


c



Two solutions ($0° \leqslant \theta \leqslant 360°$)

At point P on the unit circle $x = \cos\theta$ and $y = \sin\theta$.



For each of the following:

- **i** Use a calculator to find a value for θ between 0° and 360°.
- **ii** Find a second angle between 0° and 360° that also satisfies the given trigonometric equation.

a $\sin\theta = 0.5$ **b** $\cos\theta = 0.2$

c $\cos\theta = -0.8$ **d** $\sin\theta = -0.9$

Harder trigonometric equations

a Solve these trigonometric equations for $0° \leqslant \theta \leqslant 360°$.

i $5\sin\theta - 1 = 0$ **ii** $2\cos\theta + 3 = 0$

b Solving an equation such as $\sin(2\theta) = \frac{1}{2}$ for $0° \leqslant \theta \leqslant 360°$ means that $0° \leqslant \theta \leqslant 720°$, which includes two rotations of the point P around the unit circle. Solve these equations by first solving for 2θ and then dividing by 2 to solve for θ.

i $\sin(2\theta) = \frac{1}{2}$ **ii** $\cos(2\theta) = 0.3$

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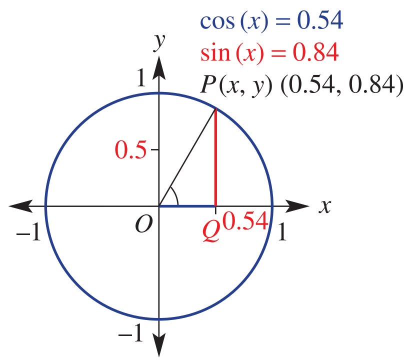
2 Building a dynamic unit circle

Use dynamic geometry software to construct a unit circle with a point $P(\cos\theta, \sin\theta)$ that can move around the circle.



a Construction steps:
- Show a grid and axes.
- Construct a unit circle with centre (0, 0) and radius 1, as shown.
- Place a point P on the unit circle to construct OP. Show the coordinates of P.
- Construct QP and OQ so that $OQ \perp QP$ and Q is on the x-axis.
- Measure the angle $\theta = \angle QOP$.

b Now drag point P around the unit circle and observe the coordinates of $P(\cos\theta, \sin\theta)$.

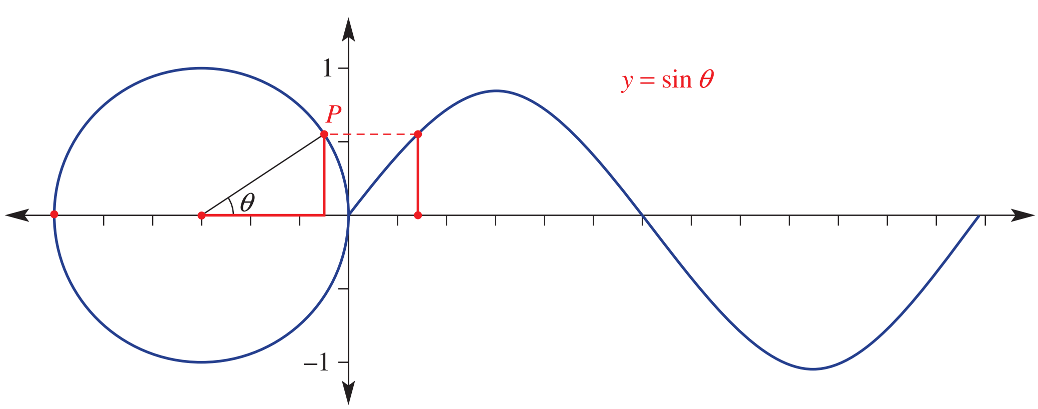
c Describe how $\cos\theta$ and $\sin\theta$ change as θ increases from 0° to 360°.

Extension

a Calculate $\tan\theta$ as $\dfrac{\sin\theta}{\cos\theta}$ and observe how the value of $\tan\theta$ changes as θ increases.

b Construct the graph of $y = \sin\theta$ by transferring the value of θ and $\sin\theta$ to a set of axes and constructing the set of coordinates $(\theta, \sin\theta)$. Trace $(\theta, \sin\theta)$ as you drag P to form the graph for $\sin\theta$.

c Repeat for $\cos\theta$ and $\tan\theta$.



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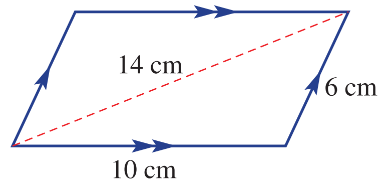
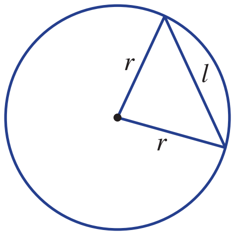
Puzzles and challenges

1 Two adjacent sides of a parallelogram have lengths of 6 cm and 10 cm. If the length of the longer diagonal is 14 cm, find:
a the size of the internal angles of the parallelogram
b the length of the other diagonal, to 1 decimal place



2 Two cyclists, Stuart and Cadel, start their ride from the same starting point. Stuart travels 30 km on a bearing of 025°, while Cadel travels the same distance but in a direction of 245°. What is Cadel's bearing from Stuart after they have travelled 30 km?

3 Show that, for a circle of radius r, the length of a chord l that subtends an angle θ at the centre of the circle is given by $l = \sqrt{2r^2(1 - \cos\theta)}$.



4 Akira measures the angle of elevation to the top of a mountain to be 20°. He walks 800 m horizontally towards the mountain and finds that the angle of elevation has doubled. What is the height of the mountain above Akira's position, to the nearest metre?

5 A walking group sets out due east from the town hall at 8 km/h. At the same time, another walking group leaves from the town hall along a different road in the direction of 030° at 5 km/h.
a How long will it be before the groups are 15 km apart? Give your answer to the nearest minute.
b What is the bearing of the second group from the first group, to the nearest degree, at any time?

6 Edwina stands due south of a building 40 m tall to take a photograph of it. The angle of elevation to the top of the building is 23°. What is the angle of elevation, correct to 2 decimal places, after she walks 80 m due east to take another photo?

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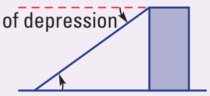
Chapter summary

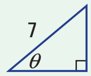
Trigonometry

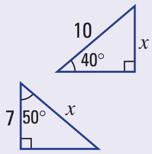
2D applications
- Draw the right-angled triangle with key information.
- Use trigonometry to find unknown.



Finding an angle
$\sin\theta = \frac{5}{7}$

$\theta = 45°35'$ (to the nearest minute)



Finding lengths


$\sin 40° = \frac{x}{10}$

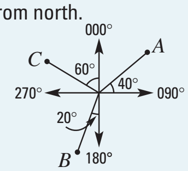
$\therefore x = 10 \times \sin 40°$

$\cos 50° = \frac{7}{x}$

$x \times \cos 50° = 7$

$x = \frac{7}{\cos 50°}$

Bearings
Bearings are measured clockwise from north.



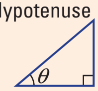
A: 050°
B: 200°
C: 300°

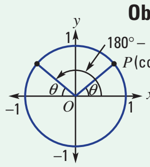
Trigonometric ratios
SOHCAHTOA

$\sin\theta = \frac{\text{O}}{\text{H}}$

$\cos\theta = \frac{\text{A}}{\text{H}}$

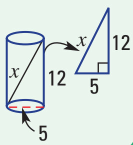
$\tan\theta = \frac{\text{O}}{\text{A}} = \frac{\sin\theta}{\cos\theta}$

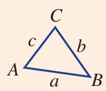


Obtuse angles


$\sin(180° - \theta) = \sin\theta$
$\cos(180° - \theta) = -\cos\theta$
$\tan(180° - \theta) = -\tan\theta$

3D applications
- Identify and redraw right-angled triangle.
- Use trigonometric ratios.
- Answer in words.

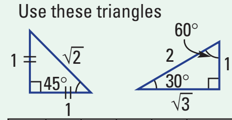


Sine rule


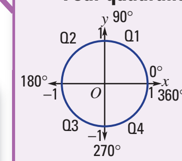
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

When finding angles, determine whether you require the acute or the obtuse angle.

Exact values
Use these triangles



θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Four quadrants of the unit circle


Q1: All are positive
Q2: Sin only positive
Q3: Tan only positive
Q4: Cos only positive

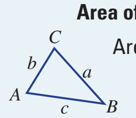
The reference angle relates an angle to its acute angle in quadrant 1.
Symmetry can then be used to find $\sin\theta$ or $\cos\theta$ or $\tan\theta$.
e.g. sin 210° in quadrant 3 is negative

$210° - 180° = 30°$

$\sin 210° = -\sin 30°$

$= -\frac{1}{2}$

To find reference angle of θ in Q1:
Q2: $180° - \theta$
Q3: $\theta - 180°$
Q4: $360° - \theta$

Area of a triangle


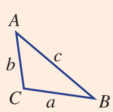
$\text{Area} = \frac{1}{2}ab\sin C$

Cosine rule
Rule is used to find:
- third side, when given two sides and included angle

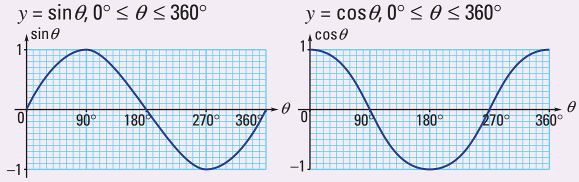
  $a^2 = b^2 + c^2 - 2bc\cos A$

  or
- an angle, when given all three sides

  $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$



Graphs of sin θ and cos θ
$y = \sin\theta, 0° \le \theta \le 360°$ $\quad$ $y = \cos\theta, 0° \le \theta \le 360°$



Symmetry of the unit circle can be observed in graphs.

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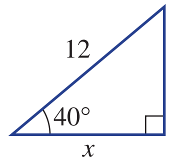
Multiple-choice questions

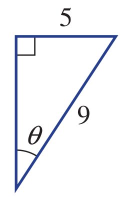
1 The value of x in the diagram shown is equal to:



A $\dfrac{12}{\cos 40°}$ **B** $12\sin 40°$ **C** $\dfrac{\sin 40°}{12}$

D $12\cos 40°$ **E** $\dfrac{12}{\tan 40°}$

2 In the diagram shown, the angle θ, correct to 1 decimal place, is:



A 56.3° **B** 33.7° **C** 29.1°

D 60.9° **E** 42.4°

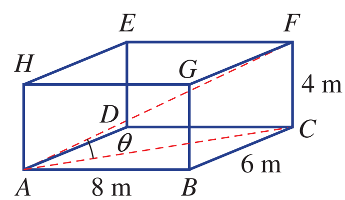
3 The angle of depression from the top of a communications tower measuring 44 m tall to the top of a communications tower measuring 31 m tall is 18°. The horizontal distance between the two towers is closest to:

A 12 m **B** 4 m **C** 14 m **D** 42 m **E** 40 m

4 A yacht sails from A to B on a bearing of 196°. To sail from B directly back to A the bearing would be:

A 074° **B** 096° **C** 164° **D** 016° **E** 286°

5 The angle θ that AF makes with the base of the rectangular prism is closest to:



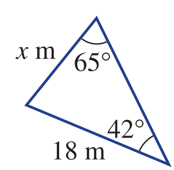
A 22° **B** 68° **C** 16°

D 24° **E** 27°

6 The obtuse angle θ such that $\cos\theta° = -\cos 35°$ is:

A $\theta = 125°$ **B** $\theta = 145°$ **C** $\theta = 140°$ **D** $\theta = 175°$ **E** $\theta = 155°$

7 In the diagram shown, the side length x, correct to 1 decimal place, is:



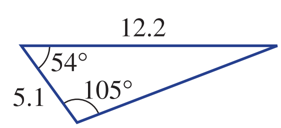
A 10.9 **B** 29.7 **C** 13.3

D 12.6 **E** 17.1

8 The smallest angle in the triangle with side lengths 8 cm, 13 cm and 19 cm, to the nearest degree, is:

A 19° **B** 33° **C** 52° **D** 24° **E** 29°

9 The area of the triangle shown can be determined by calculating:



A $\dfrac{1}{2} \times 5.1 \times 12.2 \times \cos 54°$ **B** $\dfrac{1}{2} \times 5.1 \times 12.2 \times \sin 105°$

C $\dfrac{1}{2} \times 12.2 \times 5.1$ **D** $\dfrac{1}{2} \times 12.2 \times 5.1 \times \sin 54°$

E $\dfrac{1}{2} \times 6.1 \times 5.1 \times \sin 21°$

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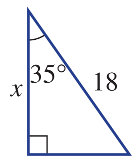
10 The incorrect statement below is:

A $\cos 110° = -\cos 70°$
B $\cos 246°$ is negative
C $\tan 130°$ is positive
D $\sin 150° = \sin 30°$
E $\sin 300°$ is negative and $\cos 300°$ is positive

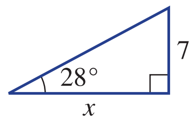
Short-answer questions

1 Find the value of each pronumeral, rounding your answer to 2 decimal places.

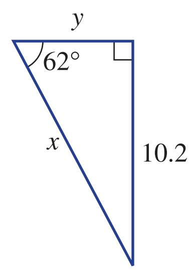
a



b

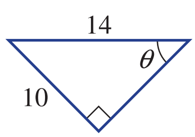


c

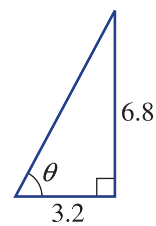


2 Find the value of θ, correct to 1 decimal place.

a

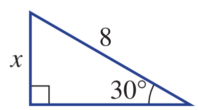


b

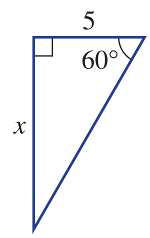


3 Use exact values to find the value of the pronumerals, without using a calculator.

a

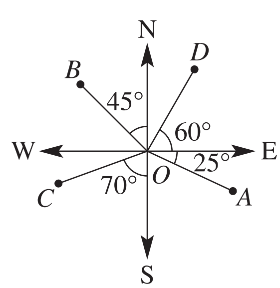


b



4 An escalator in a shopping centre from level 1 to level 2 is 22 m in length and has an angle of elevation of 16°. Determine how high level 2 is above level 1, to 1 decimal place.

5 Write down the bearings from O to A, B, C, and D.



6 A helicopter flies due south for 160 km and then on a bearing of 125° for 120 km. Answer the following to 1 decimal place.

a How far east is the helicopter from its start location?

b How far south is the helicopter from its start location?

c What bearing must it fly on to return directly to the start location?

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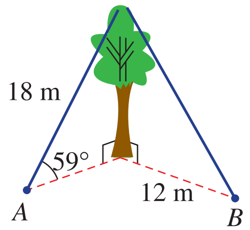
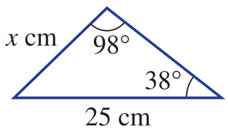
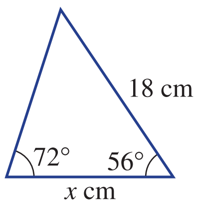
7 A tree is being supported by two ropes, as shown. The rope to point A is 18 m long and makes an angle of 59° with the ground. Point B is 12 m from the base of the tree.



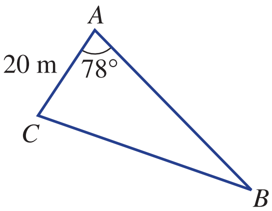
a Find the height of the tree, to 2 decimal places.
b Find the angle the rope to point B makes with the ground, to the nearest degree.

8 For these triangles, find the following, correct to 1 decimal place.
i the value of x
ii the area of the triangle

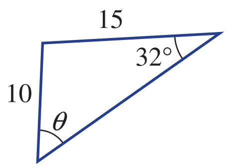
a


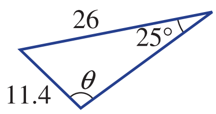
b


9 Three fences are used to form a triangular pig pen with known dimensions as shown in the diagram. If the area of the pig pen is 275 m^2, what is the length AB? Round your answer to 1 decimal place.

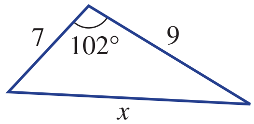


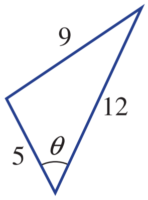
10 Use the sine rule to find the value of θ, correct to 1 decimal place.

a θ is acute


b θ is obtuse


11 Use the cosine rule to find the value of the pronumeral, to 1 decimal place.

a


b


12 a Rewrite the following using their reference angle.
i sin 120° ii cos 210° iii tan 315° iv sin 225°
b Hence, give the exact value of each part in **a**.
c For each of the following, state whether it positive or negative.
i cos 158° ii tan 231° iii sin 333° iv cos 295°

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13 Use the graph of $\sin\theta$ shown to complete the following.

a Estimate the value of:

i $\sin 130°$ **ii** $\sin 255°$

b Find the values of θ between $0°$ and $360°$ such that:

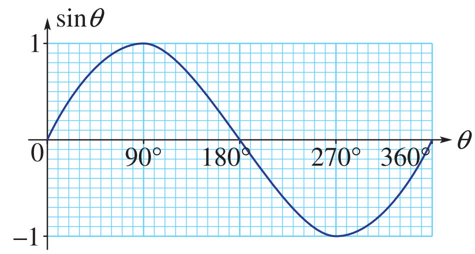
i $\sin\theta = 0.8$ **ii** $\sin\theta = -0.3$

iii $\sin\theta = 1.5$

c State whether the following are true or false.

i $\sin 90° = 1$ **ii** $\sin 75° > \sin 140°$

iii $\sin 220° < \sin 250°$



Extended-response questions

1 A group of friends set out on a hike to a waterfall in a national park. They are given the following directions to walk from the entrance to the waterfall to avoid having to cross a river.

Walk 5 km on a bearing of 325° and then 3 km due north.

Round each answer to 1 decimal place.

a Draw and label a diagram to represent this hike.

b Determine how far east or west the waterfall is from the entrance.

c Find the direct distance from the entrance to the waterfall.

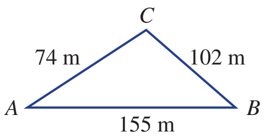
The friends set up tents on level ground at the base of the waterfall at points A (35 m from the base of the waterfall) and B (28 m from the base of the waterfall). The angle of elevation from A to the top of the waterfall is 32°.

d Determine:

i the height of the waterfall

ii the angle of elevation from B to the top of the waterfall

2 A paddock, ABC, is fenced off, as shown in the figure.



a Find $\angle A$, to 3 decimal places.

b Hence, find the area enclosed by the fences. Round your answer to 2 decimal places.

It is planned to divide the paddock into two triangular paddocks by constructing a fence from point C to meet AB at right angles at a point D.

c Determine how many metres of fencing will be required along CD, to the nearest centimetre.

d How far is point D from point A, to the nearest centimetre?

e The person who constructs the fence CD misinterprets the information and builds a fence that does not meet AB at right angles. The fence is 45 metres long.

i Determine, to 2 decimal places, the two possible angles (i.e. acute and obtuse) this fence line makes with AB.

ii Hence, find the two possible distances of fence post D from A. Round your answer to 1 decimal place.

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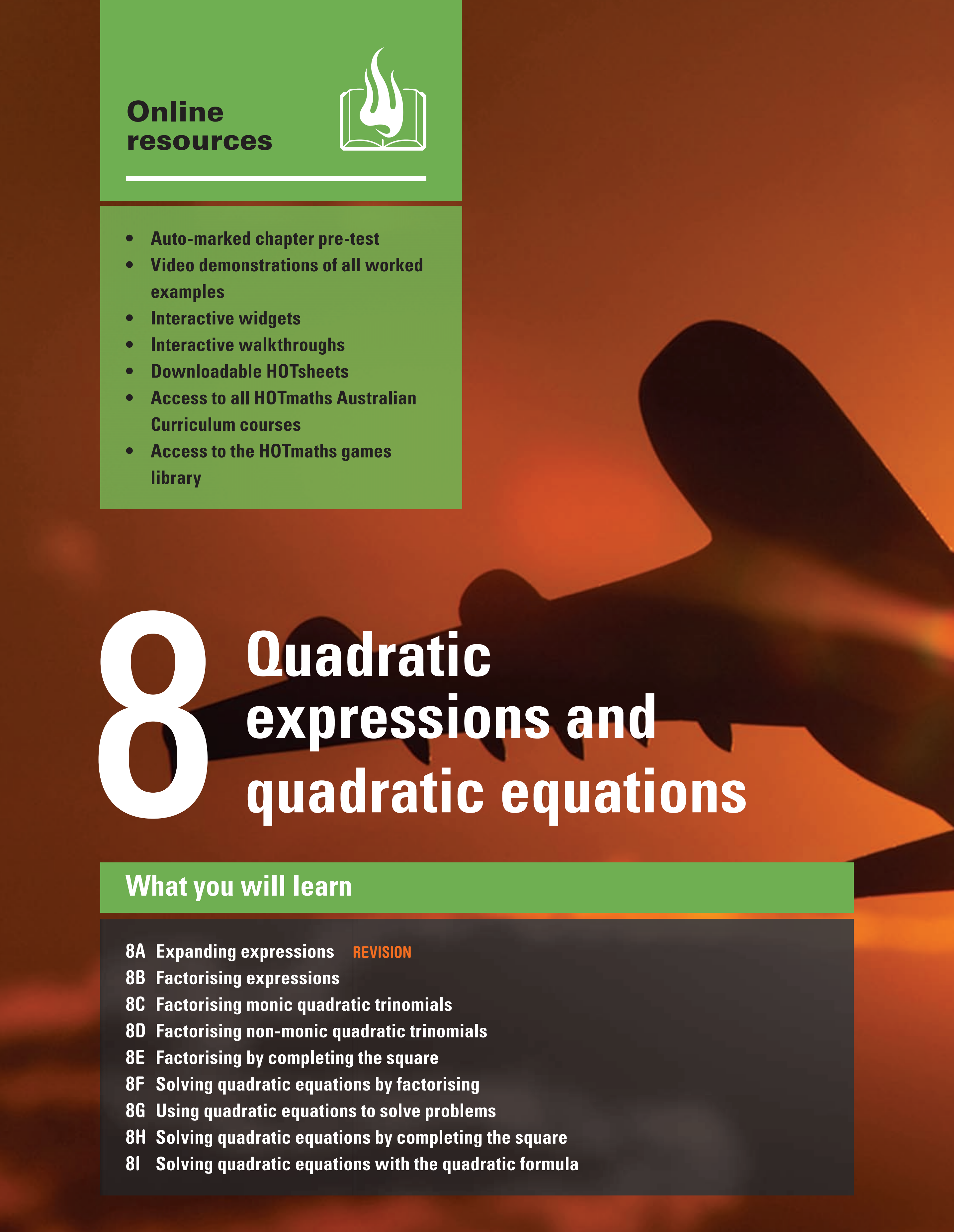

Online resources

- Auto-marked chapter pre-test
- Video demonstrations of all worked examples
- Interactive widgets
- Interactive walkthroughs
- Downloadable HOTsheets
- Access to all HOTmaths Australian Curriculum courses
- Access to the HOTmaths games library

8 Quadratic expressions and quadratic equations

What you will learn

8A Expanding expressions REVISION
8B Factorising expressions
8C Factorising monic quadratic trinomials
8D Factorising non-monic quadratic trinomials
8E Factorising by completing the square
8F Solving quadratic equations by factorising
8G Using quadratic equations to solve problems
8H Solving quadratic equations by completing the square
8I Solving quadratic equations with the quadratic formula

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NSW syllabus

STRAND: NUMBER AND ALGEBRA

SUBSTRANDS: ALGEBRAIC TECHNIQUES

EQUATIONS

Outcomes

A student simplifies algebraic fractions, and expands and factorises quadratic expressions.
(MA5.2–6NA)

A student selects and applies appropriate algebraic techniques to operate with algebraic expressions.
(MA5.3–5NA)

A student solves linear and simple quadratic equations, linear inequalities and linear simultaneous equations, using analytical and graphical techniques.
(MA5.2–8NA)

A student solves complex linear, quadratic, simple cubic and simultaneous equations, and rearranges literal equations.
(MA5.3–7NA)

Mathematics of flight

If you blow air across the top of a small piece of paper it will lift rather than be forced down. Daniel Bernoulli, an 18th century Swiss mathematician and scientist, discovered the relationship between fluid pressure and fluid speed, which is simply represented by this quadratic equation:

$p + \frac{v^2}{2} = c$, where p is internal pressure, v is speed and c is a constant.

As air is a fluid, Bernoulli's law shows us that with increased air speed there is decreased internal air pressure. This explains why, when a cyclonic wind blows across a house roof, the stronger air pressure from inside the house can push the roof off. The aerofoil shape of a bird or plane wing (i.e. concave-down on the top) causes air to flow at a higher speed over the wing's upper surface and, hence, air pressure is decreased. The air of higher pressure under the wing helps to lift the aeroplane or bird. Bernoulli's quadratic equation shows us that wing lift is proportional to the square of the air speed.

Wing loading, which is measured in kg/m^2, is calculated by dividing the fully loaded weight of an aeroplane by the wing area. Aeroplanes with a large wing loading need large wings or a high air speed or both to achieve lift. The gigantic Antonov cargo plane has a wing loading of around 660 kg/m^2 and can lift enormous weights due to its huge wing area of 905 m^2 and air speed of 800 km/h. A hang glider has large wings and a low mass, giving a wing loading of only 6 kg/m^2. Hence it can still obtain lift at very slow speeds.

Pre-test

1 Consider the expression $5 + 2ab - b$.
- **a** How many terms are there?
- **b** What is the coefficient of the second term?
- **c** What is the value of the constant term?

2 Simplify each of the following.
- **a** $7x + 2y - 3x$
- **b** $3xy + 4x - xy - 5x$
- **c** $-x^2 - 3y^2 + 4x^2$

3 Simplify:
- **a** $\frac{4a}{2}$
- **b** $\frac{-24mn}{12n}$
- **c** $6a \times 3a$
- **d** $-2x \times 3xy$
- **e** $x \times (-3) \div (9x)$
- **f** $-4x \times 3x \div (2xy)$

4 Expand and simplify by collecting like terms where possible.
- **a** $4(m + n)$
- **b** $-3(2x - 4)$
- **c** $2x(3x + 1)$
- **d** $4a(1 - 2a)$
- **e** $5 + 3(x - 4)$
- **f** $5 - 2(x + 3) + 2$
- **g** $3(x + 2) + 4(x + 1)$
- **h** $6(3x - 2) - 3(5x + 3)$

5 Expand these products and simplify by collecting like terms.
- **a** $(x + 3)(x - 1)$
- **b** $(x - 3)(x - 7)$
- **c** $(2x - 3)(3x + 4)$
- **d** $(5x + 2)(7x + 1)$
- **e** $(4x - 5)(4x + 5)$
- **f** $(10 - m)(10 + m)$
- **g** $(x + 7)^2$
- **h** $(2x - 3)^2$

6 Factorise each of the following.
- **a** $7x + 7$
- **b** $-9x - 27x^2$
- **c** $a^2 + ab + 3a$
- **d** $x^2 - 9$
- **e** $x^2 + 9x + 20$
- **f** $x^2 + 5x - 14$
- **g** $2x^2 - x - 1$
- **h** $5x^2 - 11x + 2$
- **i** $4x^2 + 4x + 1$

7 Simplify the following.
- **a** $\frac{x(x - 4)}{2x(x - 4)}$
- **b** $\frac{2x^2 + 8x}{4x}$
- **c** $\frac{3x + 9}{4x + 12}$

8 Find $b^2 - 4ac$ when:
- **a** $a = 2$, $b = 3$ and $c = 1$
- **b** $a = 4$, $b = -3$ and $c = 2$
- **c** $a = -1$, $b = 5$ and $c = -4$
- **d** $a = -2$, $b = -3$ and $c = -1$

9 Solve:
- **a** $2x + 1 = 0$
- **b** $2(x - 3) = 0$
- **c** $3(2x + 5) = 0$
- **d** $x(x - 5) = 0$
- **e** $(x + 2)(x - 3) = 0$
- **f** $(2x - 1)(x + 7) = 0$

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8A Expanding expressions REVISION

You will recall that expressions that include numerals and pronumerals are central to the topic of algebra. Sound skills in algebra are essential for solving most mathematical problems and this includes the ability to expand expressions involving brackets. This includes binomial products, perfect squares and the difference of two squares. Exploring how projectiles fly subject to the Earth's gravity, for example, can be modelled with expressions with and without brackets.

The path of this soccer ball could be modelled with the use of algebra.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Five key errors

Here are five expansion problems with incorrect answers. Discuss what error has been made and then find the correct answer.

- $-2(x - 3) = -2x - 6$
- $(x + 3)^2 = x^2 + 9$
- $(x - 2)(x + 2) = x^2 + 4x - 4$
- $5 - 3(x - 1) = 2 - 3x$
- $(x + 3)(x + 5) = x^2 + 8x + 8$

Key ideas

- **Like terms** have the same pronumeral part.
  - They can be collected (i.e. added and subtracted) to form a single term.
    For example: $7x - 11x = -4x$ and $4a^2b - 7ba^2 = -3a^2b$
- The **distributive law** is used to expand brackets.
  - $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$
  - $(a + b)(c + d) = ac + ad + bc + bd$
  - $(a + b)(c + d)$ is called a binomial product because each expression in the brackets has two terms.
- **Perfect squares**
  - $(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$
  - $(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2$
- **Difference of two squares**
  - $(a + b)(a - b) = a^2 - b^2$

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Example 1 Expanding simple expressions

Expand and simplify where possible.

a $-3(x - 5)$

b $2x(1 - x)$

c $\frac{2}{7}(14x + 3)$

d $x(2x - 1) - x(3 - x)$

SOLUTION

a $-3(x - 5) = -3x + 15$

b $2x(1 - x) = 2x - 2x^2$

c $\frac{2}{7}(14x + 3) = \frac{2}{7} \times 14x + \frac{2}{7} \times 3$
$= 4x + \frac{6}{7}$

d $x(2x - 1) - x(3 - x)$
$= 2x^2 - x - 3x + x^2$
$= 3x^2 - 4x$

EXPLANATION

a Use the distributive law $a(b - c) = ab - ac$.
$-3x \times x = -3x$ and $-3 \times -5 = 15$

b Recall that $2x \times (-x) = -2x^2$.

c When multiplying fractions, cancel before multiplying numerators and denominators. Recall that $3 = \frac{3}{1}$.

d Apply the distributive law to each set of brackets first, then simplify by collecting like terms. Recall that $-x \times (-x) = x^2$.

Example 2 Expanding binomial products, perfect squares and difference of two squares

Expand the following.

a $(x + 5)(x + 4)$

b $(x - 4)^2$

c $(2x + 1)(2x - 1)$

SOLUTION

a $(x + 5)(x + 4) = x^2 + 4x + 5x + 20$
$= x^2 + 9x + 20$

b $(x - 4)^2 = (x - 4)(x - 4)$
$= x^2 - 4x - 4x + 16$
$= x^2 - 8x + 16$
or $(x - 4)^2 = x^2 - 2(x)(4) + 4^2$
$= x^2 - 8x + 16$

c $(2x + 1)(2x - 1) = 4x^2 - 2x + 2x - 1$
$= 4x^2 - 1$
or $(2x + 1)(2x - 1) = (2x)^2 - (1)^2$
$= 4x^2 - 1$

EXPLANATION

a For binomial products use
$(a + b)(c + d) = ac + ad + bc + bd$.
Simplify by collecting like terms.

b Rewrite and expand using the distributive law.
Alternatively, for perfect squares use
$(a - b)^2 = a^2 - 2ab + b^2$.
Here, $a = x$ and $b = 4$.

c Expand, recalling that $2x \times 2x = 4x^2$.
Cancel the $-2x$ and $+2x$ terms.
Alternatively, for difference of two squares use
$(a - b)(a + b) = a^2 - b^2$. Here, $a = 2x$ and $b = 1$.

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Example 3 Expanding more binomial products

Expand and simplify.

a $(2x - 1)(3x + 5)$

b $2(x - 3)(x - 2)$

c $(x + 2)(x + 4) - (x - 2)(x - 5)$

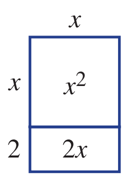
SOLUTION	EXPLANATION
a $(2x - 1)(3x + 5) = 6x^2 + 10x - 3x - 5$ $= 6x^2 + 7x - 5$	Expand using the distributive law and simplify. Note: $2x \times 3x = 2 \times 3 \times x \times x = 6x^2$.
b $2(x - 3)(x - 2) = 2(x^2 - 2x - 3x + 6)$ $= 2(x^2 - 5x + 6)$ $= 2x^2 - 10x + 12$	First expand the brackets using the distributive law, simplify and then multiply each term by 2.
c $(x + 2)(x + 4) - (x - 2)(x - 5)$ $= (x^2 + 6x + 8) - (x^2 - 7x + 10)$ $= x^2 + 6x + 8 - x^2 + 7x - 10$ $= 13x - 2$	Expand each binomial product. Remove brackets before simplifying. $-(x^2 - 7x + 10) = -1 \times x^2 - (-1) \times 7x + (-1) \times 10$ $= -x^2 + 7x - 10$

Exercise 8A REVISION

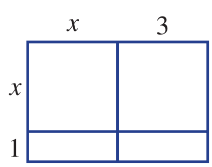
UNDERSTANDING AND FLUENCY	1–3, 4(½), 5, 6(½)	3–7(½)	4–7(½)

1 Use each diagram to help expand the expressions.

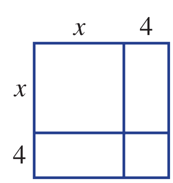
a $x(x + 2)$



b $(x + 3)(x + 1)$



c $(x + 4)^2$



2 Write expressions for the following.

a the sum of a^2 and ab

b 5 less than the product of 2 and x

c the difference between a^2 and b^2, where $a^2 > b^2$

d the square of the sum of y and x

e half of the difference between a and b, where $a > b$

3 Simplify these expressions.

a $2 \times 3x$

b $-4 \times 5x$

c $x \times 2x$

d $-x \times 4x$

e $5x \div 10$

f $3x \div 9$

g $-4x^2 \div x$

h $-6x^2 \div (2x)$

i $3x - 21x$

j $12x - 5x$

k $-3x + 8x$

l $-5x - 8x$

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Example 1a–c

4 Expand and simplify where possible.

a $2(x+5)$	**b** $3(x-4)$	**c** $-5(x+3)$	**d** $-4(x-2)$
e $3(2x-1)$	**f** $4(3x+1)$	**g** $-2(5x-3)$	**h** $-5(4x+3)$
i $x(2x+5)$	**j** $x(3x-1)$	**k** $2x(1-x)$	**l** $3x(2-x)$
m $-2x(3x+2)$	**n** $-3x(6x-2)$	**o** $-5x(2-2x)$	**p** $-4x(1-4x)$
q $\frac{2}{5}(10x+4)$	**r** $\frac{3}{4}(8x-5)$	**s** $\frac{-1}{3}(6x+1)$	**t** $\frac{-1}{2}(4x-3)$
u $\frac{-3}{8}(24x-1)$	**v** $\frac{-2}{9}(9x+7)$	**w** $\frac{3x}{4}(3x+8)$	**x** $\frac{2x}{5}(7-3x)$

Example 1d

5 Expand and simplify.

a $x(3x-1)+x(4-x)$
b $x(5x+2)+x(x-5)$
c $x(4x-3)-2x(x-5)$
d $3x(2x+4)-x(5-2x)$
e $4x(2x-1)+2x(1-3x)$
f $2x(2-3x)-3x(2x-7)$

Example 2

6 Expand the following.

a $(x+2)(x+8)$	**b** $(x+3)(x+4)$	**c** $(x+7)(x+5)$
d $(x+8)(x-3)$	**e** $(x+6)(x-5)$	**f** $(x-2)(x+3)$
g $(x-7)(x+3)$	**h** $(x-4)(x-6)$	**i** $(x-8)(x-5)$
j $(x+5)^2$	**k** $(x+7)^2$	**l** $(x+6)^2$
m $(x-3)^2$	**n** $(x-8)^2$	**o** $(x-10)^2$
p $(x+4)(x-4)$	**q** $(x+9)(x-9)$	**r** $(2x-3)(2x+3)$
s $(3x+4)(3x-4)$	**t** $(4x-5)(4x+5)$	**u** $(8x-7)(8x+7)$

Example 3a

7 Expand the following using the distributive law.

a $(2x+1)(3x+5)$	**b** $(4x+5)(3x+2)$	**c** $(5x+3)(2x+7)$
d $(3x+2)(3x-5)$	**e** $(5x+3)(4x-2)$	**f** $(2x+5)(3x-5)$
g $(4x-5)(4x+5)$	**h** $(2x-9)(2x+9)$	**i** $(5x-7)(5x+7)$
j $(7x-3)(2x-4)$	**k** $(5x-3)(5x-6)$	**l** $(7x-2)(8x-2)$
m $(2x+5)^2$	**n** $(5x+6)^2$	**o** $(7x-1)^2$

PROBLEM-SOLVING AND REASONING	8, 9, 12	9–13	10(½), 11–15

8 Write the missing number.

a $(x+\square)(x+2)=x^2+5x+6$	**b** $(x+\square)(x+5)=x^2+8x+15$
c $(x+7)(x-\square)=x^2+4x-21$	**d** $(x+4)(x-\square)=x^2-4x-32$
e $(x-6)(x-\square)=x^2-7x+6$	**f** $(x-\square)(x-8)=x^2-10x+16$

Example 3b

9 Expand the following.

a $2(x+3)(x+4)$	**b** $3(x+2)(x+7)$	**c** $-2(x+8)(x+2)$
d $-4(x+9)(x+2)$	**e** $5(x-3)(x+4)$	**f** $3(x+5)(x-3)$
g $-3(a+2)(a-7)$	**h** $-5(a+2)(a-8)$	**i** $4(a-3)(a-6)$
j $3(y-4)(y-5)$	**k** $-2(y-3)(y-8)$	**l** $-6(y-4)(y-3)$
m $3(2x+3)(2x+5)$	**n** $6(3x-4)(x+2)$	**o** $-2(x+4)(3x-7)$
p $2(x+3)^2$	**q** $4(m+5)^2$	**r** $2(a-7)^2$
s $-3(y-5)^2$	**t** $3(2b-1)^2$	**u** $-3(2y-6)^2$

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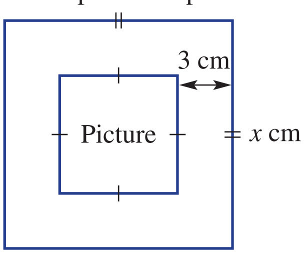
Example 3c

10 Expand and simplify the following.

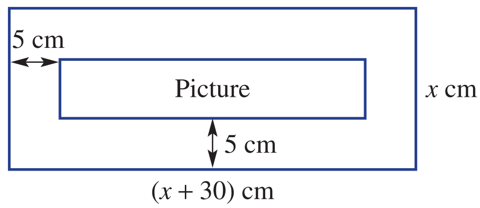
- **a** $(x + 4)(x + 3) + x - 4$
- **b** $(x - 1)(x + 5) + 5(x - 1)$
- **c** $3x^2 + (2x - 1)(x - 2)$
- **d** $(x + 1)(x + 3) + (x + 2)(x + 4)$
- **e** $(x + 8)(x + 3) + (x + 4)(x + 5)$
- **f** $(y + 3)(y - 1) + (y - 2)(y - 4)$
- **g** $(y - 7)(y + 4) + (y + 5)(y - 3)$
- **h** $(1 - x)(2 + 3x) - 2x(3x + 2)$
- **i** $(x - 5)(x + 5) + (x + 5)(x + 5)$
- **j** $3x + 9 - (2x - 1)(x + 1)$
- **k** $(2a + 3)(a - 5) - (a + 6)(2a + 5)$
- **l** $(4b + 8)(b + 5) - (3b - 5)(b - 7)$
- **m** $(x - 3)(x - 4) - (x + 2)(x - 1)$
- **n** $x^2 + 2x - 3 + (x + 2)(2 - x)$
- **o** $(x + 5)^2 - 7$
- **p** $(x - 7)^2 - 9$
- **q** $3 - (2x - 9)^2$
- **r** $14 - (5x + 3)^2$

11 Find an expanded expression for the area of the pictures centred in these frames.

a



b



12 Prove the following by expanding the left-hand side.

- **a** $(a + b)(a - b) = a^2 - b^2$
- **b** $(a + b)^2 = a^2 + 2ab + b^2$
- **c** $(a - b)^2 = a^2 - 2ab + b^2$
- **d** $(a + b)^2 - (a - b)^2 = 4ab$

13 Use the distributive law to evaluate the following without the use of a calculator.
For example: $4 \times 102 = 4 \times 100 + 4 \times 2 = 408$.

- **a** 6×103
- **b** 4×55
- **c** 9×63
- **d** 8×208
- **e** 7×198
- **f** 3×297
- **g** 8×495
- **h** 5×696

14 Each problem below has an incorrect answer. Find the error and give the correct answer.

- **a** $-x(x - 7) = -x^2 - 7x$
- **b** $3a - 7(4 - a) = -4a - 28$
- **c** $(2x + 3)^2 = 4x^2 + 9$
- **d** $(x + 2)^2 - (x + 2)(x - 2) = 0$

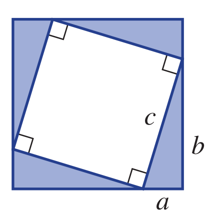
15 Expand these cubic expressions.

- **a** $(x + 2)(x + 3)(x + 1)$
- **b** $(x + 4)(x + 2)(x + 5)$
- **c** $(x + 3)(x - 4)(x + 3)$
- **d** $(x - 4)(2x + 1)(x - 3)$
- **e** $(x + 6)(2x - 3)(x - 5)$
- **f** $(2x - 3)(x - 4)(3x - 1)$

ENRICHMENT	–	–	16

Expanding to prove

16 One of the ways to prove Pythagoras' theorem is to arrange four congruent right-angled triangles around a square to form a larger square, as shown.



- **a** Find an expression for the total area of the four shaded triangles by multiplying the area of one triangle by 4.
- **b** Find an expression for the area of the four shaded triangles by subtracting the area of the inner square from the area of the outer square.
- **c** By combining your results from parts **a** and **b**, expand and simplify to prove Pythagoras' theorem: $a^2 + b^2 = c^2$.

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8B Factorising expressions

Interactive
Widgets
HOTsheets
Walkthrough

A common and key step in the simplification and solution of equations involves factorisation. Factorisation is the process of writing a number or expression as a product of its factors.

In this section we look at expressions in which each term has a common factor, expressions that are a difference of two squares and four-term expressions, which can be factorised by grouping.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: But there are no common factors!

An expression such as $xy + 4x + 3y + 12$ has no common factors across all four terms, but it can still be factorised. The method of grouping can be used.

- Complete this working to show how to factorise the expression.

$$\begin{aligned} xy + 4x + 3y + 12 &= x(\_\_\_\_) + 3(\_\_\_\_) \\ &= (\_\_\_\_)(x + 3) \end{aligned}$$

- Now repeat but rearrange the expression.

$$\begin{aligned} xy + 3y + 4x + 12 &= y(\_\_\_\_) + 4(\_\_\_\_) \\ &= (\_\_\_\_)(\_\_\_\_) \end{aligned}$$

- Are the two results equivalent? How can you prove this?

Key ideas

- **Factorise** expressions with common factors by 'taking out' the common factors.
  For example: $-5x - 20 = -5(x + 4)$ and $4x^2 - 8x = 4x(x - 2)$
- Factorise a difference of two squares using $a^2 - b^2 = (a + b)(a - b)$.
  - We use surds when a^2 or b^2 is not a perfect square, such as $1, 4, 9, \ldots$
    For example: $x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5})$, using $(\sqrt{5})^2 = 5$.
- Factorise four-term expressions, if possible, by **grouping** terms and factorising each pair.
  For example: $$\begin{aligned} x^2 + 5x - 2x - 10 &= x(x + 5) - 2(x + 5) \\ &= (x + 5)(x - 2) \end{aligned}$$

Example 4 Taking out common factors

Factorise by taking out common factors.

a $-3x - 12$ **b** $20a^2 + 30a$ **c** $2(x + 1) - a(x + 1)$

SOLUTION

a $-3x - 12 = -3(x + 4)$

b $20a^2 + 30a = 10a(2a + 3)$

c $2(x + 1) - a(x + 1) = (x + 1)(2 - a)$

EXPLANATION

-3 is common to both $-3x$ and -12.

The HCF of $20a^2$ and $30a$ is $10a$.

$(x + 1)$ is a common factor to both parts of the expression.

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Example 5 Factorising a difference of two squares

Factorise the following difference of two squares. You may need to look for a common factor first.

a $x^2 - 16$

b $9a^2 - 4b^2$

c $12y^2 - 1200$

d $(x + 3)^2 - 4$

SOLUTION

a $x^2 - 16 = (x)^2 - (4)^2$
$= (x + 4)(x - 4)$

b $9a^2 - 4b^2 = (3a)^2 - (2b)^2$
$= (3a + 2b)(3a - 2b)$

c $12y^2 - 1200 = 12(y^2 - 100)$
$= 12(y + 10)(y - 10)$

d $(x + 3)^2 - 4 = (x + 3)^2 - (2)^2$
$= (x + 3 + 2)(x + 3 - 2)$
$= (x + 5)(x + 1)$

EXPLANATION

Use $a^2 - b^2 = (a + b)(a - b)$ with $a = x$ and $b = 4$.

$9a^2 = (3a)^2$ and $4b^2 = (2b)^2$.

First, take out the common factor of 12.
$121 = (11)^2$; use $a^2 - b^2 = (a + b)(a - b)$.

Use $a^2 - b^2 = (a + b)(a - b)$ with $a = x + 3$ and $b = 2$. Simplify.

Example 6 Factorising a difference of two squares using surds

Factorise these difference of two squares using surds.

a $x^2 - 10$

b $x^2 - 24$

c $(x - 1)^2 - 5$

SOLUTION

a $x^2 - 10 = (x + \sqrt{10})(x - \sqrt{10})$

b $x^2 - 24 = (x + \sqrt{24})(x - \sqrt{24})$
$= (x + 2\sqrt{6})(x - 2\sqrt{6})$

c $(x - 1)^2 - 5 = (x - 1)^2 - (\sqrt{5})^2$
$= (x - 1 + \sqrt{5})(x - 1 - \sqrt{5})$

EXPLANATION

Recall that $(\sqrt{10})^2 = 10$.

Use $(\sqrt{24})^2 = 24$ and simplify so that $\sqrt{24} = \sqrt{4 \times 6} = 2\sqrt{6}$.

Use $a^2 - b^2 = (a + b)(a - b)$ with $a = x - 1$ and $b = \sqrt{5}$.

Example 7 Factorisation by grouping

Factorise by grouping $x^2 - x + ax - a$.

SOLUTION

$x^2 - x + ax - a = x(x - 1) + a(x - 1)$
$= (x - 1)(x + a)$

EXPLANATION

Factorise two pairs of terms and then take out the common binomial factor, in this case $(x - 1)$.

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Exercise 8B

UNDERSTANDING AND FLUENCY	1–6(½)	2–7(½)	3–7(½)

1 Determine the highest common factor of these pairs of terms.

a $7x$ and 14	**b** $12x$ and 30	**c** $-8y$ and 40	**d** $-5y$ and -25
e $4a^2$ and $2a$	**f** $12a^2$ and $9a$	**g** $-5a^2$ and $-50a$	**h** $-3x^2y$ and $-6xy$

Example 4a, b

2 Factorise by taking out the common factors.

a $3x - 18$	**b** $4x + 20$	**c** $7a + 7b$	**d** $9a - 15$
e $-5x - 30$	**f** $-4y - 2$	**g** $-12a - 3$	**h** $-2ab - bc$
i $4x^2 + x$	**j** $5x^2 - 2x$	**k** $6b^2 - 18b$	**l** $14a^2 - 21a$
m $10a - 5a^2$	**n** $12x - 30x^2$	**o** $-2x - x^2$	**p** $-4y - 8y^2$
q $ab^2 - a^2b$	**r** $2x^2yz - 4xy$	**s** $-12m^2n - 12mn^2$	**t** $6xyz^2 - 3z^2$

Example 4c

3 Factorise.

a $5(x - 1) - a(x - 1)$
b $b(x + 2) + 3(x + 2)$
c $a(x + 5) - 4(x + 5)$
d $x(x + 2) + 5(x + 2)$
e $x(x - 4) - 2(x - 4)$
f $3(x + 1) - x(x + 1)$
g $a(x + 3) + (x + 3)$
h $x(x - 2) - (x - 2)$
i $(x - 6) - x(x - 6)$

Example 5a, b

4 Factorise the following difference of two squares.

a $x^2 - 9$	**b** $x^2 - 25$	**c** $y^2 - 49$	**d** $y^2 - 1$
e $4x^2 - 9$	**f** $36a^2 - 25$	**g** $1 - 81y^2$	**h** $100 - 9x^2$
i $25x^2 - 4y^2$	**j** $64x^2 - 25y^2$	**k** $9a^2 - 49b^2$	**l** $144a^2 - 49b^2$

Example 5c, d

5 Factorise the following.

a $2x^2 - 32$	**b** $5x^2 - 45$	**c** $6y^2 - 24$	**d** $3y^2 - 48$
e $3x^2 - 75y^2$	**f** $3a^2 - 300b^2$	**g** $12x^2 - 27y^2$	**h** $63a^2 - 112b^2$
i $(x + 5)^2 - 16$	**j** $(x - 4)^2 - 9$	**k** $(a - 3)^2 - 64$	**l** $(a - 7)^2 - 1$
m $(3x + 5)^2 - x^2$	**n** $(2y + 7)^2 - y^2$	**o** $(5x + 11)^2 - 4x^2$	**p** $(3x - 5y)^2 - 25y^2$

Example 6

6 Factorise using surds.

a $x^2 - 7$	**b** $x^2 - 5$	**c** $x^2 - 19$	**d** $x^2 - 21$
e $x^2 - 14$	**f** $x^2 - 30$	**g** $x^2 - 15$	**h** $x^2 - 11$
i $x^2 - 8$	**j** $x^2 - 18$	**k** $x^2 - 45$	**l** $x^2 - 20$
m $x^2 - 32$	**n** $x^2 - 48$	**o** $x^2 - 50$	**p** $x^2 - 200$
q $(x + 2)^2 - 6$	**r** $(x + 5)^2 - 10$	**s** $(x - 3)^2 - 11$	**t** $(x - 1)^2 - 7$
u $(x - 6)^2 - 15$	**v** $(x + 4)^2 - 21$	**w** $(x + 1)^2 - 19$	**x** $(x - 7)^2 - 26$

Example 7

7 Factorise by grouping.

a $x^2 + 4x + ax + 4a$	**b** $x^2 + 7x + bx + 7b$	**c** $x^2 - 3x + ax - 3a$
d $x^2 + 2x - ax - 2a$	**e** $x^2 + 5x - bx - 5b$	**f** $x^2 + 3x - 4ax - 12a$
g $x^2 - ax - 4x + 4a$	**h** $x^2 - 2bx - 5x + 10b$	**i** $3x^2 - 6ax - 7x + 14a$

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PROBLEM-SOLVING AND REASONING	8(½), 11	8–9(½), 11(½), 12	8–11(½), 13, 14

8 Factorise fully and simplify surds.

a $x^2 - \frac{2}{9}$ b $x^2 - \frac{3}{4}$ c $x^2 - \frac{7}{16}$ d $x^2 - \frac{5}{36}$

e $(x-2)^2 - 20$ f $(x+4)^2 - 27$ g $(x+1)^2 - 75$ h $(x-7)^2 - 40$

i $3x^2 - 4$ j $5x^2 - 9$ k $7x^2 - 5$ l $6x^2 - 11$

m $-9 + 2x^2$ n $-16 + 5x^2$ o $-10 + 3x^2$ p $-7 + 13x^2$

9 Factorise by first rearranging.

a $xy - 6 - 3x + 2y$ b $ax - 12 + 3a - 4x$ c $ax - 10 + 5x - 2a$

d $xy + 12 - 3y - 4x$ e $2ax + 3 - a - 6x$ f $2ax - 20 + 8a - 5x$

10 Factorise fully.

a $5x^2 - 120$ b $3x^2 - 162$ c $7x^2 - 126$ d $2x^2 - 96$

e $2(x+3)^2 - 10$ f $3(x-1)^2 - 21$ g $4(x-4)^2 - 48$ h $5(x+6)^2 - 90$

11 Evaluate the following without the use of a calculator by first factorising.

a $16^2 - 14^2$ b $18^2 - 17^2$ c $13^2 - 10^2$ d $15^2 - 11^2$

e $17^2 - 15^2$ f $11^2 - 9^2$ g $27^2 - 24^2$ h $52^2 - 38^2$

12 a Show that $4 - (x+2)^2 = -x(x+4)$ by factorising the left-hand side.

b Now factorise these:

i $9 - (x+3)^2$ ii $16 - (x+4)^2$ iii $25 - (x-5)^2$

iv $25 - (x+2)^2$ v $49 - (x-1)^2$ vi $100 - (x+4)^2$

13 a Prove that, in general, $(x+a)^2 \neq x^2 + a^2$.

b Are there any values of x for which $(x+a)^2 = x^2 + a^2$? If so, what are they?

14 Show that $x^2 - \frac{4}{9} = \frac{1}{9}(3x+2)(3x-2)$, using two different methods.

ENRICHMENT	–	–	15, 16

Hidden difference of two squares

15 Factorise and simplify the following without initially expanding the brackets.

a $(x+2)^2 - (x+3)^2$ b $(y-7)^2 - (y+4)^2$

c $(a+3)^2 - (a-5)^2$ d $(b+5)^2 - (b-5)^2$

e $(s-3)^2 - (s+3)^2$ f $(y-7)^2 - (y+7)^2$

g $(2w+3x)^2 - (3w+4x)^2$ h $(d+5e)^2 - (3d-2e)^2$

i $(4f+3j)^2 - (2f-3j)^2$ j $(3r-2p)^2 - (2p-3r)^2$

16 a Is it possible to factorise $x^2 + 5y - y^2 + 5x$? Can you show how?

b Also try factorising:

i $x^2 + 7x + 7y - y^2$

ii $x^2 - 2x - 2y - y^2$

iii $4x^2 + 4x + 6y - 9y^2$

iv $25y^2 + 15y - 4x^2 + 6x$

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8C Factorising monic quadratic trinomials

Interactive
Widgets
HOTsheets
Walkthrough

Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

A quadratic trinomial of the form $x^2 + bx + c$ is called a monic quadratic because the coefficient of x^2 is 1.

Now consider:

$$\begin{aligned}(x + m)(x + n) &= x^2 + xn + mx + mn \\ &= x^2 + (m + n)x + mn\end{aligned}$$

We can see from this expansion that mn gives the constant term (c) and $m + n$ is the coefficient of x. This tells us that to factorise a monic quadratic trinomial, we should look for factors of the constant term (c) that add to give the coefficient of the middle term (b).

Let's start: Factorising $x^2 - 6x - 72$

Discuss what is wrong with each of these statements when trying to factorise $x^2 - 6x - 72$.

- Find factors of 72 that add to 6.
- Find factors of 72 that add to −6.
- Find factors of −72 that add to 6.
- $-18 \times 4 = -72$ so $x^2 - 6x - 72 = (x - 18)(x + 4)$
- $-9 \times 8 = -72$ so $x^2 - 6x - 72 = (x - 9)(x + 8)$

Can you write a correct statement that correctly factorises $x^2 - 6x - 72$?

Key ideas

- In **monic quadratics** the coefficient of x^2 is 1.
- Monic quadratics of the form $x^2 + bx + c$ can be factorised by finding the two numbers that multiply to give the constant term (c) and add to give the coefficient of x (b).

$$x^2 + (\underbrace{m + n}_{b})x + \underbrace{mn}_{c} = (x + m)(x + n)$$

Example 8 Factorising trinomials of the form $x^2 + bx + c$

Factorise:

a $x^2 + 8x + 15$ **b** $x^2 - 5x + 6$ **c** $2x^2 - 10x - 28$ **d** $x^2 - 8x + 16$

SOLUTION	EXPLANATION
a $x^2 + 8x + 15 = (x + 3)(x + 5)$	$3 \times 5 = 15$ and $3 + 5 = 8$ Check: $(x + 3)(x + 5) = x^2 + 5x + 3x + 15 = x^2 + 8x + 15$
b $x^2 - 5x + 6 = (x - 3)(x - 2)$	$-3 \times (-2) = 6$ and $-3 + (-2) = -5$ Check: $(x - 3)(x - 2) = x^2 - 2x - 3x + 6 = x^2 - 5x + 6$
c $2x^2 - 10x - 28 = 2(x^2 - 5x - 14)$ $= 2(x - 7)(x + 2)$	First, take out the common factor of 2. $-7 \times 2 = -14$ and $-7 + 2 = -5$
d $x^2 - 8x + 16 = (x - 4)(x - 4)$ $= (x - 4)^2$	$-4 \times (-4) = 16$ and $-4 + (-4) = -8$ $(x - 4)(x - 4) = (x - 4)^2$ is a perfect square.

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Example 9 Simplifying algebraic fractions

Use factorisation to simplify these algebraic fractions.

a $\dfrac{x^2 - x - 6}{x + 2}$ **b** $\dfrac{x^2 - 9}{x^2 - 2x - 15} \times \dfrac{x^2 - 4x - 5}{2x - 6}$

SOLUTION

a $\dfrac{x^2 - x - 6}{x + 2} = \dfrac{(x - 3)\cancel{(x + 2)}^1}{\cancel{(x + 2)}_1}$
$= x - 3$

EXPLANATION

First factorise $x^2 - x - 6$, then cancel $(x + 2)$.

SOLUTION

b $\dfrac{x^2 - 9}{x^2 - 2x - 15} \times \dfrac{x^2 - 4x - 5}{2x - 6}$
$= \dfrac{\cancel{(x + 3)}^1\cancel{(x - 3)}^1}{\cancel{(x - 5)}^1\cancel{(x + 3)}^1} \times \dfrac{\cancel{(x - 5)}^1(x + 1)}{2\cancel{(x - 3)}^1}$
$= \dfrac{x + 1}{2}$

EXPLANATION

First, factorise all expressions in the numerators and denominators. Cancel to simplify where possible.

Exercise 8C

UNDERSTANDING AND FLUENCY	1, 2, 3–6(½)	3–7(½)	4–7(½)

1 Find two integers that multiply to give the first number and add to give the second number.

a 18, 11 **b** 20, 12 **c** −15, 2 **d** −12, 1
e −24, −5 **f** −30, −7 **g** 10, −7 **h** 36, −15

2 A number divided by itself always equals 1.

For example: $\dfrac{^1\cancel{a}}{^1\cancel{a}} = 1$, $\dfrac{2\cancel{(x - 3)}^1}{\cancel{(x - 3)}^1} = 2 \times 1 = 2$, $\dfrac{\cancel{(a + 5)}^1}{2\cancel{(a + 5)}^1} = \dfrac{1}{2 \times 1} = \dfrac{1}{2}$

Invent some algebraic fractions that are equal to:

a 1 **b** 3 **c** −5 **d** $\dfrac{1}{3}$

3 Simplify by cancelling common factors. For parts **i** to **l**, first factorise the numerator.

a $\dfrac{2x}{4}$ **b** $\dfrac{7x}{21}$ **c** $\dfrac{6a}{2a}$ **d** $\dfrac{4a}{20a}$
e $\dfrac{3(x + 1)}{9(x + 1)}$ **f** $\dfrac{2(x - 2)}{8(x - 2)}$ **g** $\dfrac{15(x - 5)}{3(x - 5)}$ **h** $\dfrac{8(x + 4)}{12(x + 4)}$
i $\dfrac{x^2 + x}{x}$ **j** $\dfrac{x^2 - 2x}{x}$ **k** $\dfrac{x^2 - 3x}{2x}$ **l** $\dfrac{2x - 4x^2}{2x}$

Example 8a, b

4 Factorise these quadratic trinomials.

a $x^2 + 7x + 6$ **b** $x^2 + 5x + 6$ **c** $x^2 + 6x + 9$ **d** $x^2 + 7x + 10$
e $x^2 + 7x + 12$ **f** $x^2 + 11x + 18$ **g** $x^2 + 5x - 6$ **h** $x^2 + x - 6$
i $x^2 + 2x - 8$ **j** $x^2 + 3x - 4$ **k** $x^2 + 7x - 30$ **l** $x^2 + 9x - 22$
m $x^2 - 7x + 10$ **n** $x^2 - 6x + 8$ **o** $x^2 - 7x + 12$ **p** $x^2 - 2x + 1$
q $x^2 - 9x + 18$ **r** $x^2 - 11x + 18$ **s** $x^2 - 4x - 12$ **t** $x^2 - x - 20$
u $x^2 - 5x - 14$ **v** $x^2 - x - 12$ **w** $x^2 + 4x - 32$ **x** $x^2 - 3x - 10$

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Example 8c

5 Factorise by first taking out the common factor.

a $2x^2 + 14x + 20$ **b** $3x^2 + 21x + 36$ **c** $2x^2 + 22x + 36$ **d** $5x^2 - 5x - 10$
e $4x^2 - 16x - 20$ **f** $3x^2 - 9x - 30$ **g** $-2x^2 - 14x - 24$ **h** $-3x^2 + 9x - 6$
i $-2x^2 + 10x + 28$ **j** $-4x^2 + 4x + 8$ **k** $-5x^2 - 20x - 15$ **l** $-7x^2 + 49x - 42$

Example 8d

6 Factorise these perfect squares.

a $x^2 - 4x + 4$ **b** $x^2 + 6x + 9$ **c** $x^2 + 12x + 36$ **d** $x^2 - 14x + 49$
e $x^2 - 18x + 81$ **f** $x^2 - 20x + 100$ **g** $2x^2 + 44x + 242$ **h** $3x^2 - 24x + 48$
i $5x^2 - 50x + 125$ **j** $-3x^2 + 36x - 108$ **k** $-2x^2 + 28x - 98$ **l** $-4x^2 - 72x - 324$

Example 9a

7 Use factorisation to simplify these algebraic fractions.

a $\dfrac{x^2 - 3x - 54}{x - 9}$ **b** $\dfrac{x^2 + x - 12}{x + 4}$ **c** $\dfrac{x^2 - 6x + 9}{x - 3}$

d $\dfrac{x + 2}{x^2 + 9x + 14}$ **e** $\dfrac{x - 3}{x^2 - 8x + 15}$ **f** $\dfrac{x + 1}{x^2 - 5x - 6}$

g $\dfrac{2(x + 12)}{x^2 + 4x - 96}$ **h** $\dfrac{x^2 - 5x - 36}{3(x - 9)}$ **i** $\dfrac{x^2 - 15x + 56}{5(x - 8)}$

PROBLEM-SOLVING AND REASONING | 8(½), 11 | 8–9(½), 11, 12 | 8–10(½), 12–14

Example 9b

8 Simplify by factorising.

a $\dfrac{x^2 - 4}{x^2 + x - 6} \times \dfrac{5x - 15}{x^2 + 4x - 12}$ **b** $\dfrac{x^2 + 3x + 2}{x^2 + 4x + 3} \times \dfrac{x^2 - 9}{3x + 6}$ **c** $\dfrac{x^2 + 2x - 3}{x^2 - 25} \times \dfrac{2x - 10}{x + 3}$

d $\dfrac{x^2 - 9}{x^2 - 5x + 6} \times \dfrac{4x - 8}{x^2 + 8x + 15}$ **e** $\dfrac{x^2 - 4x + 3}{x^2 + 4x - 21} \times \dfrac{4x + 4}{x^2 - 1}$ **f** $\dfrac{x^2 + 6x + 8}{x^2 - 4} \times \dfrac{6x - 24}{x^2 - 16}$

g $\dfrac{x^2 - x - 6}{x^2 + x - 12} \times \dfrac{x^2 + 5x + 4}{x^2 - 1}$ **h** $\dfrac{x^2 - 4x - 12}{x^2 - 4} \times \dfrac{x^2 - 6x + 8}{x^2 - 36}$

9 Simplify these expressions that involve surds.

a $\dfrac{x^2 - 7}{x + \sqrt{7}}$ **b** $\dfrac{x^2 - 10}{x - \sqrt{10}}$ **c** $\dfrac{x^2 - 12}{x + 2\sqrt{3}}$

d $\dfrac{\sqrt{5}x + 3}{5x^2 - 9}$ **e** $\dfrac{\sqrt{3}x - 4}{3x^2 - 16}$ **f** $\dfrac{7x^2 - 5}{\sqrt{7}x + \sqrt{5}}$

g $\dfrac{(x + 1)^2 - 2}{x + 1 + \sqrt{2}}$ **h** $\dfrac{(x - 3)^2 - 5}{x - 3 - \sqrt{5}}$ **i** $\dfrac{(x - 6)^2 - 6}{x - 6 + \sqrt{6}}$

10 Simplify by factorising.

a $\dfrac{x^2 + 2x - 3}{x^2 - 25} \div \dfrac{3x - 3}{2x + 10}$ **b** $\dfrac{x^2 + 3x + 2}{x^2 + 4x + 3} \div \dfrac{4x + 8}{x^2 - 9}$ **c** $\dfrac{x^2 - x - 12}{x^2 - 9} \div \dfrac{x^2 - 16}{3x + 12}$

d $\dfrac{x^2 - 49}{x^2 - 3x - 28} \div \dfrac{4x + 28}{6x + 24}$ **e** $\dfrac{x^2 + 5x - 14}{x^2 + 2x - 3} \div \dfrac{x^2 + 9x + 14}{x^2 + x - 2}$ **f** $\dfrac{x^2 + 8x + 15}{x^2 + 5x - 6} \div \dfrac{x^2 + 6x + 5}{x^2 + 7x + 6}$

11 A business analyst is showing off their new formula to determine the company's profit in millions of dollars after t years.

$$\text{Profit} = \frac{t^2 - 49}{5t - 40} \times \frac{t^2 - 5t - 24}{2t^2 - 8t - 42}$$

Show that this is really the same as Profit $= \dfrac{t + 7}{10}$.

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12 Note that an expression with a perfect square can be simplified as shown.

$$\frac{(x+3)^2}{x+3} = \frac{(x+3)\cancel{(x+3)}^1}{\cancel{x+3}_1} = x+3$$

Use this idea to simplify the following.

a $\frac{x^2-6x+9}{x-3}$ **b** $\frac{x^2+2x+1}{x+1}$ **c** $\frac{x^2-16x+64}{x-8}$

d $\frac{6x-12}{x^2-4x+4}$ **e** $\frac{4x+20}{x^2+10x+25}$ **f** $\frac{x^2-14x+49}{5x-35}$

13 **a** Prove that $\frac{a^2+2ab+b^2}{a^2+ab} \div \frac{a^2-b^2}{a^2-ab} = 1$.

b Make up your own expressions which equal 1 like that in part **a**. Ask a friend to check them.

14 Simplify:

a $\frac{a^2+2ab+b^2}{a(a+b)} \div \frac{a^2-b^2}{a^2-2ab+b^2}$ **b** $\frac{a^2-2ab+b^2}{a^2-b^2} \div \frac{a^2-b^2}{a^2+2ab+b^2}$

c $\frac{a^2-b^2}{a^2-2ab+b^2} \div \frac{a^2-b^2}{a^2+2ab+b^2}$ **d** $\frac{a^2+2ab+b^2}{a(a+b)} \div \frac{a(a-b)}{a^2-2ab+b^2}$

ENRICHMENT	–	–	15

Addition and subtraction with factorisation

15 Factorisation can be used to help add and subtract algebraic fractions. Here is an example.

$$\begin{aligned}\frac{3}{x-2} + \frac{x}{x^2-6x+8} &= \frac{3}{x-2} + \frac{x}{(x-2)(x-4)} \\ &= \frac{3(x-4)}{(x-2)(x-4)} + \frac{x}{(x-2)(x-4)} \\ &= \frac{3x-12+x}{(x-2)(x-4)} \\ &= \frac{4x-12}{(x-2)(x-4)} \\ &= \frac{4(x-3)}{(x-2)(x-4)}\end{aligned}$$

Now simplify the following.

a $\frac{2}{x+3} + \frac{x}{x^2-x-12}$ **b** $\frac{4}{x+2} + \frac{3x}{x^2-7x-18}$

c $\frac{3}{x+4} - \frac{2x}{x^2-16}$ **d** $\frac{4}{x^2-9} - \frac{1}{x^2-8x+15}$

e $\frac{x+4}{x^2-x-6} - \frac{x-5}{x^2-9x+18}$ **f** $\frac{x+3}{x^2-4x-32} - \frac{x}{x^2+7x+12}$

g $\frac{x+1}{x^2-25} - \frac{x-2}{x^2-6x+5}$ **h** $\frac{x+2}{x^2-2x+1} - \frac{x+3}{x^2+3x-4}$

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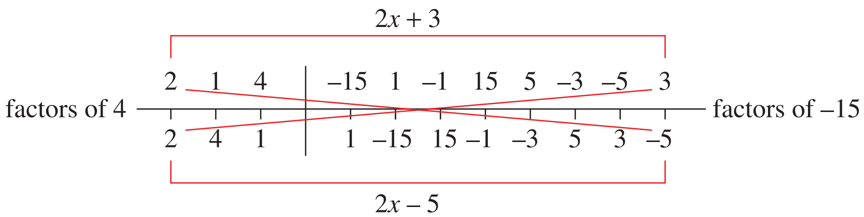
8D Factorising non-monic quadratic trinomials

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

There are a number of ways of factorising non-monic quadratic trinomials of the form $ax^2 + bx + c$, where $a \neq 1$. The cross method, for example, uses lists of factors of a and c so that a correct combination can be found. For example, to factorise $4x^2 - 4x - 15$:



$2 \times (-5) + 2 \times 3 = -4$, so choose $(2x + 3)$ and $(2x - 5)$.
$4x^2 - 4x - 15 = (2x + 3)(2x - 5)$

The method outlined in this section, however, uses grouping. Unlike the method above, grouping is more direct and involves less trial and error.

Let's start: Does the order matter?

To factorise the non-monic quadratic $4x^2 - 4x - 15$ using grouping, we multiply a by c, which is $4 \times (-15) = -60$. Then we look for numbers that multiply to give -60 and add to give -4 (the coefficient of x).

- What are the two numbers that multiply to give -60 and add to give -4?
- Complete the following using grouping.

$$\begin{aligned} 4x^2 - 4x - 15 &= 4x^2 - 10x + 6x - 15 \\ &= 2x(\_\_\_\_) + 3(\_\_\_\_) \\ &= (2x - 5)(\_\_\_\_) \end{aligned}$$

- If we changed the order of the $-10x$ and $+6x$, do you think the result would change? Copy and complete to find out.

$$\begin{aligned} 4x^2 - 4x - 15 &= 4x^2 + 6x - 10x - 15 \\ &= 2x(\_\_\_\_) - 5(\_\_\_\_) \\ &= (\_\_\_\_)(\_\_\_\_) \end{aligned}$$

Key ideas

- One method that can be used to factorise a **non-monic** trinomial of the form $ax^2 + bx + c$:
  - Find two numbers that multiply to give $a \times c$ and add to give b.
    For $15x^2 - x - 6$, $a \times c = 15 \times (-6) = -90$.
    The factors of -90 that add to -1 (b) are -10 and 9, so split the middle term and then factorise by grouping.

$$\begin{aligned} 15x^2 - x - 6 &= 15x^2 - 10x + 9x - 6 \\ &= 5x(3x - 2) + 3(3x - 2) \\ &= (3x - 2)(5x + 3) \end{aligned}$$

  - Note that this is the same as $(5x + 3)(3x - 2)$.

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Example 10 Factorising non-monic quadratics

Factorise:

a $6x^2 + 19x + 10$

b $9x^2 + 6x - 8$

SOLUTION

a $6x^2 + 19x + 10 = 6x^2 + 15x + 4x + 10$
$= 3x(2x + 5) + 2(2x + 5)$
$= (2x + 5)(3x + 2)$

b $9x^2 + 6x - 8 = 9x^2 + 12x - 6x - 8$
$= 3x(3x + 4) - 2(3x + 4)$
$= (3x + 4)(3x - 2)$

EXPLANATION

$a \times c = 6 \times 10 = 60$, choose 15 and 4 since $15 \times 4 = 60$ and $15 + 4 = 19$.
Factorise by grouping.

$a \times c = 9 \times (-8) = -72$, choose 12 and -6 since $12 \times (-6) = -72$ and $12 + (-6) = 6$.

Example 11 Simplifying algebraic fractions

Simplify $\dfrac{4x^2 - 9}{10x^2 + 13x - 3} \times \dfrac{25x^2 - 10x + 1}{10x^2 - 17x + 3}$.

SOLUTION

$$\frac{4x^2 - 9}{10x^2 + 13x - 3} \times \frac{25x^2 - 10x + 1}{10x^2 - 17x + 3}$$
$$= \frac{\cancel{(2x + 3)}^1\cancel{(2x - 3)}^1}{\cancel{(2x + 3)}^1\cancel{(5x - 1)}^1} \times \frac{\cancel{(5x - 1)}^1\cancel{(5x - 1)}^1}{\cancel{(2x - 3)}^1\cancel{(5x - 1)}^1}$$
$$= 1$$

EXPLANATION

First, factorise all quadratics.
Cancel to simplify.

Exercise 8D

UNDERSTANDING AND FLUENCY	1, 2–3(½)	2–4(½)	3–4(½)

1 Complete this table.

$ax^2 + bx + c$	$a \times c$	Two numbers that multiply to give $a \times c$ and add to give b	Factorisation
$6x^2 + 13x + 6$	36	9 and ____	
$8x^2 + 18x + 4$	32		
$12x^2 + x - 6$		-8 and ____	
$10x^2 - 11x - 6$			
$21x^2 - 20x + 4$		-6 and ____	
$15x^2 - 13x + 2$			

2 Factorise by grouping pairs.

a $x^2 + 2x + 5x + 10$
b $x^2 + 4x + 6x + 24$
c $x^2 + 3x + 7x + 21$
d $x^2 - 7x - 2x + 14$
e $x^2 - 3x - 4x + 12$
f $x^2 - 5x + 3x - 15$
g $6x^2 - 8x + 3x - 4$
h $3x^2 - 12x + 2x - 8$
i $8x^2 - 4x + 6x - 3$
j $5x^2 + 20x - 2x - 8$
k $10x^2 + 12x - 15x - 18$
l $12x^2 - 6x - 10x + 5$

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Example 10

3 Factorise the following.

a $3x^2 + 10x + 3$
b $2x^2 + 3x + 1$
c $3x^2 + 8x + 4$
d $3x^2 - 5x + 2$
e $2x^2 - 11x + 5$
f $5x^2 + 2x - 3$
g $3x^2 - 11x - 4$
h $3x^2 - 2x - 1$
i $7x^2 + 2x - 5$
j $2x^2 - 9x + 7$
k $3x^2 + 2x - 8$
l $2x^2 + 5x - 12$
m $2x^2 - 9x - 5$
n $13x^2 - 7x - 6$
o $5x^2 - 22x + 8$
p $8x^2 - 14x + 5$
q $6x^2 + x - 12$
r $10x^2 + 11x - 6$
s $6x^2 + 13x + 6$
t $4x^2 - 5x + 1$
u $8x^2 - 14x + 5$
v $8x^2 - 26x + 15$
w $6x^2 - 13x + 6$
x $9x^2 + 9x - 10$

4 Factorise the following.

a $18x^2 + 27x + 10$
b $20x^2 + 39x + 18$
c $21x^2 + 22x - 8$
d $30x^2 + 13x - 10$
e $40x^2 - x - 6$
f $28x^2 - 13x - 6$
g $24x^2 - 38x + 15$
h $45x^2 - 46x + 8$
i $25x^2 - 50x + 16$

PROBLEM-SOLVING AND REASONING	5(½), 7, 8(½)	5–6(½), 7, 8–9(½)	5–6(½), 7, 8–9(½), 10

5 Factorise by first taking out the common factor.

a $6x^2 + 38x + 40$
b $6x^2 - 15x - 36$
c $48x^2 - 18x - 3$
d $32x^2 - 88x + 60$
e $16x^2 - 24x + 8$
f $90x^2 + 90x - 100$
g $-50x^2 - 115x - 60$
h $12x^2 - 36x + 27$
i $20x^2 - 25x + 5$

6 Simplify by first factorising.

a $\dfrac{6x^2 - x - 35}{3x + 7}$
b $\dfrac{8x^2 + 10x - 3}{2x + 3}$
c $\dfrac{9x^2 - 21x + 10}{3x - 5}$
d $\dfrac{10x - 2}{15x^2 + 7x - 2}$
e $\dfrac{4x + 6}{14x^2 + 17x - 6}$
f $\dfrac{20x - 12}{10x^2 - 21x + 9}$
g $\dfrac{2x^2 + 11x + 12}{6x^2 + 11x + 3}$
h $\dfrac{12x^2 - x - 1}{8x^2 + 14x + 3}$
i $\dfrac{10x^2 + 3x - 4}{14x^2 - 11x + 2}$
j $\dfrac{9x^2 - 4}{15x^2 + 4x - 4}$
k $\dfrac{14x^2 + 19x - 3}{49x^2 - 1}$
l $\dfrac{8x^2 - 2x - 15}{16x^2 - 25}$

7 A cable is suspended across a farm channel. The height (h), in metres, of the cable above the water surface is modelled by the equation $h = 3x^2 - 21x + 30$, where x metres is the distance from one side of the channel.

a Factorise the right-hand side of the equation.
b Determine the height of the cable when $x = 3$. Interpret this result.
c Determine where the cable is at the level of the water surface.

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Example 11

8 Combine all your knowledge of factorising to simplify the following.

a $\dfrac{9x^2 - 16}{x^2 - 6x + 9} \times \dfrac{x^2 + x - 12}{3x^2 + 8x - 16}$

b $\dfrac{4x^2 - 1}{6x^2 - x - 2} \times \dfrac{9x^2 - 4}{8x - 4}$

c $\dfrac{1 - x^2}{15x + 9} \times \dfrac{25x^2 + 30x + 9}{5x^2 + 8x + 3}$

d $\dfrac{20x^2 + 21x - 5}{16x^2 + 8x - 15} \times \dfrac{16x^2 - 24x + 9}{25x^2 - 1}$

e $\dfrac{100x^2 - 25}{2x^2 - 9x - 5} \div \dfrac{5x^2 + 10x - 75}{2x^2 - 5x - 3}$

f $\dfrac{3x^2 - 12}{30x + 15} \div \dfrac{2x^2 - 3x - 2}{4x^2 + 4x + 1}$

g $\dfrac{9x^2 - 6x + 1}{6x^2 - 11x + 3} \div \dfrac{9x^2 - 1}{6x^2 - 7x - 3}$

h $\dfrac{16x^2 - 25}{4x^2 - 7x - 15} \div \dfrac{4x^2 - 17x + 15}{16x^2 - 40x + 25}$

9 Find a method to show how $-12x^2 - 5x + 3$ factorises to $(1 - 3x)(4x + 3)$. Then factorise the following.

a $-8x^2 + 2x + 15$

b $-6x^2 + 11x + 10$

c $-12x^2 + 13x + 4$

d $-8x^2 + 18x - 9$

e $-14x^2 + 39x - 10$

f $-15x^2 - x + 6$

10 Make up your own complex expression, like those in Question 8, that simplifies to 1. Check your expression with your teacher or a classmate.

ENRICHMENT	–	–	11

Non-monics with addition and subtraction

11 Factorise the quadratics in the expressions, then simplify using a common denominator.

a $\dfrac{2}{2x - 3} + \dfrac{x}{8x^2 - 10x - 3}$

b $\dfrac{3}{3x - 1} - \dfrac{x}{6x^2 + 13x - 5}$

c $\dfrac{4x}{2x - 5} + \dfrac{x}{8x^2 - 18x - 5}$

d $\dfrac{4x}{12x^2 - 11x + 2} - \dfrac{3x}{3x - 2}$

e $\dfrac{2}{4x^2 - 1} + \dfrac{1}{6x^2 - x - 2}$

f $\dfrac{2}{9x^2 - 25} - \dfrac{3}{9x^2 + 9x - 10}$

g $\dfrac{4}{8x^2 - 18x - 5} - \dfrac{2}{12x^2 - 5x - 2}$

h $\dfrac{1}{10x^2 - 19x + 6} + \dfrac{2}{4x^2 + 8x - 21}$

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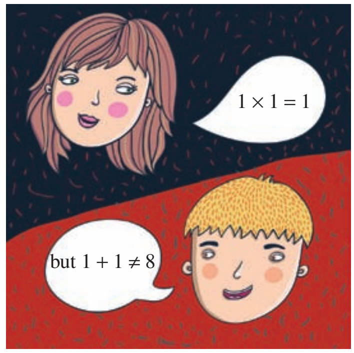
8E Factorising by completing the square

Interactive
Widgets
HOTsheets
Walkthrough

Consider the quadratic expression $x^2 + 8x + 1$. We cannot factorise this using the methods we have established in the previous exercises because there are no factors of 1 that add to 8.

We can, however, use our knowledge of perfect squares and the difference of two squares to help find factors using surds.

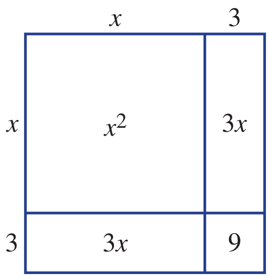


Stage

- 5.3#
- 5.3
- 5.3§
- 5.2
- 5.2◊
- 5.1
- 4

Let's start: Make a perfect square

This diagram is a square. Its sides are $x + 3$ and its area is given by $x^2 + 6x + 9 = (x + 3)^2$.



Use a similar diagram to help make a perfect square for the following and determine the missing number for each.

- $x^2 + 8x + ?$
- $x^2 + 12x + ?$

Can you describe a method for finding the missing number without drawing a diagram?

Key ideas

- To **complete the square** for $x^2 + bx$, add $\left(\frac{b}{2}\right)^2$.
  - $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$
- To factorise by completing the square:
  - Add $\left(\frac{b}{2}\right)^2$ and balance by subtracting $\left(\frac{b}{2}\right)^2$.
  - Factorise the perfect square and simplify.
  - Factorise using difference of two squares: $a^2 - b^2 = (a + b)(a - b)$; surds can be used.

$$\begin{aligned} x^2 + 8x + 1 &= x^2 + 8x + \left(\frac{8}{2}\right)^2 - \left(\frac{8}{2}\right)^2 + 1 \\ &= (x^2 + 8x + 16) - 16 + 1 \\ &= (x + 4)^2 - 15 \\ &= (x + 4)^2 - (\sqrt{15})^2 \\ &= (x + 4 + \sqrt{15})(x + 4 - \sqrt{15}) \end{aligned}$$

- Not all quadratic expressions can be factorised. This will be seen when you end up with expressions such as $(x + 3)^2 + 6$, which is *not* a difference of two squares.

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Example 12 Completing the square

Decide what number must be added to these expressions to complete the square. Then factorise the resulting perfect square.

a $x^2 + 10x$

b $x^2 - 7x$

SOLUTION	EXPLANATION
a $\left(\frac{10}{2}\right)^2 = 5^2 = 25$	$b = 10$ and evaluate $\left(\frac{b}{2}\right)^2$.
$x^2 + 10x + 25 = (x + 5)^2$	$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$
b $\left(\frac{-7}{2}\right)^2 = \frac{49}{4}$	In $x^2 - 7x$, $b = -7$ and evaluate $\left(\frac{b}{2}\right)^2$.
$x^2 - 7x + \frac{49}{4} = \left(x - \frac{7}{2}\right)^2$	Factorise the perfect square.

Example 13 Factorising by completing the square

Factorise the following by completing the square, if possible.

a $x^2 - 8x - 3$

b $x^2 + 2x + 8$

SOLUTION	EXPLANATION
a $\begin{aligned} x^2 - 8x - 3 &= x^2 - 8x + (-4)^2 - (-4)^2 - 3 \\ &= (x^2 - 8x + 16) - 16 - 3 \\ &= (x - 4)^2 - 19 \\ &= (x - 4)^2 - (\sqrt{19})^2 \end{aligned}$	Add $\left(\frac{-8}{2}\right)^2 = (-4)^2$ to complete the square and balance by subtracting $(-4)^2$ also. Factorise the perfect square and simplify.
$= (x - 4 - \sqrt{19})(x - 4 + \sqrt{19})$	Apply $a^2 - b^2 = (a + b)(a - b)$ using surds.
b $\begin{aligned} x^2 + 2x + 8 &= x^2 + 2x + 1^2 - 1^2 + 8 \\ &= (x^2 + 2x + 1) - 1 + 8 \\ &= (x + 1)^2 + 7 \end{aligned}$	Add $\left(\frac{2}{2}\right)^2 = (1)^2$ to complete the square and balance by subtracting $(1)^2$ also. Factorise the perfect square and simplify.
$\therefore x^2 + 2x + 8$ cannot be factorised.	This is not a difference of two squares.

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Example 14 Factorising with fractions

Factorise $x^2 + 3x + \frac{1}{2}$.

SOLUTION

$$\begin{aligned} x^2 + 3x + \frac{1}{2} &= x^2 + 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + \frac{1}{2} \\ &= \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{1}{2} \\ &= \left(x + \frac{3}{2}\right)^2 - \frac{7}{4} \\ &= \left(x + \frac{3}{2}\right)^2 - \left(\sqrt{\frac{7}{4}}\right)^2 \\ &= \left(x + \frac{3}{2} - \frac{\sqrt{7}}{2}\right)\left(x + \frac{3}{2} + \frac{\sqrt{7}}{2}\right) \\ &= \left(x + \frac{3 - \sqrt{7}}{2}\right)\left(x + \frac{3 + \sqrt{7}}{2}\right) \end{aligned}$$

EXPLANATION

Add $\left(\frac{3}{2}\right)^2$ to complete the square and balance by subtracting $\left(\frac{3}{2}\right)^2$. Leave in fraction form; i.e. $\frac{3}{2} \times \frac{3}{2} = \frac{9}{4}$.

Factorise the perfect square and simplify.

Recall that $\sqrt{\frac{7}{4}} = \frac{\sqrt{7}}{\sqrt{4}} = \frac{\sqrt{7}}{2}$.

Exercise 8E

UNDERSTANDING AND FLUENCY	1–5(½)	3–6(½)	4–6(½)

1 These expressions are of the form $x^2 + bx$. Evaluate $\left(\frac{b}{2}\right)^2$ for each one.

- **a** $x^2 + 6x$
- **b** $x^2 + 12x$
- **c** $x^2 + 2x$
- **d** $x^2 - 4x$
- **e** $x^2 - 8x$
- **f** $x^2 - 10x$
- **g** $x^2 + 5x$
- **h** $x^2 + 3x$
- **i** $x^2 - 9x$

2 Factorise these perfect squares.

- **a** $x^2 + 4x + 4$
- **b** $x^2 + 8x + 16$
- **c** $x^2 + 10x + 25$
- **d** $x^2 - 12x + 36$
- **e** $x^2 - 6x + 9$
- **f** $x^2 - 18x + 81$

3 Factorise using surds. Recall that $a^2 - b^2 = (a + b)(a - b)$.

- **a** $(x + 1)^2 - 5$
- **b** $(x + 2)^2 - 7$
- **c** $(x + 4)^2 - 10$
- **d** $(x - 3)^2 - 11$
- **e** $(x - 6)^2 - 22$
- **f** $(x - 5)^2 - 3$

Example 12

4 Decide what number must be added to these expressions to complete the square. Then factorise the resulting perfect square.

- **a** $x^2 + 6x$
- **b** $x^2 + 12x$
- **c** $x^2 + 4x$
- **d** $x^2 + 8x$
- **e** $x^2 - 10x$
- **f** $x^2 - 2x$
- **g** $x^2 - 8x$
- **h** $x^2 - 12x$
- **i** $x^2 + 5x$
- **j** $x^2 + 9x$
- **k** $x^2 + 7x$
- **l** $x^2 + 11x$
- **m** $x^2 - 3x$
- **n** $x^2 - 7x$
- **o** $x^2 - x$
- **p** $x^2 - 9x$

Example 13a

5 Factorise by completing the square.

- **a** $x^2 + 4x + 1$
- **b** $x^2 + 6x + 2$
- **c** $x^2 + 2x - 4$
- **d** $x^2 + 10x - 4$
- **e** $x^2 - 8x + 13$
- **f** $x^2 - 12x + 10$
- **g** $x^2 - 4x - 3$
- **h** $x^2 - 8x - 5$

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Example 13b

6 Factorise, if possible.

a $x^2 + 6x + 11$ **b** $x^2 + 4x + 7$ **c** $x^2 + 8x + 1$ **d** $x^2 + 4x + 2$
e $x^2 + 10x + 3$ **f** $x^2 + 4x - 6$ **g** $x^2 - 10x + 30$ **h** $x^2 - 6x + 6$
i $x^2 - 12x + 2$ **j** $x^2 - 2x + 2$ **k** $x^2 - 8x - 1$ **l** $x^2 - 4x + 6$

PROBLEM-SOLVING AND REASONING	7(½), 10	7–8(½), 10	7–9(½), 10, 11

Example 14

7 Factorise the following.

a $x^2 + 3x + 1$ **b** $x^2 + 7x + 2$ **c** $x^2 + 5x - 2$ **d** $x^2 + 9x - 3$
e $x^2 - 3x + \frac{1}{2}$ **f** $x^2 - 5x + \frac{1}{2}$ **g** $x^2 - 5x - \frac{3}{2}$ **h** $x^2 - 9x - \frac{5}{2}$

8 Factorise by first taking out the common factor.

a $2x^2 + 12x + 8$ **b** $3x^2 + 12x - 3$ **c** $4x^2 - 8x - 16$
d $3x^2 - 24x + 6$ **e** $-2x^2 - 4x + 10$ **f** $-3x^2 - 30x - 3$
g $-4x^2 - 16x + 12$ **h** $-2x^2 + 16x + 4$ **i** $-3x^2 + 24x - 15$

9 Factorise by first taking out the coefficient of x^2.

a $3x^2 + 9x + 3$ **b** $5x^2 + 15x - 35$ **c** $2x^2 - 10x + 4$
d $4x^2 - 28x + 12$ **e** $-3x^2 - 21x + 6$ **f** $-2x^2 - 14x + 8$
g $-4x^2 + 12x + 20$ **h** $-3x^2 + 9x + 6$ **i** $-2x^2 + 10x + 8$

10 A student factorises $x^2 - 2x - 24$ by completing the square.

a Show the student's working to obtain the factorised form of $x^2 - 2x - 24$.
b Now that you have seen the answer from part **a**, what would you suggest is a better way to factorise $x^2 - 2x - 24$?

11 **a** Explain why $x^2 + 9$ cannot be factorised using real numbers.
b Decide whether the following can or cannot be factorised.

i $x^2 - 25$ **ii** $x^2 - 10$ **iii** $x^2 + 6$ **iv** $x^2 + 11$
v $(x + 1)^2 + 4$ **vi** $(x - 2)^2 - 8$ **vii** $(x + 3)^2 - 15$ **viii** $(2x - 1)^2 + 1$

c For what values of m can the following be factorised, using real numbers?

i $x^2 + 4x + m$ **ii** $x^2 - 6x + m$ **iii** $x^2 - 10x + m$

ENRICHMENT	–	–	12

Non-monic quadratics and completing the square

12 A non-monic quadratic such as $2x^2 - 5x + 1$ can be factorised in the following way.

$$\begin{aligned}
2x^2 - 5x + 1 &= 2\left(x^2 - \frac{5}{2}x + \frac{1}{2}\right) \\
&= 2\left(x^2 - \frac{5}{2}x + \frac{25}{16} - \frac{25}{16} + \frac{1}{2}\right) \\
&= 2\left(\left(x - \frac{5}{4}\right)^2 - \frac{25}{16} + \frac{8}{16}\right) \\
&= 2\left(\left(x - \frac{5}{4}\right)^2 - \frac{17}{16}\right) \\
&= 2\left(x - \frac{5}{4} + \frac{\sqrt{17}}{4}\right)\left(x - \frac{5}{4} - \frac{\sqrt{17}}{4}\right)
\end{aligned}$$

Factorise these using a similar technique.

a $2x^2 + 5x - 12$ **b** $3x^2 + 4x - 3$ **c** $4x^2 - 7x - 16$ **d** $3x^2 - 2x + 6$
e $-2x^2 - 3x + 4$ **f** $-3x^2 - 7x - 3$ **g** $-4x^2 + 11x - 24$ **h** $-2x^2 + 3x + 4$
i $2x^2 + 5x - 7$ **j** $3x^2 + 4x - 5$ **k** $-2x^2 - 3x + 5$ **l** $-3x^2 - 7x - 4$

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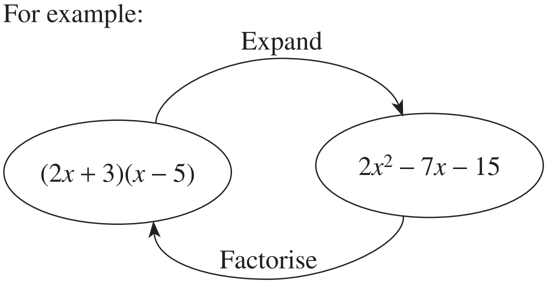
8F Solving quadratic equations by factorising

In the previous sections, **quadratic expressions** have been expanded and factorised.

For example:



Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

In the following sections, factorisation will be used as one method for solving **quadratic equations**. We will also solve them by completing the square and using a formula.

Let's start: Does factorisation beat trial and error?

Set up two teams.

Team A: Trial and error

Team B: Factorisation

Instructions:

- Team A must try to find the two solutions of $3x^2 - x - 2 = 0$ by guessing and checking for values of x that make the equation true.
- Team B must solve the same equation $3x^2 - x - 2 = 0$ by first factorising the left-hand side.

Which team is the first to find the two solutions for x? Discuss the methods used.

Key ideas

- **Quadratic expressions** can be written in the form $ax^2 + bx + c$.
  For example: $(2x + 3)(x - 5)$ can be written as $2x^2 - 7x - 15$.
- **Quadratic equations** can be written in the form $ax^2 + bx + c = 0$.
  For example: $2x^2 - 7x - 15 = 0$ is a quadratic equation. Factorisation can be used to write this equation as $(2x + 3)(x - 5) = 0$. This equation can then be solved using the null factor law. Quadratic equation can have zero, one or two real solutions.
- The **null factor law** states that if the product of two numbers is zero, then either or both of the two numbers is zero.
  - If $p \times q = 0$, then either $p = 0$ or $q = 0$.
- To solve a quadratic equation, write it in standard form (i.e. $ax^2 + bx + c = 0$) and factorise. Then use the null factor law.
  - If the coefficients of all the terms have a common factor, then first divide by that common factor.

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Example 15 Solving quadratic equations using the null factor law

Solve the following quadratic equations.

a $x^2 - 2x = 0$ **b** $x^2 - 15 = 0$ **c** $2x^2 = 50$

SOLUTION	EXPLANATION
a $x^2 - 2x = 0$	Factorise by taking out the common factor x. Apply
$x(x - 2) = 0$	the null factor law: if $a \times b = 0$, then $a = 0$ or $b = 0$.
$\therefore x = 0$ or $x - 2 = 0$	Solve for x.
$\therefore x = 0$ or $x = 2$	Check your solutions using substitution.
b $x^2 - 15 = 0$	Factorise $a^2 - b^2 = (a - b)(a + b)$ using surds.
$(x + \sqrt{15})(x - \sqrt{15}) = 0$	Apply the null factor law and solve for x.
$\therefore x + \sqrt{15} = 0$ or $x - \sqrt{15} = 0$	
$\therefore x = -\sqrt{15}$ or $x = \sqrt{15}$	Check your solutions using substitution.
c $2x^2 = 50$	First, write in standard form (i.e. $ax^2 + bx + c = 0$).
$2x^2 - 50 = 0$	Take out the common factor of 2, then factorise using
$2(x^2 - 25) = 0$	$a^2 - b^2 = (a + b)(a - b)$.
$2(x + 5)(x - 5) = 0$	Solve for x using the null factor law.
$\therefore x + 5 = 0$ or $x - 5 = 0$	
$\therefore x = -5$ or $x = 5$	Check your solutions using substitution.

Example 16 Solving $ax^2 + bx + c = 0$

Solve the following quadratic equations.

a $x^2 - 5x + 6 = 0$ **b** $x^2 + 2x + 1 = 0$ **c** $10x^2 - 13x - 3 = 0$

SOLUTION	EXPLANATION
a $x^2 - 5x + 6 = 0$	Factorise by finding two numbers that multiply to 6
$(x - 3)(x - 2) = 0$	and add to -5: $-3 \times (-2) = 6$ and $-3 + (-2) = -5$.
$\therefore x - 3 = 0$ or $x - 2 = 0$	Apply the null factor law and solve for x.
$\therefore x = 3$ or $x = 2$	Check your solutions using substitution.
b $x^2 + 2x + 1 = 0$	$1 \times 1 = 1$ and $1 + 1 = 2$
$(x + 1)(x + 1) = 0$	$(x + 1)(x + 1) = (x + 1)^2$ is a perfect square.
$(x + 1)^2 = 0$	This gives one solution for x.
$\therefore x + 1 = 0$	
$\therefore x = -1$	Check your solution using substitution.
c $10x^2 - 13x - 3 = 0$	First, factorise using grouping or another method.
$10x^2 - 15x + 2x - 3 = 0$	$10 \times (-3) = -30$, $-15 \times 2 = -30$ and
$5x(2x - 3) + (2x - 3) = 0$	$-15 + 2 = -13$.
$(2x - 3)(5x + 1) = 0$	Solve using the null factor law.
$\therefore 2x - 3 = 0$ or $5x + 1 = 0$	
$\therefore 2x = 3$ or $5x = -1$	
$\therefore x = \frac{3}{2}$ or $x = -\frac{1}{5}$	Check your solutions using substitution.

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Example 17 Solving harder quadratic equations

Solve the following by first writing in the form $ax^2 + bx + c = 0$.

a $x^2 = 4(x + 15)$

b $\dfrac{x + 6}{x} = x$

SOLUTION

a
$$x^2 = 4(x + 15)$$
$$x^2 = 4x + 60$$
$$x^2 - 4x - 60 = 0$$
$$(x - 10)(x + 6) = 0$$
$$\therefore x - 10 = 0 \text{ or } x + 6 = 0$$
$$\therefore x = 10 \text{ or } x = -6$$

EXPLANATION

First, expand and then write in standard form by subtracting $4x$ and 60 from both sides.

Factorise and apply the null factor law.

$-10 \times 6 = -60$ and $-10 + 6 = -4$.

Check your solutions.

b
$$\frac{x + 6}{x} = x$$
$$x + 6 = x^2$$
$$0 = x^2 - x - 6$$
$$0 = (x - 3)(x + 2)$$
$$\therefore x - 3 = 0 \text{ or } x + 2 = 0$$
$$\therefore x = 3 \text{ or } x = -2$$

First, multiply both sides by x and write in standard form.

Factorise and solve using the null factor law.

Check your solutions.

Exercise 8F

UNDERSTANDING AND FLUENCY	1–5(½)	3–6(½)	4–6(½)

1 Write the solutions to these equations, which are already in factorised form.

a $x(x + 1) = 0$ **b** $x(x - 5) = 0$ **c** $2x(x - 4) = 0$

d $(x - 3)(x + 2) = 0$ **e** $(x + 5)(x - 4) = 0$ **f** $(x + 1)(x - 1) = 0$

g $(x + \sqrt{3})(x - \sqrt{3}) = 0$ **h** $(x + \sqrt{5})(x - \sqrt{5}) = 0$ **i** $(x + 2\sqrt{2})(x - 2\sqrt{2}) = 0$

j $(2x - 1)(3x + 7) = 0$ **k** $(4x - 5)(5x + 2) = 0$ **l** $(8x + 3)(4x + 3) = 0$

2 Rearrange to write the following in standard form $ax^2 + bx + c = 0$. Do not solve.

a $x^2 + 2x = 3$ **b** $x^2 - 3x = 10$ **c** $x^2 - 5x = -6$

d $5x^2 = 2x + 7$ **e** $3x^2 = 14x - 8$ **f** $4x^2 = 3 - 4x$

g $x(x + 1) = 4$ **h** $2x(x - 3) = 5$ **i** $x^2 = 4(x - 3)$

j $2 = x(x - 3)$ **k** $-4 = x(3x + 2)$ **l** $x^2 = 3(2 - x)$

3 How many different solutions for x will these equations have?

a $(x - 2)(x - 1) = 0$

b $(x + 7)(x + 3) = 0$

c $(x + 1)(x + 1) = 0$

d $(x - 3)(x - 3) = 0$

e $(x + \sqrt{2})(x - \sqrt{2}) = 0$

f $(x + 8)(x - \sqrt{5}) = 0$

g $(x + 2)^2 = 0$

h $(x + 3)^2 = 0$

i $3(2x + 1)^2 = 0$

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Example 15

4 Solve the following quadratic equations.

a $x^2 - 4x = 0$	b $x^2 - 3x = 0$	c $x^2 + 2x = 0$
d $3x^2 - 12x = 0$	e $2x^2 - 10x = 0$	f $4x^2 + 8x = 0$
g $x^2 - 7 = 0$	h $x^2 - 11 = 0$	i $3x^2 - 15 = 0$
j $x^2 = 2x$	k $x^2 = -5x$	l $7x^2 = -x$
m $5x^2 = 20$	n $3x^2 = 27$	o $2x^2 = 72$

Example 16a, b

5 Solve the following quadratic equations.

a $x^2 + 3x + 2 = 0$	b $x^2 + 5x + 6 = 0$	c $x^2 - 6x + 8 = 0$
d $x^2 - 7x + 10 = 0$	e $x^2 + 4x - 12 = 0$	f $x^2 + 2x - 15 = 0$
g $x^2 - x - 20 = 0$	h $x^2 - 5x - 24 = 0$	i $x^2 - 12x + 32 = 0$
j $x^2 + 4x + 4 = 0$	k $x^2 + 10x + 25 = 0$	l $x^2 - 8x + 16 = 0$
m $x^2 - 14x + 49 = 0$	n $x^2 - 24x + 144 = 0$	o $x^2 + 18x + 81 = 0$

Example 16c

6 Solve the following quadratic equations.

a $2x^2 + 11x + 12 = 0$	b $4x^2 + 16x + 7 = 0$
c $2x^2 - 17x + 35 = 0$	d $2x^2 - 23x + 11 = 0$
e $3x^2 - 4x - 15 = 0$	f $5x^2 - 7x - 6 = 0$
g $6x^2 + 7x - 20 = 0$	h $7x^2 + 25x - 12 = 0$

PROBLEM-SOLVING AND REASONING	8(½), 10	7–8(½), 10, 11	7–9(½), 11–13

7 Solve by first taking out a common factor.

a $2x^2 + 16x + 24 = 0$
b $2x^2 - 20x - 22 = 0$
c $3x^2 - 18x + 27 = 0$
d $5x^2 - 20x + 20 = 0$
e $-8x^2 - 4x + 24 = 0$
f $18x^2 - 57x + 30 = 0$

Example 17a

8 Solve the following by first writing in the form $ax^2 + bx + c = 0$.

a $x^2 = 2(x + 12)$	b $x^2 = 4(x + 8)$	c $x^2 = 3(2x - 3)$
d $x^2 + 7x = -10$	e $x^2 - 8x = -15$	f $x(x + 4) = 4x + 9$
g $2x - 16 = x(2 - x)$	h $x^2 + 12x + 10 = 2x + 1$	i $x^2 + x - 9 = 5x - 4$
j $x^2 - 5x = -15x - 25$	k $x^2 - 14x = 2x - 64$	l $x(x + 4) = 4(x + 16)$
m $2x(x - 2) = 6$	n $3x(x + 6) = 4(x - 2)$	o $4x(x + 5) = 6x - 4x^2 - 3$

Example 17b

9 Solve the following by first writing in the form $ax^2 + bx + c = 0$.

a $\dfrac{5x + 84}{x} = x$	b $\dfrac{9x + 70}{x} = x$
c $\dfrac{18 - 7x}{x} = x$	d $\dfrac{20 - 3x}{x} = 2x$
e $\dfrac{6x + 8}{5x} = x$	f $\dfrac{7x + 10}{2x} = 3x$
g $\dfrac{3}{x} = x + 2$	h $\dfrac{1}{x} = 3 - 2x$
i $\dfrac{4}{x - 2} = x + 1$	

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10 Some equations can be 'reduced' to quadratic equations.

For example: $x^4 + 5x^2 - 36 = 0$

Replace x^2 with u: $u^2 + 5u - 36 = 0$

$(u - 4)(u + 9) = 0$

$u - 4 = 0$ or $u + 9 = 0$

$\therefore\ u = 4$ or $u = -9$

Replace u with x^2: $x^2 = 4$ or $x^2 = -9$

$x = \pm 2$ No solution to $x^2 = -9$.

Use this method to solve the following equations.

a $x^4 - 13x^2 + 36 = 0$

b $x^4 - 36 = 0$

c $x^4 + 13x^2 + 36 = 0$

11 **a** Write down the solutions to the following equations.

i $2(x - 1)(x + 2) = 0$

ii $(x - 1)(x + 2) = 0$

b What difference has the common factor of 2 made to the solutions in the first equation?

c Explain why $x^2 - 5x - 6 = 0$ and $3x^2 - 15x - 18 = 0$ have the same solutions.

12 Explain why $x^2 + 16x + 64 = 0$ has only one solution.

13 When solving $x^2 - 2x - 8 = 7$ a student writes the following.

$x^2 - 2x - 8 = 7$

$(x - 4)(x + 2) = 7$

$x - 4 = 7$ or $x + 2 = 7$

$x = 11$ or $x = 5$

Discuss the problem with this solution and then write a correct solution.

ENRICHMENT	–	–	14

More algebraic fractions with quadratics

14 Solve these equations by first multiplying by an appropriate expression.

a $x + 3 = -\frac{2}{x}$ **b** $-\frac{1}{x} = x - 2$ **c** $-\frac{5}{x} = 2x - 11$

d $\frac{x^2 - 48}{x} = 2$ **e** $\frac{x^2 + 12}{x} = -8$ **f** $\frac{2x^2 - 12}{x} = -5$

g $\frac{x - 5}{4} = \frac{6}{x}$ **h** $\frac{x - 2}{3} = \frac{5}{x}$ **i** $\frac{x - 4}{2} = \frac{-2}{x}$

j $\frac{x + 4}{2} - \frac{3}{x - 3} = 1$ **k** $\frac{x}{x - 2} - \frac{x + 1}{x + 4} = 1$ **l** $\frac{1}{x - 1} - \frac{1}{x + 3} = \frac{1}{3}$

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8G Using quadratic equations to solve problems

Interactive
Widgets
HOTsheets
Walkthrough

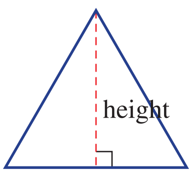
Defining variables, setting up equations, solving equations and interpreting solutions are all important elements of applying quadratic equations to problem solving. For example, the area of a rectangular paddock that can be fenced off using a limited length of fencing can be found by setting up a quadratic equation, solving it and then interpreting the solutions.

Stage

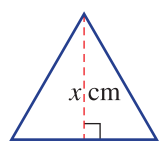
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: The 10 cm² triangle

There are many base and height measurements for a triangle that give an area of 10 cm².



- Draw three different triangles that have a 10 cm² area. Include the measurements for the base and the height.
- Do any of your triangles have a base length that is 1 cm more than the height?
- Find the special triangle with area 10 cm² that has a base 1 cm more than its height by following these steps.
  - Let x cm be the height of the triangle.
  - Write an expression for the base length.
  - Write an equation if the area is 10 cm².
  - Solve the equation to find two solutions for x.
  - Which solution is to be used to describe the special triangle? Why?



Key ideas

- When applying quadratic equations:
  - Define a variable; i.e. 'Let x be …'.
  - Write an equation.
  - Solve the equation.
  - Check that the solution(s) are reasonable.
  - Choose the solution(s) that solves the equation and answers the question in the context in which it was given.

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Example 18 Finding dimensions

The area of a rectangle is fixed at 28 m^2 and its length is 3 m more than its breadth. Find the dimensions of the rectangle.

SOLUTION

Let x m be the breadth of the rectangle.

Length $= (x + 3)$ m

$x(x + 3) = 28$

$x^2 + 3x - 28 = 0$

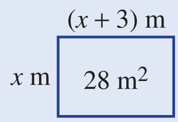
$(x + 7)(x - 4) = 0$

$x = -7$ or $x = 4$

$x > 0$, so choose $x = 4$.

Rectangle has breadth $= 4$ m and length $= 7$ m.

EXPLANATION



Write an equation using the given information.
Then write in standard form and solve for x.

Disregard $x = -7$ because $x > 0$.

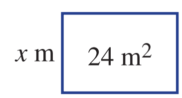
Answer the question in full.

Exercise 8G

UNDERSTANDING AND FLUENCY	1–5	2–5	4–6

Example 18

1 A rectangle has an area of 24 m^2. Its length is 5 m longer than its breadth.



- **a** Copy this sentence. ‘Let x m be the breadth of the rectangle.’
- **b** Write an expression for the rectangle’s length.
- **c** Write an equation using the rectangle’s area.
- **d** Write your equation from part **c** in standard form (i.e. $ax^2 + bx + c = 0$) and solve for x.
- **e** Find the dimensions of the rectangle.

2 Repeat all the steps in Question **1** to find the dimensions of a rectangle with the following properties.

- **a** Its area is 60 m^2 and its length is 4 m more than its breadth.
- **b** Its area is 63 m^2 and its length is 2 m less than its breadth.
- **c** Its area is 154 mm^2 and its length is 3 mm less than its breadth.

3 Find the height and base lengths of a triangle that has an area of 24 cm^2 and height 2 cm more than its base.

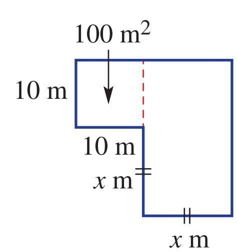
4 Find the height and base lengths of a triangle that has an area of 7 m^2 and height 5 m less than its base.

5 The product of two consecutive numbers is 72. Use a quadratic equation to find the two sets of numbers.

6 The product of two consecutive, even positive numbers is 168. Find the two numbers.

PROBLEM-SOLVING AND REASONING	7, 8, 12	7–10, 12, 13	8–11, 13, 14

7 A 100 m^2 hay shed is to be extended to give 475 m^2 of floor space in total, as shown. Find the value of x.



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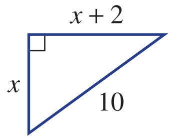
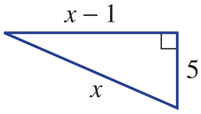
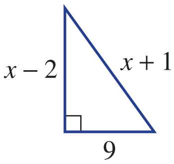
8 Use Pythagoras' theorem to solve for x in these right-angled triangles.

a


b


c


9 A square hut with side length 5 metres is to be surrounded by a veranda of width x metres. Find the width of the veranda if its area is to be 24 m^2.

10 A father's age is the square of his son's age (x). In 20 years' time the father will be three times as old as his son. What are the ages of the father and son?

11 A rectangular painting is to have a total area (including the frame) of 1200 cm^2. If the painting has length 30 cm and breadth 20 cm, find the width of the frame.

12 The sum of the first n positive integers is given by $\frac{1}{2}n(n + 1)$.

- **a** Find the sum of the first 10 positive integers (i.e. $n = 10$).
- **b** Find the value of n if the sum of the first n positive integers is:
  - **i** 28
  - **ii** 91
  - **iii** 276

13 A ball is thrust vertically upwards from a machine on the ground. The height (h metres) after t seconds is given by $h = t(4 - t)$.

- **a** Find the height after 1.5 seconds.
- **b** Find when the ball is at a height of 3 metres.
- **c** Why are there two solutions to part **b**?
- **d** Find when the ball is at ground level.
- **e** Find when the ball is at a height of 4 metres.
- **f** Why is there only one solution for part **e**?
- **g** Is there a time when the ball is at a height of 5 metres? Explain.

14 The height (h metres) of a golf ball is given by $h = -x^2 + 100x$, where x metres is the horizontal distance from where the ball was hit.

- **a** Find the values of x when $h = 0$.
- **b** Interpret your answer from part **a**.
- **c** Find how far the ball has travelled horizontally when the height is 196 metres.

ENRICHMENT	–	–	15, 16

Fixed perimeter and area

15 A small rectangular block of land has a perimeter of 100 m and an area of 225 m^2. Find the dimensions of the block of land.

16 A rectangular farm has a perimeter of 700 m and an area of 30000 m^2. Find its dimensions.

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8H Solving quadratic equations by completing the square

Interactive Widgets HOTsheets Walkthrough

Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

In Section 8E we saw that some quadratics cannot be factorised using integers but instead could be factorised by completing the square. Surds were also used to complete the factorisation. We can use this method to solve many quadratic equations.

Let' start: Where does $\sqrt{6}$ come in?

Consider the equation $x^2 - 2x - 5 = 0$ and try to solve it by discussing these points.

- Are there any common factors that can be taken out?
- Are there any integers that multiply to give -5 and add to give -2?
- Try completing the square on the left-hand side. Does this help and how?
- Show that the two solutions contain the surd $\sqrt{6}$.

Key ideas

- To solve quadratic equations of the form $ax^2 + bx + c = 0$ where you cannot factorise using integers:
  - Complete the square for the quadratic expression.
  - Solve the quadratic equation using the null factor law.

Example 19 Solving quadratic equations by completing the square

Solve these quadratic equations by first completing the square.

a $x^2 - 4x + 2 = 0$ **b** $x^2 + 6x - 11 = 0$ **c** $x^2 - 3x + 1 = 0$

SOLUTION	EXPLANATION
a $x^2 - 4x + 2 = 0$	Complete the square: $\left(\frac{-4}{2}\right)^2 = 4$
$x^2 - 4x + 4 - 4 + 2 = 0$	
$(x-2)^2 - 2 = 0$	$x^2 - 4x + 4 = (x-2)(x-2) = (x-2)^2$
$(x - 2 + \sqrt{2})(x - 2 - \sqrt{2}) = 0$	Use $a^2 - b^2 = (a+b)(a-b)$.
$\therefore x - 2 + \sqrt{2} = 0$ or $x - 2 - \sqrt{2} = 0$	Apply the null factor law and solve for x.
$\therefore x = 2 - \sqrt{2}$ or $x = 2 + \sqrt{2}$	This can also be written as $x = 2 \pm \sqrt{2}$.
Alternatively,	
$x^2 - 4x + 2 = 0$	
$x^2 - 4x = -2$	Subtract 2 from both sides
$x^2 - 4x + 4 = -2 + 4$	On LHS, complete the square and add to RHS.
$(x-2)^2 = 2$	Factorise LHS.
$x - 2 = \pm\sqrt{2}$	Take square root of both sides.
$x = 2 \pm \sqrt{2}$	Add 2 to both sides.

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b
$$x^2 + 6x - 11 = 0$$
$$x^2 + 6x + 9 - 9 - 11 = 0$$ Complete the square: $\left(\frac{6}{2}\right)^2 = 9$
$$(x+3)^2 - 20 = 0$$ Use difference of two squares with surds.
$$(x + 3 - \sqrt{20})(x + 3 + \sqrt{20}) = 0$$ Recall that $\sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}$.
$$(x + 3 - 2\sqrt{5})(x + 3 + 2\sqrt{5}) = 0$$ Apply the null factor law and solve for x.
$$\therefore x + 3 - 2\sqrt{5} = 0 \text{ or } x + 3 + 2\sqrt{5} = 0$$
$$\therefore x = -3 \pm 2\sqrt{5} \text{ or } x = -3 - 2\sqrt{5}$$ Alternatively, write solutions using $\pm$ symbol.
So $x = -3 \pm 2\sqrt{5}$.

c
$$x^2 - 3x + 1 = 0$$
$$x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 1 = 0$$ $\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$
$$\left(x - \frac{3}{2}\right)^2 - \frac{5}{4} = 0$$ $a^2 - b^2 = (a+b)(a-b)$
$$\left(x - \frac{3}{2} + \sqrt{\frac{5}{4}}\right)\left(x - \frac{3}{2} - \sqrt{\frac{5}{4}}\right) = 0$$ Using the null factor law: $x - \frac{3}{2} + \sqrt{\frac{5}{4}} = 0$ or
$$\therefore x = \frac{3}{2} - \frac{\sqrt{5}}{2} \text{ or } x = \frac{3}{2} + \frac{\sqrt{5}}{2}$$ $x - \frac{3}{2} - \sqrt{\frac{5}{4}} = 0$.
$$x = \frac{3 - \sqrt{5}}{2} \text{ or } x = \frac{3 + \sqrt{5}}{2}$$ Recall that $\sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{\sqrt{4}} = \frac{\sqrt{5}}{2}$.
So $x = \frac{3 \pm \sqrt{5}}{2}$. Combine using the $\pm$ symbol.

Exercise 8H

UNDERSTANDING AND FLUENCY	1–5(½)	3–6(½)	4–6(½)

1 What number must be added to the following expressions to form a perfect square?

a $x^2 + 2x$ **b** $x^2 + 6x$ **c** $x^2 + 20x$ **d** $x^2 + 50x$
e $x^2 - 4x$ **f** $x^2 - 10x$ **g** $x^2 + 5x$ **h** $x^2 - 3x$

2 Factorise using surds.

a $x^2 - 3 = 0$ **b** $x^2 - 7 = 0$ **c** $x^2 - 10 = 0$
d $(x + 1)^2 - 5 = 0$ **e** $(x + 3)^2 - 11 = 0$ **f** $(x - 1)^2 - 2 = 0$

3 Solve these equations.

a $(x - \sqrt{2})(x + \sqrt{2}) = 0$ **b** $(x - \sqrt{7})(x + \sqrt{7}) = 0$
c $(x - \sqrt{10})(x + \sqrt{10}) = 0$ **d** $(x - 3 + \sqrt{5})(x - 3 - \sqrt{5}) = 0$
e $(x - 4 + \sqrt{6})(x - 4 - \sqrt{6}) = 0$ **f** $(x + 5 + \sqrt{14})(x + 5 - \sqrt{14}) = 0$

Example 19a
4 Solve by first completing the square.

a $x^2 + 6x + 3 = 0$ **b** $x^2 + 4x + 2 = 0$ **c** $x^2 + 10x + 15 = 0$
d $x^2 + 4x - 2 = 0$ **e** $x^2 + 8x - 3 = 0$ **f** $x^2 + 6x - 5 = 0$
g $x^2 - 8x - 1 = 0$ **h** $x^2 - 12x - 3 = 0$ **i** $x^2 - 2x - 16 = 0$
j $x^2 - 10x + 18 = 0$ **k** $x^2 - 6x + 4 = 0$ **l** $x^2 - 8x + 9 = 0$
m $x^2 + 6x - 4 = 0$ **n** $x^2 + 20x + 13 = 0$ **o** $x^2 - 14x - 6 = 0$

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Example 19b

5 Solve by first completing the square.

a $x^2 + 8x + 4 = 0$ **b** $x^2 + 6x + 1 = 0$ **c** $x^2 - 10x + 5 = 0$
d $x^2 - 4x - 14 = 0$ **e** $x^2 - 10x - 3 = 0$ **f** $x^2 + 8x - 8 = 0$
g $x^2 - 2x - 31 = 0$ **h** $x^2 + 12x - 18 = 0$ **i** $x^2 + 6x - 41 = 0$

6 Decide how many solutions there are to these equations. Try factorising the equations if you are unsure.

a $x^2 - 2 = 0$ **b** $x^2 - 10 = 0$ **c** $x^2 + 3 = 0$
d $x^2 + 7 = 0$ **e** $(x - 1)^2 + 4 = 0$ **f** $(x + 2)^2 - 7 = 0$
g $(x - 7)^2 - 6 = 0$ **h** $x^2 - 2x + 6 = 0$ **i** $x^2 - 3x + 10 = 0$
j $x^2 + 2x - 4 = 0$ **k** $x^2 + 7x + 1 = 0$ **l** $x^2 - 2x + 17 = 0$

PROBLEM-SOLVING AND REASONING	7(½), 10, 12	7–9(½), 10, 12, 13	7–9(½), 11, 14, 15

Example 19c

7 Solve by first completing the square.

a $x^2 + 5x + 2 = 0$ **b** $x^2 + 3x + 1 = 0$ **c** $x^2 + 7x + 5 = 0$
d $x^2 - 3x - 2 = 0$ **e** $x^2 - x - 3 = 0$ **f** $x^2 + 5x - 2 = 0$
g $x^2 - 7x + 2 = 0$ **h** $x^2 - 9x + 5 = 0$ **i** $x^2 + x - 4 = 0$
j $x^2 + 9x + 9 = 0$ **k** $x^2 - 3x - \frac{3}{4} = 0$ **l** $x^2 + 5x + \frac{5}{4} = 0$

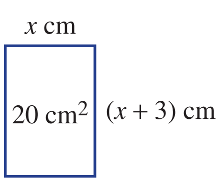
8 Solve the following if possible, by first factoring out the coefficient of x^2 and then completing the square.

a $2x^2 - 4x + 4 = 0$ **b** $4x^2 + 20x + 8 = 0$ **c** $2x^2 - 10x + 4 = 0$
d $3x^2 + 27x + 9 = 0$ **e** $3x^2 + 15x + 3 = 0$ **f** $2x^2 - 12x + 8 = 0$

9 Solve the following quadratic equations if possible.

a $x^2 + 3x = 5$ **b** $x^2 + 5x = 9$ **c** $x^2 + 7x = -15$
d $x^2 - 8x = -11$ **e** $x^2 + 12x + 10 = 2x + 5$ **f** $x^2 + x + 9 = 5x - 3$

10 A rectangle's length is 3 cm more than its breadth. Find the dimensions of the rectangle if its area is 20 cm².



11 The height, h km, of a ballistic missile launched from a submarine at sea level is given by $h = \frac{x(400 - x)}{20000}$, where x km is the horizontal distance travelled.

a Find the height of a missile that has travelled the following horizontal distances.
  i 100 km
  ii 300 km

b Find how far the missile has travelled horizontally when the height is the following.
  i 0 km
  ii 2 km

c Find the horizontal distance the missile has travelled when its height is 1 km.
Hint: Complete the square.

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12 Complete the square to show that the following have no (real) solutions.

a $x^2 + 4x + 5 = 0$

b $x^2 - 3x = -3$

13 A friend starts to solve $x^2 + x - 30 = 0$ by completing the square but you notice there is a much quicker way. What method do you describe to your friend?

14 A slightly different way to solve by completing the square is shown here. Solve the following using this method.

$$\begin{aligned} x^2 + 6x - 11 &= 0 \\ x^2 + 6x &= 11 \\ x^2 + 6x + 9 &= 11 + 9 \\ (x + 3)^2 &= 20 \\ x + 3 &= \pm\sqrt{20} \\ x &= -3 \pm \sqrt{20} \\ x &= -3 \pm 2\sqrt{5} \end{aligned}$$

a $x^2 - 6x + 2 = 0$

b $x^2 + 8x + 6 = 0$

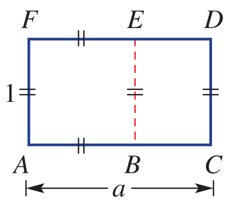
c $x^2 + 4x - 7 = 0$

d $x^2 - 2x - 5 = 0$

e $x^2 - 8x + 4 = 0$

f $x^2 + 10x + 1 = 0$

15 This rectangle is a golden rectangle.



- $ABEF$ is a square.
- Rectangle $BCDE$ is similar to rectangle $ACDF$.

a Show that $\frac{a}{1} = \frac{1}{a - 1}$.

b Find the exact value of a (which will give you the golden ratio) by completing the square.

ENRICHMENT	–	–	16

Completing the square with non-monics

16 In the Enrichment section of **Exercise 8E** we looked at a method to factorise non-monic quadratics by completing the square. It involved taking out the coefficient of x^2. Dividing both sides by that number is possible in these equations and this makes the task easier. Use this technique to solve the following equations.

a $2x^2 + 4x - 1 = 0$

b $3x^2 + 6x - 12 = 0$

c $-2x^2 + 16x - 10 = 0$

d $3x^2 - 9x + 3 = 0$

e $4x^2 + 20x + 8 = 0$

f $5x^2 + 5x - 15 = 0$

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8I Solving quadratic equations with the quadratic formula

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Interactive
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Walkthrough

A general formula for solving quadratic equations can be found by completing the square for the general case.

Consider $ax^2 + bx + c = 0$, where a, b, c are constants and $a \neq 0$. Start by dividing both sides by a.

$$
\begin{aligned}
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} &= 0 \\
\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} &= 0 \\
\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right) &= 0 \\
\left(x + \frac{b}{2a}\right)^2 &= \left(\frac{b^2 - 4ac}{4a^2}\right) \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x &= -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \\
&= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

This formula now gives us a mechanism to solve quadratic equations and to determine how many solutions the equation has.

The expression $b^2 - 4ac$ under the root sign is called the discriminant (Δ) and helps us to identify the number of solutions. A quadratic equation can have zero, one or two solutions.

Let's start: How many solutions?

Complete this table to find the number of solutions for each equation.

$ax^2 + bx + c = 0$	a	b	c	$b^2 - 4ac$	$\frac{-b + \sqrt{b^2 - 4ac}}{2a}$	$\frac{-b - \sqrt{b^2 - 4ac}}{2a}$
$2x^2 + 7x + 1 = 0$						
$9x^2 - 6x + 1 = 0$						
$x^2 - 3x + 4 = 0$						

Discuss under what circumstances a quadratic equation has:

- two solutions
- one solution
- no solutions

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Key ideas

- If $ax^2 + bx + c = 0$ (where a, b, c are constants and $a \neq 0$), then the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ may provide solutions.
  - This is called the **quadratic formula**.
- The quadratic formula is very useful when $ax^2 + bx + c$ is difficult to factorise.
- The **discriminant** is $\Delta = b^2 - 4ac$. It can be used to determine the number of real solutions.
  - $\Delta < 0$ No real solutions (since $\sqrt{\Delta}$ is undefined for real numbers when Δ is negative).
  - $\Delta = 0$ One real solution $\left(\text{at } x = -\dfrac{b}{2a}\right)$.
  - $\Delta > 0$ Two real solutions $\left(\text{at } x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$.

Example 20 Using the discriminant

Determine the number of solutions to the following quadratic equations, using the discriminant.

a $x^2 + 5x - 3 = 0$
b $2x^2 - 3x + 4 = 0$
c $x^2 + 6x + 9 = 0$

SOLUTION

a $a = 1, b = 5, c = -3$

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (5)^2 - 4(1)(-3)\\ &= 25 + 12\\ &= 37\end{aligned}$$

$\Delta > 0$, so there are two solutions.

EXPLANATION

State the values of a, b and c in $ax^2 + bx + c = 0$.
Calculate the value of the discriminant by substituting values.
Interpret the result with regard to the number of solutions.

SOLUTION

b $a = 2, b = -3, c = 4$

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-3)^2 - 4(2)(4)\\ &= 9 - 32\\ &= -23\end{aligned}$$

$\Delta < 0$, so there are no real solutions.

EXPLANATION

State the values of a, b and c and substitute to evaluate the discriminant. Recall that $(-3)^2 = -3 \times (-3) = 9$.

Interpret the result.

SOLUTION

c $a = 1, b = 6, c = 9$

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (6)^2 - 4(1)(9)\\ &= 36 - 36\\ &= 0\end{aligned}$$

$\Delta = 0$, so there is one solution.

EXPLANATION

Substitute the values of a, b and c to evaluate the discriminant and interpret the result.

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Example 21 Solving quadratic equations using the quadratic formula

Find the exact solutions to the following, using the quadratic formula.

a $x^2 + 5x + 3 = 0$ **b** $2x^2 - 2x - 1 = 0$

SOLUTION

a $a = 1, b = 5, c = 3$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-5 \pm \sqrt{(5)^2 - 4(1)(3)}}{2(1)} \\ &= \frac{-5 \pm \sqrt{25 - 12}}{2} \\ &= \frac{-5 \pm \sqrt{13}}{2} \end{aligned}$$

b $a = 2, b = -2, c = -1$

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)} \\ &= \frac{2 \pm \sqrt{4 + 8}}{4} \\ &= \frac{2 \pm \sqrt{12}}{4} \\ &= \frac{2 \pm 2\sqrt{3}}{4} \\ &= \frac{1 \pm \sqrt{3}}{2} \end{aligned}$$

EXPLANATION

Determine the values of a, b and c in $ax^2 + bx + c = 0$.

Write out the quadratic formula and substitute the values.

Simplify.

Two solutions: $x = \frac{-5 - \sqrt{13}}{2}, \frac{-5 + \sqrt{13}}{2}$

Determine the values of a, b and c.

Simplify: $\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$

Cancel:

$$\begin{aligned} \frac{2 \pm 2\sqrt{3}}{4} &= \frac{2(1 \pm \sqrt{3})}{4} \\ &= \frac{1 \pm \sqrt{3}}{2} \end{aligned}$$

You could use a calculator to check that these solutions satisfy the equation.

Exercise 8I

UNDERSTANDING AND FLUENCY	1–3, 4–5(½)	3, 4–6(½)	4–6(½)

1 For these quadratic equations in the form $ax^2 + bx + c = 0$, state the values of a, b and c.

a $3x^2 + 2x + 1 = 0$

b $2x^2 + x + 4 = 0$

c $5x^2 + 3x - 2 = 0$

d $4x^2 - 3x + 2 = 0$

e $2x^2 - x - 5 = 0$

f $-3x^2 + 4x - 5 = 0$

2 Find the value of the discriminant (i.e. $b^2 - 4ac$) for each part in Question **1** above.

3 State the number of solutions of a quadratic that has:

a $b^2 - 4ac = 0$ **b** $b^2 - 4ac < 0$ **c** $b^2 - 4ac > 0$

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Example 20

4 Using the discriminant, determine the number of solutions for these quadratic equations.

a $x^2 + 5x + 3 = 0$
b $x^2 + 3x + 4 = 0$
c $x^2 + 6x + 9 = 0$
d $x^2 + 7x - 3 = 0$
e $x^2 + 5x - 4 = 0$
f $x^2 + 4x - 4 = 0$
g $4x^2 + 5x + 3 = 0$
h $4x^2 + 3x + 1 = 0$
i $2x^2 + 12x + 9 = 0$
j $-x^2 - 6x - 9 = 0$
k $-2x^2 + 3x - 4 = 0$
l $-4x^2 - 6x + 3 = 0$

Example 21a

5 Find the exact solutions to the following quadratic equations, using the quadratic formula.

a $x^2 + 3x - 2 = 0$
b $x^2 + 7x - 4 = 0$
c $x^2 - 7x + 5 = 0$
d $x^2 - 8x + 16 = 0$
e $-x^2 - 5x - 4 = 0$
f $-x^2 - 8x - 7 = 0$
g $4x^2 + 7x - 1 = 0$
h $3x^2 + 5x - 1 = 0$
i $3x^2 - 4x - 6 = 0$
j $-2x^2 + 5x + 5 = 0$
k $-3x^2 - x + 4 = 0$
l $5x^2 + 6x - 2 = 0$

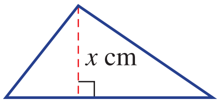
Example 21b

6 Find the exact solutions to the following quadratic equations, using the quadratic formula.

a $x^2 + 4x + 1 = 0$
b $x^2 - 6x + 4 = 0$
c $x^2 + 6x - 2 = 0$
d $-x^2 - 3x + 9 = 0$
e $-x^2 + 4x + 4 = 0$
f $-3x^2 + 8x - 2 = 0$
g $2x^2 - 2x - 3 = 0$
h $3x^2 - 6x - 1 = 0$
i $-5x^2 + 8x + 3 = 0$

PROBLEM-SOLVING AND REASONING	7, 8(½), 12	8(½), 10, 12, 13	8(½), 9, 11, 13, 14

7 A triangle's base is 5 cm more than its height of x cm. Find its height if the triangle's area is 10 cm².

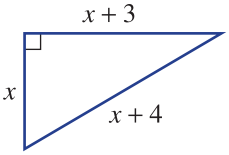


8 Solve the following using the quadratic formula.

a $3x^2 = 1 + 6x$
b $2x^2 = 3 - 4x$
c $5x = 2 - 4x^2$
d $2x - 5 = -\frac{1}{x}$
e $\frac{3}{x} = 3x + 4$
f $-\frac{5}{x} = 2 - x$
g $5x = \frac{2x + 2}{x}$
h $x = \frac{3x + 4}{2x}$
i $3x = \frac{10x - 1}{2x}$

9 Two positive numbers differ by 3 and their product is 11. Find the numbers.

10 **a** For the right-angled triangle shown, use Pythagoras' theorem to find the exact value of x.
b Hence, find the exact perimeter.
c Hence, find the exact area.



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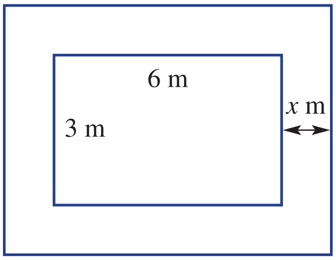
11 A pool measuring 6 m by 3 m is to have a path surrounding it. If the total area of the pool and path is to be 31 m², find the width (x m) of the path, correct to the nearest cm.



12 Explain why the rule $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ gives only one solution when $b^2 - 4ac = 0$.

13 Make up three quadratic equations that have:

a no solutions

b one solution

c two solutions

14 For what two values of k does $x^2 + kx + 9 = 0$ have only one solution?

ENRICHMENT	–	–	15

k determines the number of solutions

15 The discriminant for $x^2 + 2x + k = 0$ is $4 - 4k$, so there:

- are no solutions for $4 - 4k < 0$, $\therefore k > 1$
- is one solution for $4 - 4k = 0$, $\therefore k = 1$
- are two solutions for $4 - 4k > 0$, $\therefore k < 1$

a Determine for what values of k does $x^2 + 4x + k = 0$ have:

i no solutions

ii one solution

iii two solutions

b Determine for what values of k does $kx^2 + 3x + 2 = 0$ have:

i no solutions

ii one solution

iii two solutions

c Determine for what values of k does $x^2 + kx + 1 = 0$ have:

i no solutions

ii one solution

iii two solutions

d Determine for what values of k does $3x^2 + kx - 1 = 0$ have:

i no solutions

ii one solution

iii two solutions

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Investigation

Binomial expansions

Blaise Pascal and expansion

Blaise Pascal (1623–1662) was a French mathematician, physicist and philosopher. By the age of 16 he had proved many theorems in geometry and by 17 he had invented and made what is regarded as the first calculator.

One of his mathematical investigations involved exploring the properties and patterns of numbers in a triangular arrangement that is known today as Pascal's triangle. The triangle has many applications in mathematics, including algebraic expansion and probability. The diagram below shows part of this triangle.

Pascal's triangle

row 0							**1**						
row 1						**1**		**1**					
row 2					**1**		**2**		**1**				
row 3				**1**		**3**		**3**		**1**			
row 4			**1**		**4**		**6**		**4**		**1**		
row 5		**1**		**5**		**10**		**10**		**5**		**1**	
row 6	**1**												**1**
row 7													
row 8													
row 9													
row 10													

Expanding the triangle

a Observe and describe the pattern of numbers shown in rows 0 to 4.

b State a method that might produce the next row in the triangle.

c Complete the triangle to row 10.

Expanding brackets

Consider the expansions of binomial expressions. If you look closely, you can see how the coefficients in each term match the values in the triangle you produced in the triangle above.

$$(x + y)^0 = \mathbf{1}$$

$$(x + y)^1 = \mathbf{1}x + \mathbf{1}y$$

$$(x + y)^2 = (x + y)(x + y) = \mathbf{1}x^2 + \mathbf{2}xy + \mathbf{1}y^2$$

$$\begin{aligned}(x + y)^3 &= (x + y)(x + y)^2 = (x + y)(x^2 + 2xy + y^2)\\ &= x^3 + 2x^2y + xy^2 + yx^2 + 2xy^2 + y^3\\ &= \mathbf{1}x^3 + \mathbf{3}x^2y + \mathbf{3}xy^2 + \mathbf{1}y^3\end{aligned}$$

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Expand $(x + y)^4$, $(x + y)^5$, $(x + y)^6$, $(x + y)^7$ and $(x + y)^8$ by completing the triangle below.

$(x + y)^0$ $\quad 1$

$(x + y)^1$ $\quad 1x + 1y$

$(x + y)^2$ $\quad 1x^2 + 2xy + 1y^2$

$(x + y)^3$ $\quad 1x^3 + 3x^2y + 3xy^2 + 1y^3$

$(x + y)^4$

$(x + y)^5$

$(x + y)^6$

$(x + y)^7$

$(x + y)^8$

Factorials and combinations

Another way of generating Pascal's triangle is by using the formula for combinations.
In general, the number of ways to select r objects from a group of n objects is given by

$^nC_r = \binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$, where $n! = n \times (n - 1) \times (n - 2) \times (n - 3) \times \ldots \times 1$ and $0! = 1$.

Consider a group of two objects from which you wish to choose zero, one or two objects.

You can choose zero objects $\binom{2}{0} = \dfrac{2!}{0!(2-0)!} = 1$

one object $\binom{2}{1} = \dfrac{2!}{1!(2-1)!} = 2$

two objects $\binom{2}{2} = \dfrac{2!}{2!(2-2)!} = 1$

Use the combinations formula to copy and complete the triangle below for the number of ways of selecting objects from a group.

zero objects $\binom{0}{0} = 1$

one object $\binom{1}{0} = 1 \quad \binom{1}{1} = 1$

two objects $\binom{2}{0} = 1 \quad \binom{2}{1} = 2 \quad \binom{2}{2} = 1$

three objects

four objects

Expanding binomial expressions without the triangle

By noting the patterns above, expand the following.

a $(x + y)^5$ **b** $(x + y)^{10}$ **c** $(x + 1)^3$
d $(x + 3)^5$ **e** $(2x + 3)^4$ **f** $(3x + 1)^5$
g $(1 - x)^4$ **h** $(3 - 2x)^5$ **i** $(x^2 - 1)^6$

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Puzzles and challenges

1 Find the monic quadratic in the form $x^2 + bx + c = 0$ with solutions $x = 2 - \sqrt{3}$ and $x = 2 + \sqrt{3}$.

2 If $x + \frac{1}{x} = 7$, what is $x^2 + \frac{1}{x^2}$?

3 Find all the solutions to each equation. Hint: Consider letting $u = x^2$ in each equation.

a $x^4 - 5x^2 + 4 = 0$

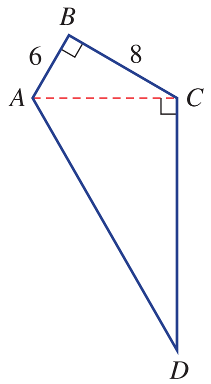
b $x^4 - 7x^2 - 18 = 0$

4 Make a substitution as you did in Question 3 to obtain a quadratic equation to help you solve each of the following.

a $3^{2x} - 4 \times 3^x + 3 = 0$

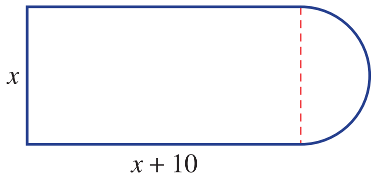
b $4 \times 2^{2x} - 9 \times 2^x + 2 = 0$

5 Quadrilateral $ABCD$ has a perimeter of 64 cm with measurements as shown. What is the area of the quadrilateral?



6 A cyclist in a charity ride rides 300 km at a constant average speed. If the average speed had been 5 km/h faster, the ride would have taken 2 hours less. What was the average speed of the cyclist?

7 Find the value of x, correct to 1 decimal place, in this diagram if the area is to be 20 square units.



8 Prove that $x^2 - 2x + 2 > 0$ for all values of x.

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Chapter summary

Quadratic expressions and quadratic equations

Factorising non–monic quadratics $ax^2 + bx + c$

Use grouping, split up bx using two numbers that multiply to give $a \times c$ and add to give b.

e.g. $6x^2 - 5x - 4$

$a \times c = 6 \times (-4) = -24$

$-8 \times 3 = -24$

$-8 + 3 = -5$

$6x^2 + 3x - 8x - 4$

$= 3x(2x + 1) - 4(2x + 1)$

$= (2x + 1)(3x - 4)$

Factorising $x^2 + bx + c$

Two numbers $\times$ to give b
$+$ to give c

e.g. $x^2 - 7x - 18 = (x - 9)(x + 2)$

$-9 \times 2 = -18$

$-9 + 2 = -7$

Difference of two squares

Always take out common factors first.

Difference of two squares

$a^2 - b^2 = (a - b)(a + b)$

e.g. $4x^2 - 9 = (2x)^2 - (3)^2$

$= (2x - 3)(2x + 3)$

$x^2 - 7 = (x - \sqrt{7})(x + \sqrt{7})$

Factorising by completing the square

e.g. $x^2 + 4x - 3$

$= \left(x^2 + 4x + \left(\frac{4}{2}\right)^2\right) - \left(\frac{4}{2}\right)^2 - 3$

$= (x + 2)^2 - 4 - 3$

$= (x + 2)^2 - 7$

$= (x + 2 - \sqrt{7})(x + 2 + \sqrt{7})$

Note, for example, $(x + 2)^2 + 5$ cannot be factorised.

Expanding brackets

$a(b + c) = ab + ac$

$(a + b)(c + d) = ac + ad + bc + bd$

$(a + b)(a - b) = a^2 - b^2$

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

Solving quadratic equations

Null factor law:

If $ab = 0$, then $a = 0$ or $b = 0$.

Write each quadratic in standard form $ax^2 + bx + c = 0$, factorise and then apply the null factor law to solve.

e.g. **1**

$x^2 - 4x = 0$

$x(x - 4) = 0$

$x = 0$ or $x - 4 = 0$

$x = 0$ or $x = 4$

2

$x^2 = 3x - 10$

$x^2 - 3x + 10 = 0$

$(x - 5)(x + 2) = 0$

$x - 5 = 0$ or $x + 2 = 0$

$x = 5$ or $x = -2$

Quadratic formula

If $ax^2 + bx + c = 0$, then

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The discriminant $\Delta = b^2 - 4ac$ and tells us how many solutions:

$\Delta > 0$ two solutions

$\Delta = 0$ one solution

$\Delta < 0$ no solutions

Applications

1. Define the variable.
2. Set up the equation.
3. Solve by factorising and using the null factor law or quadratic formula.
4. Determine the suitable answer(s).

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Multiple-choice questions

1 $(x + 5)^2$ is equivalent to:
A $x^2 + 25$ **B** $x^2 + 5x$ **C** $x^2 + 5x + 25$
D $x^2 + 10x + 25$ **E** $x^2 + 50$

2 $2(2x - 1)(x + 4)$ is equal to:
A $4x^2 + 15x - 4$ **B** $4x^2 + 14x - 8$ **C** $8x^2 + 28x - 16$
D $8x^2 + 18x - 4$ **E** $4x^2 + 10x + 8$

3 $4x^2 - 25$ in factorised form is:
A $4(x - 5)(x + 5)$ **B** $(2x - 5)^2$ **C** $(2x - 5)(2x + 5)$
D $(4x + 5)(x - 5)$ **E** $2(2x + 1)(x - 25)$

4 The fully factored form of $2x^2 - 10x - 28$ is:
A $2(x + 2)(x - 7)$ **B** $(2x + 7)(x + 4)$ **C** $2(x - 4)(x - 1)$
D $(2x - 2)(x + 14)$ **E** $(x - 2)(x + 7)$

5 $\dfrac{x^2 + x - 20}{8x} \times \dfrac{2x + 8}{x^2 - 16}$ simplifies to:
A $\dfrac{x - 20}{8}$ **B** $\dfrac{x + 5}{4x}$ **C** $\dfrac{x + 5}{x - 4}$
D $x - 5$ **E** $\dfrac{x^2 - 20}{16}$

6 The term that needs to be added to make $x^2 - 6x$ a perfect square is:
A 18 **B** -9 **C** -3 **D** 9 **E** 3

7 The solution(s) to $2x^2 - 8x = 0$ are:
A $x = 0, x = -4$ **B** $x = 2$ **C** $x = 0, x = 4$
D $x = 4$ **E** $x = 0, x = 2$

8 The solutions to $8x^2 - 14x + 3 = 0$ are x equals:
A $\frac{1}{8}, -\frac{1}{3}$ **B** $\frac{3}{4}, -\frac{1}{2}$ **C** $\frac{1}{4}, \frac{3}{2}$ **D** $\frac{3}{4}, -\frac{1}{2}$ **E** $-\frac{1}{2}, -\frac{3}{8}$

9 When written in the standard form $ax^2 + bx + c$, $\dfrac{x - 3}{x} = 2x$ is:
A $x^2 + 2x + 3 = 0$ **B** $x^2 + 3 = 0$ **C** $2x^2 + x - 3 = 0$
D $2x^2 - x - 3 = 0$ **E** $2x^2 - x + 3 = 0$

10 The product of two consecutive numbers is 72. If x is the smaller number, an equation to represent this would be:
A $x^2 + x + 72 = 0$ **B** $2x - 71 = 0$ **C** $x^2 + x - 72 = 0$
D $x^2 + 1 = 72$ **E** $x^2 = x + 72$

11 The solutions to $(x - 7)^2 - 3 = 0$ are x equals:
A $7 - \sqrt{3}, 7 + \sqrt{3}$ **B** $-7 - \sqrt{3}, -7 + \sqrt{3}$ **C** $7, -3$
D $-7 - \sqrt{3}, 7 + \sqrt{3}$ **E** 4, 10

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12 If $ax^2 + bx + c$ has exactly two solutions, then:

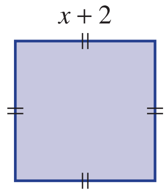
A $b^2 - 4ac = 0$ **B** $b^2 - 4ac > 0$ **C** $b^2 - 4ac \leqslant 0$
D $b^2 - 4ac \geqslant 0$ **E** $b^2 - 4ac < 0$

Short-answer questions

1 Expand the following and simplify where possible.

a $2(x + 3) - 4(x - 5)$
b $(x + 5)(3x - 4)$
c $(5x - 2)(5x + 2)$
d $(x - 6)^2$
e $(x + 4)^2 - (x + 3)(x - 2)$
f $(3x - 2)(4x - 5)$

2 Write, in expanded form, an expression for the shaded areas.

a


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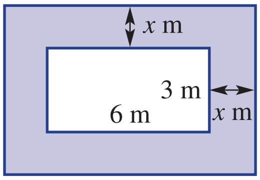

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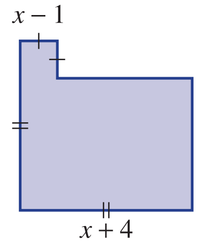

3 Factorise the following difference of two squares. Remember to look for a common factor first.

a $x^2 - 49$ **b** $9x^2 - 16$ **c** $4x^2 - 1$
d $3x^2 - 75$ **e** $2x^2 - 18$ **f** $x^2 - 11$
g $-2x^2 + 40$ **h** $(x + 1)^2 - 16$ **i** $(x - 3)^2 - 10$

4 Factorise these quadratic trinomials.

a $x^2 - 8x + 12$ **b** $x^2 + 10x - 24$ **c** $-3x^2 + 21x - 18$

5 Factorise these non-monic quadratic trinomials.

a $3x^2 + 17x + 10$ **b** $4x^2 + 4x - 15$
c $12x^2 - 16x - 3$ **d** $12x^2 - 23x + 10$

6 Simplify:

a $\dfrac{12x}{x^2 + 2x - 3} \times \dfrac{x^2 - 1}{6x + 6}$ **b** $\dfrac{4x^2 - 9}{2x^2 + x - 6} \div \dfrac{8x + 12}{x^2 - 2x - 8}$

7 Factorise the following by completing the square.

a $x^2 + 8x + 10$ **b** $x^2 + 10x - 4$ **c** $x^2 - 6x - 3$
d $x^2 + 3x - 2$ **e** $x^2 + 5x + 3$ **f** $x^2 + 7x + \dfrac{9}{2}$

8 Solve these quadratic equations by factorising and applying the null factor law.

a $x^2 + 4x = 0$ **b** $3x^2 - 9x = 0$ **c** $x^2 - 25 = 0$
d $x^2 - 10x + 21 = 0$ **e** $x^2 - 8x + 16 = 0$ **f** $x^2 + 5x - 36 = 0$
g $2x^2 + 3x - 2 = 0$ **h** $6x^2 + 11x - 10 = 0$ **i** $18x^2 + 25x - 3 = 0$

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9 Solve the following quadratic equations by first writing them in standard form.

a $3x^2 = 27$ **b** $x^2 = 4x + 5$

c $2x^2 - 28 = x(x - 3)$ **d** $\frac{3x + 18}{x} = x$

10 A rectangular sandpit is 2 m longer than its breadth. If it occupies an area of 48 m^2, determine the dimensions of the sandpit by solving a suitable equation.

11 Solve these quadratic equations by first completing the square.

a $x^2 + 4x - 3 = 0$ **b** $x^2 - 6x + 1 = 0$

c $x^2 - 3x - 2 = 0$ **d** $x^2 + 5x - 5 = 0$

12 For each quadratic equation, determine the number of solutions by finding the value of the discriminant.

a $x^2 + 2x + 1 = 0$ **b** $x^2 - 3x - 3 = 0$

c $2x^2 - 4x + 3 = 0$ **d** $-3x^2 + x + 5 = 0$

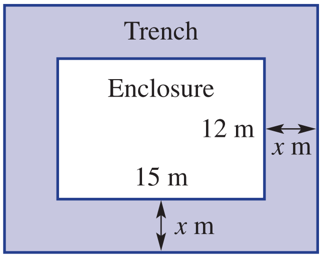
13 Use the quadratic formula to give exact solutions to these quadratic equations.

a $x^2 + 3x - 6 = 0$ **b** $x^2 - 2x - 4 = 0$

c $2x^2 - 4x - 5 = 0$ **d** $-3x^2 + x + 3 = 0$

Extended-response questions

1 A zoo enclosure for a rare tiger is rectangular in shape and has a trench of width x m all the way around it to ensure the tiger doesn't get far if it tries to escape. The dimensions are as shown.



a Write an expression in terms of x for:

i the length of the enclosure

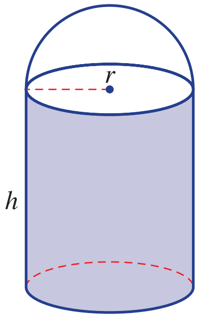
ii the width of the enclosure

b Use your answers from part **a** to find the area of the enclosure, in expanded form.

c Hence, find an expression for the area of the trench alone.

d Zoo restrictions state that the trench must have an area of at least 58 m^2. By solving a suitable equation find the minimum width of the trench.

2 The surface area A of a cylindrical tank with a hemispherical top is given by the equation $A = 3\pi r^2 + 2\pi rh$, where r is the radius and h is the height of the cylinder.



a If the radius of a tank with height 6 m is 3 m, determine its exact surface area.

b If the surface area of a tank with radius 5 m is 250 m^2, determine its height, to 2 decimal places.

c The surface area of a tank of height 6 m is found to be 420 m^2.

i Substitute the values and rewrite the equation in terms of r only.

ii Rearrange the equation and write it in the form $ar^2 + br + c = 0$.

iii Solve for r using the quadratic formula and round your answer to 2 decimal places.

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Online resources

- Auto-marked chapter pre-test
- Video demonstrations of all worked examples
- Interactive widgets
- Interactive walkthroughs
- Downloadable HOTsheets
- Access to all HOTmaths Australian Curriculum courses
- Access to the HOTmaths games library

9 Non-linear relationships, functions and their graphs

What you will learn

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NSW syllabus

STRAND: NUMBER AND ALGEBRA

SUBSTRANDS: NON-LINEAR RELATIONSHIPS

FUNCTIONS AND OTHER GRAPHS

RATIOS AND RATES

Outcomes

A student graphs simple non-linear relationships.
(MA5.1–7NA)

A student connects algebraic and graphical representations of simple non-linear relationships.
(MA5.2–10NA)

A student sketches and interprets a variety of non-linear relationships.
(MA5.3–9NA)

A student recognises direct and indirect proportion, and solves problems involving direct proportion.
(MA5.2–5NA)

A student draws, interprets and analyses graphs of physical phenomena.
(MA5.3–4NA)

A student uses function notation to describe and sketch functions.
(MA5.3–12NA)

Parabolic antennae

The graph of a quadratic equation forms a shape called a parabola. Imagine a parabola rotated horizontally about its central axis of symmetry (e.g. the y-axis) forming a parabolic 'dish'. When parallel beams of energy enter a parabolic reflective dish, they focus at a single point on the central axis called the focal point. Conversely, light or radio waves can be transmitted from the focal point outwards to the parabolic dish, which then reflects a parallel beam of this energy into space.

A parabolic antenna uses a parabolic dish to focus faint incoming radio waves into a more intense beam that can be transmitted from the focal point of the dish to computers for analysis. Satellites communicate using their own parabolic antennae and these faint signals are received on Earth using quite large parabolic antennae.

NASA's Deep Space Tracking Station located at Tidbinbilla, Canberra, has four parabolic dish antennae. The largest is 70 m in diameter, weighs 3000 tonnes and is made of 1272 aluminium reflector panels with a total surface area of 4180 m^2. The Tidbinbilla antennae receive incoming signals from many of NASA's space exploration missions, including the Mars Reconnaissance Orbiter spacecraft, which in 2015 confirmed for the first time the existence of flowing water on Mars, and the Voyager spacecraft, which was launched in 1977 and is now in interstellar space beyond our solar system.

Pre-test

1 Expand and simplify.

a $(x + 3)(x - 1)$

b $(x - 5)(2x + 1)$

c $(3x - 4)(x + 2)$

2 Complete this table and plot the graph of $y = x^2$.

x	−3	−2	−1	0	1	2	3
y	9						

3 Factorise the following.

a $x^2 - 9$ **b** $81x^2 - 49$ **c** $x^2 + 5x + 4$

d $x^2 - 9x + 20$ **e** $x^2 + 6x - 16$ **f** $3x^2 - 21x + 36$

4 Factorise the following by completing the square.

a $x^2 + 8x + 1$

b $x^2 - 2x - 9$

5 Solve the following.

a $(x - 1)(x + 2) = 0$ **b** $(x - 2)(x + 3) = 0$ **c** $(x + 5)(x + 7) = 0$

d $(2x + 3)(x - 2) = 0$ **e** $(3x - 1)(2x + 5) = 0$ **f** $(7x + 2)(3x - 4) = 0$

6 Solve these quadratic equations.

a $x^2 - x - 2 = 0$ **b** $x^2 + 2x - 15 = 0$ **c** $x^2 + x - 12 = 0$

d $x^2 + 5x + 4 = 0$ **e** $x^2 - 3x + 2 = 0$ **f** $x^2 - 7x + 12 = 0$

7 Solve the following quadratic equations.

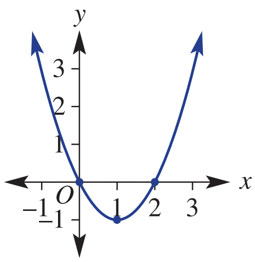
a $x^2 - 4x = 0$

b $x^2 - 8x + 16 = 0$

c $x^2 - 5 = 0$

d $x^2 + 6x + 9 = 0$

8 For the graph below:



a State the x-intercepts.

b State the y-intercept.

c State the coordinates of the turning point.

9 Sketch the following straight lines by finding the x- and y-intercepts.

a $y = 2x - 8$

b $3x - 2y = 6$

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9A Exploring parabolas

Interactive
Widgets
HOTsheets
Walkthrough

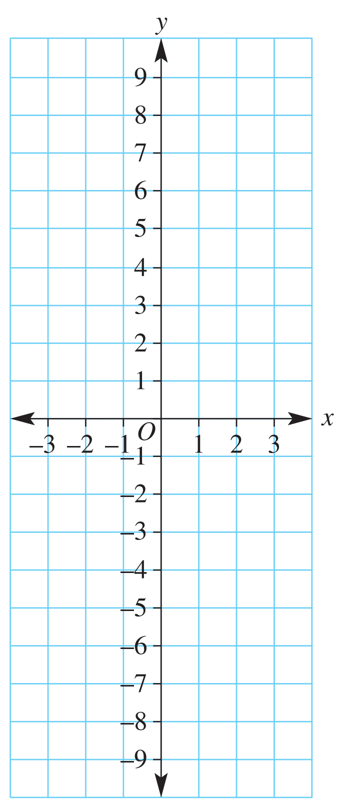
One of the simplest and most important non-linear graphs is the parabola. When a ball is thrown or water streams up and out from a garden hose or drinking fountain, the path followed has a parabolic shape. The parabola is the graph of a quadratic relation with the basic rule $y = x^2$. Quadratic rules such as $y = (x - 1)^2$, $y = 2x^2 - x - 3$ and $y = (x + 4)^2 - 7$ also give graphs that are parabolas and are transformations of the graph of $y = x^2$.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: To what effect?

To see how different quadratic rules compare to the graph of $y = x^2$, complete this table and plot the graph of each equation on the same set of axes.

x	-3	-2	-1	0	1	2	3
$y_1 = x^2$	9	4					
$y_2 = -x^2$	-9						
$y_3 = (x - 2)^2$							
$y_4 = x^2 - 3$							



- For all the graphs, find such features as the:
  - turning point
  - axis of symmetry
  - y-intercept
  - x-intercepts
- Discuss how each of the graphs of y_2, y_3 and y_4 compare to the graph of $y = x^2$. Compare the rule with the position of the graph.

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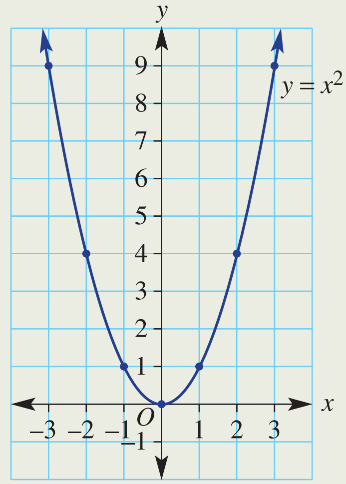
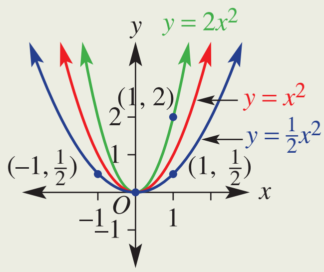
Key ideas

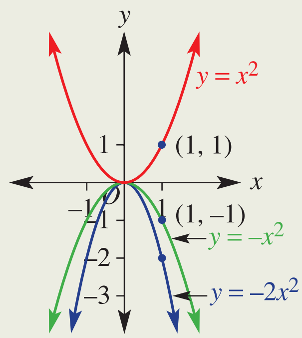
- A **parabola** is the graph of a quadratic relationship.
  The basic parabola has the equation $y = x^2$.
  - The vertex (or turning point) is (0, 0).
  - It is a minimum turning point.
  - Axis of symmetry is $x = 0$ (the y-axis).
  - y-intercept is 0.
  - x-intercept is 0.



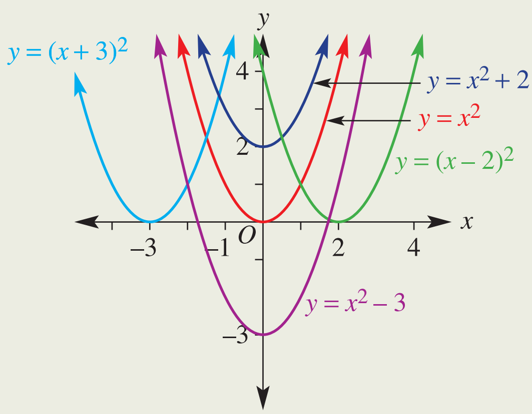
- Simple transformations of the graph of $y = x^2$ include:
  - dilation



  - reflection



  - translation

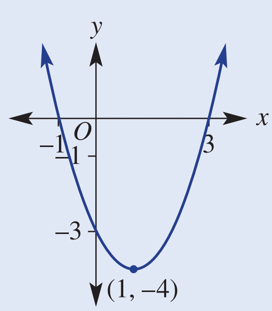


Example 1 Identifying key features of parabolas

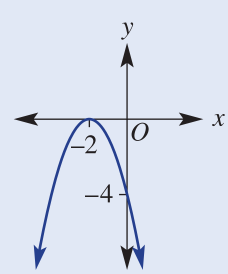
Determine the following key features of each of the given graphs.

i turning point and whether it is a maximum or minimum
ii axis of symmetry
iii x-intercepts
iv y-intercept

a



b



SOLUTION

a **i** Turning point is a minimum at $(1, -4)$.

Lowest point of graph is at $(1, -4)$.

ii Axis of symmetry is $x = 1$.

Line of symmetry is through the x-coordinate of the turning point.

iii x-intercepts are -1 and 3.

iv y-intercept is -3.

x-intercepts lie on the x-axis ($y = 0$) and the y-intercept on the y-axis ($x = 0$).

b **i** Turning point is a maximum at $(-2, 0)$.

Graph has a highest point at $(-2, 0)$.

ii Axis of symmetry is $x = -2$.

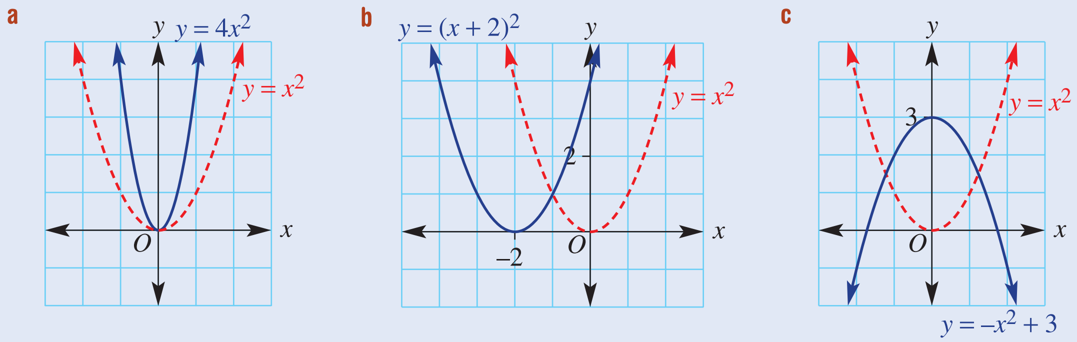
Line of symmetry is through the x-coordinate of the turning point.

iii x-intercept $= -2$

iv y-intercept $= -4$

Example 2 Transforming parabolas

Copy and complete the table for the following graphs.



	Formula	Maximum or minimum	Reflected in the x-axis (yes/no)	Turning point	y value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = 4x^2$					
b	$y = (x + 2)^2$					
c	$y = -x^2 + 3$					

SOLUTION

	Formula	Maximum or minimum	Reflected in the x-axis (yes/no)	Turning point	y value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = 4x^2$	minimum	no	(0, 0)	4	narrower
b	$y = (x + 2)^2$	minimum	no	$(-2, 0)$	9	same
c	$y = -x^2 + 3$	maximum	yes	(0, 3)	2	same

EXPLANATION

Read features from graphs and consider the effect of each change in equation on the graph.

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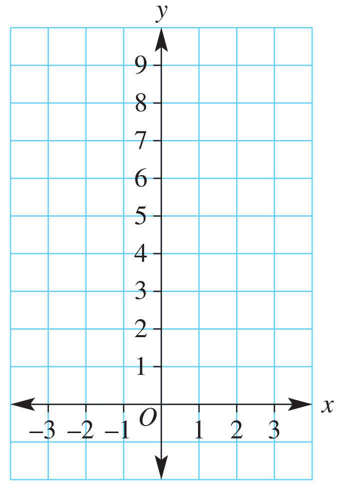
Exercise 9A

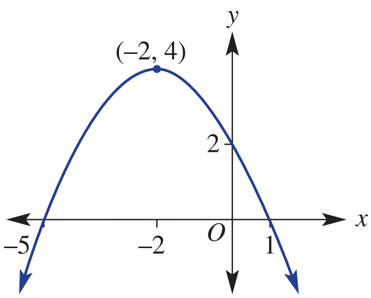
UNDERSTANDING AND FLUENCY	1–6	3–6	4–6

1 Complete this table and grid to plot the graph of $y = x^2$.

x	–3	–2	–1	0	1	2	3
y	9						



2 Write the missing features for this graph.



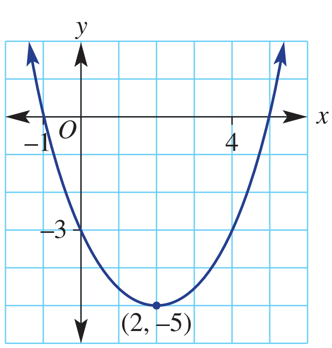
a The parabola has a __________ (maximum or minimum).
b The coordinates of the turning point are _____.
c The y-intercept is _____.
d The x-intercepts are _____ and _____.
e The axis of symmetry is ______.

Example 1

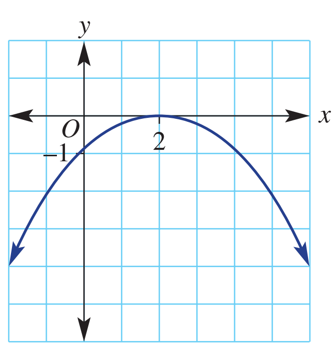
3 Determine these key features of the following graphs.

i turning point
ii axis of symmetry
iii x-intercepts
iv y-intercept

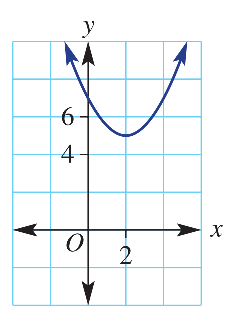
a



b



c



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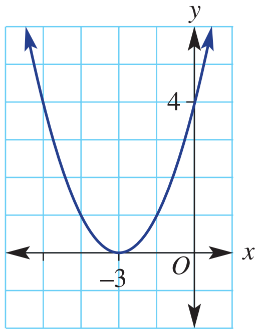
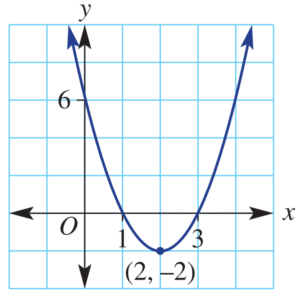
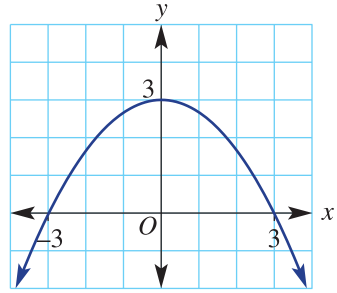
d



e



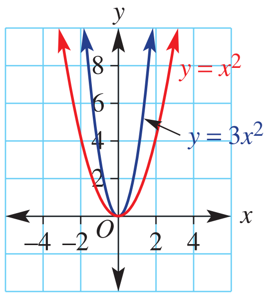
f



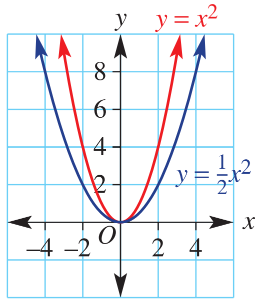
Example 2a

4 Copy and complete the table below for the following graphs.

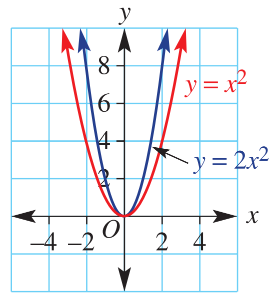
a



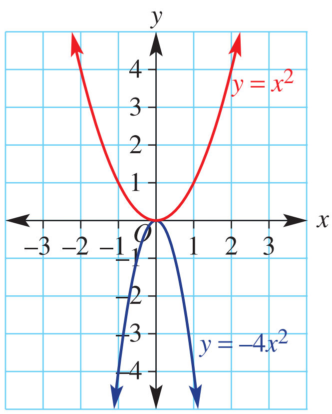
b



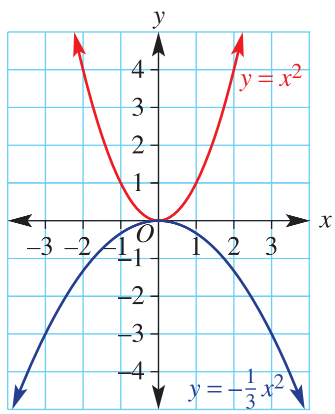
c



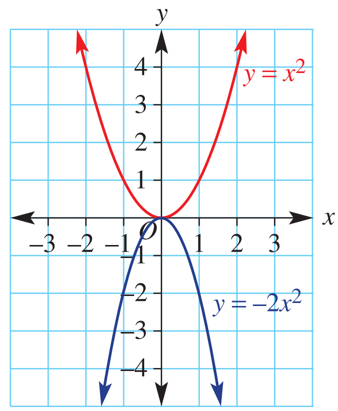
d



e



f



	Formula	Maximum or minimum	Reflected in the x-axis (yes/no)	Turning point	y value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = 3x^2$					
b	$y = \frac{1}{2}x^2$					
c	$y = 2x^2$					
d	$y = -4x^2$					
e	$y = -\frac{1}{3}x^2$					
f	$y = -2x^2$					

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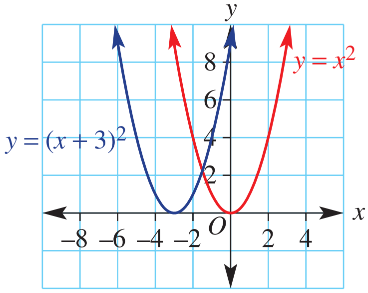
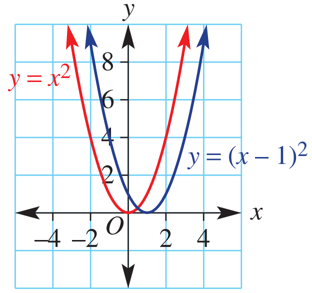
Example 2b

5 Copy and complete the table below for the following graphs.

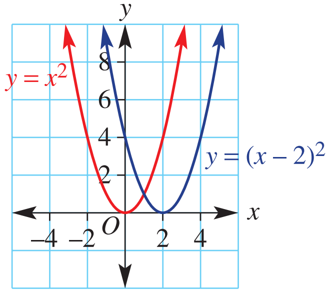
a



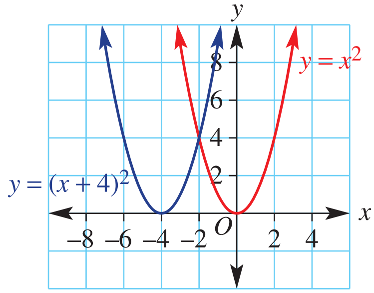
b



c



d

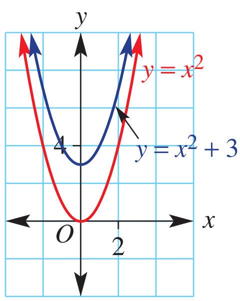


	Formula	Turning point	Axis of symmetry	y-intercept ($x = 0$)	x-intercept
a	$y = (x + 3)^2$				
b	$y = (x - 1)^2$				
c	$y = (x - 2)^2$				
d	$y = (x + 4)^2$				

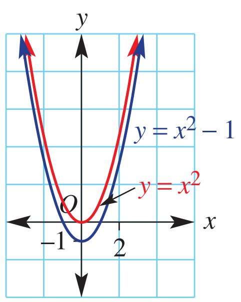
Example 2c

6 Copy and complete the table for the following graphs.

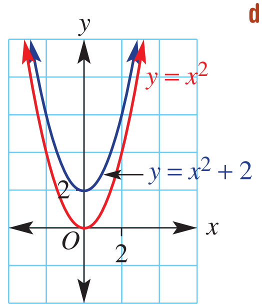
a



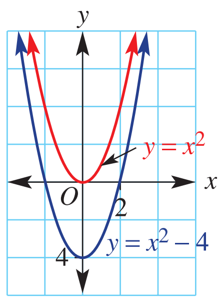
b



c



d



	Formula	Turning point	y-intercept ($x = 0$)	y value when $x = 1$
a	$y = x^2 + 3$			
b	$y = x^2 - 1$			
c	$y = x^2 + 2$			
d	$y = x^2 - 4$			

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PROBLEM-SOLVING AND REASONING	7, 8, 12, 13	7–10, 12–14	7(½), 8, 9, 11–16

7 Write down the equation of the axis of symmetry for the graphs of these rules.

a $y = x^2$ **b** $y = x^2 + 7$ **c** $y = -2x^2$

d $y = -3x^2$ **e** $y = x^2 - 4$ **f** $y = (x - 2)^2$

g $y = (x + 1)^2$ **h** $y = -(x + 3)^2$ **i** $y = -x^2 - 3$

j $y = \frac{1}{2}x^2 + 2$ **k** $y = x^2 - 16$ **l** $y = -(x + 4)^2$

8 Write down the coordinates of the turning point for the graphs of the equations in Question **7**.

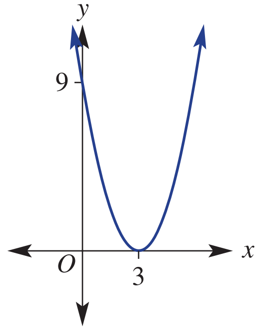
9 Find the y-intercept (i.e. when $x = 0$) for the graphs of the equations in Question **7**.

10 Match each of the following equations (**a**–**h**) to one of the graphs below (**A**–**H**).

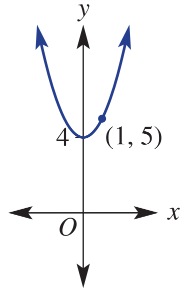
a $y = 2x^2$ **b** $y = x^2 - 6$ **c** $y = (x + 2)^2$ **d** $y = -5x^2$

e $y = (x - 3)^2$ **f** $y = \frac{1}{4}x^2$ **g** $y = x^2 + 4$ **h** $y = (x - 2)^2 + 3$

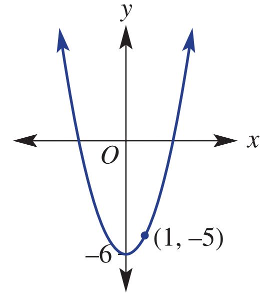
A



B



C



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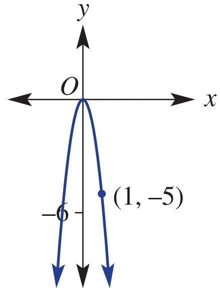

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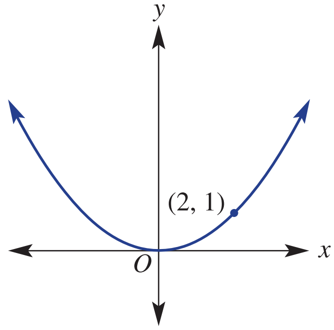

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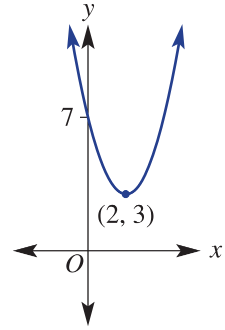

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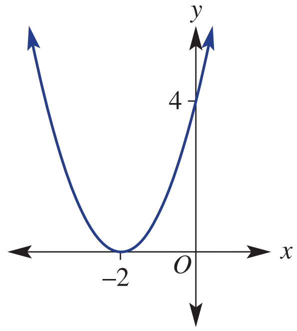

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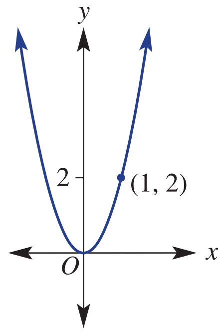

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11 The table below shows five key points on the parabola $y = x^2$.

x	−2	−1	0	1	2
y	4	1	0	1	4

What is the equation for these tables of values which form parabolas?

a

x	−2	−1	0	1	2
y	5	2	1	2	5

b

x	−2	−1	0	1	2
y	2	−1	−2	−1	2

c

x	−2	−1	0	1	2
y	8	2	0	2	8

d

x	−2	−1	0	1	2
y	5	8	9	8	5

12 Use the y values in the table to find the missing x values. Assume $x \geqslant 0$.

a $y = 2x^2 + 1$

x			
y	1	19	51

b $y = (x - 5)^2$

x			
y	0	25	49

13 **a** Using technology, plot the following pairs of graphs on the same set of axes for $-5 \leqslant x \leqslant 5$ and compare their tables of values.

i $y = x^2$ and $y = 4x^2$

ii $y = x^2$ and $y = \frac{1}{3}x^2$

iii $y = x^2$ and $y = 6x^2$

iv $y = x^2$ and $y = \frac{1}{4}x^2$

v $y = x^2$ and $y = 7x^2$

vi $y = x^2$ and $y = \frac{2}{5}x^2$

b Suggest how the constant a in $y = ax^2$ transforms the graph of $y = x^2$.

14 **a** Using technology, plot the following sets of graphs on the same set of axes for $-5 \leqslant x \leqslant 5$ and compare the turning point of each.

i $y = x^2$, $y = (x + 1)^2$, $y = (x + 2)^2$, $y = (x + 3)^2$

ii $y = x^2$, $y = (x - 1)^2$, $y = (x - 2)^2$, $y = (x - 3)^2$

b Explain how the constant h in $y = (x + h)^2$ transforms the graph of $y = x^2$.

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15 **a** Using technology, plot the following sets of graphs on the same set of axes for $-5 \leqslant x \leqslant 5$ and compare the turning point of each.

  i $y = x^2$, $y = x^2 + 1$, $y = x^2 + 2$, $y = x^2 + 3$

  ii $y = x^2$, $y = x^2 - 1$, $y = x^2 - 3$, $y = x^2 - 5$

b Explain how the constant k in $y = x^2 + k$ transforms the graph of $y = x^2$.

16 Write down an example of a quadratic equation whose graph has:

a two x-intercepts

b one x-intercept

c no x-intercepts

ENRICHMENT	–	–	17, 18

Finding the rule

17 Find a quadratic rule that satisfies the following information.

a turning point $(0, 2)$ and another point $(1, 3)$

b turning point $(0, 2)$ and another point $(1, 1)$

c turning point $(-1, 0)$ and y-intercept 1

d turning point $(2, 0)$ and y-intercept 4

e turning point $(0, 0)$ and another point $(2, 8)$

f turning point $(0, 0)$ and another point $(-1, -3)$

g turning point $(-1, 2)$ and y-intercept 3

h turning point $(4, -2)$ and y-intercept 0

18 Plot a graph of the parabola $x = y^2$ for $-3 \leqslant y \leqslant 3$ and describe its features.

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9B Sketching parabolas using transformations

Interactive

Widgets

HOTsheets

Walkthrough

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Previously we have explored simple transformations of the graph of $y = x^2$ and plotted these on a number plane. We will now formalise these transformations and sketch graphs showing key features without the need to plot every point.

The gateway arch in St Louis, USA, is a parabola measuring 192 m high.

Let's start: So where is the turning point?

Consider the quadratic rule $y = -(x - 3)^2 + 7$.

- Discuss the effect of the negative sign in $y = -x^2$ compared with $y = x^2$.
- Discuss the effect of -3 in $y = (x - 3)^2$ compared with $y = x^2$.
- Discuss the effect of $+7$ in $y = x^2 + 7$ compared with $y = x^2$.
- Now for $y = -(x - 3)^2 + 7$, find:
  - the coordinates of the turning point
  - the axis of symmetry
  - the y-intercept
- What would be the coordinates of the turning point in these quadratics?
  - $y = (x - h)^2 + k$
  - $y = -(x - h)^2 + k$

Key ideas

- Sketch parabolas by drawing a parabolic curve and labelling key features.
  - turning point
  - axis of symmetry
  - y-intercept (substitute $x = 0$)
- For $y = ax^2$, a **dilates** the graph of $y = x^2$.
  - Turning point is $(0, 0)$.
  - y-intercept and x-intercept are both 0.
  - Axis of symmetry is $x = 0$.
  - If $a > 0$ the parabola is **concave up**.
  - If $a < 0$ the parabola is **concave down**.
- For $y = (x - h)^2$, h **translates** the graph of $y = x^2$ horizontally.
  - If $h > 0$ the graph is translated h units to the right.
  - If $h < 0$ the graph is translated h units to the left.
- For $y = x^2 + k$, k translates the graph of $y = x^2$ vertically.
  - If $k > 0$ the graph is translated k units up.
  - If $k < 0$ the graph is translated k units down.
- The **turning point form** of a quadratic is $y = a(x - h)^2 + k$.
  - The turning point is (h, k).
  - The axis of symmetry is $x = h$.

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Example 3 Sketching with transformations

Sketch graphs of the following quadratic relations, labelling the turning point and the y-intercept.

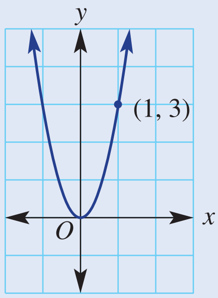
a $y = 3x^2$

b $y = -x^2 + 4$

c $y = (x - 2)^2$

SOLUTION

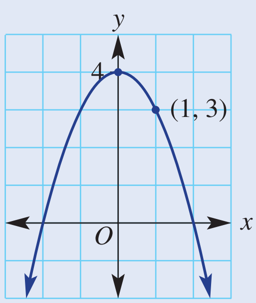
a



EXPLANATION

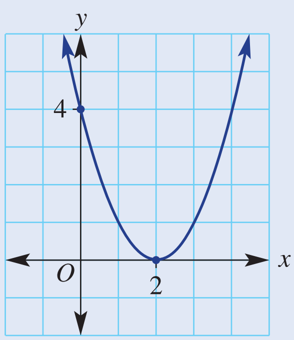
$y = 3x^2$ is concave up and narrower than $y = x^2$.
Turning point and y-intercept are at the origin (0, 0).
Substitute $x = 1$ to label a second point.
Note: $y = x^2$: $x = 1, y = 1 \rightarrow (1, 1)$
$y = 3x^2$: $x = 1, y = 3 \rightarrow (1, 3)$
(1, 1) is moved up to (1, 3).

b



$y = -x^2 + 4$ is concave down (i.e. has a maximum) and is translated 4 units up compared with $y = -x^2$.
Turning point is at (0, 4) and y-intercept (i.e. when $x = 0$) is 4.

c



$y = (x - 2)^2$ is concave up (i.e. has a minimum) and is translated 2 units right compared with $y = x^2$. Thus turning point is at (2, 0).
Substitute $x = 0$ for y-intercept: $y = (0 - 2)^2 = 4$

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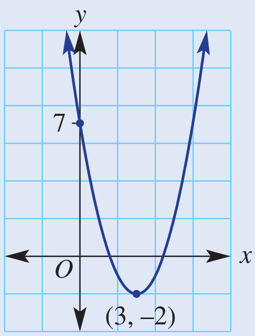
Example 4 Using turning point form

Sketch the graphs of the following, labelling the turning point and the y-intercept.

a $y = (x - 3)^2 - 2$ **b** $y = -(x + 1)^2 + 4$

SOLUTION

a $y = (x - 3)^2 - 2$



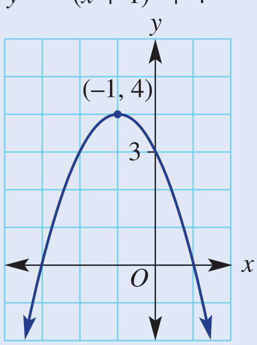
EXPLANATION

In $y = a(x - h)^2 + k$, $h = 3$ and $k = -2$, so the vertex is $(3, -2)$.

Substitute $x = 0$ to find the y-intercept:

$y = (0 - 3)^2 - 2 = 9 - 2 = 7$

b $y = -(x + 1)^2 + 4$



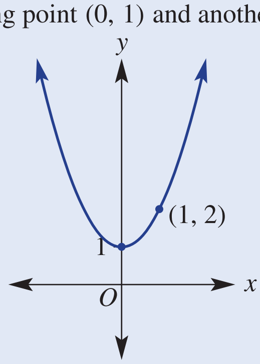
The graph is concave down since $a = -1$.

$h = -1$ and $k = 4$, so the vertex is $(-1, 4)$.

When $x = 0$, $y = -(0 + 1)^2 + 4 = 3$.

Example 5 Finding a rule from a simple graph or table

Find a rule for this parabola with turning point $(0, 1)$ and another point $(1, 2)$.



SOLUTION

$y = ax^2 + 1$

When $x = 1$, $y = 2$ so $2 = a(1)^2 + 1$.

$\therefore a = 1$

So $y = x^2 + 1$.

EXPLANATION

In $y = a(x - h)^2 + k$, $h = 0$ and $k = 1$, so the rule is $y = ax^2 + 1$.

We need $y = 2$ when $x = 1$, so $a = 1$.

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Exercise 9B

UNDERSTANDING AND FLUENCY	1–3, 4–6(½)	3, 4–6(½)	4–6(½)

1 Give the coordinates of the turning point for the graphs of these rules.

a $y = x^2$
b $y = -x^2$
c $y = x^2 + 3$
d $y = -x^2 - 3$
e $y = -x^2 + 7$
f $y = (x - 2)^2$
g $y = (x + 5)^2$
h $y = 2x^2$
i $y = -\frac{1}{3}x^2$

2 Substitute $x = 0$ to find the y-intercept of the graphs with these equations.

a $y = x^2 + 3$ **b** $y = -x^2 - 3$ **c** $y = 2x^2 - 3$
d $y = -6x^2 - 1$ **e** $y = (x - 1)^2$ **f** $y = (x + 2)^2$
g $y = -(x + 4)^2$ **h** $y = -(x - 5)^2$ **i** $y = -2(x + 1)^2$
j $y = (x + 1)^2 + 1$ **k** $y = (x - 3)^2 - 3$ **l** $y = -(x + 7)^2 - 14$

3 Choose the word *left, right, up* or *down* to suit.

a Compared with the graph of $y = x^2$, the graph of $y = x^2 + 3$ is translated ______.
b Compared with the graph of $y = x^2$, the graph of $y = (x - 3)^2$ is translated ______.
c Compared with the graph of $y = x^2$, the graph of $y = (x + 1)^2$ is translated ______.
d Compared with the graph of $y = x^2$, the graph of $y = x^2 - 6$ is translated ______.
e Compared with the graph of $y = -x^2$, the graph of $y = -x^2 - 2$ is translated ______.
f Compared with the graph of $y = -x^2$, the graph of $y = -(x + 3)^2$ is translated ______.
g Compared with the graph of $y = -x^2$, the graph of $y = -(x - 2)^2$ is translated ______.
h Compared with the graph of $y = -x^2$, the graph of $y = -x^2 + 4$ is translated ______.

Example 3

4 Sketch graphs of the following quadratics, labelling the turning point and the y-intercept.

a $y = 2x^2$ **b** $y = -3x^2$ **c** $y = \frac{1}{2}x^2$
d $y = -\frac{1}{3}x^2$ **e** $y = x^2 + 2$ **f** $y = x^2 - 4$
g $y = -x^2 + 1$ **h** $y = -x^2 - 3$ **i** $y = (x + 3)^2$
j $y = (x - 1)^2$ **k** $y = -(x + 2)^2$ **l** $y = -(x - 3)^2$

5 State the coordinates of the turning point for the graphs of these rules.

a $y = (x + 3)^2 + 1$ **b** $y = (x + 2)^2 - 4$ **c** $y = (x - 1)^2 + 3$
d $y = (x - 4)^2 - 2$ **e** $y = (x - 3)^2 - 5$ **f** $y = (x - 2)^2 + 2$
g $y = -(x - 3)^2 + 3$ **h** $y = -(x - 2)^2 + 6$ **i** $y = -(x + 1)^2 + 4$
j $y = -(x - 2)^2 - 5$ **k** $y = -(x + 1)^2 - 1$ **l** $y = -(x - 4)^2 - 10$

Example 4

6 Sketch graphs of the following quadratics, labelling the turning point and the y-intercept.

a $y = (x + 1)^2 + 1$ **b** $y = (x + 2)^2 - 1$ **c** $y = (x + 3)^2 + 2$
d $y = (x - 1)^2 + 2$ **e** $y = (x - 4)^2 + 1$ **f** $y = (x - 1)^2 - 4$
g $y = -(x - 1)^2 + 3$ **h** $y = -(x - 2)^2 + 1$ **i** $y = -(x + 3)^2 - 2$
j $y = -(x - 2)^2 + 1$ **k** $y = -(x - 4)^2 - 2$ **l** $y = -(x + 2)^2 + 2$

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PROBLEM-SOLVING AND REASONING	7, 8(½), 12(½)	7, 8(½), 10, 11, 12–13(½)	8(½), 9, 12–13(½), 14

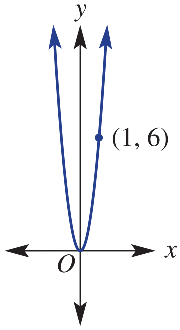
7 Write the rule for each graph if $y = x^2$ is transformed by the following.

- **a** reflected in the x-axis
- **b** translated 2 units to the left
- **c** translated 5 units down
- **d** translated 4 units up
- **e** translated 1 unit to the right
- **f** reflected in the x-axis and translated 2 units up
- **g** reflected in the x-axis and translated 3 units left
- **h** translated 5 units left and 3 units down
- **i** translated 6 units right and 1 unit up

Example 5

8 Determine the rule for the following parabolas.

a



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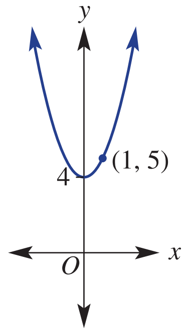

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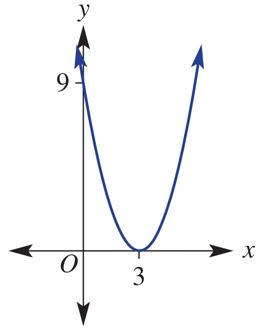

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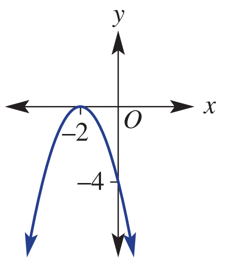

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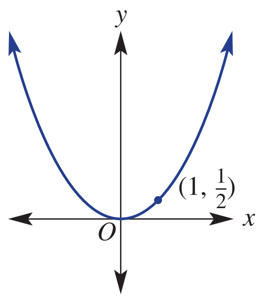

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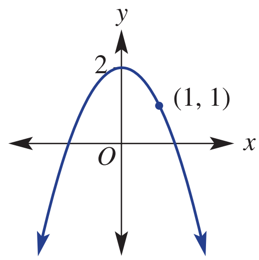

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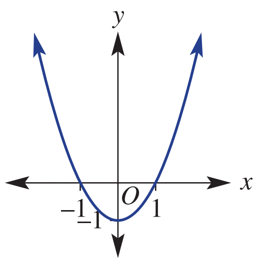

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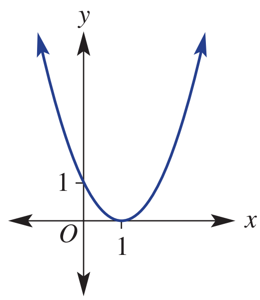
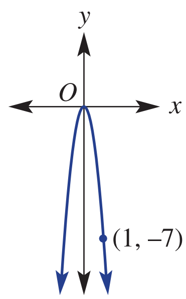


i



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9 The path of a basketball is given by $y = -(x - 5)^2 + 25$, where y metres is the height and x metres is the horizontal distance.

a Is the turning point a maximum or a minimum?
b What are the coordinates of the turning point?
c What is the y-intercept?
d What is the maximum height of the ball?
e What is the height of the ball at these horizontal distances?
i $x = 3$ **ii** $x = 7$ **iii** $x = 10$

10 Find the equation of the parabola that passes through these points.

a

x	0	1	2	3
y	−9	−8	−5	0

b

x	0	1	2	3
y	4	1	0	1

11 What points on the parabola $y = 2x^2 - 12$ have a y value of 6?

12 Recall that $y = (x - h)^2 + k$ and $y = a(x - h)^2 + k$ both have the same turning point coordinates. State the coordinates of the turning point for the graphs of these rules.

a $y = 2(x - 1)^2$ **b** $y = 3(x + 2)^2$ **c** $y = -4(x + 3)^2$
d $y = 3x^2 - 4$ **e** $y = 5x^2 - 2$ **f** $y = -2x^2 + 5$
g $y = 6(x + 4)^2 - 1$ **h** $y = 2(x + 2)^2 + 3$ **i** $y = 3(x - 5)^2 + 4$
j $y = -4(x + 2)^2 + 3$ **k** $y = -2(x + 3)^2 - 5$ **l** $y = -5(x - 3)^2 - 3$

13 Describe the transformations that take $y = x^2$ to:

a $y = (x - 3)^2$ **b** $y = (x + 2)^2$ **c** $y = x^2 - 3$
d $y = x^2 + 7$ **e** $y = -x^2$ **f** $y = (x + 2)^2 - 4$
g $y = (x - 5)^2 + 8$ **h** $y = -(x + 3)^2$ **i** $y = -x^2 + 6$

14 For $y = a(x - h)^2 + k$ write the:

a turning point
b y-intercept

ENRICHMENT	–	–	15

Sketching with many transformations

15 Sketch the graph of the following, showing the turning point and y-intercept.

a $y = 2(x - 3)^2 + 4$ **b** $y = 3(x + 2)^2 + 5$ **c** $y = -2(x - 3)^2 + 4$
d $y = -2(x + 3)^2 - 4$ **e** $y = \frac{1}{2}(x - 3)^2 + 4$ **f** $y = -\frac{1}{2}(x - 3)^2 + 4$
g $y = 4 - x^2$ **h** $y = -3 - x^2$ **i** $y = 5 - 2x^2$
j $y = 2 + \frac{1}{2}(x - 1)^2$ **k** $y = 1 - 2(x + 2)^2$ **l** $y = 3 - 4(x - 2)^2$

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9C Sketching parabolas using factorisation

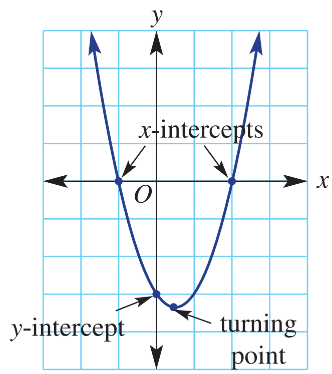
Interactive

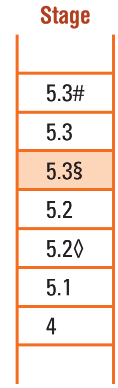
Widgets

HOTsheets

Walkthrough

A quadratic relation written in the form $y = x^2 + bx + c$ differs from that of turning point form $y = a(x - h)^2 + k$, and so the transformations of the graph of $y = x^2$ to give $y = x^2 + bx + c$ are less obvious. To address this, we have a number of options. First, we can try to factorise to find the x-intercepts and then use symmetry to find the turning point or, alternatively, we can complete the square and express the quadratic relation in turning point form. The second of these methods will be studied in the next section.



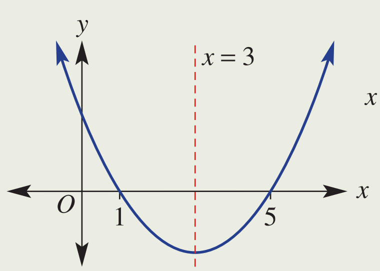


Let's start: Why does the turning point of $y = x^2 - 2x - 3$ have coordinates (1, –4)?

- Factorise $y = x^2 - 2x - 3$.
- Hence, find the x-intercepts.
- Discuss how symmetry can be used to locate the turning point.
- Hence, confirm the coordinates of the turning point.

Key ideas

- To sketch a graph of $y = x^2 + bx + c$:
  - Find the y-intercept by substituting $x = 0$.
  - Find the x-intercept(s) by substituting $y = 0$. Factorise where possible and use the null factor law.
- Once the x-intercepts are known, the turning point can be found using symmetry:
  - The axis of symmetry (also the x-coordinate of the turning point) lies halfway between the x-intercepts.



$$x = \frac{1 + 5}{2} = 3$$

  - Substitute this x-coordinate into the rule to find the y-coordinate of the turning point.

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Example 6 Using the x-intercepts to find the turning point

Sketch the graph of the quadratic $y = x^2 - 6x + 5$ and determine the coordinates of the turning point, using symmetry.

SOLUTION

y-intercept at $x = 0$: $y = 5$

x-intercepts at $y = 0$: $0 = x^2 - 6x + 5$

$$0 = (x - 5)(x - 1)$$

$$x - 5 = 0 \text{ or } x - 1 = 0$$

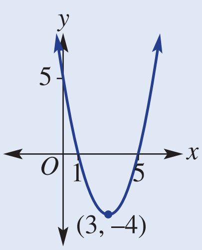
$$\therefore x = 5 \text{ or } x = 1$$

x-intercepts at 1 and 5.

Turning point at $x = \dfrac{1 + 5}{2} = 3$.

$$\begin{aligned} y &= (3)^2 - 6 \times (3) + 5 \\ &= 9 - 18 + 5 \\ &= -4 \end{aligned}$$

Turning point is a minimum at $(3, -4)$.



EXPLANATION

Identify key features of the graph: y-intercept (when $x = 0$), x-intercepts (when $y = 0$), then factorise and solve by applying the null factor law.

Using symmetry, the x-coordinate of the turning point is halfway between the x-intercepts. Substitute $x = 3$ into $y = x^2 - 6x + 5$ to find the y-coordinate of the turning point.

It is a minimum turning point since the coefficient of x^2 is positive.

Label key features on the graph and join points in the shape of a parabola.

Example 7 Sketching a perfect square

Sketch the graph of the quadratic $y = x^2 + 6x + 9$.

SOLUTION

y-intercept at $x = 0$: $y = 9$

x-intercepts at $y = 0$: $0 = x^2 + 6x + 9$

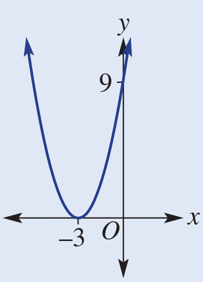
$$0 = (x + 3)^2$$

$$x + 3 = 0$$

$$\therefore x = -3$$

x-intercept at -3.

Turning point at $(-3, 0)$.



EXPLANATION

For y-intercept substitute $x = 0$.

For x-intercepts substitute $y = 0$ and factorise: $(x + 3)(x + 3) = (x + 3)^2$

Apply the null factor law to solve for x.

As there is only one x-intercept, it is also the turning point.

Label key features on graph.

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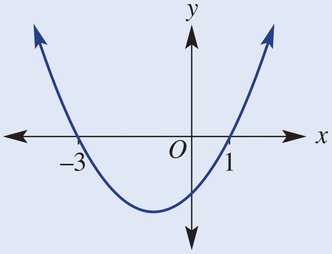
Example 8 Finding a turning point from a graph

The equation of this graph is of the form $y = (x + a)(x + b)$. Use the x-intercepts to find the values of a and b, then find the coordinates of the turning point.



SOLUTION

$a = 3$ and $b = -1$

$y = (x + 3)(x - 1)$

x-coordinate of the turning point is $\frac{-3 + 1}{2} = -1$.

y-coordinate is $(-1 + 3)(-1 - 1) = 2 \times (-2) = -4$.

Turning point is $(-1, -4)$.

EXPLANATION

Using the null factor law, $(x + 3)(x - 1) = 0$ gives $x = -3$ and $x = 1$, so $a = 3$ and $b = -1$.

Find the average of the two x-intercepts to find the x-coordinate of the turning point.

Substitute $x = -1$ into the rule to find the y value of the turning point.

Exercise 9C

UNDERSTANDING AND FLUENCY	1–3(½), 4, 5–7(½), 8	4, 5–7(½), 8, 9	5–7(½), 8, 9

1 Use the null factor law to find the x-intercepts ($y = 0$) for these factorised quadratics.

a $y = (x + 1)(x - 2)$ **b** $y = (x - 3)(x - 4)$ **c** $y = (x + 1)(x + 5)$
d $y = x(x - 3)$ **e** $y = 2x(x - 5)$ **f** $y = -3x(x + 2)$
g $y = (x + \sqrt{5})(x - \sqrt{5})$ **h** $y = (x + \sqrt{7})(x - \sqrt{7})$ **i** $y = (x + 2\sqrt{2})(x - 2\sqrt{2})$

2 Factorise and use the null factor law to find the x-intercepts for these quadratics.

a $y = x^2 + 3x + 2$ **b** $y = x^2 + 2x - 8$ **c** $y = x^2 - 8x + 16$
d $y = x^2 - 4x$ **e** $y = x^2 - 6x$ **f** $y = 2x^2 + 10x$
g $y = x^2 - 9$ **h** $y = x^2 - 25$ **i** $y = x^2 - 10$

3 Find the y-intercept for the quadratics in Question **2**.

4 A parabola has the rule $y = x^2 - 2x - 48$.

a Factorise $x^2 - 2x - 48$.
b Find the x-intercepts of $y = x^2 - 2x - 48$.
c Hence, state the equation of the axis of symmetry.
d Find the coordinates of the turning point.

Example 6

5 Sketch the graphs of the following quadratics.

a $y = x^2 - 6x + 8$ **b** $y = x^2 - 8x + 12$
c $y = x^2 + 8x + 15$ **d** $y = x^2 - 6x - 16$
e $y = x^2 - 2x - 8$ **f** $y = x^2 - 4x - 21$
g $y = x^2 + 8x + 7$ **h** $y = x^2 - 10x + 24$
i $y = x^2 - 12x + 20$

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6 Sketch graphs of the following quadratics.

a $y = x^2 - 9x + 20$
b $y = x^2 - 5x + 6$
c $y = x^2 - 13x + 12$
d $y = x^2 + 11x + 30$
e $y = x^2 + 5x + 4$
f $y = x^2 + 13x + 12$
g $y = x^2 - 4x - 12$
h $y = x^2 - x - 2$
i $y = x^2 - 5x - 14$
j $y = x^2 + 3x - 4$
k $y = x^2 + 7x - 30$
l $y = x^2 + 9x - 22$

7 Sketch by first finding x-intercepts.

a $y = x^2 + 2x$
b $y = x^2 + 6x$
c $y = x^2 - 4x$
d $y = x^2 - 5x$
e $y = x^2 + 3x$
f $y = x^2 + 7x$

Example 7

8 Sketch graphs of the following perfect squares.

a $y = x^2 + 4x + 4$
b $y = x^2 + 8x + 16$
c $y = x^2 - 10x + 25$
d $y = x^2 + 20x + 100$

9 Sketch graphs of the following quadratics, which include a difference of two squares.

a $y = x^2 - 9$
b $y = x^2 - 16$
c $y = x^2 - 4$

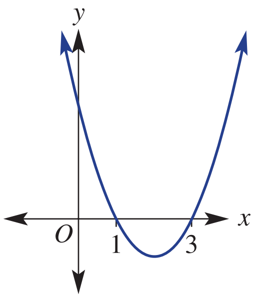
PROBLEM-SOLVING AND REASONING	10(½), 11, 15	10(½), 11–13, 15, 16	11, 12–13(½), 14, 17, 18

10 Determine the turning points of the following quadratics.

a $y = 2(x^2 - 7x + 10)$
b $y = 3(x^2 - 7x + 10)$
c $y = 3x^2 + 18x + 24$
d $y = 4x^2 + 24x + 32$
e $y = 4(x^2 - 49)$
f $y = -4(x^2 - 49)$
g $y = 3x^2 - 6x + 3$
h $y = 5x^2 - 10x + 5$

Example 8

11 The equations of these graphs are of the form $y = (x + a)(x + b)$. Use the x-intercepts to find the values of a and b, and then find the coordinates of the turning point.

a


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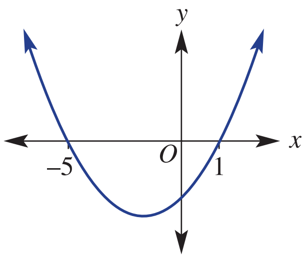

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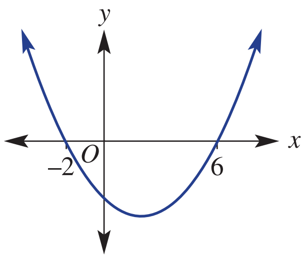

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12 State the x-intercepts and turning point for these quadratics.

a $y = x^2 - 2$
b $y = x^2 - 11$
c $y = x^2 - 50$

13 Sketch a graph of these quadratics.

a $y = 9 - x^2$
b $y = 1 - x^2$
c $y = 4x - x^2$
d $y = 3x - x^2$
e $y = -x^2 + 2x + 8$
f $y = -x^2 + 8x + 9$

14 If the graph of $y = a(x + 2)(x - 4)$ passes through the point $(2, 16)$, determine the value of a and the coordinates of the turning point for this parabola.

15 Explain why $y = (x - 3)(x - 5)$ and $y = 2(x - 3)(x - 5)$ both have the same x-intercepts.

16 **a** Explain why $y = x^2 - 2x + 1$ has only one x-intercept.
b Explain why $y = x^2 + 2$ has no x-intercepts.

17 Consider the quadratics $y = x^2 - 2x - 8$ and $y = -x^2 + 2x + 8$.

a Show that both quadratics have the same x-intercepts.
b Find the coordinates of the turning points for both quadratics.
c Compare the positions of the turning points.

18 A quadratic has the rule $y = x^2 + bx$.

a Give the y-intercept.
b Give the x-intercepts.
c Give the coordinates of the turning point.

Computer-controlled milling machines like this can be programmed with equations to create complex shapes.

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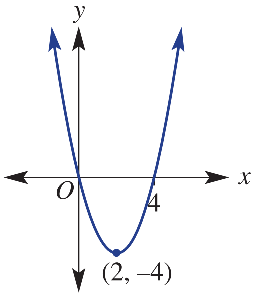
ENRICHMENT – – 19

More rules from graphs

19 Determine the equation of each of these graphs in factorised form; for example, $y = 2(x - 3)(x + 2)$.

a


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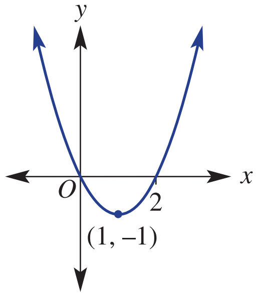

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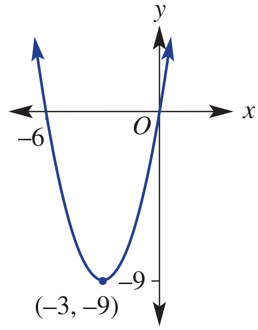

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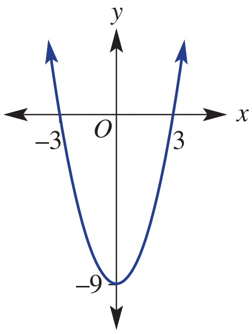

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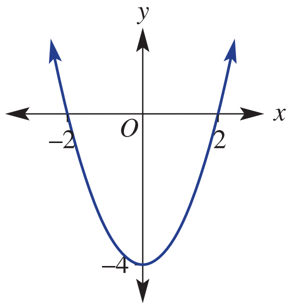

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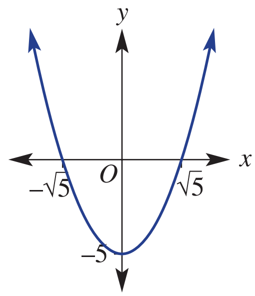

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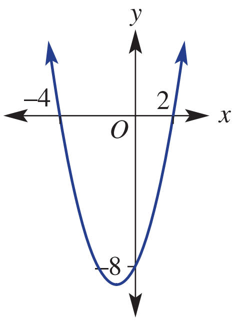

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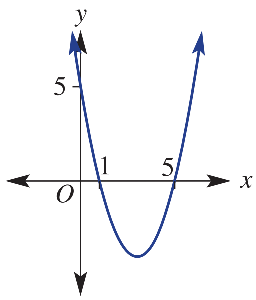

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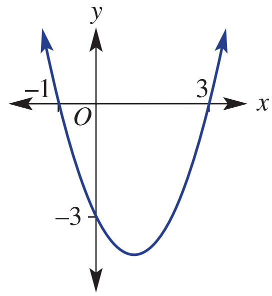

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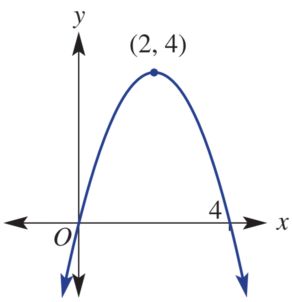

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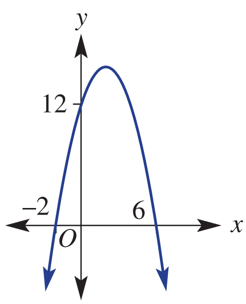

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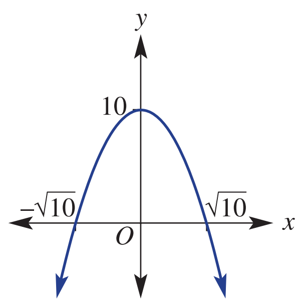

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9D Sketching parabolas by completing the square

Interactive
Widgets
HOTsheets
Walkthrough

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

We have learnt previously that the turning point of a parabola can be read directly from a rule in the form $y = a(x - h)^2 + k$. This form of quadratic can be obtained by completing the square.

Let's start: I forgot how to complete the square!

To make $x^2 + 6x$ a perfect square we need to add 9 (i.e. $\left(\frac{6}{2}\right)^2$) since $x^2 + 6x + 9 = (x + 3)^2$.

So to complete the square for $x^2 + 6x + 2$ we write $x^2 + 6x + 9 - 9 + 2 = (x + 3)^2 - 7$.

- Discuss the rules for completing the square and explain how $x^2 + 6x + 2$ becomes $(x + 3)^2 - 7$.
- What does the turning point form of $x^2 + 6x + 2$ tell us about its graph?
- How can you use the turning point form of $x^2 + 6x + 2$ to help find the x-intercepts of $y = x^2 + 6x + 2$?

Key ideas

- By **completing the square**, all quadratics in the form $y = ax^2 + bx + c$ can be expressed in turning point form: $y = a(x - h)^2 + k$.
- To sketch a quadratic in the form $y = a(x - h)^2 + k$:
  - Determine the coordinates of the turning point (h, k).
    If a is positive, the parabola has a minimum turning point.
    If a is negative, the parabola has a maximum turning point.
  - Determine the y-intercept by substituting $x = 0$.
  - Determine the x-intercepts, if any, by substituting $y = 0$ and factorising to solve the equation. Use the null factor law to help solve the equation.

Example 9 Finding key features of quadratics in turning point form

For $y = -4(x - 1)^2 + 16$:

a Determine the coordinates of its turning point and state whether it is a maximum or minimum.
b Determine the y-intercept.
c Determine the x-intercepts (if any).

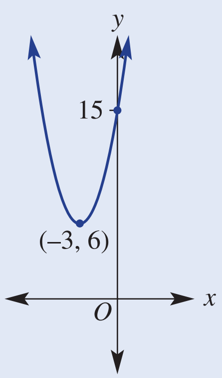
SOLUTION	EXPLANATION
a Turning point is a maximum at (1, 16).	For $y = a(x - h)^2 + k$ the turning point is at (h, k). As $a = -4$ is negative, the parabola has a maximum turning point.
b y-intercept at $x = 0$: $y = -4(0 - 1)^2 + 16$ $= -4 + 16$ $= 12$ $\therefore$ y-intercept is 12.	Substitute $x = 0$ to find the y-intercept. Recall that $(0 - 1)^2 = (-1)^2 = 1$.

Solution	Explanation
c x-intercepts at $y = 0$:	Substitute $y = 0$ for x-intercepts.
$0 = -4(x-1)^2 + 16$	Divide both sides by -4.
$0 = (x-1)^2 - 4$	Use $a^2 - b^2 = (a-b)(a+b)$ with $a = x - 1$ and
$0 = (x-1)^2 - (2)^2$	$b = 2$ to write in factorised form.
$0 = (x-1-2)(x-1+2)$	Simplify and apply the null factor law
$0 = (x-3)(x+1)$	to solve for x.
$\therefore x - 3 = 0$ or $x + 1 = 0$	Note: Check that the x-intercepts are evenly
$\therefore x - 3 = 0$ or $x + 1$	spaced either side of the turning point.
x-intercepts at -1 and 3.	

Example 10 Sketching by completing the square

Sketch these graphs by completing the square, giving the x-intercepts in exact form.

a $y = x^2 + 6x + 15$ **b** $y = x^2 - 3x - 1$

SOLUTION	EXPLANATION
a Turning point form:	To change the equation into turning point form, complete the square by adding and subtracting $\left(\frac{6}{2}\right)^2 = 9$.
$y = x^2 + 6x + 15$ $= x^2 + 6x + 9 - 9 + 15$ $= (x+3)^2 + 6$	$x^2 + 6x + 9$ is the perfect square $(x+3)^2$.
Turning point is a minimum at $(-3, 6)$.	Read off the turning point, which is a minimum, $a = 1$ is positive.
y-intercept at $x = 0$: $y = (0)^2 + 6(0) + 15$ $= 15$ $\therefore$ y-intercept at 15.	For the y-intercept, substitute $x = 0$ into the original equation.
x-intercepts at $y = 0$: $0 = (x+3)^2 + 6$	For the x-intercepts, substitute $y = 0$ into the turning point form.
$\therefore$ There is no solution and there are no x-intercepts.	This cannot be expressed as a difference of two squares and, hence, there are no factors and no x-intercepts.
	Sketch the graph, showing the key points.

Example continued over page

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b Turning point form:

$y = x^2 - 3x - 1$

$= x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 1$

$= \left(x - \frac{3}{2}\right)^2 - \frac{13}{4}$

Turning point is a minimum at $\left(\frac{3}{2}, -\frac{13}{4}\right)$.

Complete the square to write in turning point form:

$\left(\frac{3}{2}\right)^2 = \frac{9}{4}$ and $-\frac{9}{4} - 1 = -\frac{9}{4} - \frac{4}{4} = -\frac{13}{4}$

y-intercept (when $x = 0$):

$y = (0)^2 - 3(0) - 1$

$= -1$

$\therefore$ y-intercept is -1.

Substitute $x = 0$ to find the y-intercept.

x-intercepts (when $y = 0$):

$0 = \left(x - \frac{3}{2}\right)^2 - \frac{13}{4}$

$0 = \left(x - \frac{3}{2}\right)^2 - \left(\frac{\sqrt{13}}{4}\right)^2$

$0 = \left(x - \frac{3}{2} - \frac{\sqrt{13}}{2}\right)\left(x - \frac{3}{2} + \frac{\sqrt{13}}{2}\right)$

$x = \frac{3 + \sqrt{13}}{2}, x = \frac{3 - \sqrt{13}}{2}$

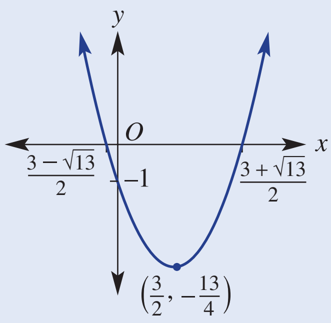
Substitute $y = 0$ to find the x-intercepts.

Factorise using difference of two squares.

$\sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{\sqrt{4}} = \frac{\sqrt{13}}{2}$

Solve for x using the null factor law:

$x - \frac{3}{2} - \frac{\sqrt{13}}{2} = 0$ or $x - \frac{3}{2} + \frac{\sqrt{13}}{2} = 0$



Label key features on graph, using exact values.

Exercise 9D

UNDERSTANDING AND FLUENCY	1–7	3–8(½)	4–8(½)

1 Copy and complete the square, then state the coordinates of the turning point (TP).

a $y = x^2 + 2x - 5$
$= x^2 + 2x + 1 - \_\_\_ - \_\_\_$
$= (\_\_\_)^2 - \_\_\_$
TP = (__, __)

b $y = x^2 + 4x - 1$
$= x^2 + 4x + \_\_\_ - \_\_\_ - 1$
$= (\_\_\_)^2 - \_\_\_$
TP = (__, __)

c $y = x^2 - 6x + 10$
$= x^2 - \_\_\_\_\_\_$
$= \_\_\_\_\_\_$
TP is (__, __).

d $y = x^2 - 3x - 7$
$= \_\_\_\_\_\_$
$= \_\_\_\_\_\_$
TP is (__, __).

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2 Solve these equations for x, giving exact answers.

a $x^2 - 9 = 0$ **b** $x^2 - 3 = 0$ **c** $(x - 1)^2 - 4 = 0$
d $(x + 1)^2 - 16 = 0$ **e** $(x + 4)^2 - 2 = 0$ **f** $(x - 6)^2 - 5 = 0$

Example 9a

3 State whether the turning points of the following are a maximum or a minimum and give the coordinates.

a $y = 2(x - 3)^2 + 5$ **b** $y = -2(x - 1)^2 + 3$
c $y = -4(x + 1)^2 - 2$ **d** $y = 6(x + 2)^2 - 5$
e $y = 3(x + 5)^2 + 10$ **f** $y = -4(x - 7)^2 + 2$
g $y = -5(x - 3)^2 + 8$ **h** $y = 2(x - 3)^2 - 7$

Example 9b

4 Determine the y-intercept of each of the following.

a $y = (x + 1)^2 + 5$ **b** $y = (x + 2)^2 - 6$ **c** $y = (x - 3)^2 - 2$
d $y = (x - 4)^2 - 7$ **e** $y = -(x + 5)^2 + 9$ **f** $y = -(x - 7)^2 - 6$
g $y = x^2 + 6x + 3$ **h** $y = x^2 + 5x + 1$ **i** $y = x^2 + 7x - 5$
j $y = x^2 + x - 8$ **k** $y = x^2 - 5x + 13$ **l** $y = x^2 - 12x - 5$

Example 9c

5 Determine the x-intercepts (if any) of the following.

a $y = (x - 3)^2 - 4$ **b** $y = (x + 4)^2 - 9$ **c** $y = (x - 3)^2 - 36$
d $y = 2(x + 2)^2 - 10$ **e** $y = -3(x - 1)^2 + 30$ **f** $y = (x - 5)^2 - 3$
g $y = (x - 4)^2$ **h** $y = (x + 6)^2$ **i** $y = 2(x - 7)^2 + 18$
j $y = -2(x - 3)^2 - 4$ **k** $y = -(x - 2)^2 + 5$ **l** $y = -(x - 3)^2 + 10$

6 Determine the x-intercepts (if any) by first completing the square and rewriting the equation in turning point form. Give exact answers.

a $y = x^2 + 6x + 5$ **b** $y = x^2 + 6x + 2$ **c** $y = x^2 + 8x - 5$
d $y = x^2 + 2x - 6$ **e** $y = x^2 - 4x + 14$ **f** $y = x^2 - 12x - 5$

7 Sketch the graphs of the following. Label the turning point and intercepts.

a $y = (x - 2)^2 - 4$ **b** $y = (x + 4)^2 - 9$ **c** $y = (x + 4)^2 - 1$
d $y = (x - 3)^2 - 4$ **e** $y = (x + 8)^2 + 16$ **f** $y = (x + 7)^2 + 2$
g $y = (x - 2)^2 + 1$ **h** $y = (x - 3)^2 + 6$ **i** $y = -(x - 5)^2 - 4$
j $y = -(x + 4)^2 - 9$ **k** $y = -(x + 9)^2 + 25$ **l** $y = -(x - 2)^2 + 4$

Example 10a

8 Sketch these graphs by completing the square. Label the turning point and intercepts.

a $y = x^2 + 4x + 3$ **b** $y = x^2 - 2x - 3$ **c** $y = x^2 + 6x + 9$
d $y = x^2 - 8x + 16$ **e** $y = x^2 - 2x - 8$ **f** $y = x^2 - 2x - 15$
g $y = x^2 + 8x + 7$ **h** $y = x^2 + 6x + 5$ **i** $y = x^2 + 12x$

PROBLEM-SOLVING AND REASONING	9(½), 12(½)	9–10(½), 12(½), 13	9–11(½), 13, 14

Example 10b

9 Sketch these graphs by completing the square. Label the turning point and intercepts, using exact values.

a $y = x^2 + 4x + 1$ **b** $y = x^2 + 6x - 5$ **c** $y = x^2 - 2x + 6$
d $y = x^2 - 8x + 20$ **e** $y = x^2 + 4x - 4$ **f** $y = x^2 - 3x + 1$
g $y = x^2 + 5x + 2$ **h** $y = x^2 - x - 2$ **i** $y = x^2 + 3x + 3$

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10 Complete the square and determine if the graphs of the following quadratics will have no, one or two x-intercepts.

a $y = x^2 - 4x + 2$ **b** $y = x^2 - 4x + 4$ **c** $y = x^2 + 6x + 9$
d $y = x^2 + 2x + 6$ **e** $y = x^2 - 3x + 4$ **f** $y = x^2 - 5x + 5$

11 Take out a common factor and complete the square to find the x-intercepts for these quadratics.

a $y = 2x^2 + 4x - 10$ **b** $y = 3x^2 - 12x + 9$ **c** $y = 2x^2 - 12x - 14$
d $y = 4x^2 + 16x - 24$ **e** $y = 5x^2 + 10x - 25$ **f** $y = 2x^2 - 6x + 2$

12 To sketch a graph of the form $y = -x^2 + bx + c$ we can complete the square by taking out a factor of -1. Here is an example.

$$\begin{aligned} y &= -x^2 - 2x + 5 \\ &= -(x^2 + 2x - 5) \\ &= -(x^2 + 2x + 1 - 1 - 5) \\ &= -((x + 1)^2 - 6) \\ &= -(x + 1)^2 + 6 \end{aligned}$$

So the turning point is a maximum at $(-1, 6)$.

Sketch the graph of these quadratics using the technique above.

a $y = -x^2 - 4x + 3$ **b** $y = -x^2 + 2x + 2$ **c** $y = -x^2 + 6x - 4$
d $y = -x^2 + 8x - 8$ **e** $y = -x^2 - 3x - 5$ **f** $y = -x^2 - 5x + 2$

13 For what values of k will the graph of $y = (x - h)^2 + k$ have:

a no x-intercepts? **b** one x-intercept? **c** two x-intercepts?

14 Show that $x^2 + bx + c = \left(x + \dfrac{b}{2}\right)^2 - \dfrac{b^2 - 4c}{4}$.

ENRICHMENT	–	–	15

Completing the square with non-monics

15 This example shows how to complete the square with non-monic quadratics of the form $y = ax^2 + bx + c$.

$$\begin{aligned} y &= 3x^2 + 6x + 1 \\ &= 3\left(x^2 + 2x + \frac{1}{3}\right) \\ &= 3\left(x^2 + 2x + 1 - 1 + \frac{1}{3}\right) \\ &= 3\left((x + 1)^2 - \frac{2}{3}\right) \\ &= 3(x + 1)^2 - 2 \end{aligned}$$

Key features:
The turning point is $(-1, -2)$.
y-intercept is 1.
x-intercepts:

$$\begin{aligned} 0 &= 3(x + 1)^2 - 2 \\ (x + 1)^2 &= \frac{2}{3} \\ x &= \pm\sqrt{\frac{2}{3}} - 1 \end{aligned}$$

Use this technique to sketch the graphs of these non-monic quadratics.

a $y = 4x^2 + 8x + 3$ **b** $y = 3x^2 - 12x + 10$ **c** $y = 2x^2 + 12x + 1$
d $y = 2x^2 + x - 3$ **e** $y = 2x^2 - 7x + 3$ **f** $y = 4x^2 - 8x + 20$
g $y = 6x^2 + 5x + 9$ **h** $y = 5x^2 - 3x + 7$ **i** $y = 5x^2 + 12x$
j $y = 7x^2 + 10x$ **k** $y = -3x^2 - 9x + 2$ **l** $y = -4x^2 + 10x - 1$

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9E Sketching parabolas using the quadratic formula and the discriminant

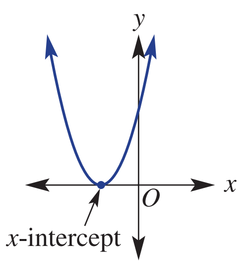
So far we have found x-intercepts for parabolas by factorising (and using the null factor law) and by completing the square. An alternative method is to use the quadratic formula, which states that if $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The discriminant $\Delta = b^2 - 4ac$ determines the number of solutions to the equation.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

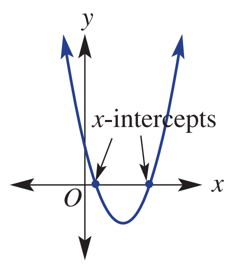
If $\Delta = 0$, then $b^2 - 4ac = 0$.
The solution to the equation becomes $x = -\dfrac{b}{2a}$.

There is one solution and one x-intercept.

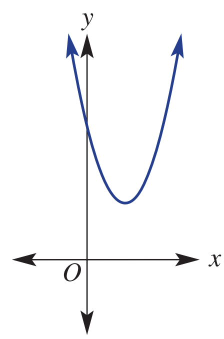


If $\Delta > 0$, then $b^2 - 4ac > 0$.
The solution to the equation becomes $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

There are two solutions and two x-intercepts.



If $\Delta < 0$, then $b^2 - 4ac < 0$.
Square roots exist for positive numbers only.
There are no solutions or x-intercepts.



Let's start: No working required

Three students set to work to find the x-intercepts for $y = x^2 - 2x + 3$:

Student A finds the intercepts by factorising.
Student B finds the intercepts by completing the square.
Student C uses the discriminant in the quadratic formula.

- Try the method used by student A. What do you notice?
- Try the method used by student B. What do you notice?
- What is the value of the discriminant found by student C? What does this tell them about the number of x-intercepts for the quadratic?
- What advice would student C give students A and B?

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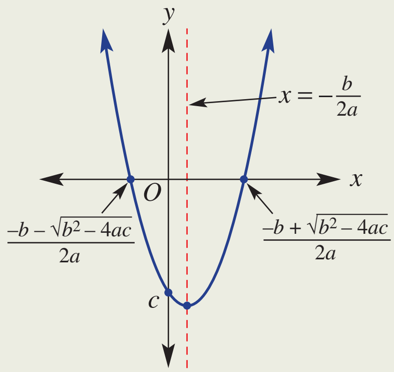
Key ideas

- To sketch the graph of $y = ax^2 + bx + c$, find the following points.
  - y-intercept at $x = 0$: $y = a(0)^2 + b(0) + c = c$
  - x-intercepts when $y = 0$: $0 = ax^2 + bx + c$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  - To determine if there are no, one or two solutions/x-intercepts, use the discriminant $\Delta = b^2 - 4ac$.
    If $\Delta < 0 \rightarrow$ no x-intercepts.
    If $\Delta = 0 \rightarrow$ one x-intercept.
    If $\Delta > 0 \rightarrow$ two x-intercepts.
  - Turning point: The x-coordinate lies halfway between the x-intercepts, so $x = -\frac{b}{2a}$. The y-coordinate is found by substituting the x-coordinate into the original equation.
  - $x = -\frac{b}{2a}$ is the axis of symmetry.



Example 11 Using the discriminant and $x = -\frac{b}{2a}$ to find the turning point

Consider the parabola given by the quadratic equation $y = 3x^2 - 6x + 5$.

a Determine the number of x-intercepts.
b Determine the y-intercept.
c Use $x = -\frac{b}{2a}$ to determine the turning point.

SOLUTION

a $\Delta = b^2 - 4ac$
$= (-6)^2 - 4(3)(5)$
$= -24$
$\Delta < 0$, so there are no x-intercepts.

b y-intercept is at 5.

c $x = -\frac{b}{2a} = -\frac{(-6)}{2(3)} = 1$
$y = 3(1)^2 - 6(1) + 5$
$= 3 - 6 + 5$
$= 2$
$\therefore$ Turning point is at $(1, 2)$.

EXPLANATION

Use the discriminant $\Delta = b^2 - 4ac$ to find the number of x-intercepts. In $3x^2 - 6x + 5$, $a = 3$, $b = -6$ and $c = 5$.

Substitute $x = 0$ for the y-intercept.

For the x-coordinate of the turning point use $x = -\frac{b}{2a}$ with $a = 3$ and $b = -6$, as above.
Substitute the x-coordinate into $y = 3x^2 - 6x + 5$ to find the corresponding y-coordinate of the turning point.

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Example 12 Sketching graphs using the quadratic formula

Sketch the graph of the quadratic $y = 2x^2 + 4x - 3$, labelling all significant points. Round the x-intercepts to 2 decimal places.

SOLUTION

y-intercept is at -3.

x-intercepts ($y = 0$): $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-4 \pm \sqrt{4^2 - 4(2)(-3)}}{2(2)}$$

$$= \frac{-4 \pm \sqrt{40}}{4}$$

$$= \frac{\cancel{-4}^{-2} \pm \cancel{2}^{1}\sqrt{10}}{\cancel{4}_{2}}$$

$$= \frac{-2 \pm \sqrt{10}}{2}$$

$x = 0.58, -2.58$ (to 2 decimal places)

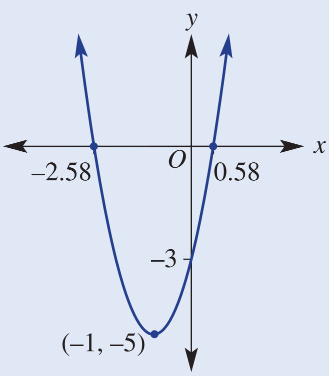
Turning point is at $x = -\dfrac{b}{2a}$

$$= -\frac{(4)}{2(2)}$$

$$= -1$$

and $\therefore y = 2(-1)^2 + 4(-1) - 3 = -5$.

$\therefore$ Turning point is at $(-1, -5)$.



EXPLANATION

Identify key features; i.e. x- and y-intercepts and the turning point. Substitute $x = 0$ for the y-intercept.

Use the quadratic formula to find the x-intercepts.

For $y = 2x^2 + 4x - 3$, $a = 2$, $b = 4$ and $c = -3$.

Simplify $\sqrt{40} = \sqrt{4 \times 10} = 2\sqrt{10}$, then cancel the common factor of 2.

Use a calculator to round to 2 decimal places.

Substitute $x = -1$ into $y = 2x^2 + 4x - 3$ to find the y-coordinate of the turning point.

Label the key features on the graph and sketch.

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Exercise 9E

UNDERSTANDING AND FLUENCY	1–3, 4–6(½)	3, 4–7(½)	4–7(½)

1 A graph has the rule $y = ax^2 + bx + c$. Determine the number of x-intercepts it will have if:

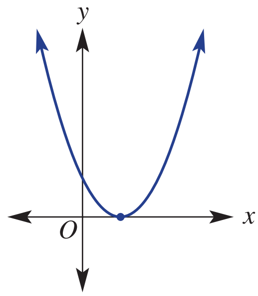
a $b^2 - 4ac > 0$ **b** $b^2 - 4ac < 0$ **c** $b^2 - 4ac = 0$

2 Give the exact value of $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ when:

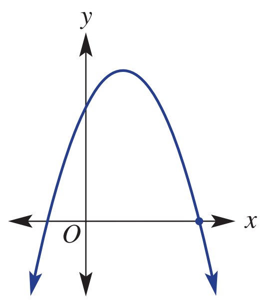
a $a = 1, b = 2, c = -1$ **b** $a = -2, b = 3, c = 5$
c $a = -2, b = -1, c = 2$ **d** $a = 3, b = 6, c = -2$

3 For the following graphs, state whether the discriminant of these quadratics would be zero, positive or negative.

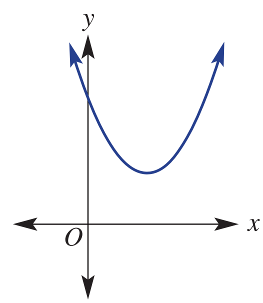
a



b



c



Example 11a

4 Use the discriminant to determine the number of x-intercepts for the parabolas given by the following quadratics.

a $y = x^2 + 4x + 4$ **b** $y = x^2 - 3x + 5$ **c** $y = -x^2 + 4x + 2$
d $y = 3x^2 - 4x - 2$ **e** $y = 2x^2 - x + 2$ **f** $y = 2x^2 - 12x + 18$
g $y = 3x^2 - 2x$ **h** $y = 3x^2 + 5x$ **i** $y = -3x^2 - 2x$
j $y = 3x^2 + 5$ **k** $y = 4x^2 - 2$ **l** $y = -5x^2 + x$

Example 11b

5 Determine the y-intercept for the parabolas given by the following quadratics.

a $y = x^2 + 2x + 3$ **b** $y = x^2 - 4x + 5$ **c** $y = 4x^2 + 3x - 2$
d $y = 5x^2 - 2x - 4$ **e** $y = -2x^2 - 5x + 8$ **f** $y = -2x^2 + 7x - 10$
g $y = 3x^2 + 8x$ **h** $y = -4x^2 - 3x$ **i** $y = 5x^2 - 7$

Example 11c

6 Using $x = -\dfrac{b}{2a}$, determine the coordinates of the turning points for the parabolas defined by the following quadratics.

a $y = x^2 + 2x + 4$ **b** $y = x^2 + 4x - 1$ **c** $y = x^2 - 4x + 3$
d $y = -x^2 + 2x - 6$ **e** $y = -x^2 - 3x + 4$ **f** $y = -x^2 + 7x - 7$
g $y = 2x^2 + 3x - 4$ **h** $y = 4x^2 - 3x$ **i** $y = -4x^2 - 9$
j $y = -4x^2 + 2x - 3$ **k** $y = -3x^2 - 2x$ **l** $y = -5x^2 + 2$

Example 12

7 Sketch the graph of these quadratics, labelling all significant points. Round the x-intercepts to 2 decimal places.

a $y = 2x^2 + 8x - 5$ **b** $y = 3x^2 + 6x - 2$ **c** $y = 4x^2 - 2x - 3$
d $y = 2x^2 - 4x - 9$ **e** $y = 2x^2 - 8x - 11$ **f** $y = 3x^2 + 9x - 10$
g $y = -3x^2 + 6x + 8$ **h** $y = -2x^2 - 4x + 7$ **i** $y = -4x^2 + 8x + 3$
j $y = -2x^2 - x + 12$ **k** $y = -3x^2 - 2x$ **l** $y = -5x^2 - 10x - 4$

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PROBLEM-SOLVING AND REASONING	8–9(½), 11	8–9(½), 11, 12	8–9(½), 10, 12, 13

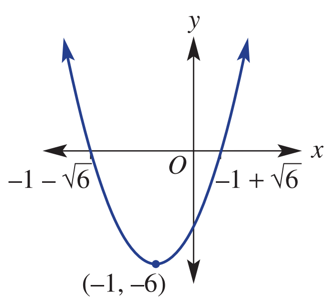
8 Sketch the graphs of these quadratics.

- **a** $y = 4x^2 + 12x + 9$
- **b** $y = 9x^2 - 6x + 1$
- **c** $y = -4x^2 - 20x - 25$
- **d** $y = -9x^2 + 30x - 25$
- **e** $y = -2x^2 + 8x - 11$
- **f** $y = -3x^2 + 12x - 16$
- **g** $y = 4x^2 + 4x + 3$
- **h** $y = 3x^2 + 4x + 2$

9 Give the exact value of the x-intercepts of the graphs of these parabolas. Simplify surds.

- **a** $y = 3x^2 - 6x - 1$
- **b** $y = -2x^2 - 4x + 3$
- **c** $y = -4x^2 + 8x + 6$
- **d** $y = 2x^2 + 6x - 3$
- **e** $y = 2x^2 - 8x + 5$
- **f** $y = 5x^2 - 10x - 1$

10 Find a rule in the form $y = ax^2 + bx + c$ that matches this graph.



11 Write down two rules in the form $y = ax^2 + bx + c$ that have:

- **a** two x-intercepts
- **b** one x-intercept
- **c** no x-intercepts

12 Explain why the quadratic formula gives only one solution when the discriminant $b^2 - 4ac = 0$.

13 Write down the quadratic formula for monic quadratic equations (i.e. where $a = 1$).

ENRICHMENT	–	–	14, 15

Some proof

14 Substitute $x = -\frac{b}{2a}$ into $y = ax^2 + bx + c$ to find the general rule for the y-coordinate of the turning point in terms of a, b and c.

15 Prove the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ by solving $ax^2 + bx + c = 0$.

Hint: Divide both sides by a and complete the square.

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9F Applications of parabolas

Interactive
Widgets
HOTsheets
Walkthrough

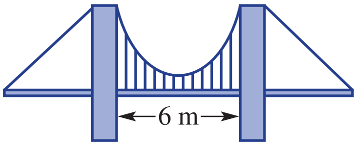
Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Quadratic equations and their graphs can be used to solve a range of practical problems. These could involve, for example, the path of a projectile or the shape of a bridge's arch. We can relate quantities with quadratic rules and use their graphs to illustrate key features. For example, x-intercepts show where one quantity (y) is equal to zero, and the turning point is where a quantity is a maximum or minimum.

Let's start: The civil engineer

Belinda, a civil engineer, designs a model for the curved cable of a 6 m suspension bridge using the equation $h = (d - 3)^2 + 2$, where h metres is the height of the hanging cables above the road for a distance d metres from the left pillar.



- What are the possible values for d?
- Sketch the graph of $h = (d - 3)^2 + 2$ for appropriate values of d.
- What is the height of the pillars above the road?
- What is the minimum height of the cable above the road?
- Discuss how key features of the graph have helped to answer the questions above.

Key ideas

- Applying quadratics to solve problems may involve:
  - defining variables
  - forming equations
  - solving equations
  - deciding on a suitable range of values for the variables
  - sketching graphs showing key features
  - finding the maximum or minimum turning point

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Example 13 Applying quadratics in problems

A piece of wire measuring 100 cm in length is bent into the shape of a rectangle. Let x cm be the breadth of the rectangle.

a Use the perimeter to write an expression for the length of the rectangle in terms of x.
b Write an equation for the area of the rectangle (A cm^2) in terms of x.
c Determine the suitable values of x.
d Sketch the graph of A versus x for suitable values of x.
e Use the graph to determine the maximum area that can be formed.
f What will be the dimensions of the rectangle to achieve its maximum area?

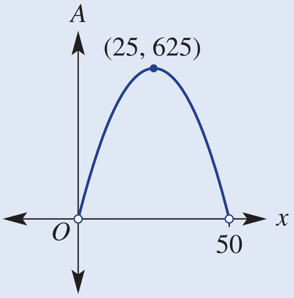
SOLUTION

a
$$\begin{aligned} 2 \times \text{length} + 2x &= 100 \\ 2 \times \text{length} &= 100 - 2x \\ \text{length} &= 50 - x \end{aligned}$$

b $A = x(50 - x)$

c Length and breadth must be positive, so we require:
$x > 0$ and $50 - x > 0$
i.e. $x > 0$ and $50 > x$
i.e. $0 < x < 50$

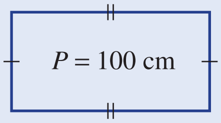
d



e The maximum area that can be formed is 625 cm^2.

f Maximum occurs when width $x = 25$ cm, so
$$\begin{aligned} \text{length} &= 50 - 25 \\ &= 25 \text{ cm} \end{aligned}$$
Dimensions that give maximum area are 25 cm by 25 cm, which is, in fact, a square.

EXPLANATION



100 cm of wire will form the perimeter.
Length is half of ($100 - 2 \times$ breadth).

Area of a rectangle = length × breadth

Require each dimension to be positive, solve for x.

Sketch the graph, labelling the intercepts and turning point, which has x-coordinate halfway between the x-intercepts; i.e. $x = 25$. Substitute $x = 25$ into the area formula to find the maximum area: $A = 25(50 - 25) = 625$.
Note open circles at $x = 0$ and $x = 50$, as these points are not included in the range of x values.

Read from the graph. The maximum area is the y-coordinate of the turning point.

From turning point, $x = 25$ gives the maximum area. Substitute to find the corresponding length.

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Exercise 9F

UNDERSTANDING AND FLUENCY	1–5	3–6	5–7

1 A ball is thrown upwards from ground level and reaches a height of h metres after t seconds, as given by the formula $h = 20t - 5t^2$.

   a Sketch a graph of the rule for $0 \leqslant t \leqslant 4$ by finding the t-intercepts (x-intercepts) and the coordinates of the turning point.

   b What maximum height does the ball reach?

   c How long does it take the ball to return to ground level?

2 The path of a javelin thrown by Jo is given by the formula $h = -\frac{1}{16}(d - 10)^2 + 9$, where h metres is the height of the javelin above the ground and d metres is the horizontal distance travelled.

   a Sketch the graph of the rule for $0 \leqslant d \leqslant 22$ by finding the intercepts and the coordinates of the turning point.

   b What is the maximum height the javelin reaches?

   c What horizontal distance does the javelin travel?

3 A wood turner carves out a bowl according to the formula $d = \frac{1}{3}x^2 - 27$, where d cm is the depth of the bowl and x cm is the distance from the centre of the bowl.

   a Sketch a graph for $-9 \leqslant x \leqslant 9$, showing x-intercepts and the turning point.

   b What is the breadth of the bowl?

   c What is the maximum depth of the bowl?

4 The equation for the arch of a particular bridge is given by $h = -\frac{1}{500}(x - 100)^2 + 20$, where h metres is the height above the base of the bridge and x metres is the distance from the left side.

   a Determine the coordinates of the turning point of the graph.

   b Determine the x-intercepts of the graph.

   c Sketch the graph of the arch for appropriate values of x.

   d What is the span of the arch?

   e What is the maximum height of the arch?

Richmond Bridge, Tasmania, is the oldest bridge in Australia that is still in use.

Example 13

5 A 20 cm piece of wire is bent to form a rectangle. Let x cm be the breadth of the rectangle.

   a Use the perimeter to write an expression for the length of the rectangle in terms of x.

   b Write an equation for the area of the rectangle (A cm^2) in terms of x.

   c Determine suitable values of x.

   d Sketch the graph of A versus x for suitable values of x.

   e Use the graph to determine the maximum area that can be formed.

   f What will be the dimensions of the rectangle to achieve its maximum area?

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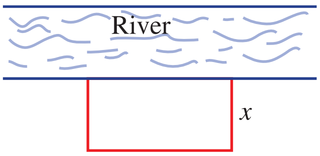
6 A farmer has 100 m of fencing to form a rectangular paddock with a river on one side (that does not require fencing), as shown.



a Use the perimeter to write an expression for the length of the paddock in terms of x.
b Write an equation for the area of the paddock ($A\,\text{m}^2$) in terms of x.
c Determine suitable values of x.
d Sketch the graph of A versus x for suitable values of x.
e Use the graph to determine the maximum paddock area that can be formed.
f What will be the dimensions of the paddock to achieve its maximum area?

7 The sum of two positive numbers is 20 and x is the smaller number.
a Write the second number in terms of x.
b Write a rule for the product, P, of the two numbers in terms of x.
c Sketch a graph of P vs x.
d Find the values of x when:
i $P = 0$ **ii** P is a maximum
e What is the maximum value of P?

PROBLEM-SOLVING AND REASONING	8, 9, 11	8, 9, 11–13	9, 10, 13–15

8 The equation for a support span is given by $h = -\frac{1}{40}(x - 20)^2$, where h metres is the distance below the base of a bridge and x metres is the distance from the left side.
a Determine the coordinates of the turning point of the graph.
b Sketch a graph of the equation for suitable values of x.
c What is the width of the support span?
d What is the maximum height of the support span?

9 A rock is tossed from the top of a 30 metre tall cliff and its height (h metres) above the sea is given by $h = 30 - 5t^2$, where t is in seconds.
a Find the exact time it takes for the rock to hit the water.
b Sketch a graph of h vs t for appropriate values of t.
c What is the exact time it takes for the rock to fall to a height of 20 metres?

10 A bird dives into the water to catch a fish. It follows a path given by $h = t^2 - 8t + 7$, where h is the height in metres above sea level and t is the time in seconds.
a Sketch a graph of h vs t, showing intercepts and the turning point.
b Find the time when the bird:
i enters the water **ii** exits the water **iii** reaches a maximum depth
c What is the maximum depth to which the bird dives?
d At what times is the bird at a depth of 8 metres?

11 Every person in a room is going to shake hands with all the other people in the room.
a Copy and complete the table below.

Number of people (P)	0	1	2	3	4	5
Number of handshakes (H)						

b The equation for the number of handshakes is $H =$ ________________.
c If there are 50 people, how many handshakes will occur?
d If 3741 handshakes occur, how many people are in the room?

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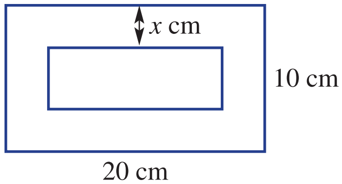
12 The height, h metres, of a flying kite is given by the rule $h = t^2 - 6t + 10$, for t seconds.

- **a** Find the minimum height of the kite during this time.
- **b** Does the kite ever hit the ground during this time? Give reasons.

13 The sum of two numbers is 64. Show that their product has a maximum of 1024.

14 A framed picture has a total length and breadth of 20 cm and 10 cm. The frame has width x cm.



- **a** Find the rule for the area (A cm^2) of the picture inside.
- **b** What are the minimum and maximum values of x?
- **c** Sketch a graph of A vs x, using suitable values of x.
- **d** Explain why there is no turning point for your graph, using suitable values of x.
- **e** Find the width of the frame if the area of the picture is 144 cm^2.

15 A dolphin jumping out of the water follows a path described by $h = -\frac{1}{2}(x^2 - 10x + 16)$, where h is the vertical height in metres and x metres is the horizontal distance travelled.

- **a** How far horizontally does the dolphin travel out of the water?
- **b** Does the dolphin ever reach a height of 5 metres above water level? Give reasons.

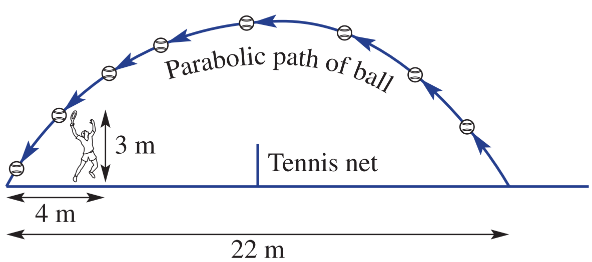
ENRICHMENT	–	–	16, 17

The highway and the river and the lobbed ball

16 The path of a river is given by the rule $y = \frac{1}{10}x(x - 100)$ and all units are given in metres. A highway is to be built near or over the river on the line $y = c$.

- **a** Sketch a graph of the path of the river, showing key features.
- **b** For the highway with equation $y = c$, determine how many bridges will need to be built if:
  - **i** $c = 0$
  - **ii** $c = -300$
- **c** Locate the coordinates of the bridge, correct to 1 decimal place, if:
  - **i** $c = -200$
  - **ii** $c = -10$
- **d** Describe the situation when $c = -250$.

17 A tennis ball is lobbed from ground level and must cover a horizontal distance of 22 m if it is to land just inside the opposite end of the court. If the opponent is standing 4 m from the baseline and he can hit any ball less than 3 m high, what is the lowest maximum height the lob must reach to win the point?



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9G Lines and parabolas

Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

We have seen previously when solving a pair of linear equations simultaneously that there is, at most, one solution. Graphically, this represents the point of intersection for the two straight lines. When the lines are parallel there will be no solution; that is, there is no point of intersection.

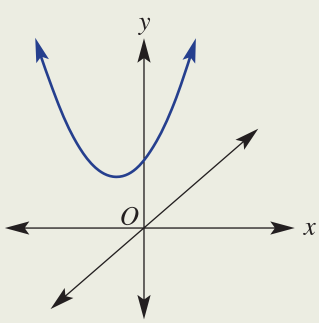
For the intersection of a parabola and a line we can have either zero, one or two points of intersection. As we have done for linear simultaneous equations, we can use the method of substitution to solve a linear equation and a non-linear equation simultaneously.

Let's start: How many times does a line cut a parabola?

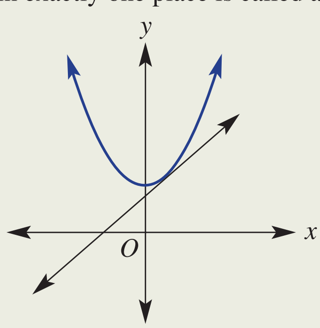
- Use computer graphing software to plot a graph of $y = x^2$.
- By plotting lines of the form $x = h$, determine how many points of intersection a vertical line will have with the parabola.
- By plotting lines of the form $y = c$, determine how many points of intersection a horizontal line could have with the parabola.
- By plotting straight lines of the form $y = 2x + k$ for various values of k, determine the number of possible intersections between a line and a parabola.
- State some values of k for which the line above intersects the parabola:
  - twice
  - never
- Can you find the value of k for which the line intersects the parabola exactly once?

Key ideas

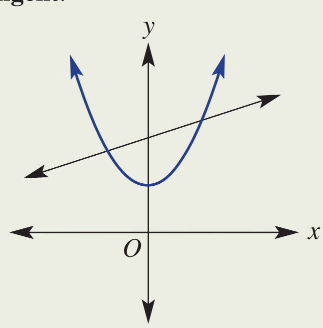
- When solving a pair of simultaneous equations involving a parabola and a line, we may obtain zero, one or two solutions. Graphically, this represents no, one or two points of intersection between the parabola and the line.
  - A line that intersects a curve twice is called a **secant**.
  - A line that intersects a curve in exactly one place is called a **tangent**.



No points of intersection



One point of intersection



Two points of intersection

- The method of substitution is used to solve the equations simultaneously.
  - Substitute one equation into the other.
  - Rearrange the resulting equation into the form $ax^2 + bx + c = 0$.
  - Solve for x by factorising and applying the null factor law or use the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
  - Substitute the x values into one of the original equations to find the corresponding y value.

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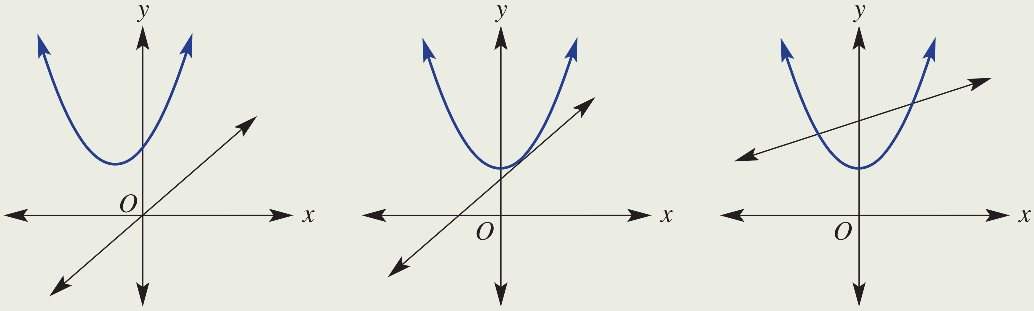
Key ideas

- After substituting the equations and rearranging, we arrive at an equation of the form $ax^2 + bx + c = 0$. Hence, the discriminant $b^2 - 4ac$ can be used to determine the number of solutions (i.e points of intersection) of the two equations.



No solutions $b^2 - 4ac < 0$ | One solution $b^2 - 4ac = 0$ | Two solutions $b^2 - 4ac > 0$

Example 14 Finding points of intersection of a parabola and a horizontal line

Find any points of intersection of these parabolas and lines.

a $y = x^2 - 3x$
$y = 4$

b $y = x^2 + 2x + 4$
$y = -2$

SOLUTION

a By substitution:

$$x^2 - 3x = 4$$
$$x^2 - 3x - 4 = 0$$
$$(x - 4)(x + 1) = 0$$
$$x - 4 = 0 \text{ or } x + 1 = 0$$
$$x = 4 \text{ or } x = -1$$

$\therefore$ The points of intersection are at $(4, 4)$ and $(-1, 4)$.

b By substitution:

$$x^2 + 2x + 4 = -2$$
$$x^2 + 2x + 6 = 0$$

Using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(6)}}{2(1)}$$
$$= \frac{-2 \pm \sqrt{-20}}{2}$$

$\therefore$ There are no real solutions.
$\therefore$ There are no points of intersection.

EXPLANATION

As both equations are of the form $y = ...$, set them equal to each other.

Write in the form $ax^2 + bx + c = 0$ by subtracting 4 from both sides.

Factorise and apply the null factor law to solve for x.

As the points are on the line $y = 4$, the y-coordinate of the points of intersection is 4.

Set the equations equal to each other by substitution.

Apply the quadratic formula to solve $x^2 + 2x + 6 = 0$, where $a = 1$, $b = 2$ and $c = 6$.

$\sqrt{-20}$ has no real solutions.

The parabola $y = x^2 + 2x + 4$ and the line $y = -2$ do not intersect.

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Example 15 Solving simultaneous equations using substitution

Solve the following equations simultaneously.

a $y = x^2$
$y = 2x$

b $y = -4x^2 - x + 6$
$y = 3x + 7$

c $y = x^2 + 1$
$2x - 3y = -4$

SOLUTION / EXPLANATION

a By substitution:

$$x^2 = 2x$$
$$x^2 - 2x = 0$$
$$x(x - 2) = 0$$
$$x = 0 \text{ or } x - 2 = 0$$
$$x = 0 \text{ or } x = 2$$

Set the equations equal to each other by substitution.
Rearrange equation so that it is equal to zero by subtracting $2x$ from both sides.
Factorise by removing the common factor x.
Apply the null factor law to solve for x.

When $x = 0, y = 2 \times (0) = 0$.
When $x = 2, y = 2 \times (2) = 4$.

Substitute the x values into $y = 2x$ to obtain the corresponding y value. Alternatively, the equation $y = x^2$ can be used to find the y values or it can be used to check the y values.

$\therefore$ The solutions are $x = 0, y = 0$ and $x = 2, y = 4$.

The points (0, 0) and (2, 4) lie on both the line $y = 2x$ and the parabola $y = x^2$.

b By substitution:

$$-4x^2 - x + 6 = 3x + 7$$
$$-x + 6 = 4x^2 + 3x + 7$$
$$6 = 4x^2 + 4x + 7$$
$$0 = 4x^2 + 4x + 1$$
$$\therefore (2x + 1)(2x + 1) = 0$$
$$2x + 1 = 0$$
$$x = -\frac{1}{2}$$

Set the equations equal to each other by substitution.
When rearranging the equation equal to zero, gather the terms on the side that makes the coefficient of x^2 positive, as this will make the factorising easier.
Hence, add $4x^2$ to both sides, then add x to both sides and subtract 6 from both sides.
Factorise and solve for x.

When $x = -\frac{1}{2}, y = 3 \times \left(-\frac{1}{2}\right) + 7$
$$= \frac{11}{2} \text{ or } 5\frac{1}{2}$$

Substitute the x value into $y = 3x + 7$ (or $y = -4x^2 - x + 6$ but $y = 3x + 7$ is a simpler equation).

$\therefore$ The only solution is $x = -\frac{1}{2}, y = \frac{11}{2}$.

Finding only one solution indicates that this line is a tangent to the parabola.

Example continued over page

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c Substitute the equation $y = x^2 + 1$ into the equation $2x - 3y = -4$.

Replace y in $2x - 3y = -4$ with $x^2 + 1$, making sure you include brackets.

$$2x - 3(x^2 + 1) = -4$$
$$2x - 3x^2 - 3 = -4$$
$$2x - 3 = 3x^2 - 4$$
$$2x = 3x^2 - 1$$

Expand the brackets and then rearrange into the form $ax^2 + bx + c = 0$.

$$\therefore 3x^2 - 2x - 1 = 0$$
$$(3x + 1)(x - 1) = 0$$
$$3x + 1 = 0 \text{ or } x - 1 = 0$$
$$x = -\frac{1}{3} \text{ or } x = 1$$

Factorise and solve for x.

When $x = -\frac{1}{3}$, $y = \left(-\frac{1}{3}\right)^2 + 1$
$= \frac{1}{9} + 1$
$= \frac{10}{9}$

When $x = 1$, $y = (1)^2 + 1$
$= 2$

Substitute the x values into one of the two original equations to solve for y.

$\therefore$ The solutions are $x = -\frac{1}{3}$, $y = \frac{10}{9}$ and $x = 1$, $y = 2$.

The line and parabola intersect at two places.

Example 16 Solving simultaneous equations with the quadratic formula

Solve the equations $y = x^2 + 5x - 5$ and $y = 2x$ simultaneously. Round your values to 2 decimal places.

SOLUTION

By substitution:
$x^2 + 5x - 5 = 2x$
$x^2 + 3x - 5 = 0$

Using the quadratic formula:

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-5)}}{2(1)}$$
$$= \frac{-3 \pm \sqrt{9 + 20}}{2}$$
$$= \frac{-3 \pm \sqrt{29}}{2}$$
$$= 1.19258\ldots \text{ or } -4.19258\ldots$$

In exact form, $y = 2x = 2 \times \left(\frac{-3 \pm \sqrt{29}}{2}\right)$
$= -3 \pm \sqrt{29}$

$\therefore$ The solutions are $x = 1.19$, $y = 2.39$ and $x = -4.19$, $y = -8.39$.

EXPLANATION

Rearrange into standard form.

$x^2 + 3x - 5$ does not factorise with whole numbers.

Quadratic formula: If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Here, $a = 1$, $b = 3$ and $c = -5$.

Use a calculator to evaluate $\frac{-3 + \sqrt{29}}{2}$ and $\frac{-3 - \sqrt{29}}{2}$.

Recall that if the number under the square root is negative, then there will be no real solutions.

Round your values to 2 decimal places, as required.

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Example 17 Determining the number of solutions of simultaneous equations

Determine the number of solutions (i.e. points of intersection) of the following pairs of equations.

a $y = x^2 + 3x - 1$
$y = x - 2$

b $y = 2x^2 - 3x + 8$
$y = 5 - 2x$

SOLUTION	EXPLANATION
a By substitution:	
$x^2 + 3x - 1 = x - 2$	Once the equation is in the form
$x^2 + 2x + 1 = 0$	$ax^2 + bx + c = 0$, the discriminant $b^2 - 4ac$ can
Using the discriminant:	be used to determine the *number* of solutions.
$b^2 - 4ac = (2)^2 - 4(1)(1)$	Here, $a = 1$, $b = 2$ and $c = 1$.
$= 4 - 4$	
$= 0$	
$\therefore$ There is one solution to the pair of equations.	Recall: $b^2 - 4ac > 0$ means two solutions.
	$b^2 - 4ac = 0$ means one solution.
	$b^2 - 4ac < 0$ means no solutions.
b By substitution:	Substitute and rearrange into the form
$2x^2 - 3x + 8 = 5 - 2x$	$ax^2 + bx + c = 0$.
$2x^2 - x + 3 = 0$	
Using the discriminant:	Calculate the discriminant. Here, $a = 2$, $b = -1$
$b^2 - 4ac = (-1)^2 - 4(2)(3)$	and $c = 3$.
$= 1 - 24$	
$= -23 < 0$	$b^2 - 4ac < 0$ means no solutions.
$\therefore$ There is no solution to the pair of equations.	

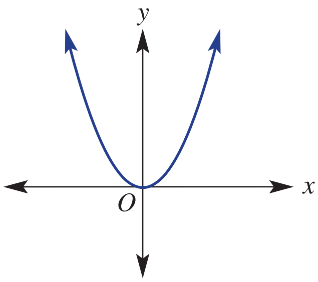
Exercise 9G

UNDERSTANDING AND FLUENCY	1–5, 6–9(½)	4, 5, 6–10(½)	6–10(½)

1 By considering the graph shown:

a How many points of intersection would a vertical line have with this parabola?

b How many possible points of intersection would a horizontal line have with this parabola?



2 **a** Find the coordinates of the point where the vertical line $x = 2$ intersects the parabola $y = 2x^2 + 5x - 6$.

b Find the coordinates of the point where the vertical line $x = -1$ intersects the parabola $y = x^2 + 3x - 1$.

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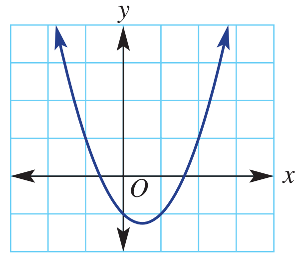
3 The graph of a parabola is shown.



a Add the line $y = x + 2$ to this graph.

b Use the grid to write down the coordinates of the points of intersection of the parabola and the line.

c The parabola has equation $y = x^2 - x - 1$. Complete the following steps to verify algebraically the coordinates of the points of intersection.

$x^2 - x - 1 =$ ____

$x^2 -$ __ $- 1 = 2$

$x^2 - 2x -$ __ $= 0$

$(x -$ __$)(x + 1) = 0$

__ $= 0$ or $x + 1 = 0$

$x =$ __ or $x =$ __

When $x =$ ___, $y =$ __ $+ 2 =$ ___.

When $x =$ ___, $y =$ __ $+ 2 =$ ___.

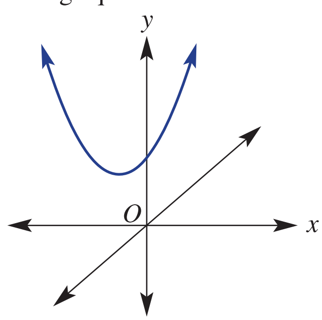
4 Rearrange the following into the form $ax^2 + bx + c = 0$, where $a > 0$.

a $x^2 + 5x = 2x - 6$ **b** $x^2 - 3x + 4 = 2x + 1$ **c** $x^2 + x - 7 = -2x + 5$

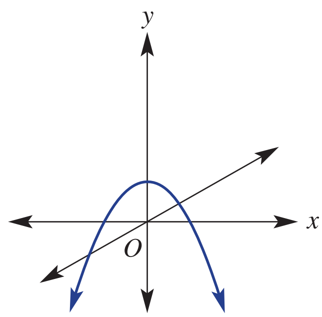
d $2x^2 - x - 4 = -2x - 3$ **e** $-x^2 + 2 = 2x + 5$ **f** $-x^2 + 2x = x + 4$

5 What do we know about the discriminant $b^2 - 4ac$ of the resulting equation from solving the following equations simultaneously?

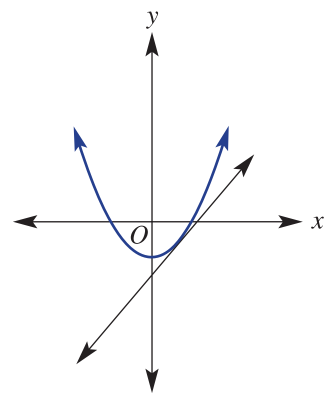
a



b



c



Example 14

6 Find the points of intersection of these parabolas and horizontal lines.

a $y = x^2 + x$
$y = 6$

b $y = x^2 - 4x$
$y = 12$

c $y = x^2 + 3x + 6$
$y = 1$

d $y = 2x^2 + 7x + 1$
$y = -2$

e $y = 4x^2 - 12x + 9$
$y = 0$

f $y = 3x^2 + 2x + 9$
$y = 5$

Example 15a, b

7 Solve these simultaneous equations using substitution.

a $y = x^2$
$y = 3x$

b $y = x^2$
$y = -2x$

c $y = x^2$
$y = 3x + 18$

d $y = x^2 - 2x + 5$
$y = x + 5$

e $y = -x^2 - 11x + 4$
$y = -3x + 16$

f $y = x^2 + 3x - 1$
$y = 4x + 5$

g $y = x^2 - 2x - 4$
$y = -2x - 5$

h $y = -x^2 + 3x - 5$
$y = 3x - 1$

i $y = 2x^2 + 4x + 10$
$y = 1 - 7x$

j $y = 3x^2 - 2x - 20$
$y = 2x - 5$

k $y = -x^2 - 4x + 3$
$y = 2x + 12$

l $y = x^2 + x + 2$
$y = 1 - x$

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Example 15c

8 Solve these simultaneous equations by first substituting.

a $y = x^2$
$2x + y = 8$

b $y = x^2$
$x - y = -2$

c $y = x^2$
$2x + 3y = 1$

d $y = x^2 + 3$
$5x + 2y = 4$

e $y = x^2 + 2x$
$2x - 3y = -4$

f $y = -x^2 + 9$
$6x - y = 7$

Example 16

9 Solve the following simultaneous equations, making use of the quadratic formula.

a Give your answers to 1 decimal place where necessary.

i $y = 2x^2 + 3x + 6$
$y = x + 4$

ii $y = x^2 + 1$
$y = 2x + 3$

iii $y = -2x^2 + x + 3$
$y = 3x + 2$

iv $y = 2x^2 + 4x + 5$
$y = 3 - 2x$

b Give your answers in exact surd form.

i $y = x^2 + 2x - 5$
$y = x$

ii $y = x^2 - x + 1$
$y = 2x$

iii $y = -x^2 - 3x + 3$
$y = -2x$

iv $y = x^2 + 3x - 3$
$y = 2x + 1$

Example 17

10 Determine the number of solutions to the following simultaneous equations.

a $y = x^2 + 2x - 3$
$y = x + 4$

b $y = 2x^2 + x$
$y = 3x - 1$

c $y = 3x^2 - 7x + 3$
$y = 1 - 2x$

d $y = x^2 + 5x + 1$
$y = 2x - 3$

e $y = -x^2$
$y = 2x + 1$

f $y = -x^2 + 2x$
$y = 3x - 1$

PROBLEM-SOLVING AND REASONING	11, 12(a), 14	11, 12(a, c), 14, 15	12, 13, 15, 16

11 A member of an indoor cricket team, playing a match in a gymnasium, hits a ball that follows a path given by $y = -0.1x^2 + 2x + 1$, where y is the height above ground, in metres, and x is the horizontal distance travelled by the ball.

The ceiling of the gymnasium is 10.6 metres high. Will this ball hit the roof? Explain.

12 Solve the following equations simultaneously.

a $y = x^2 + 2x - 1$
$y = \dfrac{x - 3}{2}$

b $y = x(x - 4)$
$y = \dfrac{1}{2}x - 5$

c $y = (x - 2)^2 + 7$
$y = 9 - x$

d $y = \dfrac{8 - x^2}{2}$
$y = 2(x - 1)$

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13 A train track is to be constructed over a section of a lake. On a map, the edge of the lake that the train track will pass over is modelled by the equation $y = 6 - 2x^2$. The segment of train track is modelled by the equation $y = x + 5$.

The section of track to be constructed will start and end at the points at which this track meets the lake.

a Determine the location (i.e. coordinates) of the points on the map where the framework for the track will start and end.

b If 1 unit represents 100 metres, determine the length of track that must be built over the lake, correct to the nearest metre.

14 Consider the parabola with equation $y = x^2 - 6x + 5$.

a Use any suitable method to determine the coordinates of the turning point of this parabola.

b Hence, state for which values of c the line $y = c$ will intersect the parabola:

i twice **ii** once **iii** not at all

15 Consider the parabola with equation $y = x^2$ and the family of lines $y = x + k$.

a Determine the discriminant, in terms of k, obtained when solving these equations simultaneously.

b Hence, determine for which values of k the line will intersect the parabola:

i twice **ii** once **iii** not at all

16 **a** Use the discriminant to show that the line $y = 2x + 1$ does not intersect the parabola $y = x^2 + 3$.

b Determine for which values of k the line $y = 2x + k$ does intersect the parabola $y = x^2 + 3$.

ENRICHMENT	–	–	17

Multiple tangents?

17 The line $y = mx$ is a tangent to the parabola $y = x^2 - 2x + 4$ (i.e the line touches the parabola at just one point).

a Find the possible values of m.

b Can you explain why there are two possible values of m? Hint: A diagram may help.

c If the value of m is changed so that the line now intersects the parabola in two places, what is the set of possible values of m?

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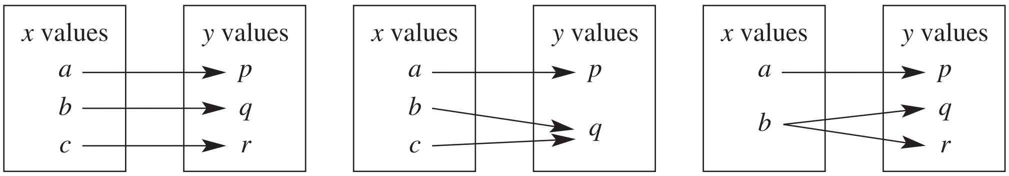
9H Functions and their notation

Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

On the Cartesian plane a relationship between the variables x and y can be shown. When this relation has a unique (i.e. only one) y value for each of its x values, it is called a function.

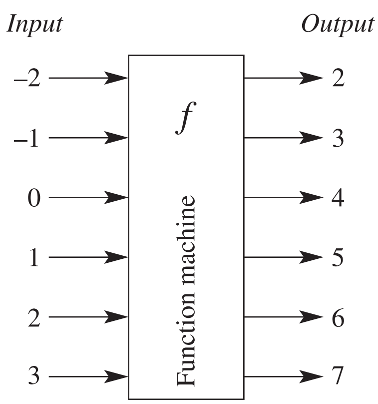


Function Function Not a function since $x = b$ gives two y values.

All functions, including parabolas, use a special notation where the y is replaced by $f(x)$.
$y = x^2$ becomes $f(x) = x^2$ (i.e. y is a function of x or the function of x obeys the rule x^2).

Let's start: A function machine

Consider the input and output of the following machine.



The name f is given to the function and it is written $f(\text{input}) = \text{output}$.

- Using this idea, complete the following.
  $f(-2) = $ _____, $f(-1) = $ _____, $f(0) = $ _____, $f(1) = $ _____, $f(2) = $ _____, $f(3) = $ _____
  and, hence, $f(x) = $ _____.

Key ideas

- Any set of ordered pairs is called a **relation**.
- A relation in which each x value produces only one y value is called a **function.**
- Function notation is another method for writing equations of graphs. For example, $y = x^2$ can be written as $f(x) = x^2$.
- Graphically, a relation that passes the vertical line test is called a function. Any vertical line drawn through the graph of a function will cut it only once.

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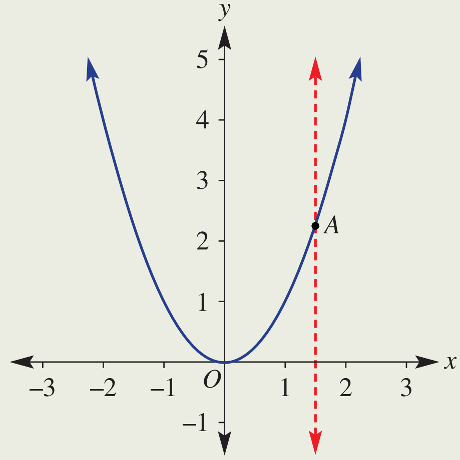
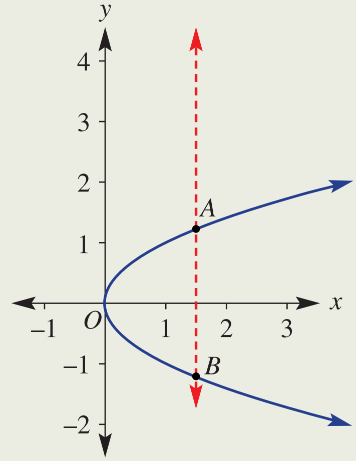
Key ideas

- For example:



This relation is a function because a vertical line cuts the curve only once.



This relation is *not* a function because a vertical line cuts the curve at *more than one* point.

- The parabola $y = x^2$ is a function, so can be written as $f(x) = x^2$. Also, $f(-2) = 4$ can be written to describe the point (2, 4).
- The set of permissible x coordinates (i.e. the input) in a relation is also called the **domain**.
- The **range** is the term given to the set of y coordinates (output) in the relation.

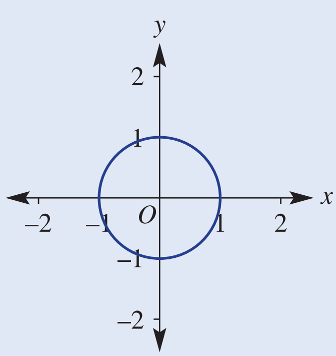
Example 18 Recognising a function

From the following, identify which are functions.

a (1, 4)(2, 8)(4, 16)(5, 20)

b $y = x^2 + 3$

c



SOLUTION

a Is a function as each x value has only one y value.

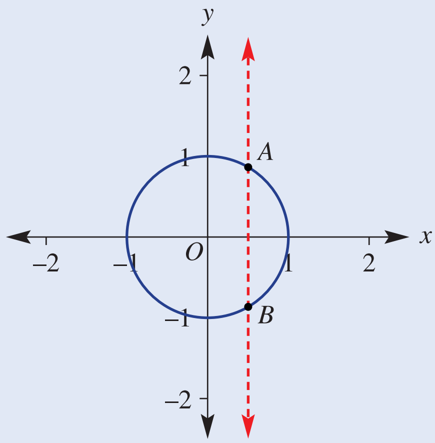
Each of the x values; i.e. $x = 1, 2, 4$, and 5, occurs only once. So the coordinates represent a function.

b Is a function as each x value will produce only one y value.

As the rule represents a parabola, each x value will produce only one y value and so it is a function.

c This is not a function because a vertical line crosses the graph more than once.

A vertical line drawn anywhere through the graph will cross in more than one place, therefore it is not a function.



Example 19 Using function notation

For $f(x) = x^2 - 3x + 1$, find:

a $f(0)$

b $f(-3)$

c $f(c)$

SOLUTION

a $f(0) = 0^2 - 3(0) + 1$
$= 1$

The input of 0 is substituted into the rule for each x value on the RHS.

b $f(-3) = (-3)^2 - 3(-3) + 1$
$= 9 + 9 + 1$
$= 19$

$x = -3$ is substituted into the rule on the RHS.

c $f(c) = c^2 - 3c + 1$

The x has been replaced by c. Therefore, replace each x with c on the RHS.

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Example 20 Determining permissible x and y values (domain and range)

Write down the permissible x and y values for each of these functions.

a $y = 4x - 1$

b $y = x^2 - 4$

SOLUTION	EXPLANATION
a Domain is the set of all real x values. Range is the set of all real y values.	The function is a straight line. The input (i.e. x values) can be any number and will produce any number as an output value.
b Domain is the set of all real x values. Range is the set of y values, where $y \geqslant -4$.	It is possible to square any value of x. As squaring a negative number makes it positive, the smallest y value possible as an output is -4.

Exercise 9H

UNDERSTANDING AND FLUENCY	1–3, 4–6(½), 8–9(½)	2, 3, 4–9(½)	4–9(½)

1 Rewrite the following functions using function notation.

a $y = 8x$ **b** $y = 9 - x^2$ **c** $y = \frac{2}{x}$

d $y = x(2x - 3)$ **e** $y = 2^x + 1$

Example 18a, b

2 For each of the following, state whether it is true or false.

a All parabolas are functions.

b A vertical line will cut $y = 2x - 1$ only once.

c Only positive x values can be used as the input in $f(x) = x^2$.

d All straight lines are functions.

e A circle is not a function.

3 Use a sketch to determine the permissible y values for $y = x^2$, given that the following x values are allowed.

a $x \geqslant 0$

b $x > 0$

c $x > 3$

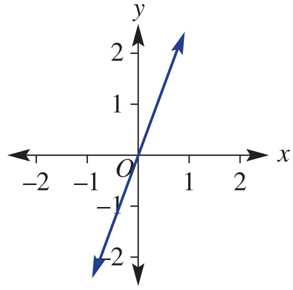
d $-1 \leqslant x \leqslant 1$

e all real x values

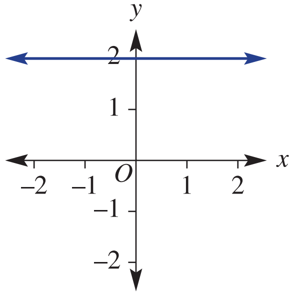
Example 18c

4 Use the vertical line test to determine which of the following graphs represents a function.

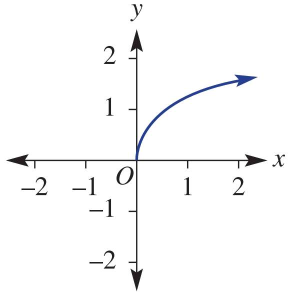
a



b



c



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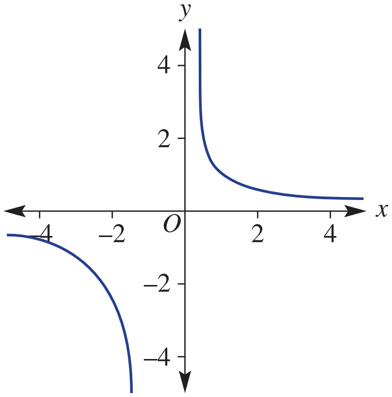
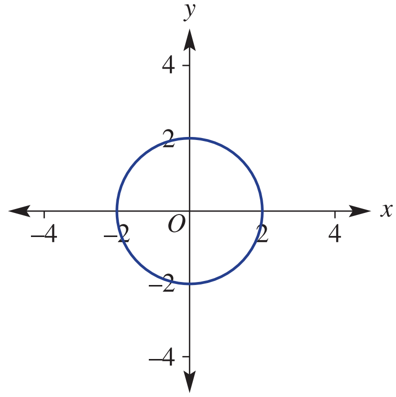
d



e



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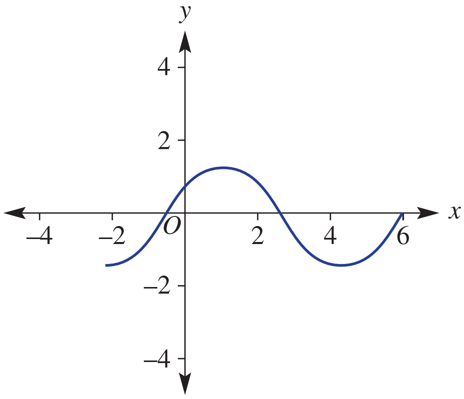

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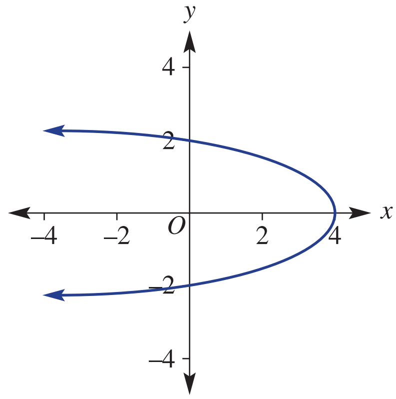
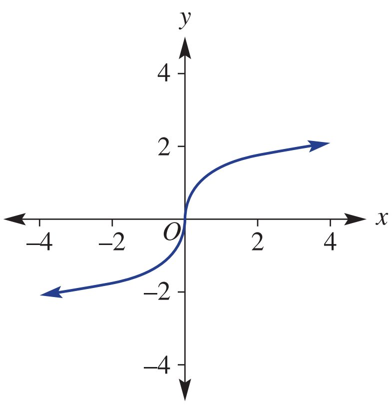
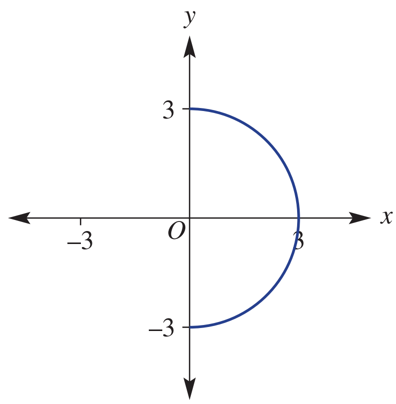


h



i



Example 19

5 Given $f(x) = 3x + 4$, find:

a $f(0)$ **b** $f(2)$ **c** $f(8)$

d $f\left(\frac{1}{2}\right)$ **e** $f(-2)$ **f** $f(a)$

6 Given $f(x) = 2x^3 - x^2 + x$, find:

a $f(0)$ **b** $f(1)$ **c** $f(-1)$

d $f(5)$ **e** $f(0.2)$ **f** $f(k)$

7 Find $f(0)$, $f(2)$, $f(-4)$, $f(a)$ and $f(a + 1)$ for each of the following functions.

a $f(x) = 4x$ **b** $f(x) = 1 - x^2$ **c** $f(x) = 2^x$

d $f(x) = \frac{2}{x}$ **e** $f(x) = (x - 2)(x + 6)$ **f** $f(x) = 4x^2 + 9$

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Example 20

8 Find the set of permissible x values for each function. (Note: This is the domain of each function.)

a $f(x) = 2 - x$
b $f(x) = x^2$
c $f(x) = 3x^2$
d $f(x) = x^3$
e $f(x) = 5^x$
f $f(x) = 5^{-x}$
g $f(x) = 2 - x^2$
h $f(x) = \dfrac{2}{x}$

9 For each function in Question **8**, write down the set of y values (i.e. the range) that it has as its permissible output.

PROBLEM-SOLVING AND REASONING	10, 13(a–c)	10, 11, 13, 14	10–16

10 Given $f(x) = 7x - 9$ and $g(x) = 6 - 2x$:

a Find:
- **i** $f(2)$
- **ii** $g(4)$
- **iii** $f(2) + g(4)$
- **iv** $f(-2) + 2g(1)$
- **v** $f(g(2))$

b The value of a is such that $f(a) = g(a)$. Explain the significance of $x = a$ in terms of the two graphs $y = f(x)$ and $y = g(x)$.

11 Answer the following as true or false for each of these functions.
- **i** $f(x) = 2x - 2$
- **ii** $f(x) = x^2 + 4$

a $f(3a) = 3f(a)$
b $f(a) = f(-a)$
c $f(a) + f(b) = f(a + b)$

12 Given the function $f(x) = 2x^2 - 3x - 1$, simplify $\dfrac{f(x+h) - f(x)}{h}$.

13 **a** Explain why all parabolas of the form $y = ax^2 + bx + c$ are functions.

b What type of straight line is not a function and why is it not a function?

c Why is finding the coordinates of the vertex of a parabola used when finding the range of the function?

d Considering your response to part **c**, find the range of the following quadratic functions.
- **i** $y = x^2 + 4x$
- **ii** $y = x^2 - 5x - 6$
- **iii** $y = 1 - x - 2x^2$
- **iv** $y = x^2 + 6x + 10$

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14 Given that $\frac{a}{0}$ is undefined, write a statement such as $x \neq 3$ to indicate values of x that are not permissible in the following functions.

a $f(x) = \frac{3}{x-1}$

b $f(x) = \frac{3}{2x+1}$

c $f(x) = \frac{-2}{1-x}$

15 Given that negative numbers cannot be square rooted, write down the domain of:

a $y = \sqrt{x}$

b $y = \sqrt{x-2}$

c $y = \sqrt{x+2}$

d $y = \sqrt{2-x}$

16 For $f(x) = x^2 + \frac{1}{x^2}$:

a Find $f(a)$ and $f(-a)$.

b Find the values of $f(-3), f(-2), f(-1)$ and $f(0)$. Hence, sketch the graph of $y = f(x)$, in the domain $-3 \leqslant x \leqslant 3$.

c Comment on any symmetry you notice.

ENRICHMENT	–	–	17

Sketching composite functions

17 **a** Sketch the following functions.

i $f(x) = \begin{cases} 2x & \text{for } x \geqslant 0 \\ -2x & \text{for } x < 0 \end{cases}$

ii $f(x) = \begin{cases} 4 & \text{for } x \geqslant 2 \\ x^2 & \text{for } -2 < x < 2 \\ 4 & \text{for } x \leqslant -2 \end{cases}$

iii $f(x) = \begin{cases} 2x+4 & \text{for } x > 0 \\ -(x+4) & \text{for } x \leqslant 0 \end{cases}$

b A function is said to be continuous if its entire graph can be drawn without lifting the pen from the page. Which of the functions in part **a** are discontinuous?

c For the functions sketched in part **a**, write down the range.

d For each of these functions, find the value of:

i $f(2) + f(0) + f(-2)$

ii $f(3) - 2f(1) + 4f(-4)$

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9I Graphs of circles

Interactive
Widgets
HOTsheets
Walkthrough

We know the circle as a common shape in geometry, but we can also describe a circle using an equation and as a graph on the Cartesian plane.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Plotting the circle

A graph has the equation $x^2 + y^2 = 9$.

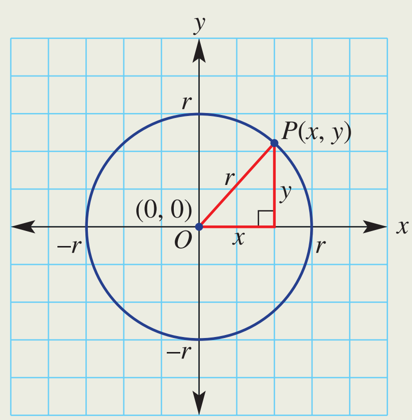
- If $x = 0$ what are the two values of y?
- If $x = 1$ what are the two values of y?
- If $x = 4$ are there any values of y? Discuss.
- Complete this table of values.

x	-3	-2	-1	0	1	2	3
y		$\pm\sqrt{5}$					

- Now plot all your points on a number plane and join them to form a smooth curve.
- What shape have you drawn and what are its features?
- How does the radius of your circle relate to the equation?

Key ideas

- The Cartesian equation of a circle with centre (0, 0) and radius r is given by $x^2 + y^2 = r^2$.
- Making x or y the subject:
  - $y = \pm\sqrt{r^2 - x^2}$
  - $x = \pm\sqrt{r^2 - y^2}$
- Circles are relations but *not* functions because there are permissable values of x that have two y values.



Using Pythagoras' theorem, $a^2 + b^2 = c^2$ gives $x^2 + y^2 = r^2$.

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Example 21 Sketching a circle

For the equation $x^2 + y^2 = 4$, complete the following.

a State the coordinates of the centre.

b State the radius.

c Find the values of y when $x = 1$.

d Find the values of x when $y = \frac{1}{2}$.

e Sketch a graph showing intercepts.

f What is the domain (i.e. the permissible x values)?

g What is the range (i.e. the permissible y values)?

SOLUTION

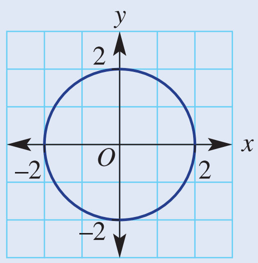
a $(0, 0)$

b $r = 2$

c
$$\begin{aligned} x^2 + y^2 &= 4 \\ 1^2 + y^2 &= 4 \\ y^2 &= 3 \\ y &= \pm\sqrt{3} \end{aligned}$$

d
$$\begin{aligned} x^2 + \left(\frac{1}{2}\right)^2 &= 4 \\ x^2 + \frac{1}{4} &= 4 \\ x^2 &= \frac{15}{4} \\ x &= \pm\frac{\sqrt{15}}{2} \end{aligned}$$

e



f $-2 \leqslant x \leqslant 2$

g $-2 \leqslant y \leqslant 2$

EXPLANATION

$(0, 0)$ is the centre for all circles $x^2 + y^2 = r^2$.

$x^2 + y^2 = r^2$, so $r^2 = 4$.

Substitute $x = 1$ and solve for y.

Recall that $(\sqrt{3})^2$ and $(-\sqrt{3})^2$ both equal 3.

Substitute $y = \frac{1}{2}$.

$4 - \frac{1}{4} = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$

$\sqrt{\frac{15}{4}} = \frac{\sqrt{15}}{\sqrt{4}} = \frac{\sqrt{15}}{2}$

Draw a circle with centre $(0, 0)$ and radius 2. Label intercepts.

As seen on the graph, x values between -2 and 2 produced real y values. Similarly, the range can be seen to be between -2 and 2.

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Example 22 Intersecting circles and lines

Find the coordinates of the points where $x^2 + y^2 = 4$ intersects $y = 2x$. Sketch a graph showing the exact intersection points.

SOLUTION

$x^2 + y^2 = 4$ and $y = 2x$

$x^2 + (2x)^2 = 4$

$x^2 + 4x^2 = 4$

$5x^2 = 4$

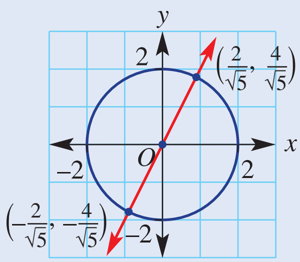
$x^2 = \frac{4}{5}$

$x = \pm\frac{2}{\sqrt{5}}$

If $y = 2x$:

$x = \frac{2}{\sqrt{5}}$ gives $y = 2 \times \left(\frac{2}{\sqrt{5}}\right) = \frac{4}{\sqrt{5}}$

$x = -\frac{2}{\sqrt{5}}$ gives $y = 2 \times \left(-\frac{2}{\sqrt{5}}\right) = -\frac{4}{\sqrt{5}}$



EXPLANATION

Substitute $y = 2x$ into $x^2 + y^2 = 4$ and solve for x.

Substitute both values of x into $y = 2x$ to find the y-coordinate.

For $x^2 + y^2 = 4$, $r = 2$.
Mark the intersection points and sketch $y = 2x$.

Exercise 9I

UNDERSTANDING AND FLUENCY	1, 2, 3–4(½), 5, 6–8(½)	3–4(½), 5, 6–8(½), 9	5, 6–9(½)

1 Draw a circle on the Cartesian plane with centre (0, 0) and radius 2 and write down its equation.

2 Give the exact solutions for these equations. There are two solutions for each.

a $x^2 + 2^2 = 9$

b $x^2 + 3^2 = 25$

c $x^2 + 7 = 10$

d $5^2 + y^2 = 36$

e $8^2 + y^2 = 121$

f $3 + y^2 = 7$

3 A circle has equation $x^2 + y^2 = r^2$. Complete these sentences.

a The centre of the circle is ____.

b The radius of the circle is ____.

4 Give the radius of the circles with these equations.

a $x^2 + y^2 = 36$ b $x^2 + y^2 = 81$ c $x^2 + y^2 = 144$

d $x^2 + y^2 = 5$ e $x^2 + y^2 = 14$ f $x^2 + y^2 = 20$

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5 Write the equation of a circle with centre (0, 0) and the following radius.

a 2 **b** 7 **c** 100 **d** 51

e $\sqrt{6}$ **f** $\sqrt{10}$ **g** 1.1 **h** 0.5

Example 21

6 A circle has equation $x^2 + y^2 = 9$. Complete the following.

a State the coordinates of the centre.

b State the radius.

c Find the values of y when $x = 2$.

d Find the values of x when $y = \frac{3}{2}$.

e Sketch a graph showing intercepts.

f What are the permissible x values?

g What are the permissible y values?

7 For the equation $x^2 + y^2 = 25$ complete the following.

a State the coordinates of the centre.

b State the radius.

c Find the values of y when $x = \frac{9}{2}$.

d Find the values of x when $y = 4$.

e Sketch a graph showing intercepts.

f What are the permissible x values?

g What are the permissible y values?

8 For the circle with equation $x^2 + y^2 = 4$, find the exact coordinates where:

a $x = 1$ **b** $x = -1$ **c** $x = \frac{1}{2}$

d $y = -\frac{1}{2}$ **e** $y = -2$ **f** $y = 0$

9 Without solving any equations, write down the x- and y-intercepts of these circles.

a $x^2 + y^2 = 1$

b $x^2 + y^2 = 16$

c $x^2 + y^2 = 3$

d $x^2 + y^2 = 11$

PROBLEM-SOLVING AND REASONING	10(½), 11, 16	10(½), 11–14, 16, 17	12–15, 17, 18

10 Write down the radius of these circles.

a $x^2 + y^2 - 8 = 0$

b $x^2 - 4 = -y^2$

c $y^2 = 9 - x^2$

d $10 - y^2 - x^2 = 0$

e $3 + x^2 + y^2 = 15$

f $17 - y^2 = x^2 - 3$

Example 22

11 Find the coordinates of the points where $x^2 + y^2 = 9$ intersects $y = x$. Sketch a graph showing the intersection points.

12 Find the coordinates of the points where $x^2 + y^2 = 10$ intersects $y = 3x$. Sketch a graph showing the intersection points.

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13 Find the coordinates of the points where $x^2 + y^2 = 6$ intersects $y = -\frac{1}{2}x$. Sketch a graph showing the intersection points.

14 Determine the exact length of the chord formed by the intersection of $y = x - 1$ and $x^2 + y^2 = 5$. Sketch a graph showing the intersection points and the chord.

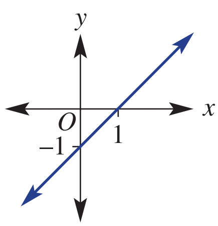
15 For the circle $x^2 + y^2 = 4$ and the line $y = mx + 4$, determine the exact values of the gradient, m, so that the line:

- **a** is a tangent to the circle
- **b** intersects the circle in two places
- **c** does not intersect the circle

16 Match equations **a–f** with graphs **A–F**.

- **a** $x^2 + y^2 = 7$
- **b** $y = x - 1$
- **c** $y = -2x + 2$
- **d** $y = x^2 - 1$
- **e** $y = -x(x + 2)$
- **f** $x^2 + y^2 = 9$

A



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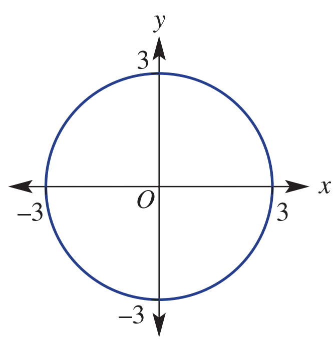

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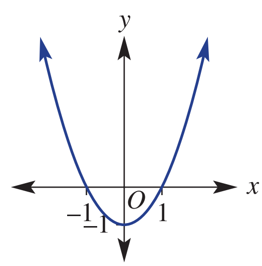

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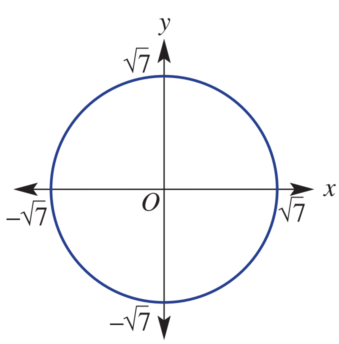

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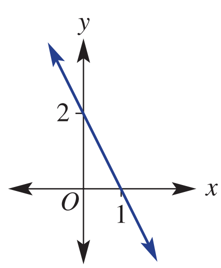

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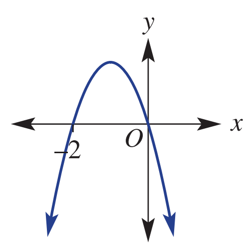

17 **a** Write $x^2 + y^2 = 16$ in the form $y = \pm\sqrt{r^2 - x^2}$.

b Write $x^2 + y^2 = 3$ in the form $x = \pm\sqrt{r^2 - y^2}$.

18 **a** Explain why the graphs of $y = 3$ and $x^2 + y^2 = 4$ do not intersect.

b Explain why the graphs of $x = -2$ and $x^2 + y^2 = 1$ do not intersect.

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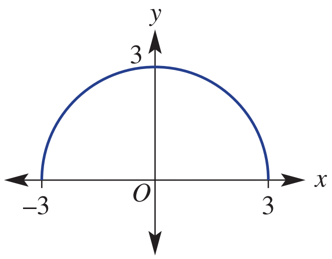
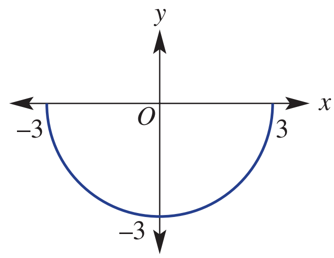
ENRICHMENT – – 19

Half circles

19 When we write $x^2 + y^2 = 9$ in the form $y = \pm\sqrt{9 - x^2}$, we define two circle halves.



$y = \sqrt{9 - x^2}$



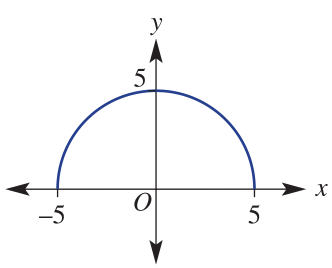
$y = -\sqrt{9 - x^2}$

a Sketch the graphs of these half circles.

i $y = \sqrt{4 - x^2}$	**ii** $y = \sqrt{25 - x^2}$	**iii** $y = -\sqrt{1 - x^2}$
iv $y = \sqrt{10 - x^2}$	**v** $y = -\sqrt{16 - x^2}$	**vi** $x = -\sqrt{36 - y^2}$
vii $x = -\sqrt{7 - y^2}$	**viii** $x = \sqrt{5 - y^2}$	**ix** $x = \sqrt{12 - y^2}$

b Write the rules for these half circles.

i



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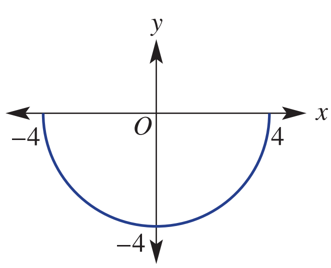

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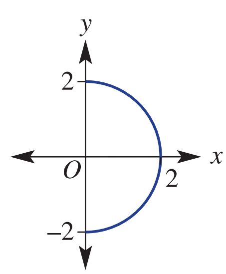

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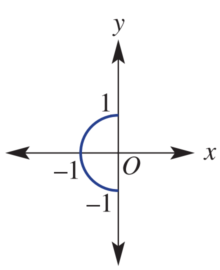

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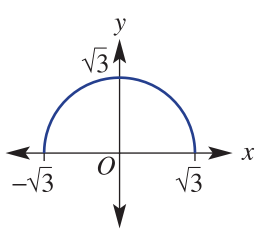

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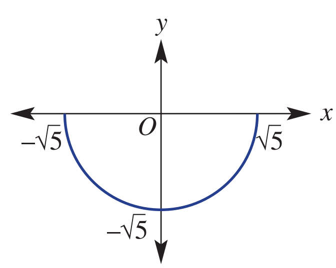

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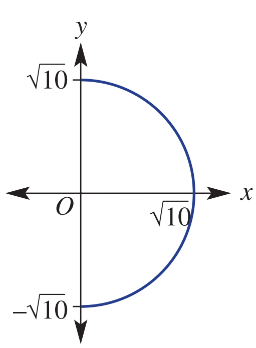

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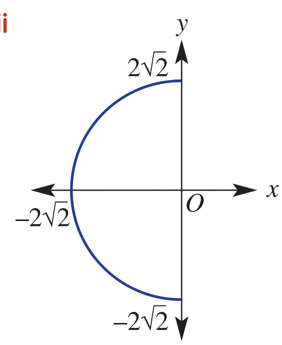

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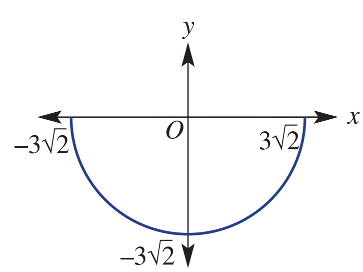

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9J Exponential functions and their graphs

Interactive
Widgets
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Walkthrough

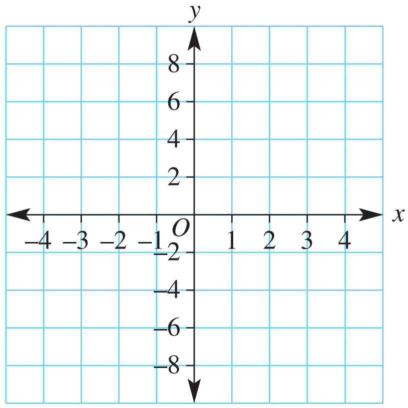
Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

We saw in Chapter 2 that indices can be used to describe some special relations. The population of the world, for example, or the balance of an investment account can be described using exponential rules that include indices. The rule $A = 100\,000(1.05)^t$ describes the account balance of \$100 000 invested at 5% p.a. compound interest for t years.

Let's start: What do $y = 2^x$, $y = -2^x$ and $y = 2^{-x}$ all have in common?

Complete this table and graph all three relations on the same set of axes before discussing the points below.

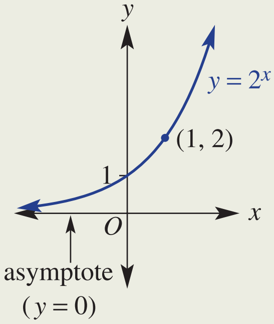
x	−3	−2	−1	0	1	2	3
$y_1 = 2^x$	$\frac{1}{8}$			1		4	
$y_2 = -2^x$							
$y_3 = 2^{-x}$							



- Discuss the shape of each graph.
- Where does each graph cut the y-axis?
- Do the graphs have x-intercepts? Why not?
- What is the one feature the graphs all have in common?

Key ideas

- $y = 2^x$, $y = 3^x$, $y = 3^{-x}$ are examples of exponential relationships.
- An **asymptote** is a line that a curve approaches by getting closer and closer to it but never reaching it.
- Exponential graphs are functions because every x value has exactly one y value. Note that $y = 2^x$ can be written as $f(x) = 2^x$.
- All x values are permissible.
- In graphs such as $y = 2^x$, y values are positive numbers; i.e. it is impossible for y to be zero or negative.



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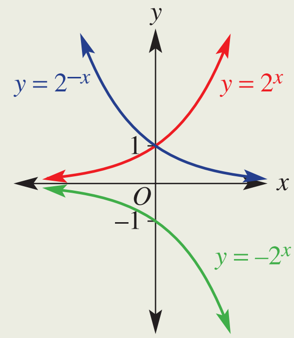
Key ideas

- A simple **exponential** rule is of the form $y = a^x$, where $a > 0$ and $a \neq 1$.
  - y-intercept is 1.
  - $y = 0$ is the equation of the asymptote.
- The graph of $y = -a^x$ is the reflection of the graph of $y = a^x$ in the x-axis.
  Note: $y = -a^x$ means $y = -1 \times a^x$.
- The graph of $y = a^{-x}$ is the reflection of the graph of $y = a^x$ in the y-axis.

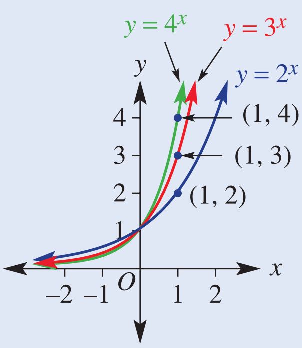


Example 23 Sketching graphs of exponentials

Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = 2^x$ **b** $y = 3^x$ **c** $y = 4^x$

SOLUTION



EXPLANATION

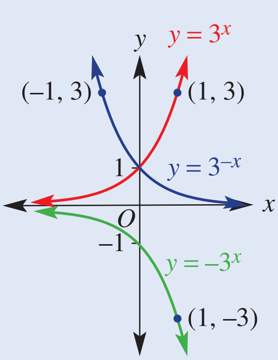
$a^0 = 1$, so all y-intercepts are at $(0, 1)$.
$y = 4^x$ is steeper than $y = 3^x$, which is steeper than $y = 2^x$.

Example 24 Sketching with reflections

Sketch the graphs of these exponentials on the same set of axes.

a $y = 3^x$ **b** $y = -3^x$ **c** $y = 3^{-x}$

SOLUTION



EXPLANATION

The graph of $y = -3^x$ is a reflection of the graph of $y = 3^x$ in the x-axis.
Check: $x = 1, y = -3^1 = -3$
The graph of $y = 3^{-x}$ is a reflection of the graph of $y = 3^x$ in the y-axis.
Check: $x = 1, y = 3^{-1} = \dfrac{1}{3}$
$x = -1, y = 3^1 = 3$

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Example 25 Solving exponential equations

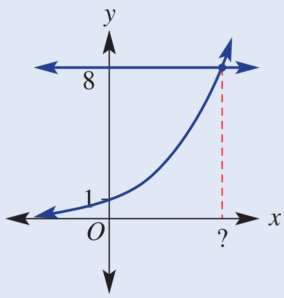
Find the intersection of the graphs of $y = 2^x$ and $y = 8$.

SOLUTION

$y = 2^x$
$8 = 2^x$
$2^3 = 2^x$
$x = 3$
$\therefore$ Intersection point is (3, 8).

EXPLANATION

Set $y = 8$ and write 8 with base 2. Since the bases are the same, equate the powers.



Exercise 9J

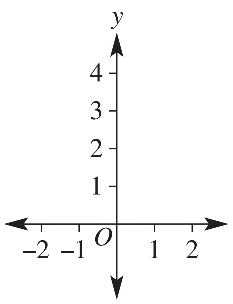
UNDERSTANDING AND FLUENCY	1–4, 5–7(a, b)	4, 5–7(a, c), 8	5–7(b), 8(½)

1 Consider the exponential rule $y = 2^x$.

a Complete this table.

x	−2	−1	0	1	2
y		$\frac{1}{2}$	1		

b Plot the points in the table to form the graph of $y = 2^x$.

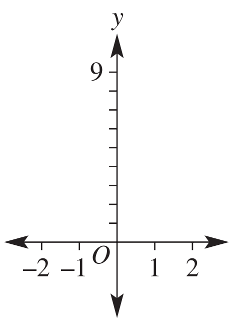


2 Consider the exponential rule $y = 3^x$.

a Complete this table.

x	−2	−1	0	1	2
y		$\frac{1}{3}$	1		

b Plot the points in the table to form the graph of $y = 3^x$.



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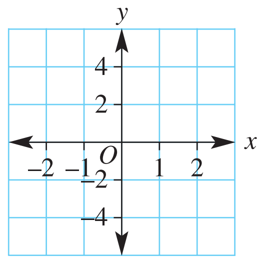
3 **a** Complete the table.

x	-2	-1	0	1	2
$y_1 = 2^x$					
$y_2 = -2^x$					
$y_3 = 2^{-x}$					

b Plot the graphs of $y = 2^x$, $y = -2^x$ and $y = 2^{-x}$ on the same set of axes.



4 **a** Explain the difference between a^{-2} and $-a^2$.

b True or false: $-3^2 = \frac{1}{3^2}$? Explain why.

c Write with negative powers: $\frac{1}{5^3}, \frac{1}{3^2}, \frac{1}{2}$.

d Simplify: -3^2, -5^3, -2^{-2}

Examples 23, 24a

5 Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = 2^x$ **b** $y = 4^x$ **c** $y = 5^x$

Example 24b

6 Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = -2^x$ **b** $y = -5^x$ **c** $y = -3^x$

Example 24c

7 Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = 2^{-x}$ **b** $y = 3^{-x}$ **c** $y = 6^{-x}$

8 **a** Find the coordinates on the graph of $y = 3^x$, where:

i $x = 0$ **ii** $x = -1$ **iii** $y = 1$ **iv** $y = 9$

b Find the coordinates on the graph of $y = -2^x$, where:

i $x = 4$ **ii** $x = -1$ **iii** $y = -1$ **iv** $y = -4$

c Find the coordinates on the graph of $y = 4^{-x}$, where:

i $x = 1$ **ii** $x = -3$ **iii** $y = 1$ **iv** $y = \frac{1}{4}$

PROBLEM-SOLVING AND REASONING	9, 10, 13	9–11, 13–15	9(½), 10–12, 14–17

Example 25

9 **a** Find the intersection of the graphs of $y = 2^x$ and $y = 4$.

b Find the intersection of the graphs of $y = 3^x$ and $y = 9$.

c Find the intersection of the graphs of $y = -4^x$ and $y = -4$.

d Find the intersection of the graphs of $y = 2^{-x}$ and $y = 8$.

10 A study shows that the population of a town is modelled by the rule $P = 2^t$, where t is in years and P is in thousands of people.

a State the number of people in the town at the start of the study.

b State the number of people in the town after:

i 1 year **ii** 3 years

c When is the town's population expected to reach:

i 4000 people? **ii** 16000 people?

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11 A single bacterium divides into two every second, so one cell becomes two in the first second and in the next second two cells become four and so on.

a Write a rule for the number of bacteria, N, after t seconds.

b How many bacteria will there be after 10 seconds?

c How long does it take for the population to exceed 10000? Round to the nearest second.

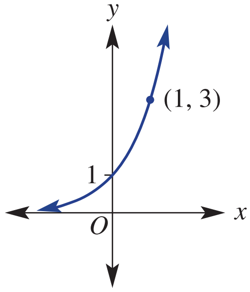
12 Use trial and error to find x when $2^x = 5$. Give the answer correct to 3 decimal places.

13 Match equations **a**–**f** with graphs **A**–**F**.

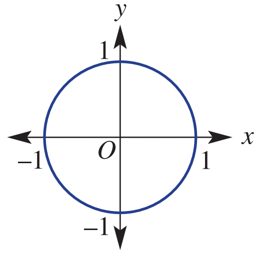
a $y = -x - 2$　　**b** $y = 3^x$　　**c** $y = 4 - x^2$

d $y = -2^x$　　**e** $y = (x + 2)(x - 3)$　　**f** $x^2 + y^2 = 1$

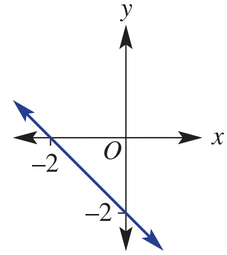
A



B



C



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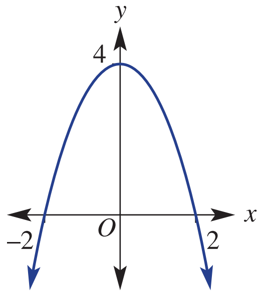
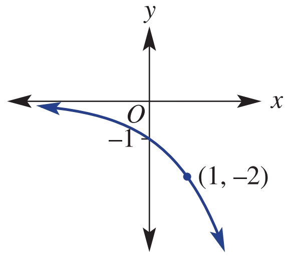
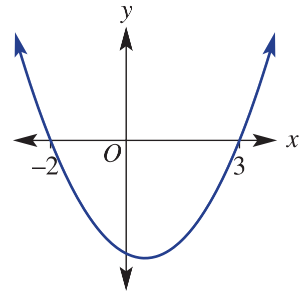


E



F



14 Explain why the point (2, 5) does not lie on the curve with equation $y = 2^x$.

15 Describe and draw the graph of the line with equation $y = a^x$ when $a = 1$.

16 Explain why $2^x = 0$ is never true for any value of x.

17 Use a graph to explain why the line $y = x + 3$ intersects the curve $y = 2^x$ twice.

ENRICHMENT	–	–	18

$y = 2^{-x}$ and $y = \left(\frac{1}{2}\right)^x$

18 Consider the exponential rule $y = 2^{-x}$ and $y = \left(\frac{1}{2}\right)^x$.

a Using $-3 \leqslant x \leqslant 3$, sketch graphs of the rules on the same set of axes. What do you notice?

b Write the following rules in the form $y = a^x$, where $0 < a < 1$.

i $y = 3^{-x}$　　**ii** $y = 5^{-x}$　　**iii** $y = 10^{-x}$

c Write the following rules in the form $y = a^{-x}$, where $a > 1$.

i $y = \left(\frac{1}{4}\right)^x$　　**ii** $y = \left(\frac{1}{7}\right)^x$　　**iii** $y = \left(\frac{1}{11}\right)^x$

d Prove that $\left(\frac{1}{a}\right)^x = a^{-x}$, for $a > 0$.

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9K Hyperbolic functions and their graphs

Interactive
Widgets
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Stage
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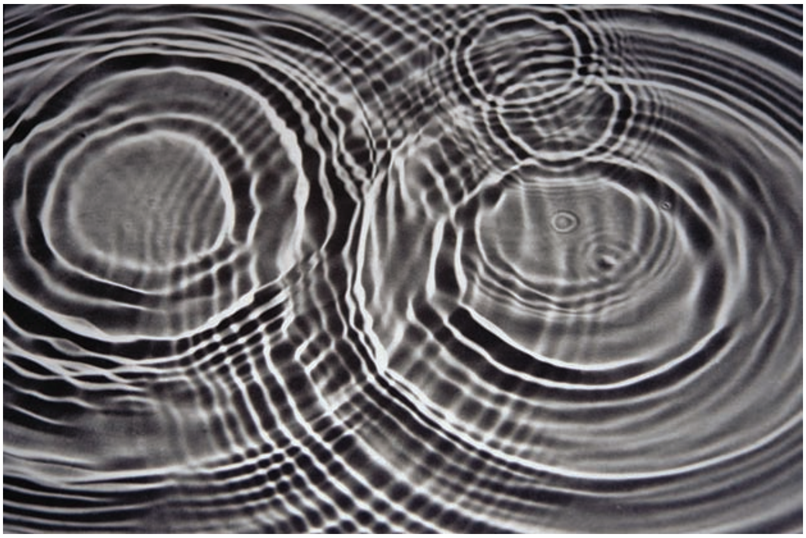
5.3§
5.2
5.2◊
5.1
4

A simple rectangular hyperbola is the graph of the equation $y = \frac{1}{x}$. These types of equations are common in many mathematical and practical situations.

When two stones are thrown into a pond, the resulting concentric ripples intersect at a set of points that together form the graph of a hyperbola.

In a similar way, when signals are received from two different satellites, a ship's navigator can map the hyperbolic shape of the intersecting signals and help to determine the ship's position.

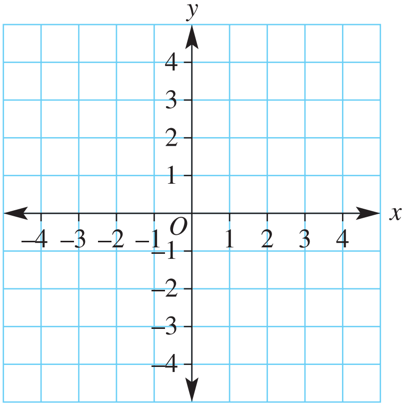


Let's start: How many asymptotes?

Consider the rule for the simple hyperbola $y = \frac{1}{x}$. First, complete the table and graph, and then discuss the points below.

x	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y										

- Discuss the shape of the graph of the hyperbola.
- What would the values of y approach as x increases to infinity or negative infinity?
- What would the values of y approach as x decreases to zero from the left or from the right?
- What are the equations of the asymptotes for $y = \frac{1}{x}$?



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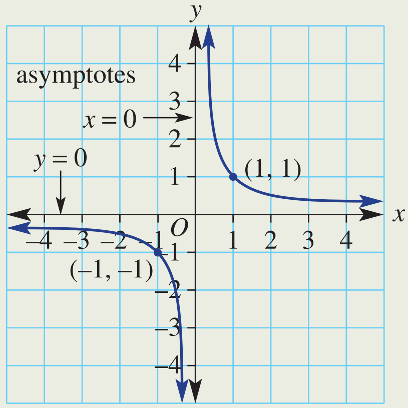
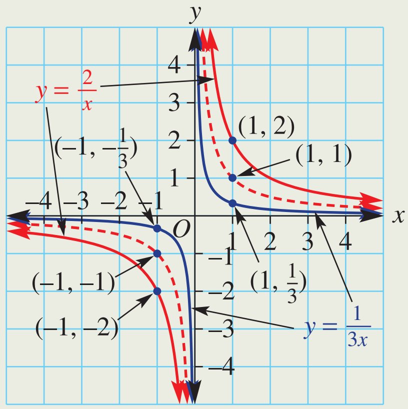
Key ideas

- An **asymptote** is a straight line that a curve approaches more and more closely but never quite reaches. Every hyperbola has a horizontal asymptote and a vertical asymptote.



- A **rectangular hyperbola** is the graph of the rule $y = \frac{a}{x}$, $a \neq 0$.



  - $y = \frac{1}{x}$ is the basic rectangular hyperbola.
  - $x = 0$ (y-axis) and $y = 0$ (x-axis) are its asymptotes.
  - For $a > 1$ the hyperbola will be further out from the asymptotes.
  - For $0 < a < 1$ the hyperbola will be closer in to the asymptotes.
- The graph of $y = -\frac{a}{x}$ is a reflection of the graph of $y = \frac{a}{x}$ in the x- (or y-) axis.
- All hyperbolas of the form $y = \frac{a}{x}$ are functions because they pass the vertical line test.
  - Note that $y = \frac{1}{x}$ can be written as $f(x) = \frac{1}{x}$.
- In the function $y = \frac{a}{x}$:
  - All x values are permissable except $x = 0$.
  - All y values are permissable except $y = 0$.

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Example 26 Sketching hyperbolas

Sketch the graphs of each hyperbola, labelling the points where $x = 1$ and $x = -1$.

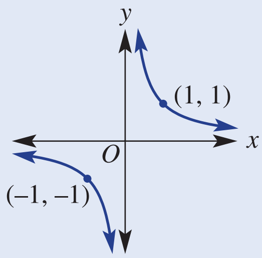
a $y = \dfrac{1}{x}$

b $y = \dfrac{2}{x}$

c $y = -\dfrac{3}{x}$

SOLUTION

a $y = \dfrac{1}{x}$

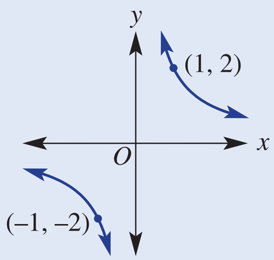


EXPLANATION

Draw the basic shape of a rectangular hyperbola $y = \dfrac{1}{x}$.

Substitute $x = 1$ and $x = -1$ to find the two points.

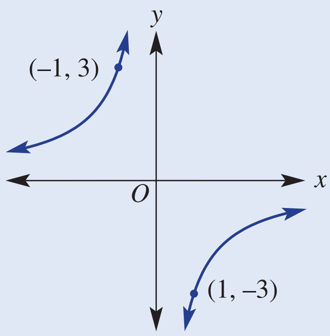
b $y = \dfrac{2}{x}$



For $x = 1$, $y = \dfrac{2}{1} = 2$.

For $x = -1$, $y = \dfrac{2}{-1} = -2$.

c $y = -\dfrac{3}{x}$



$y = -\dfrac{3}{x}$ is a reflection of $y = \dfrac{3}{x}$ in either the x-axis or y-axis.

When $x = 1$, $y = -\dfrac{3}{1} = -3$.

When $x = -1$, $y = -\dfrac{3}{-1} = 3$.

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Example 27 Intersecting with hyperbolas

Find the coordinates of the points where $y = \frac{1}{x}$ intersects these lines.

a $y = 3$ **b** $y = 4x$

SOLUTION

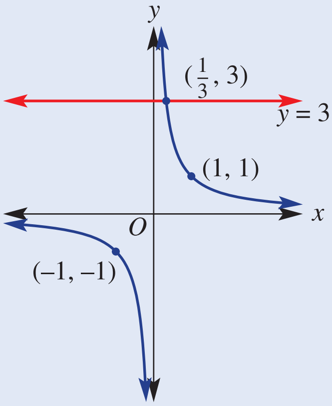
a $y = \frac{1}{x}$ and $y = 3$

$$3 = \frac{1}{x}$$
$$3x = 1$$
$$x = \frac{1}{3}$$

$\therefore$ Intersection point is $\left(\frac{1}{3}, 3\right)$.

EXPLANATION

Substitute $y = 3$ into $y = \frac{1}{x}$ and solve.

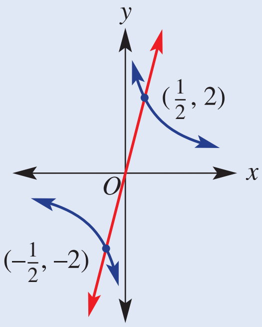


b $y = \frac{1}{x}$ and $y = 4x$

$$4x = \frac{1}{x}$$
$$4x^2 = 1$$
$$x^2 = \frac{1}{4}$$
$$x = \pm\frac{1}{2}$$
$$y = 4 \times \frac{1}{2} = 2 \text{ and } y = 4 \times \left(-\frac{1}{2}\right) = -2$$

$\therefore$ Intersection points are $\left(\frac{1}{2}, 2\right)$ and $\left(-\frac{1}{2}, -2\right)$.

Substitute and solve by multiplying both sides by x.



Exercise 9K

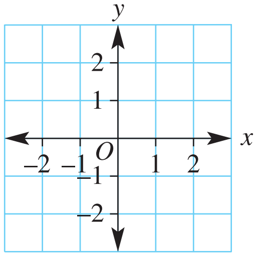
UNDERSTANDING AND FLUENCY	1–4, 5–7(½)	4, 5–8(½)	5–8(½)

1 A hyperbola has the rule $y = \frac{1}{x}$.

a Complete this table.

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y						

b Plot to form the graph of $y = \frac{1}{x}$.



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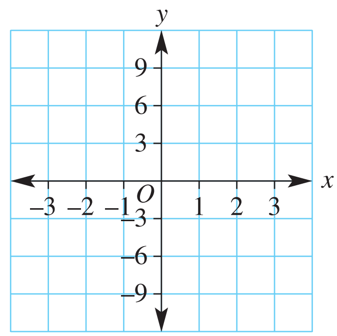
2 A hyperbola has the rule $y = \frac{3}{x}$.

a Complete this table.

x	-3	-1	$-\frac{1}{3}$	$\frac{1}{3}$	1	3
y						

b Plot to form the graph of $y = \frac{3}{x}$.

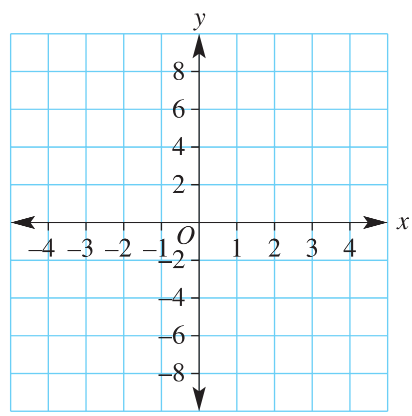


3 A hyperbola has the rule $y = -\frac{2}{x}$.

a Complete this table.

x	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y										

b Plot to form the graph of $y = -\frac{2}{x}$.



4 **a** List in ascending order: $1 \div 0.1$, $1 \div 0.001$, $1 \div 0.01$, $1 \div 0.00001$.

b For $y = \frac{1}{x}$, which of the following x values will give the largest value of y: $\frac{1}{5}, \frac{1}{10}, \frac{1}{2}$ or $\frac{1}{100}$?

c For $y = \frac{1}{x}$ calculate the difference in the y values for $x = 10$ and $x = 1000$.

d For $y = \frac{1}{x}$ calculate the difference in the y values for $x = -\frac{1}{2}$ and $x = -\frac{1}{1000}$.

Example 26

5 Sketch the graphs of these hyperbolas, labelling the points where $x = 1$ and $x = -1$.

a $y = \frac{1}{x}$ **b** $y = \frac{2}{x}$ **c** $y = \frac{3}{x}$ **d** $y = \frac{4}{x}$

e $y = -\frac{1}{x}$ **f** $y = -\frac{2}{x}$ **g** $y = -\frac{3}{x}$ **h** $y = -\frac{4}{x}$

6 Find the coordinates on the graph of $y = \frac{2}{x}$, where:

a $x = 2$ **b** $x = 4$ **c** $x = -1$ **d** $x = -6$

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7 Find the coordinates on the graph of $y = -\frac{5}{x}$, where:

a $x = 10$ **b** $x = -4$ **c** $x = -7$ **d** $x = 9$

8 Find the coordinates on the graph of $y = \frac{3}{x}$, where:

a $y = 3$ **b** $y = 1$ **c** $y = -2$ **d** $y = -6$

PROBLEM-SOLVING AND REASONING	9–10(½), 12	9–11(½), 12, 13	10–11(½), 13–15

9 **a** Determine whether the point (1, 3) lies on the hyperbola $y = \frac{3}{x}$.

b Determine whether the point (1, −5) lies on the hyperbola $y = -\frac{5}{x}$.

c Determine whether the point (2, 1) lies on the hyperbola $y = -\frac{2}{x}$.

d Determine whether the point (−3, 6) lies on the hyperbola $y = -\frac{6}{x}$.

Example 27

10 Find the coordinates of the points where $y = \frac{1}{x}$ intersects these lines.

a $y = 2$ **b** $y = 6$ **c** $y = -1$ **d** $y = -10$
e $y = x$ **f** $y = 4x$ **g** $y = 2x$ **h** $y = 5x$

11 Find the coordinates of the points where $y = -\frac{2}{x}$ intersects these lines.

a $y = -3$ **b** $y = 4$ **c** $y = -\frac{1}{2}$ **d** $y = \frac{1}{3}$

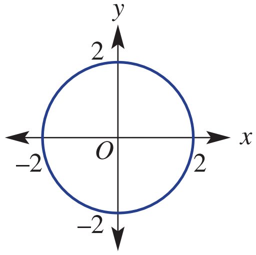
e $y = -2x$ **f** $y = -8x$ **g** $y = -\frac{1}{2}x$ **h** $y = -x$

12 Match equations **a**–**f** with graphs **A**–**F**.

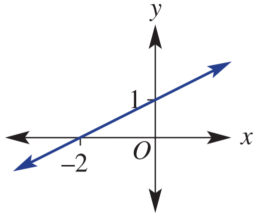
a $y = \frac{4}{x}$ **b** $y = -\frac{1}{x}$ **c** $y = 2^x$

d $y = \frac{1}{2}x + 1$ **e** $x^2 + y^2 = 4$ **f** $y = 3^{-x}$

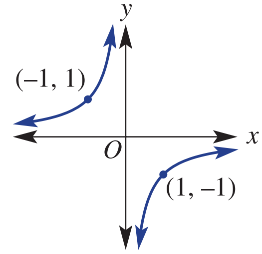
A



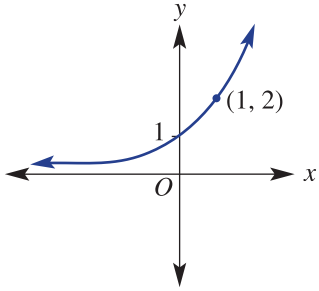
B



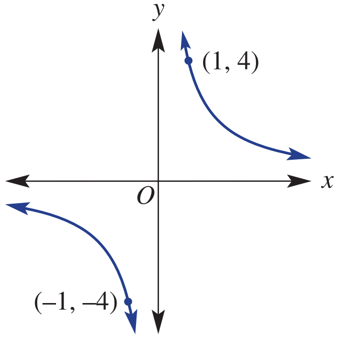
C



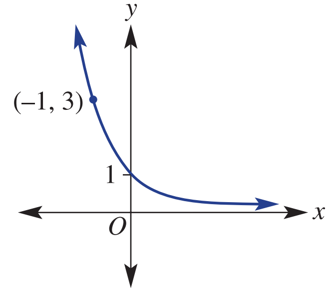
D



E



F



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13 Is it possible for a line on a number plane to not intersect the graph of $y = \frac{1}{x}$? If so, give an example.

14 Write the missing word (*zero* or *infinity*) for these sentences.

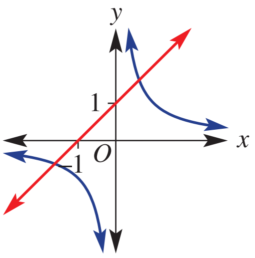
- **a** For $y = \frac{1}{x}$, when x approaches infinity, y approaches ________.
- **b** For $y = \frac{1}{x}$, when x approaches negative infinity, y approaches ________.
- **c** For $y = \frac{1}{x}$, when x approaches zero from the right, y approaches ________.
- **d** For $y = \frac{1}{x}$, when x approaches zero from the left, y approaches ________.

15 Compare the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{2x}$. Describe the effect of the coefficient of x in $y = \frac{1}{2x}$.

ENRICHMENT	–	–	16

To intersect or not!

16 The graphs of $y = \frac{1}{x}$ and $y = x + 1$ intersect at two points. To find the points, we set:



$$\frac{1}{x} = x + 1$$
$$1 = x(x + 1)$$
$$0 = x^2 + x - 1$$

Using the quadratic formula, $x = \frac{-1 \pm \sqrt{5}}{2}$ and $y = \frac{1 \pm \sqrt{5}}{2}$.

- **a** Find the exact coordinates of the intersection of $y = \frac{1}{x}$ and these lines.
  - **i** $y = x - 1$
  - **ii** $y = x - 2$
  - **iii** $y = x + 2$
- **b** Try to find the coordinates of the intersection of $y = \frac{1}{x}$ and $y = -x + 1$. What do you notice? What part of the quadratic formula confirms this result?
- **c** Write down the equations of the two straight lines (which are not horizontal or vertical) that intersect $y = \frac{1}{x}$ only once.

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9L Cubic equations, functions and graphs

Interactive
Widgets
HOTsheets
Walkthrough

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

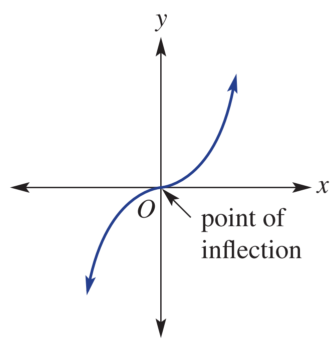
A cubic is a polynomial in which the highest power of the variable is 3 (we say it has degree 3). In this section, we will consider cubic graphs of the form $y = ax^3$ and their transformations. When solving a cubic equation of the form $ax^3 = d$, we will always obtain one distinct solution. Since $(-2)^3 = (-2) \times (-2) \times (-2) = -8$, we have $\sqrt[3]{-8} = -2$, meaning that we can take the cube root of both positive and negative values, thus obtaining one distinct solution.

Cubic functions of the form $y = ax^3$ have particular application in problems relating to volume. For example, the volume of a cube with side length x is $V = x^3$ and the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Let's start: How do a and d effect the graph of $y = ax^3 + d$?

The graph of $y = x^3$ is shown. It has a point called the **point of inflection** at the origin, O or $(0, 0)$.

- Use computer graphing software to sketch the graphs in each table on the same set of axes. Complete each table to observe the effect of changing the values of a and d.



Graphs of $y = ax^3$

Rule	Coordinates of point of inflection	Number of x-intercepts	Shape of graph	Is the graph narrower or wider than the graph of $y = x^3$?
$y = x^3$	(0, 0)	1		
$y = 2x^3$				
$y = 5x^3$				
$y = \frac{1}{2}x^3$				
$y = -x^3$				
$y = -3x^3$				
$y = -\frac{1}{2}x^3$				
$y = -\frac{1}{4}x^3$				

- What causes the graph of $y = ax^3$ to reflect in the x-axis?
- Describe the effect on the shape of the graph by different values of a.

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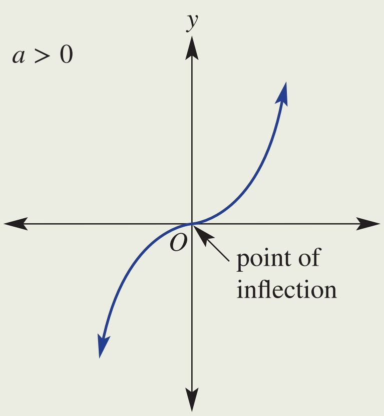
Graphs of $y = x^3 + d$

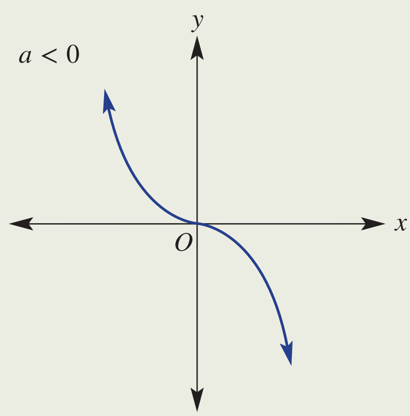
Rule	Coordinates of point of inflection	Number of x-intercepts	Position of graph compared to $y = x^3$?
$y = x^3$	(0, 0)	1	
$y = x^3 + 2$			
$y = x^3 - 3$			
$y = x^3 - 5$			
$y = x^3 + 8$			

- Describe the effect on the graph of $y = x^3 + d$ by different values of d.
- Can you explain why all graphs of the form $y = ax^3 + d$ have exactly one x-intercept?

Key ideas

- A cube root of a number, $\sqrt[3]{c}$, is a number a such that $a^3 = c$.
  - All real numbers have exactly one cube root.
  - $\sqrt[3]{8} = 2$ since $2^3 = 8$, and $\sqrt[3]{-8} = -2$ since $(-2)^3 = -8$.
- A cubic equation of the form $ax^3 = d$ has exactly one solution.
  - To solve $x^3 = c$, take the cube root of both sides; hence, $x = \sqrt[3]{c}$.
    For example: If $x^3 = 27$, then $x = \sqrt[3]{27} = 3$ since $3^3 = 27$.
  - To solve $ax^3 = d$, first solve for x^3 by dividing both sides by a and then take the cube root of both sides.
- The graphs of $y = ax^3$ are shown.





  - Each cubic is a function (i.e. it passes the vertical line test) with permissible values of x and y being all real values.
  - $y = x^3$ can be written as $f(x) = x^3$. Also, $f(2) = 8$ can be used to represent the point (2, 8).
  - The graph has a point of inflection (i.e. a point where the gradient of the graph changes from decreasing to increasing or vice versa) at (0, 0).
  - Negative values of a cause the graph to reflect in the x-axis. (For cubics this is the same as a reflection in the y-axis.)

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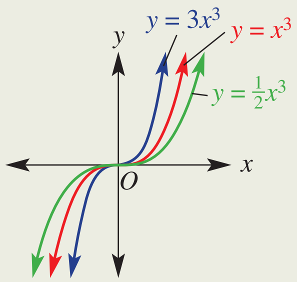
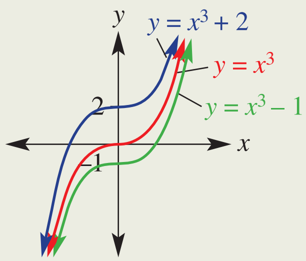
Key ideas

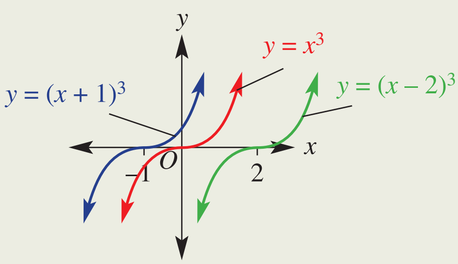
- For $a > 0$:
  $a > 1$ causes the graph to rise more quickly and makes it narrower than $y = x^3$.
  $0 < a < 1$ causes the graph to rise more slowly and makes it wider than $y = x^3$.



- The d in the rule $y = ax^3 + d$ translates the graph of $y = ax^3$ in the vertical direction.
  - $d > 0$ translates the graph up d units.
  - $d < 0$ translates the graph down d units.



- The r in the rule $y = a(x - r)^3$ translates the graph of $y = ax^3$ in the horizontal direction.
  - $r > 0$ translates the graph r units to the right.
  - $r < 0$ translates the graph r units to the left.



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Example 28 Solving simple cubic equations

Solve the following cubic equations.

a $x^3 = 27$

b $-2x^3 = 16$

c $3x^3 + 10 = 193$ (round to 1 decimal place)

SOLUTION

a $x^3 = 27$
$x = \sqrt[3]{27}$
$\therefore x = 3$

EXPLANATION

To solve for x, take the cube root of both sides.
$\sqrt[3]{27} = 3$ since $3^3 = 27$.

b $-2x^3 = 16$
$x^3 = -8$
$x = \sqrt[3]{-8}$
$\therefore x = -2$

Solve for x^3 by dividing both sides by -2.
Take the cube root of both sides.
The cube root of a negative number will be negative and $\sqrt[3]{-8} = -2$ since $(-2)^3 = -8$.

c $3x^3 + 10 = 193$
$3x^3 = 183$
$x^3 = 61$
$x = \sqrt[3]{61}$
$\therefore x = 3.9$ (to 1 decimal place)

Solve for x^3 first by subtracting 10 from both sides and then dividing both sides by 3.
Take the cube root of both sides to solve for x.
$\sqrt[3]{61}$ is the exact answer as it does not simplify to a whole number.
Use a calculator to find $\sqrt[3]{61} = 3.9$, to 1 decimal place.

Example 29 Sketching cubic functions of the form $y = ax^3$

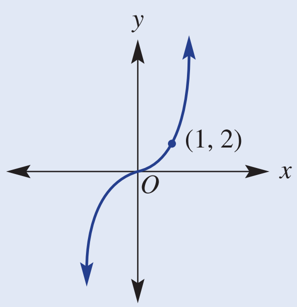
Sketch the following cubic graphs.

a $y = 2x^3$

b $y = -\frac{1}{2}x^3$

SOLUTION

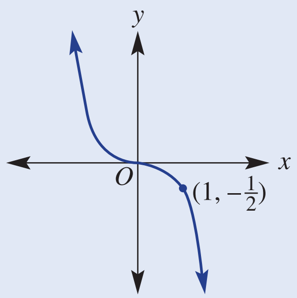
a



EXPLANATION

The graph is a positive cubic graph with point of inflection at (0, 0).
Mark in the point at $x = 1$ to show the effect of the scale factor 2.

b



The graph is a negative cubic graph with point of inflection at (0, 0). It is the graph of $y = \frac{1}{2}x^3$ and is reflected in the x-axis.
Mark in the point at $x = 1$ to show the effect of the scale factor $\frac{1}{2}$.

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Example 30 Sketching cubic graphs involving translations

Sketch the following cubic functions, labelling the point of inflection and axes intercepts.

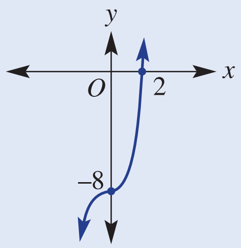
a $y = x^3 - 8$ **b** $y = -2x^3 + 6$ **c** $y = (x - 1)^3$

SOLUTION

a Point of inflection is $(0, -8)$.

x-intercept (when $y = 0$):

$$x^3 - 8 = 0$$
$$x^3 = 8$$
$$x = 2$$



EXPLANATION

Locate the point of inflection at $(0, -8)$ since the graph of $y = x^3 - 8$ is the graph of $y = x^3$ translated down 8 units.

The point of inflection is also the y-intercept.

Determine the x-intercept by substituting $y = 0$.

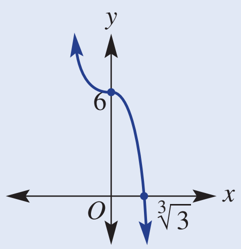
$\sqrt[3]{8} = 2$

Mark in the point of inflection and intercepts and join them to form a positive cubic curve.

b Point of inflection is $(0, 6)$.

x-intercept (when $y = 0$):

$$-2x^3 + 6 = 0$$
$$2x^3 = 6$$
$$x^3 = 3$$
$$x = \sqrt[3]{3}$$



The graph is a negative cubic graph.

The graph is the graph of $y = -2x^3$ translated 6 units up.

The point of inflection is also the y-intercept.

Substitute $y = 0$ to find the x-intercept. Solve the remaining equation by first solving for x^3 and then taking the cube root of both sides.

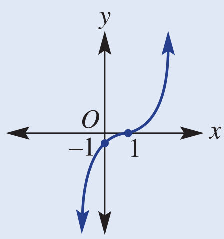
Leave the answer in exact form, although it is useful to consider the decimal approximation when marking on the axis (i.e. $\sqrt[3]{3} \approx 1.4$).

Mark the key points on the graph and join them to form a negative cubic curve.

c Point of inflection is $(1, 0)$.

y-intercept ($x = 0$):

$$y = (0 - 1)^3$$
$$y = -1$$



The graph of $y = (x - 1)^3$ is the graph of $y = x^3$ translated 1 unit to the right.

The point of inflection is also the x-intercept.

Locate the y-intercept by substituting $x = 0$.

Recall that $(-1)^3 = -1$.

Mark points on the graph and join them to form a cubic curve.

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Exercise 9L

UNDERSTANDING AND FLUENCY	1–4, 5(½), 6–10	4, 5–6(½), 7–10	5–6(½), 8–10(½)

1 Evaluate:

a 3^3 b 5^3 c -4^3

d $(-10)^3$ e $(-6)^3$ f $\sqrt[3]{8}$

g $\sqrt[3]{1}$ h $\sqrt[3]{-27}$ i $\sqrt[3]{-125}$

2 Find the value $2x^3$ for the following values of x.

a $x = 0$ b $x = 3$ c $x = -1$ d $x = -4$

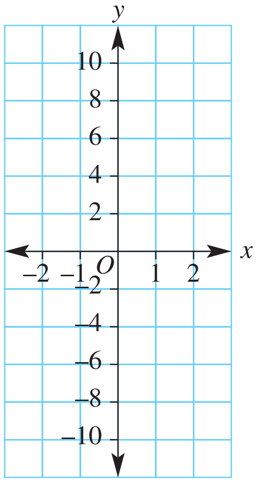
3 Complete the following tables for the given rules and plot on the same Cartesian plane.

a $y = x^3$

x	−2	−1	0	1	2
y					

b $y = -x^3$

x	−2	−1	0	1	2
y					



4 The graph of a cubic function is shown.

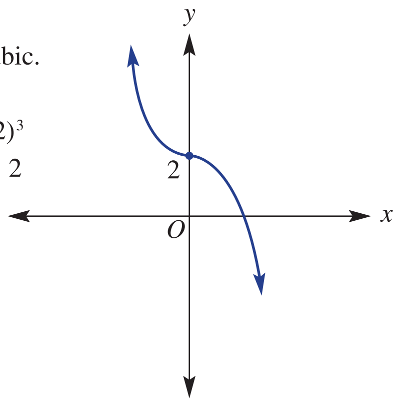
a Write down the coordinates of the point of inflection of this cubic.

b A possible equation for this cubic function is:

A $y = 2x^3$ B $y = (x + 2)^3$

C $y = -2x^3$ D $y = -x^3 + 2$

E $y = x^3 + 2$



Example 28a, b

5 Solve the following cubic equations.

a $x^3 = 64$ b $x^3 = 125$

c $x^3 = -27$ d $x^3 = -8$

e $2x^3 = 2000$ f $-3x^3 = 81$

g $\frac{x^3}{2} = 108$ h $-\frac{x^3}{3} = 9$

i $x^3 = \frac{1}{8}$ j $x^3 = \frac{1}{27}$

k $25x^3 = \frac{1}{5}$ l $-24x^3 = -3$

Example 28c

6 Solve these cubic equations, rounding your answer to 1 decimal place where necessary.

a $x^3 + 4 = 5$ b $x^3 - 12 = 15$ c $2x^3 - 3 = -9$ d $3x^3 + 5 = 86$

e $\frac{1}{2}x^3 - 10 = 22$ f $\frac{1}{3}x^3 - 43 = 32$ g $4 - x^3 = 28$ h $3 - 2x^3 = 165$

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7 Give exact solutions to these cubic equations.

a $x^3 - 15 = 6$ **b** $2x^3 + 10 = 22$

c $1 - 2x^3 = 5$ **d** $\frac{x^3}{2} + 9 = 18$

Example 29

8 Sketch a graph of the following cubic functions.

a $y = 3x^3$ **b** $y = 4x^3$ **c** $y = -2x^3$

d $y = \frac{1}{2}x^3$ **e** $y = \frac{1}{10}x^3$ **f** $y = -\frac{1}{4}x^3$

Example 30a, b

9 Sketch these cubic graphs involving vertical translations. Label the axes intercepts with exact values.

a $y = x^3 + 1$ **b** $y = x^3 - 8$ **c** $y = -x^3 - 27$

d $y = \frac{1}{2}x^3 + 32$ **e** $y = 2x^3 + 10$ **f** $y = -\frac{1}{3}x^3 + 8$

Example 30c

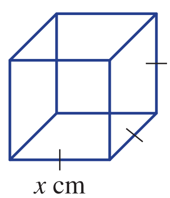
10 Sketch these cubic graphs involving horizontal translations.

a $y = (x + 1)^3$ **b** $y = (x - 2)^3$ **c** $y = (x - 3)^3$

d $y = -(x + 2)^3$ **e** $y = -2(x - 1)^3$ **f** $y = \frac{1}{2}(x + 4)^3$

PROBLEM-SOLVING AND REASONING	11, 12, 15	11, 12, 15, 16	11–17

11 A cube with side lengths x cm has a volume of 512 cm³. Determine its surface area.



12 A seedling is planted and its height, h millimetres, is recorded for a number of weeks. It is found that the height of the plant t weeks after planting can be modelled by the equation $h = 2t^3$.

a What was the height of the plant after 3 weeks?

b After how many weeks was the plant 25 cm tall?

c The plant has an expected maximum height of 1 metre. After how many weeks would it have reached this height?

d Plot a graph of h against t for $0 \leqslant t \leqslant 8$.

13 Solve the following cubic equations.

a $(x + 1)^3 = 8$ **b** $(x - 2)^3 = 27$

c $(x + 3)^3 = -64$ **d** $2(x + 2)^3 = 250$

e $-3(x - 2)^3 = 192$ **f** $-\frac{1}{2}(x + 4)^3 = 500$

14 If the Earth is taken to be spherical with volume 108.321×10^{10} km³, determine the mean radius of the Earth, to the nearest kilometre. Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.

15 **a** Explain why $(-x)^2 \neq -x^2$ for $x \neq 0$.

b Explain why $(-x)^3 = -x^3$ for all values of x.

c Generalise your results from parts **a** and **b** and use them to explain for which values of n will $(-x)^n$ equal $-x^n$ for all values of x.

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16 A cylindrical tank is such that its height is equal to its radius.

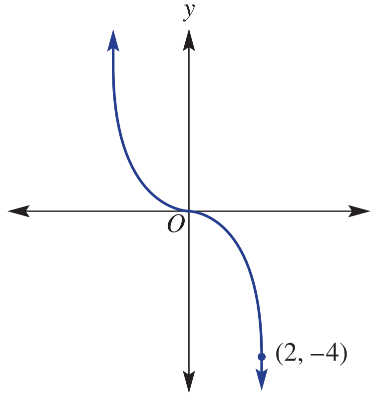
a Plot a graph of the volume, V, of the tank against the radius, r, for suitable values of r.

b Determine the exact radius of the cylinder if the volume is 8000 units^3.

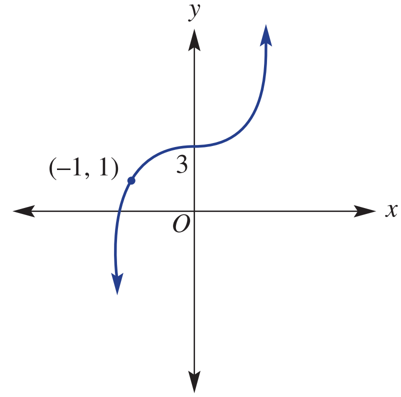
c If the radius of the tank is doubled (and the height is equal to this new radius) what is the resulting change in volume? Can you explain this?

17 The following graphs have rules of the form $y = ax^3 + d$. Use the points given to find the values of a and d in each one.

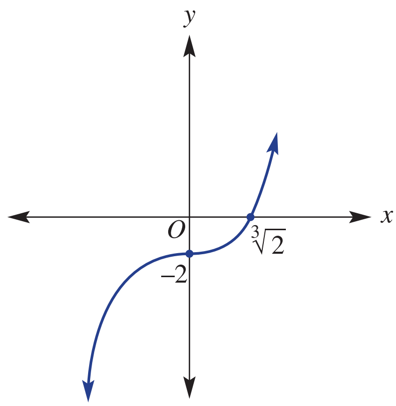
a



b



c



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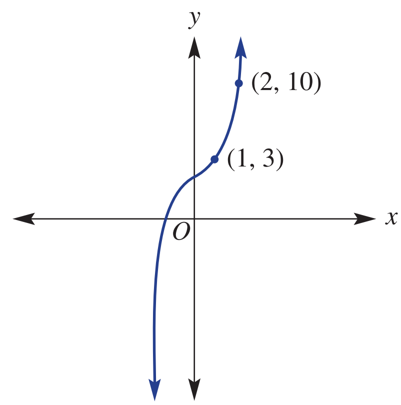

ENRICHMENT	–	–	18

Combining transformations

18 Combine your knowledge of transformations to sketch the following cubic graphs. Label the point of inflection and axes intercepts.

For example, the point of inflection of the graph of $y = (x - 1)^3 - 8$ is at $(1, -8)$.

a $y = (x - 1)^3 - 8$

b $y = (x - 2)^3 + 27$

c $y = 2(x + 3)^3 + 2$

d $y = \frac{1}{2}(x + 4)^3 - 4$

e $y = -(x + 2)^3 + 1$

f $y = -(x + 1)^3 - 8$

g $y = -\frac{1}{3}(x + 3)^3 - 9$

h $y = -2(x - 1)^3 + 16$

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9M Further transformations of graphs

Earlier in this chapter we considered a wide range of transformations of parabolas but a limited number of transformations of circles, exponentials and hyperbolas. We will now look more closely at translations of these relations and the key features of their graphs.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Interactive

Widgets

HOTsheets

Walkthrough

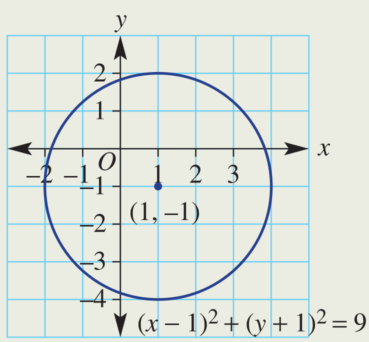
Let's start: Translations, translations, translations

Use technology to assist in the discussion of these questions.

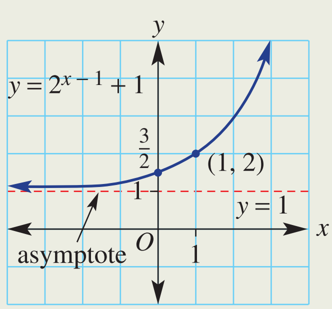
- How does the graph of $(x-1)^2+(y+2)^2=9$ compare with that of $x^2+y^2=9$?
- What is the effect of h, k and r in $(x-h)^2+(y-k)^2=r^2$?
- How does the graph of $y=2^{x-2}+1$ compare with that of $y=2^x$?
- What is the effect of h and k in $y=2^{x-h}+k$?
- How does the graph of $y=\dfrac{1}{x+2}-1$ compare with that of $y=\dfrac{1}{x}$?
- What is the effect of h and k in $y=\dfrac{1}{x+h}+k$?

Key ideas

- The equation of a **circle** in standard form is $(x-h)^2+(y-k)^2=r^2$.
  - (h, k) is the centre.
  - r is the radius.



- For the graph of the **exponential** equation $y=a^{x-h}+k$ the graph of $y=a^x$ is:
  - translated h units to the right
  - translated k units up
  - The equation of the asymptote is $y=k$.



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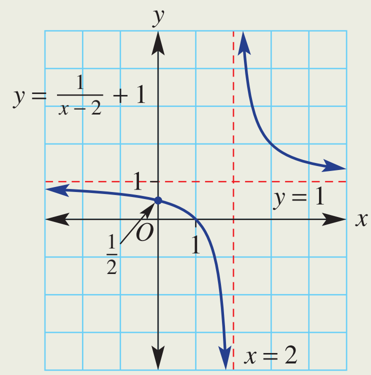
Key ideas

- For the graph of the **hyperbola** $y = \frac{1}{x-h} + k$ the graph of $y = \frac{1}{x}$ is:
  - translated h units to the right
  - translated k units up
  - The asymptotes are $x = h$ and $y = k$.

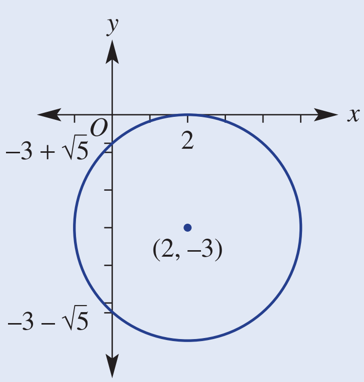


- Sometimes $y = f(x)$ is used as a generic name for an unknown graph; that is, $y = f(x)$ could be any function.
  That being the case, if $y = f(x)$ is graphed, then:
  - $y = f(x) + k$ is the same graph translated k units up or down.
  - $y = f(x - k)$ is the same graph translated k units horizontally.
    For example: If $y = x^2$ is translated 2 units to the right, it becomes $y = (x - 2)^2$.

Example 31 Sketching with transformations

Sketch the graphs of the following relations. Label important features.

a $(x - 2)^2 + (y + 3)^2 = 9$ **b** $y = 2^{x+2} - 3$ **c** $y = \frac{1}{x+1} + 2$

SOLUTION	EXPLANATION
a $(x - 2)^2 + (y + 3)^2 = 9$ Centre $(2, -3)$ Radius $= 3$ x-intercept is 2.	For $(x - h)^2 + (y - k)^2 = r^2$, (h, k) is the centre and r is the radius.
y-intercepts at $x = 0$: $(0 - 2)^2 + (y + 3)^2 = 9$ $4 + (y + 3)^2 = 9$ $(y + 3)^2 = 5$ $y + 3 = \pm\sqrt{5}$ $y = -3 \pm \sqrt{5}$	Find the y-intercepts by substituting $x = 0$ and solving for y.
	Draw the graph.

Example continued over page

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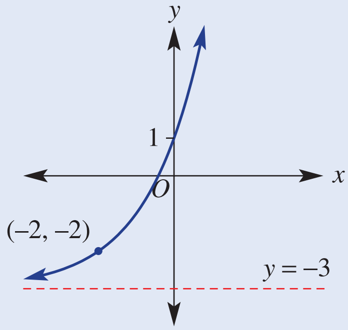
b $y = 2^{x+2} - 3$



For $y = 2^{x-h} + k$, $y = k$ is the equation of the asymptote.

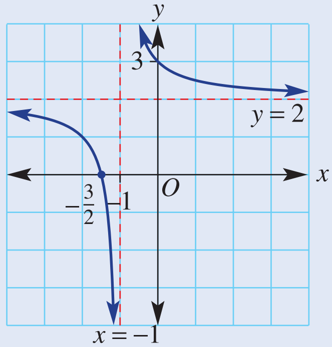
$(0, 1)$ in $y = 2^x$ is translated 2 units to the left and 3 units down to $(-2, -2)$.

Substitute $x = 0$ to find the y-intercept. (We will calculate the x-intercept when we study logarithms in Chapter 10.)

y-intercept is 1.

At $x = 0$, $y = 2^2 - 3 = 1$.

c $y = \dfrac{1}{x+1} + 2$



For $y = \dfrac{1}{x-h} + k$, $h = -1$ and $k = 2$, so the asymptotes are $x = -1$ and $y = 2$.

Substitute to find the x- and y-intercepts.

Find x-intercepts (when $y = 0$):

$$\text{At } y = 0, \quad 0 = \frac{1}{x+1} + 2$$
$$-2 = \frac{1}{x+1}$$
$$x + 1 = -\frac{1}{2}$$
$$x = -\frac{3}{2}$$

At $x = 0$, $y = \dfrac{1}{1} + 2 = 3$.

y-intercept is 3.

x-intercept is $-\dfrac{3}{2}$.

Exercise 9M

UNDERSTANDING AND FLUENCY	1–4(½)	2–4(½)	2–4(½)

1 Choose the word *left, right, up* or *down* to complete each sentence.

a The graph of $y = 2^x + 1$ is the translation of the graph of $y = 2^x$ _____ by 1 unit.

b The graph of $y = 2^x - 3$ is the translation of the graph of $y = 2^x$ _____ by 3 units.

c The graph of $y = 2^{x-4}$ is the translation of the graph of $y = 2^x$ _____ by 4 units.

d The graph of $y = 2^{x+1}$ is the translation of the graph of $y = 2^x$_____ by 1 unit.

e The graph of $y = \dfrac{1}{x-3}$ is the translation of the graph of $y = \dfrac{1}{x}$ _____ by 3 units.

f The graph of $y = \dfrac{1}{x+2}$ is the translation of the graph of $y = \dfrac{1}{x}$ _____ by 2 units.

g The graph of $y = \dfrac{1}{x} + 3$ is the translation of the graph of $y = \dfrac{1}{x}$ _____ by 3 units.

h The graph of $y = \dfrac{1}{x} - 6$ is the translation of the graph of $y = \dfrac{1}{x}$ _____ by 6 units.

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i The graph of $(x - 1)^2 + y^2 = 1$ is the translation of the graph of $x^2 + y^2 = 1$ _____ by 1 unit.
j The graph of $(x + 3)^2 + y^2 = 1$ is the translation of the graph of $x^2 + y^2 = 1$ _____ by 3 units.
k The graph of $x^2 + (y + 3)^2 = 1$ is the translation of the graph of $x^2 + y^2 = 1$ _____ by 3 units.
l The graph of $x^2 + (y - 2)^2 = 1$ is the translation of the graph of $x^2 + y^2 = 1$ _____ by 2 units.

Example 31a

2 Sketch the graph of the following circles. Label the coordinates of the centre and find the x- and y-intercepts, if any.
a $(x - 3)^2 + (y + 1)^2 = 1$
b $(x + 2)^2 + (y - 3)^2 = 4$
c $(x - 1)^2 + (y + 3)^2 = 25$
d $(x + 3)^2 + (y - 2)^2 = 25$
e $(x + 2)^2 + (y - 1)^2 = 9$
f $x^2 + (y - 4)^2 = 36$
g $(x + 1)^2 + y^2 = 9$
h $(x - 2)^2 + (y - 5)^2 = 64$
i $(x + 3)^2 + (y - 1)^2 = 5$

Example 31b

3 Sketch the graph of these exponentials. Label the asymptote and find the y-intercept.
a $y = 2^x - 2$
b $y = 2^x + 1$
c $y = 2^x - 5$
d $y = 2^x - 1$
e $y = 2^{x+3}$
f $y = 2^{x+1}$
g $y = 2^{x-1} + 1$
h $y = 2^{x+2} - 3$
i $y = 2^{x-3} - 4$

Example 31c

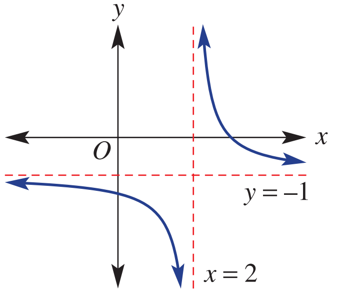
4 Sketch the graph of these hyperbolas. Label the asymptotes and find the x- and y-intercepts.
a $y = \frac{1}{x} + 2$
b $y = \frac{1}{x} - 1$
c $y = \frac{1}{x + 3}$
d $y = \frac{1}{x - 2}$
e $y = \frac{1}{x + 1} + 1$
f $y = \frac{1}{x - 1} - 3$
g $y = \frac{1}{x - 3} + 2$
h $y = \frac{1}{x + 4} - 1$
i $y = \frac{1}{x - 5} + 6$

PROBLEM-SOLVING AND REASONING	5(½), 6, 9	5(½), 6, 7, 9, 10	7, 8, 10, 11

5 The graphs of these exponentials involve a number of transformations. Sketch their graphs, labelling the y-intercept and the equation of the asymptote.
a $y = -2^x + 1$
b $y = -2^x - 3$
c $y = -2^{x+3}$
d $y = -2^{x-2}$
e $y = -2^{x+1} - 1$
f $y = -2^{x+2} + 5$

6 Sketch these hyperbolas, labelling asymptotes and intercepts.
a $y = \frac{-1}{x + 1} + 2$
b $y = \frac{-2}{x + 2} - 1$
c $y = \frac{-2}{x - 3} - 2$

7 The following hyperbolas are of the form $y = \frac{1}{x - h} + k$. Write the rule for each graph.

a


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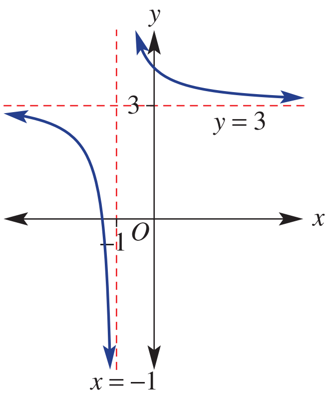
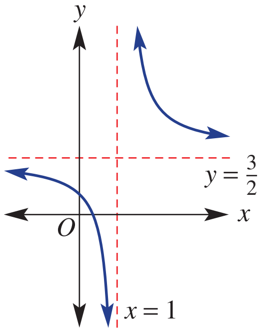


c


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8 Find the coordinates of the intersection of the graphs of these equations.

a $y = \dfrac{1}{x+1}$ and $y = x + 2$

b $y = \dfrac{1}{x-2} + 1$ and $y = x + 3$

c $y = \dfrac{-1}{x+2} - 3$ and $y = -2x - 1$

d $(x - 1)^2 + y^2 = 4$ and $y = 2x$

e $(x + 2)^2 + (y - 3)^2 = 16$ and $y = -x - 3$

f $x^2 + (y + 1)^2 = 10$ and $y = \dfrac{1}{3}x - 1$

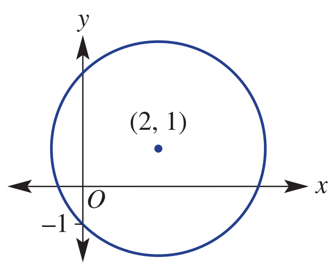
9 A circle has equation $(x - 3)^2 + (y + 2)^2 = 4$. Without sketching a graph, state the minimum and maximum values for:

a x

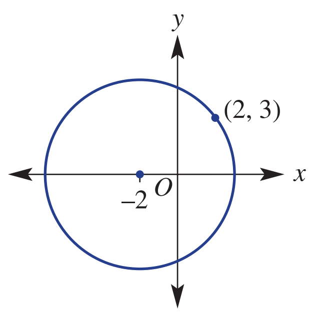
b y

10 Find a rule for each graph.

a



b



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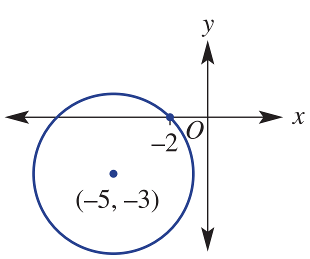

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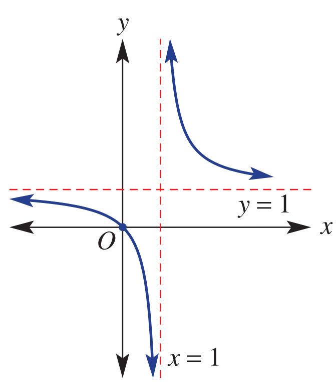
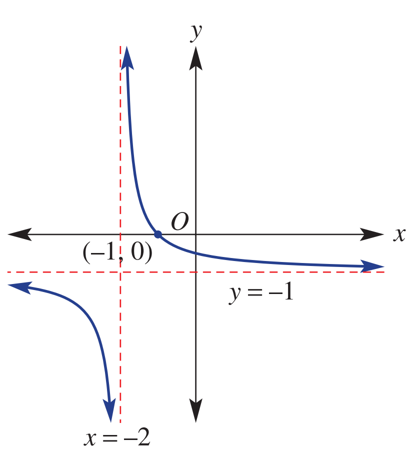
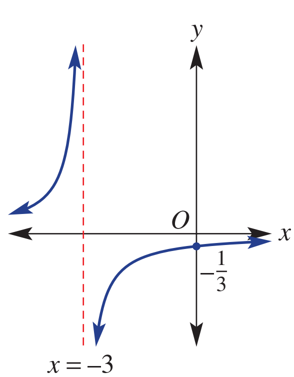


e



f



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11 Explain why the graphs of the following pairs of relations do not intersect.

a $y = \frac{1}{x}$ and $y = -x$

b $(x - 1)^2 + (y + 2)^2 = 4$ and $y = 1$

c $y = 2^{x-1} + 3$ and $y = x - 3$

d $y = \frac{2}{x+3} - 1$ and $y = \frac{1}{3x}$

ENRICHMENT	–	–	12

Circles by completing the square

12 By expanding brackets of an equation in the standard form of a circle, we can write:

$(x - 1)^2 + (y + 2)^2 = 4 \quad (1)$
$x^2 - 2x + 1 + y^2 + 4y + 4 = 4$
$x^2 - 2x + y^2 + 4y + 1 = 0 \quad (2)$

Note that in equation (2) it is not obvious that the centre is $(1, -2)$ and that the radius is 2. It is therefore preferable to write the equation of a circle in standard form (i.e. as in equation (1)).

If given an equation such as (2), we can complete the square in both x and y to write the equation of the circle in standard form.

$x^2 - 2x + y^2 + 4y + 1 = 0$
$x^2 - 2x + 1 - 1 + y^2 + 4y + 4 - 4 + 1 = 0$
$(x - 1)^2 - 1 + (y + 2)^2 - 4 + 1 = 0$
$(x - 1)^2 + (y + 2)^2 = 4$
The radius is 2 and centre $(1, -2)$.

a Write these equations of circles in standard form. Then state the coordinates of the centre and the radius.

i $x^2 + 4x + y^2 - 2y + 1 = 0$
ii $x^2 + 8x + y^2 + 10y + 5 = 0$
iii $x^2 - 6x + y^2 - 4y - 3 = 0$
iv $x^2 - 2x + y^2 + 6y - 5 = 0$
v $x^2 + 10x + y^2 + 8y + 17 = 0$
vi $x^2 + 6x + y^2 + 6y = 0$
vii $x^2 + 3x + y^2 - 6y + 4 = 0$
viii $x^2 + 5x + y^2 - 4y - 2 = 0$
ix $x^2 - x + y^2 + 3y + 1 = 0$
x $x^2 - 3x + y^2 - 5y - 4 = 0$

b Give reasons why $x^2 + 4x + y^2 - 6y + 15 = 0$ is not the equation of a circle.

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9N Using graphs to describe change

Interactive
Widgets
HOTsheets
Walkthrough

Stage
5.3#
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5.1
4

Two variables are said to be directly related if they are in a constant (i.e. unchanged) ratio. If two variables are in direct proportion, as one variable increases so does the other. For example, consider the relationship between speed and distance travelled in a given time. In 1 hour, a car can travel 50 km at 50 km/h, 100 km at 100 km/h, etc.

For two variables in inverse or indirect variation, as one variable increases the other decreases. For example, consider a beach house that costs \$2000 per week to rent. As the number of people renting the house increases, then the cost per person decreases.

The shape of a graph shows how y varies and also how the rate of change of y (i.e. the gradient) varies. For example, when a car brakes to stop at traffic lights, it decelerates. The distance travelled per second is increasingly smaller until the car stops.

Let's start: Movement graphs

This is a whole class activity. Two volunteers are needed: the 'walker' who completes a journey between the front and back of the classroom and the 'grapher' who graphs the journey on the whiteboard.

1. The 'walker' will complete a variety of journeys but these are not stated to the class. For example:
   - Walk slowly from the front of the room, stop halfway for a few seconds and then walk steadily to the back of the room. Stop and then return to the front at a fast steady pace.
   - Start quickly from the back of the room and gradually slow down until stopping at the front.
   - Start slowly from the front and gradually increase walking speed all the way to the back. Stop for a few seconds and then return at a steady speed to the front of the room.
2. The 'grapher' draws a graph of distance versus time on the whiteboard at the same time as the 'walker' moves. The distance is measured from the front of the room. No numbers are needed.
3. The class members also each draw their own distance–time graph as the 'walker' moves.
4. After each walk, discuss how well the 'grapher' has modelled the 'walker's' movement.
5. This activity can also be done in reverse. The 'grapher' draws a distance–time graph on the board and the 'walker' moves to match the graph. The class checks that the 'walker' is following the graph correctly.

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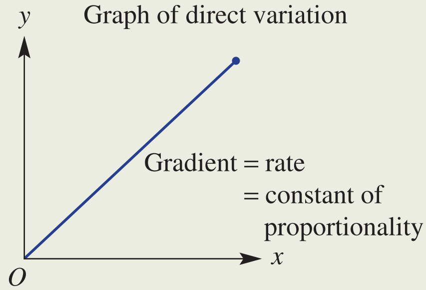
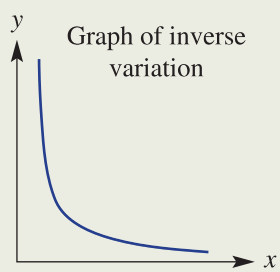
Key ideas

- y varies directly with x if their relationship is $y = kx$ or $\frac{y}{x} = k$.



  - k is a constant, and is called the constant of proportionality.
  - The graph of y versus x gives a straight line that passes through the origin, O or $(0, 0)$, where k is the gradient.
  - We write: $y \propto x$, which means that $y = kx$.
  - We say: y varies directly as x or y is directly proportional to x.
- y varies inversely with x if their relationship is $y = \frac{k}{x}$ or $xy = k$.
  - The graph of y versus x gives a hyperbola.
  - We write: $y \propto \frac{1}{x}$, which means that $y = \frac{k}{x}$ or $xy = k$.
  - We say: y varies inversely as x or y is inversely proportional to x.



- The shape of a graph shows how both y and the rate of change of y (i.e. the gradient) varies.
  - If y increases as x increases, then the rate of change (the gradient) is positive.
  - If y decreases as x increases, then the rate of change (the gradient) is negative.
  - If y does not change as x increases, then the rate of change (the gradient) is zero.
  - Straight lines have a constant rate of change. There is a fixed change in y for each unit increase in x.
  - Curves have a varying rate of change. The change in y varies for each unit increase in x.
  - Analysing a graph and describing how both y and the rate of change of y varies allows us to check whether a given graph models a situation accurately.

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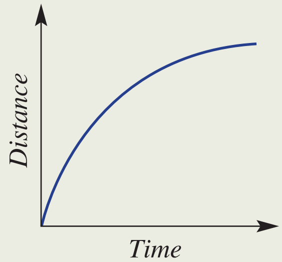
Key ideas

For example, these distance–time graphs show various journeys from 'home' (distance = 0 at home).

Journey A

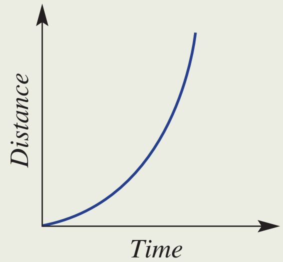
Decelerating away from home.



Distance from home is increasing at a decreasing rate.

Journey B

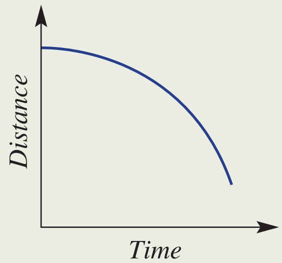
Accelerating away from home.



Distance from home is increasing at an increasing rate.

Journey C

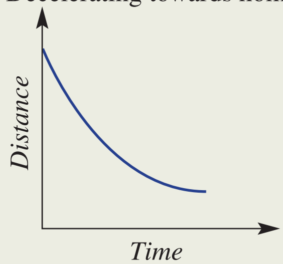
Accelerating towards home.



Distance from home is decreasing at an increasing rate.

Journey D

Decelerating towards home.

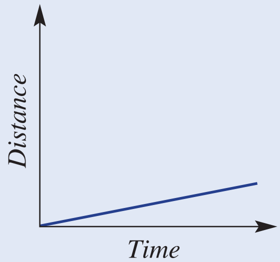


Distance from home is decreasing at a decreasing rate.

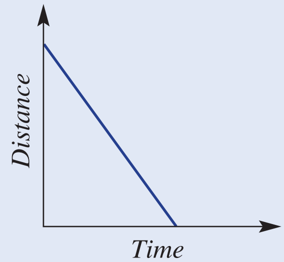
Example 32 Matching descriptions to distance–time graphs

These distance–time graphs show various journeys, each with distance measured from 'home' (i.e. distance = 0). For each graph **A** to **F**, select a correct description from each of the three lists supplied.

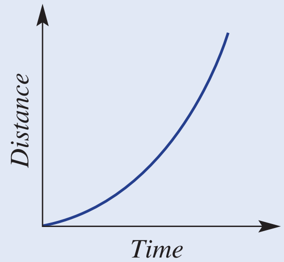
A



B



C



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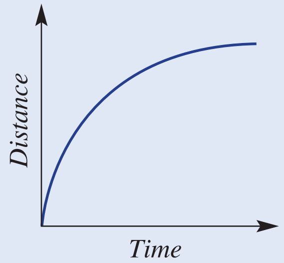
D



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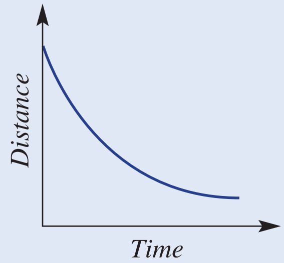

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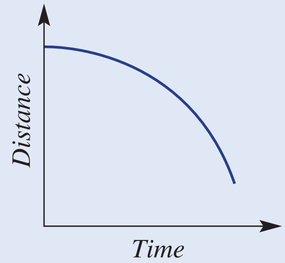

Distance from home	**Gradient of graph**	**Speed**
Increasing distance from home	Positive constant gradient	Stationary
Decreasing distance from home	Positive varying gradient	Lower constant speed
Fixed distance from home	Negative constant gradient	Higher constant speed
	Negative varying gradient	Decreasing speed, decelerating
	Zero gradient	Increasing speed, accelerating

SOLUTION	EXPLANATION
Graph A	
Increasing distance from home. Positive constant gradient. Lower constant speed.	Distance from home increases at a constant rate (speed), so an upward sloping straight line has a positive constant gradient. Lines that are less steep mean a lower constant speed.
Graph B	
Decreasing distance from home. Constant negative gradient. Higher constant speed.	Distance from home decreases at a constant rate (speed), so a downward sloping straight line has a constant negative gradient. Steeper lines mean a higher constant speed.
Graph C	
Increasing distance from home. Positive variable gradient. Increasing speed, accelerating.	Distance from home is increasing at an increasing rate, so the curve has a positive varying gradient. As the curve becomes steeper the rate of change (speed) increases (i.e. more distance in a given time). Increasing speed indicates accelerating.
Graph D	
Increasing distance from home. Positive varying gradient. Decreasing speed, decelerating.	Distance from home is increasing at a decreasing rate, so the curve has a positive varying gradient. As the curve becomes flatter the rate of change (speed) decreases (i.e. less distance in a given time). Decreasing speed indicates decelerating.
Graph E	
Decreasing distance from home. Negative varying gradient. Decreasing speed, decelerating.	Distance from home is decreasing at a decreasing rate, so the curve has a negative varying gradient. As the curve becomes flatter the rate of change (speed) decreases (i.e. less distance in a given time). Decreasing speed indicates decelerating.
Graph F	
Decreasing distance from home. Negative varying gradient. Increasing speed, accelerating.	Distance from home is decreasing at an increasing rate, so the curve has a negative varying gradient. As the curve becomes steeper the rate of change (speed) increases (i.e. more distance in a given time). Increasing speed indicates accelerating.

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Example 33 Finding and using a direct variation rule

If m is directly proportional to l and $m = 90$ when $l = 20$, determine:

a the relationship between m and l **b** m when $l = 16$

c l when $m = 10$

SOLUTION	EXPLANATION
a $m \propto l$	First, write the variation statement.
$m = kl$	Write the equation, including k.
$90 = k \times 20$	Substitute $m = 90$ and $l = 20$.
$k = \frac{90}{20}$	Divide both sides by 20 to find k.
$k = \frac{9}{2}$	Simplify.
$\therefore m = \frac{9}{2}l$	Write the rule with the found value of k.
b $m = \frac{9}{2} \times 16$	Substitute $l = 16$.
$\therefore m = 72$	Simplify to find m.
c $10 = \frac{9}{2} \times l$	Substitute $m = 10$.
$10 \times \frac{2}{9} = l$	Solve for l.
$\therefore l = \frac{20}{9}$	

Example 34 Finding and using an inverse variation rule

If x and y are inversely proportional and $y = 6$ when $x = 10$, determine:

a the constant of proportionality, k, and write the rule **b** y when $x = 15$

c x when $y = 12$

SOLUTION	EXPLANATION
a $y \propto \frac{1}{x}$	Write the variation statement.
$y = \frac{k}{x}$	Write the equation, including k.
$k = xy$	The constant of proportionality, $k = xy$:
$k = 60$	$k = 10 \times 6 = 60$
$y = \frac{60}{x}$	Substitute $k = 60$ into the rule $y = \frac{k}{x}$.
b $y = \frac{60}{15}$	Substitute $x = 15$ into the rule $y = \frac{60}{x}$.
$y = 4$	
c $12 = \frac{60}{x}$	Substitute $y = 12$ into the rule $y = \frac{60}{x}$.
$12x = 60$	Multiply both sides by x. Divide both sides by 12.
$x = 5$	

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Exercise 9N

UNDERSTANDING AND FLUENCY	1–6	3–7	4–8

1 For each pair of variables, state whether they are in direct or inverse proportion or neither.

a The number of *hours* worked and *wages* earned at a fixed rate per hour.

b The *volume* of remaining fuel in a car and the *cost* of filling the fuel tank.

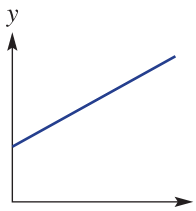
c The *speed* and *time* taken to drive a certain distance.

d The *size* of a movie file and the *time* for downloading it to a computer at a constant rate of kB/s.

e The *cost* of a taxi ride and the *distance* travelled. The cost includes flag fall (i.e. a starting charge) and a fixed $/km.

f The *rate* of typing in words per minute and the *time* needed to type a particular assignment.

2 State the main features of each graph and whether it shows direct proportion or inverse (i.e. indirect) proportion or neither.

a


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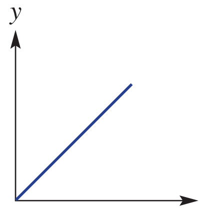

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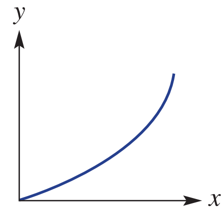

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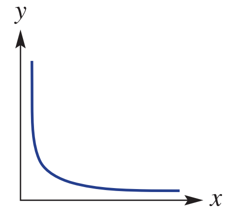
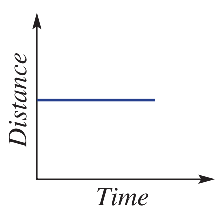


Example 32

3 The distance–time graphs below show various journeys, each with distance measured from 'home' (i.e. distance = 0). For each graph, select and copy the correct descriptions of how the distance from home, the gradient and the speed are varying.

a


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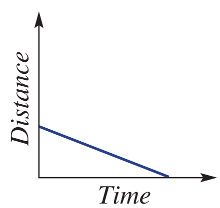

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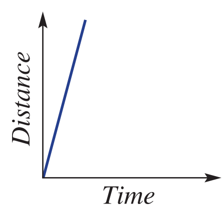

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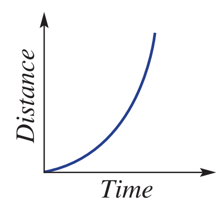

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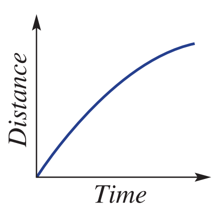

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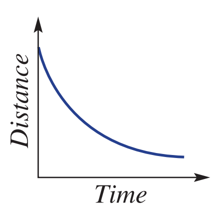
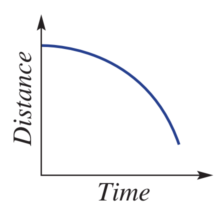


g


Distance from home	**Gradient of graph**	**Speed**
Increasing distance from home	Positive constant gradient	Stationary
Decreasing distance from home	Positive variable gradient	Lower constant speed
Fixed distance from home	Negative constant gradient	Higher constant speed
	Negative variable gradient	Decreasing speed, decelerating
	Zero gradient	Increasing speed, accelerating

Example 33

4 **a** If $p = k \times q$ and $p = 40$ when $q = 10$, determine:

i the relationship between p and q **ii** p when $q = 15$ **iii** q when $p = 100$

b If $p = k \times q$ and $p = 100$ when $q = 2$, determine:

i the relationship between p and q **ii** p when $q = 15$ **iii** q when $p = 200$

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Example 34

5 **a** If $y = \frac{k}{x}$ and $y = 12$ when $x = 6$, determine:

i the constant of proportionality, k, and write the rule

ii y when $x = 36$

iii x when $y = 3$

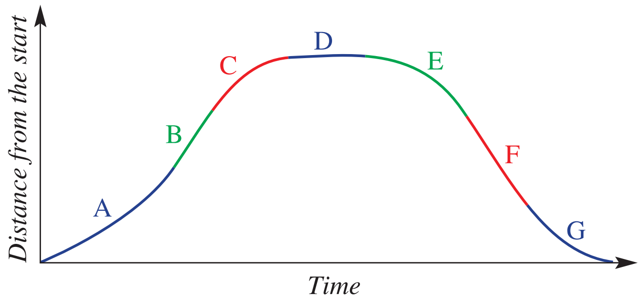
b If y varies inversely as x and $y = 10$ when $x = 5$, determine:

i the constant of proportionality, k, and write the rule

ii y when $x = 100$

iii x when $y = 100$

6 From the lists below, select and copy the correct description for the rate and speed of each segment of this distance–time graph. The rate of change of distance with respect to time is the gradient.

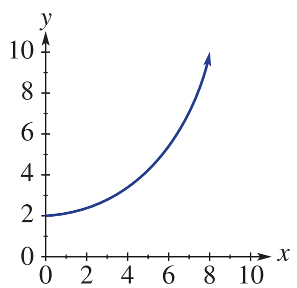


Rate of change of distance with respect to time	**Speed**
Positive constant rate of change	Stationary
Positive varying rate of change	Constant speed
Negative constant rate of change	Decreasing speed, decelerating
Negative varying rate of change	Increasing speed, accelerating
Zero rate of change	

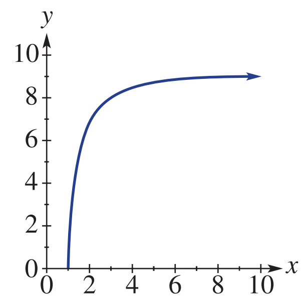
7 For each graph below, copy and complete this sentence by supplying the words *increasing* or *decreasing*.

y is ____________ at a ____________ rate.

a



b



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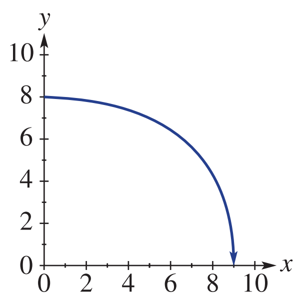

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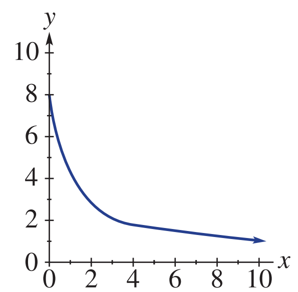

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8 The amount that a farmer earns from selling wheat is in direct proportion to the number of tonnes harvested.

a Find the constant of proportionality, k, given that a farmer receives \$8296 for 34 tonnes of wheat.
b Write the direct proportion equation relating selling price, P, and number of tonnes, n.
c Calculate the selling price of 136 tonnes of wheat.
d Calculate the number of tonnes of harvested wheat that is sold for \$286 700.

PROBLEM-SOLVING AND REASONING	9, 10, 13	9–11, 13, 14	10–12, 13–15

9 A 30-seater school bus costs 20 students \$3.70 each to hire for a day. The overall cost of the bus remains the same regardless of the number of students.

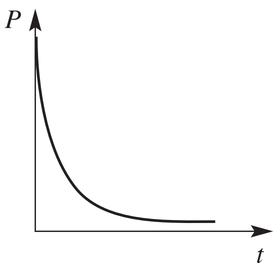
a Write a relationship between the cost per student (c) and the number of students (s).
b If only 15 students use the bus, what would be their individual cost, to the nearest cent?
c What is the minimum a student would be charged, to the nearest cent?

10 For each relationship described below:

i Write a suitable equation.
ii Sketch the graph, choosing sensible values for the initial and final points on the graph.

a The distance that a car travels in 1 hour is directly proportional to the speed of the car. The roads have a 100 km/h speed limit.
b The cost per person of hiring a yacht is inversely proportional to the number of people sharing the total cost. A yacht in the Whitsunday Islands can be hired for \$320 per day for a maximum of eight people on board.
c There is a direct proportional relationship between a measurement given in metric units and in imperial units. A weight measured in pounds is 2.2 times the value of the weight in kilograms.
d The time taken to type 800 words is inversely proportional to the typing speed in words per minute.

11 Each of the following graphs shows how a population, P, is changing over time, t. For each graph, state whether the population is increasing or decreasing and if it is changing at an increasing, decreasing or constant rate.

a


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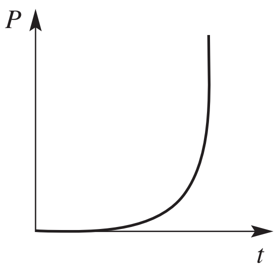
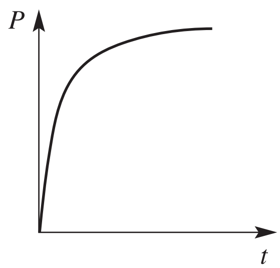


c


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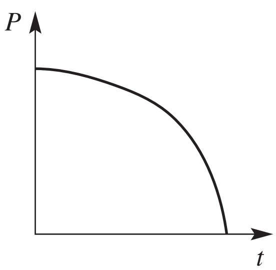
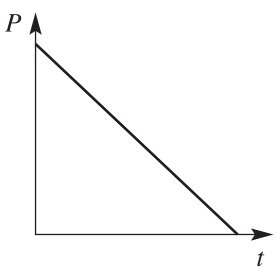


e


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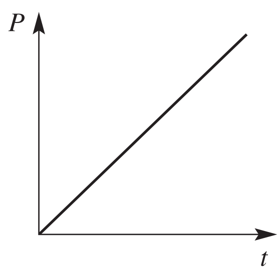

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12 Sketch a population–time graph from each of these descriptions.

a A population of bilbies is decreasing at a decreasing rate.

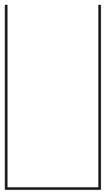
b A population of Tasmanian devils is decreasing at an increasing rate.

c A population of camels is increasing at a decreasing rate.

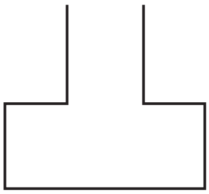
d A population of rabbits is increasing at an increasing rate.

13 Water is poured at a constant rate into each of the containers shown below. For each container, draw a graph of the water depth versus time. Numbers are not required.

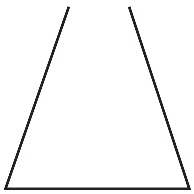
a



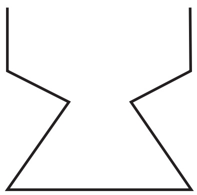
b



c



d



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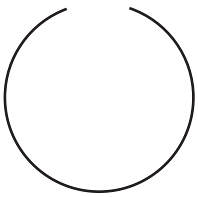

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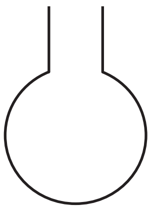
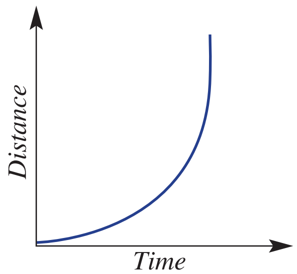
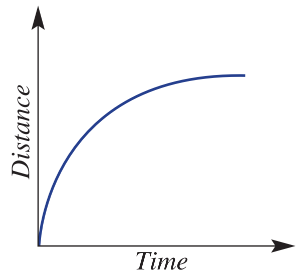


14 Which of the following graphs doesn't match the journey description correctly or is not physically possible? For each graph, explain the feature that is incorrect and redraw it correctly.

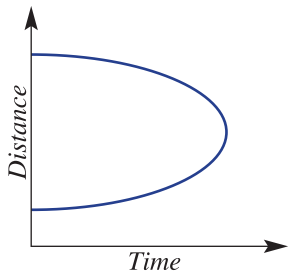
a Stopped, then accelerating to a very high speed.



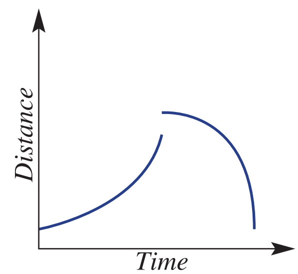
b Moving at very high speed and decelerating to a stop.



c Accelerating, changing direction, then decelerating.



d Accelerating, changing direction, then decelerating.



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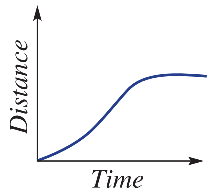
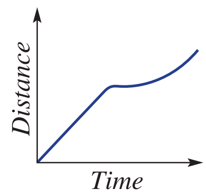
15 Match each distance–time graph (**A**–**D**) with the correct journey described (**a**–**d**). Give reasons for each choice, describing how the changing distance and varying rate relates to the movement of the object.

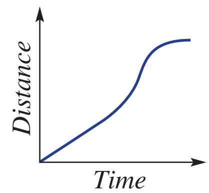
A



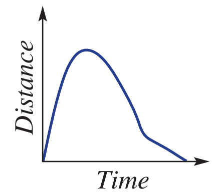
B



C



D



- **a** A soccer player runs at a steady speed across the field, stops briefly to avoid a tackle and then accelerates farther away.
- **b** A rocket stage 1 booster accelerates to huge speed, then detaches and quickly decelerates, then accelerates as it falls towards Earth. Finally, a parachute opens and it slows, falling to Earth at a steady speed.
- **c** A motorbike moves at a steady speed, then accelerates to pass a car, then brakes and decelerates, coming to a stop at traffic lights.
- **d** A school bus accelerates away from the bus stop, then moves at a steady speed and then decelerates and stops as it arrives at the next bus stop.

ENRICHMENT	–	–	16

Creating distance–time graphs

16 Work in small groups to develop distance–time graphs from recorded data.
Equipment: 100 m tape measure, stopwatch, recording materials, video camera.

- **a** Determine a suitable method for recording the distance a student has moved after every 5 seconds over a 30 second period.
- **b** Select a variety of activities for the moving student to do in each 30 second period. For example, walking slowly, running fast, starting slowly then speeding up, etc.
- **c** Record and graph distance versus time for each 30 second period.
- **d** For each graph, using sentences with appropriate vocabulary, describe how the distance and rate of change of distance is varying.
- **e** Analyse how accurately each graph has modelled that student's movement.

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90 Literal equations and restrictions on variables

Interactive
Widgets
HOTsheets
Walkthrough

Many mathematical equations consist primarily of variables (or pronumerals). These are called literal equations and include formulas. Mathematicians, engineers, physicists, economists and programmers, for example, use literal equations to help describe a situation or solve for an unknown. An example is a formula linking distance (s), time (t), acceleration (a) and initial velocity (u), which is:

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

$$s = ut + \frac{1}{2}at^2$$

When the formula is rearranged to make a the subject, then it becomes:

$$a = \frac{2(s - ut)}{t^2}$$

When using or rearranging literal equations, we should consider what restrictions are placed on particular variables, which will be studied in this section.

Let's start: Restrictions with square roots

Consider the rule $a = 3c^2$.

- Find the value of a when $c = -2$ and $+2$.
- Are there any restrictions on the values of c?
- What are the permissible values of a?
- Can you rearrange the formula to make c the subject? How might you use the $\pm$ symbol?
- Are there any values of a that are not permissible?
- A similar formula linking the area of a circle with its radius is $A = \pi r^2$. Rearrange this formula to make r the subject. Discuss the permissible values of r and A.

Key ideas

- A **literal equation** is an equation consisting primarily of variables (or pronumerals). These equations include formulas.
- The **subject** of a rule or formula sits by itself, usually on the LHS of the equation.
- For example: a is the subject of $a = \sqrt{c^2 - b^2}$.
- Literal equations can be rearranged (or solved or transposed) to make a letter the subject of the equation. This is sometimes called 'solving a literal equation'.

  For example: $P = 2l + 2b$ can be rearranged (or transposed or solved) to give $l = \dfrac{P - 2b}{2}$.
- Some literal equations will have **restrictions** on some or all of their variables. The values allowed for the variables are called the **permissible values**.
  - Further restrictions sometimes need to be placed on variables after equations are rearranged.

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Example 35 Rearranging literal equations

Rearrange to make the letter in the brackets the subject.

a $y = mx + b \quad (x)$

b $y^2 = a^2 + b^2 \quad (a)$

SOLUTION

a
$$y = mx + b$$
$$y - b = mx$$
$$\frac{y-b}{m} = x$$
So $x = \dfrac{y-b}{m}$.

b
$$y^2 = a^2 + b^2$$
$$y^2 - b^2 = a^2$$
$$\pm\sqrt{y^2 - b^2} = a$$
So $a = \pm\sqrt{y^2 - b^2}$.

EXPLANATION

Like solving other linear equations, solve for x by subtracting b from both sides and dividing both sides by m.

First, subtract b^2 from both sides. Then take the square root with $\pm$ since there is no given restriction on the values of a.

Note: $\sqrt{y^2 - b^2} \neq y - b$

Example 36 Restricting variables

Consider the equation $y = ab^2$.

a Are there any restrictions on a, b or y?

b Rearrange to make b the subject and assume $b \geqslant 0$.

c If $y > 0$, are there any further restrictions on a after rearranging to make b the subject? If yes, what are they?

SOLUTION

a no

b
$$y = ab^2$$
$$\frac{y}{a} = b^2$$
$$\pm\sqrt{\frac{y}{a}} = b$$
$$\therefore b = +\sqrt{\frac{y}{a}} \text{ since } b \geqslant 0.$$

c Yes, $a > 0$.

EXPLANATION

y, a and b can take any value.

Divide both sides by a then square root with $\pm$.
Since $b \geqslant 0$, take the positive square root.

$a \neq 0$ because you can't divide by zero and $a > 0$ because $\dfrac{y}{a} \geqslant 0$ inside the square root.

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Exercise 90

UNDERSTANDING AND FLUENCY	1–3, 4(½), 5–7	3, 4(½), 5–8	4(½), 5, 6, 7(½), 8

1 The formula for the area of a triangle is $A = \frac{1}{2}bh$.

a Realistically, what are the permissible values of A, b and h?
b Rearrange the equation to make b the subject.
c Rearrange the equation to make h the subject.
d Using the formula $b = \frac{2A}{h}$, find the value of b when:
  i $A = 10$ and $h = 4$ **ii** $A = 5$ and $h = 2$
e Is it possible to substitute $h = 0$ into $b = \frac{2A}{h}$? Why?

2 Use the equation $y = x^2$ to answer the following.

a Find the value of y for these values of x.
  i 0 **ii** 4 **iii** 1
  iv -2 **v** -9
b What are the permissible values of x?
c What are the permissible values of y?
d Rearrange to make x the subject.
e For your rearranged formula, determine whether the permissible values of x and y have changed.

3 The rule for a particular hyperbola is $y = \frac{1}{x}$.

a Try to find the value of y when x equals:
  i -3 **ii** 2 **iii** 0
b Try to find the value of x when y equals:
  i $\frac{1}{4}$ **ii** 4 **iii** 0
c What restrictions are there on the variables x and y?

Example 35

4 Rearrange to make the letter in the brackets the subject.

a $P = 2a + b$ (a) **b** $P = 2l + 2b$ (b) **c** $M = \frac{a+b}{2}$ (a)
d $D = b^2 - 4ac$ (c) **e** $d = st$ (t) **f** $A = \pi r^2$ if $r > 0$ (r)
g $s = ut + \frac{1}{2}at^2$ (a) **h** $c^2 = a^2 + b^2$ (b) **i** $A = \frac{1}{2}(a + b)h$ (a)
j $y = \sqrt{9 - x^2}$ if $x > 0$ (x) **k** $Q = \sqrt{2gh}$ (h) **l** $V = \pi r^2 h$ (h)

5 If there is no restriction on the variable a, make a the subject of these equations.

a $b = a^2$
b $a^2 + b^2 = c^2$
c $(a - 1)^2 = b$

Example 36

6 Consider the equation $a = bc^2$.

a Are there any restrictions on a, b or c in this equation?
b Rearrange to make b the subject.
c Are there any restrictions on a, b or c in your equation from part **b**?
d Rearrange to make c the subject.
e Are there any restrictions on b in your equation from part **d**?

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7 Write down the restrictions on x and y in these equations; i.e. state the permissible values of x and y.

a $y = \dfrac{1}{x}$ **b** $y = \dfrac{1}{x} + 2$ **c** $y = \dfrac{1}{x} - 3$

d $y = \dfrac{1}{x - 4}$ **e** $y = \dfrac{1}{x + 2}$ **f** $y = \dfrac{2}{x - 3} + 2$

8 State any new restrictions to be placed on variables after rearranging them to make the letter in the brackets the subject.

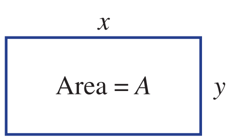
a $P = VT \quad (T)$

b $s = kt^2 \quad (k)$

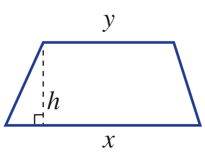
PROBLEM-SOLVING AND REASONING	9, 12	9, 10, 12, 13	9–14

9 Find a formula to link the variables in these shapes and make x the subject.

a

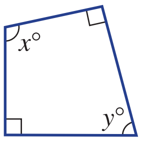


b

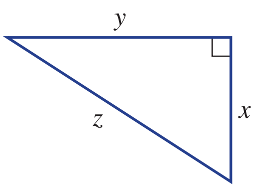


Area = A

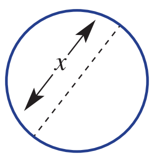
c



d

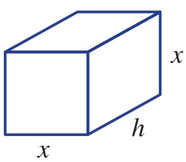


e



Area = A

f



Volume = V

10 The formula used to convert degrees Fahrenheit to degrees Celsius is $C = \dfrac{5}{9}(F - 32)$.

a Find the value of C when:

i $F = 41$

ii $F = 14$

b What temperature in degrees Fahrenheit is equivalent to $0°C$?

c Rearrange to make F the subject.

d What temperature in $°F$ gives $100°C$?

e Are there any restrictions on C or F? Explain.

11 Consider $x^2 + y^2 = 4$, an equation of a circle.

a Rearrange to make x the subject.

b Rearrange to make y the subject.

c State any restrictions on x and y.

d Write down any restrictions for the following circle equations.

i $x^2 + y^2 = 9$

ii $x^2 + y^2 = 25$

e Describe the graph of $y = -\sqrt{25 - x^2}$.

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12 This question looks at the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$.

a Do you think that $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$ is equivalent to $x + y = z$? Check by substituting $x = 2$, $y = 3$ and $z = 5$ into both equations. What do you notice?

b Make z the subject in the equation given.

c Make x the subject in the equation given.

13 To answer the following questions, use the fact that if $x^3 = k$, then $x = \sqrt[3]{k}$.

a Solve for x when:

i $x^3 = 8$ **ii** $x^3 = -27$ **iii** $x^3 = y$

b Are there any restrictions on x and y in the following?

i $x^3 = y$ **ii** $x = \sqrt[3]{y}$

c Find the rule for the radius r of a sphere if $V = \frac{4}{3}\pi r^3$.

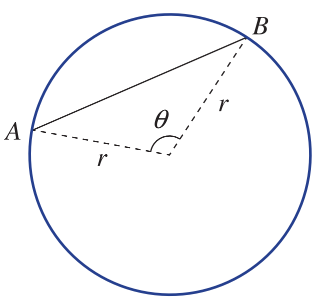
14 Is there any difference in the restrictions for $y\sqrt{x} = k$ and $y = \frac{k}{\sqrt{x}}$? Explain.

ENRICHMENT	–	–	15

Chord length

15 In circle mensuration the length of a chord AB is given by the formula

$AB = \sqrt{2r^2(1 - \cos\theta)}$.



a Find the chord length AB, correct to 1 decimal place, when:

i $r = 2$ and $\theta = 80°$

ii $r = 1.7$ and $\pi = 135°$

b Use the formula to prove that:

i $AB = 0$ when $\theta = 0°$

ii $AB = 2r$ when $\theta = 180°$

c Rearrange to make r the subject.

d Find the radius of a circle when $AB = 9$ and $\theta = 60°$. Round your answer to 1 decimal place.

e State any restrictions on AB, r or θ in the formula from part **c**.

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9P Inverse functions

Interactive Widgets HOTsheets Walkthrough

Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Inverse operations are often used in mathematics. When we solve equations we employ inverse operations to isolate the pronumeral in the equation to find its solution.

Operation		Inverse operation
add x	→	subtract x
multiply by x	→	divide by x
square x	→	square root of x
cube of x	→	cube root of x

When dealing with functions it is often the output (i.e. the y values) that is known and the input values (i.e. the x values) that need to be found. This is when the process of finding the inverse function is helpful. Inverse functions exist when a function has a one-to-one correspondence between its x and y values. A function that passes the horizontal line test has an inverse function.

Let's start: Interchanging x and y

1 Consider the table of values.

x	–1	0	1	2
y	3	4	5	6

a Write down the rule for the table of values shown.
b Complete the table below by interchanging the x and y values of the table above.

x	3			
y	–1			

c Write down the rule for this second table of values.
d Graph the rules for both tables of values on the same number plane, as well as the line $y = x$.
e What is the relationship between the two functions and the line $y = x$?

2 Now consider this table of values.

x	–1	0	1	2
y	–2	0	2	4

a Write down the rule for the table of values shown.
b Complete the table below by interchanging the x and y values of the table above.

x				
y				

c Write down the rule for this second table of values.
d Graph the rules for both tables of values on the same number plane, as well as the line $y = x$.
e What is the relationship between the two functions and the line $y = x$?

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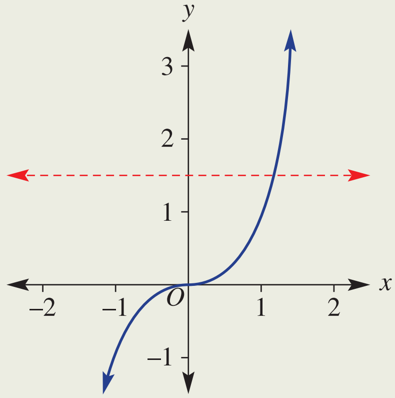
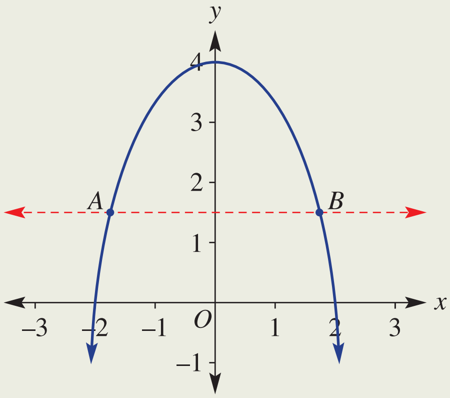
Key ideas

- Graphically, a function has an inverse function when it passes the horizontal line test (i.e. a horizontal line cuts the graph only once for an inverse to exist).



This function has an inverse function because there are no horizontal lines that cut the curve more than once.



This function does not have an inverse function because some horizontal lines cut more than once.

- An inverse function uses the notation $f^{-1}(x)$. (Note: This is not to be confused with a negative index, which creates a reciprocal.)
- $y = f(x)$ and $y = f^{-1}(x)$ are reflections in the line $y = x$.
- To find the equation for the inverse function algebraically, follow these steps:
  - Interchange the x and y in the given function.
  - Make y the subject of this new equation.
  - Now write this equation using the inverse function notation $f^{-1}(x)$.
  - Note:
    - The domain (i.e. permissible x values) of the function $y = f(x)$ becomes the range of the inverse function $f^{-1}(x)$.
    - The range (i.e. permissible y values) of the function $y = f(x)$ becomes the domain of the inverse function $y = f^{-1}(x)$.
- Restricting the domain of a quadratic function allows an inverse function to be created.

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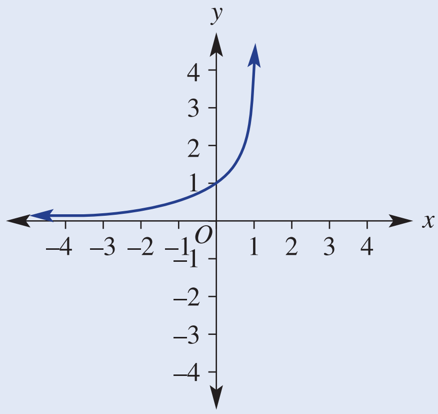
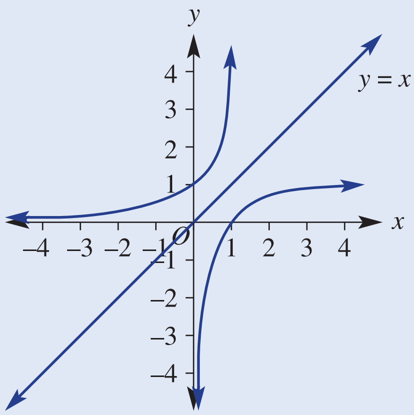
Example 37 Graphing an inverse function as a reflection in the line $y = x$

Reflect the following function in the line $y = x$ to show its inverse.



SOLUTION



EXPLANATION

The reflection in the line $y = x$ changes the point (0, 1) to (1, 0) on the inverse, and instead of approaching the x-axis the inverse approaches the y-axis.

The roles of x and y have interchanged.

Example 38 Finding the inverse function algebraically

Find the inverse function for $y = 3x + 4$.

SOLUTION

Creating inverse of $y = 3x + 4$ gives:

$$\begin{aligned} x &= 3y + 4 \\ x - 4 &= 3y \\ 3y &= x - 4 \\ y &= \frac{x-4}{3} \\ \therefore f^{-1}(x) &= \frac{x-4}{3} \end{aligned}$$

EXPLANATION

Interchange the x and y in the given function.
Make y the subject of this new equation.
Now write this equation using the inverse function notation $f^{-1}(x)$.

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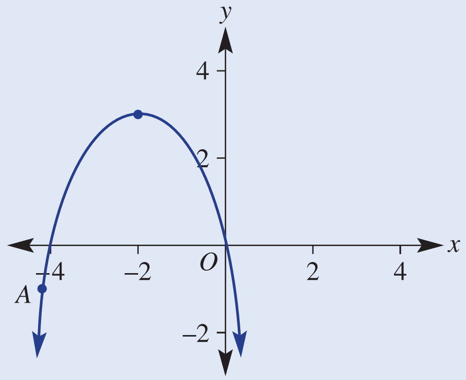
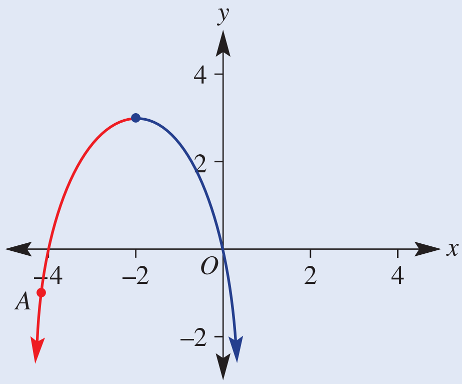
Example 39 Restricting the domain to create an inverse function

For this quadratic function, write down the largest set of x values (i.e. the domain) that include the point A, for which an inverse exists.



SOLUTION

$x \leqslant -2$



EXPLANATION

The red section of curve *will* have an inverse function and it contains the point A.

Exercise 9P

UNDERSTANDING AND FLUENCY	1–5	3, 4, 5–6(½)	4, 5–6(½)

1 State whether each of the following is true or false.

a $y = 3x$ and $y = \dfrac{3}{x}$ are inverse functions.

b $y = 2x + 1$ and $y = \dfrac{x - 1}{2}$ are inverse functions.

c $y = x$ is its own inverse.

d Reflecting any function in the line $y = x$ will result in an inverse function.

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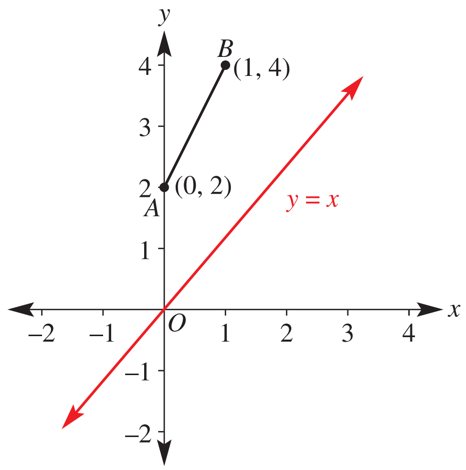
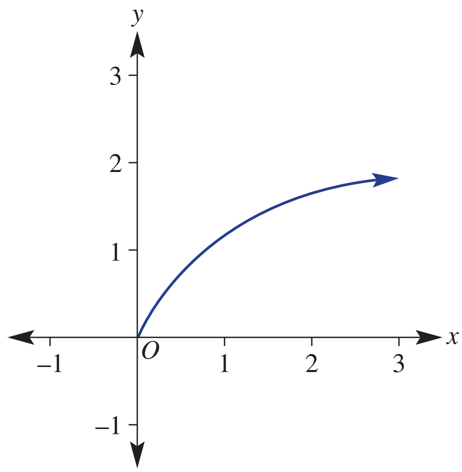
2 Given that the interval AB has been reflected in the line $y = x$ to produce the interval CD, write down the coordinates of C and D.

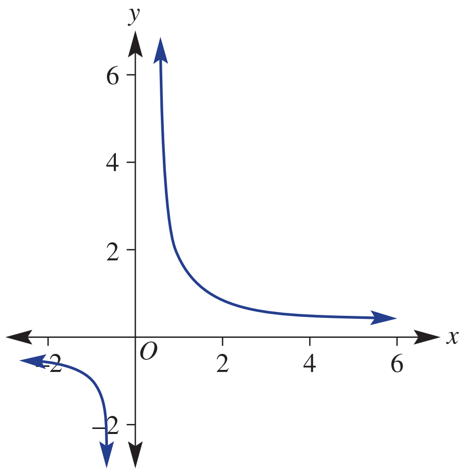


3 Explain why the following graphs have an inverse function.

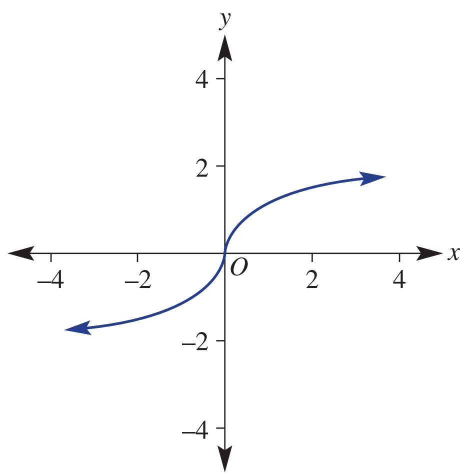
a



b



c



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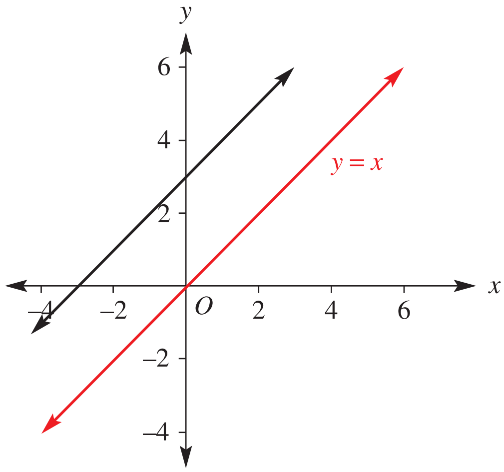
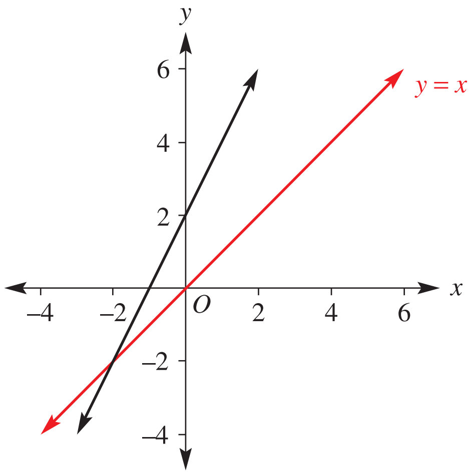
Example 37

4 Reflect each of the following functions in the line $y = x$.

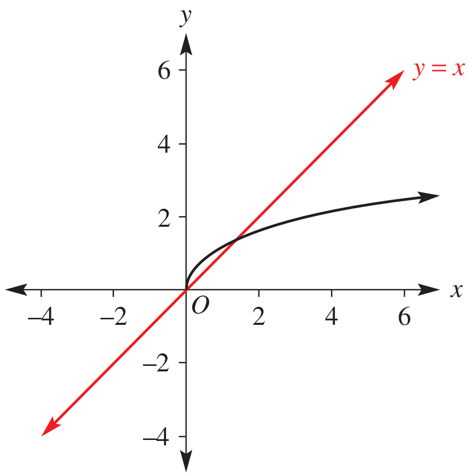
a



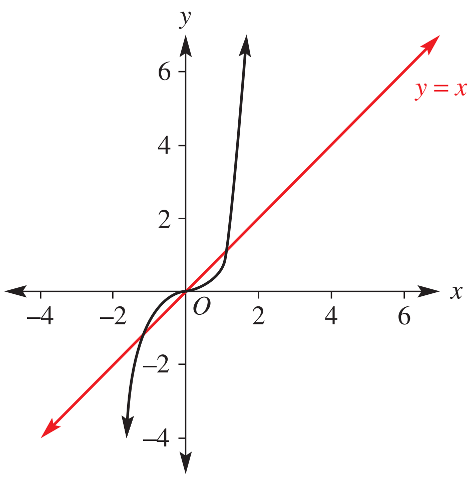
b



c



d



Example 37

5 Sketch each of the following functions and their inverse functions on the same set of axes. Make sure you include the line $y = x$ on each diagram and state the equation of $y = f^{-1}(x)$ for each.

a $y = x - 4$ **b** $y = 6x$ **c** $y = 2x + 4$ **d** $y = \frac{x}{4}$

e $y = 2 - x$ **f** $y = 6 - 3x$ **g** $y = \frac{x+1}{2}$ **h** $y = \frac{1}{x}$

Example 38

6 Find the inverse function for each function below. Sketch the graph of each function and its inverse on the same set of axes and include the line $y = x$.

a $y = x^3$ **b** $y = 2x^3$ **c** $y = -x^3$ **d** $y = \frac{1}{x+2}$

e $y = x^3 - 1$ **f** $y = \frac{1}{x+2}$ **g** $y = \sqrt{x}$ **h** $y = \frac{1}{x} + 1$

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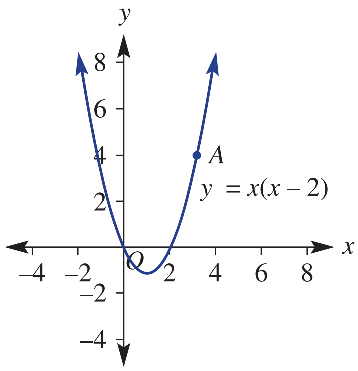
PROBLEM-SOLVING AND REASONING | 7, 9 | 7, 9, 10 | 7–12

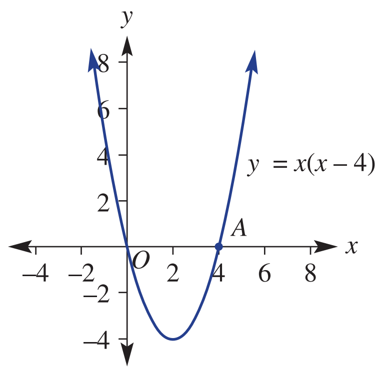
Example 39

7 For each of the quadratic equations below, write down the largest possible set of x values (i.e. the domain) that include the point A, for which an inverse exists.

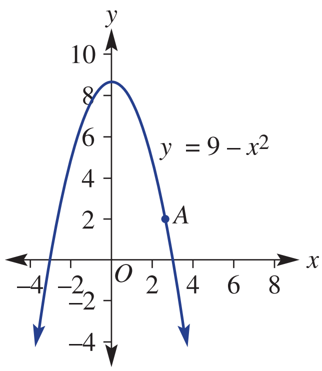
a



b

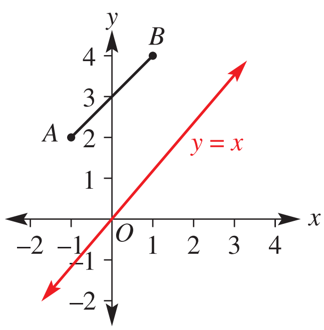


c

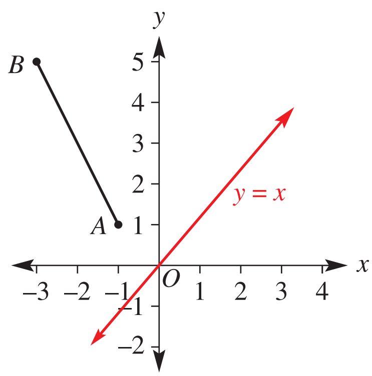


8 Sketch the inverse of each of the following and state the domain and range of the inverse.

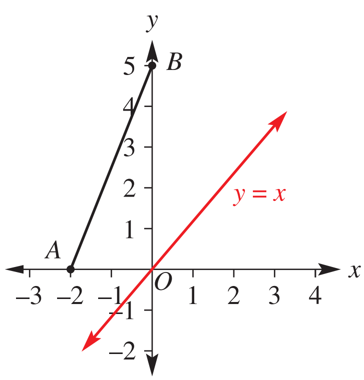
a



b



c



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9 **a** Sketch the graph of $y = x^2$.
b Explain why $y = x^2$ does not have an inverse.
c Show that $y = x^2$, with the restricted x values of $0 \leqslant x \leqslant 3$, has an inverse.
d Find this inverse and state its domain (i.e. permissible x values) and range (i.e. permissible y values).

10 Consider the quadratic function $y = x^2 - 6x$.
a What are the coordinates of the x-intercepts?
b Write down the coordinates of the vertex.
c Explain why $y = x^2 - 6x$ does not have an inverse unless the x values are restricted.
d Write down the largest possible domain that includes an x-intercept, for which an inverse exists.
e Explain why there can be more than one possible domain given in part **d**.

11 Explain why the coordinates of the vertex are important when deciding on the domain for which a quadratic function will have an inverse.

12 Write down the largest possible domain that includes the value of $x = 0$, for which each of the quadratic functions below will have an inverse.
a $y = x^2 + 2x + 1$
b $y = x^2 + 2x$
c $y = x^2 + x - 12$
d $y = x^2 - x - 6$
e $y = x^2 - x - 1$

ENRICHMENT	–	–	13

Finding the inverse function of a quadratic

13 For each of the following, find the equation of $y = f^{-1}(x)$. You may need to complete the square.
a $y = x^2 + 6$ for $x \geqslant 0$
b $y = x^2 - 2x$ for $x \geqslant 1$
c $y = x^2 + 4x$ for $x \geqslant -2$
d $y = x^2 - x - 2$ for $x \geqslant \dfrac{1}{2}$
e $y = x^2 + 2x - 15$ for $x \geqslant -1$

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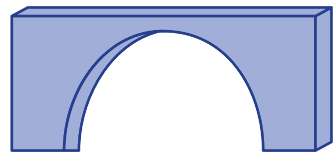
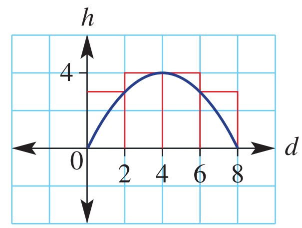
1 Painting bridges



A bridge is 6 m tall and has an overall width of 12 m and an archway underneath, as shown.

We need to determine the actual surface area of the face of the arch so that we can order paint to refurbish it. The calculation for this area would be:

$$\begin{aligned}\text{Bridge area} &= 12 \times 6 - \text{area under arch}\\ &= 72 - \text{area under parabola}\end{aligned}$$



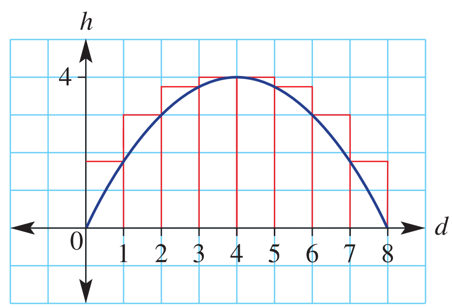
Consider an archway modelled by the formula $h = -\frac{1}{4}(d - 4)^2 + 4$, where h metres is the height of the arch and d metres is the distance from the left. To estimate the area under the arch, divide the area into rectangular regions. If we draw rectangles above the arch and calculate their areas, we will have an estimate of the area under the arch even though it is slightly too large. Use the rule for h to obtain the height of each rectangle.

$$\begin{aligned}\text{Area} &= (2 \times 3) + (2 \times 4) + (2 \times 4) + (2 \times 3)\\ &= 6 + 8 + 8 + 6 = 28 \text{ m}^2\end{aligned}$$

$\therefore$ Area is approximately 28 m^2.

We could obtain a more accurate answer by increasing the number of rectangles; i.e. by reducing the width of each rectangle (called the strip width).

Overestimating the area under the arch

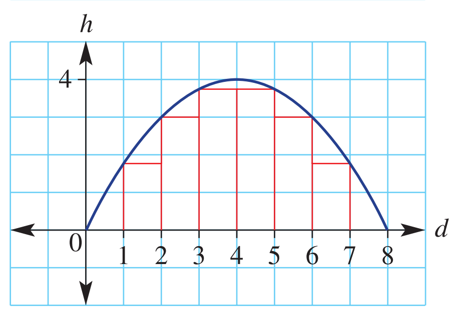


a Construct an accurate graph of the parabola and calculate the area under it using a strip width of 1.

b Repeat your calculations using a strip width of 0.5.

c Calculate the surface area of the face of the arch using your answer from part **b**.

Underestimating the area under the arch



a Estimate the area under the arch by drawing rectangles under the graph with a strip width of 1.

b Repeat the process for a strip width of 0.5.

c Calculate the surface area of the face of the arch using your answer from part **b**.

Improving accuracy

a Suggest how the results from parts 1 and 2 could be combined to achieve a more accurate result.

b Explore how a graphics or CAS calculator or a spreadsheet can give accurate results for finding areas under curves.

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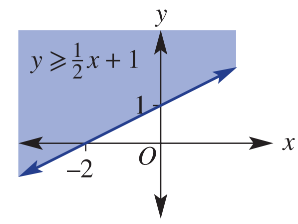
2 Sketching regions in the plane

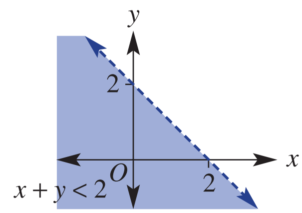
A region in a plane is an infinite set of points defined on one side of a line or other curve. We describe the region with an inequality like $y < 2x + 1$, $x + 2y \geqslant 3$ or $x^2 + y^2 < 1$. All points that satisfy the inequality form the region.

Half planes

A half plane is the region on one side of a line. Here are two examples.



For $y \geqslant \frac{1}{2}x + 1$ we sketch $y \geqslant \frac{1}{2}x + 1$. Draw the full line (since $y \geqslant \frac{1}{2}x + 1$ is included in the region) and then check the point (0, 0). As (0, 0) does not satisfy $y \geqslant \frac{1}{2}x + 1$, the region must lie on the other side of the line.



a For the given graph of $x + y < 2$, explain why the line is dashed.

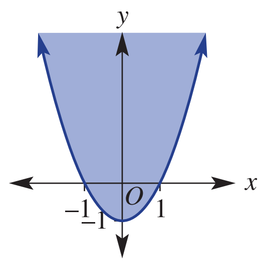
b Show that (0, 0) is inside the region.

c Sketch these half planes.
- **i** $y < 2x + 2$
- **ii** $y \geqslant -x - 3$
- **iii** $3x - y \geqslant -2$
- **iv** $x + 2y < 1$

d Sketch the half planes that satisfy these inequalities.
- **i** $x \leqslant 3$
- **ii** $x < -1$
- **iii** $y > -2$
- **iv** $y \leqslant 6$

e Sketch the common intersecting region for each pair of inequalities.
- **i** $y < x + 2$ and $y \geqslant -x + 1$
- **ii** $x + 2y \geqslant 4$ and $3x - y < -2$

Parabolic regions

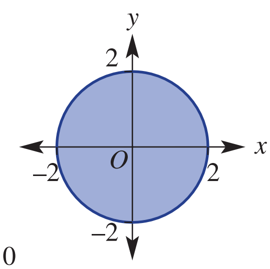


The region in this graph satisfies $y \geqslant x^2 - 1$.

a Sketch these parabolic regions.
- **i** $y \geqslant x^2 - 4$
- **ii** $y < x^2 + 2$
- **iii** $y \geqslant (x + 1)(x - 2)$
- **iv** $y > 2(x + 3)(x + 1)$
- **v** $y < x^2 - x - 6$
- **vi** $y \leqslant x^2 + 3x - 40$

b Sketch the intersecting region defined by $y \geqslant x^2 - 4$ and $y < x + 2$. Find all significant points.

Circular regions



The region shown satisfies $x^2 + y^2 \leqslant 4$.

Sketch these regions.
- **a** $x^2 + y^2 \leqslant 9$
- **b** $x^2 + y^2 < 16$
- **c** $x^2 + y^2 > 4$
- **d** $(x - 1)^2 + y^2 \geqslant 9$
- **e** $(x + 2)^2 + (y - 1)^2 < 3$
- **f** $(x - 3)^2 + (y - 4)^2 \geqslant 10$

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Puzzles and challenges

1 Solve these inequalities for x.

a $6x^2 + x - 2 \leqslant 0$

b $12x^2 + 5x - 3 > 0$

c $x^2 - 7x + 2 < 0$

2 Sketch the region defined by $x^2 - 4x + y^2 - 6y - 3 \leqslant 0$.

3 Prove the following.

a The graphs of $y = x - 4$ and $y = x^2 - x - 2$ do not intersect.

b The graphs of $y = -3x + 2$ and $y = 4x^2 - 7x + 3$ touch at one point.

c The graphs of $y = 3x + 3$ and $y = x^2 - 2x + 4$ intersect at two points.

4 Find the coordinates of the intersection of the graph of $y = x + 1$ and the following curves.

a $y = -x^2 - 3x + 2$

b $x^2 + y^2 = 9$

c $y = \dfrac{2}{x}$

d $y = \dfrac{1}{x + 3} - 2$

5 Prove that there are no points (x, y) that satisfy $x^2 - 4x + y^2 + 6y + 15 = 0$.

6 For what values of k does the graph of $y = kx^2 - 2x + 3$ have:

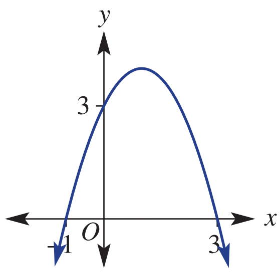
a one x-intercept? **b** two x-intercepts? **c** no x-intercepts?

7 For what values of k does the graph of $y = 5x^2 + kx + 1$ have:

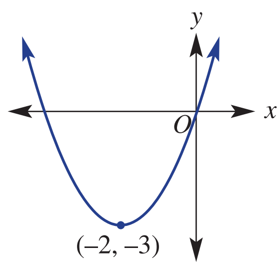
a one x-intercept? **b** two x-intercepts? **c** no x-intercepts?

8 Find the rules for these parabolas.

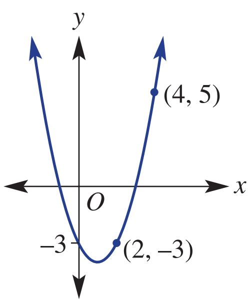
a



b



c



9 A graph of $y = ax^2 + bx + c$ passes through the points $A(0, -8)$, $B(-1, -3)$ and $C(1, -9)$. Use your knowledge of simultaneous equations to find the values of a, b and c and, hence, find the turning point for this parabola, stating the answer using fractions.

10 Determine the maximum vertical distance between these two parabolas at any given x value: $y = x^2 + 3x - 2$ and $y = -x^2 - 5x + 10$

11 Search the internet for information about the 'Towers of Hanoi'.

a Copy and complete:

Number of discs (D)	1	2	3	4	5	6
Number of moves (M)						

b The equation is $M =$ ______________.

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Chapter summary

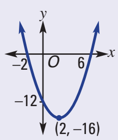
Parabolas

Sketching $y = x^2 + bx + c$

e.g. $y = x^2 - 4x - 12$
y-intercept: $x = 0$, $y = -12$
x-intercepts:
$y = 0$, $x^2 - 4x - 12 = 0$
Factorise $(x - 6)(x + 2) = 0$
Null factor law
$x - 6 = 0$ or $x + 2 = 0$
$x = 6$ or $x = -2$
Turning point: halfway between x-intercepts

$$x = \frac{6 + (-2)}{2} = 2$$

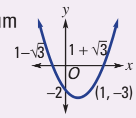
Substitute $x = 2$:
$y = 2^2 - 4(2) - 12$
$= -16$
$(2, -16)$ minimum



Sketching $y = a(x - h)^2 + k$

Any quadratic can be written in turning point form by completing the square.
e.g. Sketch $y = (x - 1)^2 - 3$.

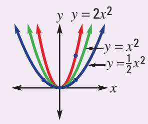
Turning point $(1, -3)$ minimum
y-intercept: $x = 0$, $y = 1 - 3$.
$= -2$



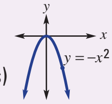
x-intercepts: $y = 0$, $0 = (x - 1)^2 - (\sqrt{3})^2$
$0 = (x - 1 - \sqrt{3})(x - 1 + \sqrt{3})$
$\therefore x = 1 + \sqrt{3}, 1 - \sqrt{3}$

Transformations and turning point form

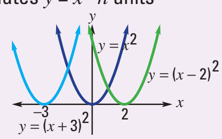
$y = ax^2$ dilates graph compared to $y = x^2$
$0 < a < 1$ graph is wider
$a > 1$ graph is narrower



$a < 0$ graph is inverted (reflected in x-axis)

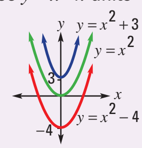


$y = (x - h)^2$ translates $y = x^2$ h units horizontally
$h > 0$ move right
$h < 0$ move left



$y = x^2 + k$ translates $y = x^2$ k units vertically

$k > 0$ move up
$k < 0$ move down



Turning point form
$y = a(x - h)^2 + k$ has turning point at (h, k)

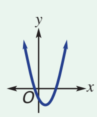
Quadratic formula

For $y = ax^2 + bx + c$, the quadratic formula can be used to find x-intercepts.
i.e. if $ax^2 + bx + c = 0$

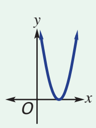
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant $\Delta = b^2 - 4ac$ tells how many x-intercepts:

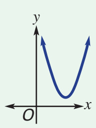
$\Delta > 0$
two x-intercepts



$\Delta = 0$
one x-intercept

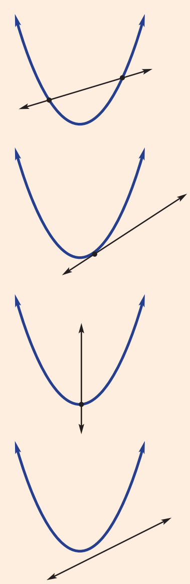


$\Delta < 0$
no x-intercepts

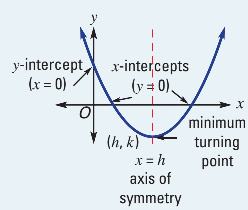


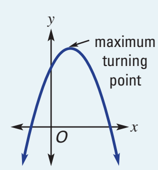
Lines and parabolas

It is possible for a line to meet a parabola twice, once or not at all.



Parabola arises from a quadratic equation





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Relation vs function

Any collection of points is a relation.
A function is a special relation.

Relations and functions

$y = x^2$ is an example of a function because every x value gives only one y value.

Domain

The set of all permissible x values

Range

The set of all permissible y values

Function notation

$y = x^2$ can be written as $f(x) = x^2$.

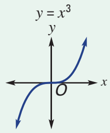
Vertical line test

A relation is a function when any vertical line drawn through the graph cuts it only once.

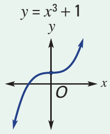
Other graphs

Cubic functions

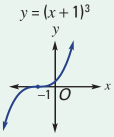
$y = x^3$



$y = x^3 + 1$



$y = (x + 1)^3$



Using graphs to describe change

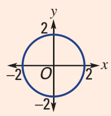
Direct variation: $y = kx$
- Both variables increase or decrease together.
- Graph is a straight line through (0, 0).

Inverse (indirect) variation: $y = \frac{k}{x}$ or $xy = k$
- If one variable increases, the other decreases.
- Graph is a hyperbola shape.

Description of graphs:
- y is increasing or decreasing or constant.
- Rate of change is positive or negative or zero.
- Rate of change is increasing or decreasing or constant.

Circles

$x^2 + y^2 = r^2$ is a circle centred at (0, 0) with radius r.

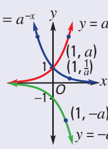


$x^2 + y^2 = 4$
$r^2 = 4$
$\therefore r = 2$

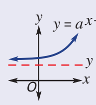
$y = \pm\sqrt{r^2 - x^2}$
$x = \pm\sqrt{r^2 - y^2}$

Circle with equation $(x - h)^2 + (y - k)^2 = r^2$ is centred at (h, k) with radius r.

Exponential graphs

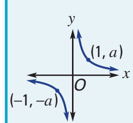


asymptote $y = 0$



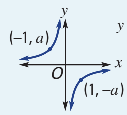
Hyperbolas

$y = \frac{a}{x}$

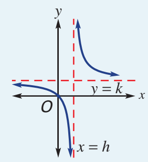


Asymptotes
$x = 0$ and $y = 0$

$y = -\frac{a}{x}$ (reflect in x- or y-axis)



$y = \frac{a}{x - h} + k$
find asymptotes and intercepts to sketch



Inverse functions

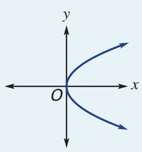
Notation

$f^{-1}(x)$ is the inverse of $f(x)$.

Restricting the domain

Sometimes the inverse is not a function unless the domain is restricted.

The inverse of $y = x^2$ is $x = y^2$.

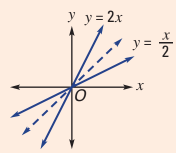


This is not a function.

An inverse is created by reflecting every point across the line $y = x$. This swaps the x value and y value of every point.
e.g. (3, 2) becomes (2, 3).

$y = 2x$ and $y = \frac{x}{2}$

Consider $y = 2x$.
Swapping x and y gives:
$x = 2y$
$2y = x$
$y = \frac{x}{2}$



$f(x) = 2x$
$f^{-1}(x) = \frac{x}{2}$
inverse functions

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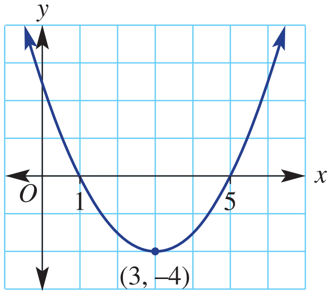
Chapter review

Multiple-choice questions

1 The equation of the axis of symmetry of the graph shown is:



A $y = -4$ B $x = 3$ C $x = -4$ D $y = 3$ E $y = 3x$

2 Compared to the graph of $y = x^2$, the graph of $y = (x - 3)^2$ is:

A translated 3 units down
B translated 3 units left
C dilated by a factor of 3
D translated 3 units right
E translated 3 units up

3 The coordinates and type of turning point of $y = -(x + 2)^2 + 1$ is:

A a minimum at $(-2, -1)$
B a maximum at $(2, 1)$
C a minimum at $(2, 1)$
D a maximum at $(2, -1)$
E a maximum at $(-2, 1)$

4 The parabola $y = 3(x - 1)^2 + 4$ intersects with the y-axis at:

A $(0, 1)$ B $(0, 4)$ C $(3, 4)$ D $(0, 7)$ E $(3, 0)$

5 The graph of $y = x^2 + 3x - 10$ meets the x-axis at:

A $(2, 0)$ and $(-5, 0)$
B $(0, -10)$
C $(5, 0)$ and $(-2, 0)$
D $(0, -5)$ and $(0, -2)$
E $(5, 0)$ and $(2, 0)$

6 A quadratic graph has x-intercepts at $x = -7$ and $x = 2$. The x-coordinate of the turning point is:

A $x = -\frac{5}{2}$ B $x = -\frac{7}{2}$ C $x = \frac{9}{3}$ D $x = \frac{5}{2}$ E $x = -\frac{9}{2}$

7 The quadratic rule $y = x^2 - 4x - 3$, when written in turning point form is:

A $y = (x - 2)^2 - 3$
B $y = (x - 4)^2 + 1$
C $y = (x - 2)^2 - 7$
D $y = (x + 4)^2 - 19$
E $y = (x + 2)^2 - 1$

8 The number of points of intersection of the parabola $y = x^2 + 6x + 1$ and the line $y = 2x - 2$ is:

A 0 B 1 C 3 D 2 E 4

9 A quadratic graph $y = ax^2 + bx + c$ has two x-intercepts. This tells us that:

A The graph has a maximum turning point.
B $-\frac{b}{2a} < 0$
C There is no y-intercept.
D $b^2 - 4ac > 0$
E $b^2 - 4ac = 0$

10 A toy rocket follows the path given by $h = -t^2 + 4t + 6$, where h is the height above ground, in metres, t seconds after launch. The maximum height reached by the rocket is:

A 10 metres B 2 metres C 6 metres D 8 metres E 9 metres

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11 The equation of a circle centred at the origin with radius 4 units is:

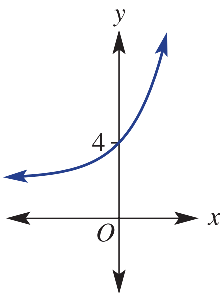
A $y = 4x^2$ **B** $x^2 + y^2 = 4$ **C** $x^2 + y^2 = 8$ **D** $y = 4^x$ **E** $x^2 + y^2 = 16$

12 The graph of $y = 3^x$ has y-intercept with coordinates:

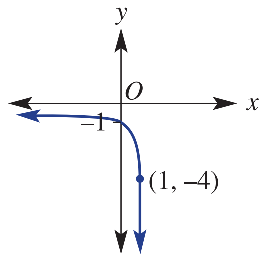
A $(0, 3)$ **B** $(3, 0)$ **C** $(0, 1)$ **D** $(1, 3)$ **E** $\left(0, \frac{1}{3}\right)$.

13 The graph of $y = 4^{-x}$ is:

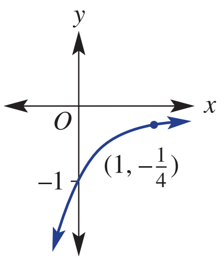
A



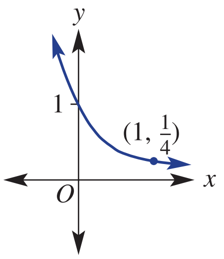
B



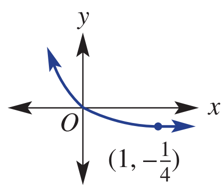
C



D



E



14 The graphs of $y = 3^x$ and $y = \frac{1}{3}$ intersect at the point:

A $(1, 3)$ **B** $\left(-1, \frac{1}{3}\right)$ **C** $(-1, 3)$ **D** $\left(\frac{1}{9}, \frac{1}{3}\right)$ **E** $\left(1, \frac{1}{3}\right)$

15 The graph of $y = \frac{1}{x-1} + 2$ has asymptote(s) at:

A $x = 1, y = 2$ **B** $y = 2$ **C** $x = -1$
D $x = 2, y = 1$ **E** $x = -1, y = 2$

16 The circle with equation $(x + 1)^2 + (y - 3)^2 = 16$ has centre coordinates and radius:

A $(1, -3), r = 4$ **B** $(1, -3), r = 16$ **C** $(-1, 3), r = 4$
D $(-1, 3), r = 16$ **E** $(-1, -3), r = 4$

17 If y is inversely proportional to x, the equation is of the form:

A $y = kx$ **B** $y = kx + c$ **C** $y = \frac{x}{k}$ **D** $y = kx^2$ **E** $y = \frac{k}{x}$

18 The restrictions on x and y in the rule $y = \frac{1}{x+1}$ are:

A $x \neq 1, y \neq 0$ **B** $x \neq 1, y \neq 1$ **C** $x \neq 0, y \neq 0$
D $x \neq -1, y \neq 0$ **E** $x \neq -1, y \neq -1$

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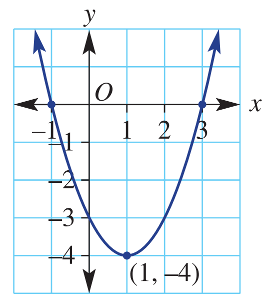
Short-answer questions



1 State the following features of the quadratic graph shown.
  a turning point and whether it is a maximum or a minimum
  b axis of symmetry
  c x-intercepts
  d y-intercept

2 State whether the graphs of the following quadratics have a maximum or a minimum turning point and give its coordinates.
  a $y = (x - 2)^2$ b $y = -x^2 + 5$
  c $y = -(x + 1)^2 - 2$ d $y = 2(x - 3)^2 + 4$

3 Sketch the quadratics below by first finding:
  i the y-intercept
  ii the x-intercepts, using factorisation
  iii the turning point
  a $y = x^2 - 4$ b $y = x^2 + 8x + 16$ c $y = x^2 - 2x - 8$

4 Complete the following for each quadratic below.
  i State the coordinates of the turning point and whether it is a maximum or a minimum.
  ii Find the y-intercept.
  iii Find the coordinates of the x-intercepts (if any).
  iv Sketch the graph, labelling the features above.
  a $y = -(x - 1)^2 - 3$ b $y = 2(x + 3)^2 - 8$

5 Sketch the following quadratics by completing the square. Label all key features with exact coordinates.
  a $y = x^2 - 4x + 1$ b $y = x^2 + 3x - 2$

6 State the number of x-intercepts of the following quadratics either by using the discriminant or by inspection where applicable.
  a $y = (x + 4)^2$ b $y = (x - 2)^2 + 5$ c $y = x^2 - 2x - 5$ d $y = 2x^2 + 3x + 4$

7 For the following quadratics:
  i Find the y-intercept.
  ii Use $x = -\frac{b}{2a}$ to find the coordinates of the turning point.
  iii Use the quadratic formula to find the x-intercepts, rounding your answer to 1 decimal place where necessary.
  iv Sketch the graph.
  a $y = 2x^2 - 8x + 5$ b $y = -x^2 + 3x + 4$

8 200 m of fencing is to be used to form a rectangular paddock. Let x m be the breadth of the paddock.
  a Write an expression for the length of the paddock in terms of x.
  b Write an equation for the area of the paddock (A m^2) in terms of x.
  c Determine on the suitable values of x.
  d Sketch the graph of A versus x for suitable values of x.
  e Use the graph to determine the maximum paddock area that can be formed.
  f What will be the dimensions of the paddock to achieve its maximum area?

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9 Sketch these circles. Label the centre and axes intercepts.

a $x^2 + y^2 = 25$ **b** $x^2 + y^2 = 7$

10 Find the exact coordinates of the points of intersection of the circle $x^2 + y^2 = 9$ and the line $y = 2x$. Sketch the graphs, showing the points of intersection.

11 Sketch the following graphs, labelling the y-intercept and the point where $x = 1$.

a $y = 4^x$ **b** $y = -3^x$ **c** $y = 5^{-x}$

12 Sketch these hyperbolas, labelling the points where $x = 1$ and $x = -1$.

a $y = \frac{2}{x}$ **b** $y = -\frac{3}{x}$

13 Find the points of intersection of the hyperbola $y = \frac{4}{x}$ and these lines.

a $y = 3$ **b** $y = 2x$

14 Sketch the following graphs, labelling key features with exact coordinates.

a $(x + 1)^2 + (y - 2)^2 = 4$ **b** $y = 2^{x-1} + 3$ **c** $y = \frac{1}{x + 2} - 3$

15 Solve these equations simultaneously.

a $y = x^2 + 4x - 2$
$y = 10$

b $y = 2x^2 + 5x + 9$
$y = -x + 4$

c $y = -x^2 + 6x$
$y = 2x$

d $y = x^2 + 1$
$2x + 3y = 4$

16 **a** Use the discriminant to show that the line $y = 2x + 3$ does not intersect the parabola $y = x^2 + x + 7$.

b Use the discriminant to show that the line $y = x + 4$ intersects the parabola $y = x^2 - x + 5$ at just one place.

c Find the value of k such that the line $y = x + k$ intersects the parabola $y = 2x^2 + 5x$ at just one place.

17 Solve these cubic equations for x. Round your answers to 1 decimal place where necessary.

a $2x^3 = 54$ **b** $3x^3 = -24$ **c** $x^3 + 2 = 8$ **d** $x^3 + 5 = 0$

e $\frac{1}{2}x^3 - 20 = 12$ **f** $\frac{1}{3}x^3 - 2 = 7$ **g** $1 - x^3 = 9$ **h** $14 - 4x^3 = 74$

18 Sketch the following cubic functions, labelling the axes intercepts with exact values.

a $y = 3x^3$ **b** $y = x^3 - 8$ **c** $y = -2x^3 + 10$ **d** $y = \frac{1}{2}(x - 2)^3$

19 If $f(x) = x^3 - 2x^2 + x - 3$, find:

a $f(0)$ **b** $f(-1)$ **c** $f(0.5)$ **d** $f(4)$ **e** $f(k)$

20 For each of the following, state the domain (i.e. the set of permissible x values) and range (i.e. permissible y values).

a $y = 2x - 8$ **b** $y = 4$ **c** $f(x) = 4 - 2x^2$ **d** $f(x) = \frac{2}{3x}$ **e** $f(x) = \sqrt{x}$

21 Find the inverse function for:

a $y = 4x - 5$ **b** $y = 5x$ **c** $y = 4x^3$ **d** $y = \frac{3}{x}$ **e** $y = 1 - x$

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22 Find a domain that includes the vertex for which the parabola $y = x^2 - 4$ has an inverse.

23 **a** If $y = \frac{k}{x}$ and $x = 6$ when $y = 12$, determine:

- **i** the relationship between x and y
- **ii** y when $x = 4$
- **iii** x when $y = 0.7$

b At any given time, an amount of money in a foreign currency is in direct proportion to the corresponding amount in Australian dollars. If \$24 Australian dollars (AUD) is equal to 155.28 Chinese yuan (CNY), write the rule for CNY in terms of AUD and determine how many Australian dollars are equal to 1 million Chinese yuan.

c Given that y varies inversely with x for each given table of values, determine the constant of proportionality, k, then write the rule and calculate the missing values.

i

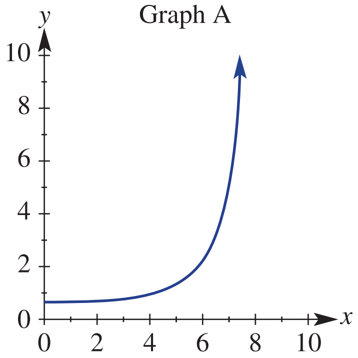
x	30	80	100
y	40		12

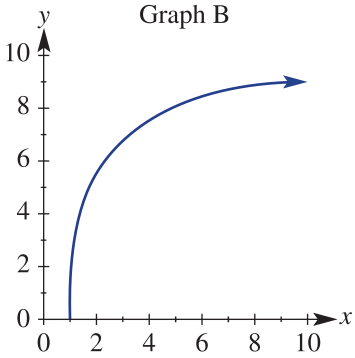
ii

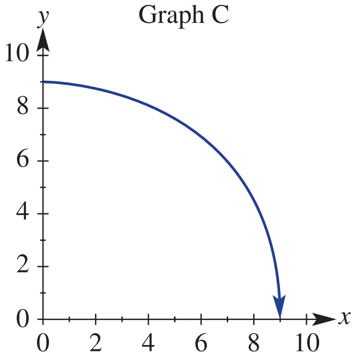
x		24	30
y	12	10	8

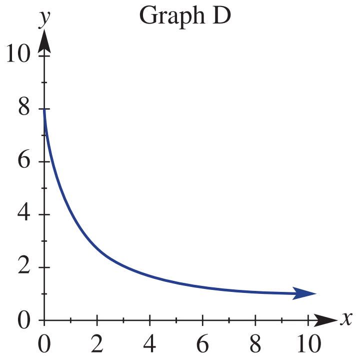
24 **a** For each of the graphs below, copy and complete this sentence by inserting *increasing* or *decreasing* or *constant* for the missing words.

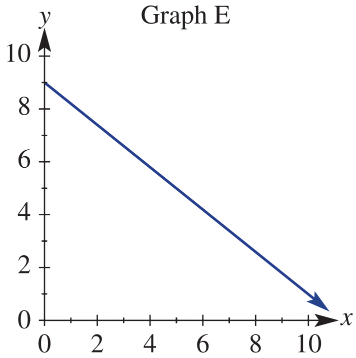
y is ____________ at a ____________ rate.











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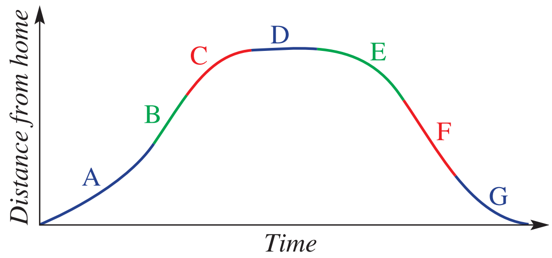
b This distance–time graph shows a car's journey from home. Match each segment of the graph with all of the descriptions that are correct for that segment.



i accelerating
ii decelerating
iii Rate of change of distance is zero.
iv Distance from home is decreasing and rate of change of distance is increasing.
v Positive rate of change of distance
vi Distance from home is decreasing and rate of change of distance is decreasing.
vii Negative rate of change of distance
viii Distance from home is increasing and rate of change of distance is increasing.
ix Constant rate of change of distance

25 For each of the following, rearrange the equation to make the letter in the brackets the subject of the formula.

a $P = 2b + a \ (b)$ **b** $s = \frac{d}{t} \ (t)$ **c** $A = \pi r^2$ if $A > 0 \ (r)$
d $A = \frac{1}{2}(a + b)h \ (b)$ **e** $c^2 = a^2 + b^2$ if $a > 0 \ (a)$ **f** $y = kx^2 \ (x)$

26 Consider the equation $y = ax^2$.
a Are there any restrictions on the variables a, x or y?
b Rearrange the equation to make a the subject.
c Are there any restrictions on your rule from part **b**? If so, what are they?
d Rearrange the equation to make x the subject.
e What restrictions are there on the variable a in your rule from part **d**?

Extended-response questions

1 The cable for a suspension bridge is modelled by the equation $h = \frac{1}{800}(x - 200)^2 + 30$, where h metres is the distance above the base of the bridge and x metres is the distance from the left side of the bridge.
a Determine the turning point of the graph of the equation.
b Determine the suitable values of x.
c Determine the range of values of h.
d Sketch a graph of the equation for the suitable values of x.
e What horizontal distance does the cable span?
f What is the closest distance of the cable from the base of the bridge?
g What is the greatest distance of the cable from the base of the bridge?

2 The amount, A grams, of a radioactive element remaining after t years is given by the rule $A = 450(3^{-t})$.
a What is the initial amount (i.e. at $t = 0$) of radioactive material?
b How much material remains after:
i 1 year? **ii** 4 years? (Round your answer to 1 decimal place.)
c Determine when the amount will reach 50 grams.
d Sketch a graph of A vs t for $t \geqslant 0$.
e Use trial and error to find (to 1 decimal place) when less than 0.01 grams will remain.

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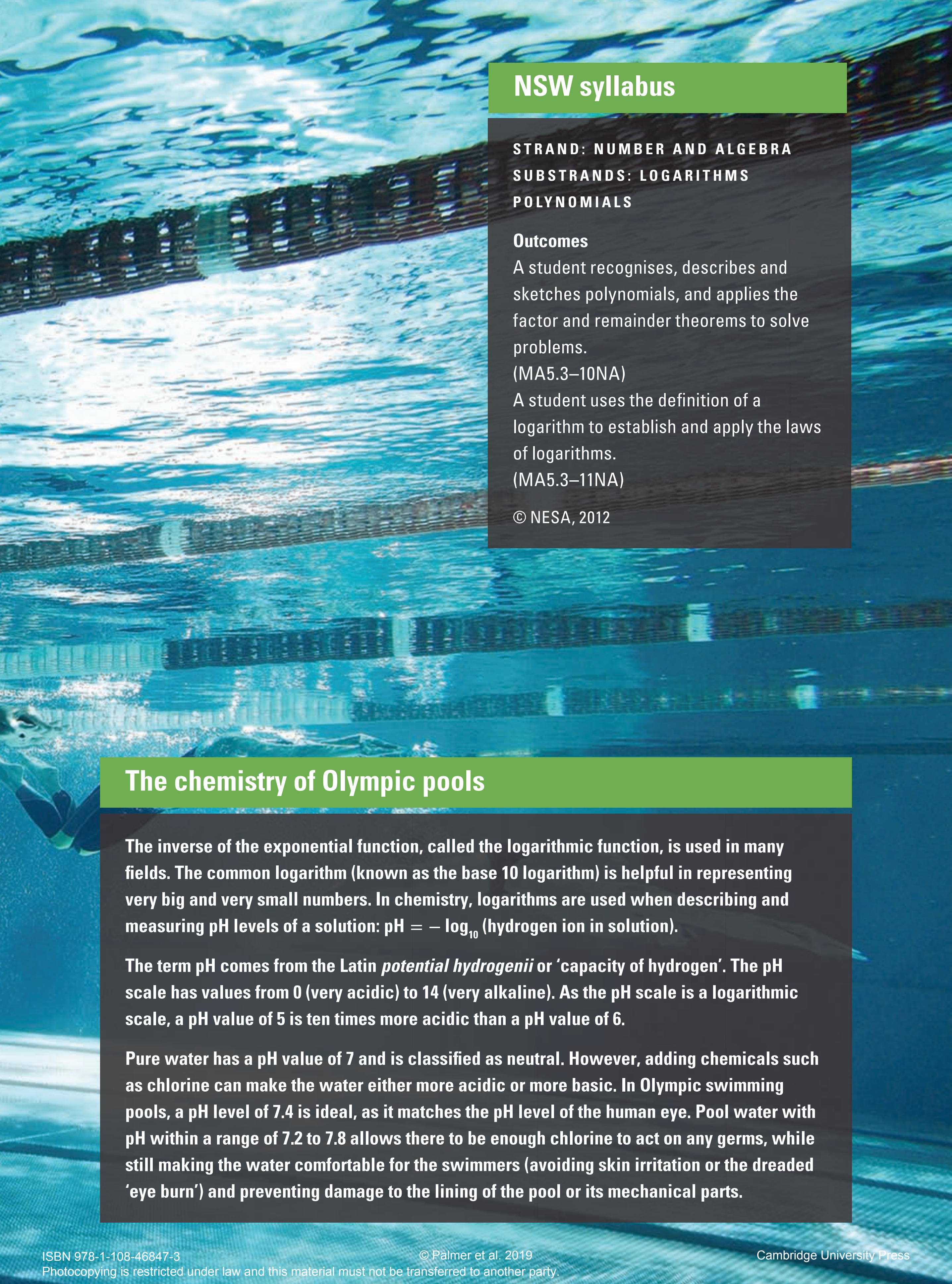
Online resources

- Auto-marked chapter pre-test
- Video demonstrations of all worked examples
- Interactive widgets
- Interactive walkthroughs
- Downloadable HOTsheets
- Access to all HOTmaths Australian Curriculum courses
- Access to the HOTmaths games library

10 Logarithms and polynomials

What you will learn



NSW syllabus

STRAND: NUMBER AND ALGEBRA
SUBSTRANDS: LOGARITHMS
POLYNOMIALS

Outcomes

A student recognises, describes and sketches polynomials, and applies the factor and remainder theorems to solve problems.
(MA5.3–10NA)

A student uses the definition of a logarithm to establish and apply the laws of logarithms.
(MA5.3–11NA)

The chemistry of Olympic pools

The inverse of the exponential function, called the logarithmic function, is used in many fields. The common logarithm (known as the base 10 logarithm) is helpful in representing very big and very small numbers. In chemistry, logarithms are used when describing and measuring pH levels of a solution: pH $= -\log_{10}$ (hydrogen ion in solution).

The term pH comes from the Latin *potential hydrogenii* or 'capacity of hydrogen'. The pH scale has values from 0 (very acidic) to 14 (very alkaline). As the pH scale is a logarithmic scale, a pH value of 5 is ten times more acidic than a pH value of 6.

Pure water has a pH value of 7 and is classified as neutral. However, adding chemicals such as chlorine can make the water either more acidic or more basic. In Olympic swimming pools, a pH level of 7.4 is ideal, as it matches the pH level of the human eye. Pool water with pH within a range of 7.2 to 7.8 allows there to be enough chlorine to act on any germs, while still making the water comfortable for the swimmers (avoiding skin irritation or the dreaded 'eye burn') and preventing damage to the lining of the pool or its mechanical parts.

Pre-test

1 Evaluate:

a 2^3
b $4^{\frac{1}{2}}$
c $\frac{5}{\sqrt[3]{125}}$
d $10^4 - 10^3$
e 3^{-3}
f $2^2 \times 2^3$
g $\frac{8^6}{8^4}$
h $\frac{2^2 \times 3^2}{6^3}$

2 Use index laws to simplify.

a $x^3 \times x^4$
b $3x \times 4x^7$
c $14x^4 \div (7x^3)$
d $2(3x)^0$
e $3x(2x)^2$
f $(x^3y)^3$
g $\left(\frac{2x}{y}\right)^3$
h $\frac{x^2}{2y} \times \left(\frac{y}{x}\right)^2$
i $2\left(\frac{xy}{4}\right)^3 \div \left(3\left(\frac{x^2y}{2}\right)^2\right)$

3 Find the value of x in these exponential equations.

a $2^x = 8$
b $3^x = 81$
c $10^x = 10000$
d $3^x = \frac{1}{9}$
e $10^x = 0.001$
f $6^{2x} = 216$

4 An amount of \$10 000 is invested at 5% p.a., earning compound interest.

a Write a rule connecting the investment balance \$$A$ and the number of years invested (n years).
b Use your rule to find the investment balance after 10 years. Give your answer correct to the nearest dollar.

5 Expand and simplify.

a $(x + 1)(x + 3)$
b $(2x - 1)(3x + 2)$
c $(2x - 5)(2x + 5)$
d $(5x - 1)(3x + 1)$
e $(x - 1)(x + 1) - (x^2 - 1)$
f $(x + 2)^2 - x^2 + 4$

6 Factorise:

a $5 - 10x$
b $x^2 - 2x - 48$
c $x^2 + 3x - 10$
d $2x^2 - x - 3$
e $15x^2 - 18x - 24$
f $6 - x - x^2$

7 Evaluate these expressions when $a = 3$ and $b = -2$.

a $a - 2b$
b $a^2 - b^2$
c $a^3 + 2b^2 - ab$

8 Use long division to find the quotient and remainder.

a $9\overline{)147}$
b $12\overline{)471}$
c $31\overline{)1724}$

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10A Introducing logarithms

Interactive
Widgets
HOTsheets
Walkthrough

Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Logarithms ('logical arithmetic') are an important idea in mathematics and were invented by John Napier in the 17th century to simplify arithmetic calculations. Logarithms are linked directly to exponentials and can be used to solve a range of exponential equations.

Recall that $2^3 = 8$ (2 to the power 3 equals 8). We can also say that the logarithm of 8 to the base 2 equals 3, which is written as $\log_2 8 = 3$. So for exponential equations such as $y = 2^x$, a logarithm finds x for a given value of y. A logarithm can often be evaluated by hand, but calculators can also be used.

Logarithms can also be used to create logarithmic scales, which are commonly used in science, economics and engineering. For example, the Richter scale, and the moment magnitude scale that replaced it, are logarithmic scales that measure the strength of an earthquake.

Let's start: Can you work out logarithms?

We know that $3^2 = 9$, so $\log_3 9 = 2$. This means that $\log_3 9$ is equal to the index that makes 3 to the power of that index equal 9. Similarly $10^3 = 1000$, so $\log_{10} 1000 = 3$.

Now find the value of the following.

- $\log_{10} 100$
- $\log_{10} 10\,000$
- $\log_2 16$
- $\log_2 64$
- $\log_3 27$
- $\log_4 64$

Key ideas

- **A logarithm** of a number to a given base is the power (or index) to which the base is raised to give the number.
  - For example: $\log_2 16 = 4$ since $2^4 = 16$
  - The base a is written as a subscript to the operator word 'log'; i.e. $\log_a \ldots$
  - The expression can be spoken as 'log base 2 of 16 is 4'.
- In general, if $a^x = y$ then $\log_a y = x$ with $a > 0$ and $y > 0$.
  - We say 'log to the base a of y equals x'.

Example 1 Writing equivalent statements

Write an equivalent statement to the following.

a $\log_{10} 1000 = 3$ **b** $2^5 = 32$

SOLUTION	EXPLANATION
a $10^3 = 1000$	$\log_a y = x$ is equivalent to $a^x = y$.
b $\log_2 32 = 5$	$a^x = y$ is equivalent to $\log_a y = x$.

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Example 2 Evaluating logarithms

a Evaluate the following logarithms.

i $\log_2 8$

ii $\log_5 625$

b Evaluate the following.

i $\log_3 \frac{1}{9}$

ii $\log_{10} 0.001$

c Evaluate, correct to 3 decimal places, using a calculator.

i $\log_{10} 7$

ii $\log_{10} 0.5$

SOLUTION	EXPLANATION
a i $\log_2 8 = 3$	Ask the question '2 to what power gives 8?' Note: $2^3 = 8$
ii $\log_5 625 = 4$	$5^4 = 5 \times 5 \times 5 \times 5 = 625$
b i $\log_3 \frac{1}{9} = -2$	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$
ii $\log_{10} 0.001 = -3$	$10^{-3} = \frac{1}{10^3} = \frac{1}{1000} = 0.001$
c i $\log_{10} 7 = 0.845$	Use the log button on a calculator and use base 10. (Some calculators will give log base 10 by pressing the log button.)
ii $\log_{10} 0.5 = -0.301$	Use the log button on a calculator.

Example 3 Solving simple logarithmic equations

Find the value of x in these equations.

a $\log_4 64 = x$

b $\log_2 x = 6$

SOLUTION	EXPLANATION
a $\log_4 64 = x$ $4^x = 64$ $x = 3$ (by inspection)	If $\log_a y = x$ then $a^x = y$. $4^3 = 64$
b $\log_2 x = 6$ $2^6 = x$ $x = 64$	Write in index form: $2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$

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Exercise 10A

UNDERSTANDING AND FLUENCY	1–5, 6–7(½)	4, 5, 6–8(½)	6–8(½)

1 Complete this table.

x	0	1	2	3	4	5
2^x						
3^x						243
4^x					256	
5^x		5				
10^x			100			

2 State the value of the unknown number for each statement.
- **a** 2 to the power of what number gives 16?
- **b** 3 to the power of what number gives 81?
- **c** 7 to the power of what number gives 343?
- **d** 10 to the power of what number gives 10 000?

3 Write these numbers as fractions.
- **a** 0.0001
- **b** 0.5
- **c** 2^{-2}
- **d** 2^{-5}
- **e** 3^{-3}
- **f** 5^{-2}
- **g** 4^{-3}
- **h** 6^{-2}

Example 1a

4 Write the following in index form.
- **a** $\log_2 16 = 4$
- **b** $\log_{10} 100 = 2$
- **c** $\log_3 27 = 3$
- **d** $\log_2 \frac{1}{4} = -2$
- **e** $\log_{10} 0.1 = -1$
- **f** $\log_3 \frac{1}{9} = -2$

Example 1b

5 Write the following in logarithmic form.
- **a** $2^3 = 8$
- **b** $3^4 = 81$
- **c** $2^5 = 32$
- **d** $4^2 = 16$
- **e** $10^{-1} = \frac{1}{10}$
- **f** $5^{-3} = \frac{1}{125}$

Example 2a

6 Evaluate the following logarithms.
- **a** $\log_2 16$
- **b** $\log_2 4$
- **c** $\log_2 64$
- **d** $\log_3 27$
- **e** $\log_3 3$
- **f** $\log_4 16$
- **g** $\log_5 125$
- **h** $\log_{10} 1000$
- **i** $\log_7 49$
- **j** $\log_{11} 121$
- **k** $\log_{10} 100\,000$
- **l** $\log_9 729$
- **m** $\log_2 1$
- **n** $\log_5 1$
- **o** $\log_{37} 1$
- **p** $\log_1 1$

Example 2b

7 Evaluate the following.
- **a** $\log_2 \frac{1}{8}$
- **b** $\log_2 \frac{1}{4}$
- **c** $\log_3 \frac{1}{9}$
- **d** $\log_{10} \frac{1}{1000}$
- **e** $\log_7 \frac{1}{49}$
- **f** $\log_3 \frac{1}{81}$
- **g** $\log_5 \frac{1}{625}$
- **h** $\log_8 \frac{1}{8}$
- **i** $\log_{10} 0.1$
- **j** $\log_{10} 0.001$
- **k** $\log_{10} 0.00001$
- **l** $\log_2 0.5$
- **m** $\log_2 0.125$
- **n** $\log_5 0.2$
- **o** $\log_5 0.04$
- **p** $\log_3 0.\dot{1}$

Example 2c

8 Evaluate, correct to 3 decimal places, using a calculator.
- **a** $\log_{10} 5$
- **b** $\log_{10} 47$
- **c** $\log_{10} 162$
- **d** $\log_{10} 0.8$
- **e** $\log_{10} 0.17$
- **f** $\log_{10} \frac{1}{27}$

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PROBLEM-SOLVING AND REASONING	9(½), 10, 12	9(½), 10, 12, 13	9(½), 11, 13, 14

Example 3

9 Find the value of x in these equations.

a $\log_3 27 = x$ **b** $\log_2 32 = x$ **c** $\log_2 64 = x$ **d** $\log_5 625 = x$
e $\log_{10} 1000 = x$ **f** $\log_6 36 = x$ **g** $\log_2 x = 4$ **h** $\log_3 x = 4$
i $\log_{10} x = 3$ **j** $\log_3 x = -2$ **k** $\log_4 x = -1$ **l** $\log_7 x = -3$
m $\log_x 27 = 3$ **n** $\log_x 32 = 5$ **o** $\log_x 64 = 3$ **p** $\log_x 64 = 2$
q $\log_x 81 = 4$ **r** $\log_x 10000 = 4$ **s** $\log_x 0.5 = -1$ **t** $\log_4 0.25 = x$

10 A single bacterium cell divides into two every minute.

a Complete this cell population table.

Time (minutes)	0	1	2	3	4	5
Population	1	2				

b Write a rule for the population P after t minutes.
c Use your rule to find the population after 8 minutes.
d Use trial and error to find the time (correct to the nearest minute) for the population to rise to 10000.
e Write the exact answer to part **d** as a logarithm.

11 Is it possible for a logarithm (of the form $\log_a b$) to give a negative result? If so, give an example and reasons.

12 If $\log_a y = x$ and $a > 0$, is it possible for $y \leqslant 0$ for any value of x? Give examples and reasons.

13 Evaluate:

a $\log_2 4 \times \log_3 9 \times \log_4 16 \times \log_5 25$
b $2 \times \log_3 27 - 5 \times \log_8 64 + 10 \times \log_{10} 1000$
c $\dfrac{4 \times \log_5 125}{\log_2 64} + \dfrac{2 \times \log_3 9}{\log_{10} 10}$

14 Explain why $\dfrac{1}{\log_a 1}$ does not exist.

ENRICHMENT	–	–	15

Fractional logarithms

15 We know that we can write $\sqrt{2} = 2^{\frac{1}{2}}$, so $\log_2 \sqrt{2} = \dfrac{1}{2}$ and $\log_2 \sqrt[3]{2} = \dfrac{1}{3}$. Now evaluate the following without the use of a calculator.

a $\log_2 \sqrt[4]{2}$ **b** $\log_2 \sqrt[5]{2}$ **c** $\log_3 \sqrt{3}$ **d** $\log_3 \sqrt[3]{3}$
e $\log_7 \sqrt{7}$ **f** $\log_{10} \sqrt[3]{10}$ **g** $\log_{10} \sqrt[3]{100}$ **h** $\log_2 \sqrt[3]{16}$
i $\log_3 \sqrt[4]{9}$ **j** $\log_5 \sqrt[4]{25}$ **k** $\log_2 \sqrt[5]{64}$ **l** $\log_3 \sqrt[7]{81}$

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10B Logarithmic graphs

Interactive
Widgets
HOTsheets
Walkthrough

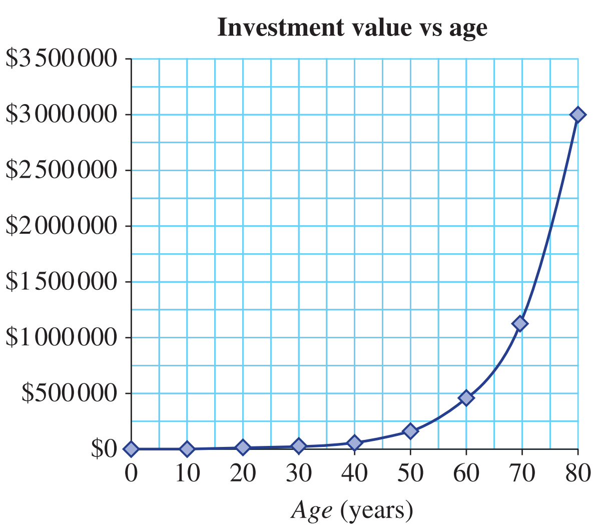
The graph of the exponential equation $y = a^x$ is roughly a concave-up shape with the x-axis being an asymptote. If a mirror were placed along the line $y = x$, the reflection of $y = a^x$ would be the graph of $y = \log_a x$, roughly a concave-down shape with the y-axis being an asymptote. Each point on the exponential graph, when reflected, has its coordinates interchanged and the new point is on the logarithmic graph. Logarithmic and exponential functions are the inverse of each other. Understanding logarithmic functions and their relationship to exponential functions enables mathematical techniques for solving a wide variety of problems, especially in engineering, science and computer technology.

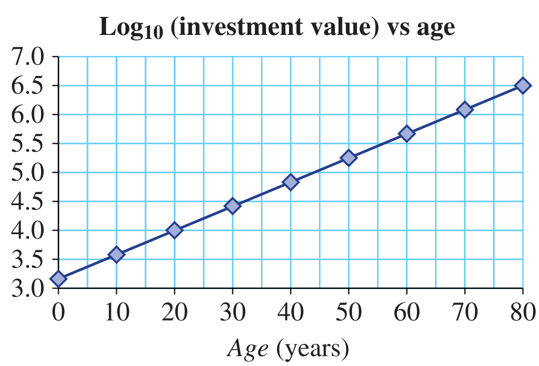
Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Log scales are used to measure physical quantities that increase exponentially. For example, the Richter scale is used to measure earthquake intensity; decibels measure the loudness of sound and pH measures the acidity of a solution.

Let's start: Log scales

Imagine that when you were born your grandparents invested \$1500, compounding at a fixed rate per annum for life. The value of your investment could be graphed either using a normal linear scale or a log scale, as shown by these graphs. Discuss the answers to the questions below.





- How do the coordinates of a point on the graph with the log scale relate to the corresponding point on the linear scale graph? Give an example.
- What investment value does 3.6 correspond to on the log scale?
- Which graph can be used to find the investment value at age 20 years and what is this value?
- By what factor has the investment value increased from age 20 to age 30?
- By what factor has the investment value increased when the log scale increases by 1 unit?
- If a graph using a log scale is linear, what form is the graph of the original data?
- Discuss the advantages and disadvantages of linear and logarithmic scales.

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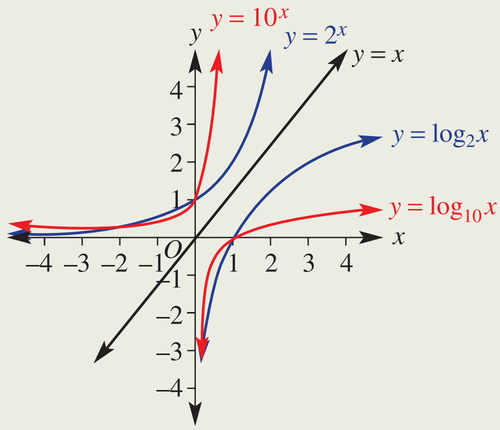
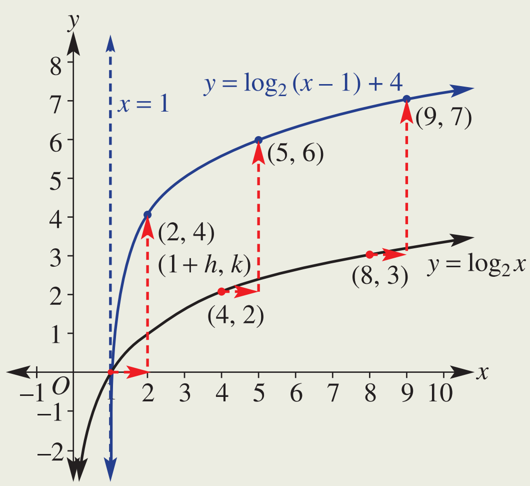
Key ideas

- A simple logarithmic relationship is of the form $y = \log_a x$, where $a > 1$.
  - $x > 0$
  - As $x \to 0$, $y \to -\infty$.
  - Asymptote is $x = 0$.
  - x-intercept is 1.



- The function $y = \log_a x$ is the inverse of $y = a^x$.
  - $y = \log_a x$ is the reflection of $y = a^x$ in the line $y = x$.
  - A point on $y = a^x$ has coordinates in the reverse order to a point on $y = \log_a x$. For example, when (0, 1) on $y = a^x$ is reflected in the line $y = x$, it gives (1, 0) on $y = \log_a x$.
  - For $\log_a a^x = x$, we say 'log to the base a, of a to the power of x, equals x'.
  - For $a^{\log_a x} = x$, we say 'a to the power of log x to the base a, equals x'.

- For the graph of $y = \log_a(x - h) + k$:
  - $x - h > 0$, so $x > h$
  - As $x \to h$, $y \to -\infty$.
  - Asymptote is $x = h$; i.e. when $(x - h) = 0$.
  - translated h units to the right
  - translated k units up
  - Each x value is increased by h and each y value is increased by k, so $(x, y) \to (x + h, y + k)$.
  - For example: $(1, 0) \to (1 + h, k)$
  - The x-intercept is found by solving $\log_a(x - h) + k = 0$.



- A 'log scale' is used for physical values that increase exponentially from very small to very large numbers. The logarithm (usually to the base 10) of the physical value is used on the scale instead of the actual value, to enable ease of graphing and the comparison of values.
  For example:

Physical values	0.01	0.1	1	10	100	1000	10 000
log scale $\log_{10}$ (physical value)	−2	−1	0	1	2	3	4

  - The log scale values increase by adding 1 unit.
  - The physical values increase by a factor of 10 for each 1 unit increase in the log scale.
  - For example, an increase of 2 units on the log scale means that the physical value has increased by a factor of 100.
  - Applications of log scales include the measurement of acidity using the pH scale ($\text{pH} = -\log_{10}[H^+]$); the loudness of sound, in decibels ($\text{dB} = 10 \log\left(\frac{I}{I_0}\right)$); and earthquake intensity using Richter scale magnitudes ($M = \log_{10}\left(\frac{I}{I_0}\right)$).

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Example 4 Comparing the features of exponential and log graphs

a Copy and complete these tables for $y = 3^x$ and $y = \log_3 x$.

x	-2	-1	0	1	2
$y = 3^x$					

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$y = \log_3 x$					

b Plot the graphs of $y = 3^x$, $y = \log_3 x$ and also $y = x$ on the same axes with $-3 \leqslant x \leqslant 3$ and $-3 \leqslant y \leqslant 3$.

c State two features of $y = \log_3 x$ that show that it is the inverse function of $y = 3^x$.

d For each graph, state the equation of its asymptote.

e State any limitations for the input values for $y = 3^x$ and $y = \log_3 x$.

f State any limitations for the output values for $y = 3^x$ and $y = \log_3 x$.

SOLUTION

a

x	-2	-1	0	1	2
$y = 3^x$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

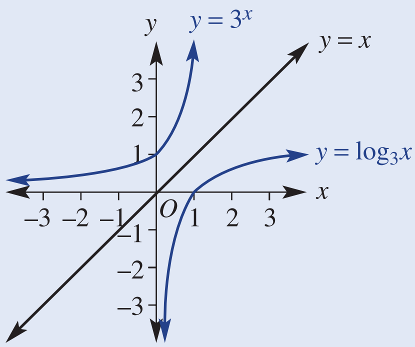
x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$y = \log_3 x$	-2	-1	0	1	2

EXPLANATION

$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

$\log_3\left(\frac{1}{9}\right) = -2$ since $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$.

b



Plot points and join them with a smooth curve.

c **i** $y = \log_3 x$ is the mirror image of $y = 3^x$, reflected in the line $y = x$.

ii When the coordinates of a point on $y = 3^x$ are reversed, it gives the coordinates of a point on $y = \log_3 x$.

Inverse functions are reflected in the line $y = x$. By drawing lines perpendicular to $y = x$, the mirror image from each point on $y = 3^x$ can be found and it will have coordinates in the reverse order; e.g. $(1, 3)$ is on $y = 3^x$ and $(3, 1)$ is its mirror image on $y = \log_3 x$.

Example continued over page

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d $y = 3^x$ has asymptote $y = 0$.
$y = \log_3 x$ has asymptote $x = 0$.

As $x \to -\infty$, $3^x \to 0$; i.e. the curve approaches the x-axis ($y = 0$) but never reaches it.
As $x \to 0$, $\log_3 x \to -\infty$; i.e. the curve approaches the y-axis ($x = 0$) but never reaches it.

e $y = 3^x$ has no limitations on x values.
$x > 0$ for $y = \log_3 x$.

Any x value input into $y = 3^x$ produces a valid output. Only positive x values input into $y = \log_3 x$ give a valid output.

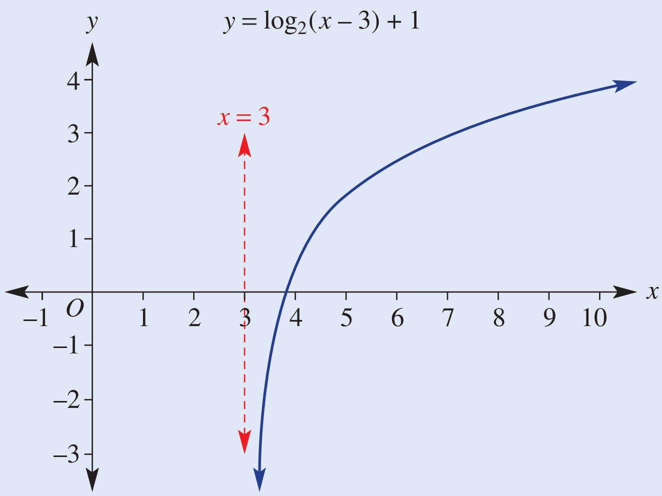
f $y > 0$ for $y = 3^x$.
There are no limitations on the y values for $y = \log_3 x$.

The output values (y) from $y = 3^x$ are always positive. The output values (y) from $y = \log_3 x$ can result in any real number value.

Example 5 Identifying transformations, asymptotes and intercepts of log graphs

The log equation $y = \log_2(x - 3) + 1$ is graphed here.

a State any limitations on the input values (i.e. x values).
b State the equation of the asymptote.
c State the horizontal and the vertical translations that have been applied to $y = \log_2 x$.
d Determine the axes intercepts, if any.



SOLUTION

a $x > 3$

$(x - 3) > 0$, $x > 3$
If $x \leqslant 3$ the log calculation will be invalid as there is no result for the log of a negative number.

b $x = 3$

Asymptote equation when $(x - 3) = 0$, so $x = 3$.
An asymptote is a line that the curve approaches by getting closer and closer to it, but never reaching it.

c Horizontal translation of 3 units to the right.
Vertical translation of 1 unit up.

The asymptote equation $x = 3$ shows that the curve has moved 3 units to the right, increasing each x value by 3 units. Adding 1 to the log function means the curve moves up by 1 unit so each y value is increased by 1 unit.

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d x-intercept:

$\log_2(x-3)+1=0$
$\log_2(x-3)=-1$
$x-3=2^{-1}$
$x=3.5$

Find x-intercept when $y=0$.
Solve $\log_2(x-3)+1=0$ for x.
Isolate the log, then convert to the exponential form; i.e. $\log_a m = p \rightarrow m = a^p$

x-intercept is 3.5.

Solve for x.

y-intercept:
$y=\log_2(0-3)+1$
$y=\log_2(-3)+1$
No y-intercept.

Find y-intercept when $x=0$.
The log of a negative number is invalid.

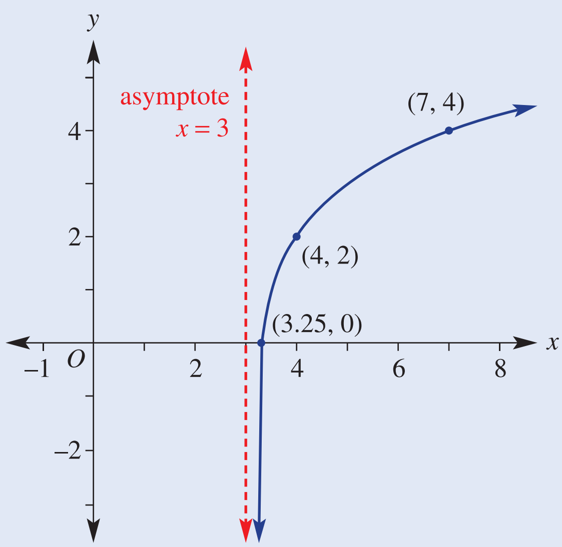
Example 6 Sketching log graphs with transformations

Sketch a graph for each of the following log equations, labelling any axes intercepts and at least one other point. Also sketch and write the equation of the asymptote.

a $y=\log_2(x-3)+2$ **b** $y=\log_5(x+2)-1$

SOLUTION

a



EXPLANATION

Asymptote is when $(x-3)=0$ so $x=3$, showing the curve moves 3 units to the right; $h=3$ and $k=2$.
A point on the graph is $(1+h, k)=(4, 2)$.
Also $(7, 4)$ is on the graph. Choosing $x=7$ makes $(x-3)$ a power of the base, simplifying the calculation.
To find the x-intercept, solve the equation when $y=0$.

$\log_2(x-3)+2=0$
$\log_2(x-3)=-2$
$x-3=2^{-2}$
$x=3.25$

Example continued over page

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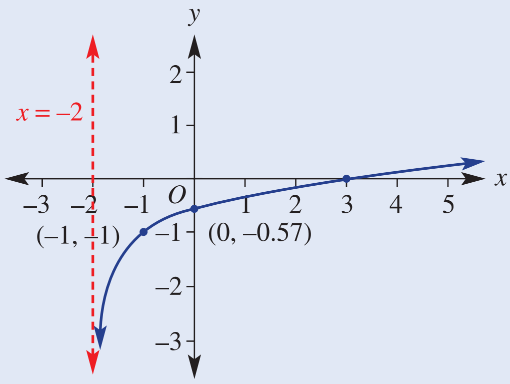
b $y = \log_5(x + 2) - 1$



Asymptote is when $(x + 2) = 0$ so $x = -2$, showing the curve moves 2 units to the left; $h = -2$ and $k = -1$.

A point on the graph is $(1 + h, k) = (-1, -1)$

$$\begin{aligned} \log_5(x + 2) - 1 &= 0 \\ \log_5(x + 2) &= 1 \\ x + 2 &= 5^1 \\ x &= 3 \end{aligned}$$

x-intercept is 3.

$y = \log_5(0 + 2) - 1$

$y = \log_5 2 - 1$

$y = \dfrac{\log_{10} 2}{\log_{10} 5} - 1$ (#See note below.)

$y = -0.5693...$

y-intercept is -0.57 (to 2 decimal places).

#In this step we used the change of base formula, which is explained in Section 10C.

Rule: $\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$

Example: $\log_5 2 = \dfrac{\log_{10} 2}{\log_{10} 5}$

Exercise 10B

UNDERSTANDING AND FLUENCY	1–7	4–7, 8(½)	5–7, 8(½)

1 For the graph of $y = \log_2 x$, state:

a any limitations on the input values (i.e. x values)

b the domain (permissable x values)

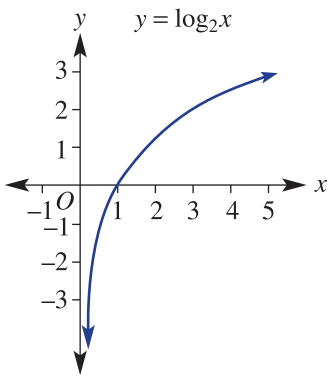
c any limitations on the output values (i.e. y values)

d the range (permissable y values)

e how y changes as x approaches zero

f the equation of the asymptote

g the x-intercept



2 State two features of $y = \log_2 x$ which show that it is the inverse function of $y = 2^x$.

3 For the graph of $y = \log_2(x - 4) + 5$, state:

a any limitations on the value of x

b the equation of the asymptote

c the horizontal translation of $y = \log_2 x$

d the vertical translation of $y = \log_2 x$

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4 **a** Copy and complete this table, showing the log scale values that would be used for the given physical values.

Physical values	1	10	100	1000	10 000	100 000
log scale $\log_{10}$ (physical value)						

b When the physical value increases by a factor of 10, what is the increase in the log scale?
c By what factor has the physical value increased when the log scale increases by 2 units?
d What log scale value would be used for a physical value of 16 000?
e What physical value would be represented by 0.301 on the log scale?
f What physical value would be represented by 0.602 on the log scale?
g What log scale value would be used for a physical value of 2^3?
h If the physical value doubles, by how much does the log scale increase?
i What log scale value would be used for a physical value of 2^{10}?

Example 4

5 **a** Copy and complete these tables for $y = 2^x$ and $y = \log_2 x$.

x	−3	−2	−2	0	1	2	3
$y = 2^x$							

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$y = \log_2 x$							

b Plot the graphs of $y = 2^x$, $y = \log_2 x$ and also $y = x$ on the same axes with $-4 \leqslant x \leqslant 4$ and $-4 \leqslant y \leqslant 4$.
c State two features of $y = \log_2 x$ which show that it is the inverse function of $y = 2^x$.
d For each graph, state the axes intercepts, if any.
e For each graph, state the equation of its asymptote.
f State any limitations for the input values for $y = 2^x$ and $y = \log_2 x$.
g State any limitations for the output values for $y = 2^x$ and $y = \log_2 x$.

6 **a** Copy and complete these tables for $y = 10^x$ and $y = \log_{10} x$.

x	−3	−2	−2	0	1	2	3
$y = 10^x$							

x	0.001	0.01	0.1	1	10	100	1000
$y = \log_{10} x$							

b Graph $y = 10^x$, $y = \log_{10} x$ and also $y = x$ on the same axes with $-2 \leqslant x \leqslant 10$ and $-2 \leqslant y \leqslant 10$.
c For each graph, state the axes intercepts, if any.
d The points (1, 10) and (0, 1) are on the graph $y = 10^x$. State the mirror image of each point when reflected in the line $y = x$ and show that they are on the graph of $y = \log_{10} x$.
e State the equations of any asymptotes for each function: $y = 10^x$ and $y = \log_{10} x$.
f State any limitations for the input values for each function: $y = 10^x$ and $y = \log_{10} x$.
g State any limitations on the output values for each function: $y = 10^x$ and $y = \log_{10} x$.

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Example 5

7 For each of the log equations graphed below, complete the following.

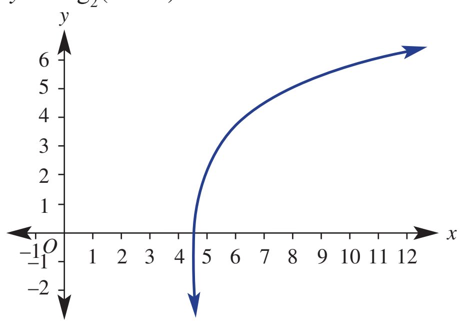
a State any limitations on the input values (i.e. x values).

b State the equation of the asymptote.

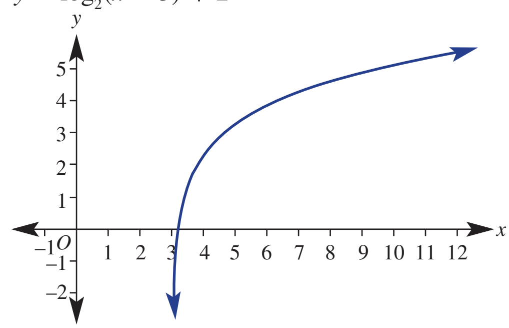
c State the horizontal and the vertical translations that have been applied to $y = \log_2 x$.

d Determine any axes intercepts. Round to 2 decimal places in parts **iii** and **iv**.

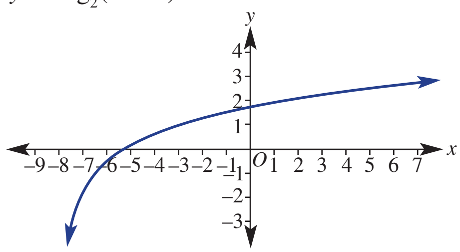
i $y = \log_2(x - 4) + 3$



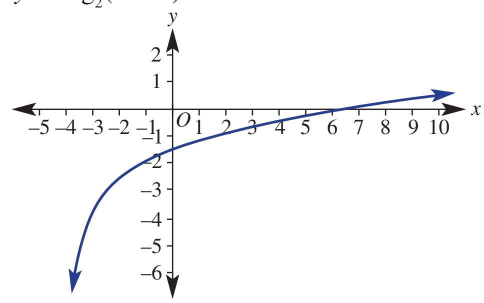
ii $y = \log_2(x - 3) + 2$



iii $y = \log_2(x + 8) - 1.2$



iv $y = \log_2(x + 4) - 3.5$



Example 6

8 Sketch a graph for each of the following log equations, labelling any axes intercepts and at least one other point. Also sketch and write the equation of the asymptote. Use a new set of axes for each graph.

a $y = \log_2(x - 3)$

b $y = \log_2 x - 3$

c $y = \log_2(x + 4)$

d $y = \log_2 x + 4$

e $y = \log_2(x - 4) + 3$

f $y = \log_5(x - 2) + 1$

g $y = \log_4(x + 3) - 2$

h $y = \log_3(x + 4) - 1$

PROBLEM-SOLVING AND REASONING	9, 10, 13	9, 10, 13, 14	9–15

9 Prove that $y = \log_a x$ and $y = a^x$ are inverse functions by showing mathematically that $\log_a a^x = x$ and $a^{\log_a x} = x$.

10 Earthquake intensity is measured by a logarithmic scale called the Richter scale. It is named after Charles Richter, who developed it in 1935 as a way of comparing different-sized earthquakes.

$M = \log_{10}\left(\dfrac{I}{I_0}\right)$, where M is the magnitude of the earthquake on the Richter scale, I is the earthquake intensity and I_0 is the intensity of the smallest earthquake that can be measured.

a Calculate the Richter scale magnitude of an earthquake with intensity $I = 10\,000I_0$.

b How much more intense than I_0 is an earthquake measuring 7 on the Richter scale?

c Rearrange the equation to make I the subject.

d Show mathematically that an earthquake of magnitude 7 is 1000 times more intense than an earthquake of magnitude 4.

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e Show mathematically that when one earthquake is double the intensity of another, their difference in magnitude on the Richter scale is 0.301.

f How much stronger in intensity was the 2011 East Japan earthquake, of magnitude 9, compared with the 2011 Christchurch, New Zealand earthquake, of magnitude 6.3?

g The most damaging Australian earthquake was in 1989, Newcastle, NSW, measuring 5.6. It killed 13 people, injured 160 and did $4 billion worth of damage. The strongest onshore Australian earthquake was in 1941, Meeberrie, WA, measuring 7.2 but caused only minor damage. How much more intense was the energy of the Meeberrie earthquake and why did it cause so much less damage than the Newcastle earthquake?

Rescue workers hurry to find survivors amongst the rubble of the Kent Hotel in the aftermath of the 1989 Newcastle earthquake.

11 A logarithmic scale is used to measure the loudness of sound with units in decibels. $\text{dB} = 10\log_{10}\left(\frac{I}{I_0}\right)$, where dB is in decibels, I is the intensity of the sound and I_0 is the threshold of sound intensity detected by the human ear.

a What is the decibel rating of sound at the threshold of hearing (i.e. $I = I_0$)?

b Find the decibel rating of normal conversation, which is 1 million times louder than the threshold of human hearing.

c Calculate how many times louder a hair dryer is at 80 dB compared with the threshold of human hearing.

d How much greater is the noise intensity of an iPod playing at a peak volume of 115 dB compared with a half volume level of 85 dB?

e By what factor does the noise intensity increase for each 3 dB increase? (Round your answer to the nearest whole number.)

f The human ear can withstand 85 dB for 8 hours before permanent hearing damage occurs. For every 3 dB increase over 85 dB the safe exposure time is halved. Calculate the safe exposure time to these noises: Band practise 91 dB; belt sander 94 dB; hand drill 97 dB; chainsaw 100 dB; rock concert 112 dB; iPod earphones at peak volume 115 dB; jet engine at take-off 140 dB.

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12 The pH of a solution is a measure of its acidity or alkalinity. Pure water has a pH of 7 and is neutral, whereas acids have pH < 7 and alkaline (basic) solutions have pH > 7. $\text{pH} = -\log_{10} \rightleftharpoons H^+]$, where $[H^+]$ is the concentration of hydrogen ions, in moles/litre.

a Calculate the pH of the following liquids and state whether they are acidic or alkaline. Round to 1 decimal place where necessary.

i lemon juice $[H^+] = 3.9 \times 10^{-3}$ moles/litre

ii beer $[H^+] = 1 \times 10^{-4}$ moles/litre

iii blood $[H^+] = 4 \times 10^{-8}$ moles/litre

iv hand soap $[H^+] = 1 \times 10^{-10}$ moles/litre

b Rearrange the pH equation, making $[H^+]$ the subject.

c By what factor has the hydrogen ion concentration increased in black coffee (pH = 5) compared with pure water (pH = 7)?

d By what factor does the hydrogen ion concentration change when the pH increases by 1?

e The pH of a cola drink is 2.525. If a soft drink has half the acidity of cola, what is its $[H^+]$ and pH?

f If the ocean absorbs enough CO_2 to cause the pH of sea water to change from 8.1 to 7.9, what percentage increase in acidity (i.e.$[H^+]$) would this cause? Round your answer to the nearest whole number.

13 **a** Using technology, graph each pair of functions on the same axes.

i $y = 2^x$, $y = \log_2 x$

ii $y = 5^x$, $y = \log_5 x$

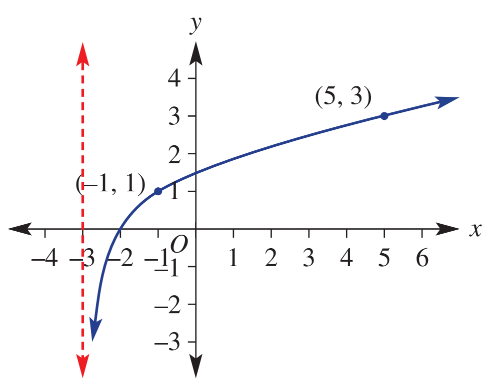
iii $y = 12^x$, $y = \log_{12} x$

b Compare logarithmic graphs to exponential graphs; that is, state any similarities between the two graph types.

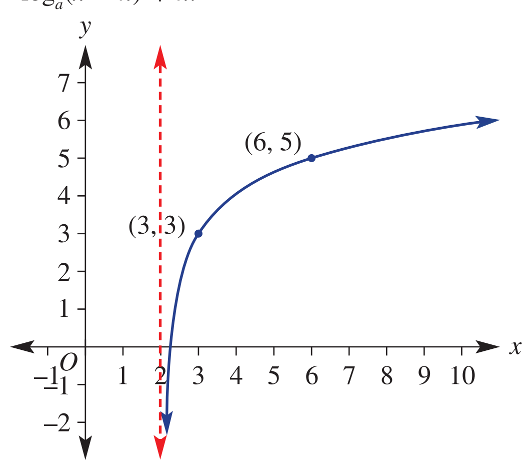
c Contrast logarithmic graphs to exponential graphs; that is, state any differences between the two graph types.

14 Find the rule for each logarithmic graph in the form $y = \log_a(x - h) + k$.

a



b



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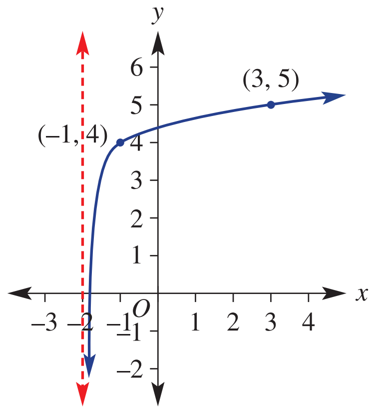
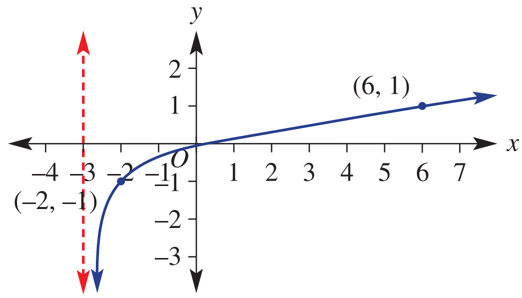
c



d



15 Describe the effects of increasing the value of the base a on the graph of $y = \log_a x$ (where $a > 1$). Explain the variations when a is increased for the regions: $0 < x < 1$, $x = 1$ and $x > 1$. Include examples in your explanation.

ENRICHMENT	–	–	16

Rules for inverse function

16 The rule for the inverse of a function can be found by interchanging x and y and then making y the subject of the equation.
For example, the rule for the inverse of $y = 3^{x-1} - 2$ can be found as follows.

$$\begin{aligned} x &= 3^{y-1} - 2 \\ x + 2 &= 3^{y-1} \\ y - 1 &= \log_3(x + 2) \\ y &= \log_3(x + 2) + 1 \end{aligned}$$

Follow this procedure to find the inverse of the following functions.

a $y = 5^{x-2} - 4$
b $y = 4^{x-3} + 1$
c $y = 5 + 2^{x+3}$
d $y = 7 + 3^{2x-4}$
e $y = \log_3(x - 1) + 4$
f $y = \log_2(x + 5) + 3$
g $y = 2 + \log_5(x - 8)$
h $y = 5 + \log_4(x + 6)$
i $y = 3 + \log_2(4x + 2)$
j $y = 8 + 5 \times 3^{2x-7}$

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10C Laws of logarithms

Interactive
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Walkthrough

Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

From the study of indices you will recall a number of index laws that can be used to manipulate expressions involving powers. Similarly, we have laws for logarithms and these can be derived using the index laws.

Recall this index law: $a^m \times a^n = a^{m+n}$

Now let $x = a^m$ and $y = a^n$ (1)

so $m = \log_a x$ and $n = \log_a y$ (2)

From equation (1): $xy = a^m \times a^n$
$= a^{m+n}$

So: $m + n = \log_a(xy)$

From equation (2): $m + n = \log_a x + \log_a y$

So: $\log_a(xy) = \log_a x + \log_a y$

This is a proof for one of the logarithmic laws. We will develop the others later in this section.

Let's start: Proving a logarithmic law

The introduction above presents a proof of the first logarithmic law, which is considered in this section. It uses the index law for multiplication.

- Now complete a similar proof for the second logarithmic law, $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$, using the index law for division.

Key ideas

- $\log_a(xy) = \log_a x + \log_a y$
  - This relates to the index law $a^m \times a^n = a^{m+n}$.
- $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
  - This relates to the index law $a^m \div a^n = a^{m-n}$.
- $\log_a(x^n) = n\log_a x$
  - This relates to the index law $(a^m)^n = a^{m \times n}$.
- Other properties of logarithms:
  - $\log_a 1 = 0$, using $a^0 = 1$
  - $\log_a a = 1$, using $a^1 = a$
  - $\log_a \frac{1}{x} = \log_a x^{-1} = -\log_a x$.

Example 7 Simplifying logarithmic expressions

Simplify the following.

a $\log_a 4 + \log_a 5$ **b** $\log_a 22 - \log_a 11$ **c** $3\log_a 2$

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SOLUTION

a $\log_a 4 + \log_a 5 = \log_a 20$

This uses the law $\log_a(xy) = \log_a x + \log_a y$.

b $\log_a 22 - \log_a 11 = \log_a 2$

This uses the law $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.

Note: $\log_a \frac{22}{11} = \log_a 2$

c $3\log_a 2 = \log_a 2^3$
$= \log_a 8$

This uses the law $n\log_a x = \log_a x^n$.

Example 8 Evaluating logarithmic expressions

Simplify and evaluate the following expressions.

a $\log_2 1$

b $\log_5 5$

c $\log_6 \frac{1}{36}$

d $\log_2 6 - \log_2 3$

SOLUTION

a $\log_2 1 = 0$

$2^0 = 1$

b $\log_5 5 = 1$

$5^1 = 5$

c $\log_6 \frac{1}{36} = \log_6 6^{-2}$
$= -2 \times \log_6 6$
$= -2 \times 1$
$= -2$

Alternatively, use the rule $\log_a \frac{1}{x} = -\log_a x$.

So $\log_6 \frac{1}{36} = -\log_6 36$
$= -2$

d $\log_2 6 - \log_2 3 = \log_2 2$
$= 1$

$\log_2\left(\frac{6}{3}\right) = \log_2 2$ and $2^1 = 2$

Exercise 10C

UNDERSTANDING AND FLUENCY	1, 2–6(½)	3, 4–7(½)	4–7(½)

1 Copy and complete the rules for logarithms using the given pronumerals.

a $\log_b(xy) = \log_b x +$ ______

b $\log_b\left(\frac{x}{y}\right) =$ ______ $-$ ______

c $\log_a b^m = m \times$ ______

d $\log_a a =$ ______

e $\log_c 1 =$ ______

f $\log_a \frac{1}{b} =$ ______

2 Write the missing numbers.

a $\log_2 \square = 1$

b $\log_3 \square = 0$

c $\log_{\square} 32 = 5$

d $\log_a 2 + \log_a \square = \log_a 8$

e $\log_a 36 - \log_a \square = \log_a 3$

f $\square \log_a 5 = \log_a 5^3$

g $\log_a 3^4 = \square \log_a 3$

h $\log_a \frac{1}{7} = \log_a 7^{\square}$

i $\square \log_a 3 = \log_a \frac{1}{3}$

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3 Evaluate:

a $\log_{10} 100$ **b** $\log_2 32$ **c** $\log_3 27$
d $-2\log_5 25$ **e** $4\log_{10} 1000$ **f** $-6\log_5 1$

Example 7a

4 Simplify:

a $\log_a 3 + \log_a 2$ **b** $\log_a 5 + \log_a 3$ **c** $\log_a 7 + \log_a 4$
d $\log_b 6 + \log_b 3$ **e** $\log_b 15 + \log_b 1$ **f** $\log_b 1 + \log_b 17$

Example 7b

5 Simplify:

a $\log_a 10 - \log_a 5$ **b** $\log_a 36 - \log_a 12$ **c** $\log_a 100 - \log_a 10$
d $\log_b 28 - \log_b 14$ **e** $\log_b 3 - \log_b 2$ **f** $\log_b 7 - \log_b 5$

Example 7c

6 Simplify:

a $2\log_a 3$ **b** $2\log_a 5$ **c** $3\log_a 3$
d $4\log_a 2$ **e** $5\log_a 2$ **f** $3\log_a 10$

Example 8a, b

7 Evaluate:

a $\log_3 1$ **b** $\log_7 1$ **c** $\log_x 1$
d $\log_4 4$ **e** $\log_{18} 18$ **f** $\log_a a$
g $5\log_2 1$ **h** $3\log_4 4$ **i** $\frac{1}{3}\log_7 7$
j $\frac{2}{3}\log_{10} 10$ **k** $\frac{\log_{15} 225}{2}$ **l** $\frac{\log_3 243}{10}$

PROBLEM-SOLVING AND REASONING	8–9(½), 11	8–9(½), 11, 12	8–10(½), 11–13

Example 8c

8 Simplify and evaluate.

a $\log_2 \frac{1}{4}$ **b** $\log_3 \frac{1}{27}$ **c** $\log_4 \frac{1}{64}$
d $\log_5 \frac{1}{5}$ **e** $\log_{10} \frac{1}{100}$ **f** $\log_{10} \frac{1}{100\,000}$

Example 8d

9 Simplify and evaluate.

a $\log_2 10 - \log_2 5$ **b** $\log_3 30 - \log_3 10$ **c** $\log_4 128 - \log_4 2$
d $\log_4 8 + \log_4 2$ **e** $\log_8 16 + \log_8 4$ **f** $\log_{10} 50 + \log_{10} 2$

10 Simplify using a combination of logarithmic laws.

a $2\log_3 2 + \log_3 5$ **b** $4\log_{10} 2 + \log_{10} 3$
c $3\log_{10} 2 - \log_{10} 4$ **d** $5\log_7 2 - \log_7 16$
e $\frac{1}{2}\log_3 4 + 2\log_3 2$ **f** $\log_5 3 - \frac{1}{2}\log_5 9$
g $\frac{1}{3}\log_2 27 - \frac{1}{3}\log_2 64$ **h** $\frac{1}{4}\log_5 16 + \frac{1}{5}\log_5 243$

11 Recall that $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[3]{x} = x^{\frac{1}{3}}$ and, in general, $\sqrt[n]{x} = x^{\frac{1}{n}}$. Use this to simplify the following.

a $\log_2 \sqrt{8}$ **b** $\log_2 \sqrt{32}$ **c** $\log_2 \sqrt[3]{16}$
d $\log_{10} \sqrt{1000}$ **e** $\log_7 \sqrt[3]{7}$ **f** $\log_5 \sqrt[5]{625}$

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12 Prove that:

a $\log_a \frac{1}{x} = -\log_a x$ using the logarithm law for subtraction

b $\log_a \frac{1}{x} = -\log_a x$ by first rewriting $\frac{1}{x}$ in index form

13 Prove that $\log_a \sqrt[n]{x} = \frac{\log_a x}{n}$.

ENRICHMENT	–	–	14

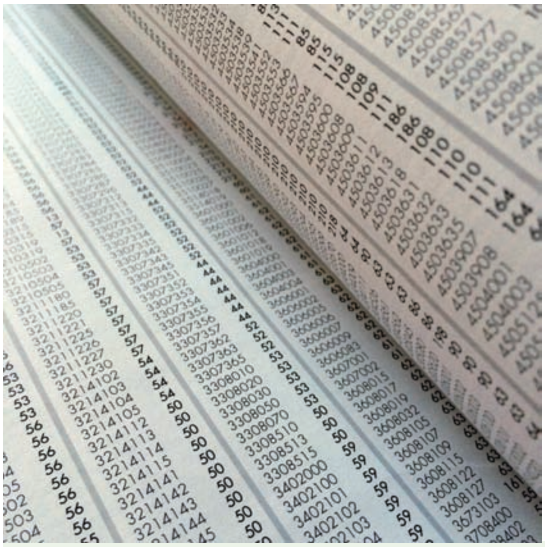
Proving the laws for logarithms

14 Read the proof for the logarithmic law for multiplication in the introduction and then complete the following tasks.

a Complete a proof giving all reasons for the law $\log_a(xy) = \log_a x + \log_a y$.

b Complete a proof for the law $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.

c Complete a proof for the law $\log_a x^n = n\log_a x$.



Before the invention of the electronic calculator, multiplication and division of numbers with many digits was done with tables of logarithms or slide rules with logarithmic scales.

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10D Solving equations using logarithms

Interactive
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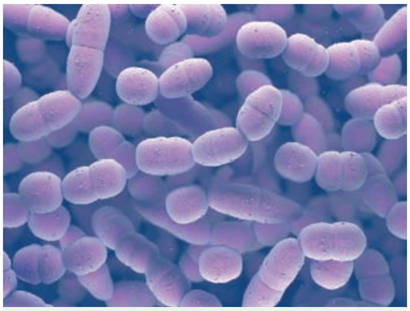
Walkthrough

When solving a simple exponential equation like $2^x = 16$ we know that the solution is $x = 4$ because $2^4 = 16$. Solving $2^x = 10$, however, is much more difficult and requires the use of logarithms. Depending on what calculator functions you have, one of two different methods can be chosen. These methods can be used to solve many types of problems in science and finance.



The growth of bacteria can be modelled by exponential equations.

Stage
5.3#
5.3
5.3§
5.2
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5.1
4

Let's start: Trial and error versus logarithms

Consider the equation $10^x = 20$.

- First, use a calculator and trial and error to find a value of x (correct to 3 decimal places) that satisfies the equation.
- Now write $10^x = 20$ in logarithmic form and use the log function on your calculator to find the value of x.
- Check the accuracy of your value of x obtained by trial and error.

Key ideas

- Solving for x if $a^x = y$
  - Using the given base: $x = \log_a y$
  - Using base 10:

$$\begin{aligned} a^x &= y \\ \log_{10} a^x &= \log_{10} y && \text{(taking } \log_{10} \text{ of both sides)} \\ x\log_{10} a &= \log_{10} y && \text{(using the law } \log_a(x^n) = n\log_a x\text{)} \\ x &= \frac{\log_{10} y}{\log_{10} a} && \text{(dividing by } \log_{10} a\text{)} \end{aligned}$$

- Most calculators can evaluate using log base 10, but only some calculators can work with any base.

Example 9 Solving using the given base

Solve the following using the given base. Round your answer to 3 decimal places.

a $2^x = 7$ **b** $50 \times 1.1^x = 100$

SOLUTION

a $2^x = 7$

$x = \log_2 7$

$= 2.807$ (to 3 decimal places)

EXPLANATION

If $a^x = y$ then $x = \log_a y$.
This method can be used on calculators that have a log function $\log_a y$, where both a and y can be entered.

SOLUTION

b $50 \times 1.1^x = 100$

$1.1^x = 2$

$x = \log_{1.1} 2$

$= 7.273$ (to 3 decimal places)

EXPLANATION

Divide both sides by 50.
Write in logarithmic form, then use a calculator for the approximation.

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Example 10 Solving using base 10

Solve using base 10 and evaluate, correct to 3 decimal places.

a $3^x = 5$

b $1000 \times 0.93^x = 100$

SOLUTION	EXPLANATION
a $3^x = 5$	
$\log_{10} 3^x = \log_{10} 5$	Take $\log_{10}$ of both sides.
$x \log_{10} 3 = \log_{10} 5$	Use the law $\log_a x^n = n \log_a x$.
$x = \dfrac{\log_{10} 5}{\log_{10} 3}$	Divide by $\log_{10} 3$.
$= 1.465$ (to 3 decimal places)	Use the log function on a calculator.
b $1000 \times 0.93^x = 100$	
$0.93^x = 0.1$	Divide by 1000.
$\log_{10} 0.93^x = \log_{10} 0.1$	Take $\log_{10}$ of both sides.
$x \log_{10} 0.93 = \log_{10} 0.1$	Use the law $\log_a x^n = n \log_a x$.
$x = \dfrac{\log_{10} 0.1}{\log_{10} 0.93}$	Divide by $\log_{10} 0.93$.
$= 31.729$ (to 3 decimal places)	Use the log function on a calculator.

Exercise 10D

UNDERSTANDING AND FLUENCY	1–3, 4–5(½)	2, 3–6(½)	4–6(½)

1 Write the logarithmic form of these equations.

a $2^3 = 8$ b $5^2 = 25$ c $4^{\frac{1}{2}} = 2$

d $3^x = 10$ e $7^x = 2$ f $1.1^x = 7$

2 State the missing number.

a $5^{\square} = 125$ b $10^{\square} = 10\,000$ c $8^{\square} = 64$

d $\log_2 \square = 3$ e $\log_4 \square = \dfrac{1}{2}$ f $\log_{10} \square = -1$

3 Use a calculator to evaluate the following, correct to 3 decimal places.

a $\log_{10} 7$ b $\log_{10} 0.6$ c $\log_{10} \dfrac{3}{4}$

d $\dfrac{\log_{10} 12}{\log_{10} 7}$ e $\dfrac{\log_{10} 1.3}{\log_{10} 1.4}$ f $\dfrac{\log_{10} 37}{\log_{10} 56}$

Example 9a

4 Solve the following using the given base and round to 3 decimal places where necessary.

a $3^x = 5$ b $2^x = 11$ c $5^x = 13$

d $1.2^x = 3.5$ e $2.9^x = 3.5$ f $0.2^x = 0.04$

Example 9b

5 Solve the following using the given base and round to 3 decimal places where necessary.

a $10 \times 2^x = 20$ b $25 \times 3^x = 75$ c $4 \times 1.5^x = 20$

d $3.8 \times 1.7^x = 9.5$ e $300 \times 0.9^x = 150$ f $7.3 \times 0.4^x = 1.8$

Example 10

6 Solve using base 10 and evaluate, correct to 3 decimal places.

a $2^x = 6$ b $3^x = 8$ c $5^x = 7$

d $11^x = 15$ e $1.8^x = 2.5$ f $0.9^x = 0.5$

g $10 \times 2^x = 100$ h $7 \times 3^x = 28$ i $130 \times 7^x = 260$

j $4 \times 1.5^x = 20$ k $100 \times 0.8^x = 50$ l $30 \times 0.7^x = 20$

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PROBLEM-SOLVING AND REASONING	7, 8, 10	7, 8, 10, 11	8, 9, 11, 12

7 The rule modelling a population (P) of mosquitoes is given by $P = 8^t$, where t is measured in days. Find the number of days, correct to 3 decimal places where necessary, required for the population to reach:

a 64 b 200 c 1000

8 An investment of \$10 000 is expected to grow by 5% p.a. The balance \$$A$ is given by the rule $A = 10\,000 \times 1.05^n$, where n is the number of years. Find the time (to 2 decimal places) for the investment to grow to:

a \$20 000 b \$32 000 c \$100 000

9 50 kg of a radioactive isotope in a set of spent nuclear fuel rods is decaying at a rate of 1% per year. The mass of the isotope (m kg) is therefore given by $m = 50 \times 0.99^n$, where n is the number of years. Find the time (to 2 decimal places) when the mass of the isotope reduces to:

a 45 kg b 40 kg c 20 kg

10 The value of a bank balance increases by 10% per year. The initial amount is \$2000.
a Write a rule connecting the balance \$$A$ with the time (n years).
b Find the time, correct to the nearest year, when the balance is double the original amount.

11 The value of a Ferrari is expected to reduce by 8% per year. The original cost is \$300 000.
a Find a rule linking the value of the Ferrari (\$$F$) and the time ($n$ years).
b Find the time it takes for the value of the Ferrari to reduce to \$150 000. Round your answer to 1 decimal place.

12 The half-life of a substance is the time it takes for the substance to reduce to half its original mass. Round your answers to the nearest year.
a Find the half-life of a 10 kg rock if its mass reduces by 1% per year.
b Find the half-life of a 20 gram crystal if its mass reduces by 0.05% per year.

ENRICHMENT	–	–	13

Change of base formula

13 If $a^x = y$ then we can write $x = \log_a y$. Alternatively, if $a^x = y$ we can find the logarithm of both sides, as shown here.

$$\begin{aligned} a^x &= y \\ \log_b a^x &= \log_b y \\ x\log_b a &= \log_b y \\ x &= \frac{\log_b y}{\log_b a} \\ \therefore \log_a y &= \frac{\log_b y}{\log_b a} \end{aligned}$$

This is the change of base formula.

a Use the change of base formula to write the following with base 10.
i $\log_2 7$ ii $\log_3 16$ iii $\log_5 1.3$

b Change to log base 10 and simplify.
i $\log_5 10$ ii $\log_2 1000$ iii $\log_3 0.1$

c Make x the subject and then change to base 10. Round your answer to 3 decimal places.
i $3^x = 6$ ii $9^x = 13$ iii $2 \times 1.3^x = 1.9$

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10E Polynomials

Interactive
Widgets
HOTsheets
Walkthrough

We are familiar with linear expressions such as $3x - 1$ and $4 + \frac{x}{2}$ and with quadratic expressions such as $x^2 - 3$ and $-4x^2 + 2x - 4$. These expressions are in fact part of a larger group called polynomials, which are sums of powers of a variable using whole number powers $\{0, 1, 2, \ldots\}$. For example, $2x^3 - 3x^2 + 4$ is a cubic polynomial and $1 - 4x^3 + 3x^7$ is a polynomial of degree 7. The study of polynomials opens up many ideas in the analysis of functions and graphing that are studied in many senior mathematics courses.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Is it a polynomial?

A polynomial is an expression that includes sums of powers of x with whole number powers $\{0, 1, 2, \ldots\}$. Decide, with reasons, if the following are polynomials.

- $5 + 2x + x^2$
- $\sqrt{x} + x^2$
- $\frac{2}{x} + 3$
- $4x^4 - x^2 - 6$
- $4x^{\frac{1}{3}} + 2x^2 + 1$
- 5

Key ideas

- A **polynomial** is an expression of the form $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0 x^0$, where:
  - n is a whole number $\{0, 1, 2, \ldots\}$
  - $a_n, a_{n-1}, \ldots, a_0$ are **coefficients**
  - $a_0 x^0 = a_0$ is the **constant term**
  - $a_n x^n$ is the **leading term**
- Naming polynomials
  Polynomials are named by the highest power of x. This is called the **degree** of the polynomial.
  - **constant** For example: 2
  - **linear** For example: $3x - 7$
  - **quadratic** For example: $2x^2 - 4x + 11$
  - **cubic** For example: $-4x^3 + 6x^2 - x + 3$
  - **quartic** For example: $\frac{1}{2}x^4 - x^4 - x^2 - 2$
  - **of degree 8** For example: $3x^8 - 4x^5 + x - 3$
- **Function notation** is used for polynomials.
  - A polynomial in x can be called $P(x)$.
    For example, the curve $y = 2x^3 - x$ is a cubic polynomial that can be written as $P(x) = 2x^3 - x$.
  - $P(k)$ is the value of the polynomial at $x = k$.
    For example: If $P(x) = 2x^3 - x$, then:

$$\begin{aligned} P(3) &= 2(3)^3 - (3) \\ &= 51 \end{aligned} \quad \text{and} \quad \begin{aligned} P(-1) &= 2(-1)^3 - (-1) \\ &= -2 + 1 \\ &= -1 \end{aligned}$$

$\therefore$ The point $(3, 51)$ lies on the curve.

$\therefore$ The point $(-1, -1)$ lies on the curve.

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Example 11 Determining if an expression is a polynomial

Determine if the following expressions are polynomials.

a $4x^2 - 1 + 7x^4$

b $2x^2 - \sqrt{x} + \dfrac{2}{x}$

SOLUTION

a yes

b no

EXPLANATION

a Powers of x are whole numbers $\{0, 1, 2, \ldots\}$.

b $2x^2 - \sqrt{x} + \dfrac{2}{x} = 2x^2 - x^{\frac{1}{2}} + 2x^{-1}$

Powers include $\dfrac{1}{2}$ and -1, which are not allowed.

Example 12 Evaluating polynomials

If $P(x) = x^3 - 3x^2 - x + 2$, find:

a $P(2)$

b $P(-3)$

SOLUTION

a $P(2) = (2)^3 - 3(2)^2 - (2) + 2$
$= 8 - 12 - 2 + 2$
$= -4$

b $P(-3) = (-3)^3 - 3(-3)^2 - (-3) + 2$
$= -27 - 27 + 3 + 2$
$= -49$

EXPLANATION

a Substitute $x = 2$ and evaluate.

b Substitute $x = -3$ and note that $(-3)^3 = -27$ and $(-3)^2 = 9$.

Exercise 10E

UNDERSTANDING AND FLUENCY	1, 2, 3–4(½), 5, 6	2, 3–4(½), 5–7	4(½), 5–7

1 A polynomial expression is given by $3x^4 - 2x^3 + x^2 - x + 2$.

a How many terms does the polynomial have?

b State the coefficient of:

i x^4 ii x^3 iii x^2 iv x

c What is the value of the constant term?

2 Determine if these polynomials are constant, linear, quadratic, cubic or quartic.

a $2x - 5$

b $x^2 - 3$

c $x^4 + 2x^3 + 1$

d $1 + x + 3x^2$

e 6

f $3 - \dfrac{2}{5}x$

g -2

h $1 - x^4$

i $4x - x^3 + x^2$

3 State the coefficient of x^2 in each of the following.

a $2x^3 + 4x^2 - 2x + 1$

b $5x^4 + 3x^2 - 1$

c $x^4 - 2x^2 - 2$

d $-7x^5 + x^3 - 2x^2 + 8$

e $-3x^6 + 2x^4 - 9x^2 + 1$

f $-3x^5 + 2x^2 - 3x^3 + 10$

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Example 11

4 Determine if the following are polynomials.

a $3x^3 + x^2 - x + 3$ **b** $2x^4 - x^2 - 4$ **c** $\frac{2}{x} - \frac{3}{x} + 2$

d $\frac{7}{x^2} - \frac{1}{x} + 2$ **e** $x^4 - x^3 + \frac{2}{x^3}$ **f** $4 - 7x^8$

g $\sqrt{x} + 2 - x^2$ **h** $\sqrt[4]{x} + \sqrt[3]{x} + \sqrt{x}$ **i** $x^3 + \frac{1}{\sqrt{x}}$

5 Evaluate the quadratic polynomial $x^2 - x + 2$, using:

a $x = 4$ **b** $x = 10$ **c** $x = -2$ **d** $x = -1$

Example 12

6 If $P(x) = x^3 - 3x^2 - 2x + 3$, find:

a $P(2)$ **b** $P(4)$ **c** $P(-1)$ **d** $P(-3)$

7 If $P(x) = 2x^4 - 3x^3 + 5x - 4$, find:

a $P(1)$ **b** $P(3)$ **c** $P(-1)$ **d** $P(-2)$

PROBLEM-SOLVING AND REASONING	8, 9, 12	8-10, 12, 13(½)	9-11, 13–14(½)

8 If $P(x) = x^3 - x^2$ and $Q(x) = 4 - 3x$, find:

a $P(1) + Q(2)$ **b** $P(3) + Q(-1)$ **c** $P(-2) - Q(-2)$

d $Q(1) - P(3)$ **e** $(P(2))^2 + (Q(1))^2$ **f** $(P(-1))^3 - (Q(-1))^3$

9 Find the coefficient of x^2 in these polynomials.

a $P(x) = \frac{4 - 2x^2}{4}$

b $P(x) = \frac{x^3 + 7x^2 + x - 3}{-7}$

c $P(x) = \frac{x^3 + 4x^2}{-8}$

10 Evaluate $P(-2)$ for these polynomials.

a $P(x) = (x + 2)^2$

b $P(x) = (x - 2)(x + 3)(x + 1)$

c $P(x) = x^2(x + 5)(x - 7)$

11 The height (P metres) of a roller coaster track above a platform is given by $P(x) = x^3 - 12x^2 + 35x$, where x metres is the horizontal distance from the beginning of the platform.

a Find the height of the track using:

i $x = 2$

ii $x = 3$

iii $x = 7$

b Does the track height ever fall below the level of the platform? If so, find a value of x for which this occurs.

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12 **a** What is the maximum number of terms in a polynomial of degree 7?
b What is the maximum number of terms in a polynomial of degree n?
c What is the minimum number of terms in a polynomial of degree 5?
d What is the minimum number of terms in a polynomial of degree n?

13 If $P(x) = x^3 - x^2 - 2x$, evaluate and simplify these without the use of a calculator.

a $P\left(\frac{1}{2}\right)$
b $P\left(\frac{1}{3}\right)$
c $P\left(-\frac{1}{2}\right)$
d $P\left(-\frac{1}{4}\right)$
e $P\left(-\frac{2}{3}\right)$
f $P\left(\frac{4}{5}\right)$
g $P\left(-\frac{1}{2}\right) + P\left(\frac{1}{2}\right)$
h $P\left(-\frac{3}{4}\right) + P\left(\frac{3}{4}\right)$

14 If $P(x) = 2x^3 - x^2 - 5x - 1$, find the following and simplify where possible.

a $P(k)$ **b** $P(b)$ **c** $P(2a)$ **d** $P(-a)$
e $P(-2a)$ **f** $P(-3k)$ **g** $P(ab)$ **h** $P(-ab)$

ENRICHMENT	–	–	15

Finding unknown coefficients

15 If $P(x) = x^3 - 2x^2 + bx + 3$ and $P(1) = 4$, we can find the value of b as follows.

$$\begin{aligned} P(1) &= 4 \\ (1)^3 - 2(1)^2 + b(1) + 3 &= 4 \\ 2 + b &= 4 \\ b &= 2 \end{aligned}$$

a Use this method to find the value of b if $P(x) = x^3 - 4x^2 + bx - 2$ and if:
i $P(1) = 5$
ii $P(2) = -6$
iii $P(-1) = -8$
iv $P(-2) = 0$
v $P(-1) = 2$
vi $P(-3) = -11$

b If $P(x) = x^4 - 3x^3 + kx^2 - x + 2$, find k if:
i $P(1) = 2$
ii $P(-2) = 0$
iii $P(-1) = -15$

c If $P(x) = x^3 + ax^2 + bx - 3$ and $P(1) = -1$ and $P(-2) = -1$, find the values of a and b.

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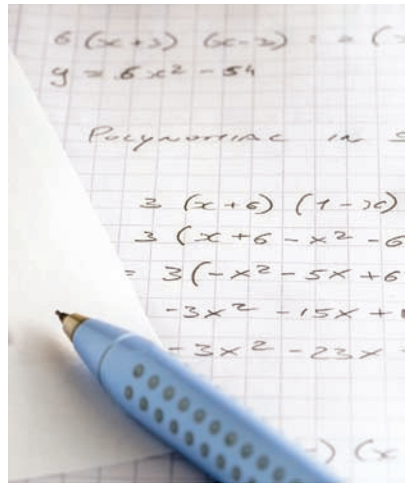
10F Expanding and simplifying polynomials

Interactive
Widgets
HOTsheets
Walkthrough

From your work on quadratics, you will remember how to use the distributive law to expand brackets. For example, $(2x - 1)(x + 5)$ expands to $2x^2 + 10x - x - 5$, and after collecting like terms this simplifies to $2x^2 + 9x - 5$. In a similar way we can expand the product of two or more polynomials of any degree. To do this we also multiply every term in one polynomial with every term in the next polynomial.



Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: The product of two quadratics

The equation $(x^2 - x + 3)(2x^2 + x - 1) = 2x^4 - x^3 + 4x^2 + 4x - 3$ is written on the board.

- Is the equation true for $x = 1$?
- Is the equation true for $x = -2$?
- How can you prove the equation to be true for all values of x?

Key ideas

- Expand products of polynomials by multiplying each term in one polynomial by each term in the next polynomial.
- Simplify by collecting like terms.

Example 13 Expanding polynomials

Expand and simplify.

a $x^3(x - 4x^2)$

b $(x^2 + 1)(x^3 - x + 1)$

SOLUTION

a $x^3(x - 4x^2) = x^4 - 4x^5$

b $(x^2 + 1)(x^3 - x + 1)$
$= x^2(x^3 - x + 1) + 1(x^3 - x + 1)$
$= x^5 - x^3 + x^2 + x^3 - x + 1$
$= x^5 + x^2 - x + 1$

EXPLANATION

$x^3 \times x^1 = x^4$ and $x^3 \times (-4x^2) = -4x^5$

$(x^2 + 1)(x^3 - x + 1)$

$-x^3$ cancels with x^3.

Example 14 Expanding $P(x) \times Q(x)$

If $P(x) = x^2 + x - 1$ and $Q(x) = x^3 + 2x + 3$, expand and simplify the following.

a $P(x) \times Q(x)$ **b** $(Q(x))^2$

SOLUTION

a $P(x) \times Q(x)$
$= (x^2 + x - 1)(x^3 + 2x + 3)$
$= x^2(x^3 + 2x + 3) + x(x^3 + 2x + 3) - 1(x^3 + 2x + 3)$
$= x^5 + 2x^3 + 3x^2 + x^4 + 2x^2 + 3x - x^3 - 2x + 3$
$= x^5 + x^4 + x^3 + 5x^2 + x + 3$

EXPLANATION

Each term in the first polynomial is multiplied by each term in the second polynomial.

b $(Q(x))^2$
$= (x^3 + 2x + 3)^2$
$= (x^3 + 2x + 3)(x^3 + 2x + 3)$
$= x^3(x^3 + 2x + 3) + 2x(x^3 + 2x + 3) + 3(x^3 + 2x + 3)$
$= x^6 + 2x^4 + 3x^3 + 2x^4 + 4x^2 + 6x + 3x^3 + 6x + 9$
$= x^6 + 4x^4 + 6x^3 + 4x^2 + 12x + 9$

$(Q(x))^2 = Q(x) \times Q(x)$
Expand to gain nine terms, then collect and simplify.

Exercise 10F

UNDERSTANDING AND FLUENCY	1–3, 4–5(½), 6	3, 4–5(½), 6	4–5(½), 6, 7

1 Expand and simplify these quadratics.
- **a** $x(x + 2)$
- **b** $x(1 - 3x)$
- **c** $(x + 1)(x - 1)$
- **d** $(x - 5)(x + 11)$
- **e** $(2x - 5)(3x + 1)$
- **f** $(4x - 3)(2x - 5)$

2 Collect like terms to simplify.
- **a** $2x^4 - 3x^3 + x^2 - 1 - x^4 - 2x^3 + 3x^2 - 2$
- **b** $5x^6 + 2x^4 - x^2 + 5 - 5x^4 + x^3 + 8 - 6x^6$
- **c** $7x^8 - 1 + 2x^6 + x - 10x^8 - 3x^6 + 4 - 7x$

3 Use substitution to confirm that this equation is true for the given x values.
$(x^3 - x + 3)(x^2 + 2x - 1) = x^5 + 2x^4 - 2x^3 + x^2 + 7x - 3$

a $x = 1$ **b** $x = 0$ **c** $x = -2$

Example 13a

4 Expand and simplify.
- **a** $x^2(x - 3)$
- **b** $x^2(x^2 - 1)$
- **c** $2x^2(1 + 3x)$
- **d** $x^3(1 - x)$
- **e** $x^3(x^2 + 3x)$
- **f** $-3x^2(x^4 - x)$
- **g** $-2x^3(x^2 + x)$
- **h** $-x^2(x^5 - x^2)$
- **i** $-4x^3(x^4 - 2x^7)$

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Example 13b

5 Expand and simplify.

a $(x^2 + 1)(x^3 + 2)$ b $(x^2 - 1)(x^3 + x)$ c $(x^2 - x)(x^3 - 3x)$
d $(x^2 - 2)(x^3 + x - 2)$ e $(x^3 - x)(x^2 + 2x + 3)$ f $(x^3 - x^2)(x^2 - x + 4)$
g $(x^3 - x^2 - 1)(x^3 + x - 2)$ h $(x^3 - 5x^2 + 2)(x^3 - x + 1)$ i $(x^4 - x^2 + 1)(x^4 + x - 3)$

Example 14

6 If $P(x) = x^2 - 2x + 1$ and $Q(x) = x^3 + x - 1$, expand and simplify:

a $P(x) \times Q(x)$ b $(Q(x))^2$ c $(P(x))^2$

7 If $P(x) = x^3 + 2x^2 - x - 4$ and $Q(x) = x^2 + x - 2$, expand and simplify:

a $P(x) \times Q(x)$ b $(Q(x))^2$ c $(P(x))^2$

PROBLEM-SOLVING AND REASONING	8(½), 11	8(½), 9, 11, 12	8(½), 9, 10, 12, 13

8 If $P(x) = x^2 - 5x + 1$ and $Q(x) = x^3 + x$, expand and simplify:

a $P(x) + Q(x)$ b $Q(x) - P(x)$ c $5P(x) + 2Q(x)$
d $1 - P(x)Q(x)$ e $4 - (Q(x))^2$ f $(P(x))^2 - (Q(x))^2$

9 Find the square of $P(x)$ if $P(x) = (x^2 + x - 1)^2$.

10 Show that $(x^2 - x - 1)^2 - (x^2 - x + 1)^2 = 4x - 4x^2$.

11 If $P(x)$ and $Q(x)$ are polynomials, does $P(x)Q(x) = Q(x)P(x)$ for all values of x?

12 Give the degree of the polynomial $P(x) \times Q(x)$ when:

a $P(x)$ is quadratic and $Q(x)$ is linear
b $P(x)$ is quadratic and $Q(x)$ is cubic
c $P(x)$ is cubic and $Q(x)$ is quartic
d $P(x)$ is of degree 7 and $Q(x)$ is of degree 5

13 If $P(x)$ is of degree m and $Q(x)$ is of degree n and $m > n$, what is the highest possible degree of the following polynomials?

a $P(x) + Q(x)$
b $P(x) - Q(x)$
c $P(x) \times Q(x)$
d $(P(x))^2$
e $(P(x))^2 - Q(x)$
f $(Q(x))^3$

ENRICHMENT	–	–	14

Triple expansions

14 Expand and simplify.

a $x(x^2 + 1)(x - 1)$
b $x^3(x + 3)(x - 1)$
c $(x + 2)(x - 1)(x + 3)$
d $(x + 4)(2x - 1)(3x + 1)$
e $(5x - 2)(x - 2)(3x + 5)$
f $(x^2 + 1)(x^2 - 2)(x + 3)$

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10G Dividing polynomials

Stage: 5.3#, 5.3, 5.3§, 5.2, 5.2◊, 5.1, 4

Division of polynomials requires the use of the long division algorithm. You may have used this algorithm for the division of whole numbers in primary school.

Recall that 7 divided into 405 can be calculated in the following way.

	57	
	$7\overline{)405}$	7 into 4 does not go.
5×7	$\underline{35}$	7 into 40 gives 5 and $5 \times 7 = 35$.
	55	Then subtract $405 - 350$.
7×7	$\underline{49}$	7 into 55 gives 7 and $7 \times 7 = 49$.
	6	Subtract to give remainder 6.

Division of polynomials occurs often in the field of signal processing.

So $405 \div 7 = 57$ and 6 remainder. The 57 is called the quotient.

Another way to write this is $405 = 7 \times 57 + 6$.

We use this technique to divide polynomials.

Let's start: Recall long division

Use long division to find the quotient and remainder for the following.

- $832 \div 3$
- $2178 \div 7$

Key ideas

- We use the long division algorithm to divide polynomials.
- The result is not necessarily a polynomial.

  For example:

  $$\underbrace{\frac{\overbrace{x^3 - x^2 + x - 1}^{\text{dividend}}}{\underbrace{x + 2}_{\text{divisor}}}} = \underbrace{x^2 - 3x + 7}_{\text{quotient}} - \frac{\overbrace{15}^{\text{remainder}}}{x + 2}$$

  We can write this as:

  $$\underbrace{x^3 - x^2 + x - 1}_{\text{dividend}} = \underbrace{(x + 2)}_{\text{divisor}}\underbrace{(x^2 - 3x + 7)}_{\text{quotient}} - \underbrace{15}_{\text{remainder}}$$

  $$P(x) = A(x)Q(x) + R(x)$$

- The degree of the remainder $R(x)$ must be less than the degree of the divisor $A(x)$.

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Example 15 Dividing polynomials

a Divide $P(x) = x^3 + 2x^2 - x + 3$ by $(x - 2)$ and write in the form $P(x) = (x - 2)Q(x) + R$, where R is the remainder.

b Divide $P(x) = 2x^3 - x^2 + 3x - 1$ by $(x + 3)$ and write in the form $P(x) = (x + 3)Q(x) + R$.

SOLUTION

a

$$
\begin{array}{r r}
 & x^2 + 4x + 7 \\
x - 2\,\big) & \overline{x^3 + 2x^2 - x + 3} \\
x^2(x - 2) & \underline{x^3 - 2x^2 \qquad\qquad} \\
 & 4x^2 - x + 3 \\
4x(x - 2) & \underline{4x^2 - 8x \qquad} \\
 & 7x + 3 \\
7(x - 2) & \underline{7x - 14} \\
 & 17
\end{array}
$$

$\therefore x^3 + 2x^2 - x + 3 = (x - 2)(x^2 + 4x + 7) + 17$

EXPLANATION

First, divide x from $(x - 2)$ into the leading term (i.e. x^3). So divide x into x^3 to give x^2.

After subtraction, divide x into $4x^2$ to give $4x$.

After subtraction, divide x into $7x$ to give 7.

Subtract to give the remainder 17.

SOLUTION

b

$$
\begin{array}{r r}
 & 2x^2 - 7x + 24 \\
x + 3\,\big) & \overline{2x^3 - x^2 - 3x - 1} \\
2x^2(x + 3) & \underline{2x^3 + 6x^2 \qquad\qquad} \\
 & -7x^2 + 3x - 1 \\
-7x(x + 3) & \underline{-7x^2 - 21x \qquad} \\
 & 24x - 1 \\
24(x + 3) & \underline{24x + 72} \\
 & -73
\end{array}
$$

$\therefore 2x^3 - x^2 + 3x - 1 = (x + 3)(2x^2 - 7x + 24) - 73$

EXPLANATION

First, divide x from $(x + 3)$ into the leading term. So divide x into $2x^3$ to give $2x^2$.

After subtraction, divide x into $-7x^2$ to give $-7x$.

After subtraction, divide x into $24x$ to give 24.

Subtract to give the remainder -73.

Exercise 10G

UNDERSTANDING AND FLUENCY	1–4	2–4, 5(½)	3, 4, 5(½)

1 Use long division to find the remainder.

a $208 \div 9$ **b** $143 \div 7$ **c** $2184 \div 3$

2 Copy and complete the equation with the missing numbers.

a If $182 \div 3 = 60$ remainder 2, then $182 = \_\_ \times 60 + \_\_$.

b If $2184 \div 5 = 436$ remainder 4, then $2184 = \_\_ \times 436 + \_\_$.

c If $617 \div 7 = 88$ remainder 1, then $617 = 7 \times \_\_ + \_\_$.

Example 15 **3** Divide $P(x) = x^3 + x^2 - 2x + 3$ by $(x - 1)$ and write in the form $P(x) = (x - 1)Q(x) + R$, where R is the remainder.

4 Divide $P(x) = 3x^3 - x^2 + x + 2$ by $(x + 1)$ and write in the form $P(x) = (x + 1)Q(x) + R$.

5 For each of the following, express in this form:

Dividend = divisor × quotient + remainder (as in the examples)

a $(2x^3 - x^2 + 3x - 2) \div (x - 2)$ **b** $(2x^3 + 2x^2 - x - 3) \div (x + 2)$

c $(5x^3 - 2x^2 + 7x - 1) \div (x + 3)$ **d** $(-x^3 + x^2 - 10x + 4) \div (x - 4)$

e $(-2x^3 - 2x^2 - 5x + 7) \div (x + 4)$ **f** $(-5x^3 + 11x^2 - 2x - 20) \div (x - 3)$

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PROBLEM-SOLVING AND REASONING	6, 8	6–9	6, 7, 9–11

6 Divide and write in this form:
Dividend = divisor × quotient + remainder

a $(6x^4 - x^3 + 2x^2 - x + 2) \div (x - 3)$
b $(8x^5 - 2x^4 + 3x^3 - x^2 - 4x - 6) \div (x + 1)$

7 Divide the following and express in the usual form.

a $(x^3 - x + 1) \div (x + 2)$
b $(x^3 + x^2 - 3) \div (x - 1)$
c $(x^4 - 2) \div (x + 3)$
d $(x^4 - x^2) \div (x - 4)$

8 There are three values of k for which $P(x) = x^3 - 2x^2 - x + 2$ when divided by $(x - k)$ gives a remainder of zero. Find the three values of k.

9 Prove that $(6x^3 - 37x^2 + 32x + 15) \div (x - 5)$ leaves remainder 0.

10 Find the remainder when $P(x)$ is divided by $(2x - 1)$ given that:

a $P(x) = 2x^3 - x^2 + 4x + 2$
b $P(x) = -3x^3 + 2x^2 - 7x + 5$

11 Find the remainder when $P(x) = -3x^4 - x^3 - 2x^2 - x - 1$ is divided by these expressions.

a $x - 1$
b $2x + 3$
c $-3x - 2$

ENRICHMENT	–	–	12

When the remainder is not a constant

12 Divide the following and express in the form $P(x) = \text{divisor} \times Q(x) + R$, where R is a function of x.

a $(x^3 - x^2 + 3x + 2) \div (x^2 - 1)$
b $(2x^3 + x^2 - 5x - 1) \div (x^2 + 3)$
c $(5x^4 - x^2 + 2) \div (x^3 - 2)$

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10H Remainder theorem and factor theorem

Interactive

Widgets

HOTsheets

Walkthrough

Stage

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Using long division we can show, after dividing $(x-2)$ into $P(x) = x^3 - x^2 + 2x - 3$, that $P(x) = x^3 - x^2 + 2x - 3 = (x-2)(x^2 + x + 4) + 5$, where 5 is the remainder.

Using the right-hand side to evaluate $P(2)$, we have:

$$\begin{aligned} P(2) &= (2-2)(2^2 + 2 + 4) + 5 \\ &= 0 \times (2^2 + 2 + 4) + 5 \\ &= 0 + 5 \\ &= 5 \end{aligned}$$

This shows that the remainder when $P(x)$ is divided by $(x-2)$ is $P(2)$.

More generally, when $P(x)$ is divided by $(x-a)$ we obtain:

$P(x) = (x-a)Q(x) + R$

$$\begin{aligned} \text{So } P(a) &= 0 \times Q(x) + R \\ &= R \end{aligned}$$

So the remainder is $P(a)$ and this result is called the remainder theorem. This means that we can find the remainder when dividing $P(x)$ by $(x-a)$ simply by evaluating $P(a)$.

We also know that when factors are divided into a number there is zero remainder. So if $P(x)$ is divided by $(x-a)$ and the remainder $P(a)$ is zero, then $(x-a)$ is a factor of $P(x)$. This result is called the factor theorem.

Let's start: Which way is quicker?

A polynomial $P(x) = x^3 - 3x^2 + 6x - 4$ is divided by $(x-2)$.

- Show, using long division, that the remainder is 4.
- Find $P(2)$. What do you notice?
- Explain how you can find the remainder when $P(x)$ is divided by:
  - **a** $x - 3$
  - **b** $x - 5$
- Show that when $P(x)$ is divided by $(x+1)$ the remainder is -14.
- What would be the remainder when $P(x)$ is divided by $(x-1)$? What do you notice and what does this say about $(x-1)$ in relation to $P(x)$?

Key ideas

- **Remainder theorem**: When a polynomial $P(x)$ is divided by $(x-a)$ the remainder is $P(a)$.
  - When dividing by $(x-3)$ the remainder is $P(3)$.
  - When dividing by $(x+2)$ the remainder is $P(-2)$.
- **Factor theorem:** When $P(x)$ is divided by $(x-a)$ and the remainder is zero (i.e. $P(a) = 0$), then $(x-a)$ is a factor of $P(x)$.

$$P(x) = x^3 - 3x^2 - 3x + 10 = \underbrace{(x-2)}_{\text{factor}}\underbrace{(x^2 - x - 5)}_{\text{quotient}}$$

$P(2) = 0$

$(x-2)$ is a factor with zero remainder.

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Example 16 Using the remainder theorem

Find the remainder when $P(x) = x^3 - 5x^2 - x + 4$ is divided by:

a $x - 2$

b $x + 1$

SOLUTION

a $P(x) = x^3 - 5x^2 - x + 4$
$P(2) = 8 - 20 - 2 + 4$
$= -10$
The remainder is -10.

b $P(x) = x^3 - 5x^2 - x + 4$
$P(-1) = -1 - 5 + 1 + 4$
$= -1$
The remainder is -1.

EXPLANATION

For $(x - 2)$ substitute $x = 2$.
Using the remainder theorem, $P(2)$ gives the remainder.

For $(x + 1)$ substitute $x = -1$.
Note: $(-1)^3 = -1$, $(-1)^2 = 1$ and $-(-1) = +1$.

Example 17 Finding a linear factor

Determine whether each of the following are factors of $P(x) = x^3 + x^2 - 3x - 6$.

a $x + 1$

b $x - 2$

SOLUTION

a $P(-1) = -1 + 1 + 3 - 6$
$= -3$
$\therefore (x + 1)$ is not a factor.

b $P(2) = 8 + 4 - 6 - 6$
$= 0$
$\therefore (x - 2)$ is a factor.

EXPLANATION

If $(x + 1)$ is a factor of $P(x)$, then $P(-1) = 0$. But this is not true because the remainder is -3.

Substitute $x = 2$ to evaluate $P(2)$.
Since $P(2) = 0$, $(x - 2)$ is a factor of $P(x)$.

Example 18 Applying the remainder theorem

Consider the polynomial $P(x) = x^3 - x^2 + 2x + k$. For what value of k will $P(x) \div (x - 1)$ have remainder 5?

SOLUTION

Require $P(1) = 5$
$\therefore (1)^3 - (1)^2 + 2(1) + k = 5$
$2 + k = 5$
$k = 3$

EXPLANATION

The remainder is $P(1)$, which is 5.
Substitute $x = 1$ and solve for k.

Exercise 10H

UNDERSTANDING AND FLUENCY	1–3, 4–6(½)	3, 4–7(½)	4–7(½)

1 If $P(x) = 2x^3 - x^2 - x - 1$, find the value of the following.

a $P(1)$ b $P(3)$ c $P(-2)$ d $P(-4)$

2 What value of x do you substitute into $P(x)$ to find the remainder when a polynomial $P(x)$ is divided by:

a $x - 3$? b $x + 2$?

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3 What is the remainder when an expression is divided by one of its factors?

Example 16

4 Find the remainder when $P(x) = x^3 - 2x^2 + 7x - 3$ is divided by:

a $x - 1$ b $x - 2$ c $x - 3$ d $x - 4$

e $x + 4$ f $x + 2$ g $x + 1$ h $x + 3$

5 Find the remainder when $P(x) = x^4 - x^3 + 3x^2$ is divided by:

a $x - 1$ b $x - 2$ c $x + 2$ d $x + 1$

Example 17

6 Determine which of the following are factors of $P(x) = x^3 - 4x^2 + x + 6$.

a $x - 1$ b $x + 1$ c $x - 2$ d $x + 2$

e $x - 3$ f $x + 3$ g $x - 4$ h $x + 4$

7 Determine which of the following are factors of $P(x) = x^4 - 2x^3 - 25x^2 + 26x + 120$.

a $x - 2$ b $x + 2$ c $x + 3$ d $x - 3$

e $x - 4$ f $x + 4$ g $x - 5$ h $x + 5$

PROBLEM-SOLVING AND REASONING	8(½), 10	8–9(½), 10–12	8–9(½), 11–13

8 Use the factor theorem and trial and error to find a linear factor of these polynomials.

a $P(x) = x^3 + 2x^2 + 7x + 6$ b $P(x) = x^3 + 2x^2 - x - 2$

c $P(x) = x^3 + x^2 + x + 6$ d $P(x) = x^3 - 2x - 4$

9 Use the factor theorem to find all three linear factors of each of these polynomials.

a $P(x) = x^3 - 2x^2 - x + 2$ b $P(x) = x^3 - 2x^2 - 5x + 6$

c $P(x) = x^3 - 4x^2 + x + 6$ d $P(x) = x^3 - 2x^2 - 19x + 20$

Example 18

10 For what value of k will $(x^3 - 2x^2 + 5x + k) \div (x - 1)$ have the following remainders?

a 0 b 2 c -10 d 100

11 For what value of k will $(x^4 - 2x^3 + x^2 - x + k) \div (x + 2)$ have zero remainder?

12 Use this technique to find the value of k in these polynomials.

a $P(x) = x^3 + 2x^2 + kx - 4$ and when divided by $(x - 1)$ the remainder is 4.

b $P(x) = x^3 - x^2 + kx - 3$ and when divided by $(x + 1)$ the remainder is -6.

c $P(x) = 2x^3 + kx^2 + 3x - 4$ and when divided by $(x + 2)$ the remainder is -6.

d $P(x) = kx^3 + 7x^2 - x - 4$ and when divided by $(x - 2)$ the remainder is -2.

13 Find the value of k when:

a $(x + 2)$ is a factor of $x^3 - kx^2 - 2x - 4$ b $(x - 3)$ is a factor of $2x^3 + 2x^2 - kx - 3$

ENRICHMENT	–	–	14

Simultaneous coefficients

14 Use simultaneous equations and the given information to find the value of a and b in these cubics.

a $P(x) = x^3 + ax^2 + bx - 3$ and $P(1) = -1$ and $P(2) = 5$

b $P(x) = 2x^3 - ax^2 - bx - 1$ and $P(-1) = -10$ and $P(-2) = -37$

ISBN 978-1-108-46847-3

10I Factorising polynomials to find zeros

Interactive Widgets HOTsheets Walkthrough

Stage

5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

We know from our work with quadratics that the null factor law can be used to solve a quadratic that has been factorised.

For example: $x^2 - 3x - 40 = 0$

$(x - 8)(x + 5) = 0$

Using the null factor law:

$x - 8 = 0$ or $x + 5 = 0$

$x = 8$ or $x = -5$

We can also apply this method to polynomials.

If a polynomial is not in a factorised form, we need to use the remainder and factor theorems to help find its factors. Long division can also be used.

Let's start: Solving a cubic

Consider the cubic equation $P(x) = 0$, where $P(x) = x^3 + 6x^2 + 5x - 12$.

- Explain why $(x - 1)$ is a factor of $P(x)$.
- Use long division to find $P(x) \div (x - 1)$.
- Write $P(x)$ in the form $(x - 1)Q(x)$.
- Now complete the factorisation of $P(x)$.
- Show how the null factor law can be used to solve $P(x) = 0$. Why are there three solutions?

Key ideas

- A **polynomial equation** of the form $P(x) = 0$ can be solved by:
  - factorising $P(x)$
  - using the null factor law: If $a \times b \times c = 0$, then $a = 0$, $b = 0$ or $c = 0$.
- To factorise a polynomial, follow these steps:
  - Find one factor using the remainder and factor theorems. Start with $(x - 1)$ using $P(1)$ or $(x + 1)$ using $P(-1)$. If required, move to $(x - 2)$ or $(x + 2)$.
  - Use long division to find the quotient after dividing by the factor.
  - Factorise the quotient (if possible).
  - Continue until $P(x)$ is fully factorised.

Example 19 Using the null factor law

Solve for x.

a $(x - 1)(x + 2)(x + 5) = 0$

b $(2x - 3)(x + 7)(3x + 1) = 0$

SOLUTION

a $(x - 1)(x + 2)(x + 5) = 0$

$x - 1 = 0$ or $x + 2 = 0$ or $x + 5 = 0$

$x = 1$ $\quad x = -2$ $\quad x = -5$

EXPLANATION

Using the null factor law, if $a \times b \times c = 0$ then $a = 0$ or $b = 0$ or $c = 0$.

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b $(2x-3)(x+7)(3x+1)=0$

$2x-3=0$ or $x+7=0$ or $3x+1=0$

$2x=3$ $\quad x=-7$ $\quad 3x=-1$

$x=\frac{3}{2}$ $\quad x=-7$ $\quad x=-\frac{1}{3}$

Equate each factor to zero and solve for the three values of x.

Example 20 Factorising and solving

Solve $x^3+2x^2-5x-6=0$.

SOLUTION

Let $P(x)=x^3+2x^2-5x-6$.

$P(1)=1+2-5-6\neq 0$

$P(-1)=-1+2+5-6=0$

$\therefore x+1$ is a factor.

$$
\begin{array}{r l}
 & x^2+x-6 \\
x+1\,\big) & x^3+2x^2-5x-6 \\
x^2(x+1) & \underline{x^3+x^2\qquad\qquad} \\
 & x^2-5x-6 \\
x(x+1) & \underline{x^2+x\qquad} \\
 & -6x-6 \\
-6(x+1) & \underline{-6x-6} \\
 & 0
\end{array}
$$

$\therefore P(x)=(x+1)(x^2+x-6)$

$=(x+1)(x+3)(x-2)$

Solve $P(x)=0$:

$(x+1)(x+3)(x-2)=0$

$x+1=0$ or $x+3=0$ or $x-2=0$

$x=-1$ $\quad x=-3$ $\quad x=2$

EXPLANATION

Try to find a factor using the remainder and factor theorems. Start with $(x-1)$ using $P(1)$ or $(x+1)$ using $P(-1)$. If required, move to $(x-2)$ or $(x+2)$. $P(-1)=0$ so $(x+1)$ is a factor.

Divide $(x+1)$ into $P(x)$ to find the quotient using long division.

Note: The remainder is 0 as expected $(P(-1)=0)$. $P(x)=(x+1)Q(x)+R$ but $R=0$. x^2+x-6 factorises to $(x+3)(x-2)$. Use the null factor law to now solve for x.

Exercise 10I

UNDERSTANDING AND FLUENCY	1, 2, 3–4(½)	2, 3–4(½)	3–4(½)

1 Give a reason why $(x+1)$ is a factor of $P(x)=x^3-7x-6$. Hint: Find $P(-1)$.

2 Solve these quadratic equations.

a $(x-1)(x+3)=0$ **b** $(x+2)(x-2)=0$ **c** $(2x+1)(x-4)=0$

d $x^2-x-12=0$ **e** $x^2+6x+9=0$ **f** $x^2+7x+12=0$

Example 19

3 Solve for x.

a $(x+3)(x-2)(x-1)=0$ **b** $(x+2)(x+7)(x-1)=0$

c $(x-4)(x+4)(x-3)=0$ **d** $\left(x+\frac{1}{2}\right)(x-3)\left(x+\frac{1}{3}\right)=0$

e $(2x+1)(x-3)(3x+2)=0$ **f** $(4x-1)(5x-2)(7x+2)=0$

g $\left(x+\frac{1}{2}\right)(3x+1)(11x+12)=0$ **h** $(5x+3)(19x+2)\left(x-\frac{1}{2}\right)=0$

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Example 20

4 For each of the following cubic equations, follow these steps.
- Use the factor theorem to find a factor.
- Use long division to find the quotient.
- Factorise the quotient.
- Write the polynomial in a fully factorised form.
- Solve for x.

a $x^3 - 4x^2 + x + 6 = 0$
b $x^3 + 6x^2 + 11x + 6 = 0$
c $x^3 - 6x^2 + 11x - 6 = 0$
d $x^3 - 8x^2 + 19x - 12 = 0$
e $x^3 - 3x^2 - 16x - 12 = 0$
f $x^3 + 6x^2 - x - 30 = 0$

PROBLEM-SOLVING AND REASONING	5, 8	5, 6, 8, 9	6, 7, 9–11

5 Use the quadratic formula to solve for x, expressing your answers in exact form.
a $(x - 1)(x^2 - 2x - 4) = 0$
b $(x + 2)(x^2 + 6x + 10) = 0$

6 Solve by first taking out a common factor.
a $2x^3 - 14x^2 + 14x + 30 = 0$
b $3x^3 + 12x^2 + 3x - 18 = 0$

7 Solve for x.
a $x^3 - 13x + 12 = 0$
b $x^3 - 7x - 6 = 0$

8 What is the maximum number of solutions to $P(x) = 0$ when $P(x)$ is of degree:
a 3?
b 4?
c n?

9 Show that the following equations can be factorised easily without the use of long division, and then give the solutions.
a $x^3 - x^2 = 0$
b $x^3 + x^2 = 0$
c $x^3 - x^2 - 12x = 0$
d $2x^5 + 4x^4 + 2x^3 = 0$

10 Explain why $x^4 + x^2 = 0$ has only one solution.

11 Explain why $(x - 2)(x^2 - 3x + 3) = 0$ has only one solution.

ENRICHMENT	–	–	12

Quartics with four factors

12 Factorising a quartic may require two applications of the factor theorem and long division. Solve these quartics by factorising the left-hand side.
a $x^4 + 8x^3 + 17x^2 - 2x - 24 = 0$
b $x^4 - 2x^3 - 11x^2 + 12x + 36 = 0$
c $x^4 + x^3 - 11x^2 - 9x + 18 = 0$
d $2x^4 - 3x^3 - 7x^2 + 12x - 4 = 0$

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10J Graphs of polynomials

Interactive
Widgets
HOTsheets
Walkthrough

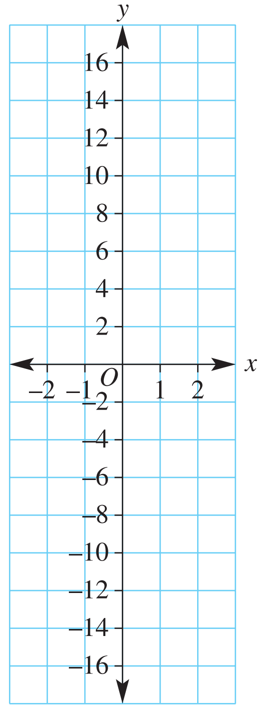
So far in Year 10 we have studied graphs of linear equations (straight lines) and graphs of quadratic equations (parabolas). We have also looked at graphs of circles, exponentials, cubics and hyperbolas. In this section we introduce the graphs of polynomials by focusing on those of degree 3 and 4. We start by considering the basic cubic $y = x^3$ and the quartic $y = x^4$, and then explore other cubics and quartics in factorised form.

Stage
5.3#
5.3
5.3§
5.2
5.2◊
5.1
4

Let's start: Plotting $y = x^3$ and $y = x^4$

Complete the table before plotting points and considering the discussion points below.

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
$y = x^2$							
$y = x^3$							
$y = x^4$							

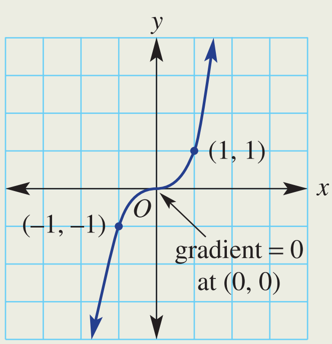


- Describe the features and shape of each graph.
- Describe the differences between the graphs of $y = x^2$ and $y = x^4$. Where do they intersect?

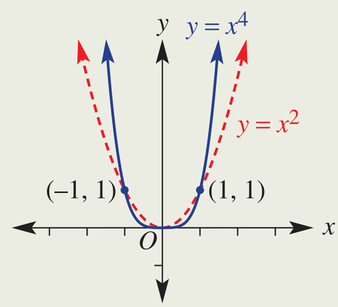
Key ideas

■ Graphs of basic polynomials

- $y = x^3$



- $y = x^4$



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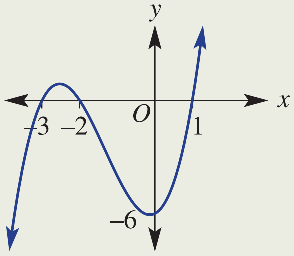
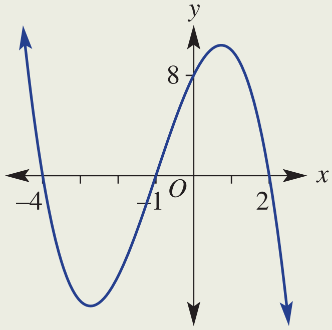
Key ideas

- To sketch cubic graphs in factorised form with three different factors:
  - Find the three x-intercepts.
  - Find the y-intercept.
  - Connect the intercepts to sketch a positive or negative cubic graph.

Positive cubic
(The coefficient of x^3 is positive.)
$y = (x - 1)(x + 2)(x + 3)$



Negative cubic
(The coefficient of x^3 is negative.)
$y = -(x + 4)(x - 2)(x + 1)$

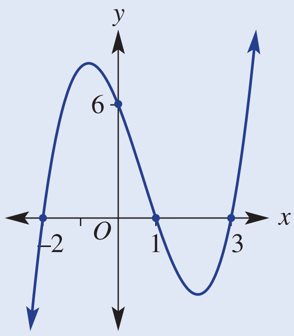


- The turning points are usually not located halfway between x-intercepts. You will learn how to locate turning points in future years.

Example 21 Sketching cubic graphs

Sketch the graphs of the following by finding x- and y-intercepts.

a $y = (x + 2)(x - 1)(x - 3)$ **b** $y = -x(x + 3)(x - 2)$

SOLUTION	EXPLANATION
a $y = (x + 2)(x - 1)(x - 3)$	Substitute $x = 0$ to find the y-intercept.
y-intercept at $x = 0$.	
$y = (2)(-1)(-3) = 6$	
x-intercepts at $y = 0$:	Substitute $y = 0$ to find the x-intercepts.
$0 = (x + 2)(x - 1)(x - 3)$	Use the null factor law.
$\therefore x + 2 = 0$ or $x - 1 = 0$ or $x - 3 = 0$	
$x = -2$ $\quad$ $x = 1$ $\quad$ $x = 3$	
	Mark the four intercepts and connect them to form a positive cubic graph. The coefficient of x^3 in the expansion of y is positive, so the graph points upwards at the right.

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b $y = -x(x + 3)(x - 2)$

Find the y-intercept using $x = 0$.

y-intercept at $x = 0$:

$y = -0(3)(-2) = 0$

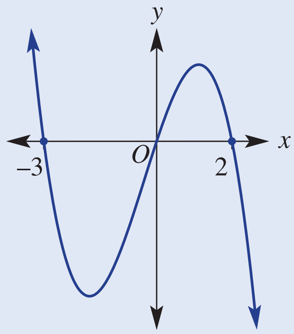
x-intercepts at $y = 0$.

$0 = -x(x + 3)(x - 2)$

The three factors are $-x$, $x + 3$ and $x - 2$.

$\therefore -x = 0 \quad \text{or} \quad x + 3 = 0 \quad \text{or} \quad x - 2 = 0$

$\quad\; x = 0 \qquad\qquad x = -3 \qquad\qquad x = 2$



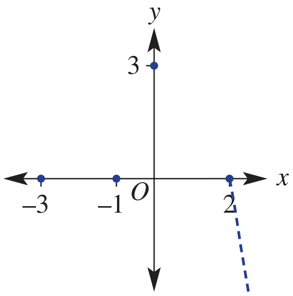
The coefficient of x^3 in the expansion of y is negative, so the graph points downwards at the right.

Exercise 10J

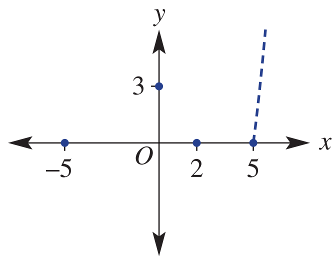
UNDERSTANDING AND FLUENCY	1, 3, 4(½)	2, 4(½)	4(½)

1 Join the given x- and y-intercepts to form a smooth cubic curve. Each graph has been started for you on the right-hand side.

a



b



2 Find the x- and y-intercepts of the graphs of these cubics.

a $y = (x + 1)(x - 3)(x - 4)$ **b** $y = (x + 3)(x - 7)(x + 1)$

c $y = -x(x + 2)(x - 4)$ **d** $y = -2x(x + 7)(x - 5)$

3 On the same set of scaled axes, plot graphs of the following. Use this table to help you.

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
$y = x^2$							
$y = x^3$							
$y = x^4$							

a $y = x^2$

b $y = x^3$

c $y = x^4$

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Example 21

4 Sketch the graphs of the following by finding x- and y-intercepts.

a $y = (x+2)(x-1)(x-3)$
b $y = (x-3)(x-4)(x+1)$
c $y = (x-5)(x-1)(x+2)$
d $y = \frac{1}{2}(x+3)(x-2)(x-1)$
e $y = x(x-2)(x+3)$
f $y = x(x-5)(x+1)$
g $y = -2x(x-1)(x+3)$
h $y = -\frac{1}{3}x(x+1)(x-3)$
i $y = -(x+2)(x+4)(x-1)$
j $y = -(x+3)\left(x-\frac{1}{2}\right)(x+1)$

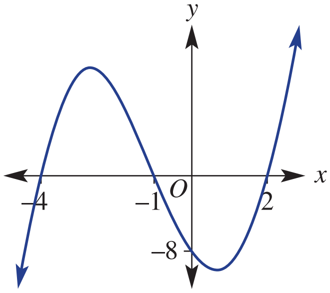
PROBLEM-SOLVING AND REASONING	5, 6, 8	5, 6, 8–9(½)	5–7, 8–9(½)

5 Sketch the graph of:

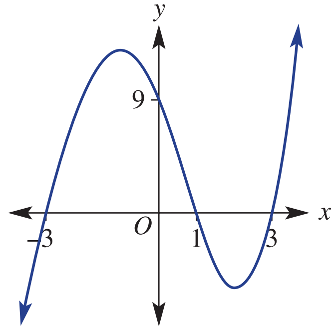
a $y = -x^3$
b $y = -x^4$

6 Find a cubic rule for these graphs.

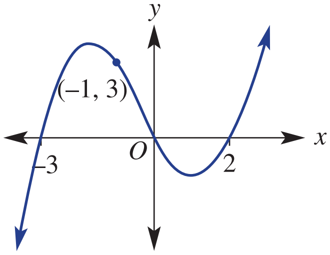
a



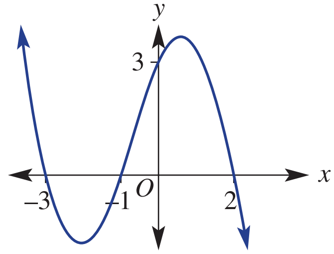
b



c



d



7 Sketch these quartics.

a $y = (x-5)(x-3)(x+1)(x+2)$
b $y = -x(x+4)(x+1)(x-4)$

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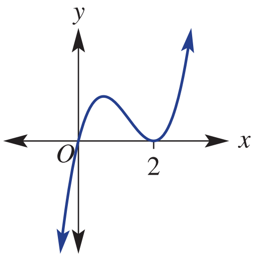
8 We know that the graph of $y = (x - 2)^2 - 1$ is the graph $y = x^2$ translated 2 units to the right and 1 unit down. Use this idea to sketch graphs of the following.

- **a** $y = (x - 2)^3 - 1$
- **b** $y = (x + 2)^3$
- **c** $y = x^3 - 2$
- **d** $y = x^4 - 1$
- **e** $y = (x + 3)^4$
- **f** $y = (x - 2)^4 - 3$

9 If a polynomial has a repeated factor $(x - a)$, then the point at $x = a$ is an x-intercept and also a turning point; e.g. $y = x(x - 2)^2$.



Now sketch these polynomials.

- **a** $y = x(x - 3)^2$
- **b** $y = -2x(x + 1)^2$
- **c** $y = -(x + 2)^2(x - 3)$
- **d** $y = (x + 4)(x + 1)^2$
- **e** $y = (2 - x)(x + 1)^2$
- **f** $y = -x^2(x + 2)(x - 2)$

ENRICHMENT	–	–	10

Polynomial with the lot

10 To sketch a graph of a polynomial that is not in factorised form, you must factorise the polynomial to help find the x-intercepts.
Consider the following polynomials.

- **i** Find the y-intercept.
- **ii** Factorise the polynomial using the factor theorem and long division.
- **iii** Find the x-intercepts.
- **iv** Sketch the graph.

- **a** $y = x^3 + 4x^2 + x - 6$
- **b** $y = x^3 - 7x^2 + 7x + 15$
- **c** $y = x^4 + 2x^3 - 9x^2 - 2x + 8$
- **d** $y = x^4 - 34x^2 + 225$

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Investigation

Logarithmic scales – the moment magnitude scale

The moment magnitude scale, like the Richter scale it replaced, is a logarithmic scale with base 10. It is used to measure the strength of an earthquake. The moment magnitude is obtained by multiplying a calculation of the area and the amount of displacement in the slip. Because it is a logarithmic scale, an earthquake of magnitude 4.0, for example, is 10 times more powerful than one of magnitude 3.0.

An earthquake of magnitude 4.0 is considered 'light', and the shaking of the ground is noticeable but unlikely to cause damage. An earthquake of magnitude 6.0 would be 100 times more powerful than that of magnitude 4.0 and is likely to cause significant damage in populated areas. The earthquake in Christchurch, New Zealand in 2011 was of magnitude 6.3. The Great East Japan Earthquake in 2011 was of magnitude 9.0, nearly 1000 times as strong.

a Explain why an earthquake of magnitude 4.0 is 10 times more powerful than one of magnitude 3.0.

b How much stronger is an earthquake of magnitude 7.0 than an earthquake of magnitude:

- **i** 6.0?
- **ii** 5.0?
- **iii** 2.0?

c Investigate some of the most powerful earthquakes ever recorded. Describe their strength, using the moment magnitude scale, and the damage they caused.

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1 Simplify the following without the use of a calculator.

a $2\log_3 4 - \log_3 \frac{16}{9}$

b $-\log_2 \frac{1}{4} + 3\log_2 4$

c $\log_5 \sqrt{125} + \log_3 \frac{1}{3}$

d $2\log_2 27 \div \log_2 9$

2 Solve these equations using log base 10.

a $5^{x-1} = 2$

b $0.2^x = 10$

c $2^x = 3^{x+1}$

d $0.5^x \leqslant 7$

3 If $y = a \times 2^{bx}$ and the graph of y passes through $(-1, 2)$ and $(3, 6)$, find the exact values of a and b.

4 An amount of money is invested at 10% p.a., earning compound interest. How long will it take for the money to double? Give an exact value.

5 Find the remainder when $x^4 - 3x^3 + 6x^2 - 6x + 6$ is divided by $(x^2 + 2)$.

6 $x^3 + ax^2 + bx - 24$ is divisible by $(x + 3)$ and $(x - 2)$. Find the values of a and b.

7 Prove using division.

a $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

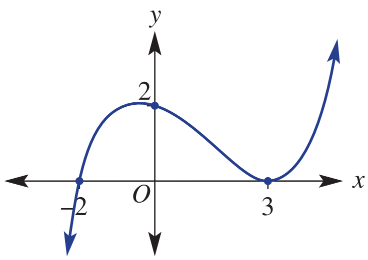
b $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

8 Solve for x.

a $(x + 1)(x - 2)(x - 5) \leqslant 0$

b $x^3 - x^2 - 16x + 16 > 0$

9 A cubic graph has a y-intercept of 2, a turning point at $(3, 0)$ and another x-intercept at -2. Find the rule for the graph.



10 A quartic graph has a turning point at $(0, 0)$ and two x-intercepts at 3 and -3. Find the rule for the graph if it also passes through $(2, 2)$.

Puzzles and challenges

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Chapter summary

Logarithms

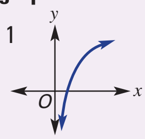
Definition

$\log_a y = x$ is equivalent to $a^x = y$.

- $\log_2 8 = 3$ since $2^3 = 8$
- $\log_{10} 100 = 2$ since $10^2 = 100$
- $\log_3 \frac{1}{27} = -3$ since $3^{-3} = \frac{1}{27}$
- $\log_{10} 0.001 = -3$ since $10^{-3} = 0.001$

Logarithmic graphs

$y = \log_a x$ where $a > 1$



- $x > 0$
- Asymptote is $x = 0$.
- Translated h units right, k units up: $y = \log_a(x - h) + k$
- log scales are $\log_{10}$ (physical value)

Other laws

- $\log_a 1 = 0$ $(a^0 = 1)$
- $\log_a a = 1$ $(a^1 = a)$
- $\log_a \frac{1}{x} = -\log_a x$ $\left(\frac{1}{x} = x^{-1}\right)$

$\log_a(xy) = \log_a x + \log_a y$

e.g. $\log_a 5 + \log_a 6 = \log_a 30$

$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

e.g. $\log_3 18 - \log_3 2 = \log_3 9$
$= 2$

$\log_a x^n = n \log_a x$

e.g. $\log_2 8^{\frac{1}{2}} = \frac{1}{2}\log_2 8$
$= \frac{1}{2} \times 3$
$= \frac{3}{2}$

Solving exponential equations

$2^x = 5$
$x = \log_2 5$

or

$2^x = 5$
$\log_{10} 2^x = \log_{10} 5$
$x \log_{10} 2 = \log_{10} 5$
$x = \frac{\log_{10} 5}{\log_{10} 2}$

Polynomials

Polynomial of degree *n*

$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 x^0$

e.g. cubic $x^3 - 2x^2 + 3x - 4$

Powers of x are whole numbers.

Expanding polynomials

Example:

$(x^2 - 1)(x^2 + 2x - 3)$
$= x^2(x^2 + 2x - 3) - 1(x^2 + 2x - 3)$
$= x^4 + 2x^3 - 3x^2 - x^2 - 2x + 3$
$= x^4 + 2x^3 - 4x^2 - 2x + 3$

Remainder theorem

If $P(x)$ is divided by $(x - a)$, the remainder is $P(a)$.

Factor theorem

If $P(x)$ is divided by $(x - a)$ and $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.

Division

$(x^3 - 2x^2 + 5x + 1) \div (x - 1)$

$$
\begin{array}{rl}
 & x^2 - x + 4 \\
x - 1 \,\big) & x^3 - 2x^2 + 5x + 1 \\
x^2(x - 1) & x^3 - x^2 \\
 & -x^2 + 5x + 1 \\
-x(x - 1) & -x^2 + x \\
 & 4x + 1 \\
4(x - 1) & 4x - 4 \\
 & 5
\end{array}
$$

$\therefore\ \underbrace{x^3 - 2x^2 + 5x + 1}_{\text{dividend}} = \underbrace{(x - 1)}_{\text{divisor}}\underbrace{(x^2 - x + 4)}_{\text{quotient}} + \underbrace{5}_{\text{remainder}}$

Null factor law

$(x + 2)(x - 3)(2x + 1) = 0$

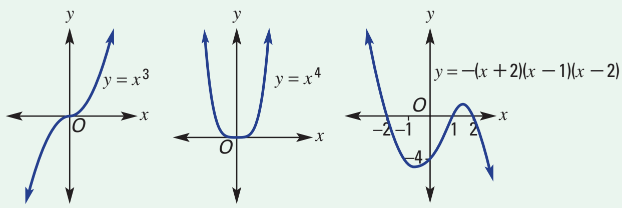
$x + 2 = 0$ or $x - 3 = 0$ or $2x + 1 = 0$

$x = -2$ or $x = 3$ or $x = -\frac{1}{2}$

Factorising

- Find one factor using the factor theorem.
- Use long division to find the quotient.
- Factorise the quotient.

Graphs



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Multiple-choice questions

1 Which of the following is equivalent to $5^3 = 125$?

A $\log_3 125 = 5$ **B** $\log_3 5 = 125$ **C** $\log_5 125 = 3$
D $125^3 = 5$ **E** $\log_{125} 3 = 5$

2 If $\log_2 64 = x$, then x is equal to:

A 5 **B** 6 **C** 32 **D** 128 **E** 64^2

3 If $5^x = 7$, then x is equal to:

A $\log_{10}\frac{7}{5}$ **B** $\log_x 5$ **C** $\log_{10} 7$ **D** $\log_7 5$ **E** $\log_5 7$

4 $\log_6 \frac{1}{6}$ simplifies to:

A -1 **B** 1 **C** 36 **D** 6 **E** 0

5 $x^6 - 2x^2 + 1$ is a polynomial of degree:

A 1 **B** -2 **C** 2 **D** 6 **E** 0

6 Which of these is a polynomial?

A $x^{\frac{1}{2}} + x^2$ **B** $\sqrt[3]{x} + x^2$ **C** $x^3 + x^2 + \frac{1}{x^2}$ **D** $4x^5 - 2x^3 - 1$ **E** $\frac{1}{x} - x$

7 If $P(x) = x^3 - x$ and $Q(x) = -2x^2 + 1$, then $P(-1) - Q(1)$ is equal to:

A 1 **B** -1 **C** 2 **D** 3 **E** -3

8 The remainder when $P(x) = 2x^3 + 4x^2 - x - 5$ is divided by $(x + 1)$ is:

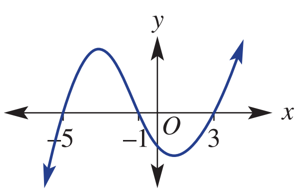
A -4 **B** -2 **C** -10 **D** 0 **E** -1

9 The three solutions to $(x - 3)(x + 5)(2x - 1) = 0$ are:

A $\frac{1}{3}, -\frac{1}{5}, 2$ **B** $5, -3, -\frac{1}{2}$ **C** $-5, 3, -\frac{1}{2}$ **D** $-5, 3, 1$ **E** $-5, 3, \frac{1}{2}$

10 The equation of this graph could be:

A $y = x(x + 5)(x - 1)$
B $y = (x + 5)(x + 1)(x + 3)$
C $y = (x - 5)(x - 1)(x + 3)$
D $y = (x - 5)(x + 1)(x - 3)$
E $y = (x + 5)(x + 1)(x - 3)$



11 The equation of the asymptote of $y = \log_{10}(x + 5) - 3$ is:

A $y = -5$ **B** $y = -3$ **C** $x = 5$ **D** $x = -5$ **E** $x = -3$

Short-answer questions

1 Write the following in logarithmic form.

a $2^4 = 16$ **b** $10^3 = 1000$ **c** $3^{-2} = \frac{1}{9}$

2 Write the following in index form.

a $\log_3 81 = 4$ **b** $\log_4 \frac{1}{16} = -2$ **c** $\log_{10} 0.1 = -1$

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3 Evaluate the following.

a $\log_{10} 1000$ **b** $\log_3 81$ **c** $\log_2 16$

d $\log_7 1$ **e** $\log_3 \frac{1}{27}$ **f** $\log_5 \frac{1}{125}$

g $\log_4 0.25$ **h** $\log_{10} 0.0001$ **i** $\log_3 0.\dot{1}$

4 State two features of $y = \log_2 x$ which show that it is the inverse function of $y = 2^x$.

5 Answer the questions below for each of these logarithmic functions.

a $y = \log_2(x - 2) - 1$ **b** $y = \log_3(x + 4) + 1$

i State any limitations for the input values.

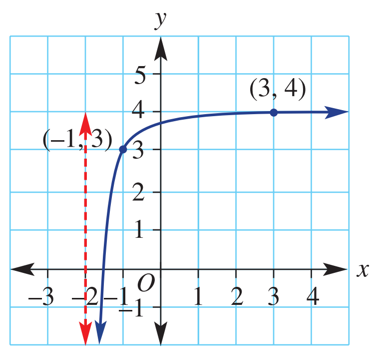
ii State the equation of the asymptote.

iii State any transformations that have been applied to $y = \log_2 x$.

iv Determine the axes intercepts.

v Sketch the graph, labelling all relevant features.

6 For this logarithmic graph, find the rule in the form $y = \log_a(x - h) + k$.



7 **a** Write the rule for pH, which is defined as the logarithm to the base 10 of the reciprocal of the hydrogen ion concentration, $[H^+]$, in moles/litre.

b What is the hydrogen ion concentration of acid rain that has a pH of 4.2? State the answer in scientific notation, using three significant figures.

c What is the percentage increase in acidity if soil with pH = 5.5 changes to a pH level of 5?

d The Richter scale for measuring the magnitude of an earthquake is a $\log_{10}$ scale. How much stronger in intensity was the 2004 Indian Ocean earthquake near Sumatra, which was of magnitude 9.2, compared with the 2011 Christchurch, New Zealand earthquake, which was of magnitude 6.3?

8 Simplify using the laws for logarithms.

a $\log_a 4 + \log_a 2$ **b** $\log_b 7 + \log_b 3$ **c** $\log_b 24 + \log_b 6$

d $\log_a 1000 - \log_a 100$ **e** $2 \log_a 2$ **f** $3 \log_a 10$

g $\log_{10} 25 + \log_{10} 4$ **h** $\log_3 60 - \log_3 20$ **i** $\log_2 \sqrt{8}$

j $3 \log_a 2 + \log_a 2$

9 Solve these equations using logarithms with the given base.

a $3^x = 6$ **b** $20 \times 1.2^x = 40$

10 Solve using base 10.

a $2^x = 13$ **b** $100 \times 0.8^x = 200$

11 If $P(x) = x^3 - x^2 - x - 1$, find:

a $P(0)$ **b** $P(2)$ **c** $P(-1)$ **d** $P(-3)$

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12 Expand and simplify.
- **a** $(x^2 + 2)(x^2 + 1)$
- **b** $x^3(x^2 - x - 3)$
- **c** $(x^2 + x - 3)(x^3 - 1)$
- **d** $(x^3 + x - 3)(x^3 + x - 1)$

13 Use long division to express each of the following in this form:

Dividend = divisor × quotient + remainder
- **a** $(x^3 + x^2 + 2x + 3) \div (x - 1)$
- **b** $(x^3 - 3x^2 - x + 1) \div (x + 1)$
- **c** $(2x^3 - x^2 + 4x - 7) \div (x + 2)$
- **d** $(-2x^3 - x^2 - 3x - 4) \div (x - 3)$

14 Use the remainder theorem to find the remainder when $P(x) = 2x^3 - 2x^2 + 4x - 7$ is divided by:
- **a** $x - 1$
- **b** $x + 2$
- **c** $x + 3$
- **d** $x - 3$

15 Using the factor theorem, determine whether the following are factors of $P(x) = x^3 - 2x^2 - 11x + 12$.
- **a** $x + 1$
- **b** $x - 1$
- **c** $x - 4$
- **d** $x + 3$

16 Solve these cubic equations.
- **a** $(x - 3)(x - 1)(x + 2) = 0$
- **b** $(x - 5)(2x - 3)(3x + 1) = 0$

17 Factorise and solve these cubic equations.
- **a** $x^3 + 4x^2 + x - 6 = 0$
- **b** $x^3 - 9x^2 + 8x + 60 = 0$

18 Sketch the graphs of these polynomials.
- **a** $y = x^3$
- **b** $y = (x + 1)(x - 1)(x - 4)$
- **c** $y = -x(x - 3)(x + 2)$
- **d** $y = x^4$

Extended-response questions

1 A share portfolio initially valued at \$100 000 is expected to grow at a rate of 10%, compounded annually.
- **a** If \$$A$ is the value of the investment and n is the number of years, which of the following is the correct rule linking A and n?
  - **A** $A = 100\,000 \times 0.1^n$
  - **B** $A = 100\,000 \times 1.1^n$
  - **C** $A = \dfrac{1.1^n}{100\,000}$
- **b** Find the value of the investment, correct to the nearest dollar, after:
  - **i** 2 years
  - **ii** 18 months
  - **iii** 10.5 years
- **c** Find the time, correct to 2 decimal places, when the investment is expected to have increased in value to:
  - **i** \$200 000
  - **ii** \$180 000
  - **iii** \$0.5 million

2 A cubic polynomial has the rule $P(x) = x^3 - 5x^2 - 17x + 21$.
- **a** Find:
  - **i** $P(-1)$
  - **ii** $P(1)$
- **b** Explain why $(x - 1)$ is a factor of $P(x)$.
- **c** Divide $P(x)$ by $(x - 1)$ to find the quotient.
- **d** Factorise $P(x)$ completely.
- **e** Solve $P(x) = 0$.
- **f** Find $P(0)$.
- **g** Sketch a graph of $P(x)$, labelling x- and y-intercepts.

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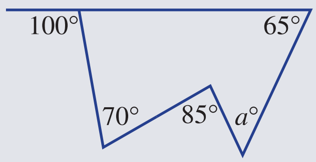
Semester review 2

Chapter 6: Geometrical figures and circle geometry

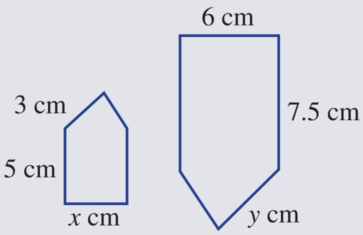
Multiple-choice questions

1 The value of a in the diagram is:



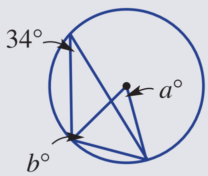
A 40 B 25 C 30
D 50 E 45

2 The values of x and y in these similar figures are:



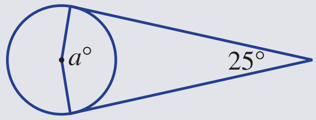
A $x = 2.6, y = 5$
B $x = 4, y = 4.5$
C $x = 4, y = 7.5$
D $x = 3, y = 6$
E $x = 3.5, y = 4.5$

3 The values of the pronumerals in this diagram are:



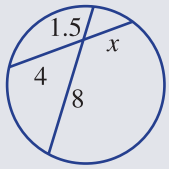
A $a = 17, b = 56$
B $a = 34, b = 73$
C $a = 68, b = 56$
D $a = 34, b = 34$
E $a = 68, b = 34$

4 The value of angle a in this diagram is:



A 115 B 165 C 140
D 130 E 155

5 The value of x in this diagram is:

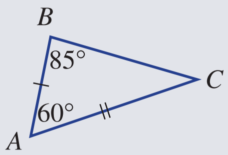


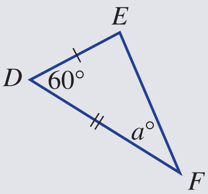
A 5.5 B 0.75 C 3
D 4 E 8

Short-answer questions

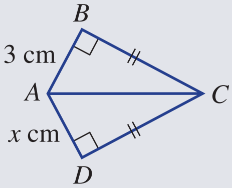
1 Prove the following congruence statements, giving reasons, and use this to find the value of the pronumerals.

a $\triangle ABC \equiv \triangle DEF$





b $\triangle ABC \equiv \triangle DEF$



2 Use congruence to prove that a parallelogram (with opposite parallel sides) has equal opposite sides.

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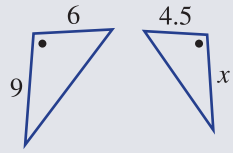
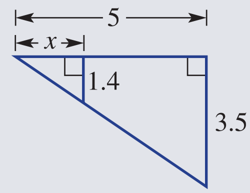
3 Find the value of the pronumeral, given that these pairs of triangles are similar.

a

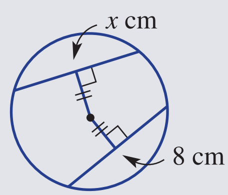


b

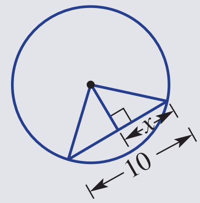


4 Use the chord and circle theorems to find the value of each pronumeral.

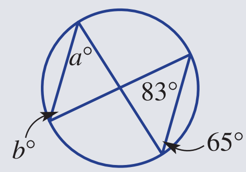
a



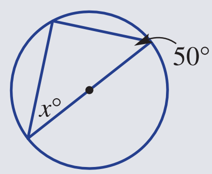
b



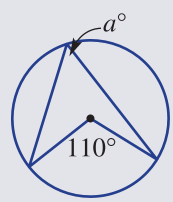
c



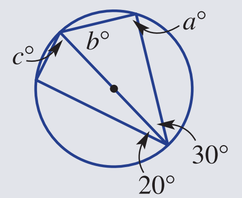
d



e

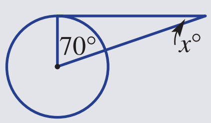


f

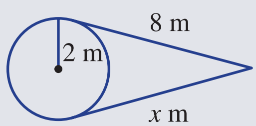


5 Use tangent properties to find the value of the pronumerals.

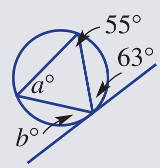
a



b

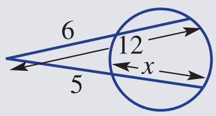


c

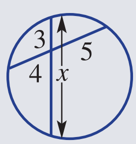


6 Find the value of x in each figure.

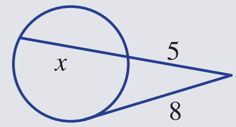
a



b

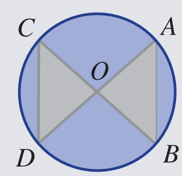


c



Extended-response question

1 A logo for a car manufacturing company is silver and violet and shaped as shown, with O indicating the centre of the circle.



The radius of the logo is 5 cm and chord AB is 6 cm. Given that the two chords are equidistant from the centre of the circle, complete the following.

a What is the length of CD? Give a reason.

b Hence, prove that $\triangle OAB \equiv \triangle OCD$.

c By first finding the length of OM, where M is the point such that $OM \perp AB$, find the area of $\triangle OAB$.

d Hence, determine what percentage of the logo is occupied by the silver portion, given the area of a circle is πr^2. Answer correct to 1 decimal place.

e Given that $\angle OCD = 53.1°$, what is the angle between the two triangles (i.e. $\angle BOD$)?

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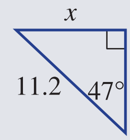
Chapter 7: Trigonometry

Multiple-choice questions

1 The value of x in the triangle shown is approximately:



A 7.6 B 12.0 C 10.4
D 6.5 E 8.2

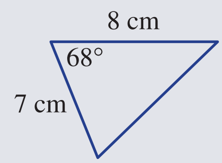
2 A bird 18 m up in a tree spots a worm on the ground 12 m from the base of the tree. The angle of depression from the bird to the worm is closest to:

A 41.8° B 56.3° C 61.4°
D 33.7° E 48.2°

3 A walker travels due south for 10 km and then on a bearing of 110° for 3 km. The total distance south from the starting point to the nearest km is:

A 11 km B 1 km C 9 km
D 13 km E 15 km

4 The area of the triangle shown is closest to:



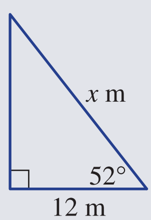
A 69 cm^2 B 52 cm^2 C 28 cm^2
D 26 cm^2 E 10 cm^2

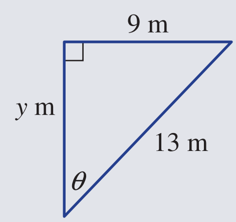
5 Choose the incorrect statement.

A $\theta = 290°$ is in quadrant 4
B $\sin 120° = \frac{\sqrt{3}}{2}$
C $\cos 110° = -\cos 20°$
D $\tan \theta$ is positive for $200° < \theta < 250°$
E $\sin 230° = -\sin 50°$

Short-answer questions

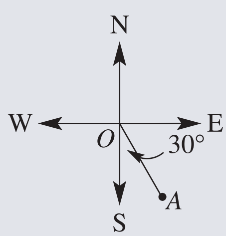
1 Find the value of the pronumeral in these right-angled triangles, correct to 1 decimal place.

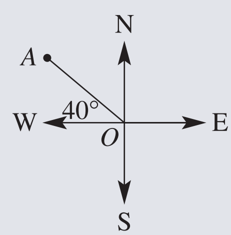
a


b


2 For the following, give the bearing of:

i A from O
ii O from A

a


b


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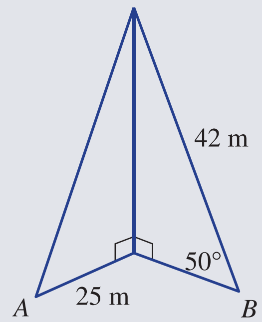
3 Two wires reach from the top of an antenna to points A and B on the ground, as shown. Point A is 25 m from the base of the antenna, and the wire from point B is 42 m long and makes an angle of 50° with the ground.

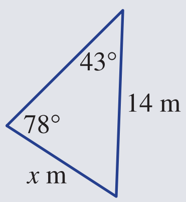
a Find the height of the antenna, to 3 decimal places.

b Find the angle that the wire at point A makes with the ground, to 1 decimal place.

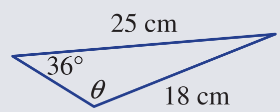


4 Find the value of the pronumeral, correct to 1 decimal place.

a



b θ is obtuse



5 Find the largest angle, correct to 1 decimal place, in a triangle with side lengths 8 m, 12 m and 15 m.

6 **a** If $\theta = 223°$, state which of $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive.

b Choose the angle θ to complete each statement.

i $\sin 25° = \sin\theta$, where θ is obtuse.

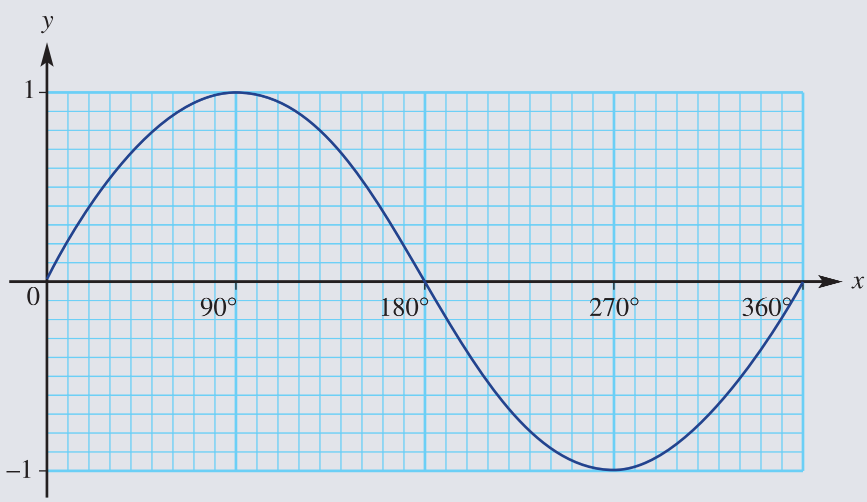
ii $\tan 145° = -\tan\theta$, where θ is acute.

iii $\cos 318° = \cos\theta$, where θ is the reference angle.

c State the exact value of:

i $\cos 60°$ **ii** $\sin 135°$ **iii** $\tan 330°$

7 Use this graph of $y = \sin x$ to answer the following.



a Estimate the value of $\sin x$ for $x = 160°$.

b Estimate the two values of x for which $\sin x = -0.8$.

c Is $\sin 40° < \sin 120°$?

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Extended-response question

1 A group of walkers sets out on a trek to get to the base of a mountain range. The mountains have two peaks, 112 m and 86 m, above ground level at the base. The angle of elevation from the peak of the smaller mountain to the peak of the higher mountain is 14°.

a Find the horizontal distance between the two mountain peaks, correct to 1 decimal place.

To get to the base of the mountain range, the walkers set out from the national park entrance on a bearing of 52° for a distance of 13 km and then turn on a bearing of 340° for the last 8 km of the trek.

b Draw a diagram representing the trek. Label all known measurements.

c If the walkers were able to trek directly from their start location to their end point, what distance would they cover? Round your answer to 3 decimal places.

d After they have explored the mountains, the group will be taken by bus back along the direct path from their end location to the park entrance. Determine the bearing on which they will travel. Round your answer to the nearest degree.

Chapter 8: Quadratic expressions and quadratic equations

Multiple-choice questions

1 The expanded form of $2(2x - 3)(3x + 2)$ is:

A $12x^2 - 5x - 6$
B $12x^2 - 12$
C $12x^2 - 10x - 12$
D $24x^2 - 20x - 24$
E $12x^2 - x - 6$

2 The factorised form of $25y^2 - 9$ is:

A $(5y - 3)^2$
B $(5y - 3)(5y + 3)$
C $(25y - 3)(y + 3)$
D $(5y - 9)(5y + 1)$
E $5(y + 1)(y - 9)$

3 $\dfrac{x^2 - 4}{x^2 - x - 6} \times \dfrac{x^2 - 4x + 3}{4x - 8}$ simplifies to:

A $x - 2$
B $\dfrac{x + 3}{12}$
C $\dfrac{x^2 + 1}{2x}$
D $\dfrac{x - 1}{4}$
E $\dfrac{x^2}{x - 2}$

4 The solution(s) to the quadratic equation $x^2 - 4x + 4 = 0$ is/are:

A $x = 0, 4$
B $x = 2$
C $x = 1, 4$
D $x = 2, -2$
E $x = -1, 4$

5 A quadratic equation $ax^2 + bx + c = 0$ has a discriminant equal to 17. This tells us that:

A The equation has a solution $x = 17$.
B The equation has no solutions.
C $a + b + c = 17$
D The equation has two solutions.
E The equation has one solution.

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Short-answer questions

1 Expand and simplify.

a $(3x + 1)(3x - 1)$ **b** $(2x - 5)^2$ **c** $(2x + 3)(x + 5) - (3x - 5)(x - 4)$

2 Factorise fully these quadratics. Remember to take out any common factors first.

a $4x^2 - y^2$ **b** $(x + 2)^2 - 7$ **c** $3x^2 - 48$

d $x^2 + 5x - 14$ **e** $x^2 - 10x + 25$ **f** $2x^2 - 16x + 24$

3 Factorise these non-monic quadratics.

a $3x^2 - 2x - 8$ **b** $6x^2 + 7x - 3$ **c** $10x^2 - 23x + 12$

4 Solve these quadratic equations.

a $2x(x - 3) = 0$ **b** $(x + 4)(2x - 1) = 0$

c $x^2 + 5x = 0$ **d** $x^2 - 16 = 0$

e $x^2 - 7 = 0$ **f** $x^2 - 4x + 4 = 0$

g $x^2 - 5x - 24 = 0$ **h** $3x^2 + 5x - 2 = 0$

5 Solve these quadratic equations by first writing them in the form $ax^2 + bx + c = 0$.

a $x^2 = 40 - 3x$ **b** $x(x - 6) = 4x - 21$ **c** $\frac{x + 20}{x} = x$

6 a Factorise by completing the square.

i $x^2 - 6x + 4$ **ii** $x^2 + 4x + 7$ **iii** $x^2 + 3x + 1$

b Use your answers to part **a** to solve these, if possible.

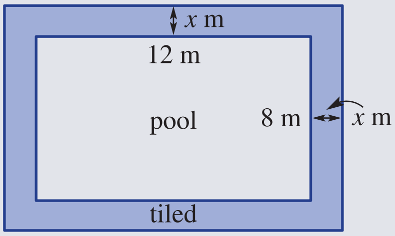
i $x^2 - 6x + 4 = 0$ **ii** $x^2 + 4x + 7 = 0$ **iii** $x^2 + 3x + 1 = 0$

7 Solve these quadratic equations using the quadratic formula. Leave your answers in exact surd form.

a $2x^2 + 3x - 6 = 0$ **b** $x^2 - 4x - 6 = 0$

Extended-response question

1 A backyard swimming pool, measuring 12 m by 8 m, is surrounded by a tiled path of width x metres, as shown.



a Find a simplified expression for the area of the tiled path.

b If $x = 1$, what is the tiled area?

c Solve an appropriate equation to determine the width x m if the tiled area is 156 m^2.

d Find the width x m if the tiled area is 107.36 m^2. Use the quadratic formula.

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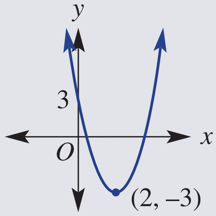
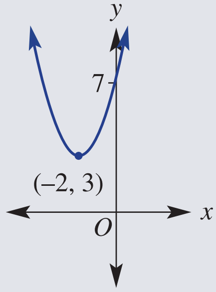
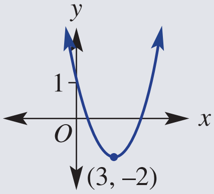
Chapter 9: Non-linear relationships, functions and their graphs

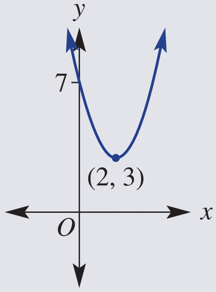
Multiple-choice questions

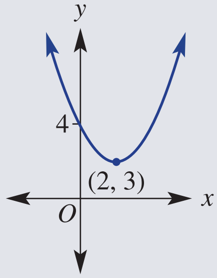
1 The graph of $y = (x - 2)^2 + 3$ could be:

A


B


C


D


E


2 The graph of $y = x^2 - 4x$ has a turning point with coordinates:
A $(2, -4)$ **B** $(0, -4)$ **C** $(4, 0)$ **D** $(-2, 12)$ **E** $(1, -3)$

3 For the quadratic $y = ax^2 + bx + c$, $b^2 - 4ac < 0$, so we know that the graph has:
A a maximum turning point
B two x-intercepts
C no y-intercept
D no x-intercepts
E a minimum turning point

4 The graph with equation $x^2 + y^2 = 9$ is:
A a circle with radius 9
B a parabola with turning point $(0, 9)$
C a circle with radius 3
D a hyperbola with asymptote at $x = 3$
E an exponential curve with y-intercept at 3

5 The equation of the asymptote of $y = 2^x + 3$ is:
A $x = 3$ **B** $y = 3$ **C** $x = 2$ **D** $y = 2$ **E** $y = 2x$

6 Which of the following does not represent a function?
A $y = 4 - x^2$ **B** $y = x^4 + x^2$ **C** $x = y^2 - 4$
D $x + 2y - 8 = 0$ **E** $y = 3^x$

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Short-answer questions

1 Sketch the following parabolas and state the transformations from $y = x^2$.

a $y = 3x^2$ **b** $y = -(x + 2)^2$ **c** $y = x^2 + 5$

2 Consider the quadratic $y = x^2 + 4x - 5$.

a Find the y-intercept.
b Find the x-intercepts by factorising.
c Use symmetry to find the turning point.
d Sketch the graph.

3 Consider the quadratic $y = -2(x - 3)^2 + 8$.

a State the coordinates of the turning point and whether it is a maximum or minimum.
b Find the y-intercept.
c Find the x-intercepts by factorising.
d Sketch the graph.

4 Sketch the following quadratics by first completing the square.

a $y = x^2 + 6x + 2$ **b** $y = x^2 - 5x + 8$

5 Consider the quadratic $y = 2x^2 - 4x - 7$.

a Use the discriminant to determine the number of x-intercepts of the graph.
b Sketch its graph using the quadratic formula. Round x-intercepts to 1 decimal place.

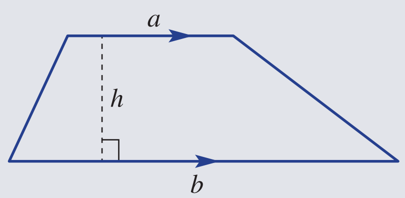
6 Sketch the following graphs, labelling key features.

a $x^2 + y^2 = 4$ **b** $x^2 + y^2 = 10$ **c** $y = 3^x$
d $y = 5^{-x}$ **e** $y = \frac{2}{x}$ **f** $y = -\frac{6}{x}$

7 $f(x) = 2x^2 - 4$ and $g(x) = 3 - x$, find:

a $f(0)$ **b** $f(-1)$ **c** $g(-3)$
d $f(4) + g(1)$ **e** $f(k) - g(k)$

8 The rule for the area of a trapezium is $A = \frac{1}{2}h(a + b)$.



a Realistically, what restrictions are there on A, a, b and h?
b Rearrange the formula to make h the subject.
c Find the height of a trapezium when $A = 10$, $a = 2$ and $b = 3$.
d Rearrange to make a the subject.
e Find the value of a when $A = 25$, $b = 2$ and $h = 10$.

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9 State the domain (i.e. the set of permissible x values) and range (i.e. the set of permissible y values) for each of the following.

a $y = 3x$
b $y = 4 - x$
c $f(x) = 4 - x^2$
d $f(x) = -\frac{2}{x}$
e $f(x) = \sqrt{x + 2}$

10 Find the inverse function for:

a $y = 4x$
b $y = 5 - x$
c $y = 4 - x^3$
d $y = \frac{3}{2x}$
e $y = 1 - \frac{1}{x}$

11 Find the largest domain that includes the vertex, for which the parabola $y = x^2 - 2x + 1$ has an inverse.

12 Find the coordinates of the points of intersection of these graphs.

a $x^2 + y^2 = 15$ and $y = 2x$
b $y = 2^x$ and $y = 16$
c $y = \frac{2}{x}$ and $y = 8x$

13 Sketch the graphs of the following relations. Label important features.

a $(x–2)^2 + (y + 1)^2 = 16$
b $y = 2^{x+3} + 1$
c $y = \frac{1}{x + 2} - 3$

14 Solve these equations simultaneously.

a $y = x^2 + 5x - 2$
$y = 3x + 13$
b $y = 3x^2 + x + 5$
$y = -2x + 1$
c $y = x^2$
$2x + y = -1$

15 Sketch graphs of these cubic functions, labelling axes intercepts.

a $y = 2x^3$
b $y = -x^3 + 8$
c $y = 2(x - 1)^3$

Extended-response questions

1 A rollercoaster has a section modelled by the equation $h = \frac{1}{40}(x^2 - 120x + 1100)$, where h is the height above the ground and x is the horizontal distance from the start of the section. All distances are measured in metres and x can take all values between 0 and 200 metres.

a Sketch the graph of h vs x for $0 \leqslant x \leqslant 200$, labelling the end points.
b What is the height above ground at the start of the section?
c The rollercoaster travels through an underground tunnel. At what positions from the start will it enter and leave the tunnel?
d What is the maximum height the rollercoaster reaches?
e What is the maximum depth the rollercoaster reaches?

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2 **a** A boat-hire company charges \$2.50 for 15 minutes' use.
- **i** Explain why this is an example of direct variation.
- **ii** Write the equation for cost (C) in terms of number (n) of hours hired.
- **iii** Determine the cost to hire the boat for 3.7 hours.

b If $y = \frac{k}{x}$ and $y = 12$ when $x = 6$, find:
- **i** y when $x = 36$
- **ii** x when $y = 3$

c **i** Explain the type of proportional relationship that exists between speed and the time taken to drive a certain distance.
- **ii** If it takes 81 minutes to drive to a destination at 90 km/h, determine how long it would take in heavier traffic at 75 km/h.

d Sketch the shape of a first quadrant graph from each of the descriptions below. Numbers are not required.
- **i** y varies directly as x.
- **ii** y varies inversely as x.
- **iii** y decreases at an increasing rate.
- **iv** y increases at a decreasing rate.
- **v** A distance–time graph showing acceleration, steady speed, then deceleration and coming to a stop.

Chapter 10: Logarithms and polynomials

Multiple-choice questions

1 The value of $\log_2 16$ is:

A 8 **B** 256 **C** 2^{16} **D** $\frac{1}{4}$ **E** 4

2 An equivalent statement to $3^x = 20$ is:

A $x = \log_3 20$
B $20 = \log_3 x$
C $x = \log_{20} 3$
D $3 = \log_x 20$
E $x = \log_{10}\left(\frac{20}{3}\right)$

3 The expression that is not a polynomial is:

A $3x^2 + 1$ **B** $2 - 5x^5 + x$ **C** $7x - 5$
D $4x^3 + 2x - \frac{1}{x}$ **E** $5x^6 + 2x^4 - 3x^2 - x$

4 The remainder when $P(x) = x^4 - 3x^3 + 2x + 1$ is divided by $(x - 2)$ is:

A 37 **B** 2 **C** -3 **D** 13 **E** 5

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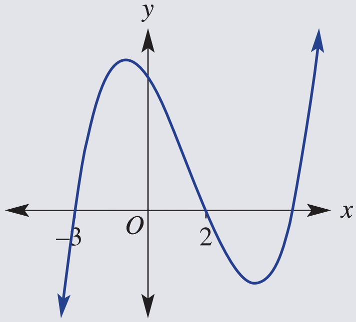
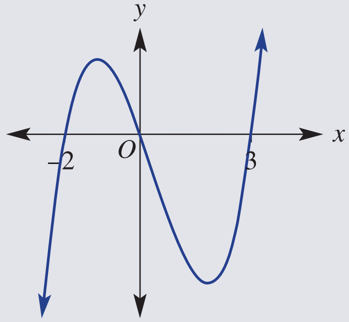
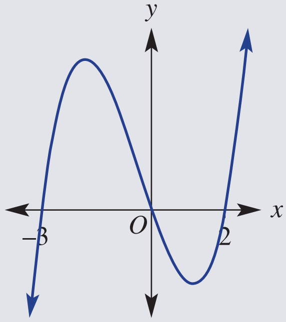
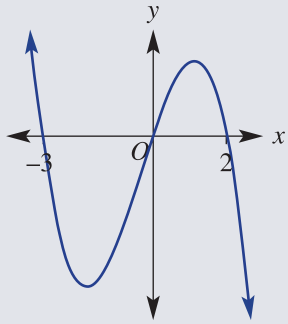
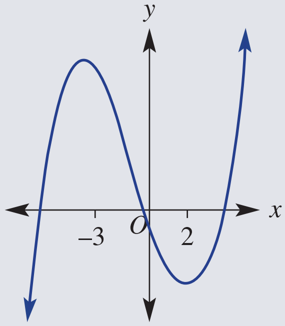
5 A possible graph of $y = -x(x + 3)(x - 2)$ is:

A


B


C


D


E


Short-answer questions

1 Simplify where necessary and evaluate, without using a calculator.

a $\log_4 64$ b $\log_5 \frac{1}{25}$ c $\log_{10} 0.1$

d $\log_7 1$ e $\log_4 2 + \log_4 8$ f $\log_3 54 - \log_3 6$

g $\log_8 8$ h $\log_a a^3$

2 Solve for x.

a $\log_6 216 = x$ b $\log_x 27 = 3$ c $\log_3 x = 4$

3 a Solve for x using the given base.

i $3^x = 30$ ii $15 \times 2.4^x = 60$

b Solve for x using base 10 and evaluate, correct to 3 decimal places.

i $7^x = 120$ ii $2000 \times 0.87^x = 500$

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4 Consider the polynomials $P(x) = x^3 + 3x^2 - 4x - 6$ and $Q(x) = 2x^3 - 3x - 4$.

a Find:

i $P(2)$ **ii** $P(-1)$

iii $Q(-3)$ **iv** $Q(1)$

b Expand and simplify.

i $P(x) \times Q(x)$ **ii** $(Q(x))^2$

5 Divide $P(x) = x^3 - 4x^2 + 2x + 7$ by $(x - 3)$ and write in the form $P(x) = (x - 3)Q(x) + R$, where $Q(x)$ is the quotient and R is the remainder.

6 Find the remainder when $P(x) = x^3 - 2x^2 - 13x - 10$ is divided by each of the following and, hence, state if it is a factor.

a $x - 1$ **b** $x + 2$ **c** $x - 3$

7 Solve for x.

a $(x + 1)(x - 3)(x + 6) = 0$ **b** $-x(2x - 5)(3x + 2) = 0$

c $x^3 + 5x^2 + 2x - 8 = 0$ **d** $2x^3 - 3x^2 - 3x + 2 = 0$

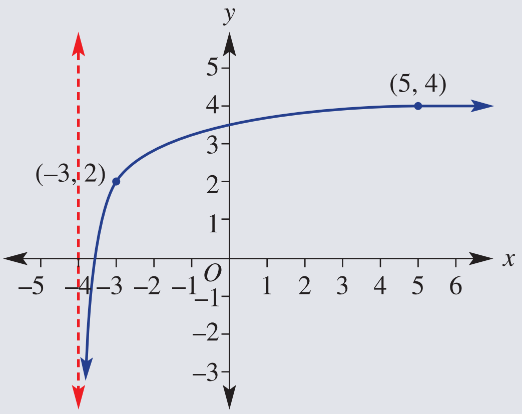
8 For the logarithmic graph shown:

a State any limitations for the input values.

b State the equation of the asymptote.

c Find the rule in the form $y = \log_3(x - h) + k$.

d Determine the axes intercepts, to 2 decimal places



9 A section of a train track that heads through a valley and then over a mountain is modelled by the equation $P(x) = -2x^3 + 3x^2 + 23x - 12$ for $-5 \leqslant x \leqslant 6$.

a Show that $(x + 3)$ is a factor of $P(x)$.

b Hence, factorise $P(x)$ using division.

c Sketch a graph of this section of the track, labelling axes intercepts and end points.

Extended response question

1 The loudness of sound (I) is measured in decibels (dB), where $\text{dB} = 10\log_{10}\left(\frac{I}{I_0}\right)$ and I_0 is the loudness level at the threshold of hearing.

a How much louder than the threshold of human hearing is the sound of rain at 50 dB?

b What is the increase in decibels when the loudness of sound triples?

c Hearing damage can occur above 85 dB. If an iPod is played at a decibel level of 30% above 85 dB, how much louder is the sound compared to the maximum safe level?

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Answers

Chapter 1

Pre-test

1 a 200 b 0.05 c 2.3 d 430
e 62.9 f 1 380 000

2 a 21 b 30 c 3.8 d 10

3 a 12 cm b 7 m c 32 cm

4 a $C = 12.6$ m, $A = 12.6$ m^2
b $C = 37.7$ cm, $A = 113.1$ cm^2
c $C = 24.5$ km, $A = 47.8$ km^2

5 a 5 b 2.2 c 5

6 a Surface area = 76 cm^2, Volume = 40 cm^3
b Surface area = 24 m^2, Volume = 8 m^3
c Surface area = 100.5 cm^2, Volume = 75.4 cm^3

Exercise 1A

1 a milliseconds b millimetres
c kilometres d megatonnes
e megagrams f milligrams
g nanometre h decimetre

2 a \$40 000 b 8 gigabytes
c 2 megatonnes d 3 nanoseconds

3
$$\begin{tikzcd}
\text{day} \arrow[r, bend left, "\times 24"] & \text{hour} \arrow[l, bend left, "\div 24"] \arrow[r, bend left, "\times 60"] & \text{min} \arrow[l, bend left, "\div 60"] \arrow[r, bend left, "\times 60"] & \text{s} \arrow[l, bend left, "\div 60"] \arrow[r, bend left, "\times 1000"] & \text{ms} \arrow[l, bend left, "\div 1000"] \arrow[r, bend left, "\times 1000"] & \mu\text{s} \arrow[l, bend left, "\div 1000"] \arrow[r, bend left, "\times 1000"] & \text{ns} \arrow[l, bend left, "\div 1000"]
\end{tikzcd}$$

4
$$\begin{tikzcd}
\text{TB} \arrow[r, bend left, "\times 1000"] & \text{GB} \arrow[l, bend left, "\div 1000"] \arrow[r, bend left, "\times 1000"] & \text{MB} \arrow[l, bend left, "\div 1000"] \arrow[r, bend left, "\times 1000"] & \text{kB} \arrow[l, bend left, "\div 1000"] \arrow[r, bend left, "\times 1000"] & \text{B} \arrow[l, bend left, "\div 1000"]
\end{tikzcd}$$

5 a 0.002 b 5000
c 5000 d 2
e 1 f 3×10^6
g 3.2×10^4, 3.2×10^{10} h 0.06
i 0.2 j 5000
k 60 l 500 000
m 1000 n 10^6
o 2×10^3 p 0.035
q 2×10^{21} r 10^9
s 10^3, 10^6, 10^9 t 400

6 a \$5258.88 b \$43.82 c 164 MB

7 1:54.27 means 1 min and 54.27 s or 114 270 ms.

8 nearest millisecond

9 8.64×10^7

10 0.00024

11 171

12 10^9

13 11.574

14 a 9002 kB b 9.002 MB
c No, at least two emails are needed, sending the 1.002 MB picture separately.

15 a

millisecond	1000
microsecond	1 000 000
nanosescond	1 000 000 000
minute	$0.01\dot{6}$
hour	$0.0002\dot{7}$
day	0.000011574
week	0.000001653
month	0.000000381
year	3.2×10^{-8}
century	$3.1709\ldots \times 10^{-10}$
millennium	$3.1709\ldots \times 10^{-11}$

b

millisecond	3.1536×10^{10}
microsecond	3.1536×10^{13}
nanosescond	3.1536×10^{16}
second	31 536 000
minute	525 600
hour	8760
day	365
week	52.142857
month	12
year	1
century	0.01
millennium	0.001

16 a ^{213}Po, ^{216}Po, O, Ba, Zn, Ar, Na, Au, Cr, H, U, C, Ca
b i 3.6×10^5 ii 441.75
c 13.48 days d Answers will vary.

Exercise 1B

1 Some examples are 3.35, 3.44 and 3.4499

2 a 347 cm b 3 m

3 6.65

4 a i 1 cm ii 44.5 cm to 45.5 cm
b i 0.1 mm ii 6.75 mm to 6.85 mm
c i 1 m ii 11.5 m to 12.5 m
d i 0.1 kg ii 15.55 kg to 15.65 kg
e i 0.1 g ii 56.75 g to 56.85 g
f i 1 m ii 9.5 m to 10.5 m
g i 1 h ii 672.5 h to 673.5 h
h i 0.01 m ii 9.835 m to 9.845 m
i i 0.01 km ii 12.335 km to 12.345 km
j i 0.001 km ii 0.9865 km to 0.9875 km

5 a 4.5 m to 5.5 m b 7.5 cm to 8.5 cm
c 77.5 mm to 78.5 mm d 4.5 ns to 5.5 ns
e 1.5 km to 2.5 km f 34.15 cm to 34.25 cm
g 3.85 kg to 3.95 kg h 19.35 kg to 19.45 kg
i 457.85 t to 457.95 t j 18.645 m to 18.655 m
k 7.875 km to 7.885 km l 5.045 s to 5.055 s

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6 a \$4450 to \$4550 b \$4495 to \$4505
c \$4499.50 to \$4500.50

7 a 30 m b 145 g
c 4.6 km d 9.0 km
e 990 g f 990 g (nearest whole)

8 a 149.5 cm to 150.5 cm b 145 cm to 155 cm
c 149.95 cm to 150.05 cm

9 a 24.5 cm to 25.5 cm b 245 cm
c 255 cm

10 a 9.15 cm b 9.25 cm
c 36.6 cm to 37 cm
d 83.7225 cm^2 to 85.5625 cm^2

11 a 9.195 cm b 9.205 cm
c 36.78 cm to 36.82 cm
d 84.732025 cm^2 to 84.7832025 cm^2
e Increasing the level of accuracy lowers the difference between the upper and lower limits of any subsequent working.

12 a different rounding (level of accuracy being used)
b Cody used to the nearest kg, Jacinta used to the nearest 100 g and Lachlan used to the nearest 10 g.
c yes

13 a distances on rural outback properties, distances between towns, length of wires and pipes along roadways
b building plans, measuring carpet and wood
c giving medicine at home to children, paint mixtures, chemical mixtures by students
d buying paint, filling and costing filling a pool

14 a 1.8% b 5.6% c 0.56%
d 0.056% e 0.28% f 0.056%
g 0.12% h 0.071%

Exercise 1C

1 a $\sqrt{55}$ b $\sqrt{11}$ c $\sqrt{77}$
d $\sqrt{2}$ e $\sqrt{8} = 2\sqrt{2}$ f $\sqrt{50} = 5\sqrt{2}$

2 a $x^2 + y^2 = z^2$ b $2a^2 = b^2$ c $2x^2 = c^2$

3 a 5 cm b 11.18 m c 16.55 km
d 1.81 mm e 0.43 km f 77.10 cm

4 a 4.58 m b 7.94 m c 0.63 m
d 1.11 cm e 14.60 cm f 0.09 cm

5 a i $\sqrt{34}$ ii 6.16
b i $\sqrt{80}$ (or $4\sqrt{5}$) ii 16.61
c i $\sqrt{10}$ ii 7.68
d i $\sqrt{68}$ (or $2\sqrt{17}$) ii 12.21

6 a no b yes c no
d no e yes f yes

7 8.3 cm

8 a 2.86 m b 2.11 cm c 26.38 m
d 4.59 cm e 0.58 km f 3.65 km

9 a 13.19 mm b 13.62 m c 4.53 cm
d 2.61 m e 12.27 km f 5.23 cm

10 a $2\sqrt{13}$ b $4\sqrt{2}$ c $\sqrt{181}$

11 a i 22.4 ii 24.5
b Student's own investigation.

12 $\frac{\sqrt{5}}{2}$ cm

13 a $4\sqrt{5}$ cm by $2\sqrt{5}$ cm b $3\sqrt{10}$ cm by $\sqrt{10}$ cm
c $\sqrt{\frac{100}{101}}$ cm by $10\sqrt{\frac{100}{101}}$ cm

14 a i 5.41 m ii 4.61 m iii 5.70 m
iv 8.70 m v 8.91 m vi 6.44 m
b 8.70 m

15 Student's own research.

Exercise 1D

1 a πr^2 b $\frac{\theta}{360} \times \pi r^2$ c s^2 d $l \times b$
e $\frac{1}{2}xy$, where x and y are the diagonals
f $\frac{1}{2}h(a + b)$ g $\frac{1}{2}bh$ h $\frac{1}{2}xy$
i bh j $\frac{1}{2}\pi r^2$ k $\frac{1}{4}\pi r^2$

2 a 30 cm^2 b 2.98 m^2
c 0.205 km^2 d 5000 cm^2
e 5 000 000 m^2 f 100 m^2
g 230 cm^2 h 53 700 mm^2
i 2700 m^2 j 10 000 000 mm^2
k 2 200 000 cm^2 l 1 450 000 cm^2

3 a 25 cm^2 b 54.60 m^2 c 1.82 km^2
d 0.03 mm^2 e 153.94 m^2 f 75 cm^2
g 1 472 m^2 h 0.05 mm^2 i 0.17 km^2
j 2.36 km^2 k 1.12 m^2 l 3.97 cm^2

4 a 2.88 cm b 14.35 m c 1.44 km
d 1.05 m e 1.91 mm f 8.89 m
g 1.26 cm h 0.52 m i 5.75 km

5 a 9π cm^2, 28.27 cm^2 b $\frac{25}{2}\pi$ m^2, 39.27 m^2
c $\frac{49}{3}\pi$ m^2, 51.31 m^2 d $\frac{26}{9}\pi$ m^2, 9.08 m^2
e 21π km^2, 65.97 km^2 f $\frac{7}{8}\pi$ mm^2, 2.75 mm^2

6 43.2 m^2

7 a $\frac{25}{8}\pi + 25$ cm^2, 34.82 cm^2
b 49 m^2
c $\frac{289}{200}\pi + \frac{104}{25}$ m^2, 8.70 m^2
d $\frac{(3969 - 441\pi)}{25}$ mm^2, 103.34 mm^2
e $81\pi + 324$ km^2, 578.47 km^2
f $\frac{49}{200}\pi - \frac{99}{400}$ m^2, 0.52 m^2

8 a 66 m^2 b 14 bags

9 a 1 : 3 b $1 : \sqrt{2}$

10 a 100 ha b 200 000 m^2 c 0.4 ha d 2.5 acres

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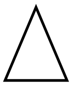
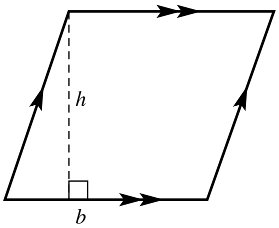
11 a $a = \frac{2A}{h} - b$

b i $3\frac{1}{3}$ **ii** 4.7 **iii** 0

c  triangle

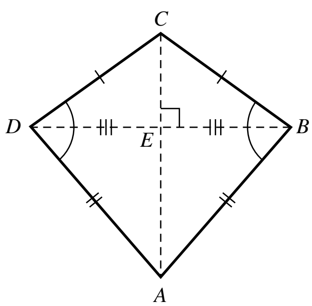
12 a



Let x be the base of the triangle.

$A = (b - x) \times h + \frac{1}{2}xh + \frac{1}{2}xh$
$A = bh - xh + xh$
$A = bh$

b



Let $x = AC$ and $y = BD$.

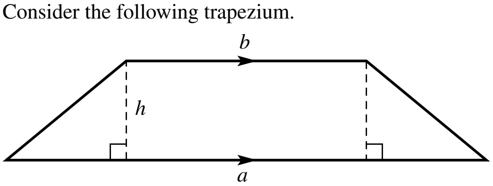
AC bisects BD, hence $DE = EB = \frac{1}{2}y$.

$A = \frac{1}{2} \times x \times \frac{1}{2}y + \frac{1}{2} \times x \times \frac{1}{2}y$

$A = \frac{1}{4}xy + \frac{1}{4}xy$

$A = \frac{1}{2}xy$

c Consider the following trapezium.



$A = bh + \frac{1}{2}(a - b)h$

$A = bh + \frac{1}{2}ah - \frac{1}{2}bh$

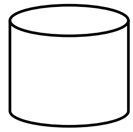
$A = h\left(\frac{1}{2}b + \frac{1}{2}a\right)$

$A = \frac{1}{2}h(a + b)$

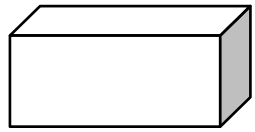
13 a 63.7% **b** 78.5%
c 50% **d** 53.9%

Exercise 1E

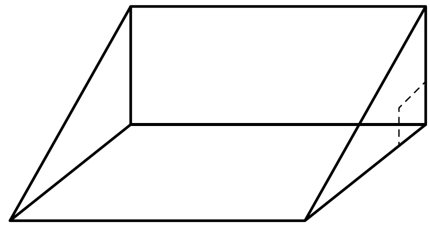
1 a



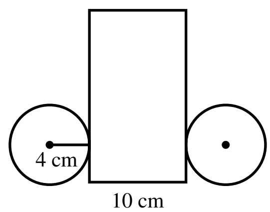
b



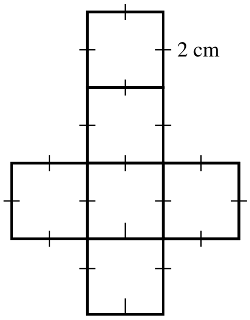
c



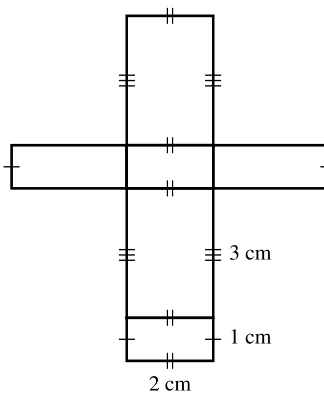
2 a



b



c



3 a 90 cm^2 **b** 47.82 mm^2 **c** 111.3 cm^2
d 920 cm^2 **e** 502.91 m^2 **f** 168.89 m^2

4 a 8.64 cm^2 **b** 96 mm^2 **c** 836.6 m^2
d 872 mm^2 **e** 4.74 cm^2 **f** 43.99 m^2

5 24.03 m^2

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6 3880 cm^2

7 a 121.3 cm^2 b 10.2 m^2
c 236.5 m^2 d 2437.8 cm^2

8 a 66.2 b 17.9
c 243.1 d 207.3
e 2308.7 f 65.0

9 a 144.5 cm^2 b 851.3 m^2
c 1192.7 cm^2 d 4170.8 m^2

10 33.5 m^2

11 a $6x^2$
b $2(ab + ac + bc)$
c $\pi\left(\frac{1}{2}d\right)^2 + \frac{1}{2}\pi dh + dh$
d $\frac{1}{2}\pi r^2 + 2rh + \frac{1}{2}\pi rh$

12 a 6π b $5\frac{1}{2}\pi$

13 a 0.79 m b 7.71 m

14 1 cm

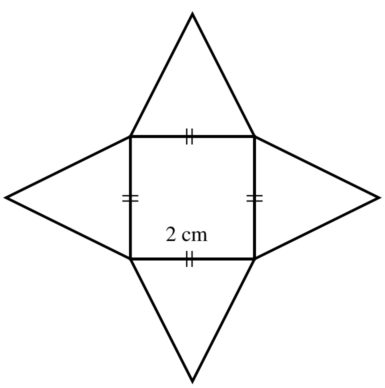
15 a $4\pi r^2$ b $2x(x + 2y)$
c $2rh + \pi r(h + r)$ d $2rh + \frac{\theta}{180}\pi r(h + r)$

Exercise 1F

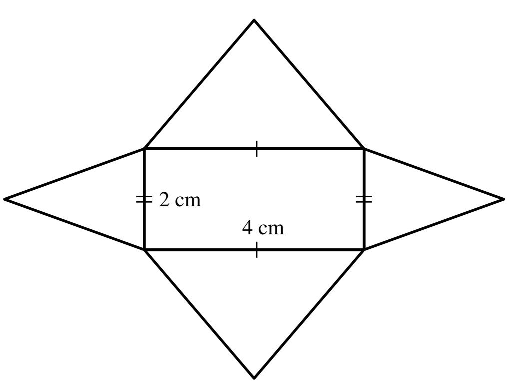
1 a $\frac{1}{2}bh$ b πr^2 c πrl

2 a $\sqrt{29}$ cm b $\sqrt{221}$ m c $\sqrt{109}$ cm

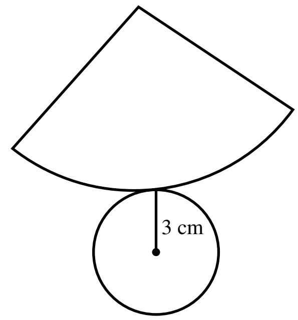
3 a



b



c



4 a 593.76 mm^2 b 0.82 m^2 c 435.90 km^2

5 a 64 m^2 b 105 cm^2 c 0.16 m^2

6 a 62.83 m^2 b 5.18 cm^2 c 1960.35 mm^2

7 a 10.44 cm b 126.7 cm^2

8 a 25.5 cm b 25.0 cm

9 a 18.9 cm b 17.8 cm

10 a 6.3 m b 66.6 m^2

11 hat B

12 a 105 cm^2 b 63 cm^2 c 163.3 cm^2
d 299.4 m^2 e 502.8 mm^2 f 76.6 m^2

13 Slant height $l = \sqrt{r^2 + h^2}$,
so $\pi r(r + l) = \pi r\left(r + \sqrt{r^2 + h^2}\right)$

14 Substitute $h = r$ into the equation given in Question 13.

$$\begin{aligned}\pi r\left(r + \sqrt{r^2 + h^2}\right) &= \pi r\left(r + \sqrt{r^2 + r^2}\right)\\ &= \pi r\left(r + \sqrt{2r^2}\right) = \pi r\left(r + \sqrt{2}r\right)\\ &= \pi r^2\left(1 + \sqrt{2}\right) \text{ as required}\end{aligned}$$

15 182.3 cm^2

16 a $4\sqrt{26}$ cm b 306.57 cm^2 c $4\sqrt{2}$ cm
d 20.199 cm e 260.53 cm^2 f 85%

Exercise 1G

1 a 80 cm^3 b 32 m^3 c 108 mm^3

2 a 2000 mm^3 b $200\,000\text{ cm}^3$ c $15\,000\,000\text{ m}^3$
d 5.7 cm^3 e 0.0283 km^3 f 0.762 m^3
g $130\,000\text{ cm}^3$ h 1000 m^3 i 2094 mm^3
j 2700 mL k 0.342 ML l 0.035 kL
m 5720 kL n 74.25 L o 18 440 L

3 a 40 cm^3 b $10\,500\text{ m}^3$ c 259.7 mm^3

4 a 785.40 m^3 b 18.85 cm^3 c 1583.36 m^3

5 a 12 cm^3 b 1570.8 m^3 c 2.448 mm^3

6 a 30 km^3 b 196 cm^3 c 30 m^3
d 10 cm^3 e 0.0021 m^3 f 4752.51 cm^3
g 0.157 m^3 h 1357.168 cm^3 i 24 m^3

7 1000

8 480 L

9 a 379.33 cm^3 b 223.17 m^3 c 6.81 m^3
d 716.46 mm^3 e 142.36 cm^3 f 42.85 cm^3

10 a 27 cm^3 b $3\sqrt{3}\text{ m}^3$

11 846.7 cm^2

12 0.5 cm

13 The student must use the perpendicular height of the oblique prism instead of 5.

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14 $V = \frac{\theta}{360}\pi r^2 h$

15 yes; 69.3 m³

16 a $\frac{1}{\sqrt{2}}$ b 5.8 m³

Exercise 1H

1 4 cm³

2 15 m³

3 a 10 m³ b $\frac{8}{3}$ cm³ c $58\frac{1}{3}$ mm³

4 a 4 cm³ b 585 m³ c 50 km³
d $\frac{8}{3}$ cm³ e 8 cm³ f 0.336 mm³

5 a 0.82 m³ b 9.38 mm³ c 25 132.74 m³
d 25.13 m³ e 0.12 m³ f 523.60 cm³

6 47 mL

7 a 282.74 m³ b 276 cm³ c 48 m³
d 56.88 mm³ e 10.35 m³ f 70.79 m³

8 11.11 cm

9 $\frac{2}{3}$

Wood wasted = volume of cylinder − volume of cone

Wood wasted $= \pi r^2 h - \frac{1}{3}\pi r^2 h$

Wood wasted $= \frac{2}{3}\pi r^2 h$

Wood wasted $= \frac{2}{3}$ of the volume of cylinder

10 a i $V = \frac{1}{3}x^2 h$ ii $V = \frac{1}{12}\pi x^2 h$
b $\frac{\pi}{4}$

11 a 3.7 cm
b i $h = \frac{3V}{\pi r^2}$ ii $r = \sqrt{\frac{3V}{\pi h}}$

12 a Similar triangles are formed, so corresponding sides are in the same ratio.
b $\frac{1}{3}\pi\left(r_1^2 h_1 - r_2^2 h_2\right)$
c i 18.3 cm³ ii 14.7 cm³

Exercise 1I

1 a 314.16 b 60.82 c 3.14
d 33.51 e 91.95 f 1436.76

2 a Calculate the surface area of a sphere with radius 5 cm.
b Calculate the surface area of a sphere with radius 2.2 cm.
c Calculate the surface area of a sphere with radius 0.5 cm.
d Calculate the volume of a sphere with radius 2 cm.
e Calculate the volume of a sphere with radius 2.8 cm.
f Calculate the volume of a sphere with radius 7 cm.

3 a 113.10 cm², 113.10 cm³ b 201.06 m², 268.08 m³
c 688.13 m², 1697.40 m³ d 15.71 mm², 5.85 mm³
e 21.99 m², 9.70 m³ f 15.21 km², 5.58 km³

4 a $\frac{1}{2}$ b $V = \frac{2}{3}\pi r^3$ c $A = 2\pi r^2$

5 a 50.27 cm², 33.51 cm³ b 3.14 m², 0.52 m³
c 18 145.84 mm², 229 847.30 mm³
d 1017.88 cm², 3053.63 cm³
e 2.66 km², 0.41 km³ f 5.81 m², 1.32 m³

6 a $r = \sqrt{\frac{A}{4\pi}}$ b $\sqrt[3]{\frac{3V}{4\pi}}$

7 a i 1.53 cm ii 3.50 cm iii 0.50 km
b i 0.89 m ii 3.09 cm iii 0.18 mm

8 a 113.10 cm³ b 5654.9 cm³ c 21 345.1 cm³

9 11.5 cm

10 52%

11 a 32.7 cm³ b 39.3 cm² c 58.9 cm²

12 1570.8 cm²

13 a 4 m b 234.6 m³

14 a 235.62 m² b 5.94 cm² c 138.23 mm²
d 94.25 m² e 27.14 m² f 26.85 cm²

15 a 5.24 m³ b 942.48 m³ c 10.09 cm³
d 1273.39 cm³ e 4.76 m³ f 0.74 cm³

16 a i 523.60 cm³ ii 4188.79 cm³
iii 14 137.17 cm³
b 61.2 cm

17 a 5 cm b $5\sqrt{5}$ cm c 332.7 cm²

18 a $r = \sqrt{\frac{A}{4\pi}}$ b $r = \sqrt[3]{\frac{3V}{4\pi}}$

19 a 4 times b 8 times

20 $V = \frac{4}{3} \times \pi r^3$

Substitute $\frac{d}{2}$ into r, giving

$V = \frac{4}{3} \times \pi\left(\frac{d}{2}\right)^3$

$V = \frac{4}{3} \times \frac{\pi d^3}{8} = \frac{1}{3} \times \frac{\pi d^3}{2}$

$V = \frac{1}{6}\pi d^3$

21 $h = \frac{4}{3}r$

22 a i $\sqrt[3]{\frac{3}{4\pi}}$ ii $\sqrt[3]{36\pi}$ iii 1
iv 6 units² v 80.6%
b i $4\pi r^2$ ii $\sqrt[3]{\frac{4\pi}{3}}r$ iii $6\left(\frac{4\pi}{3}\right)^{\frac{2}{3}} r^2$
c Proof required. Example:

$$\frac{4\pi r^2}{6\left(\frac{4\pi}{3}\right)^{\frac{2}{3}} r^2} = \frac{2\pi}{3^{\frac{1}{3}}(4\pi)^{\frac{2}{3}}} = \frac{2\pi^{\frac{1}{3}}}{8^{\frac{1}{3}} \times 6^{\frac{1}{3}}} = \sqrt[3]{\frac{\pi}{6}} \text{ as required.}$$

d They are the same.

Puzzles and challenges

1 6

2 1.3 m

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3 As the sphere touches the top, bottom and curved surface, the height of the cylinder is $2r$ and the radius of the base is r. So the curved surface area $= 2 \times \pi \times r \times h$ and $h = 2r$, therefore this equals $4\pi r^2$, which is equal to the surface area of the sphere.

4 $h = 4r$ 5 $(4 - \pi)r^2$ 6 $\sqrt{2} : 1$ 7 67%

Multiple-choice questions

1 D 2 E 3 A 4 D
5 C 6 A 7 B 8 D
9 E 10 C 11 D 12 E

Short-answer questions

1 a 23 cm b 2.7 cm^2 c 2600000 cm^3
d 8372 mL e 0.63825 m^2 f 3000000 cm^2
g 6 h 777600 i 1000
j 1000000 k 125 l 0.089
2 a 5.5 m to 6.5 m b 8.85 g to 8.95 g
c 12.045 min to 12.055 min
3 a 32 m b 28.6 m c 20.4 cm
4 a $\frac{7}{\pi}$ m b 15.60 m^2
5 a $\sqrt{65}$ b 8.31
6 a 13.02 m^2 b 216 m^2 c 38.5 m^2
d 78.54 cm^2 e 100.43 m^2 f 46.69 m^2
7 a 4.8 m b 25.48 m
8 a i 236 m^2 ii 240 m^3
b i 184 cm^2 ii 120 cm^3
c i 1407.43 cm^2 ii 4021.24 cm^3
d i 360 cm^2 ii 400 cm^3
e i 201.06 m^2 ii 268.08 m^3
f i 282.74 cm^2 ii 314.16 cm^3
9 a $\frac{175}{3\pi}$ cm b 17.6 cm
10 a 18 cm b $3\sqrt{61}$ cm c 2305.8 cm^2
11 12 m
12 a i 414.25 cm^2 ii 535.62 cm^3
b i 124 m^2 ii 88 m^3
c i 19.67 mm^2 ii 6.11 mm^3
13 a i 117.27 cm^2 ii 84.94 cm^3
b i 104 cm^2 ii 75 cm^3
c i 25.73 cm^2 ii 9.67 cm^3
14 a 4950π cm^3 b 1035π cm^2

Extended-response questions

1 a 72 m^3 b $\sqrt{37}$ m
c 138.7 m^2 d 6 L, $120
2 a 100 m b $50\sqrt{2}$ m
c 5000 m^2 d 36%
e athlete A, 0.01 seconds

Chapter 2

Pre-test

1 a 25 b 25 c -25 d -25
e 27 f $\frac{1}{9}$ g $\frac{1}{16}$ h $-\frac{1}{2}$
i 4 j 13 k 2 l 4
2 a 2^5 b $3^2 \times 4^3$ c $2a^2b$
3 a $5 \times 5 \times 5$ b $4 \times 4 \times 3$
c $m \times m \times m \times m$ d $7 \times x \times x \times x \times y \times y$
4 a y^7 b b^3 c a^{15}
d 1 e $10d^7e^3$ f $\frac{3s^2}{2}$
g $8g^3h^{12}$ h 5 i 1
5 a rational b rational c irrational
d rational e rational f irrational
6 a x b $4a - 2b$ c $-10ab + a$
7 a $3x + 12$ b $-2x - 6$
c $-2x - 10$ d $-4x + 28$
8 a $x^2 + 4x + 3$ b $x^2 + 4x - 5$
c $2x^2 - 5x - 12$ d $x^2 - 4x + 4$
9 a i $2205 ii $3258
b 15

Exercise 2A

1 a irrational b root
c non-recurring d rational
2 a irrational b rational
c rational d rational
e rational f irrational
g irrational h irrational
3 a $\frac{8}{9}$ b $\frac{17}{90}$ c $\frac{2}{11}$ d $\frac{5}{12}$
4 a yes b yes c no d no
e no f yes g yes h no
5 a 4 b 9 c 25 d 4
e 16 f 16 g 36 h 36
6 a $2\sqrt{3}$ b $3\sqrt{5}$ c $2\sqrt{6}$ d $4\sqrt{3}$
e $5\sqrt{3}$ f $10\sqrt{5}$ g $7\sqrt{2}$ h $3\sqrt{10}$
i $8\sqrt{2}$ j $6\sqrt{10}$ k $9\sqrt{2}$ l $4\sqrt{5}$
7 a $6\sqrt{2}$ b $6\sqrt{5}$ c $16\sqrt{3}$ d $6\sqrt{7}$
e $21\sqrt{2}$ f $20\sqrt{5}$ g $\sqrt{5}$ h $\sqrt{7}$
i $\frac{\sqrt{6}}{2}$ j $\frac{\sqrt{6}}{4}$ k $\frac{\sqrt{5}}{5}$ l $\frac{\sqrt{11}}{6}$
m $3\sqrt{11}$ n $2\sqrt{2}$ o $2\sqrt{2}$ p $\frac{2\sqrt{17}}{7}$
q $\frac{3\sqrt{3}}{2}$ r $4\sqrt{6}$ s $\frac{2\sqrt{3}}{3}$ t $\frac{3\sqrt{3}}{2}$

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8 **a** $\frac{2\sqrt{2}}{3}$ **b** $\frac{2\sqrt{3}}{7}$ **c** $\frac{3\sqrt{2}}{5}$ **d** $\frac{\sqrt{11}}{5}$
e $\frac{\sqrt{10}}{3}$ **f** $\frac{\sqrt{21}}{12}$ **g** $\frac{\sqrt{13}}{4}$ **h** $\frac{\sqrt{14}}{5}$
i $\frac{\sqrt{5}}{3}$ **j** $\frac{3\sqrt{3}}{2}$ **k** $\frac{\sqrt{5}}{2\sqrt{2}}$ **l** $\frac{\sqrt{14}}{\sqrt{19}}$

9 **a** $\sqrt{12}$ **b** $\sqrt{32}$ **c** $\sqrt{50}$ **d** $\sqrt{27}$
e $\sqrt{45}$ **f** $\sqrt{108}$ **g** $\sqrt{128}$ **h** $\sqrt{700}$
i $\sqrt{810}$ **j** $\sqrt{125}$ **k** $\sqrt{245}$ **l** $\sqrt{363}$

10 **a** $15\sqrt{3}$ **b** $13\sqrt{7}$ **c** $19\sqrt{5}$ **d** $31\sqrt{3}$

11 **a** $4\sqrt{2}$ m **b** $2\sqrt{30}$ cm **c** $4\sqrt{15}$ mm

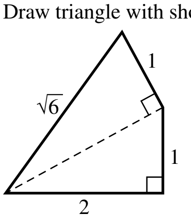
12 **a** radius $= 2\sqrt{6}$ cm, diameter $= 4\sqrt{6}$ cm
b radius $= 3\sqrt{6}$ m, diameter $= 6\sqrt{6}$ m
c radius $= 8\sqrt{2}$ m, diameter $= 16\sqrt{2}$ m

13 **a** $2\sqrt{5}$ cm **b** $3\sqrt{5}$ m **c** $\sqrt{145}$ mm
d $\sqrt{11}$ m **e** $\sqrt{11}$ mm **f** $2\sqrt{21}$ cm

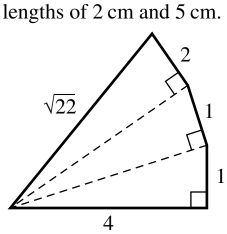
14 $\sqrt{72} = \sqrt{36} \times 2$ (i.e. 36 is highest factor of 72)
$= 6\sqrt{2}$

15 **a** 9, 25, 225 **b** $15\sqrt{2}$

16 **a** Draw triangle with shorter side lengths of 1 cm and 3 cm.
b Draw triangle with shorter side lengths of 2 cm and 5 cm.
c



d



17 Check with your teacher.

Exercise 2B

1 **a** yes **b** no **c** no **d** yes
e yes **f** no **g** yes **h** yes

2 **a** $6x$, $6\sqrt{2}$ **b** $7y$, $7\sqrt{2}$ **c** y, $\sqrt{2}$

3 **a** $4\sqrt{3}$
b **i** $5\sqrt{3}$ **ii** $-3\sqrt{3}$ **iii** $17\sqrt{3}$

4 **a** $10\sqrt{2}$
b **i** $5\sqrt{2}$ **ii** $-16\sqrt{2}$ **iii** 0

5 **a** $6\sqrt{5}$ **b** $3\sqrt{3}$ **c** $4\sqrt{2}$ **d** $3\sqrt{2}$
e $11\sqrt{5}$ **f** $\sqrt{3}$ **g** $6\sqrt{10}$ **h** $5\sqrt{2}$
i $-2\sqrt{21}$ **j** $-6\sqrt{11}$ **k** $-\sqrt{13}$ **l** $-7\sqrt{30}$

6 **a** $\sqrt{3} + 5\sqrt{2}$ **b** $3\sqrt{6} + 7\sqrt{11}$ **c** $4\sqrt{5} - 7\sqrt{2}$
d $-2\sqrt{2} + \sqrt{5}$ **e** $4\sqrt{3}$ **f** 0
g $-3\sqrt{2} - 3\sqrt{10}$ **h** $-2\sqrt{5} + 3\sqrt{15}$

7 **a** $\sqrt{2}$ **b** $5\sqrt{2}$ **c** $4\sqrt{3}$ **d** $\sqrt{5}$
e $7\sqrt{2}$ **f** $12\sqrt{3}$ **g** $8\sqrt{11}$ **h** $3\sqrt{2}$
i $5\sqrt{6}$ **j** $\sqrt{5}$ **k** $32\sqrt{2}$ **l** $20\sqrt{2}$

8 **a** $13\sqrt{2}$ **b** $9\sqrt{6}$
c $2\sqrt{5} - \sqrt{7}$ **d** $5\sqrt{5} + 6\sqrt{7}$
e $\sqrt{6} - 3\sqrt{2}$ **f** $2\sqrt{3} + 11\sqrt{5} - 5\sqrt{2}$
g $9\sqrt{3} + 2\sqrt{2}$ **h** $11 - 9\sqrt{3}$
i $9 + 18\sqrt{2}$ **j** $-9\sqrt{2} + 15\sqrt{5}$

9 **a** $\frac{5\sqrt{3}}{6}$ **b** $\frac{7\sqrt{5}}{12}$ **c** $\frac{\sqrt{2}}{30}$
d $\frac{\sqrt{7}}{6}$ **e** $\frac{-\sqrt{2}}{10}$ **f** $\frac{13\sqrt{3}}{14}$
g $\frac{13\sqrt{5}}{18}$ **h** $\frac{-7\sqrt{3}}{30}$ **i** $\frac{-11\sqrt{10}}{24}$

10 **a** $4\sqrt{3} + 2\sqrt{5}$ cm **b** $14\sqrt{2}$ cm
c $\sqrt{10} + 3\sqrt{2}$ cm **d** $2\sqrt{10} + 4\sqrt{5}$ cm
e $4\sqrt{3} + \sqrt{30}$ cm **f** $12\sqrt{3}$ cm

11 **a** $\sqrt{20} = 2\sqrt{5}$
b $3\sqrt{72} = 18\sqrt{2}$, $\sqrt{338} = 13\sqrt{2}$

12 **a** $5\sqrt{3} - 6\sqrt{3} + \sqrt{3} = 0$ **b** $\sqrt{6} + 2\sqrt{6} - 3\sqrt{6} = 0$
c $6\sqrt{2} - 8\sqrt{2} + 2\sqrt{2} = 0$ **d** $2\sqrt{2} - 3\sqrt{2} + \sqrt{2} = 0$
e $4\sqrt{5} - 7\sqrt{5} + 3\sqrt{5} = 0$
f $3\sqrt{2} - 6\sqrt{3} - 5\sqrt{2} + 6\sqrt{3} + 2\sqrt{2} = 0$

13 **a** $6\sqrt{3} - 3\sqrt{2}$, unlike surds
b $8\sqrt{2} + 2\sqrt{5}$, unlike surds
c $5\sqrt{2} - 6\sqrt{5}$, unlike surds
d $10\sqrt{10} + 10\sqrt{3}$, unlike surds
e $20\sqrt{2} + 30\sqrt{3}$, unlike surds
f $4\sqrt{5} - 6\sqrt{6}$, unlike surds

14 **a** $\frac{7\sqrt{2}}{15}$ **b** $\frac{2\sqrt{3}}{3}$ **c** $\frac{\sqrt{5}}{12}$ **d** $\frac{-3\sqrt{2}}{4}$
e $\frac{\sqrt{3}}{2}$ **f** $\frac{-7\sqrt{7}}{15}$ **g** $-\sqrt{2}$ **h** $\frac{29\sqrt{6}}{28}$
i 0 **j** $8\sqrt{3}$ **k** $\frac{6\sqrt{6}}{35}$ **l** $\frac{29\sqrt{5}}{42}$

Exercise 2C

1 **a** 3 **b** 5 **c** 1 **d** 5 **e** 12 **f** 18

2 **a** $\sqrt{15}$ **b** $\sqrt{21}$ **c** $\sqrt{26}$ **d** $\sqrt{35}$ **e** $-\sqrt{30}$
f $-\sqrt{30}$ **g** $\sqrt{66}$ **h** $\sqrt{6}$ **i** $\sqrt{70}$

3 **a** $\sqrt{10}$ **b** $\sqrt{6}$ **c** $-\sqrt{3}$ **d** $-\sqrt{5}$ **e** $\sqrt{3}$
f $\sqrt{10}$ **g** $\sqrt{5}$ **h** $-\sqrt{13}$ **i** $-\sqrt{5}$

4 **a** $\sqrt{21}$ **b** $\sqrt{10}$ **c** $\sqrt{30}$ **d** 3
e 5 **f** 9 **g** $7\sqrt{2}$ **h** $2\sqrt{11}$
i $3\sqrt{6}$ **j** $5\sqrt{2}$ **k** $4\sqrt{6}$ **l** 10

5 **a** $10\sqrt{3}$ **b** $21\sqrt{2}$ **c** $12\sqrt{14}$ **d** $-50\sqrt{3}$
e $-18\sqrt{3}$ **f** $15\sqrt{5}$ **g** $42\sqrt{6}$ **h** $-60\sqrt{10}$
i $-20\sqrt{10}$ **j** $42\sqrt{2}$ **k** $24\sqrt{30}$ **l** $216\sqrt{7}$

6 **a** $2\sqrt{2}$ **b** $3\sqrt{6}$ **c** $\frac{\sqrt{5}}{2}$
d $\frac{-4}{\sqrt{13}}$ **e** $\frac{-1}{3\sqrt{7}}$ **f** $\frac{2\sqrt{5}}{3}$

7 **a** $\sqrt{6} + \sqrt{15}$ **b** $\sqrt{14} - \sqrt{10}$
c $-\sqrt{55} - \sqrt{65}$ **d** $-2\sqrt{15} - 2\sqrt{21}$
e $6\sqrt{26} - 3\sqrt{22}$ **f** $20 - 20\sqrt{2}$
g $30\sqrt{2} + 15\sqrt{30}$ **h** $-12\sqrt{3} + 12\sqrt{2}$
i $42 + 63\sqrt{2}$ **j** $90\sqrt{3} - 24\sqrt{10}$
k $-16 + 24\sqrt{10}$ **l** $42\sqrt{2} + 30$

8 **a** 28 **b** 18 **c** -75
d $\sqrt{2} - \sqrt{6}$ **e** $3\sqrt{3} + 4$ **f** $-\sqrt{10} + \sqrt{5}$
g 2 **h** $8\sqrt{2}$ **i** $\sqrt{2} - 6$

9 **a** $2\sqrt{6}$ **b** $\sqrt{30}$ **c** 6

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10 a $\frac{3}{4}$ cm^2 b $2\sqrt{6}$ cm

11 a $\sqrt{6} \times \sqrt{6} = \sqrt{6 \times 6} = \sqrt{36} = 6$
b $-\sqrt{8} \times \sqrt{8} = -\sqrt{8 \times 8} = -\sqrt{64} = -8$
c $-\sqrt{5} \times -\sqrt{5} = +\sqrt{5 \times 5} = \sqrt{25} = 5$

12 a Simplify each surd before multiplying.
b Allows for the multiplication of smaller surds, which is simpler.
c i $3\sqrt{2} \times 3\sqrt{3} = 9\sqrt{6}$
ii $2\sqrt{6} \times 2\sqrt{5} = 4\sqrt{30}$
iii $5\sqrt{2} \times 3\sqrt{5} = 15\sqrt{10}$
iv $3\sqrt{6} \times 5\sqrt{3} = 45\sqrt{2}$
v $6\sqrt{2} \times 4\sqrt{3} = 24\sqrt{6}$
vi $6\sqrt{3} \times -10\sqrt{5} = -60\sqrt{15}$
vii $-12\sqrt{3} \times -2\sqrt{7} = 24\sqrt{21}$
viii $7\sqrt{2} \times 10\sqrt{3} = 70\sqrt{6}$
ix $12\sqrt{2} \times 12\sqrt{5} = 144\sqrt{10}$

13 a 3 b 2 c -9 d $-\frac{1}{5}$ e $\frac{2}{5}$ f 3

14 a $54\sqrt{2}$ b $375\sqrt{3}$ c $162\sqrt{3}$ d 25
e 9 f $128\sqrt{2}$ g $-120\sqrt{5}$ h $-108\sqrt{2}$
i 720 j $14\sqrt{7}$ k $\frac{27\sqrt{2}}{2}$ l 81
m $100\sqrt{3}$ n 144 o $-96\sqrt{15}$ p $\frac{81\sqrt{3}}{25}$
q $\frac{5}{3\sqrt{3}}$ r $\frac{9\sqrt{6}}{2}$

Exercise 2D

1 a $\sqrt{21}$ b $-\sqrt{10}$ c $6\sqrt{6}$
d 11 e 13 f 12
g 125 h 147 i 162

2 a 0 b 0 c 0
d 0 e $2\sqrt{3}$ f $6\sqrt{3}$

3 a $x^2 + 5x + 6$ b $x^2 - 4x - 5$
c $x^2 + x - 12$ d $2x^2 - 9x - 5$
e $6x^2 - 11x - 10$ f $6x^2 - 17x - 28$
g $x^2 - 16$ h $4x^2 - 9$
i $25x^2 - 36$ j $x^2 + 4x + 4$
k $4x^2 - 4x + 1$ l $9x^2 - 42x + 49$

4 a $-\sqrt{2} - 4$ b $2\sqrt{5} - 3$ c $4 + \sqrt{6}$
d $7 + 3\sqrt{3}$ e $5 + \sqrt{7}$ f $-13 - 2\sqrt{2}$
g $6\sqrt{5} - 13$ h $30 - 9\sqrt{10}$ i $23 - 8\sqrt{7}$

5 a $27 + 12\sqrt{2}$ b $21 + 2\sqrt{3}$ c $25 + 32\sqrt{5}$
d $2\sqrt{6} - 118$ e $35 - 13\sqrt{10}$ f $18\sqrt{7} - 65$
g $23\sqrt{3} - 46$ h $43 - 19\sqrt{2}$ i $24\sqrt{5} - 89$

6 a $14 - 6\sqrt{5}$ b $10 - 4\sqrt{6}$ c $23 + 8\sqrt{7}$
d $15 + 4\sqrt{11}$ e $28 + 10\sqrt{3}$ f $54 - 14\sqrt{5}$
g $9 + 2\sqrt{14}$ h $13 - 2\sqrt{22}$ i $13 - 2\sqrt{30}$
j $32 + 2\sqrt{247}$ k $40 + 2\sqrt{391}$ l $60 - 2\sqrt{899}$

7 a 7 b 19 c 13 d 6 e 4
f -6 g 3 h 6 i -4

8 a $118 + 28\sqrt{10}$ b $139 + 24\sqrt{21}$ c $195 + 30\sqrt{30}$
d $176 - 64\sqrt{6}$ e $207 - 36\sqrt{33}$ f $87 - 12\sqrt{42}$
g $66 + 36\sqrt{2}$ h $140 + 60\sqrt{5}$ i $107 - 40\sqrt{6}$

9 a 97 b 17 c 41 d 163 e 26
f 10 g 0 h -33 i -40

10 a $7 + 4\sqrt{3}$ cm^2
b 2 m^2
c $15\sqrt{6} - 5\sqrt{2} - 18 + 2\sqrt{3}$ mm^2
d $5 + 6\sqrt{5}$ m^2
e $\frac{7\sqrt{6} - 7\sqrt{2} - \sqrt{3} + 1}{2}$ cm^2
f $81 - 30\sqrt{2}$ mm^2

11 a -5 b 7 c 128

12 a $11\sqrt{21} - 26\sqrt{3}$ b $2\sqrt{5} + \sqrt{30}$
c $5\sqrt{35} + 31\sqrt{7}$ d $19\sqrt{7} + \sqrt{2}$
e $\sqrt{2} - 2\sqrt{6}$ f $\sqrt{10} + 3\sqrt{5}$

13 a $a = 4, b = 2$ b $a = 21, b = -8$
c $a = 14, b = 4$ d $a = 9, b = 32$

14 Yes. Possible example: $a = \sqrt{12}, b = \sqrt{3}$

15 a $19 - 2\sqrt{6}$ b 16 c $2\sqrt{15} - 85$
d $10\sqrt{3} - 37$ e $30 - 10\sqrt{2}$ f 0
g $4\sqrt{3} - 14$ h $47\sqrt{2} - 10\sqrt{30} + 11$

Exercise 2E

1 a 1 b 1 c $\frac{1}{2}$ d $-\frac{1}{2}$
e -2 f -9 g 6 h -1

2 a $\sqrt{3}$ b $\sqrt{5}$ c 10
d $\sqrt{5}$ e $\sqrt{3}$ f $\sqrt{7}$
g $\frac{\sqrt{3}}{\sqrt{3}}$ h $\frac{\sqrt{7}}{\sqrt{7}}$ i $\frac{\sqrt{13}}{\sqrt{13}}$

3 a 0.377... b 2.886... c 16.31...
Notice that all pairs of numbers are equal.

4 a $\frac{\sqrt{2}}{2}$ b $\frac{\sqrt{7}}{7}$ c $\frac{3\sqrt{11}}{11}$ d $\frac{4\sqrt{5}}{5}$
e $\frac{5\sqrt{3}}{3}$ f $4\sqrt{2}$ g $\frac{\sqrt{15}}{3}$ h $\frac{\sqrt{14}}{7}$

5 a $\frac{\sqrt{6}}{3}$ b $\frac{\sqrt{35}}{7}$ c $\frac{\sqrt{66}}{11}$ d $\frac{\sqrt{10}}{5}$
e $\frac{\sqrt{21}}{3}$ f $\frac{\sqrt{42}}{7}$ g $\frac{\sqrt{30}}{3}$ h $\frac{\sqrt{34}}{2}$

6 a $\frac{4\sqrt{14}}{7}$ b $\frac{5\sqrt{6}}{3}$ c $\frac{3\sqrt{10}}{2}$
d $\frac{3\sqrt{42}}{7}$ e $\frac{7\sqrt{30}}{10}$ f $\frac{2\sqrt{105}}{15}$

7 a $\frac{4\sqrt{21}}{15}$ b $\frac{\sqrt{6}}{3}$ c $\frac{\sqrt{35}}{3}$ d $\frac{2\sqrt{2}}{5}$
e $\frac{2\sqrt{5}}{15}$ f $\frac{10}{9}$ g $\frac{9\sqrt{2}}{2}$ h $\frac{3\sqrt{7}}{2}$

8 a $\frac{\sqrt{3} + \sqrt{6}}{3}$ b $\frac{3\sqrt{7} + \sqrt{35}}{7}$ c $\frac{2\sqrt{5} - \sqrt{15}}{5}$
d $\frac{\sqrt{6} - \sqrt{10}}{2}$ e $\frac{\sqrt{35} + \sqrt{14}}{7}$ f $\frac{\sqrt{30} - \sqrt{21}}{3}$
g $\frac{2\sqrt{3} + \sqrt{42}}{6}$ h $\frac{5\sqrt{2} + 2\sqrt{5}}{10}$ i $\frac{\sqrt{30} - 5\sqrt{2}}{5}$
j $\frac{8\sqrt{3} - 15\sqrt{2}}{6}$ k $\frac{3\sqrt{2} + 2\sqrt{5}}{2}$ l $\frac{6\sqrt{5} + 5\sqrt{6}}{2}$

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9 a $\frac{5\sqrt{3}}{3}$ cm^2 **b** $\frac{2}{3}$ m^2 **c** $\frac{\sqrt{10}+\sqrt{15}}{10}$ mm^2

10 a $\frac{2\sqrt{3}+3\sqrt{2}}{6}$ **b** $\frac{6\sqrt{5}+5\sqrt{2}}{10}$ **c** $\frac{9\sqrt{7}-14\sqrt{3}}{21}$

d $\frac{5\sqrt{3}-2\sqrt{2}}{6}$ **e** $\frac{2\sqrt{2}+5\sqrt{3}}{12}$ **f** $\frac{9\sqrt{5}+4\sqrt{3}}{30}$

g $\frac{-2\sqrt{14}}{15}$ **h** $\frac{6\sqrt{30}+4\sqrt{6}}{9}$ **i** $\frac{3\sqrt{10}-2\sqrt{42}}{9}$

11 Does not change because $\frac{\sqrt{x}}{\sqrt{x}}$ is equal to1.

12 a $\frac{\sqrt{21}+\sqrt{7a}}{7}$ **b** $\frac{\sqrt{30}+\sqrt{5a}}{5}$ **c** $\frac{2\sqrt{3}+\sqrt{6a}}{6}$

d $1-\sqrt{3a}$ **e** $1-\sqrt{5a}$ **f** $1-\sqrt{7a}$

g $\frac{4\sqrt{10a}+5\sqrt{2}}{10}$ **h** $\frac{\sqrt{6a}+\sqrt{2}}{2}$ **i** $\frac{2\sqrt{14a}+7\sqrt{2}}{14}$

13 a i 14 **ii** 2 **iii** 47

b Each question is a difference of two squares, and each answer is an integer.

c $\frac{4+\sqrt{2}}{4+\sqrt{2}}$

d i $\frac{12+3\sqrt{2}}{14}$ **ii** $\frac{-3\sqrt{3}-3}{2}$

iii $2\sqrt{2}+\sqrt{6}$ **iv** $\frac{-(6+2\sqrt{30})}{7}$

14 a $\frac{5\sqrt{3}-5}{2}$ **b** $2\sqrt{3}+2$ **c** $3\sqrt{5}+6$

d $-4-4\sqrt{2}$ **e** $\frac{-3-3\sqrt{3}}{2}$ **f** $\frac{42+7\sqrt{7}}{29}$

g $-12-4\sqrt{10}$ **h** $-14-7\sqrt{5}$ **i** $\frac{2\sqrt{11}+2\sqrt{2}}{9}$

j $2\sqrt{5}-2\sqrt{2}$ **k** $\sqrt{7}-\sqrt{3}$ **l** $\frac{\sqrt{14}-\sqrt{2}}{6}$

m $\frac{6+\sqrt{6}}{5}$ **n** $\sqrt{14}+2\sqrt{2}$ **o** $10-4\sqrt{5}$

p $\frac{b\sqrt{a}-b\sqrt{b}}{a-b}$ **q** $\frac{a\sqrt{a}+a\sqrt{b}}{a-b}$ **r** $\frac{a+b-2\sqrt{ab}}{a-b}$

s $\frac{a-\sqrt{ab}}{a-b}$ **t** $\frac{a\sqrt{b}+b\sqrt{a}}{a-b}$

Exercise 2F

1 a 8 **b** 27 **c** 16 **d** 125 **e** 100 **f** 12

g 36 **h** 1 **i** $\frac{4}{9}$ **j** 50 **k** 100 **l** 72

2 a 2^8 **b** 2^2 **c** 2^{15} **d** 5^5 **e** 5^3 **f** 5^{-3}

3 a $x=5$ **b** $x=3$ **c** $x=5$

d $x=0$ **e** $x=1$ **f** $x=2$

4 a a^9 **b** x^5 **c** b^6 **d** $14m^5$

e $6s^7$ **f** $2t^{16}$ **g** $\frac{p^3}{5}$ **h** $\frac{c^7}{6}$

i $\frac{9}{25}s^2$ **j** $6x^3y^3$ **k** $15a^3b^6$ **l** $18v^9w^2$

m $150x^5y^6$ **n** $12r^7s^6$ **o** $20m^8n^{10}$

5 a x^3 **b** a **c** q^3 **d** b^4 **e** y^5

f d^5 **g** j **h** m^6 **i** $2xy^3$ **j** $3r^2s$

k $2p^2$ **l** $2m^4x$ **m** $5b^3$ **n** $4st$ **o** $\frac{1}{4}y^2$

p $\frac{1}{2}a$ **q** $-\frac{x}{3}$ **r** $\frac{-y^2}{2}$

6 a x^{10} **b** t^6 **c** $4a^6$ **d** $5y^{15}$

e $64t^6$ **f** $4u^4$ **g** $27r^9$ **h** $81p^{16}$

i $\frac{a^4}{b^6}$ **j** $\frac{x^9}{y^{12}}$ **k** $\frac{x^4y^6}{z^8}$ **l** $\frac{u^{16}w^8}{v^8}$

m $\frac{27f^6}{125g^3}$ **n** $\frac{9a^4b^2}{4p^2q^6}$ **o** $\frac{a^3t^9}{27g^{12}}$ **p** $\frac{256p^8q^{12}}{81r^4}$

7 a 8 **b** 3 **c** 1 **d** 1

e 5 **f** 3 **g** -5 **h** 3

8 a x^8 **b** x^2y^2 **c** x^6n^8 **d** xy^2

e m **f** r^4s^7 **g** $\frac{9x^8y^2}{2}$ **h** $2y^4$

i $2a^2b^2$ **j** $27m^7n^{14}$ **k** $-45a^8b^5$ **l** $\frac{16}{3}f^3$

m $2m^6n^3$ **n** $21y^3z^2$ **o** 1 **p** $-6m^2n^7$

9 a -27 **b** -27 **c** 81 **d** -81

10 a x^{12} **b** a^{105} **c** $\frac{a^{30}}{b^{15}}$

11 a 13 **b** 18 **c** 81 **d** 64

e 1 **f** 1 **g** 9 **h** 8

12 Billy has not included the minus sign inside the brackets; i.e. has applied it only afterwards. Need $(-2)^4$ not -2^4.

13 a 3 **b** 4 **c** 1 **d** 3 **e** 4 **f** -1

14 a 9 **b** 2 **c** 162 **d** -18

15 a ± 2 **b** 5 **c** 2 **d** $\frac{7}{2}$

16 a $x=2, y=4$ or $x=4, y=2$ or $x=16, y=1$

b $x=8, y=2$ or $x=4, y=3$ or $x=64, y=1$ or $x=2, y=6$

c $x=9, y=2$ or $x=3, y=4$ or $x=81, y=1$

d $x=1$, $y=$ any real number, but in this situation $y=$ any positive integer.

Exercise 2G

1

10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}
1000	100	10	1	0.1	0.01	0.001

2 a $\frac{1}{4}$ **b** $\frac{1}{10}$ **c** $\frac{1}{x}$ **d** 2

e $\frac{2}{3}$ **f** 4 **g** $\frac{4}{3}$ **h** $\frac{y}{x}$

3 a $\frac{1}{9}$ **b** $\frac{1}{16}$ **c** $\frac{1}{25}$ **d** $\frac{1}{16}$

e 9 **f** $\frac{9}{4}$ **g** $\frac{4}{9}$ **h** 32

4 a $\frac{1}{x^5}$ **b** $\frac{1}{a^4}$ **c** $\frac{2}{m^4}$ **d** $\frac{3}{y^7}$

e $\frac{3a^2}{b^3}$ **f** $\frac{4m^3}{n^3}$ **g** $\frac{10y^5z}{x^2}$ **h** $\frac{3z^3}{x^4y^2}$

i $\frac{q^3r}{3p^2}$ **j** $\frac{d^2f^5}{5e^4}$ **k** $\frac{3u^2w^7}{8v^6}$ **l** $\frac{2b^3}{5c^5d^2}$

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5 **a** x^2 **b** $2y^3$ **c** $4m^7$ **d** $3b^5$
e $2b^4d^3$ **f** $3m^2n^4$ **g** $\frac{4b^4a^3}{3}$ **h** $\frac{5h^3g^3}{2}$

6 **a** x **b** a^3 **c** $\frac{2}{b^4}$ **d** $\frac{3}{y^3}$
e $\frac{6}{a^2}$ **f** $\frac{12}{x}$ **g** $\frac{-10}{m^6}$ **h** $\frac{-18}{a^{10}}$
i $\frac{2x}{3}$ **j** $\frac{7d^2}{10}$ **k** $\frac{1}{2}$ **l** $\frac{3b^2}{4}$
m $\frac{5}{3s^3}$ **n** $\frac{4}{3f^2}$ **o** $\frac{1}{2d^2}$ **p** $\frac{5}{6t^2}$

7 **a** $\frac{16}{x^4}$ **b** $\frac{1}{64m^6}$ **c** $\frac{2}{x^{21}}$ **d** $\frac{4}{d^6}$
e $\frac{9}{t^8}$ **f** $\frac{5}{x^4}$ **g** $\frac{81}{x^{20}}$ **h** $\frac{-8}{x^{15}}$
i $\frac{y^4}{16}$ **j** $\frac{h^{12}}{81}$ **k** $7j^8$ **l** $2t^6$

8 **a** $\frac{1}{xy}$ **b** $\frac{4y^2}{a^3}$ **c** $\frac{6}{a^5b^2}$ **d** $\frac{18b^4}{a^2}$
e $\frac{a}{b^3}$ **f** $\frac{q}{p^5}$ **g** $\frac{a^2}{b^2}$ **h** $\frac{m^2}{n}$
i $\frac{p}{q^2r}$ **j** $\frac{x}{2y}$ **k** $\frac{4m}{7n^3}$ **l** $\frac{4r^3s^7}{3}$
m $\frac{f^5}{g^5}$ **n** $\frac{1}{r^6s^2}$ **o** $\frac{wx^5}{2}$ **p** $\frac{5c^5d^4}{4}$

9 **a** a^7b^2 **b** $\frac{16p^4}{9q^2}$ **c** $54x^7y^{10}$
d $4a^8b^3$ **e** $\frac{324r^{11}}{s}$ **f** $\frac{2y^{14}}{x^3}$
g a^2b^{18} **h** $\frac{m^{14}}{n^8}$ **i** $\frac{27x}{2y}$

10 **a** $\frac{1}{25}$ **b** $\frac{1}{64}$ **c** $\frac{2}{49}$ **d** $\frac{-5}{81}$
e $\frac{1}{9}$ **f** 1 **g** 98 **h** −48
i $\frac{9}{4}$ **j** $\frac{-64}{125}$ **k** $\frac{1}{16}$ **l** 100

11 0.0041 cm

12 **a** **i** $\frac{3}{2}$ **ii** $\frac{7}{5}$ **iii** $\frac{y}{2x}$
b $\frac{b}{a}$

13 The negative index should be applied to x only, not to 2:
$2x^{-2} = \frac{2}{x^2}$

14 **a** $\frac{5}{6}$ **b** $\frac{5}{18}$ **c** $\frac{1}{3}$
d $-\frac{7}{12}$ **e** $\frac{71}{48}$ **f** $\frac{106}{9}$

15 Proof: $\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-1\times x} = 2^{-x}$

16 **a** −2 **b** −5 **c** −3 **d** −1
e −2 **f** −3 **g** −3 **h** −4
i 0 **j** 0 **k** 1 **l** 2
m −2 **n** 1 **o** −2 **p** 2

Exercise 2H

1 **a** 3 **b** 4 **c** 3 **d** 2

2 **a** 10^3 **b** 10^7 **c** 10^{-6} **d** 10^{-3}

3 **a** 4.3×10^4 **b** 7.12×10^5 **c** 9.012×10^5
d 1.001×10^4 **e** 7.8×10^{-4} **f** 1.01×10^{-3}
g 3×10^{-5} **h** 3.00401×10^{-2}

4 **a** 3120 **b** 54 293
c 710 500 **d** 8 213 000
e 59 500 **f** 800 200
g 10 120 **h** 9 990 000
i 210 500 000 **j** 55 000
k 2 350 000 000 **l** 1 237 000 000 000

5 **a** 0.0045 **b** 0.0272
c 0.0003085 **d** 0.00783
e 0.000092 **f** 0.265
g 0.0001002 **h** 0.000006235
i 0.98 **j** 0.000000000545
k 0.000000000003285 **l** 0.000000875

6 **a** 6.24×10^3 **b** 5.73×10^5 **c** 3.02×10^4
d 4.24×10^5 **e** 1.01×10^4 **f** 3.50×10^7
g 7.25×10^4 **h** 3.56×10^5 **i** 1.10×10^8
j 9.09×10^5 **k** 4.56×10^6 **l** 9.83×10^9

7 **a** 2.42×10^{-3} **b** 1.88×10^{-2} **c** 1.25×10^{-4}
d 7.87×10^{-3} **e** 7.08×10^{-4} **f** 1.14×10^{-1}
g 6.40×10^{-6} **h** 7.89×10^{-5} **i** 1.30×10^{-4}
j 7.01×10^{-7} **k** 9.89×10^{-9} **l** 5.00×10^{-4}

8 **a** 2.4×10^4 **b** 5.71×10^6 **c** 7.0×10^8
d 4.88×10^3 **e** 1.9×10^{-3} **f** 7.05×10^{-4}
g 9.8×10^{-6} **h** 3.571×10^{-1} **i** 5.00×10^{-5}

9 **a** 7.7×10^6 km^2 **b** 2.5×10^6 **c** 7.4×10^9 km
d 1×10^{-2} cm **e** 1.675×10^{-27} kg **f** 9.5×10^{-13} g

10 **a** 2.85×10^{-3} **b** 1.55×10^{-3} **c** 4.41×10^{-8}
d 6.38×10^{-3} **e** 8.00×10^7 **f** 3.63×10^8
g 1.80×10^{-3} **h** 3.42×10^{15} **i** 8.31×10^{-2}

11 328 min

12 3^8 is larger than 10.

13 **a** 2.1×10^4 **b** 3.94×10^9 **c** 6.004×10^1
d 1.79×10^{-4} **e** 2×10^3 **f** 7×10^{-1}
g 1×10^7 **h** 6×10^6 **i** 4×10^{-3}
j 3.1×10^{-14} **k** 2.103×10^{-4} **l** 9.164×10^{-21}

14 **a** 9×10^4 **b** 8×10^9 **c** 6.4×10^9
d 1.44×10^{-8} **e** 4×10^4 **f** 6.25×10^{-12}
g 2.25×10^{-6} **h** 1.25×10^7 **i** 1×10^{-5}
j 1.275×10^{-4} **k** 1.8×10^{-1} **l** 2×10^2
m 8×10^{-1} **n** 4×10^{-14} **o** 2.5×10^4

15 **a** **i** 9×10^{17} J **ii** 2.34×10^{21} J
iii 2.7×10^{15} J **iv** 9×10^{11} J
b **i** 1.11×10^8 kg **ii** 4.2×10^{-1} kg
iii 9.69×10^{-13} kg **iv** 1.89×10^{-19} kg
c 5.4×10^{41} J

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Exercise 2I

1 **a** 3 **b** 5 **c** 11 **d** 25 **e** 2 **f** 3
g 5 **h** 4 **i** 2 **j** 3 **k** 2 **l** 10

2 **a** 1.91, 1.91 **b** 1.58, 1.58 **c** 1.43, 1.43

3 **a** $x = \frac{1}{3}$ **b** $x = \frac{1}{4}$ **c** $x = \frac{1}{2}$
d $x = \frac{1}{5}$ **e** $x = \frac{1}{2}$ **f** $x = \frac{1}{3}$

4 **a** $29^{\frac{1}{2}}$ **b** $35^{\frac{1}{3}}$ **c** $x^{\frac{2}{5}}$ **d** $b^{\frac{3}{4}}$
e $2^{\frac{1}{2}}a^{\frac{1}{2}}$ **f** $4^{\frac{1}{3}}t^{\frac{7}{3}}$ **g** $10^{\frac{1}{5}}t^{\frac{2}{5}}$ **h** $8^{\frac{1}{8}}m^{\frac{1}{2}}$

5 **a** $7x^{\frac{5}{2}}$ **b** $6n^{\frac{7}{3}}$ **c** $3y^3$
d $5p^{\frac{2}{3}}r^{\frac{1}{3}}$ **e** $2a^{\frac{4}{3}}b^{\frac{2}{3}}$ **f** $2g^{\frac{3}{4}}h^{\frac{5}{4}}$
g $5^{\frac{3}{2}}$ **h** $7^{\frac{3}{2}}$ **i** $4^{\frac{4}{3}}$

6 **a** $\sqrt[5]{2}$ **b** $\sqrt[7]{8}$ **c** $\sqrt[3]{6}$ **d** $\sqrt[10]{11}$
e $\sqrt[3]{3}$ **f** $\sqrt[3]{49}$ **g** $\sqrt[5]{8}$ **h** $\sqrt[7]{81}$

7 **a** 6 **b** 3 **c** 4 **d** 7
e 2 **f** 5 **g** $\frac{1}{3}$ **h** $\frac{1}{2}$
i $\frac{1}{3}$ **j** $\frac{1}{10}$ **k** $\frac{1}{20}$ **l** $\frac{1}{10}$

8 **a** 4 **b** 8 **c** 216 **d** 32
e $\frac{1}{8}$ **f** $\frac{1}{9}$ **g** $\frac{1}{16}$ **h** $\frac{1}{125}$
i $\sqrt[5]{5}$ **j** $\frac{1}{16}$ **k** $\frac{1}{81}$ **l** $\frac{1}{100}$

9 **a** a^2 **b** m^3 **c** x **d** $b^{\frac{1}{2}}$
e $s^{\frac{6}{7}}$ **f** $y^{\frac{1}{9}}$ **g** 1 **h** $\frac{a^{\frac{1}{2}}}{b}$

10 **a** $5s^2$ **b** $3t^2$ **c** $2t^2$ **d** $5t^4$
e x **f** b^4 **g** t^3 **h** m^2
i $4ab^4$ **j** $6m^2n$ **k** $2x^2y^3$ **l** $7r^3t^2$
m $\frac{5}{7}$ **n** $\frac{2x}{3}$ **o** $\frac{2}{x^2}$ **p** $10x$

11 **a** method B
b **i** 32 **ii** 216 **iii** 128
iv 81 **v** 625 **vi** $\frac{1}{27}$
vii $\frac{32}{3125}$ **viii** $\frac{81}{10000}$

12 It equals 2 since $2^6 = 64$.

13 **a** **i** -3 **ii** -10 **iii** -2 **iv** -3
b **i** no **ii** yes **iii** yes **iv** no
c y is a real number when n is odd, for $x < 0$.

Exercise 2J

1 **a** **i** 4 **ii** 8 **iii** 16 **iv** 32
b **i** 3 **ii** 5 **iii** 6

2 **a** $x = 2$ **b** $x = -1$ **c** $x = 4$
d $x = 5$ **e** $x = -3$ **f** $x = 0$

3 **a** 3^2 **b** 5^3 **c** 3^5 **d** 2^7 **e** 3^6

4 **a** 3 **b** 3 **c** 2 **d** 2 **e** 3 **f** 3
g 4 **h** 3 **i** 4 **j** 5 **k** 4 **l** 3

5 **a** -2 **b** -2 **c** -2 **d** -4 **e** -5
f 3 **g** 2 **h** 6 **i** 3

6 **a** $\frac{3}{2}$ **b** $\frac{4}{3}$ **c** $\frac{3}{2}$ **d** $\frac{3}{2}$
e $\frac{1}{2}$ **f** $\frac{1}{3}$ **g** $\frac{1}{5}$ **h** $\frac{1}{4}$
i -2 **j** -4 **k** $-\frac{3}{2}$ **l** $-\frac{3}{2}$

7 **a** 1
b **i** 2 **ii** 32 **iii** 2^{60} **iv** 2^{1440}
c **i** 3 min **ii** 8 min **iii** 10 min

8 **a** $\frac{1}{2}$ **b** 1 **c** 3 **d** 1
e $\frac{3}{4}$ **f** 2 **g** 9 **h** $\frac{6}{7}$
i $\frac{15}{4}$ **j** $-\frac{11}{2}$ **k** 4 **l** $-\frac{3}{2}$

9 Prefer 1 cent doubled every second for 30 seconds because receive 2^{29} cents, which is more than 1 million dollars.

10 **a** **i** 1 **ii** 1 **iii** 1
b No solutions. If $a = 1$, then $a^x = 1$ for all values of x.

11 **a** 2 **b** 1 **c** $\frac{2}{3}$ **d** $\frac{3}{4}$
e $\frac{5}{4}$ **f** $\frac{1}{3}$ **g** $\frac{3}{10}$ **h** $-\frac{1}{2}$

12 **a** **i** 0.25 **ii** 0.125
iii 0.001 **iv** 0.0016
b **i** 5^{-2} **ii** 2^{-4} **iii** 2^{-1} **iv** 5^{-4}

13 **a** -4 **b** -6 **c** -5
d $\frac{1}{2}$ **e** -1.5 **f** $-\frac{3}{4}$

14 **a** 1 **b** -1 **c** 8 **d** $-\frac{3}{2}$
e $-\frac{5}{2}$ **f** -2 **g** 3 **h** $\frac{2}{3}$
i $\frac{1}{5}$ **j** 2 **k** 0 **l** -2

Exercise 2K

1 **a** \$50 **b** \$1050 **c** \$52.50
d \$55.13 **e** \$1276.28

2 **a** 4.9 kg **b** $\frac{2}{100}$, 0.98 **c** 4.52 kg **d** 4.52 kg

3 **a** growth **b** growth **c** decay
d decay **e** growth **f** decay

4 **a** A = amount of money at any time, n = number of years of investment
$A = 200000 \times 1.17^n$
b A = house value at any time, n = number of years since initial valuation
$A = 530000 \times 0.95^n$
c A = car value at any time, n = number of years since purchase
$A = 14200 \times 0.97^n$
d A = population at any time, n = number of years since initial census
$A = 172500 \times 1.15^n$

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e A = litres in tank at any time, n = number of hours elapsed
$A = 1200 \times 0.9^n$
f A = cell size at any time, n = number of minutes elapsed
$A = 0.01 \times 2^n$
g A = size of oil spill at any time, n = number of minutes elapsed
$A = 2 \times 1.05^n$
h A = mass of substance at any time, n = number of hours elapsed
$A = 30 \times 0.92^n$

5 b 1.1
c i \$665 500 **ii** \$1 296 871.23
iii \$3 363 749.97
d After 7.3 years.

6 a 300 000
b i \$216 750 **ii** \$96 173.13 **iii** \$42 672.53
c 3.1 years

7 a $V = 15000 \times 0.94^t$
b i 12 459 L **ii** 9727 L
c 769.53 L **d** 55.0 h

8 a $V = 50000 \times 1.11^n$
b i \$75 903.52 **ii** \$403 115.58
c 6.65 years

9 a 3000
b i 3000 **ii** 20 280 **iii** 243 220
c 10 h 11 min

10 a $D = 10 \times 0.875^t$, where t = number of 10 000 km travelled
b 100 000 **c** yes

11 a $T = 90 \times 0.92^t$
b i 79.4°C **ii** 76.2°C
c 3.2 minutes = 3 minutes 12 seconds

12 a i \$1610.51 **ii** \$2143.59 **iii** \$4177.25
b i \$1645.31 **ii** \$2218.18 **iii** \$4453.92

13 a \$2805.10 **b** \$2835.25 **c** \$2837.47

14 a i 90 g **ii** 72.9 g **iii** 53.1 g
b 66 years

15 a 60 L **b** 22.8 min

16 0.7%

Puzzles and challenges

1 3^n

2 a 5 **b** $\left(\frac{4}{9}\right)^x$

3 $\frac{1}{5}$

4 a -8 **b** 2^{2-a}

5 length = $10\sqrt{2}$ cm, breadth = 10 cm

6 $\frac{-3 - \sqrt{2} + 7\sqrt{3}}{7}$

7 a $\frac{x-y}{xy}$ **b** $\sqrt{xy}(x-y)$

8 $12 + 8\sqrt{2}$

Multiple-choice questions

1 C	**2** D	**3** B	**4** E	**5** A	**6** D
7 D	**8** C	**9** B	**10** D	**11** C	**12** D

Short-answer questions

1 a $2\sqrt{6}$ **b** $6\sqrt{2}$ **c** $30\sqrt{2}$ **d** $12\sqrt{6}$
e $\frac{2}{7}$ **f** $\frac{2\sqrt{2}}{3}$ **g** $5\sqrt{7}$ **h** $\frac{2\sqrt{5}}{5}$

2 a $4 + 7\sqrt{3}$ **b** $2\sqrt{5} + 2\sqrt{7}$ **c** $5\sqrt{2}$
d $4\sqrt{3} + 2\sqrt{2}$ **e** $2\sqrt{30}$ **f** $-12\sqrt{5}$
g $2\sqrt{5}$ **h** $\frac{\sqrt{7}}{3}$

3 a $2\sqrt{6} + 4\sqrt{2}$ **b** $12\sqrt{5} - 6$ **c** $12\sqrt{3} - 4$
d $6\sqrt{5}$ **e** 6 **f** -9
g $16 + 6\sqrt{7}$ **h** $56 - 16\sqrt{6}$

4 a $\frac{\sqrt{6}}{6}$ **b** $5\sqrt{2}$ **c** $3\sqrt{6}$
d $2\sqrt{14}$ **e** $\frac{3\sqrt{2}}{4}$ **f** $\frac{5\sqrt{2}}{8}$
g $\frac{\sqrt{10} + 2\sqrt{2}}{2}$ **h** $\frac{4\sqrt{6} - 3}{3}$

5 a $\frac{25}{y^6}$ **b** 6 **c** $\frac{20x^3}{y^4}$
d $\frac{3y^4}{2x^3}$ **e** $\frac{1}{6t^3}$ **f** $\frac{27}{4b^8}$

6 a $21^{\frac{1}{2}}$ **b** $x^{\frac{1}{3}}$ **c** $m^{\frac{5}{3}}$ **d** $a^{\frac{2}{5}}$
e $10^{\frac{1}{2}}x^{\frac{3}{2}}$ **f** $2^{\frac{1}{3}}a^{\frac{1}{3}}b^{\frac{1}{3}}$ **g** $7^{\frac{3}{2}}$ **h** $4^{\frac{4}{3}}$

7 a 5 **b** 4 **c** $\frac{1}{2}$
d $\frac{1}{7}$ **e** $\frac{1}{10}$ **f** $\frac{1}{5}$

8 a i 3210 **ii** 4 024 000
iii 0.00759 **iv** 0.0000981
b i 3.08×10^{-4} **ii** 7.18×10^{-6}
iii 5.68×10^6 **iv** 1.20×10^8

9 a 3 **b** 2 **c** 1 **d** 6
e -2 **f** -3 **g** $\frac{3}{2}$ **h** $\frac{2}{3}$
i 4 **j** 3 **k** -4 **l** 0

10 a $V = 800 \times 1.07^t$ **b** $V = 3000 \times 0.82^t$

Extended-response questions

1 a $36\sqrt{15} + 3\sqrt{45} = 36\sqrt{15} + 9\sqrt{5}$ cm²
b $360\sqrt{3} + 144\sqrt{15} + 90 + 36\sqrt{5}$ cm³
c $4\sqrt{3} + 1$
d i 10 000 cm² **ii** 1.6%

2 a $A = 10000 \times 1.065^n$
b i \$11 342.25 **ii** \$13 700.87
c 11.1 years
d i \$14 591 **ii** $V = 14591 \times 0.97^t$
iii \$12 917; profit of \$2917

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Chapter 3

Pre-test

1 a i 14 ii 25 iii 11
b i $\frac{18}{25}$ ii $\frac{7}{25}$ iii $\frac{7}{25}$

2 0, 1 in 5, 39%, 0.4, $\frac{1}{2}$, 0.62, 71%, $\frac{3}{4}$, $\frac{9}{10}$, 1

3 a $\frac{1}{8}$ b $\frac{1}{2}$ c $\frac{1}{2}$ d $\frac{1}{4}$
e $\frac{5}{8}$ f $\frac{7}{8}$ g $\frac{1}{4}$

4 a 11
b i $\frac{1}{11}$ ii $\frac{2}{11}$ iii $\frac{4}{11}$
iv $\frac{7}{11}$ v $\frac{3}{11}$ vi $\frac{8}{11}$

5 a $\frac{7}{16}$ b $\frac{9}{16}$

6 a

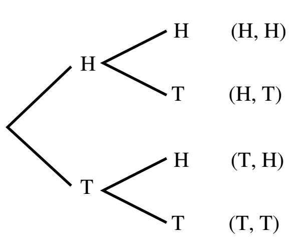
		Roll 1			
		1	2	3	4
Roll 2	1	2	3	4	5
	2	3	4	5	6
	3	4	5	6	7
	4	5	6	7	8

b 16
c i $\frac{1}{16}$ ii $\frac{3}{16}$ iii $\frac{3}{8}$
iv $\frac{5}{8}$ v $\frac{13}{16}$ vi $\frac{3}{16}$

7

	A	$\overline{A}$	
B	2	8	10
$\overline{B}$	5	2	7
	7	10	17

8 a *Toss 1* *Toss 2* *Outcome*



b 4
c i $\frac{1}{4}$ ii $\frac{1}{4}$ iii $\frac{1}{2}$ iv $\frac{3}{4}$ v $\frac{1}{2}$ vi 1

Exercise 3A

1 a 2 b {H, T} c yes
d $\frac{1}{2}$ e $\frac{1}{2}$ f 1

2 a $\frac{1}{4}$ b $\frac{1}{6}$ c $\frac{1}{4}$ d $\frac{3}{8}$ e $\frac{2}{3}$ f 0

3 a $\frac{1}{7}$ b $\frac{2}{7}$ c $\frac{5}{7}$ d $\frac{3}{7}$

4 a $\frac{3}{10}$ b $\frac{2}{5}$ c $\frac{3}{5}$ d $\frac{1}{2}$

5 a $\frac{1}{10}$ b $\frac{1}{2}$ c $\frac{1}{2}$ d $\frac{1}{2}$
e $\frac{2}{5}$ f $\frac{1}{5}$ g $\frac{3}{10}$

6 a 0.09 b 0.43 c 0.47 d 0.91

7 a 0.62 b 0.03 c 0.97 d 0.38

8 a $\frac{1}{50}$ b $\frac{3}{10}$ c $\frac{49}{50}$

9 a $\frac{1}{2}$ b $\frac{3}{8}$ c $\frac{1}{4}$ d $\frac{5}{24}$ e 1 f 0

10 a $\frac{6}{25}$ b $\frac{1}{50}$ c $\frac{21}{25}$ d $\frac{2}{5}$ e $\frac{2}{25}$ f $\frac{4}{25}$

11 a i $\frac{7}{10}$ ii $\frac{1}{5}$ iii $\frac{1}{20}$ iv 0 v $\frac{1}{20}$
b $\frac{1}{10}$

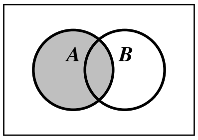
12 a 59
b 4, as $\frac{41}{100}$ of 10 is closest to 4.
c 8, as $\frac{41}{100}$ of 20 is closest to 8.

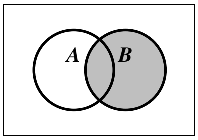
13 a $\frac{1}{4}$ b $\frac{1}{13}$ c $\frac{1}{52}$ d $\frac{1}{2}$
e $\frac{2}{13}$ f $\frac{4}{13}$ g $\frac{12}{13}$ h $\frac{9}{13}$

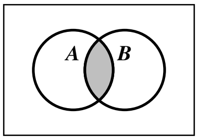
14 a $\frac{7}{15}$
b 15; any multiple of 15 is a possibility as 3 and 5 must be factors.

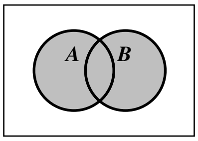
15 a 625π
b i 25π ii 200π iii 400π
c i $\frac{1}{25}$ ii $\frac{8}{25}$ iii $\frac{16}{25}$
iv $\frac{9}{25}$ v $\frac{24}{25}$ vi $\frac{17}{25}$
vii 1 viii $\frac{17}{25}$
d Yes. Hint: Try changing 10, 10, 10 to 14, 14, 14. Also try changing 10, 10, 10 to $(10 + x)$, $(10 + x)$, $(10 + x)$.

Exercise 3B

1 a


b


c


d


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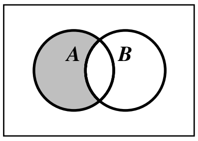
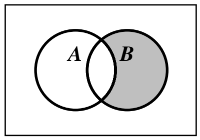
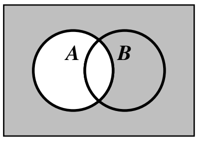
e



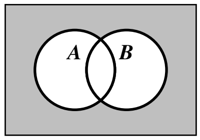
f



g



h

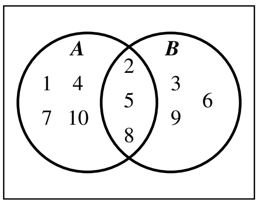


2 **a** $\varnothing$ **b** $A \cap B$ **c** $A \cup B$ **d** $\varnothing$

e $E \cap F$ **f** $W \cup Z$ **g** $A \cup B \cup C$ **h** $A \cap B \cap C$

3 **a** no **b** yes **c** no

4 **a**



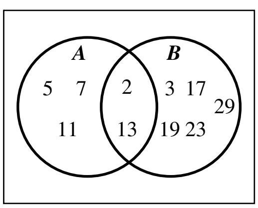
b **i** $A \cap B = \{2, 5, 8\}$

ii $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

c **i** $\frac{7}{10}$ **ii** $\frac{3}{10}$ **iii** 1

d No, since $A \cap B \neq \varnothing$.

5 **a**

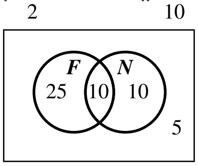


b **i** $A \cap B = \{2, 13\}$

ii $A \cup B = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$

c **i** $\frac{1}{2}$ **ii** $\frac{7}{10}$ **iii** $\frac{1}{5}$ **iv** 1

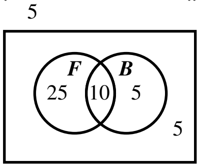
6 **a**



b **i** 25 **ii** 5

c **i** $\frac{2}{5}$ **ii** $\frac{1}{5}$ **iii** $\frac{1}{5}$

7 **a**



b **i** 25 **ii** 5

c **i** $\frac{7}{9}$ **ii** $\frac{2}{9}$ **iii** $\frac{8}{9}$ **iv** $\frac{2}{9}$ **v** $\frac{1}{9}$

8 **a**

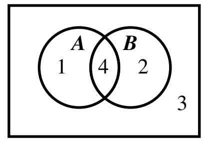
	A	$\overline{A}$	
B	2	6	8
$\overline{B}$	5	3	8
	7	9	16

b **i** 2 **ii** 6 **iii** 5 **iv** 3

v 7 **vi** 8 **vii** 13

c **i** $\frac{1}{8}$ **ii** $\frac{9}{16}$ **iii** $\frac{5}{16}$

9 **a**



b

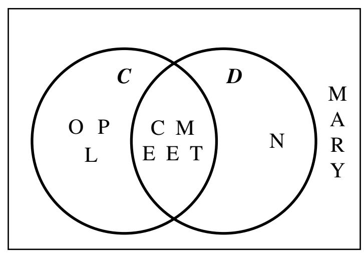
	A	$\overline{A}$	
B	4	2	6
$\overline{B}$	1	3	4
	5	5	10

c **i** 2 **ii** 3 **iii** $\frac{2}{5}$ **iv** $\frac{7}{10}$

10 **a** 4 **b** 10, 12

c a, c, e **d** nothing

11 **a**



b **i** $\frac{8}{13}$ **ii** $\frac{5}{13}$ **iii** $\frac{9}{13}$

iv $\frac{5}{13}$ **v** $\frac{4}{13}$

12 **a**

	A	$\overline{A}$	
B	3	3	6
$\overline{B}$	4	1	5
	7	4	11

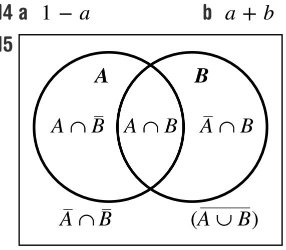
b

	A	$\overline{A}$	
B	2	7	9
$\overline{B}$	2	1	3
	4	8	12

13 3

14 **a** $1 - a$ **b** $a + b$ **c** 0

15



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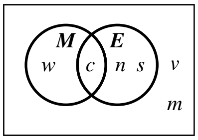
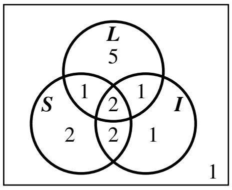
16 a



b **i** $\frac{1}{3}$ **ii** $\frac{2}{3}$ **iii** $\frac{1}{6}$ **iv** $\frac{2}{3}$ **v** $\frac{1}{3}$

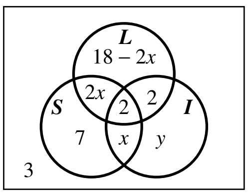
17 a



b 1

c **i** $\frac{3}{5}$ **ii** $\frac{1}{3}$ **iii** $\frac{13}{15}$ **iv** $\frac{1}{15}$

18 a



L = Own State
S = Interstate
I = Overseas

b **i** 4 **ii** 10

c **i** $\frac{5}{19}$ **ii** $\frac{1}{19}$ **iii** $\frac{7}{38}$ **iv** $\frac{35}{38}$ **v** $\frac{25}{38}$

Exercise 3C

1 a **i** $\{4, 5, 6\}$ **ii** $\{2, 4, 6\}$
iii $\{2, 4, 5, 6\}$ **iv** $\{4, 6\}$

b No, $A \cap B \neq \varnothing$ **c** $\frac{2}{3}$

2 a 0.8 **b** 0.8 **c** 0.7 **d** 1

3 0.05

4 a **i** 13 **ii** 4 **iii** 1

b **i** $\frac{1}{4}$ **ii** $\frac{3}{4}$ **iii** $\frac{1}{52}$

c $\frac{4}{13}$ **d** $\frac{3}{52}$

5 a **i** $\{3, 6, 9, 12, 15, 18\}$
ii $\{2, 3, 5, 7, 11, 13, 17, 19\}$

b **i** $\frac{1}{20}$ **ii** $\frac{13}{20}$

c $\frac{7}{20}$

6 a $\frac{1}{8}$ **b** $\frac{5}{24}$

7 a 0.1 **b** 0.2

8 a 0.3 **b** 0.1

9 a $\frac{3}{8}$ **b** $\frac{5}{32}$

10 a $\frac{4}{13}$ **b** $\frac{4}{13}$ **c** $\frac{7}{13}$ **d** $\frac{7}{13}$

e $\frac{49}{52}$ **f** $\frac{10}{13}$ **g** $\frac{10}{13}$ **h** $\frac{25}{26}$

11 a 0.4 **b** 0.45

12 Because $P(A \cap B) = 0$ for mutually exclusive events.

13 a $P(A) < P(A \cap B)$ **b** $P(A) + P(B) < P(A \cup B)$

14 $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)$
$- P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

15 a $\frac{3}{10}$ **b** $\frac{1}{4}$ **c** $\frac{3}{20}$ **d** $\frac{13}{20}$ **e** $\frac{9}{20}$ **f** $\frac{3}{5}$

16 a $\frac{1}{4}$ **b** $\frac{71}{500}$ **c** $\frac{33}{500}$ **d** $\frac{7}{100}$ **e** $\frac{1}{25}$ **f** $\frac{7}{500}$

Exercise 3D

1 a **i** 2 **ii** 9

b $\frac{2}{9}$

2 a **i** 7 **ii** 10

b $\frac{7}{10}$ **c** $\frac{7}{12}$

3 a $\frac{1}{3}$ **b** $\frac{1}{2}$

4 a **i** $\frac{9}{13}$ **ii** $\frac{3}{13}$ **iii** $\frac{3}{7}$ **iv** $\frac{1}{3}$

b **i** $\frac{14}{17}$ **ii** $\frac{4}{17}$ **iii** $\frac{4}{7}$ **iv** $\frac{2}{7}$

c **i** $\frac{3}{4}$ **ii** $\frac{5}{8}$ **iii** $\frac{5}{7}$ **iv** $\frac{5}{6}$

d **i** $\frac{7}{16}$ **ii** $\frac{1}{8}$ **iii** $\frac{1}{4}$ **iv** $\frac{2}{7}$

5 a **i** $\frac{7}{18}$ **ii** $\frac{1}{9}$ **iii** $\frac{1}{5}$ **iv** $\frac{2}{7}$

b **i** $\frac{4}{9}$ **ii** $\frac{1}{9}$ **iii** $\frac{1}{5}$ **iv** $\frac{1}{4}$

c **i** $\frac{8}{17}$ **ii** $\frac{7}{17}$ **iii** $\frac{7}{10}$ **iv** $\frac{7}{8}$

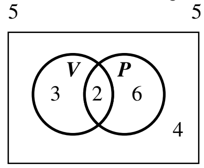
d **i** $\frac{3}{4}$ **ii** $\frac{1}{4}$ **iii** $\frac{2}{3}$ **iv** $\frac{1}{3}$

6 a

	A	$\overline{A}$	
B	9	6	15
$\overline{B}$	4	1	5
	13	7	20

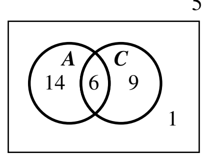
b $\frac{1}{5}$ **c** $\frac{3}{5}$ **d** $\frac{9}{13}$

7 a



b 4 **c** $\frac{2}{5}$ **d** $\frac{1}{4}$

8 a



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	A	$\overline{A}$	
C	6	9	15
$\overline{C}$	14	1	15
	20	10	30

b **i** $\frac{3}{10}$ **ii** $\frac{7}{15}$

c $\frac{2}{5}$ **d** $\frac{3}{10}$

9 **a**

	A	$\overline{A}$	
B	2	2	4
$\overline{B}$	3	1	4
	5	3	8

i 1 **ii** $\frac{2}{5}$ **iii** $\frac{1}{2}$

b

	A	$\overline{A}$	
B	3	13	16
$\overline{B}$	5	6	11
	8	19	27

i 6 **ii** $\frac{3}{8}$ **iii** $\frac{3}{16}$

10 **a** $\frac{1}{13}$ **b** $\frac{1}{13}$ **c** $\frac{1}{4}$ **d** $\frac{1}{2}$

11 **a** $\frac{1}{3}$ **b** $\frac{1}{2}$

12 $P(A|B) = P(B|A) = 0$ as $P(A \cap B) = 0$

13 **a** 1 **b** $\frac{1}{5}$

14 **a** $P(A \cap B) = P(A) \times P(B|A)$

b 0.18

15 **a** 329 **b** $\frac{174}{329}$ **c** $\frac{81}{329}$

d $\frac{24}{155}$ **e** $\frac{31}{231}$ **f** $\frac{18}{31}$

Exercise 3E

1 **a** **i**

		1st		
		D	O	G
2nd	D	(D, D)	(O, D)	(G, D)
	O	(D, O)	(O, O)	(G, O)
	G	(D, G)	(O, G)	(G, G)

ii

		1st		
		D	O	G
2nd	D	×	(O, D)	(G, D)
	O	(D, O)	×	(G, O)
	G	(D, G)	(O, G)	×

b **i** 9 **ii** 6

c **i** $\frac{1}{3}$ **ii** $\frac{5}{9}$ **iii** $\frac{4}{9}$ **iv** $\frac{8}{9}$ **v** $\frac{2}{9}$

d **i** 0 **ii** $\frac{2}{3}$ **iii** $\frac{1}{3}$ **iv** 1 **v** $\frac{1}{3}$

2 **a** 9 **b** 6

3 **a**

		1st roll			
		1	2	3	4
2nd roll	1	(1, 1)	(2, 1)	(3, 1)	(4, 1)
	2	(1, 2)	(2, 2)	(3, 2)	(4, 2)
	3	(1, 3)	(2, 3)	(3, 3)	(4, 3)
	4	(1, 4)	(2, 4)	(3, 4)	(4, 4)

b 16 **c** $\frac{1}{16}$

d **i** $\frac{1}{4}$ **ii** $\frac{5}{8}$ **iii** $\frac{13}{16}$

4 **a**

		1st toss	
		H	T
2nd toss	H	(H, H)	(T, H)
	T	(H, T)	(T, T)

b 4 **c** $\frac{1}{4}$

d **i** $\frac{1}{2}$ **ii** $\frac{3}{4}$

e 250

5 **a**

		1st		
		S	E	T
2nd	S	×	(E, S)	(T, S)
	E	(S, E)	×	(T, E)
	T	(S, T)	(E, T)	×

b **i** $\frac{1}{6}$ **ii** $\frac{2}{3}$ **iii** $\frac{2}{3}$ **iv** $\frac{1}{3}$ **v** 1

6 **a**

		1st				
		L	E	V	E	L
2nd	L	×	(E, L)	(V, L)	(E, L)	(L, L)
	E	(L, E)	×	(V, E)	(E, E)	(L, E)
	V	(L, V)	(E, V)	×	(E, V)	(L, V)
	E	(L, E)	(E, E)	(V, E)	×	(L, E)
	L	(L, L)	(E, L)	(V, L)	(E, L)	×

b 20

c **i** 8 **ii** 12 **iii** 12

d **i** $\frac{2}{5}$ **ii** $\frac{3}{5}$ **iii** $\frac{3}{5}$

e $\frac{1}{5}$

7 **a**

		Die 2					
		1	2	3	4	5	6
Die 1	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

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b 36

c i 2 ii 6 iii 15

d i $\frac{1}{6}$ ii $\frac{1}{6}$ iii $\frac{35}{36}$ iv $\frac{1}{12}$

e $\frac{1}{6}$. Min's guess is wrong.

8 a

		1st		
		O	L	D
2nd	C	(O, C)	(L, C)	(D, C)
	O	(O, O)	(L, O)	(D, O)
	L	(O, L)	(L, L)	(D, L)
	L	(O, L)	(L, L)	(D, L)
	E	(O, E)	(L, E)	(D, E)
	G	(O, G)	(L, G)	(D, G)
	E	(O, E)	(L, E)	(D, E)

b 21 c $\frac{1}{7}$

9 a i 100 ii 90

b i $\frac{1}{10}$ ii $\frac{1}{10}$ iii $\frac{4}{5}$

c $\frac{19}{100}$

10 a i $\frac{1}{4}$ ii $\frac{5}{8}$

b i $\frac{2}{5}$ ii $\frac{1}{10}$ iii $\frac{2}{3}$

11 a without b with c with d without

12 a 30

b i $\frac{1}{15}$ ii $\frac{1}{15}$ iii $\frac{2}{15}$ iv $\frac{4}{15}$

c $\frac{1}{18}$

13 a

		1st			
		2.5	5	10	20
2nd	2.5	5	7.5	12.5	22.5
	5	7.5	10	15	25
	10	12.5	15	20	30
	20	22.5	25	30	40

b 16

c i 1 ii 8 iii 8

d i $\frac{1}{16}$ ii $\frac{1}{8}$ iii $\frac{1}{4}$ iv $\frac{3}{16}$

e $\frac{7}{16}$

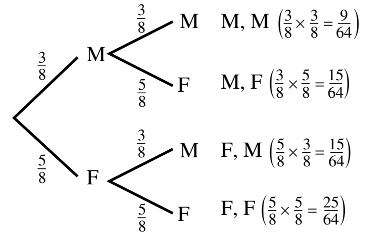
Exercise 3F

1 a i $\frac{2}{5}$ ii $\frac{3}{5}$

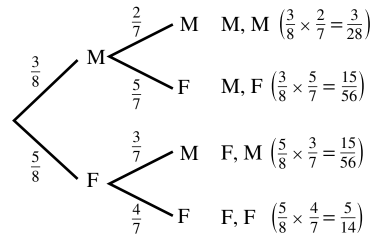
b i $\frac{2}{5}$ ii $\frac{3}{5}$

c i $\frac{1}{4}$ ii $\frac{3}{4}$

2 a

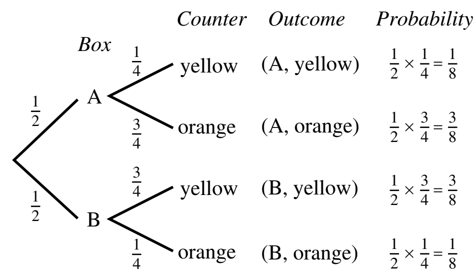


b



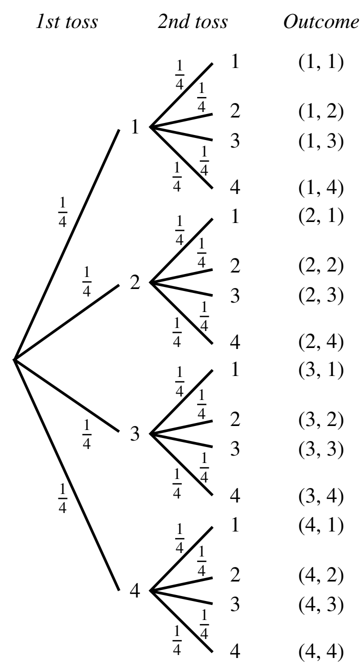
3 a $\frac{1}{4}$ b $\frac{3}{4}$

c



d $\frac{3}{8}$ e $\frac{1}{2}$

4 a

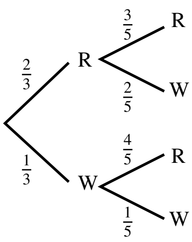


b 16

c i $\frac{1}{16}$ ii $\frac{1}{4}$

d i $\frac{1}{16}$ ii $\frac{1}{4}$ iii $\frac{5}{8}$

5 a

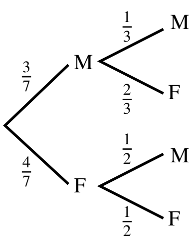


Outcome	Probability
(R, R)	$\frac{2}{3}\times\frac{3}{5}=\frac{2}{5}$
(R, W)	$\frac{2}{3}\times\frac{2}{5}=\frac{4}{15}$
(W, R)	$\frac{1}{3}\times\frac{4}{5}=\frac{4}{15}$
(W, W)	$\frac{1}{3}\times\frac{1}{5}=\frac{1}{15}$

b i $\frac{4}{15}$ **ii** $\frac{2}{5}$ **iii** $\frac{8}{15}$

c i $\frac{2}{9}$ **ii** $\frac{4}{9}$ **iii** $\frac{4}{9}$

6 a

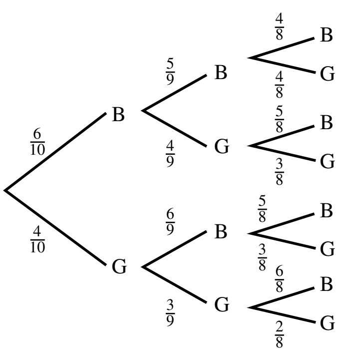


Outcome	Probability
(M, M)	$\frac{3}{7}\times\frac{1}{3}=\frac{1}{7}$
(M, F)	$\frac{3}{7}\times\frac{2}{3}=\frac{2}{7}$
(F, M)	$\frac{4}{7}\times\frac{1}{2}=\frac{2}{7}$
(F, F)	$\frac{4}{7}\times\frac{1}{2}=\frac{2}{7}$

i $\frac{1}{7}$ **ii** $\frac{2}{7}$ **iii** $\frac{4}{7}$ **iv** $\frac{3}{7}$

b i $\frac{9}{49}$ **ii** $\frac{16}{49}$ **iii** $\frac{24}{49}$ **iv** $\frac{25}{49}$

7 a



Outcome
(B, B, B)
(B, B, G)
(B, G, B)
(B, G, G)
(G, B, B)
(G, B, G)
(G, G, B)
(G, G, G)

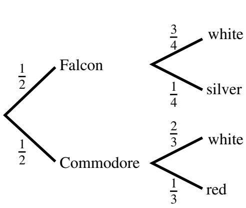
b i $P(B,B,G)+P(B,G,B)+P(G,B,B)$

$$= 3\times P(B,B,G)$$
$$= 3\times\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}$$
$$= \frac{1}{2}$$

ii $P(\text{at least 1 girl}) = 1 - P(B,B,B)$

$$= 1-\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}$$
$$= \frac{5}{6}$$

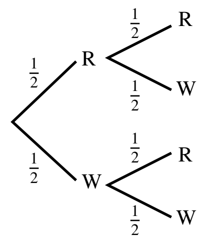
8 a



Outcome	Probability
(Falcon, white)	$\frac{1}{2}\times\frac{3}{4}=\frac{3}{8}$
(Falcon, silver)	$\frac{1}{2}\times\frac{1}{4}=\frac{1}{8}$
(Commodore, white)	$\frac{1}{2}\times\frac{2}{3}=\frac{1}{3}$
(Commodore, red)	$\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$

b i $\frac{3}{8}$ **ii** $\frac{1}{6}$ **iii** $\frac{17}{24}$ **iv** $\frac{7}{24}$ **v** $\frac{5}{6}$ **vi** $\frac{1}{3}$

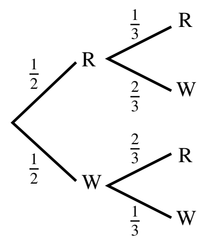
9 a



Outcome
(R, R)
(R, W)
(W, R)
(W, W)

i $\frac{1}{4}$ **ii** $\frac{1}{2}$ **iii** $\frac{3}{4}$ **iv** $\frac{3}{4}$

b

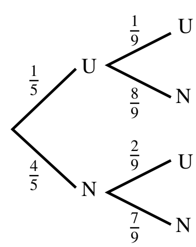


Outcome	Probability
(R, R)	$\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$
(R, W)	$\frac{1}{2}\times\frac{2}{3}=\frac{1}{3}$
(W, R)	$\frac{1}{2}\times\frac{2}{3}=\frac{1}{3}$
(W, W)	$\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$

i $\frac{1}{6}$ **ii** $\frac{2}{3}$ **iii** $\frac{5}{6}$ **iv** $\frac{5}{6}$

10 a i $\frac{1}{5}$ **ii** $\frac{4}{5}$

b



Outcome	Probability
(U, U)	$\frac{1}{5}\times\frac{1}{9}=\frac{1}{45}$
(U, N)	$\frac{1}{5}\times\frac{8}{9}=\frac{8}{45}$
(N, U)	$\frac{4}{5}\times\frac{2}{9}=\frac{8}{45}$
(N, N)	$\frac{4}{5}\times\frac{7}{9}=\frac{28}{45}$

i $\frac{1}{45}$ **ii** $\frac{16}{45}$ **iii** $\frac{44}{45}$

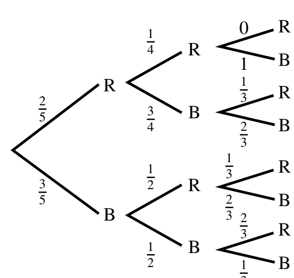
c 62.2%

11 a i 0.17 **ii** 0.11 **iii** 0.83

b i 0.1445 **ii** 0.0965 **iii** 0.8555

12 a $\frac{3}{7}$ **b** $\frac{4}{7}$

13 a



Outcome	Probability
(R, R, R)	0
(R, R, B)	$\frac{1}{10}$
(R, B, R)	$\frac{1}{10}$
(R, B, B)	$\frac{1}{5}$
(B, R, R)	$\frac{1}{10}$
(B, R, B)	$\frac{1}{5}$
(B, B, R)	$\frac{1}{5}$
(B, B, B)	$\frac{1}{10}$

i $\frac{1}{10}$ **ii** $\frac{3}{10}$ **iii** 0 **iv** $\frac{9}{10}$ **v** $\frac{9}{10}$

b i 1 **ii** $\frac{2}{5}$

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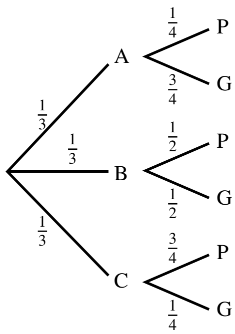
14 a



Outcome	*Probability*
(A, P)	$\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$
(A, G)	$\frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$
(B, P)	$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$
(B, G)	$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$
(C, P)	$\frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$
(C, G)	$\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$

b 6

c i $\frac{1}{12}$ ii $\frac{1}{6}$ iii $\frac{1}{4}$

d $\frac{1}{2}$

15 a i $\frac{7}{8}$ ii $\frac{1}{8}$

b \$87.50 to player A, \$12.50 to player B

c i A\$68.75, B\$31.25 ii A\$50, B\$50

iii A\$81.25, B\$18.75 iv A\$34.375, B\$65.625

Exercise 3G

1 a i $\frac{1}{2}$ ii $\frac{1}{2}$

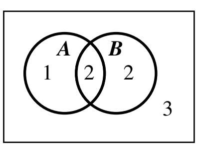
b yes c $\frac{1}{2}$

2 a i $\frac{3}{10}$ ii $\frac{1}{3}$

b no c no

3 a with b without

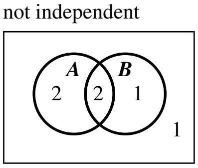
4 a



b i $\frac{3}{8}$ ii $\frac{1}{2}$

c not independent

5 a



b i $\frac{2}{3}$ ii $\frac{2}{3}$

c independent

6 a i $\frac{3}{4}, \frac{1}{2}$ ii not independent

b i $\frac{1}{4}, \frac{1}{4}$ ii independent

c i $\frac{1}{3}, \frac{1}{3}$ ii independent

d i $\frac{2}{7}, 0$ ii not independent

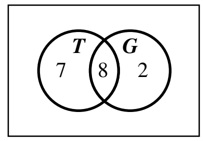
7 a $P(A) = \frac{1}{2}, P(A|B) = \frac{1}{2}$, independent

b $P(A) = \frac{3}{10}, P(A|B) = \frac{1}{4}$, not independent

c $P(A) = \frac{5}{12}, P(A|B) = \frac{3}{20}$, not independent

d $P(A) = \frac{1}{9}, P(A|B) = \frac{1}{9}$, independent

8 a



	T	$\overline{T}$	
G	8	2	10
$\overline{G}$	7	0	7
	15	2	17

i $\frac{15}{17}$ ii $\frac{7}{17}$ iii $\frac{4}{5}$

b no

9 a $\frac{1}{32}$ b $\frac{31}{32}$ c $\frac{31}{32}$

10 a $\frac{1}{216}$ b $\frac{1}{216}$ c $\frac{1}{72}$ d $\frac{1}{36}$

11 False; $P(A|B) = 0$ but $P(A) = \frac{2}{9}$.

12 a 6 b 22 c 49 d 2

13 a 0.24 b 0.76

14 $\frac{5}{6}$

Puzzles and challenges

1 a 0.16 b 0.192 c 0.144

2 0.593 75

3 a $\frac{7}{8}$ b 1 c $\frac{4}{7}$

4 a $\frac{1}{12}$ b $\frac{1}{2}$ c $\frac{3}{4}$ d $\frac{2}{3}$

5 $\frac{63}{64}$

6 $\frac{1}{13\,983\,816}$

7 $\frac{3}{5}$

8 $\frac{1}{12}$

9 true

Multiple-choice questions

1 A	2 B	3 C	4 E	5 A
6 B	7 D	8 C	9 A	10 E

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Short-answer questions

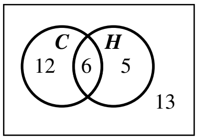
1 **a** $\frac{1}{8}$ **b** $\frac{1}{4}$ **c** $\frac{3}{8}$ **d** $\frac{5}{8}$ **e** $\frac{1}{2}$

2 **a** $\frac{5}{8}$ **b** $\frac{1}{2}$ **c** $\frac{5}{8}$

3 **a** **i** $\frac{2}{5}$ **ii** $\frac{1}{4}$ **iii** $\frac{1}{5}$ **iv** $\frac{1}{10}$ **v** $\frac{1}{20}$

b **i** $\frac{3}{5}$ **ii** $\frac{17}{20}$

4 **a**



b

	C	$\overline{C}$	
H	6	5	11
$\overline{H}$	12	13	25
	18	18	36

c 13

d **i** $\frac{1}{6}$ **ii** $\frac{5}{36}$ **iii** $\frac{1}{2}$

5 **a** 6 **b** $\frac{6}{13}$

6 **a** **i** 13 **ii** 4 **iii** 1

b **i** $\frac{3}{4}$ **ii** $\frac{1}{52}$

c $\frac{4}{13}$ **d** $\frac{10}{13}$

7 **a** 0.1 **b** 0.5

8 **a** $\frac{2}{5}$ **b** $\frac{1}{5}$

9 **a** **i** $\frac{4}{11}$ **ii** $\frac{5}{11}$ **iii** $\frac{1}{5}$

b No, $P(A|B) \neq P(A)$

c **i** $\frac{1}{2}$ **ii** $\frac{1}{4}$ **iii** $\frac{1}{2}$

d Yes, $P(A|B) = P(A)$

10 **a**

		1st				
		H	A	P	P	Y
2nd	H	(H, H)	(A, H)	(P, H)	(P, H)	(Y, H)
	E	(H, E)	(A, E)	(P, E)	(P, E)	(Y, E)
	Y	(H, Y)	(A, Y)	(P, Y)	(P, Y)	(Y, Y)

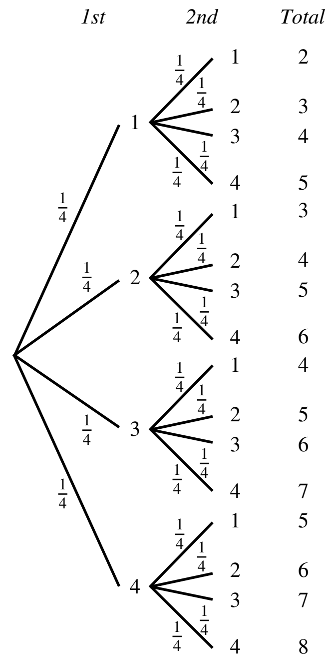
b 15

c **i** $\frac{1}{15}$

ii $\frac{2}{15}$

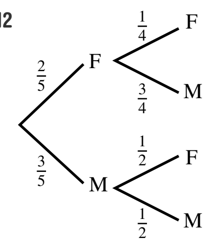
iii $\frac{13}{15}$

11 **a**



b **i** $\frac{1}{16}$ **ii** $\frac{1}{4}$ **iii** 0 **iv** 1

12



a $\frac{2}{5}$ **b** $\frac{3}{10}$ **c** $\frac{3}{10}$

d $\frac{3}{5}$ **e** $\frac{7}{10}$

13 **a** 0.12 **b** 0.58

Extended-response questions

1 **a** 3

b **i** $\frac{7}{15}$ **ii** $\frac{1}{15}$

c

	R	$\overline{R}$	
S	3	1	4
$\overline{S}$	3	8	11
	6	9	15

d **i** $\frac{1}{2}$ **ii** $\frac{3}{4}$

2 **a**

		1st		
		R	S	W
2nd	R	(R, R)	(S, R)	(W, R)
	S	(R, S)	(S, S)	(W, S)
	W	(R, W)	(S, W)	(W, W)

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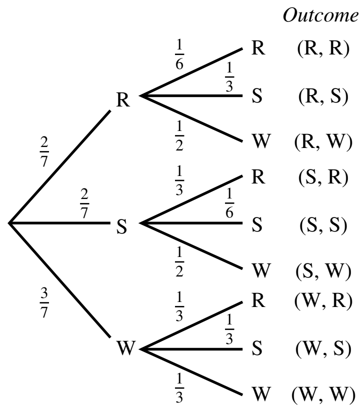
b **i** $\frac{1}{9}$ **ii** $\frac{1}{3}$ **iii** $\frac{5}{9}$ **iv** $\frac{4}{9}$

c 4 **d** $\frac{5}{9}$

e



i $\frac{1}{21}$ **ii** $\frac{10}{21}$ **iii** $\frac{6}{7}$ **iv** $\frac{16}{21}$

Chapter 4

Pre-test

1 **a** **i** mean = 5.9 **ii** median = 6
iii modes = 2, 6, 9
b **i** mean = 44.6 **ii** median = 41
iii mode = 41
c **i** mean = 0.7 **ii** median = 0.65
iii mode = 0.3

2 **a** 6 **b** **i** 19 **ii** 23
c 30 **d** 10%

3 **a** 8 **b** 40 **c** 82 **d** 32.5%

4 **a** 15 **b** 123 g
c 111, 139 are most frequent. **d** 47

5 $Q_1 = 4$, $Q_3 = 12$, IQR = 8

Exercise 4A

1 Answers will vary and should be discussed in class.

2 **a** E **b** F **c** A **d** B
e H **f** D **g** G **h** C

3 **a** B **b** E **c** C **d** D
e F **f** A

4 C

5 D

6 **a** numerical and discrete
b numerical and discrete
c categorical and nominal
d numerical and continuous
e categorical and ordinal

7 D **8** C **9** D

10 **A** The first 30 students are likely to be keen, conscientious students.
B The students in the highest Maths class are probably good at mathematics and they probably like it too.
C Some classes might be bigger than others, so the number of students chosen from each class should be related to the number of students in that class.

11 Answers will vary and should be discussed in class.

12 Answers will vary and should be discussed in class.

13 Answers will vary and should be discussed in class.

14 Answers will vary and should be discussed in class.

15 Answers will vary and should be discussed in class.

16 Answers will vary and should be discussed in class.

Exercise 4B

1 **a** 10 **b** 1.4 **c** 1 **d** 1 **e** 90%

2 **a** numerical **b** categorical (nominal)
c categorical (ordinal) **d** numerical

3 **a**

Class interval	Frequency	Percentage frequency
0–9	2	20%
10–19	1	10%
20–29	5	50%
30–40	2	20%
Total	**10**	**100%**

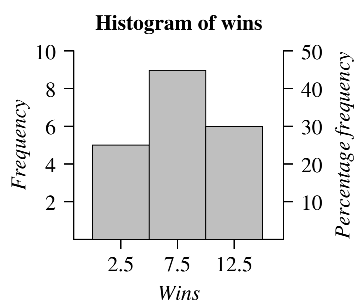
b

Class interval	Frequency	Percentage frequency
80–84	8	16%
85–89	23	46%
90–94	13	26%
95–100	6	12%
Total	**50**	**100%**

4 **a**

Class interval	Frequency	Percentage frequency
0–4	5	25%
5–9	9	45%
10–15	6	30%
Total	**20**	**100%**

b



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c

Stem	Leaf
0	0 1 3 4 4 5 5 6 7 7 8 9 9 9
1	0 1 2 2 3 5

d 7.5

5 a

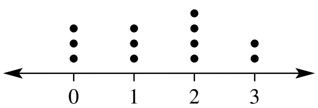
Type of transport	Frequency	Percentage frequency
Car	16	40%
Train	6	15%
Tram	8	20%
Walking	5	12.5%
Bicycle	2	5%
Bus	3	7.5%
Total	**40**	**100%**

b i 6 ii car iii 40%
iv 17.5% v 42.5%

6 a symmetrical b negatively skewed
c positively skewed d symmetrical

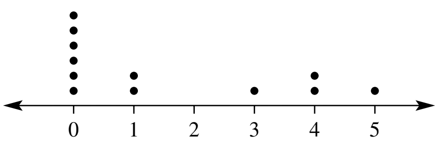
7 a i 34.3 ii 38 iii 39
b i 19.4 ii 20 iii no mode

8 a



Nick's goal scoring

b



Jack's goal scoring

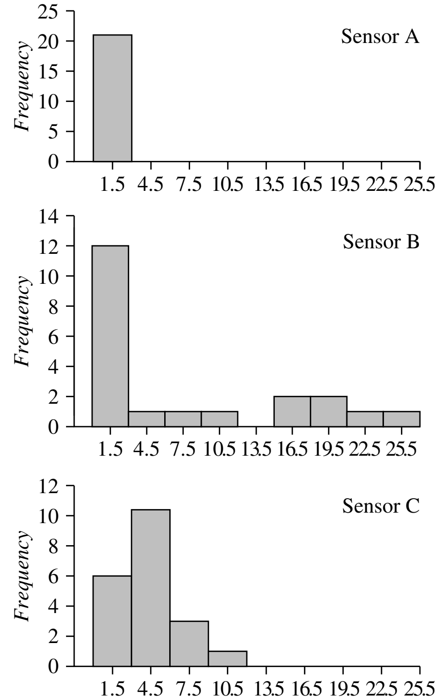
c Well-spread performance.

d Irregular performance, positively skewed.

9 a

	Sensor A frequency	Sensor B frequency	Sensor C frequency
0–2	21	12	6
3–5	0	1	11
6–8	0	1	3
9–11	0	1	1
12–14	0	0	0
15–17	0	2	0
18–20	0	2	0
21–23	0	1	0
24–27	0	1	0
Total	**21**	**21**	**21**

b



c i insensitive ii very sensitive
iii moderately sensitive

10 a

Mass	Frequency	Percentage frequency
10–14	3	6%
15–19	6	12%
20–24	16	32%
25–29	21	42%
30–35	4	8%
Total	**50**	**100%**

b 50

c 32%

d At least 25 g but less than 30 g.

e 42%

f 94%

11 a

Section	Frequency	Percentage frequency
Strings	21	52.5%
Woodwind	8	20%
Brass	7	17.5%
Percussion	4	10%
Total	**40**	**100%**

b 40 c 52.5% d 47.5%
e 9.3% f 65.6%

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12 Eight students scored between 20 and 30 and there are 32 students all together, so this class interval makes up 25% of the class.

13 No discrete information; only intervals are given, not individual values.

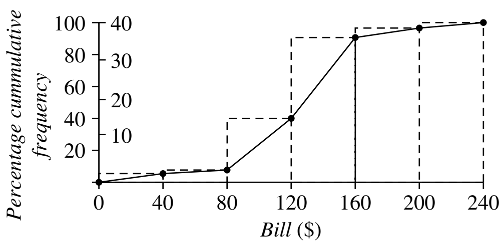
14 $3 \leqslant a \leqslant 7$, $0 \leqslant b \leqslant 4$, $c = 9$

15 a

Bill ($)	Frequency	Cumulative frequency	Percentage cumulative frequency
0–	2	2	5.4%
40–	1	3	8.1%
80–	12	15	40.5%
120–	18	33	89.2%
160–	3	36	97.3%
200–240	1	37	100%

b 15

c



d i \$130 **ii** \$100 **iii** \$150

e \$180 **f** approx. 20%

Exercise 4C

1 a Min, lower quartile (Q_1), median (Q_2), upper quartile (Q_3), max

b Range is max − min; IQR is $Q_3 - Q_1$. Range is the spread of all the data; IQR is the spread of the middle 50% of data.

c An outlier is a data point (element) outside the vicinity of the rest of the data.

d When the data point is greater than $Q_3 + 1.5 \times \text{IQR}$ or less than $Q_1 - 1.5 \times \text{IQR}$.

2 a 0, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 4, 8

b 2

c i 1 **ii** 3

d 2

e −2, 6

f Yes, the data value 8 is an outlier.

3 a i 10.5 **ii** 7.5 **iii** 12

b 4.5

c 0.75, 18.75

d No

4 a min = 3, $Q_1 = 4$, median = 8, $Q_3 = 10$, max = 13, range = 10, IQR = 6

b min = 10, $Q_1 = 10.5$, median = 14, $Q_3 = 15.5$, max = 18, range = 8, IQR = 5

c min = 1.2, $Q_1 = 1.85$, median = 2.4, $Q_3 = 3.05$, max = 3.4, range = 2.2, IQR = 1.2

d min = 41, $Q_1 = 53$, median = 60.5, $Q_3 = 65$, max = 68, range = 27, IQR = 12

5 a median = 8, mean = 8 **b** median = 4.5, mean = 4.2

c median = 4, mean = 4.3

6 a min = 0, max = 17 **b** median = 13

c $Q_1 = 10$, $Q_3 = 15$ **d** IQR = 5

e 0

f Road may have been closed that day.

7 a i min = 4, max = 14 **ii** 7.5

iii $Q_1 = 5$, $Q_3 = 9$ **iv** IQR = 4

v no outliers

b i min = 16, max = 31 **ii** 25

iii $Q_1 = 21$, $Q_3 = 27$ **iv** IQR = 6

v no outliers

8 a i min = 25, max = 128 **ii** 47

iii $Q_1 = 38$, $Q_3 = 52.5$ **iv** IQR = 14.5

v Yes; 128 **vi** 51.25

b Median, as it is not affected dramatically by the outlier.

c A more advanced calculator was used.

9 a no outliers **b** Outlier is 2.

c Outliers are 103, 182. **d** Outliers are 2, 8.

10 a IQR = 12 **b** no outliers

c 24 **d** 22

11 1, 2, 3

12 a Increases by 5. **b** It is doubled.

c It is divided by 10.

13 a It stays the same. **b** It doubles.

c It is reduced by a scale factor of 10.

14 Answers may vary. Examples:

a 3, 4, 5, 6, 7

b 2, 4, 6, 6, 6

c 7, 7, 7, 10, 10

15 It is not greatly affected by outliers.

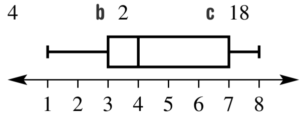
Exercise 4D

1 a 15 **b** 5 **c** 25 **d** 20

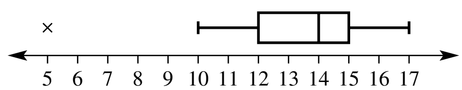
e 10 **f** 20 **g** 10

2 a 4 **b** 2 **c** 18 **d** 20 **e** It is.

3 a

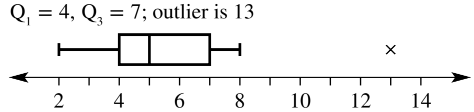


b



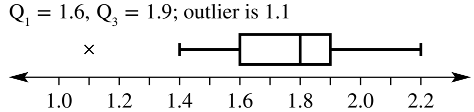
4 a i $Q_1 = 4$, $Q_3 = 7$; outlier is 13

ii



b i $Q_1 = 1.6$, $Q_3 = 1.9$; outlier is 1.1

ii



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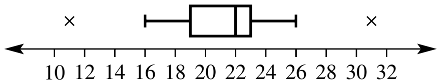
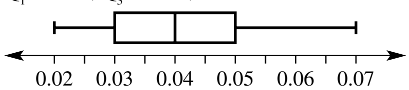
c **i** $Q_1 = 19$, $Q_3 = 23$; outliers are 11 and 31

ii

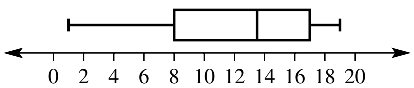


d **i** $Q_1 = 0.03$, $Q_3 = 0.05$; no outliers

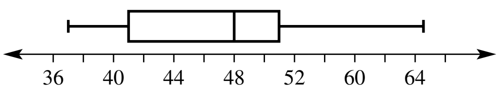
ii



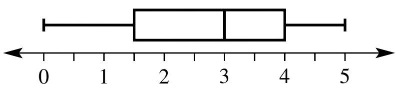
5 **a**



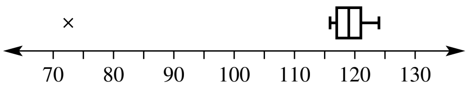
b



c



d



6 **a** Same minimum of 1.

b B

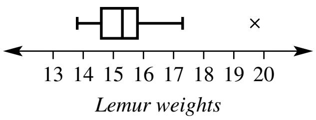
c **i** 5 **ii** 10

d Data points for B are more evenly spread than those for A.

7 **a** $Q_1 = 14.6$, $Q_2 = 15.3$, $Q_3 = 15.8$

b 19.7 kg

c **Box plot of lemur weights**



8 **a** They have the same lower quartile.

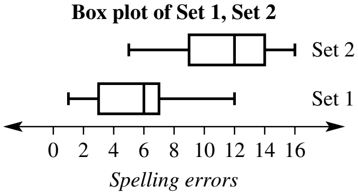
b B

c **i** 4 **ii** 5

d Set B is more spread out.

9 **a** A **b** B **c** B

10 **a** **Box plot of Set 1, Set 2**



b Yes, examiner 2 found more errors.

11 Answers may vary. Examples:

a **i, ii** Class results have a smaller spread in the top 25%, and bottom 25% performed better.

iii State results have a larger IQR.

b The class did not have other results close to zero but the school did.

Exercise 4E

1 **a** larger **b** smaller

2 **a** B **b** A

3 **a** A

b The data values in A are spread farther from the mean than the data values in B.

4 **a** Gum Heights **b** Gum Heights

5 **a** mean = 6, $\sigma = 2$ **b** mean = 3.6, $\sigma = 2.3$

c mean = 8, $\sigma = 3.5$ **d** mean = 32.5, $\sigma = 3.3$

6 **a** mean = 2.7, $\sigma = 0.9$ **b** mean = 14.5, $\sigma = 6.0$

7 **a** The outer suburb has more data values in the higher range.

b There is less spread. Data values are closer to the mean.

c Students at outer-suburb schools may live some distance from the school.

8 **a** false **b** true **c** true

9 **a** mean = 2, $\sigma = 0.9$ **b** mean = 5.25, $\sigma = 0.7$

10 **a** no **b** no **c** yes

d Yes, one of the deviations would be calculated using the outlier.

11 **a** No; standard deviation reflects the spread of the data values from the mean, not the size of the data values.

b No. As for part **a**.

12 The IQRs would be the same, making the data more comparable.

13 **a** **i** 85.16 **ii** 53.16 **iii** 101.16

iv 37.16 **v** 117.16 **vi** 21.16

b **i** 66% **ii** 96% **iii** 100%

c **i** Student's own research required.

ii One SD from the mean = 68%

Two SDs from the mean = 95%

Three SDs from the mean = 99.7%

Close to answers found.

Exercise 4F

1 **a** linear **b** no trend

c non-linear **d** linear

2 **a** **i** 28°C **ii** 33°C

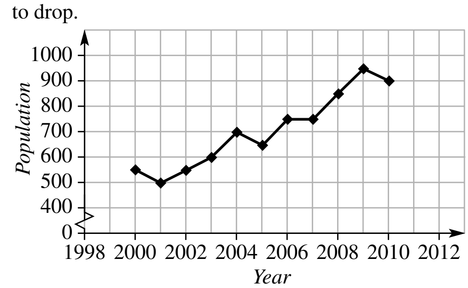
iii 33°C **iv** 35°C

b 36°C

c **i** noon to 1 p.m. **ii** 3–4 p.m.

d Temperature is increasing from 8 a.m. to 3 p.m. in a generally linear way. At 3 p.m. the temperature starts to drop.

3 **a**



b Generally linear in a positive direction.

c **i** 500 **ii** 950

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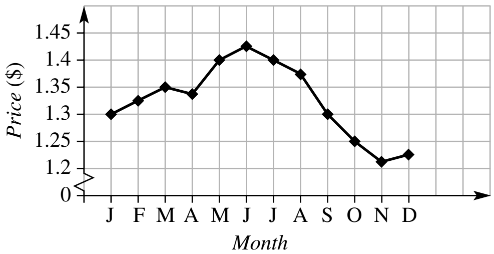
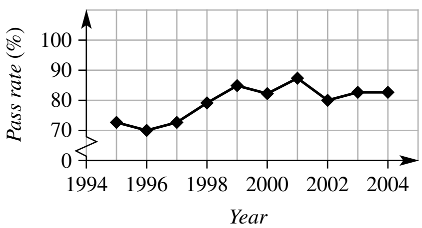
4 a



b The share price generally increased until it peaked in June and then continually decreased to a yearly low in November before trending upwards again in the final month.

c $0.21

5 a



b The pass rate for the examination has increased marginally over the 10 years, with a peak in 2001.

c 2001 **d** 11%

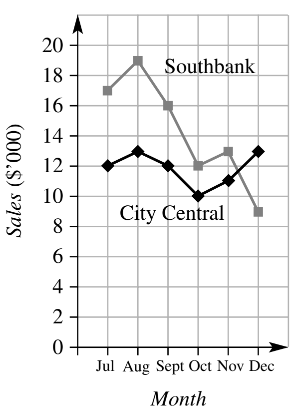
6 a linear

b i $650 000 **ii** $750 000

7 a i $6000 **ii** $4000

b 1

c



d i The sales trend for City Central for the 6 months is fairly constant.

ii Sales for Southbank peaked in August before taking a downturn.

e about $5000

8 a i 5.84 km **ii** 1.7 km

b i Blue Crest slowly gets closer to the machine.

ii Green Tail starts near the machine and gets farther from it.

c 8.30 p.m.

9 a The yearly temperature is cyclical and January is the next month after December and both are in the same season.

b no

c Northern Hemisphere, as the seasons are opposite; June is summer.

10 a Increases continually, rising more rapidly as the years progress.

b Compound interest due to exponential growth.

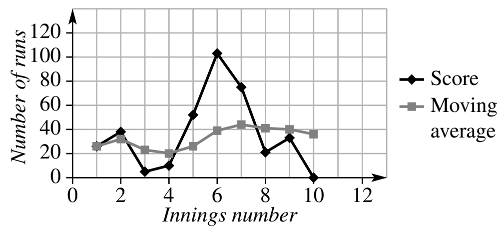
11 a Graphs may vary, but it should decrease from room temperature to the temperature of the fridge.

b No. Drink cannot cool to a temperature *lower* than that of the internal environment of the fridge.

12 a

Innings	1	2	3	4	5	6	7	8	9	10
Score	26	38	5	10	52	103	75	21	33	0
Moving average	26	32	23	20	26	39	44	41	40	36

b



c Innings number.

i The score fluctuates wildly.

ii The graph is fairly constant with small increases and decreases.

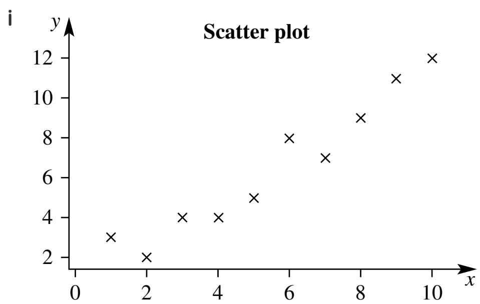
d The moving average graph follows the trend of the score graph but the fluctuations are much less significant.

Exercise 4G

1 a unlikely **b** likely **c** unlikely

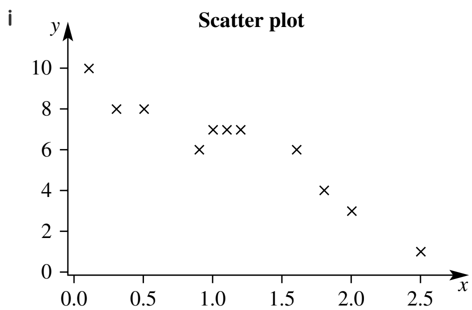
d likely **e** likely **f** likely

2 a i



ii y generally increases as x increases.

b i



ii y generally decreases as x increases.

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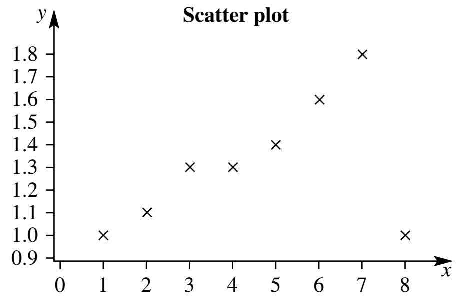
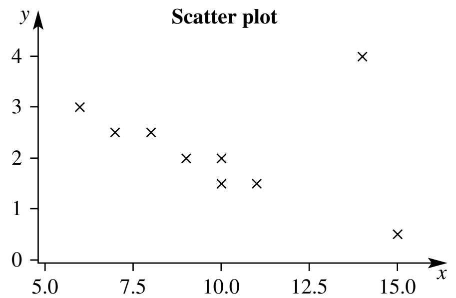
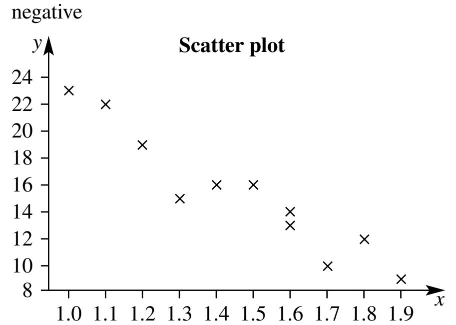
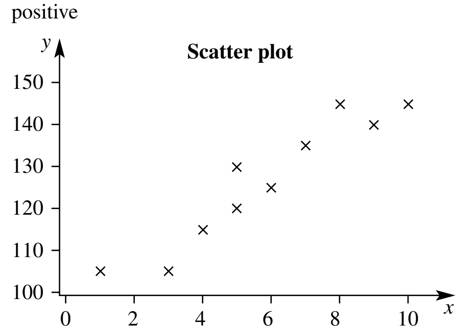
3 a


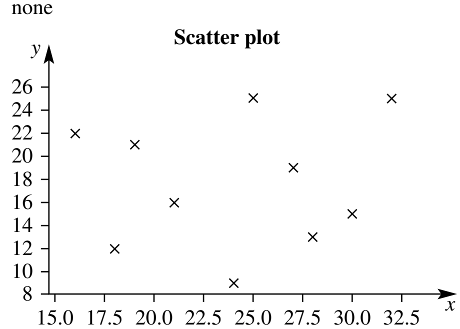
b positive c strong d (8, 1.0)

4 a


b negative c strong d (14, 4)

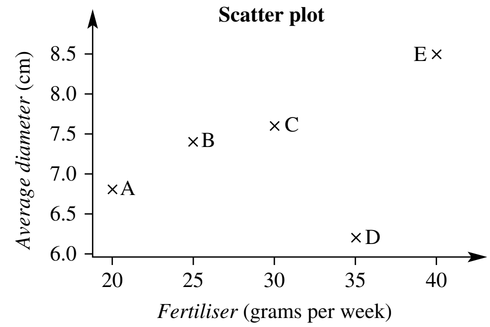
5 a negative


b positive


c none


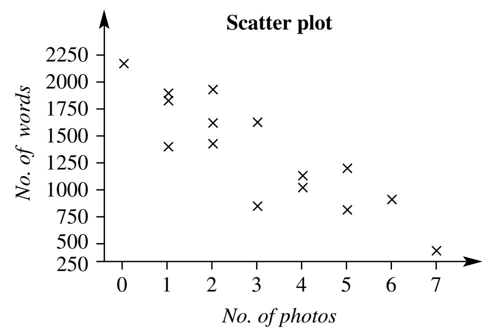
6 a none b weak negative
c positive d strong positive

7 a yes b decrease
c i yes ii car H

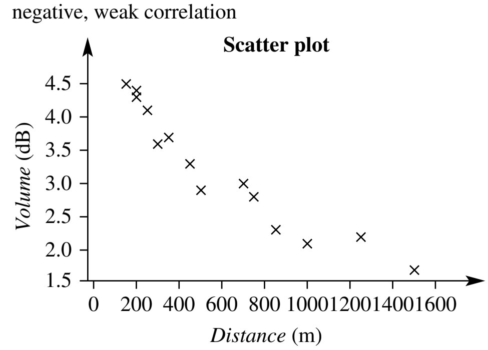
8 a


b D

c Yes, although small sample size does lead to doubt.

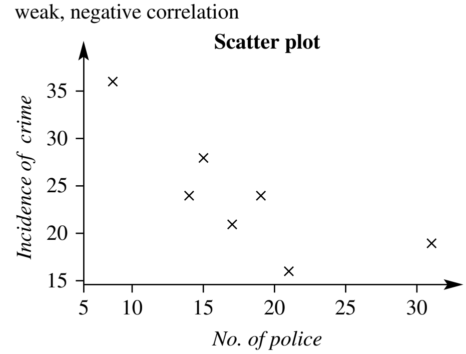
9 a


b negative, weak correlation

10 a


b negative

c As *Distance* increases, *Volume* decreases.

11 a i weak, negative correlation


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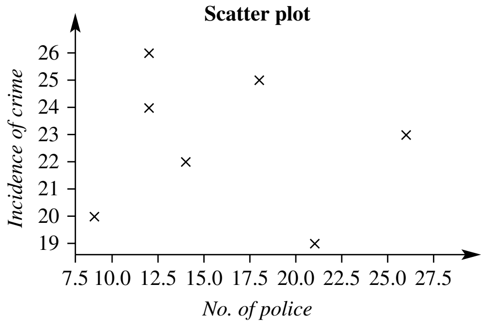
ii no correlation



b Survey 1, as this shows an increase in the number of police has seen a decrease in the incidence of crime.

12 The positive correlation shows that as height increases, the ability to play tennis increases.

13 Each axis needs a better scale. All data lie between 6 and 8 hours sleep and shows only a minimum change in exam marks.

14 a i students I, T ii students G, S

b i students H, C ii students B, N

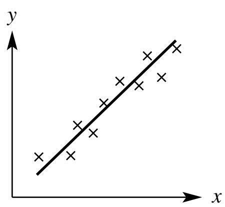
c students G, S

d students B, N

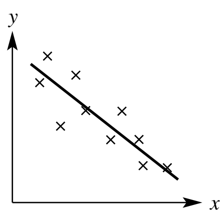
e no

Exercise 4H

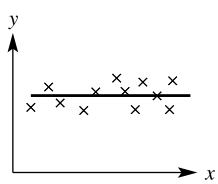
1 a



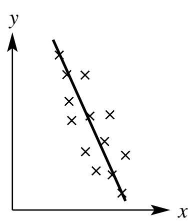
b



c



d

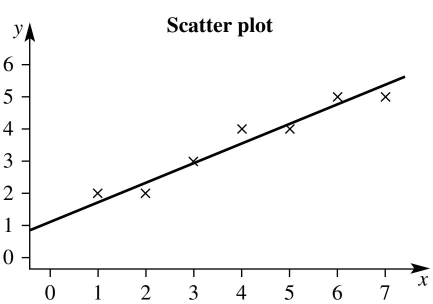


2 a $y = \frac{1}{2}x + \frac{7}{2}$ b $y = -\frac{2}{3}x + \frac{17}{3}$

3 a i 17 ii $\frac{23}{4}$

b i $\frac{28}{5}$ ii $\frac{14}{5}$

4 a



b positive correlation

c See part a

d All answers are approximate.

i 3.2 ii 0.9 iii 1.8 iv 7.4

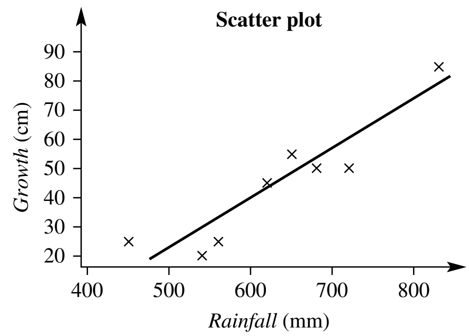
5 a ≈ 4.5 b ≈ 6 c ≈ 0.5 d ≈ 50

6 a $y = \frac{3}{5}x + 18$

b i 42 ii 72

c i 30 ii 100

7 a, b



c i ≈ 25 cm ii ≈ 85 cm

d i ≈ 520 mm ii ≈ 720 mm

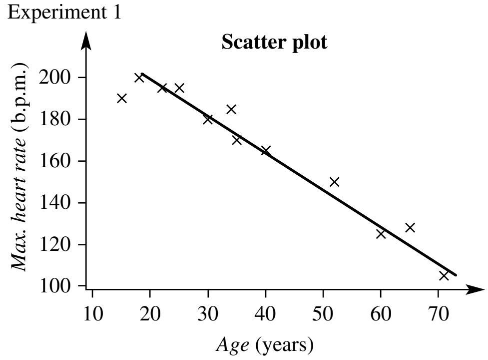
8 a $y = 5x - 5$ b 85 cm c 21 kg

9 a B, C b B, C c B d A

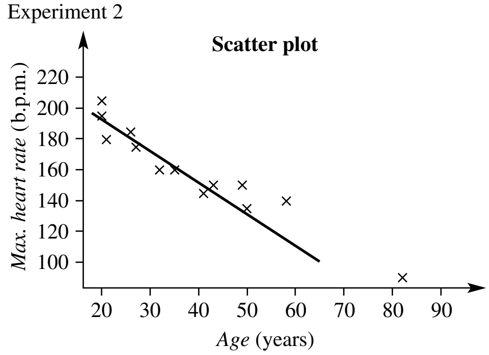
10 a i 50 ii 110

b It is possible to obtain scores of greater than 100%.

11 a Experiment 1



Experiment 2



b i ≈ 140 ii ≈ 125

c i ≈ 25 ii ≈ 22

d experiment 2

e Student's own research required.

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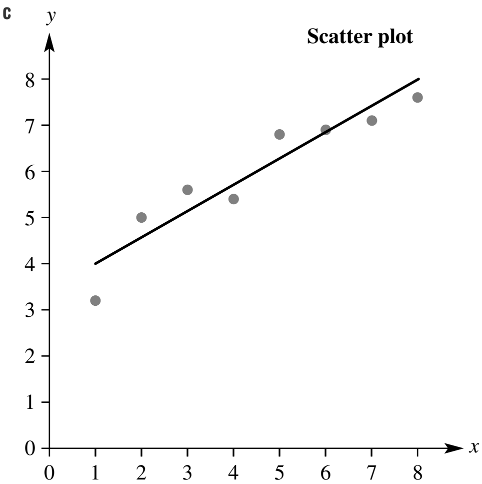
Exercise 4I

1 **a** **i** 12 **ii** 3.26
b **i** 7 **ii** 2

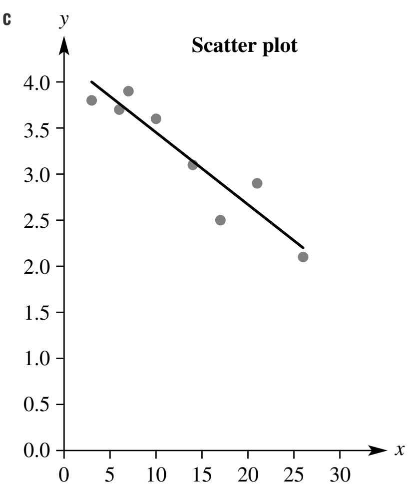
2 **a** There is no linear correlation.
b The correlation shown is not a linear shape.

3 **A** **a, c**



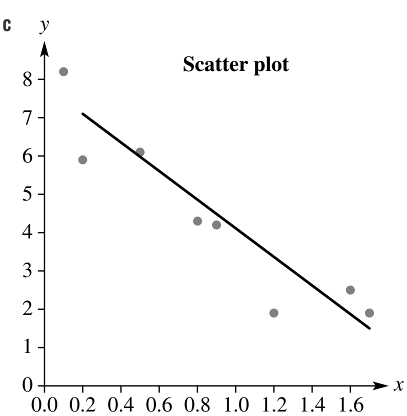
b $y = 0.554762x + 3.45357$
d **i** 7.3 **ii** 10.1

B **a, c**



b $y = -0.077703x + 4.21014$
d **i** 3.7 **ii** 3.3

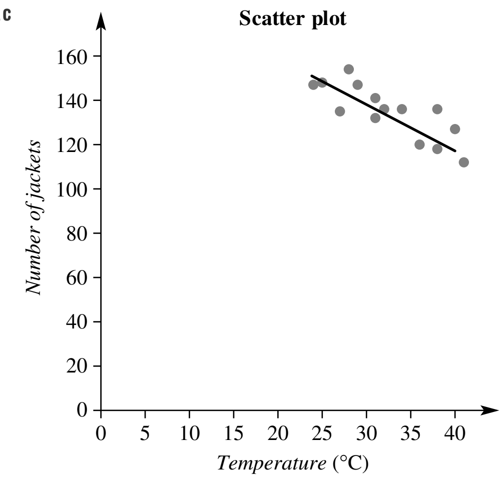
C **a, c**



b $y = -3.45129x + 7.41988$
d **i** -16.7 **ii** -34.0

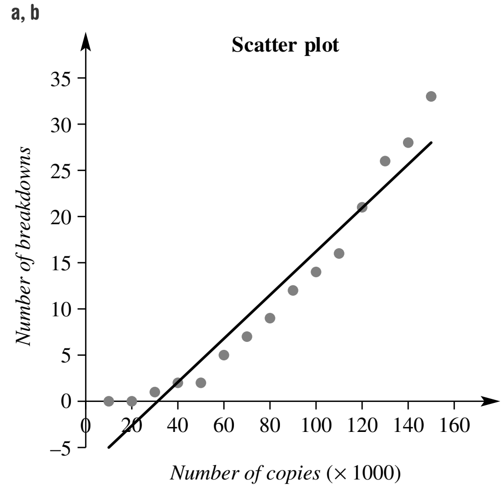
4 **a** $y = -3.54774x + 43.0398$
b \$32 397 **c** 8 years

5 **a, c**



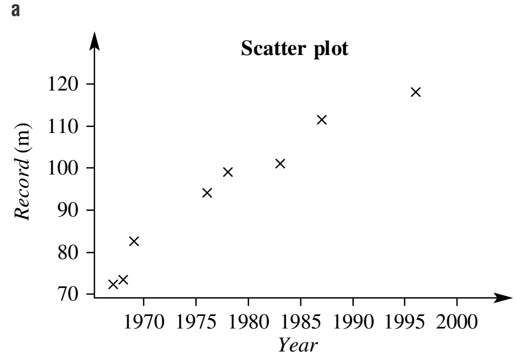
b $y = -1.72461x + 190.569$
d **i** 139 **ii** 130 **iii** 113

6 **a, b**



c 241 000 copies
d No, the percentage of breakdowns is still very low.

7 **a**



b **i** 129 m **ii** 153 m
c No, records are not likely to continue to increase at this rate.

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8 All deviations are used in the calculation of the least squares regression.
9 Line A, as it has been affected by the outlier.
10 Student's own research required.

Puzzles and challenges

1 66 kg
2 88%
3 19
4 1.1
5 a larger by 3 b larger by 3 c no change
d no change e no change
6 $y = x^2 - 3x + 5$
7 $5.8 \leqslant a < 6.2$

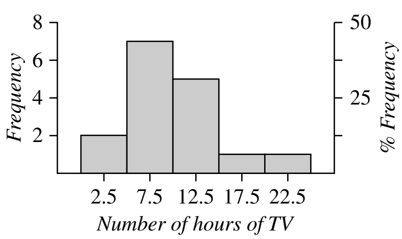
Multiple-choice questions

1 B	2 E	3 D	4 C	5 A
6 C	7 D	8 E	9 C	10 B

Short-answer questions

1 a

Class interval	Frequency	Percentage frequency
0–4	2	12.5
5–9	7	43.75
10–14	5	31.25
15–19	1	6.25
20–25	1	6.25
Total	**16**	**100**

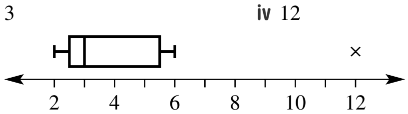
b


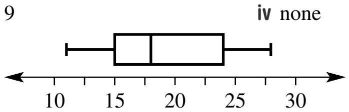
c It is positively skewed.

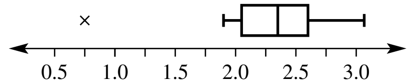
d

Stem	Leaf
0	1 3 5 6 6 7 8 8 9
1	0 1 2 3 4 6
2	4

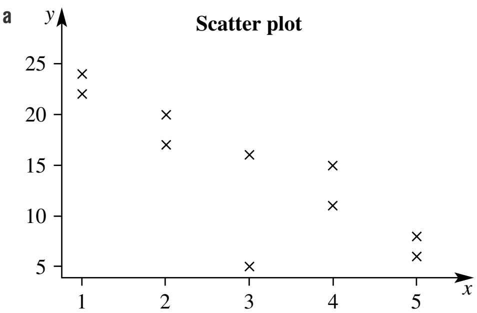
e 8.5 hours

2 a i 10 ii $Q_1 = 2.5, Q_3 = 5.5$
iii 3 iv 12
v


b i 17 ii $Q_1 = 15, Q_3 = 24$
iii 9 iv none
v


c i 2.4 ii $Q_1 = 2.1, Q_3 = 2.6$
iii 0.50 iv 0.7
v


3 a false b true c true d true

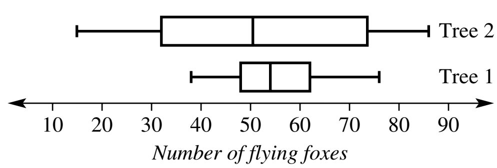
4 a


b negative c weak d (3, 5)

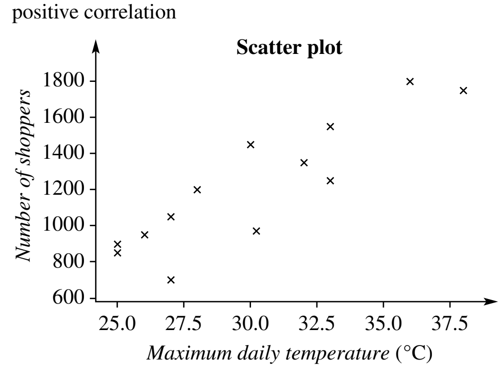
5 a $y = \frac{3}{5}x + \frac{7}{5}$
b i 3.8 ii 7.4
c i $2\frac{2}{3}$ ii $17\frac{2}{3}$

6 a non-linear b linear
7 a mean = 7, $\sigma = 2.3$ b mean = 4, $\sigma = 2.8$
8 a The Eagles b The Eagles
c The Eagles d The Eagles
9 $y = -3.75x + 25.65$

Extended-response questions

1 a i 14 ii 41
b i no outliers ii no outliers
c


d More flying foxes regularly take refuge in tree 1 than in tree 2, for which the spread is much greater.

2 a positive correlation


b $y = 74.5585x - 1010.06$
c i 779 ii 33.7°C

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Chapter 5

Pre-test

1 **a** $2x + 1$ **b** $5(x - 1)$ **c** $x + \frac{1}{2}x$
d $2x - 3$ **e** $\frac{1}{3}(x + 4)$ **f** $3x - 7$

2 **a** $7x$ **b** $-5ab$ **c** $-5x^2$

3 **a** $8y$ **b** $-21x$ **c** $15a^2$

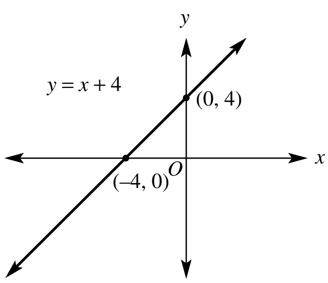
4 **a** $3x + 3$ **b** $-4x + 4$ **c** $-10x + 2x^2$

5 **a** $4m$ **b** 3 **c** a **d** $\frac{2x}{3}$

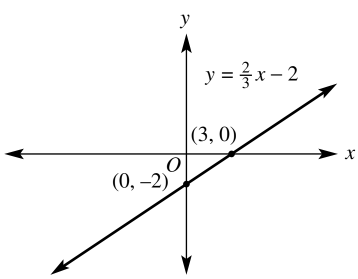
6 **a** $\frac{7}{6}$ **b** $\frac{7}{12}$ **c** $\frac{4}{7}$ **d** $\frac{19}{72}$
e $\frac{4}{7}$ **f** $\frac{21}{20}$ **g** $\frac{3}{4}$ **h** $\frac{49}{9}$

7 **a** 4 **b** 7 **c** 10

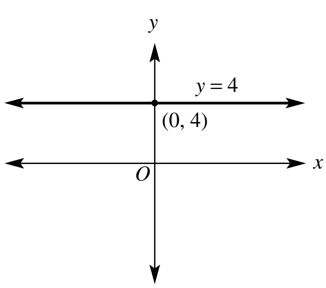
8 **a** **i** $x = -4$, $y = 4$ **ii** 1
iii



b **i** $x = 3$, $y = -2$ **ii** $\frac{2}{3}$
iii



c **i** No x-intercept, $y = 4$ **ii** 0
iii



9 **a** 3 **b** -2 **c** 1

10 **a** $y = 2x + 1$ **b** $y = -x + 5$

Exercise 5A

1 C

2 D

3 **a** 7 **b** 1 **c** -4 **d** -9
e $\frac{1}{2}$ **f** $-\frac{1}{5}$ **g** $\frac{2}{7}$ **h** $-\frac{7}{3}$

4 **a** yes **b** yes **c** no

5 **a** 9 **b** -8 **c** -8 **d** -9

6 **a** $10a$ **b** $15d$ **c** 0
d $5xy$ **e** $4ab$ **f** $9t$
g $9b$ **h** $-st^2$ **i** $-3m^2n$
j $-0.7a^2b$ **k** $2gh + 5$ **l** $12xy - 3y$
m $3a + 7b$ **n** $8jk - 7j$ **o** $ab^2 + 10a^2b$
p $2mn - m^2n$ **q** $5st - s^2t$ **r** $3x^3y^4 + 2xy^2$

7 **a** $12ab$ **b** $25ab$ **c** $-6ad$ **d** $-10hm$
e $30ht$ **f** $30bl$ **g** $12s^2t$ **h** $-21b^2d^5$
i $8a^2b^4$ **j** $24p^3q$ **k** $-18h^5i^5$ **l** $63m^2pr$
m x **n** $3ab$ **o** $-\frac{a}{3}$ **p** $-\frac{ab}{4}$
q $2b$ **r** $-3x$ **s** $-\frac{y}{2}$ **t** $-\frac{a}{2}$

8 **a** $5x + 5$ **b** $2x + 8$
c $3x - 15$ **d** $-20 - 5b$
e $-2y + 6$ **f** $-7a - 7c$
g $6m + 18$ **h** $4m - 12n + 20$
i $-2p + 6q + 4$ **j** $2x^2 + 10x$
k $6a^2 - 24a$ **l** $-12x^2 + 16xy$
m $15y^2 + 3yz - 24y$ **n** $36g - 18g^2 - 45gh$
o $-8ab + 14a^2 - 20a$ **p** $14y^2 - 14y^3 - 28y$
q $-6a^3 + 3a^2 + 3a$ **r** $-5t^4 - 6t^3 - 2t$

9 **a** $5x + 23$ **b** $10a + 26$ **c** $21y + 3$ **d** $15m + 6$
e 10 **f** $11t - 1$ **g** $3x^2 + 15x$ **h** $15z - 7$
i $-11d^3$ **j** $9q^4 - 9q^3$

10 **a** $3(x - 3)$ **b** $4(x - 2)$ **c** $10(y + 2)$
d $6(y + 5)$ **e** $x(x + 7)$ **f** $2a(a + 4)$
g $5x(x - 1)$ **h** $9y(y - 7)$ **i** $xy(1 - y)$
j $x^2y(1 - 4y)$ **k** $8a^2(b + 5)$ **l** $ab(7a + 1)$
m $-5t(t + 1)$ **n** $-6mn(1 + 3n)$ **o** $-y(y + 8z)$
p $-3a(ab + 2b + 1)$

11 **a** -32 **b** 7 **c** 61 **d** 12
e $-\frac{1}{2}$ **f** $\frac{13}{5}$ **g** $-\frac{7}{5}$ **h** 1

12 **a** $2x^2 + 6x$ **b** $x^2 - 5x$

13 **a** $P = 4x - 4$, $A = x^2 - 2x - 4$
b $P = 4x + 2$, $A = 3x - 1$
c $P = 4x + 14$, $A = 7x + 12$

14 **a** $(-2)(-2) = 4$, negative signs cancel
b $a^2 > 0 \quad \therefore -a^2 < 0$
c $(-2)^3 = (-2)(-2)(-2) = -8$

15 **a** true **b** false, $1 - 2 \neq 2 - 1$
c true **d** false, $\frac{1}{2} \neq \frac{2}{1}$
e true **f** false, $3 - (2 - 1) \neq (3 - 2) - 1$
g true **h** false, $8 \div (4 \div 2) \neq (8 \div 4) \div 2$

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16 a $\frac{x+y}{2}$ or $x+\frac{y}{2}$

b It could refer to either of the above, depending on interpretation.

c 'Half of the sum of a and b' or 'a plus b all divided by 2'.

17 a $P=\left(4+\frac{\pi}{2}\right)x+2,\ A=\left(1+\frac{\pi}{4}\right)x^2+x$

b $P=\left(6+\frac{\pi}{2}\right)x-6,\ A=\left(3-\frac{\pi}{4}\right)x^2-3x$

c $P=2\pi x,\ A=\left(1+\frac{\pi}{2}\right)x^2$

Exercise 5B

1 a $\frac{11}{8}$ **b** $\frac{3}{8}$ **c** $\frac{1}{24}$ **d** 3

2 a $\frac{2}{3}$ **b** $\frac{3}{7a}$ **c** $-\frac{7t}{4xy}$ **d** $-\frac{b^2c}{8x^2a}$

3 a 12 **b** 6 **c** 14 **d** $2x$

4 a $5x$ **b** $4x$ **c** $\frac{a}{4}$ **d** $\frac{1}{3a}$

e $5x$ **f** $-2x$ **g** $-9b$ **h** $-2y$

i $-\frac{1}{2p}$ **j** $-\frac{4}{9st}$ **k** $-\frac{3x}{y}$ **l** $\frac{6b}{7}$

5 a $x+2$ **b** $a-5$ **c** $3x-9$ **d** $1-3y$

e $1+6b$ **f** $1-3x$ **g** $3-t$ **h** $x-4$

i $x+2$ **j** $3-2x$ **k** $a-1$ **l** $\frac{1+2a}{3}$

6 a $\frac{x-1}{2x}$ **b** $\frac{x+4}{5x}$ **c** -4

d $\frac{4}{9}$ **e** 5 **f** $\frac{5a}{2}$

g 2 **h** 15 **i** $-\frac{1}{2}$

7 a 3 **b** 3 **c** $\frac{18}{5}$

d $\frac{3}{4}$ **e** $\frac{4}{3}$ **f** $\frac{1}{25}$

g $-\frac{5}{3}$ **h** $\frac{2}{5}$ **i** $-\frac{1}{3}$

8 a $\frac{3a+14}{21}$ **b** $\frac{4a+3}{8}$ **c** $\frac{3-15b}{10}$ **d** $\frac{4x+6}{15}$

e $\frac{1-6a}{9}$ **f** $\frac{3a+8}{4a}$ **g** $\frac{7a-27}{9a}$ **h** $\frac{16-3b}{4b}$

i $\frac{4b-21}{14b}$ **j** $\frac{27-14y}{18y}$ **k** $\frac{-12-2x}{3x}$ **l** $\frac{-27-2x}{6x}$

9 a $\frac{9x+23}{20}$ **b** $\frac{7x+11}{12}$ **c** $\frac{-x-7}{4}$

d $\frac{2x+15}{9}$ **e** $\frac{4x+7}{6}$ **f** $\frac{8x+3}{10}$

g $\frac{7x+2}{24}$ **h** $\frac{5x-1}{5}$ **i** $\frac{5-3x}{14}$

10 a $\frac{7x+22}{(x+1)(x+4)}$ **b** $\frac{7x-13}{(x-7)(x+2)}$

c $\frac{3x-1}{(x-3)(x+5)}$ **d** $\frac{x-18}{(x+3)(x-4)}$

e $\frac{-21}{(2x-1)(x-4)}$ **f** $\frac{14x-26}{(x-5)(3x-4)}$

g $\frac{41-7x}{(2x-1)(x+7)}$ **h** $\frac{3x+17}{(x-3)(3x+4)}$

i $\frac{14-17x}{(3x-2)(1-x)}$

11 a i a^2 **ii** x^2

b i $\frac{2a-3}{a^2}$ **ii** $\frac{a^2+a-4}{a^2}$ **iii** $\frac{3x+14}{4x^2}$

12 The 2 in the second numerator needs to be subtracted; i.e. $\frac{x-2}{6}$.

13 a $-(3-2x)=-3+2x$ (since $-1\times-2x=2x$)

b i $\frac{2}{x-1}$ **ii** $\frac{2x}{3-x}$ **iii** $\frac{x+3}{7-x}$

iv $-\frac{7}{3}$ **v** $-\frac{12}{x}$ **vi** $\frac{7}{2}$

14 a -1 **b** $\frac{5a+2}{a^2}$ **c** $\frac{3x+5}{(x+1)^2}$

d $\frac{3x-x^2}{(x-2)^2}$ **e** $\frac{21x-9x^2}{14(x-3)^2}$ **f** $\frac{yz-xz-xy}{xyz}$

15 a 2 **b** 1

Exercise 5C

1 a no **b** no **c** yes **d** yes

2 a true **b** false **c** false

3 a false **b** true **c** true

4 a 5 **b** 8 **c** -3 **d** 4

e $\frac{5}{2}$ **f** $\frac{11}{4}$ **g** $-\frac{1}{3}$ **h** $-\frac{11}{6}$

i -4 **j** $\frac{3}{2}$ **k** $-\frac{9}{2}$ **l** $-\frac{4}{3}$

m -2 **n** 7 **o** -2 **p** $\frac{11}{9}$

5 a 1 **b** 9 **c** $\frac{23}{2}$ **d** $-\frac{5}{6}$

e $-\frac{9}{11}$ **f** $\frac{2}{3}$ **g** 1 **h** 2

i 4 **j** 7 **k** -9 **l** 5

m 19 **n** 23

6 a 10 **b** 13 **c** -22 **d** 4

e -5 **f** 6 **g** 16 **h** 4

i -9 **j** 8 **k** 6 **l** -7

m 20 **n** 15 **o** -9

7 a $x+3=7,\ x=4$ **b** $x+8=5,\ x=-3$

c $x-4=5,\ x=9$ **d** $15-x=22,\ x=-7$

e $2x+5=13,\ x=4$ **f** $2(x-5)=-15,\ x=-\frac{5}{2}$

g $3x+8=23,\ x=5$ **h** $2x-5=x-3,\ x=2$

8 a 1 **b** 0 **c** -17

d $\frac{7}{2}$ **e** $\frac{27}{23}$ **f** $\frac{28}{5}$

g $\frac{13}{14}$ **h** $\frac{2}{5}$ **i** $\frac{94}{11}$

9 a 1 **b** 6 **c** 2 **d** 25

10 17 cm

11 17 and 18

12 24 km

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13 a \$214 **b** \$582
c i 1 **ii** 10.5 **iii** 21

14 a 41 L **b** 90 s = 1 min 30 s **c** 250 s = 4 min 10 s

15 $x = 9$. Method 2 is better; expanding the brackets is unnecessary, given that 2 is a factor of 8.

16 a $5 - a$ **b** $\frac{a}{6}$ **c** $\frac{5}{a}$
d $\frac{2a+1}{a}$ **e** $\frac{3a+1}{a}$ **f** $\frac{c-b}{a}$

17 a 6 **b** 4 **c** −15
d 20 **e** 3 **f** 6
g 1 **h** −26 **i** −10

18 a $a = \frac{c}{b+1}$ **b** $a = \frac{b}{b+1}$ **c** $a = \frac{1}{c-b}$
d $a = 3b$ **e** $a = -b$ **f** $a = \frac{bc}{b-c}$

Exercise 5D

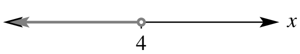
1 a 3, 6, 10 (Answers may vary.)
b −4, −3, −2 (Answers may vary.)
c 5, 6, 7 (Answers may vary.)
d −8.5, −8.4, −8.3 (Answers may vary.)

2 a B **b** C **c** A

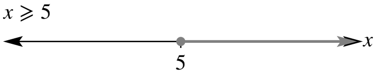
3 11, 12 or 13 rabbits

4 a $x \geqslant 1$ **b** $x < 7$ **c** $x \leqslant 4$
d $x > -9$ **e** $-2 < x \leqslant 1$ **f** $8 < x \leqslant 11$
g $-9 < x < -7$ **h** $1.5 \leqslant x \leqslant 2.5$ **i** $-1 \leqslant x < 1$

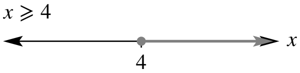
5 a $x < 4$



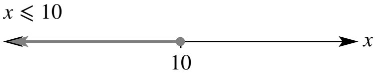
b $x \geqslant 5$



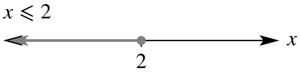
c $x \geqslant 4$



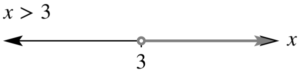
d $x \leqslant 10$



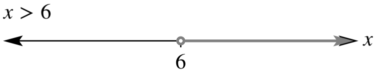
e $x \leqslant 2$



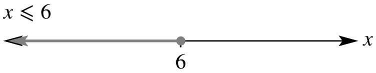
f $x > 3$



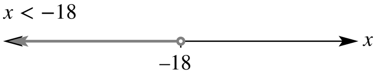
g $x > 6$



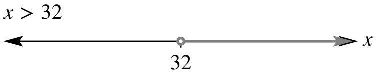
h $x \leqslant 6$



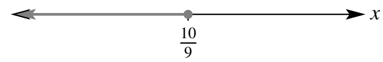
i $x < -18$



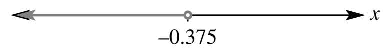
j $x > 32$



k $x \leqslant \frac{10}{9}$



l $x < -\frac{3}{8}$



6 a $x \geqslant -\frac{2}{5}$ **b** $x < 2$ **c** $x \leqslant -5$ **d** $x \leqslant -7$
e $x < -8$ **f** $x \geqslant 4$ **g** $x \geqslant -10$ **h** $x < -21$

7 a $x > 6$ **b** $x \leqslant 2$ **c** $x < \frac{5}{2}$
d $x \geqslant 10$ **e** $x \leqslant \frac{1}{16}$ **f** $x < \frac{11}{4}$

8 a $2x + 7 < 12, x < \frac{5}{2}$ **b** $4 - \frac{x}{2} \geqslant -2, x \leqslant 12$
c $3(x+1) \geqslant 2, x \geqslant -\frac{1}{3}$ **d** $x + (x+2) \leqslant 24, x \leqslant 11$
e $(x-6) + (x-4) + (x-2) + x \leqslant 148, x \leqslant 40$

9 a i $C <$ \$1.30 **ii** $C >$ \$2.30
b i less than 9 min **ii** 16 min or more

10 a $x < -5$ **b** $x \geqslant \frac{11}{4}$ **c** $x \geqslant \frac{11}{29}$
d $x \leqslant \frac{14}{5}$ **e** $x \geqslant \frac{27}{29}$ **f** $x < \frac{1}{2}$

11 a An infinite number of whole numbers (i.e. all the ones greater than 8).
b Only one as 3 is the only whole number.

12 a $x \geqslant \frac{a+3}{10}$ **b** $x < 2 - 4a$
c $x < 1 - \frac{7}{a}$ or $x < \frac{a-7}{a}$

13 a $-4 \leqslant x < 5$ **b** $-9.5 < x \leqslant -7$ **c** $x = 10$

14 a $3 \leqslant x \leqslant 9$

b $-7 \leqslant x \leqslant 3$

c $-9 \leqslant x < -7$

d $-\frac{3}{2} \leqslant x \leqslant 2$

e $-3 \leqslant x \leqslant \frac{7}{3}$

f $-4 \leqslant x \leqslant -2$

g $11 \leqslant x \leqslant 12$

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h $1 \leqslant x \leqslant \frac{15}{4}$

15 a $x \geqslant 23$ **b** $x < \frac{19}{5}$ **c** $x \leqslant 1$

Exercise 5E

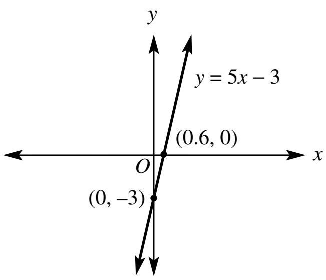
1 a $y = -2x + 5$, $m = -2$, $b = 5$
b $y = 2x - 3$, $m = 2$, $b = -3$
c $y = x - 7$, $m = 1$, $b = -7$
d $y = -\frac{2x}{5} - \frac{3}{5}$, $m = -\frac{2}{5}$, $b = -\frac{3}{5}$

2 a i 3 **ii** 6 **iii** $\frac{21}{2}$
b i 2 **ii** 6 **iii** $\frac{8}{3}$

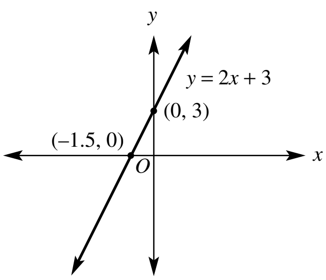
3 a i **b** iv **c** ii
d iii **e** v **f** vi

4 a yes **b** yes **c** no
d no **e** yes **f** no

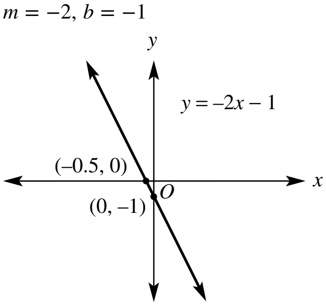
5 a $m = 5$, $b = -3$



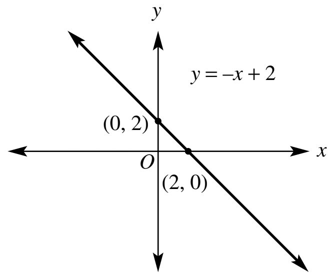
b $m = 2$, $b = 3$



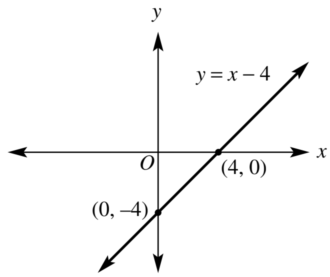
c $m = -2$, $b = -1$



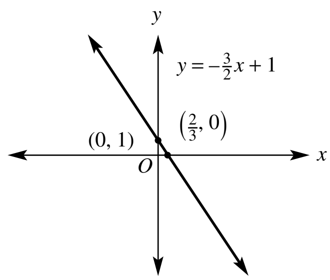
d $m = -1$, $b = 2$



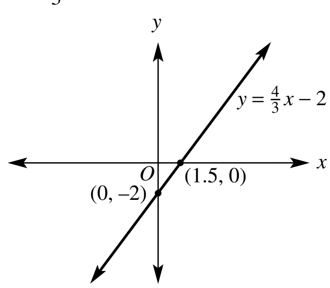
e $m = 1$, $b = -4$



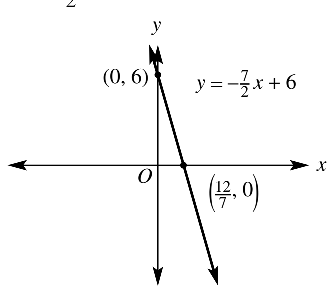
f $m = -\frac{3}{2}$, $b = 1$



g $m = \frac{4}{3}$, $b = -2$



h $m = -\frac{7}{2}$, $b = 6$



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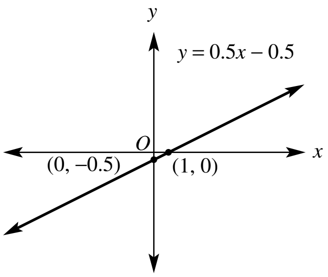
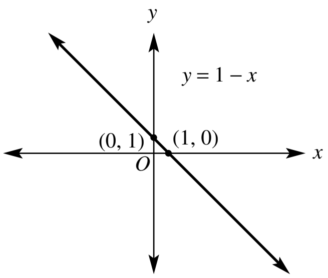
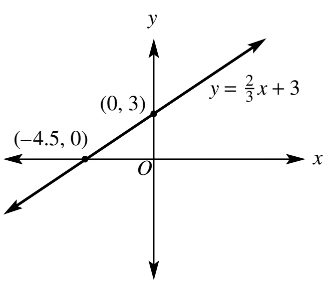
i $m = 0.5, b = -0.5$



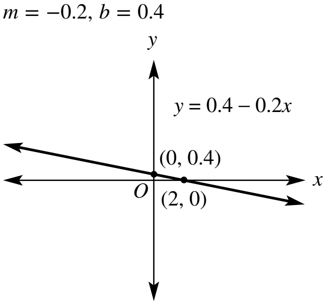
j $m = -1, b = 1$



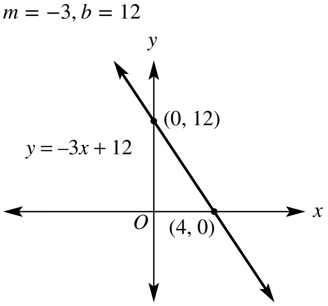
k $m = \frac{2}{3}, b = 3$



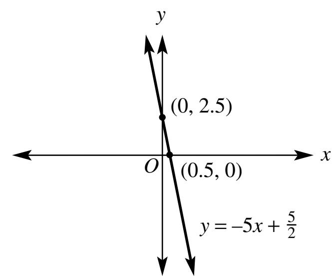
l $m = -0.2, b = 0.4$



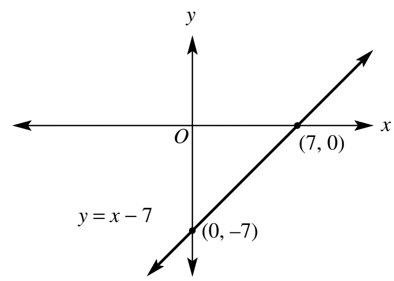
6 a $m = -3, b = 12$



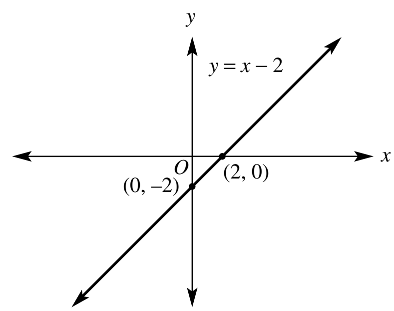
b $m = -5, b = \frac{5}{2}$



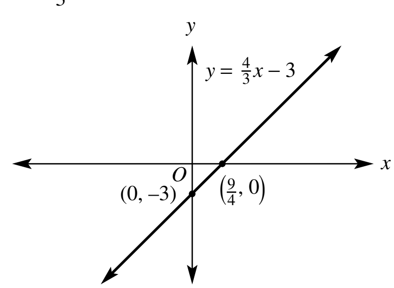
c $m = 1, b = -7$



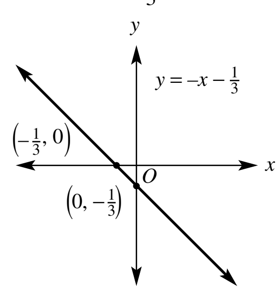
d $m = 1, b = -2$



e $m = \frac{4}{3}, b = -3$



f $m = -1, b = -\frac{1}{3}$



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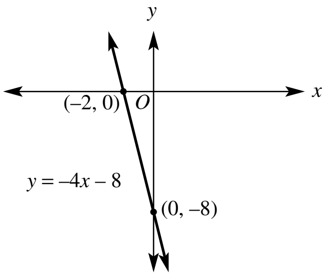
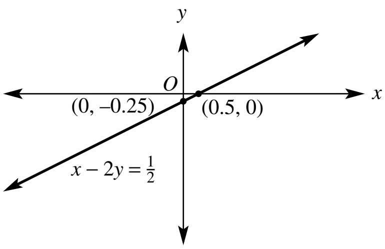
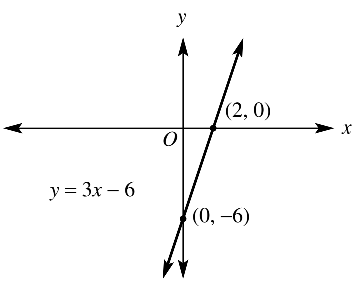
g $m = -4$, $b = -8$



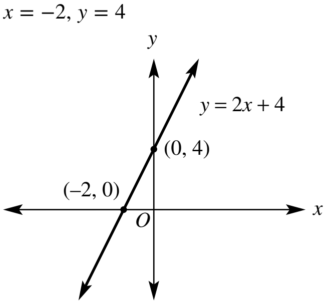
h $m = \frac{1}{2}$, $b = -\frac{1}{4}$



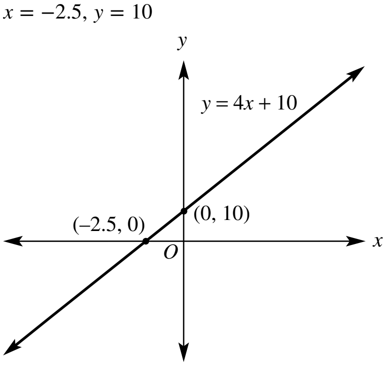
7 a $x = 2, y = -6$



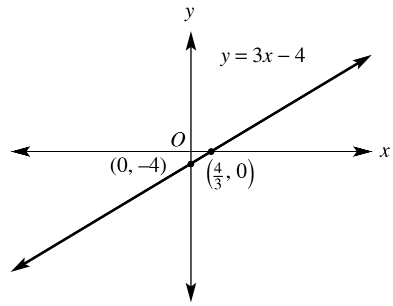
b $x = -2$, $y = 4$



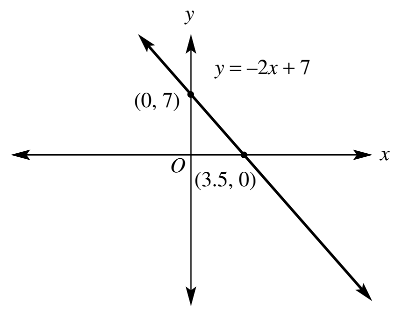
c $x = -2.5$, $y = 10$



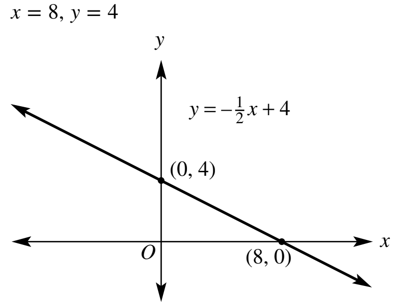
d $x = \frac{4}{3}$, $y = -4$



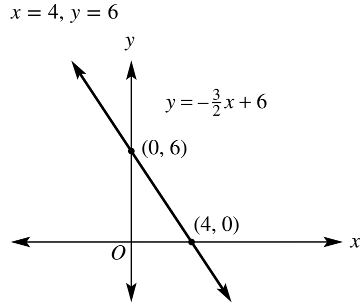
e $x = 3.5$, $y = 7$



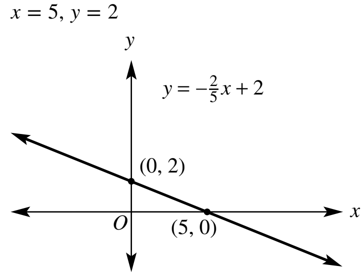
f $x = 8$, $y = 4$



g $x = 4$, $y = 6$



h $x = 5$, $y = 2$



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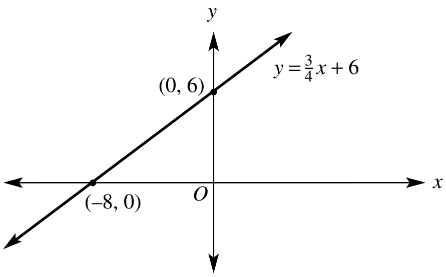
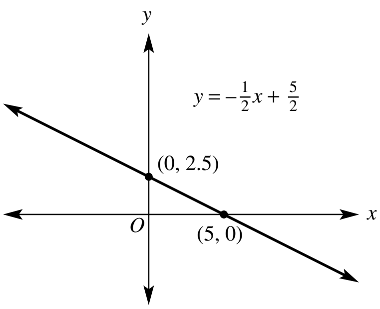
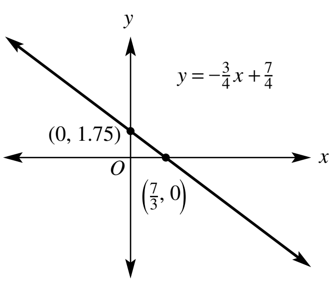
i $x = -8, y = 6$



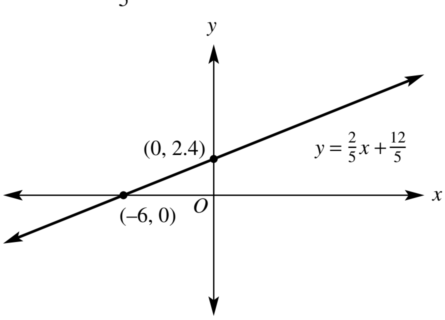
j $x = 5, y = 2.5$



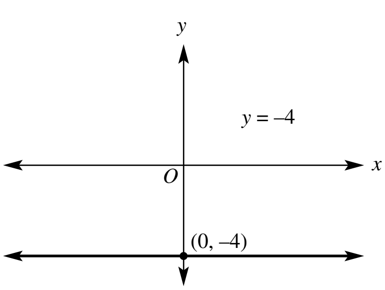
k $x = \frac{7}{3}, y = \frac{7}{4}$



l $x = -6, y = \frac{12}{5}$



8 a



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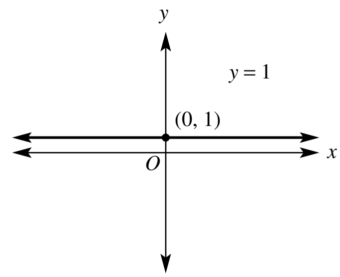

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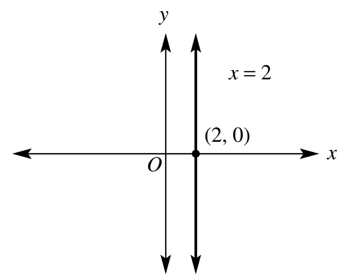

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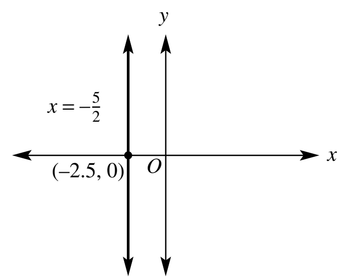

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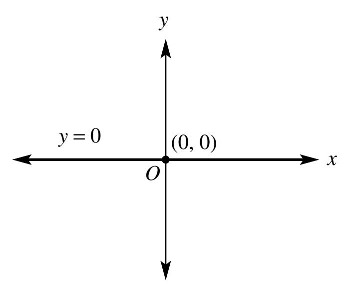

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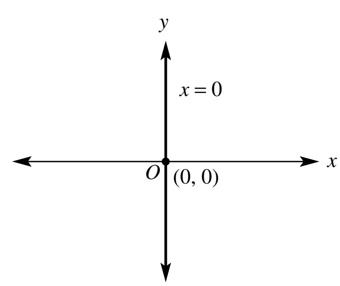

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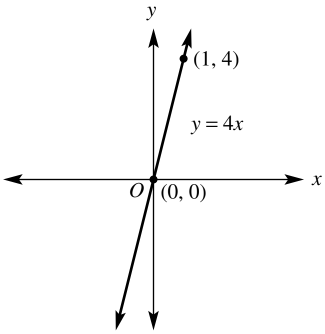

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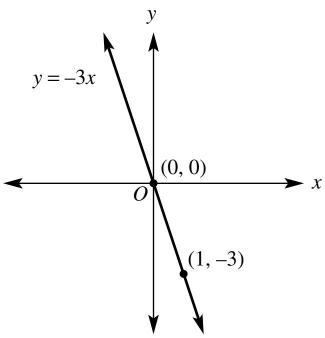

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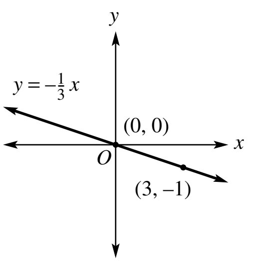

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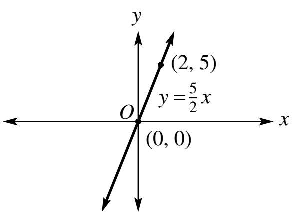

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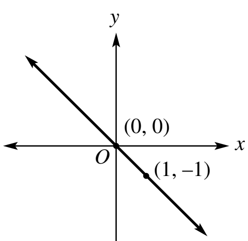
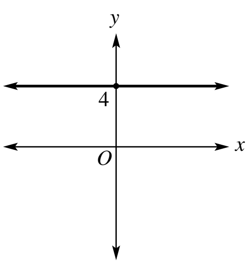
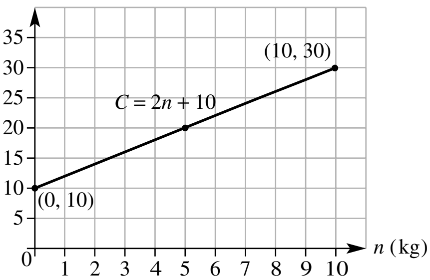


l



9 a $C = 2n + 10$

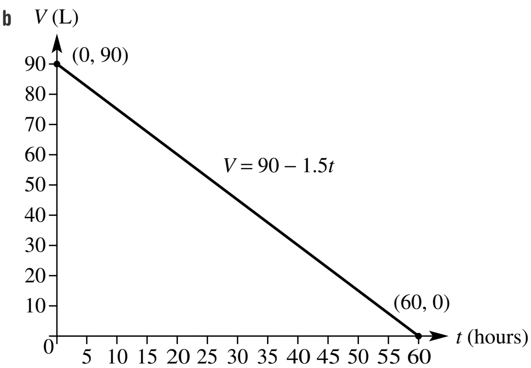
b



c i \$28 **ii** 23.5 kg

10 a $V = 90 - 1.5t$

b



c i 82.5 L **ii** 60 hours

11 a \$7 per hour **b** $P = 7t$

12 a \$0.05/km **b** $C = 0.05k$

c $C = 1200 + 0.05k$

13 a $m = 25$

b The cyclist started 30 km from home.

c (0, 30)

14 a $y = x + \frac{1}{2}$, gradient $= 1$

b $y = 0.5x + 1.5$, y-intercept $= 1.5$

c $y = -3x + 7$, gradient $= -3$

d $y = \frac{1}{2}x - 2$, gradient $= \frac{1}{2}$

15 a gradient $= \frac{3}{a}$, y-intercept $= \frac{7}{a}$

b gradient $= a$, y-intercept $= -b$

c gradient $= -\frac{a}{b}$, y-intercept $= \frac{3}{b}$

16 a $\frac{d}{a}$ **b** $\frac{d}{b}$ **c** $-\frac{a}{b}$

17 a 12 sq. units **b** 9 sq. units **c** $\frac{121}{4}$ sq. units

d $\frac{121}{5}$ sq. units **e** $\frac{32}{3}$ sq. units

Exercise 5F

1 a $y = 4x - 10$ **b** $y = -x + 3$

c $y = -x - 7$ **d** $y = \frac{1}{2}x + \frac{11}{2}$

2 a 2 **b** 3 **c** 0

d -4 **e** -3 **f** undefined

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3 **a** $\frac{1}{4}$ **b** 2 **c** $\frac{5}{2}$

d 3 **e** 0 **f** 0

g −1 **h** $\frac{5}{2}$ **i** −1

j undefined **k** $\frac{3}{2}$ **l** $-\frac{3}{2}$

4 **a** $y = x + 3$ **b** $y = x - 2$ **c** $y = 3x + 6$

d $y = -3x + 4$ **e** $y = 4$ **f** $y = -7x - 10$

5 **a** $y = 2x + 4$ **b** $y = 4x - 5$ **c** $y = x - 4$

d $y = -2x + 12$ **e** $y = -3x - 4$ **f** $y = -3x - 2$

6 **a** $y = 3x + 5$ **b** $y = -2x + 4$

c $y = \frac{1}{2}x - \frac{3}{2}$ **d** $y = -2x - 2$

7 **a** $m_{AB} = \frac{2}{3}$

b $m_{BC} = \frac{2}{3}$

c Yes; they are collinear.

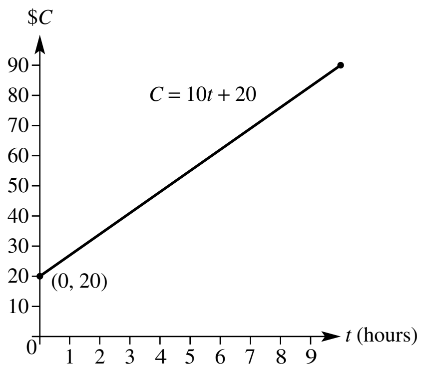
8 **a** $A = 500t + 15000$

b \$15000

c 4 years more; i.e. 10 years from investment

d \$21250

9 **a**



b $C = 10t + 20$

c **i** \$10 per hour **ii** \$20 up-front fee

10 **a** **i** $V = 4t$ **ii** $V = 3t$

iii $V = t + 1$ **iv** $V = 1.5t + 2$

b 1 L, 2 L

c Initially, the container has b litres and it is losing 1 litre per minute.

11 **a** $m = \frac{-5}{5} = -1$ **b** $m = \frac{5}{-5} = -1$

c It doesn't matter which pair of points is (x_1, y_1) and which is (x_2, y_2).

d $x_2 - x_1$ and $x_1 - x_2$ have the same value but opposite sign. Likewise for $y_2 - y_1$ and $y_1 - y_2$. If $\frac{y_2 - y_1}{x_2 - x_1}$ is changed to $\frac{y_1 - y_2}{x_1 - x_2}$, both the numerator and denominator will change sign; e.g. $\frac{-3}{4}$ becomes $\frac{4}{-3}$. These are equal to each other.

12 **a** $-\frac{4}{3}$ **b** $y = -\frac{4x}{3} + \frac{13}{3}$ **c** $y = -\frac{4x}{3} + \frac{13}{3}$

d The results from parts **b** and **c** are the same (when simplified). So it doesn't matter which point on the line is used in the formula $y - y_1 = m(x - x_1)$.

13 **a** **i** $\frac{1}{50} = 0.02$ **ii** $\frac{2}{50} = 0.04$

b **i** $y = 0.02x + 1.5$ **ii** $y = 0.04x + 1.5$

c The archer needs m to be between 0.02 and 0.04 to hit the target.

Exercise 5G

1 **a** 4 **b** 5 **c** $\sqrt{41}$ **d** $\left(3, \frac{9}{2}\right)$

2 **a** 4 **b** 4 **c** $\sqrt{32} = 4\sqrt{2}$ **d** $(0, -3)$

3 **a** $d = \sqrt{20} = 2\sqrt{5}, M = (2, 5)$

b $d = \sqrt{97}, M = (2, 3.5)$

c $d = \sqrt{41}, M = (-1, 1.5)$

d $d = \sqrt{37}, M = (-1, -1.5)$

4 **a** $\sqrt{29}$ **b** $\sqrt{58}$ **c** $\sqrt{37}$

d $\sqrt{65}$ **e** $\sqrt{37}$ **f** 15

g $\sqrt{101}$ **h** $\sqrt{193}$ **i** $\sqrt{37}$

5 **a** (1, 6.5) **b** (1.5, 2.5) **c** (−0.5, 1)

d (−1, 4.5) **e** (1, −1.5) **f** (−3.5, 3)

g (−3, −0.5) **h** (2, 2.5) **i** (−7, 10.5)

6 **a** $\sqrt{68} = 2\sqrt{17}$ **b** (5, 3) **c** −4

d $y = -4x + 23$

7 B and C are both 5 units away from (2, 3).

8 **a** $a = 3, b = 5$ **b** $a = -4, b = 5$

c $a = -2, b = 2$ **d** $a = 11, b = 2$

9 **a** 3, 7 **b** −1, 3 **c** −1, 9 **d** −6, 0

10 **a** 1478 m **b** 739 m

11 **a** (−0.5, 1)

b (−0.5, 1)

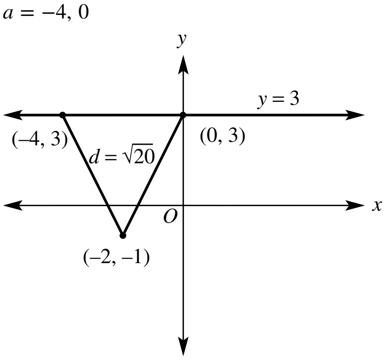
c These are the same. The order of the points doesn't matter $(x_1 + x_2 = x_2 + x_1)$.

d 5

e 5

f The order of the points doesn't matter $(x - y)^2 = (y - x)^2$.

12 $a = -4, 0$



13 **a** $\left(\frac{1}{2}, 2\right)$ **b** $\left(-\frac{1}{3}, \frac{4}{3}\right)$ **c** $\left(\frac{4}{3}, \frac{8}{3}\right)$

d $\left(2, \frac{16}{5}\right)$ **e** $\left(-\frac{3}{4}, 1\right)$ **f** $\left(0, \frac{8}{5}\right)$

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14 a $C(x_1, 0)$ **b** $D(x_2, 0)$
c $x_2 - x_1$ **d** yes
e $F(0, y_1)$ **f** $G(0, y_2)$
g $y_2 - y_1$ **h** yes
i $AB^2 = AE^2 + BE^2$
$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
j In the diagram, $x_2 - x_1$ will be positive. $x_1 - x_2$ will have equal value but opposite sign; e.g. if $x_2 - x_1 = 3$, then $x_1 - x_2 = -3$. When they are squared they have equal value and equal sign.

15 a $\sqrt{(x - 7)^2 + y^2}$
b $\sqrt{(x - 7)^2 + (x + 3)^2}$
c i 721 m **ii** 707 m **iii** 721 m **iv** 762 m
d $x = 2$
e The distance will be a minimum when the dotted line joining Sarah to the fence is perpendicular to the fence (i.e. when it has gradient -1). The closest point is $(2, 5)$.

Exercise 5H

1 a 4 **b** -7 **c** $-\frac{3}{4}$ **d** $\frac{8}{7}$

2 a $-\frac{1}{3}$ **b** $\frac{1}{2}$ **c** $-\frac{8}{7}$ **d** $\frac{9}{4}$

3 a 5 **b** 4 **c** $y = 5x + 4$

4 a parallel **b** parallel **c** neither **d** neither
e perpendicular **f** perpendicular
g parallel **h** parallel
i perpendicular **j** perpendicular

5 a $y = x + 4$ **b** $y = -x - 6$ **c** $y = -4x - 1$
d $y = \frac{2}{3}x - 6$ **e** $y = -\frac{4}{5}x + 7$ **f** $y = -\frac{1}{2}x + 6$
g $y = \frac{1}{4}x - 2$ **h** $y = -\frac{3}{2}x + 5$ **i** $y = -\frac{3}{4}x - 5$
j $y = \frac{7}{2}x + 31$

6 a $x = 6$ **b** $x = 0$ **c** $y = 11$ **d** $y = 8.4$
e $y = 3$ **f** $y = -3$ **g** $x = \frac{2}{3}$ **h** $x = -\frac{4}{11}$

7 a $y = \frac{2}{3}x + 5$ **b** $y = -\frac{5}{7}x + \frac{54}{7}$
c $y = \frac{2}{3}x + \frac{16}{3}$ **d** $y = 7x + 20$

8 a $y = -\frac{3}{2}x + 5$ **b** $y = \frac{7}{5}x + \frac{28}{5}$
c $y = -\frac{3}{2}x + 1$ **d** $y = -\frac{1}{7}x - \frac{10}{7}$

9 The second line has equation $y = -\frac{2}{3}x - \frac{5}{3}$. It cuts the x-axis at $x = -\frac{5}{2}$.

10 a 14 **b** -2 **c** 5 **d** $\frac{9}{7}$

11 a m **b** $-\frac{a}{b}$ **c** $-\frac{1}{m}$ **d** $\frac{b}{a}$

12 a $y = 2x + d - 2c$ **b** $y = mx + d - mc$
c $y = x + d - c$ **d** $y = -\frac{1}{m}x + d + \frac{c}{m}$

13 a i 1 **ii** -3 **iii** 1 **iv** $-\frac{1}{3}$
b AB is parallel to CD, BD is parallel to AC; i.e. opposite sides are parallel.
c parallelogram

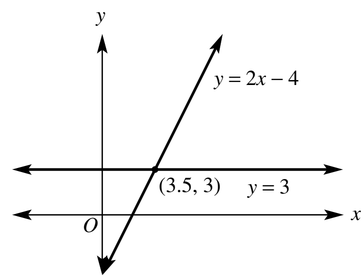
14 a i $\frac{4}{3}$ **ii** $-\frac{3}{4}$ **iii** 0
b Right-angled triangle (AB is perpendicular to BC).
c 20

15 $y = -\frac{1}{2}x + 4$, x-intercept $= 8$

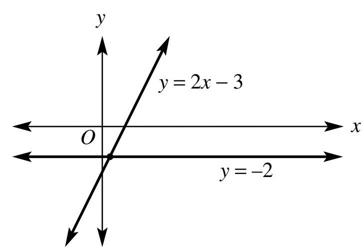
Exercise 5I

1 a yes **b** yes **c** no **d** no
e no **f** no **g** yes **h** no

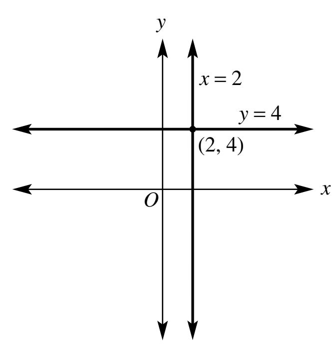
2 a $\left(\frac{7}{2}, 3\right)$



b $\left(\frac{1}{2}, -2\right)$



c $(2, 4)$



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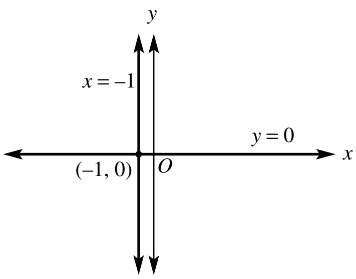
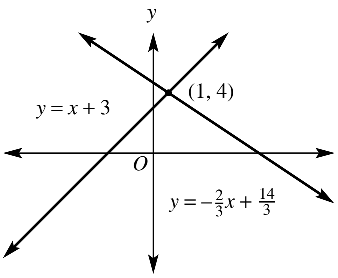
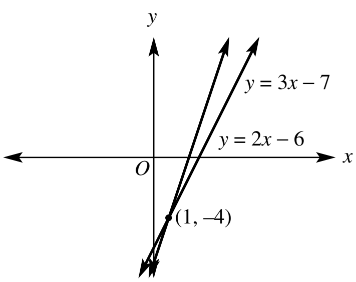
d $(-1, 0)$



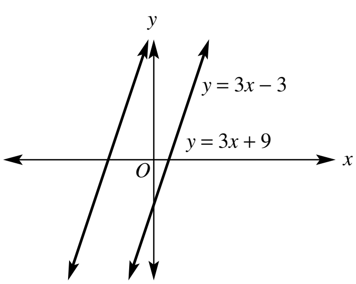
e $(1, 4)$



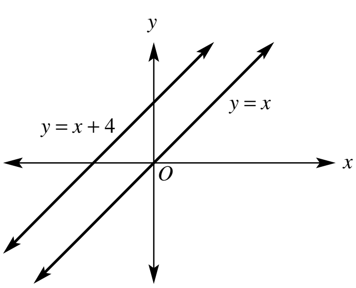
f $(1, -4)$



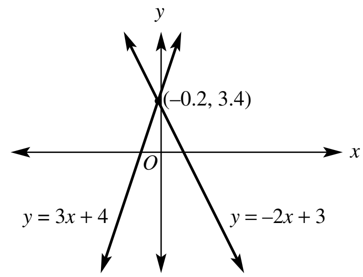
g No intersection (lines are parallel).



h No intersection (lines are parallel).



i $\left(-\frac{1}{5}, \frac{17}{5}\right)$



3 a i Joe's: \$60, Paul's: \$150
ii Joe's: \$0.20 per km, Paul's: \$0.10 per km
iii Joe's: $C = 0.2k + 60$, Paul's: $C = 0.1k + 150$
iv 900 km
b Joe's Car Rental **c** Paul's Motor Mart

4 a $(2, 7)$ **b** $(2, 5)$ **c** $(3, 1)$ **d** $(2, 1)$
e $(1, 1)$ **f** $(1, 1)$ **g** $(5, 1)$ **h** $(10, 4)$
i $(1, 2)$ **j** $(9, 2)$

5 a $(2, 10)$ **b** $(1, -5)$ **c** $(-3, 3)$ **d** $(13, -2)$
e $(3, 1)$ **f** $(2, 1)$ **g** $(1, 4)$ **h** $(1, 3)$

6 a i $E = 20t$ **ii** $E = 15t + 45$
b $t = 9, E = 180$ **c i** 9 hours **ii** \$180

7 a i $V = 62\,000 - 5000t$ **ii** $V = 40\,000 - 3000t$
b $t = 11, V = 7000$ **c i** 11 years **ii** \$7000

8 18 years

9 $197\,600\ \text{m}^2$

10 a no **b** no **c** yes **d** yes
e no **f** yes **g** yes **h** no

11 a -4 **b** $\frac{3}{2}$ **c** 12

12 a $\left(\frac{k}{3}, \frac{2k}{3}\right)$ **b** $\left(\frac{k}{2}, -\frac{k}{2}\right)$
c $(-1 - k, -2 - k)$ **d** $\left(\frac{-2k-1}{3}, \frac{-2k-4}{3}\right)$

13 a $x = \frac{b}{a-b}, y = \frac{b^2}{a-b}$ **b** $x = \frac{-b}{a+b}, y = \frac{a}{a+b}$
c $x = \frac{a}{1+b}, y = \frac{-a}{1+b}$ **d** $x = \frac{b}{b-a}, y = \frac{b^2}{b-a}$
e $x = \frac{1}{a-2b}, y = \frac{(a-b)}{a-2b}$ **f** $x = \frac{c(1-b)}{a(b+1)}, y = \frac{2c}{b+1}$
g $x = \frac{ab}{a^2+b}, y = \frac{a^2 b}{a^2+b}$

Exercise 5J

1 a $4x = 16$ **b** $2y = 6$

2 a subtract **b** subtract **c** add **d** add
e add **f** add **g** subtract **h** subtract

3 a $4x - 6y = 8$ **b** $6x - 9y = 12$
c $8x - 12y = 16$ **d** $20x - 30y = 40$

4 a $(2, 5)$ **b** $(2, 3)$ **c** $(4, 2)$ **d** $(2, 2)$
e $(1, 1)$ **f** $(2, 1)$ **g** $(2, -1)$ **h** $(2, 2)$
i $(1, 2)$ **j** $(2, 1)$ **k** $(2, 1)$ **l** $(-1, 2)$

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5 **a** $(1, 1)$ **b** $(4, 2)$ **c** $(2, 1)$ **d** $(4, -3)$

e $\left(\frac{1}{2}, 1\right)$ **f** $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

6 **a** $(4, -3)$ **b** $(1, 1)$ **c** $(3, 4)$

d $(2, 2)$ **e** $\left(\frac{1}{2}, -1\right)$ **f** $\left(-3, \frac{1}{3}\right)$

7 799 and 834

8 \$0.60

9 $A = \$15$, $C = \$11$

10 Should have been $(1) - (2)$ to eliminate y: $-2y - (-2y) = 0$. The correct solution is $(1, -1)$.

11 **a** $\left(\frac{1}{a}, -1\right)$ **b** $\left(\frac{13}{3}, \frac{1}{3b}\right)$ **c** $\left(-\frac{2}{a}, \frac{2}{b}\right)$

d $\left(\frac{a+b}{2a}, \frac{a-b}{2b}\right)$ **e** $\left(\frac{c}{a+b}, \frac{c}{a+b}\right)$

12 The two lines are parallel; they have the same gradient.

13 **a** $\frac{2}{x-1} - \frac{2}{x+1}$ **b** $\frac{2}{2x-3} - \frac{1}{x+2}$

c $\frac{3}{3x+1} - \frac{2}{2x-1}$ **d** $\frac{3}{3x-1} + \frac{2}{x+2}$

e $\frac{1}{x+3} + \frac{1}{x-4}$ **f** $\frac{1}{7(2x-1)} - \frac{3}{7(4-x)}$

Exercise 5K

1 **a** $x + y = 16$, $x - y = 2$; 7 and 9

b $x + y = 30$, $x - y = 10$; 10 and 20

c $x + y = 7$, $2x + y = 12$; 5 and 2

d $2x + 3y = 11$, $4x - 3y = 13$; 4 and 1

2 7 cm × 21 cm

3 Nikki is 16, Travis is 8.

4 Cam is 33, Lara is 30.

5 Bolts cost \$0.10, washers cost \$0.30.

6 There were 2500 adults and 2500 children.

7 Thickshakes cost \$5, juices cost \$3.

8 There are 36 ducks and 6 sheep.

9 \$6.15 (mangoes cost \$1.10, apples cost \$0.65)

10 43

11 70

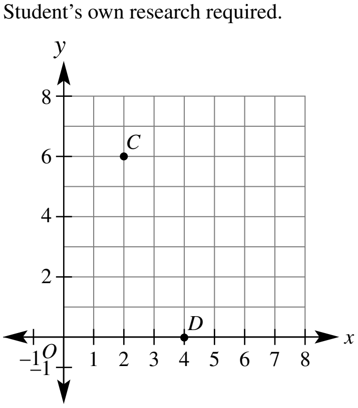
12 1 hour and 40 minutes

13 $\frac{1}{7}$ of an hour

14 200 m

15 **a** Student's own research required.

b



c $A(2, 0)$ **d** $x = 2$

e $E(3, 3)$ **f** $y = x$

g $B(2, 2)$ **h** $AB = 2$, $BC = 4$

i $1:2$ **j** yes

Exercise 5L

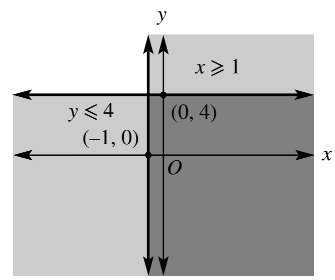
1 **a** no **b** yes **c** yes

d no **e** no **f** no

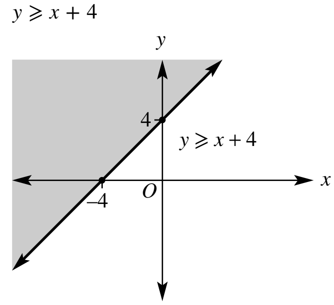
g no **h** yes **i** yes

2 **a** B **b** C **c** A

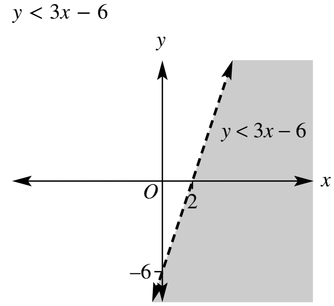
3 **a–d** $x \geqslant -1$, $y \leqslant 4$



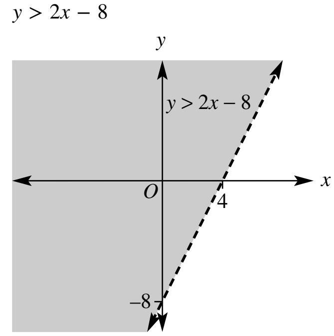
4 **a** $y \geqslant x + 4$



b $y < 3x - 6$



c $y > 2x - 8$



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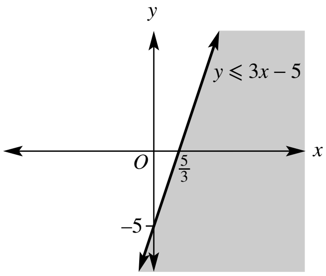
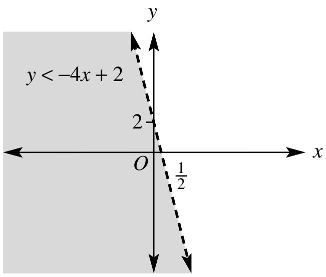
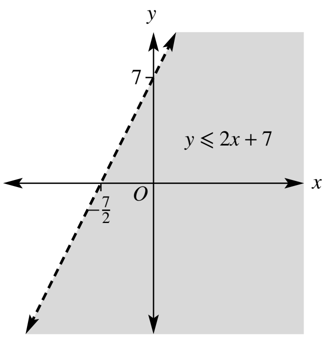
d $y \leqslant 3x - 5$



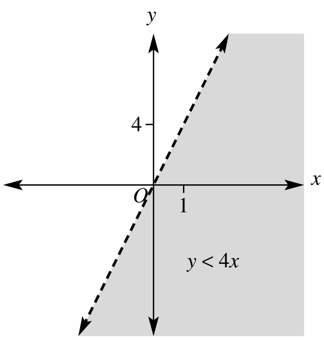
e $y < -4x + 2$



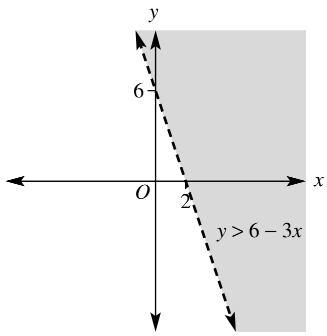
f $y \leqslant 2x + 7$



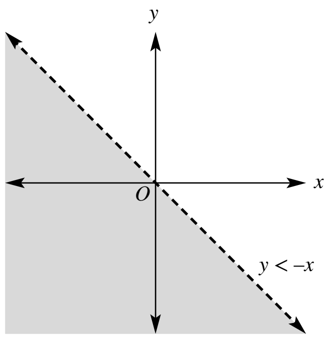
g $y < 4x$



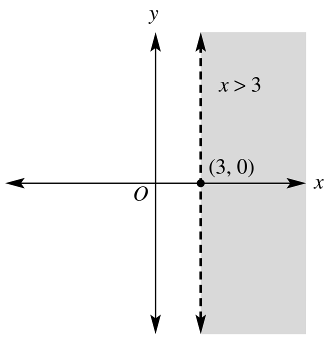
h $y > -3x + 6$



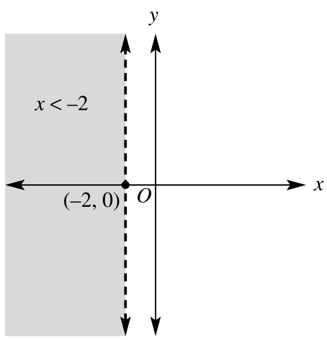
i $y < -x$



j $x > 3$



k $x < -2$



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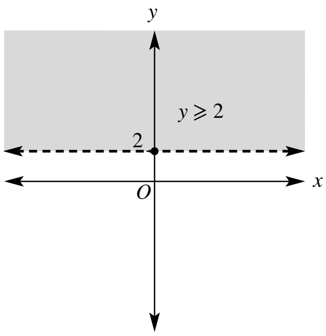
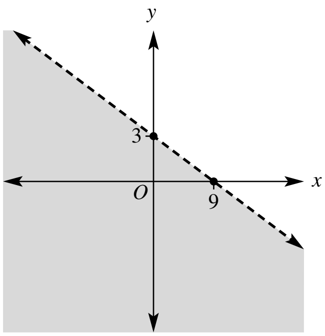
l $y \geqslant 2$



5 a



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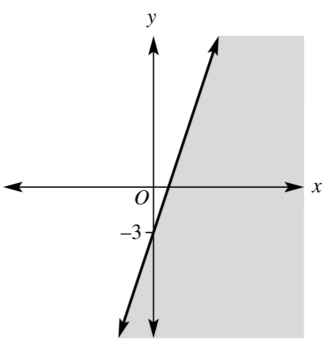

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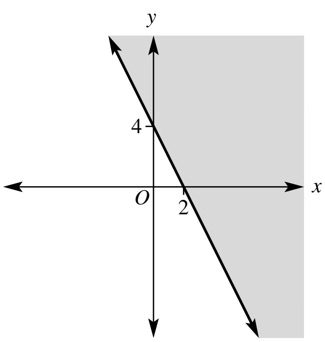

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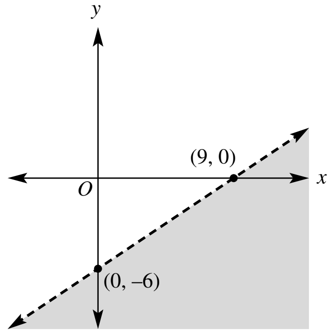

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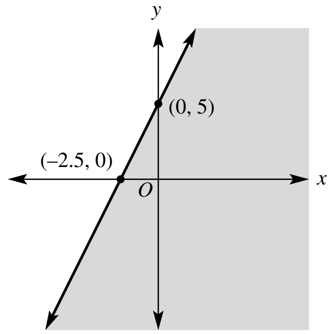

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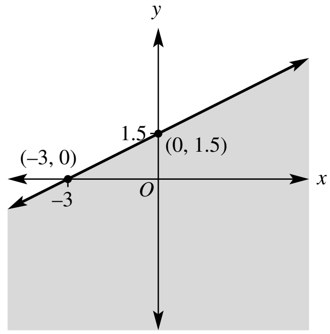

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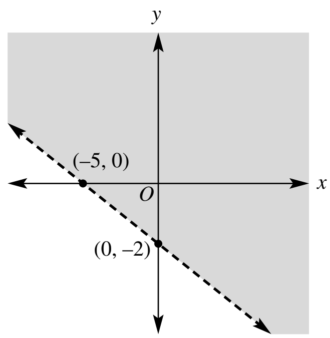

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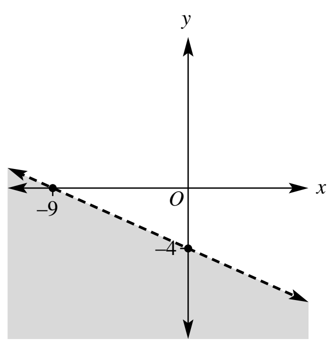

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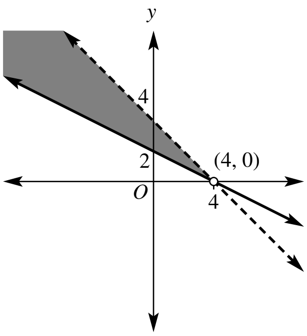
6 **a** yes **b** no **c** no **d** yes

7 **a** no **b** yes **c** no **d** no

8 **a** $y \leqslant x + 3$ **b** $y \geqslant -2x + 2$

c $y < -\frac{3}{2}x - 3$ **d** $y > \frac{2}{5}x - 2$

9 **a**



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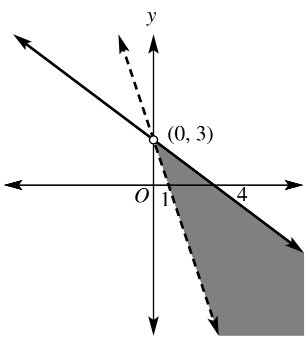

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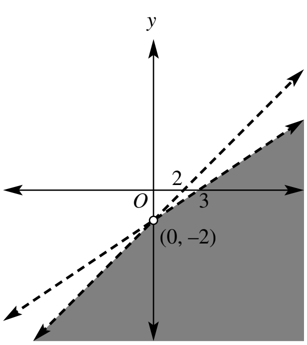

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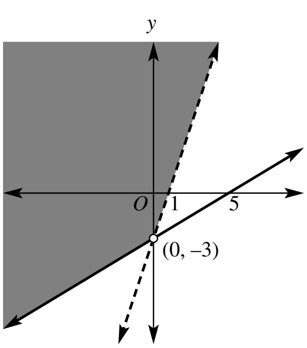

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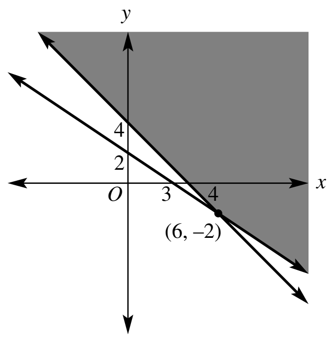

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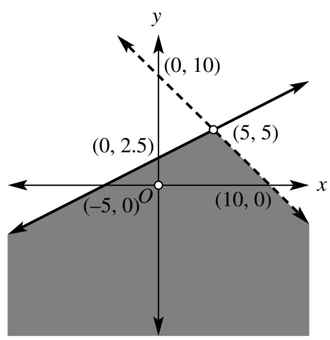

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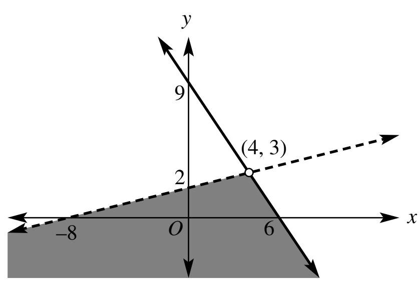

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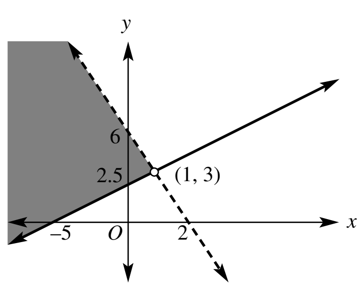
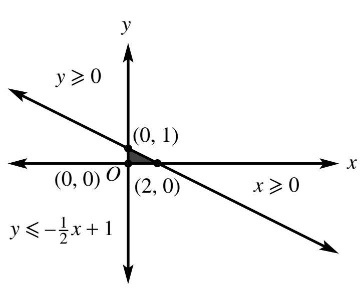


10 **a**



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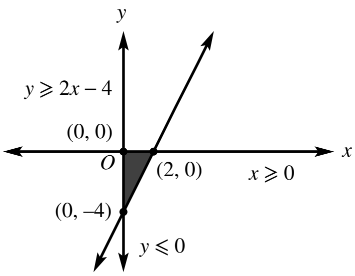

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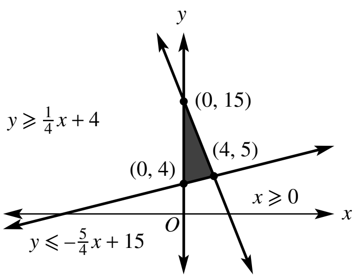

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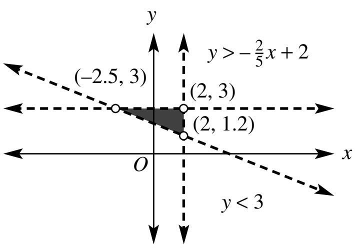

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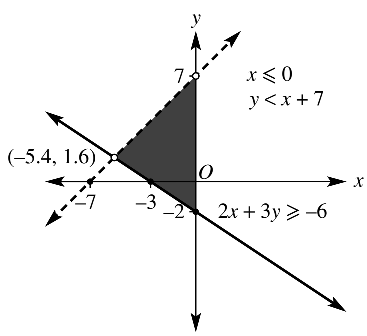

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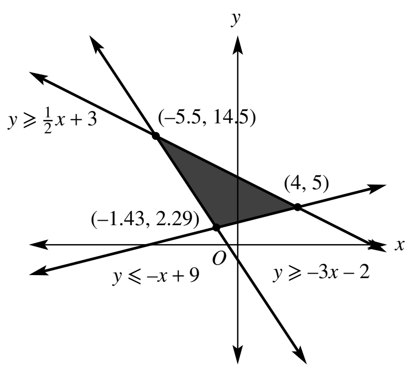

11 a $y \geqslant 0$, $y < 2x + 4$, $y \leqslant -x + 7$

b $y > -\frac{1}{2}x + 6$, $y \leqslant x + 3$, $x < 8$

12 a 1 **b** 4 **c** 22 **d** $\frac{81}{20}$

13 a i $\frac{115}{6}$ **ii** $\frac{578}{15}$

b Answers may vary; e.g. $x > 0$, $x < 3$, $y > 0$, $y < 2$.

Puzzles and challenges

1 0.75 km

2 $\frac{6}{8}$

3 a The gradient from (2, 12) to (−2, 0) = the gradient from (−2, 0) to (−5, −9) = 3.

b The gradient from $(a, 2b)$ to $(2a, b)$ = the gradient from $(2a, b)$ to $(-a, 4b) = -\frac{b}{a}$.

4 The gradient of AC is $\frac{3}{5}$ and the gradient of AB is $-\frac{5}{3}$.
So $\triangle ABC$ is a right-angled triangle, as AC is perpendicular to AB.
Can also show that side lengths satisfy Pythagoras' theorem.

5 The missiles are travelling at $\frac{4840}{9}$ km/h and $\frac{9680}{9}$ km/h.

6 The distance between the two points and (2, 5) is 5 units.

7 The diagonals have equations $x = 0$ and $y = 3$. These lines are perpendicular and intersect at the midpoint (0, 3) of the diagonals. It is not a square, since the angles at the corners are not 90°. In particular, AB is not perpendicular to $BC\left(m_{AB} \neq m_{BC}\right)$.

8 $x = 2$, $y = -3$, $z = -1$

9 24 square units

10 24 players, 15 years old

Multiple-choice questions

1 E	**2** D	**3** B	**4** C
5 D	**6** B	**7** C	**8** A
9 C	**10** D	**11** E	**12** B
13 A	**14** A	**15** D	**16** C

Short-answer questions

1 a $5xy + 6x$ **b** $12a^2b$ **c** $\frac{3}{2}x$

d $3b + 21$ **e** $-2m^2 + 12m$ **f** $x + 2$

2 a $\frac{6 - 7a}{14}$ **b** $\frac{5a + 18}{6a}$

c $\frac{7x + 26}{30}$ **d** $\frac{11 - x}{(x + 1)(x - 3)}$

3 a $3x - 1$ **b** $\frac{2}{x + 2}$ **c** $\frac{3}{4}$

4 a $x = -3$ **b** $x = -\frac{3}{4}$ **c** $x = \frac{1}{5}$ **d** $x = 2$

5 a $x < 1$ **b** $x \geqslant -4$ **c** $-1 < x \leqslant 3$

6 a $x > 5$ **b** $x \geqslant 10$ **c** $x > -3$ **d** $x \leqslant \frac{2}{7}$

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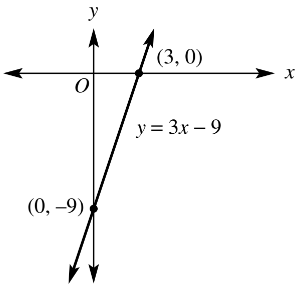
7 a $V = 2 - 0.4t$ **b** 1.4 L
c 5 min **d** $t \leqslant 3.5$ min

8 a



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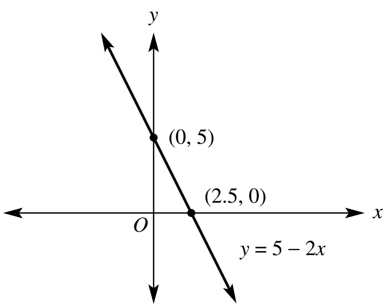

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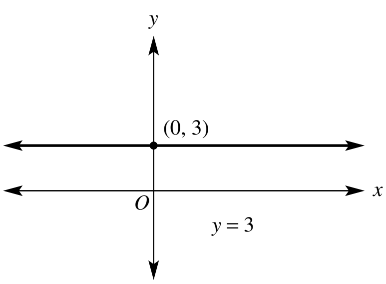

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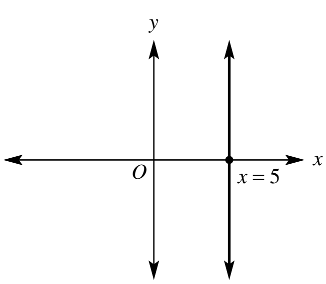

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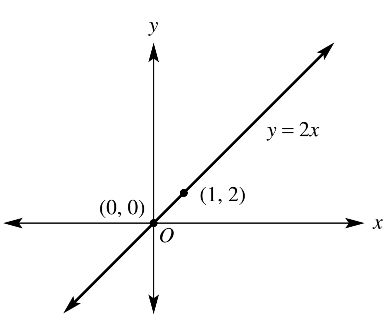

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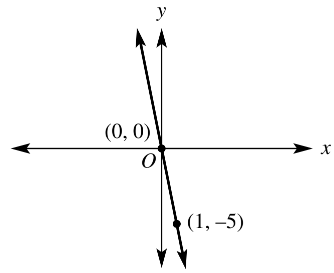

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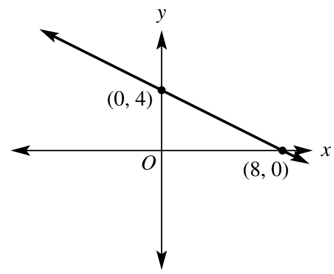

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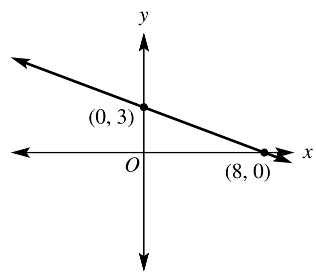


9 a $y = \frac{1}{2}x + 3$ **b** $y = -\frac{3}{2}x + \frac{15}{2}$ **c** $y = 2x - 3$

10 a $m = -\frac{3}{5}$ **b** $y = -\frac{3}{5}x + \frac{34}{5}$

11 a $M = (4, 8)$, $d = \sqrt{52} = 2\sqrt{13}$
b $M = \left(\frac{11}{2}, 1\right)$, $d = \sqrt{61}$
c $M = \left(\frac{1}{2}, -\frac{5}{2}\right)$, $d = \sqrt{18} = 3\sqrt{2}$

12 a $y = 3x - 2$ **b** $y = -1$
c $y = -\frac{1}{2}x + 5$ **d** $y = 3x - 1$

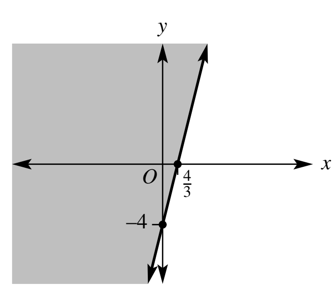
13 a $a = 7$ **b** $b = -8$ **c** $c = 0$ or 4

14 a $(-3, -1)$ **b** $(-8, -21)$

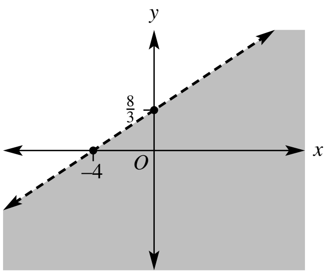
15 a $(-3, -1)$ **b** $(0, 2)$

16 A regular popcorn costs \$4 and a small drink costs \$2.50.

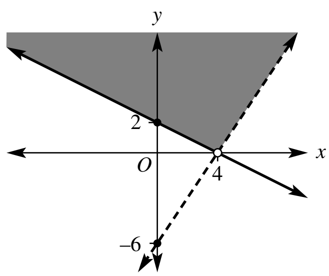
17 a



b



18 The point of intersection is (4, 0).



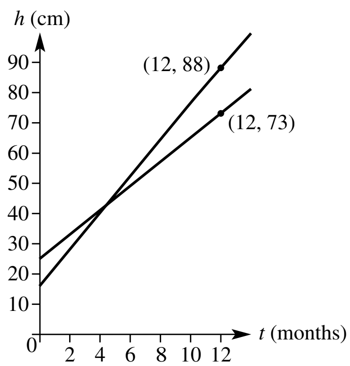
Extended-response questions

1 a i $h = 4t + 25$ **ii** $h = 6t + 16$

b 16 cm

c Shrub B is growing at a faster rate because its gradient is greater.

d



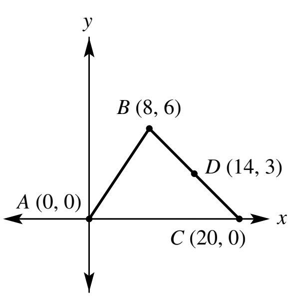
e after 4.5 months

f i 1.24 m

ii 26.25 months

iii between 8.75 and 11.25 months

2 a $A(0, 0)$, $B(8, 6)$, $C(20, 0)$



b 43.4 km

c The drink station is at (14, 3).

d i $y = \frac{3}{4}x$ **ii** $y = -\frac{1}{2}x + 10$ **iii** $y = 0$

e $y > 0$, $y < \frac{3}{4}x$, $y < -\frac{1}{2}x + 10$

f $y = -\frac{4}{3}x + \frac{80}{3}$

Semester review 1

Chapter 1: Measurement

Multiple-choice questions

1 B **2** D **3** B **4** A **5** D

Short-answer questions

1 a 23000 mm **b** 8000 ms **c** 7.8×10^9 ns
d 0.008 Mt **e** 2.3×10^6 TB

2 a 6.5 s to 7.5 s **b** 8.985 g to 8.995 g
c 699.5 km to 700.5 km **d** 695 km to 705 km

3 a 6.75 m and 6.85 m **b** 4.25 m to 4.35 m
c $22 \text{ m} \leqslant P \leqslant 22.4$ m and $28.6875 \text{ m}^2 \leqslant A \leqslant 29.7975 \text{ m}^2$

4 a 36 cm, 52 cm^2 **b** 1.3 m, 0.1 m^2
c 220 mm, 2100 mm^2

5 a 188.5 m^2, 197.9 m^3 **b** 50.3 cm^2, 23.7 cm^3
c 6.8 m^2, 1.3 m^3

6 a 1.8 cm **b** 58.8 cm^2

7 $\sqrt{\frac{27}{\pi}}$ cm

Extended-response question

1 a 753.98 cm^3 **b** 206.02 cm^3
c 17 cm **d** 1.79 cm

Chapter 2: Indices and surds

Multiple-choice questions

1 B **2** D **3** E **4** E **5** C

Short-answer questions

1 a $3\sqrt{6}$ **b** $20\sqrt{3}$ **c** $3\sqrt{6}$ **d** $\sqrt{10}$ **e** 21
f $48\sqrt{3}$ **g** $\sqrt{3}$ **h** $\frac{\sqrt{5}}{3}$ **i** $\frac{10\sqrt{2}}{7}$

2 a $7\sqrt{5} - \sqrt{7}$ **b** 0 **c** $-\sqrt{2} - 4$

3 a $2\sqrt{15} - 4\sqrt{3}$ **b** $11\sqrt{5} - 62$ **c** 4
d $59 + 24\sqrt{6}$

4 a $\frac{3\sqrt{2}}{2}$ **b** $\frac{\sqrt{2}}{5}$ **c** $\frac{2\sqrt{5} - 5}{5}$

5 a $24x^{10}y^2$ **b** $3a^2b^2$ **c** $\frac{3b^2}{a^5}$ **d** $\frac{2x^2}{5y^3}$

6 a i 37200 **ii** 0.0000049
b i 7.30×10^{-5} **ii** 4.73×10^9

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7 **a** **i** $10^{\frac{1}{2}}$ **ii** $7^{\frac{1}{2}}x^3$ **iii** $4x^{\frac{3}{5}}$ **iv** $15^{\frac{3}{2}}$

b **i** $\sqrt{6}$ **ii** $\sqrt[5]{20}$ **iii** $\sqrt[4]{7^3}$ or $(\sqrt[4]{7})^3$

8 **a** $\frac{1}{5}$ **b** $\frac{1}{16}$ **c** 3 **d** $\frac{1}{2}$

9 **a** $x = 3$ **b** $x = 2$ **c** $x = \frac{3}{2}$ **d** $-\frac{1}{2}$

Extended-response question

1 **a** $V = 80000(1.08)^n$

b **i** \$86400 **ii** \$108839

c 11.91 years **d** 6% per year

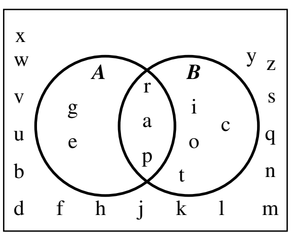
Chapter 3: Probability

Multiple-choice questions

1 C **2** E **3** B **4** D **5** C

Short-answer questions

1 **a**



b **i** $\frac{5}{26}$ **ii** $\frac{3}{26}$ **iii** $\frac{9}{26}$ **iv** $\frac{19}{26}$

c No, $A \cap B \neq \varnothing$

2 **a**

	B	$\overline{B}$	
A	3	1	4
$\overline{A}$	4	4	8
	7	5	12

b **i** 3 **ii** 4 **iii** 5 **iv** 8

c **i** $\frac{1}{4}$ **ii** $\frac{1}{12}$ **iii** $\frac{7}{12}$ **iv** $\frac{3}{4}$

3 **a** 0.18 **b** 0.37

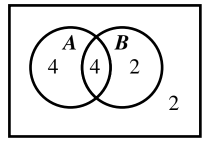
4 **a**

	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

b 16

c **i** $\frac{3}{16}$ **ii** $\frac{5}{8}$ **iii** $\frac{1}{5}$

5 **a**

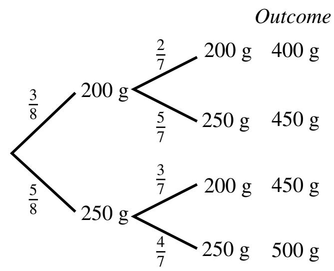


b **i** $\frac{2}{3}$ **ii** $\frac{2}{3}$

c Yes they are because $P(A|B) = P(A)$.

Extended-response question

1



a **i** $\frac{3}{28}$ **ii** $\frac{15}{28}$ **iii** $\frac{5}{14}$

b $\frac{9}{14}$ **c** $\frac{3}{5}$

Chapter 4: Single variable and bivariate statistics

Multiple-choice questions

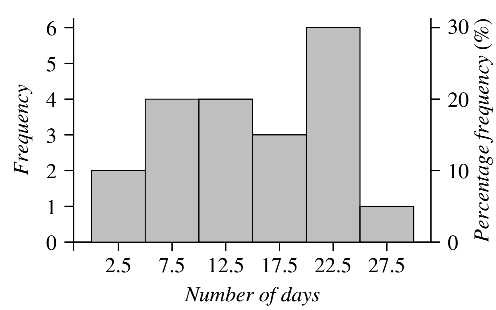
1 E **2** B **3** C **4** B **5** A

Short-answer questions

1 **a**

Class interval	Frequency	Percentage frequency
0–4	2	10%
5–9	4	20%
10–14	4	20%
15–19	3	15%
20–24	6	30%
25–30	1	5%
Total	**20**	**100%**

b



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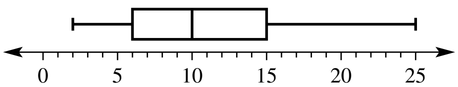
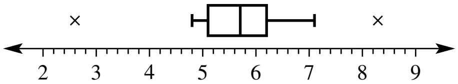
c **i** 14 **ii** 50%

iii 20–24 days, those that maybe catch public transport to work or school each week day.

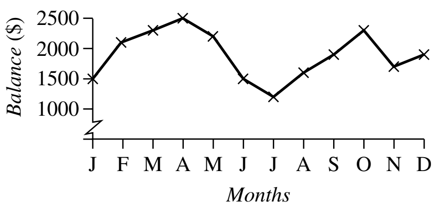
2 **a**



b



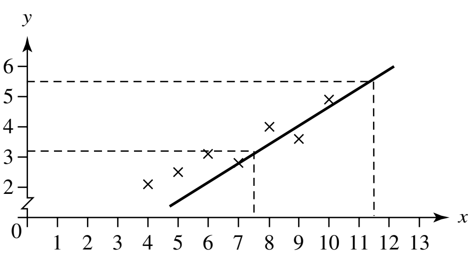
3 **a**



b Balance fluctuated throughout the year but ended up with more money after 12 months.

c May and June **d** increase of $500

4 **a, c**



b positive

d **i** ≈ 3.2 **ii** ≈ 11.5

5 **a** **i** under 40 years **ii** over 40 years

b Over 40 years: mean = 11, standard deviation (σ) = 7.0

Under 40 years: mean = 24.1, standard deviation (σ) = 12.2

Extended-response question

1 **a** $y = 1.50487x + 17.2287$

b 41 cm

Chapter 5: Expressions, equations and linear relationships

Multiple-choice questions

1 A **2** C **3** D **4** E **5** C

Short-answer questions

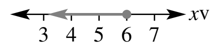
1 **a** $3 - 2x$ **b** 20

c $\dfrac{3a-8}{4a}$ **d** $\dfrac{9x-2}{(x+2)(x-3)}$

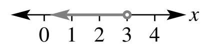
2 **a** **i** $x = -4$ **ii** $x = 2$

iii $x = 13$ **iv** $x = 2$

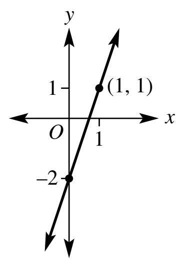
b **i** $x \leqslant 6$,

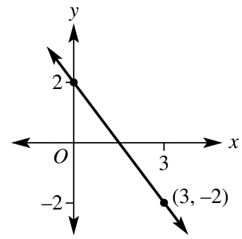


ii $x < 3$,

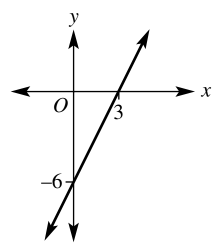


3 **a** **i** $m = 3, b = -2$ **ii** $m = -\frac{4}{3}, b = 2$

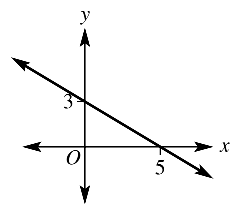




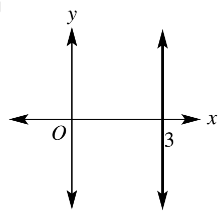
b **i**



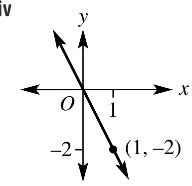
ii



iii



iv



4 **a** $y = -x + 3$ **b** $y = \frac{8}{5}x - \frac{9}{5}$

5 **a** $a = -3$ **b** $a = -4$

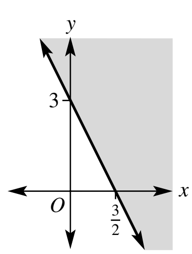
c $a = 1$ or $a = 7$ **d** $a = -4$

6 **a** $x = -3, y = -7$ **b** $x = -2, y = -4$

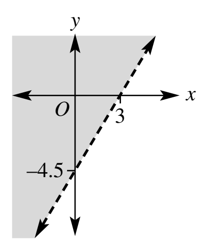
c $x = -1, y = 4$ **d** $x = 3, y = -5$

7 A hot dog costs $3.50 and a can of soft drink costs $2.

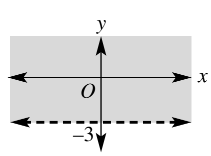
8 **a**



b



c



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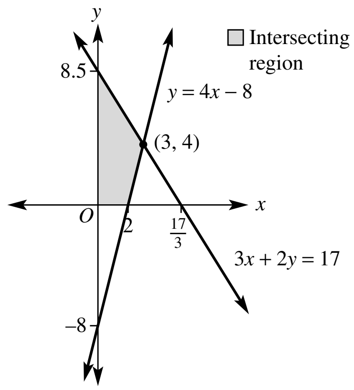
Extended-response question

1 a $x = 3, y = 4$

b, c



d 167 500 m²

Chapter 6

Pre-test

1 a All side lengths and angles are equal.
b Two side lengths and two angles are equal.
c All side lengths and angles are different.
d One angle of the triangle is 90°.
e Opposite pairs of sides are parallel and of equal length.
f Opposite pairs of sides are parallel and of equal length and all angles are 90°.
g All sides are of equal length and all angles are 90°.
h One pair of opposite sides are parallel.
i All sides are of equal length, opposite pairs of sides are parallel.
j Two pairs of adjacent sides with the same length.

2 a b, j b f, i

3 a 60° b 55° c 190° d 40° e 40°
f 60° g 90° h 130° i 70°

4 a supplementary b complementary
c neither d supplementary

5 $a = 120°, b = 60°, c = 120°, d = 60°$

6 triangle, quadrilateral, pentagon, hexagon, heptagon, octagon, nonagon, decagon

7 a 105 b 69 c $\frac{8}{3}$
d 8.8 e $\frac{25}{2}$ f 6

Exercise 6A

1 a triangle, 180°, 360°, 60°
b quadrilateral, 360°, 360°, 90°
c pentagon, 540°, 360°, 108°
d hexagon, 720°, 360°, 120°
e octagon, 1080°, 360°, 135°
f decagon, 1440°, 360°, 144°
g dodecagon, 1800°, 360°, 150°

2 a false b true c true
d true e false f false
g true h false i true

3 a $a = 110$ (angles on a straight line), $b = 70$ (vertically opposite)
b $a = 140$ (angles in a revolution)
c $a = 19$ (angles in a right angle)
d $a = 113$ (cointerior angles on parallel lines), $b = 67$ (alternate angles on parallel lines), $c = 67$ (vertically opposite to b)
e $a = 81$ (equal angles are opposite sides of equal length), $b = 18$ (angle sum of a triangle)
f $a = 17$ (angle sum of a triangle), $b = 102$ (angles on a straight line)
g $a = 106$ (cointerior angles on parallel lines), $b = 74$ (opposite angles in a parallelogram)
h $a = 17$ (angle sum of a triangle), $b = 17$ (angles in a right angle)
i $a = 90$ (vertically opposite), $b = 60$ (angle sum of a triangle)

4 a 72 b 60 c 56

5 a 60 b 60 c 110 d 80 e 10 f 20
g 109 h 28 i 23 j 121 k 71 l 60

6 a 50 (angle sum of a quadrilateral)
b 95 (angle sum of a quadrilateral)
c 125 (angle sum of a pentagon)
d 30 (angle sum of a pentagon)
e 45 (angle sum of a hexagon)
f 15 (angle sum of a quadrilateral)

7 a 108° b 135° c 144°

8 a 95 b 113 c 85
d 106 e 147 f 292

9 a 176.4° b 3.6°

10 a 12 b 20 c 48

11 $x = 36, y = 144$

12 115

13 a Expand the brackets. b $n = \frac{S + 360}{180}$
c $I = \frac{S}{n} = \frac{180(n - 2)}{n}$ d $E = 180 - I = \frac{360}{n}$

14 a $\angle BCA = 180° - a° - b°$ (angles in a triangle)
b $c = 180 - \angle BCA = a + b$ (angles on a straight line)

15 a alternate angles ($BA \parallel CD$)
b $\angle ABC + \angle BCD = 180°$ (cointerior angles on parallel lines), so $a + b + c = 180$.
c Angle sum of a triangle is 180°.

16 $\angle ACB = \angle DCE$ (vertically opposite), so $\angle CAB = \angle CBA = \angle CDE = \angle CED$ (isosceles), since $\angle CAB = \angle CED$ (alternate), $AB \parallel DE$.

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17 Answers may vary.

18 a 15° (alternate angles in parallel lines)

b 315° (angle sum in an octagon)

19 $\angle AMB \equiv \angle ABM = 60°$ ($\triangle AMB$ is equilateral)

$\angle AMB + \angle BMC = 180°$ (angles on a straight line)

$\therefore \angle BMC = 120°$

$\angle MCB \equiv \angle MBC = 30°$ (angle sum of isosceles $\triangle MCB$)

$\therefore \angle ABC = 60° + 30° = 90°$

20 Let $\angle AOB = x$ and $\angle COD = y$. $2x + 2y = 180°$ (angles on a straight line), so $\angle BOD = x + y = 90°$.

Exercise 6B

1 a SAS b SSS c AAS

d SAS e RHS f RHS

2 a 5 b 4 c 3 d 5 e 2 f 2

3 a In $\triangle ABC$ and $\triangle DEF$,

$AB = DE$ (given)

$\angle ABC = \angle DEF$ (given)

$BC = EF$ (given)

$\therefore \triangle ABC \equiv \triangle DEF$ (SAS test)

b In $\triangle FED$ and $\triangle CBA$,

$\angle FED = \angle ABC = 90°$ (given)

$FD = AC$ (given)

$EF = BC$ (given)

$\therefore \triangle FED \equiv \triangle CBA$ (RHS test)

c In $\triangle ABC$ and $\triangle DEF$,

$AC = DF$ (given)

$BC = EF$ (given)

$AB = DE$ (given)

$\therefore \triangle ABC \equiv \triangle DEF$ (SSS test)

d In $\triangle DEF$ and $\triangle ABC$,

$\angle EDF = \angle BAC$ (given)

$\angle DFE = \angle ACB$ (given)

$EF = BC$ (given)

$\therefore \triangle DEF \equiv \triangle ABC$ (AAS test)

4 a $x = 7.3, y = 5.2$ b $x = 12, y = 11$

c $a = 2.6, b = 2.4$ d $x = 16, y = 9$

5 a In $\triangle ADC$ and $\triangle CBA$,

$\angle DAC = \angle ACB$ (alternate angles, $AD \parallel BC$)

$\angle DCA = \angle CAB$ (alternate angles, $AB \parallel DC$)

AC is common

$\therefore \triangle ADC \equiv \triangle CBA$ (AAS test)

b In $\triangle ABD$ and $\triangle CDB$,

$\angle ABD = \angle BDC$ (given)

$\angle ADB = \angle DBC$ (given)

BD is common

$\therefore \triangle ABD \equiv \triangle CDB$ (AAS test)

c In $\triangle ABC$ and $\triangle EDC$,

$\angle BAC = \angle CED$ (alternate angles, $AB \parallel DE$)

$\angle ABC = \angle CDE$ (alternate angles, $AB \parallel DE$)

$BC = CD$

$\therefore \triangle ABC \equiv \triangle EDC$ (AAS test)

d In $\triangle ABD$ and $\triangle CBD$,

$AD = DC$ (given)

$\angle ADB = \angle CDB$ (given)

BD is common

$\therefore \triangle ABD \equiv \triangle CBD$ (SAS test)

e In $\triangle OAB$ and $\triangle OCD$,

$OA = OC$ (equal radii)

$OB = OD$ (equal radii)

$AB = CD$ (given)

$\therefore \triangle OAB \equiv \triangle OCD$ (SSS test)

f In $\triangle ADC$ and $\triangle ABC$,

$\angle ADC = \angle ABC = 90°$ (given)

AC is common

$DC = BC$ (given)

$\therefore \triangle ADC \equiv \triangle ABC$ (RHS test)

6 a In $\triangle AOB$ and $\triangle COB$,

$AO = CO$ (equal radii)

$\angle AOB = \angle COB$ (given)

OB is common

$\therefore \triangle AOB \equiv \triangle COB$ (SAS test)

b AB and BC are matching sides in congruent triangles,

$\therefore AB = BC$.

c $AB = 10$ mm

7 a In $\triangle ABC$ and $\triangle EDC$,

$BC = DC$ (given)

$\angle ACB = \angle DCE$ (vertically opposite angles)

$AC = EC$ (given)

$\therefore \triangle ABC \equiv \triangle EDC$ (SAS test)

b AB and DE are matching sides in congruent triangles,

$\therefore AB = DE$.

c $\angle ABC = \angle CDE$ (matching angles in congruent triangles)

$\angle ABC$ and $\angle CDE$ are alternate angles, $\therefore AB \parallel DE$.

d $DE = 5$ cm

8 a In $\triangle ABD$ and $\triangle CDB$,

$AB = CD$ (given)

$AD = BC$ (given)

BD is common

$\therefore \triangle ABD \equiv \triangle CDB$ (SSS test)

b $\angle DBC = \angle BDA$ are matching angles in congruent triangles, $\therefore \angle DBC = \angle BDA$.

c $\angle ADB$ and $\angle DBC$ are equal alternate angles, $\therefore AD \parallel BC$.

9 a In $\triangle ABC$ and $\triangle EDC$,

$BC = DC$ (given)

$\angle BCA = \angle ECD$ (vertically opposite angles)

$AC = EC$ (given)

$\therefore \triangle ABC \equiv \triangle EDC$ (SAS test)

$\angle ABC = \angle CDE$ (matching angles in congruent triangles) and they are alternate.

$\therefore AB \parallel DE$

b ABC is straight.

$\therefore \angle ABO = \angle OBC = 90°$

In $\triangle ABO$ and $\triangle CBO$,

$\angle ABO = \angle OBC$ (see above)

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$OA = OC$ (equal radii)
OB is common
$\therefore \triangle ABO \equiv \triangle CBO$ (RHS test)
$\therefore AB = BC$ (matching sides in congruent triangles)
$\therefore OB$ bisects AC.

c In $\triangle ACD$ and $\triangle CAB$,
$CD = AB$ (given)
$AD = BC$ (given)
AC is common
$\therefore \triangle ACD \equiv \triangle CAB$ (SSS test)
$\therefore \angle DAC = \angle ACB$ (matching angles in congruent triangles) and they are alternate.
$\therefore AD \parallel BC$

d $AE = AB$ (given)
$\therefore \triangle AEB$ is isosceles.
$\angle ABE = \angle AEB$ (equal angles are opposite equal sides)
In $\triangle AED$ and $\triangle ABC$,
$AE = AB$ (given)
$\angle ABE = \angle AEB$ (see above)
$DE = BC$ (given)
$\therefore \triangle AED \equiv \triangle ABC$ (SAS test)
$\therefore AD = AC$ (matching sides in congruent triangles)

e In $\triangle OAD$ and $\triangle OBC$,
$OD = OC$ (given)
$\angle DOA = \angle COB$ (vertically opposite angles)
$OA = OB$ (given)
$\therefore \triangle OAD \equiv \triangle OBC$ (SAS test)
$\therefore \angle OAD = \angle OBC$ (matching angles in congruent triangles)

f In $\triangle ADC$ and $\triangle ABC$,
AC is common
$\angle DAC = \angle BAC$ (given)
$AD = AB$ (given)
$\therefore \triangle ADC \equiv \triangle ABC$ (SAS test)
$\therefore \angle ACD = \angle ACB$ (matching angles in congruent triangles) and they are supplementary (since BD is straight).
$\therefore \angle ACD = \angle ACB = 90°$
$\therefore AC \perp BD$

10 a In $\triangle ABD$ and $\triangle ACD$:
$\angle BDA = \angle CDA = 90° (AD \perp BC)$
$AB = AC$ (given)
AD is common
$\therefore \triangle ABD \equiv \triangle ACD$ (RHS test)

b They are matching angles in congruent triangles.

c If two sides of a triangle are equal, then the angles opposite those sides are also equal.

11 In line 2 $\angle ABC$ should be $\angle ACD$.
In line 4 AB should be AD.
In line 5 RHS should be AAS.

12 a In $\triangle OAM$ and $\triangle OBM$,
$OA = OB$ (equal radii)
OM is common
$AM = BM$ (M is the midpoint of AB)
$\therefore \triangle OAM \equiv \triangle OBM$ (SSS test)
$\therefore \angle OMA = \angle OMB$ (matching angles in congruent triangles) and they are supplementary (since AB is straight).
$\therefore \angle OMA = \angle OMB = 90°$
$\therefore OM \perp AB$

b In $\triangle OAC$ and $\triangle OBC$,
$OA = OB$ (equal radii)
$AC = BC$ (equal radii)
OC is common
$\therefore \triangle OAC \equiv \triangle OBC$ (SSS test)
$\therefore \angle AOC = \angle BOC$ (matching angles in congruent triangles)

c $\angle CAB = \angle CBA = x$ (equal angles are opposite sides of equal length)
$\therefore \angle EAB = \angle DBA = \frac{x}{2}$
$\therefore AF = BF$ (sides of equal length are opposite equal angles in $\triangle AFB$)

Exercise 6C

1 a rectangle **b** parallelogram
c square **d** rhombus

2 a rectangle, square **b** rectangle, square
c parallelogram, rhombus, rectangle, square
d rhombus, square **e** rhombus, square

3 a A trapezium does not have both pairs of opposite sides parallel.
b A kite does not have two pairs of opposite sides parallel.

4 a $\angle BAC = \angle DCA$ (alternate angles), $\angle BCA = \angle DAC$ (alternate angles), AC is common. Therefore, $\triangle ABC \equiv \triangle CDA (AAS)$.
b As $\triangle ABC \equiv \triangle CDA, AD = CB, AB = CD$ (corresponding sides).

5 a $\angle ABE = \angle CDE$ (alternate angles, $AB \parallel DC$)
$\angle EAB = \angle ECD$ (alternate angles, $AB \parallel DC$)
$AB = DC$ (opposite sides of a parallelogram)
$\therefore \triangle ABE \equiv \triangle CDE$ (AAS test)
b $AE = CE$ (corresponding sides), $BE = DE$ (corresponding sides).

6 a $AB = CB$ (given), $AD = CD$ (given), BD is common. Therefore, $\triangle ABD \equiv \triangle CDB$ (SSS).
b $\angle ABD = \angle ADB = \angle CBD = \angle CDB$ (equal angles in congruent isosceles triangles). Therefore, BD bisects $\angle ABC$ and $\angle CDA$.

7 a $AE = CE$ (given), $BE = DE$ (given), $\angle AEB = \angle CED$ (vertically opposite angles). Therefore, $\triangle ABE \equiv \triangle CDE$ (SAS).
b $\angle ABE = \angle CDE$ (corresponding angles), $\angle BAE = \angle DCE$ (corresponding angles). Therefore, $AB \parallel DC$ (alternate angles are equal). $\angle ADE = \angle CBE$ (corresponding angles),
$\angle DAE = \angle BCE$ (corresponding angles). Therefore, $AD \parallel BC$ (alternate angles are equal).

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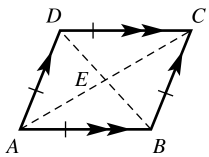
8 a $AD = CB$ (given), $\angle DAC = \angle BCA$ (alternate angles), AC is common. Therefore, $\triangle ABC \equiv \triangle CDA$ (SAS).

b $\angle BAC = \angle DCA$ (corresponding angles), therefore $AB \parallel DC$ (alternate angles are equal).

9 a $\triangle ABE \equiv \triangle CBE \equiv \triangle ADE \equiv \triangle CDE$(SAS)

b $\angle ABE = \angle CDE$ (corresponding angles), $\angle BAE = \angle DCE$ (corresponding angles), therefore $AB \parallel CD$.
$\angle ADE = \angle CBE$ (corresponding angles), $\angle DAE = \angle BCE$ (corresponding angles), therefore $AD \parallel CB$.
Also, $AB = AD = CB = CD$ (corresponding sides).
Therefore, $ABCE$ is a rhombus.

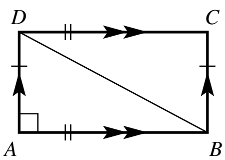
10 a



$\angle CAB = \angle ACD$ and $\angle CAD = \angle ACB$ (alternate angles).
So $\angle ACB = \angle ACD$.
$\triangle CDE \equiv \triangle CBE$ (SAS)
So $\angle CED = \angle CEB = 90°$.

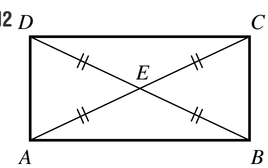
b From part **a**, $\angle ECD = \angle ECB$.

11



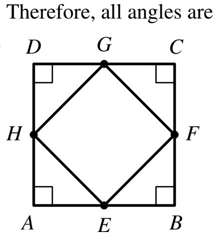
As $ABCD$ is a parallelogram $\angle BDC = \angle DBA$ and $\angle DBC = \angle BDA$.
So $\triangle CBD \equiv \triangle ADB$ (AAS).
So $\angle BAD = \angle DCB = 90°$.
Similarly, $\angle ADC = \angle CBA = 90°$.

12



First, prove $\triangle AED \equiv \triangle BEC$ (SAS).
Hence, corresponding angles in the isosceles triangles are equal and $\triangle CED \equiv \triangle BEA$ (SAS).
Hence, corresponding angles in the isosceles triangles are equal.
So $\angle ADC = \angle DCB = \angle CBA = \angle BAC$, which sum to 360°.
Therefore, all angles are 90° and $ABCD$ is a rectangle.

13



First, prove all four corner triangles are congruent (SAS).
So $EF = FG = GH = HE$, so $EFGH$ is a rhombus.

14 The simplest way to do this is to prove that the midpoint of one diagonal is the same as the midpoint of the other diagonal.

Exercise 6D

1 a Yes, both squares have all angles 90° and all sides of equal length.
b 3 **c** 15 cm **d** 1 : 3 **e** 1 : 9

2 a 2 **b** $\frac{8}{5}$ **c** $\frac{4}{3}$ **d** $\frac{3}{2}$

3 a A **b** $\angle C$
c FD **d** $\triangle ABC \mathbin{|||} \triangle EFD$

4 a $ABCDE \mathbin{|||} FGHIJ$ **b** $\frac{AB}{FG} = \frac{DE}{IJ}$
c $\frac{3}{2}$ **d** $\frac{3}{2}$ cm **e** $\frac{4}{3}$ cm

5 a $ABCD \mathbin{|||} EFGH$ **b** $\frac{EF}{AB} = \frac{GH}{CD}$
c $\frac{4}{3}$ **d** 12 m **e** 10.5 m

6 a 1.2 **b** 12.5 **c** 4.8
d 3.75 **e** 11.5 **f** 14.5

7 1.7 m

8 a 1.6 **b** 62.5 cm

9 a 2 **b** 1 **c** 1.875 **d** 4.3

10 a BC **b** $\triangle ABC \mathbin{|||} \triangle EDC$
c 1 **d** 4.5

11 a true **b** true **c** false **d** false **e** false
f false **g** false **h** false **i** true **j** true

12 Yes, the missing angle in the first triangle is 20° and the missing angle in the second triangle is 75°. So all three angles are equal.

13 a $x : y = 2 : 6 = 1 : 3$
b i 4 **ii** 36
c i 8 **ii** 216
d

Cube	Length	Area	Volume
Small	2	4	8
Large	6	36	216
Scale factor (fraction)	1 : 3	1 : 3^2	1 : 3^3

e i 1 : 4 **ii** 1 : 16 or 1 : 4^2
iii 1 : 64 or 1 : 4^3
f i 1 : k^2 **ii** 1 : k^3

Exercise 6E

1 a E **b** $\angle C$
c AB **d** $\triangle ABC \mathbin{|||} \triangle DEF$

2 a $\angle D$ (alternate angles, $AB \parallel DE$)
b $\angle A$ (alternate angles, $AB \parallel DE$)
c $\angle ECD$
d CA
e $\triangle ABC \mathbin{|||} \triangle EDC$

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3 a sides about equal angles are in proportion
b matching angles are equal
c sides about equal angles are in proportion
d three pairs of sides are in proportion

4 a $\angle ABC = \angle DEF = 65°$, $\angle BAC = \angle EDF = 70°$. Therefore, $\triangle ABC \mathbin{|||} \triangle DEF$ (matching angles are equal).
b $\frac{DE}{AB} = \frac{2}{1} = 2$, $\frac{EF}{BC} = \frac{6}{3} = 2$ (ratio of corresponding sides), $\angle ABC = \angle DEF = 120°$. Therefore, $\triangle ABC \mathbin{|||} \triangle DEF$ (sides about equal angles are in proportion).
c $\frac{DF}{CA} = \frac{10}{5} = 2$, $\frac{DE}{CB} = \frac{8}{4} = 2$ (ratio of corresponding sides), $\angle ABC = \angle FED = 90°$. Therefore, $\triangle ABC \mathbin{|||} \triangle FED$ (hypotenuse and side are in proportion).
d $\frac{AB}{DE} = \frac{28}{7} = 4$, $\frac{BC}{EF} = \frac{16}{4} = 4$, $\frac{AC}{DF} = \frac{32}{8} = 4$. Therefore, $\triangle ABC \mathbin{|||} \triangle DEF$ (three pairs of sides are in proportion).

5 a $\frac{3}{2}$
b 19.5
c 2.2
d $a = 4$, $b = 15$
e $x = 0.16$, $y = 0.325$
f $a = 43.2$, $b = 18$

6 a $\angle ABC = \angle EDC$ (alternate angles), $\angle BAC = \angle DEC$ (alternate angles), $\angle ACB = \angle ECD$ (vertically opposite angles). Therefore, $\triangle ABC \mathbin{|||} \triangle EDC$ (matching angles are equal).
b $\angle ABE = \angle ACD$ (corresponding angles), $\angle AEB = \angle ADC$ (corresponding angles), $\angle BAE = \angle CAD$ (common). Therefore, $\triangle ABE \mathbin{|||} \triangle ACD$ (matching angles are equal).
c $\angle DBC = \angle AEC$ (given), $\angle BCD = \angle ECA$ (common). Therefore, $\triangle BCD \mathbin{|||} \triangle ECA$ (matching angles are equal).
d $\frac{AB}{CB} = \frac{3}{7.5} = 0.4$, $\frac{EB}{DB} = \frac{2}{5} = 0.4$ (ratio of corresponding sides), $\angle ABE = \angle CBD$ (vertically opposite angles). Therefore, $\triangle AEB \mathbin{|||} \triangle CDB$ (sides about equal angles are in proportion).

7 a $\angle EDC = \angle ADB$ (common), $\angle CED = \angle BAD = 90°$. Therefore, $\triangle EDC \mathbin{|||} \triangle ADB$ (matching angles are equal).
b $\frac{4}{3}$ cm

8 a $\angle ACB = \angle DCE$ (common), $\angle BAC = \angle EDC = 90°$. Therefore, $\triangle BAC \mathbin{|||} \triangle EDC$ (matching angles are equal).
b 1.25 m

9 1.90 m

10 4.5 m

11 a Yes, matching angles are equal for both.
b 20 m
c 20 m
d Less working is required for Jenny's triangles.

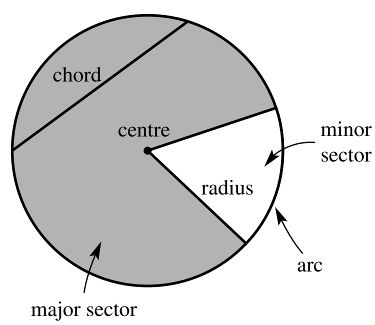
12 The missing angle in the smaller triangle is 47°, and the missing angle in the larger triangle is 91°. Therefore, the two triangles are similar (matching angles are equal).

13 a $\angle AOD = \angle BOC$ (common), $\angle OAD = \angle OBC$ (corresponding angles), $\angle ODA = \angle OCB$ (corresponding angles). So $\triangle OAD \mathbin{|||} \triangle OBC$ (matching angles are equal). $\frac{OC}{OD} = \frac{3}{1} = 3$ (ratio of corresponding sides), therefore $OB = 3OA$.
b $\angle ABC = \angle EDC$ (alternate angles), $\angle BAC = \angle DEC$ (alternate angles), $\angle ACB = \angle ECD$ (vertically opposite). So $\triangle ABC \mathbin{|||} \triangle EDC$ (matching angles are equal). $\frac{BD}{BC} = \frac{7}{5}$, therefore $\frac{AE}{AC} = \frac{7}{5}$ and $AE = \frac{7}{5}AC$.

14 a $\angle BAD = \angle BCA = 90°$, $\angle ABD = \angle CBA$ (common). So $\triangle ABD \mathbin{|||} \triangle CBA$ (matching angles are equal). Therefore $\frac{AB}{CB} = \frac{BD}{AB}$, $AB^2 = CB \times BD$.
b $\angle BAD = \angle ACD = 90°$, $\angle ADB = \angle CDA$ (common). So $\triangle ABD \mathbin{|||} \triangle CAD$ (matching angles are equal). Therefore $\frac{AD}{CD} = \frac{BD}{AD}$, $AD^2 = CD \times BD$.
c Adding the two equations: $AB^2 + AD^2 = CB \times BD + CD \times BD = BD(CB + CD) = BD \times BD = BD^2$.

Exercise 6F

1 a–f



2 a 55° **b** 90° **c** 75° **d** 140°

3 a 85° each
b $\angle AOB = \angle COD$ (equal chords subtend equal angles at the centre)
c 0.9 cm each
d $OE = OF$ (equal chords subtend equal angles at the centre)

4 a 1 cm each **b** 52° each
c $AM = BM$ and $\angle AOM = \angle BOM$

5 a $\angle DOC = 70°$ (equal chords subtend equal angles at the centre)
b $OE = 7.2$ cm (equal chords subtend equal angles at the centre)
c $XZ = 4$ cm and $\angle XOZ = 51°$
d $\angle OBC = 90°$ (line through centre of circle that bisects a chord is perpendicular to the chord)

6 The perpendicular bisectors of two different chords of a circle intersect at the centre of the circle.

7 a 3.5 m **b** 9 m **c** 90° **d** 90°

8 a $140°$ b $40°$ c $19°$
d $72°$ e $30°$ f $54°$

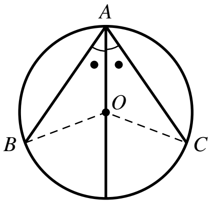
9 6 m

10 $3 + \sqrt{128}$ mm $= 3 + 8\sqrt{2}$ mm

11 a Triangles are congruent (SSS), so angles at the centre of the circle are corresponding, and therefore equal.
b Triangles are congruent (SAS), so chords are corresponding sides, and therefore equal.

12 a Triangles are congruent (SSS), so the angles formed by the chord and radius are corresponding, and therefore equal. Since these angles are also supplementary, they must be 90°.
b Triangles are congruent (SAS), so the angles formed by the chord and radius are corresponding, and therefore equal. Since these angles are also supplementary, they must be 90°.

13



First, prove $\triangle OAB \equiv \triangle OAC$ (AAS), which are isosceles. So $AB = AC$, corresponding sides in congruent triangles.

14 a $AD = BD$ (equal radii), $AC = BC$ (equal radii), CD is common. Therefore, $\triangle ACD \equiv \triangle BCD$ (SSS).
b $AC = BC$, $\angle ACE = \angle BCE$ (corresponding angles), CE is common. Therefore, $\triangle ACE \equiv \triangle BCE$ (SAS).
c $\angle BEC = \angle CEA = 90°$ (equal angles on a straight line). Therefore, $CD \perp AB$ and $BE = EA$ (matching sides in congruent triangles).

15 a It is the angle between a tangent and a radius.
b yes c 180
d $\angle OCP$ is a straight angle.
e yes f collinear

Exercise 6G

1 a $\angle ADC$ b $\angle ADC$ c $\angle ADC$
d $\angle AFC$ e $\angle AEC$ f $\angle AEC$

2 a $\angle AOB$ b $\angle ACB$ c $80°$ d $61°$

3 a $180°$ b $90°$ c $60°$ d $7°$

4 a $50°$ b $40°$ c $80°$ d $60°$ e $250°$
f $112.5°$ g $38°$ h $120°$ i $18°$

5 a $70°$ b $25°$ c $10°$

6 a $\angle ABC = 72°, \angle ABD = 22°$
b $\angle ABC = 70°, \angle ABD = 45°$
c $\angle ABC = 72°, \angle ABD = 35°$

7 a $\angle ADC = 75°, \angle ABC = 75°$
b $\angle ABC = 57.5°, \angle ADC = 57.5°$
c $\angle AOD = 170°, \angle ABD = 85°$

8 a $100°$ b $94.5°$ c $100°$
d $119°$ e $70°$ f $66°$

9 a $58°$ b $53°$ c $51°$
d $45°$ e $19°$ f $21°$

10 a $70°$ b $90°$
c The angle in a semicircle is 90°.
d The second circle is the specific case of the first circle when the angle at the centre is 180°.

11 a i false ii true iii true iv false
b i false ii true iii true iv false

12 a $2x$ b $360 - 2x$

13 a $\angle AOC = 180° - 2x°$ ($\triangle AOC$ is isosceles)
b $\angle BOC = 180° - 2y°$ ($\triangle BOC$ is isosceles)
c $\angle AOB = 360° - \angle AOC - \angle BOC = 2x° + 2y°$
d $\angle AOB = (2x° + 2y°) = 2\angle ACB$

14 a $\angle BOC = 180 - 2x$ ($\triangle BOC$ is isosceles),
$\angle AOB = 180 - \angle BOC = 180 - (180 - 2x) = 2x$
b $\angle AOC = 180 - 2x$ ($\triangle AOC$ is isosceles),
$\triangle BOC = 180 - 2y$ ($\triangle BOC$ is isosceles).
Reflex $\angle AOB = 360 - \angle AOC - \angle BOC$
$= 360 - (180 - 2x) - (180 - 2y) = 2x + 2y = 2(x + y)$
$= 2\angle ACB$.
c $\angle OBC = x + y$, $\angle COB = 180 - 2(x + y)$,
$\angle AOB = 180 - 2x - (180 - 2(x + y)) = 2y$

15 $\angle AOB = 180 - 2x$ ($\triangle AOB$ is isosceles), $\angle BOC = 180 - 2y$ ($\triangle BOC$ is isosceles)
$\angle AOB + \angle BOC = 180$ (supplementary angles), therefore
$(180 - 2x) + (180 - 2y) = 180, 360 - 2x - 2y = 180,$
$2x + 2y = 180, 2(x + y) = 180, x + y = 90.$

Exercise 6H

1 a $\angle ACD$ b $\angle ACD$ c $\angle ACD$

2 a $\angle ABD$ and $\angle ACD$ b $85°$
c $\angle BAC$ and $\angle BDC$ d $17°$

3 a Supplementary angles sum to 180°.
b $117°$
c $109°$
d Yes, $117° + 109° + 63° + 71° = 360°$

4 a $x = 37$ b $x = 20$ c $x = 110$
d $x = 40$ e $x = 22.5$ f $x = 55$

5 a $x = 60$ b $x = 90$ c $x = 30$
d $x = 88$ e $x = 72, y = 108$ f $x = 123$

6 a 72 b 43 c 69 d 57 e 52
f 48 g 30 h 47 i 108

7 a $a = 30, b = 100$
b $a = 54, b = 90$
c $a = 105, b = 105, c = 75$
d $a = 55, b = 70$
e $a = 118, b = 21$
f $a = 45, b = 35$

8 a $80°$
b $71°$
c $\angle CBE + \angle ABE = 180°$ (angles on a straight line)
$\angle CBE + \angle CDE = 180°$ (opposite angles of cyclic quadrilateral $BCDE$)

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$\therefore \angle CBE + \angle ABE = \angle CBE + \angle CDE$

$\therefore \angle ABE = \angle CDE$

9 a $\angle ACD = \angle ABD = x$ and $\angle DAC = \angle DBC = y$ (angles on the same arc)

b Using angle sums of $\angle CAD$ and $\angle ACD$, $\angle ADC = 180 - (x° + y°)$

c $\angle ABC$ and $\angle ADC$ are supplementary.

10 a i 80° **ii** 100° **iii** 80°

b $\angle BAF + \angle DCB = 180°$, therefore $AF \parallel CD$ (cointerior angles are supplementary).

11 a $\angle PCB = 90°$ (angle in a semicircle)

b $\angle A = \angle P$ (angles on the same arc)

c $\sin P = \frac{a}{2r}$

d As $\angle A = \angle P$, $\sin A = \frac{a}{2r}$; therefore, $2r = \frac{a}{\sin A}$.

Exercise 6I

1 a once **b** 90° **c** 5 cm

2 a $\angle BAP$ **b** $\angle BPX$ **c** $\angle ABP$ **d** $\angle APY$

3 a 70° **b** 40° **c** 70°

4 a $a = 19$ **b** $a = 62$ **c** $a = 70$

5 a $a = 30$ **b** $a = 18$ **c** $a = 25$ **d** $a = 63$

6 a 50° **b** 59°

7 a $a = 73, b = 42, c = 65$

b $a = 26, b = 83, c = 71$

c $a = 69, b = 65, c = 46$

8 a 5 cm **b** 11.2 cm

9 a $a = 115$ **b** $a = 163$ **c** $a = 33$

d $a = 28$ **e** $a = 26$ **f** $a = 26$

g $a = 36$ **h** $a = 26$ **i** $a = 30$

10 a $a = 70$ **b** $a = 50$ **c** $a = 73$

d $a = 40$ **e** $a = 19$ **f** $a = 54$

11 4 cm

12 a OA and OB are radii of the circle.

b $\angle OAP = \angle OBP = 90°$

c $\angle OAP = \angle OBP = 90°$ OP is common $OA = OB$

$\therefore \triangle OAP \equiv \triangle OBP$ (RHS)

d AP and BP are corresponding sides in congruent triangles.

13 a $\angle OPB = 90° - x°$ (tangent meets radii at right angles)

b $\angle BOP = 2x°$ (using angle sum in an isosceles triangle)

c $\angle BAP = x°$ (angle at centre is twice the angle at the circumference)

14 $\angle BAP = \angle BPY$ (angle in alternate segment)

$\angle BPY = \angle DPX$ (vertically opposite angles)

$\angle DPX = \angle DCP$ (angle in alternate segment)

$\therefore \angle BAP = \angle DCP$, so $AB \parallel DC$ (alternate angles are equal).

15 $AP = TP$ and $TP = BP$; hence, $AP = BP$.

16 a Let $\angle ACB = x$, therefore $\angle ABC = 90 - x$. Construct OP. $OP \perp PM$ (tangent). $\angle OPC = x$ ($\triangle OPC$ is isosceles). Construct OM. $\triangle OAM \equiv \triangle OPM$ (RHS), therefore, $AM = PM$. $\angle BPM = 180 - 90 - x = 90 - x$. Therefore, $\triangle BPM$ is isosceles with $PM = BM$. Therefore, $AM = BM$.

Exercise 6J

1 a 3 **b** 6 **c** 7 **d** 8

2 a $\frac{21}{2}$ **b** $\frac{5}{2}$ **c** $\frac{33}{7}$ **d** $\frac{27}{7}$

3 a $AP \times CP = BP \times DP$

b $AP \times BP = DP \times CP$

c $AP \times BP = CP^2$

4 a 5 **b** 10 **c** $\frac{112}{15}$

5 a $\frac{143}{8}$ **b** $\frac{178}{9}$ **c** $\frac{161}{9}$

6 a $\frac{32}{3}$ **b** $\frac{16}{3}$ **c** $\frac{35}{2}$

7 a $\sqrt{65}$ **b** $\sqrt{77}$

8 a $\frac{64}{7}$ **b** $\frac{209}{10}$ **c** $\frac{81}{7}$

d $\frac{74}{7}$ **e** $\frac{153}{20}$ **f** $(\sqrt{65}) - 1$

9 a $x(x + 5) = 7 \times 8$, $x^2 + 5x = 56$, $x^2 + 5x - 56 = 0$

b $x(x + 11) = 10 \times 22$, $x^2 + 11x = 220$, $x^2 + 11x - 220 = 0$

c $x(x + 23) = 41^2$, $x^2 + 23x = 1681$, $x^2 + 23x - 1681 = 0$

10 The third secant rule states that, for this diagram, $AP^2 = DP \times CP$ and $BP^2 = DP \times CP$, so $BP = AP$.

11 $AP \times BP = DP \times CP$

$AP \times BP = AP \times CP$

$BP = CP$

$\therefore AB = DC$

12 a $\angle A = \angle D$ and $\angle B = \angle C$ (angles on the same arc)

b $\angle P$ is the same for both triangles (vertically opposite), so $\triangle ABP \mathrel{|||} \triangle DCP$ (matching angles are equal).

c $\frac{AP}{DP} = \frac{BP}{CP}$

d $\frac{AP}{DP} = \frac{BP}{CP}$, cross multiplying gives $AP \times CP = BP \times DP$

13 a $\angle B = \angle C$ (angles on the same arc)

b $\triangle PBD \mathrel{|||} \triangle PCA$ (matching angles are equal)

c $\frac{AP}{DP} = \frac{CP}{BP}$ so $AP \times BP = DP \times CP$.

14 a yes

b alternate segment theorem

c $\triangle BPC \mathrel{|||} \triangle CPA$ (matching angles are equal)

d $\frac{BP}{CP} = \frac{CP}{AP}$, so $CP^2 = AP \times BP$.

15 $d = \sqrt{(4r_1r_2)} = 2\sqrt{r_1r_2}$

Puzzles and challenges

1 21 units²

2 $BD = 5$ cm, $CE = 19$ cm

3 $\angle ADE = \angle ABE$, $\angle EFD = \angle BFA$, $\angle DEB = \angle DAB$, $\angle DFB = \angle EFA$

4 42.5%

5 In $\triangle ABC$, $AB = AC$ (given), $CE = EB$ (given), AE is common.

$\therefore \triangle ABE \equiv \triangle ACE$ (SSS)

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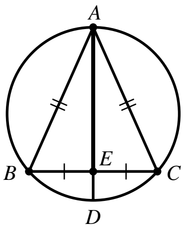
$\therefore \angle AEB = \angle AEC = 90°$ (equal angles on a straight line)



Using Pythagoras' theorem in $\triangle AEB$:

$AB^2 = AE^2 + BE^2$

$AB^2 - AE^2 = BE^2$

$AB^2 - AE^2 = BE \times BE$

$AB^2 - AE^2 = BE \times CE$ ($BE = CE$)

6 a $\angle FDE = \angle DFC = \angle ABC$ (alternate and corresponding angles in parallel lines)

$\angle FED = \angle EFB = \angle ACB$ (alternate and corresponding angles in parallel lines)

$\angle DFE = \angle BAC$ (angle sum of a triangle)

$\triangle ABC \parallel\!| \triangle FDE$ (matching angles are equal)

b i 4 : 1 **ii** 16 : 1

c $4^{n-1} : 1$

Multiple-choice questions

1 C	**2** B	**3** B	**4** C	**5** B
6 A	**7** E	**8** C	**9** D	**10** B

Short-answer questions

1 a 65 **b** 120

c $x = 62, y = 118$ **d** 46

2 a 148° **b** 112°

3 a $\triangle ABC \equiv \triangle DEF$ (AAS) **b** $\triangle ABC \equiv \triangle ADC$ (SAS)

c $\triangle ABD \equiv \triangle CDB$ (SSS)

4 a $AB = CD$ (given), $\angle BAC = \angle DCA$ (alternate angles), AC is common. Therefore, $\triangle ABC \equiv \triangle CDA$(SAS).

b $\angle BAC = \angle DCA$ (alternate angles), therefore $AB \parallel DC$ (alternate angles are equal).

5 a $\triangle ABC \parallel\!| \triangle DEF$ (sides about equal angles are in proportion), $x = 19.5$

b $\triangle ABE \parallel\!| \triangle ACD$ (matching angles are equal), $x = 6.25$

c $\triangle ABC \parallel\!| \triangle DEC$ (matching angles are equal), $x = 8.82$

d $\triangle ABD \parallel\!| \triangle DBC$ (matching angles are equal), $x = \frac{100}{7}$

6 a 65 **b** 7 **c** 6

7 a 25 **b** $a = 50, b = 40$

c 70 **d** $a = 30, b = 120$

e 115 **f** 54

8 a $x = 26, y = 58, z = 64$

b $a = 65, b = 130, c = 50, d = 8$

c $t = 63$

9 a 5 **b** 6 **c** $\frac{40}{3}$

Extended-response questions

1 a $\angle BAC = \angle BDE = 90°$

$\angle B$ is common

$\triangle ABC \parallel\!| \triangle DBE$ (matching angles are equal)

b 1.2 km

c i $\frac{AC}{DE} = \frac{3}{2}$

$\therefore \frac{AB}{DB} = \frac{3}{2}$ (ratio of sides in similar triangles)

$\therefore \frac{x+1}{x} = \frac{3}{2}$

$\therefore 2(x + 1) = 3x$

ii 2

d 44.4%

2 a i 132.84° **ii** 47.16°

b 12 cm **c** 25 cm

d $(2\sqrt{525} + 20)$ cm

3 a Angle in a semicircle is 90°.

b $\angle BCP + \angle BCA = 180°$ (angles on a straight line)

$\therefore \angle BCP = 90°$

now $\angle BCP + \angle QPB = 180°$

and $BCQP$ is cyclic as the opposite angles are supplementary

c $\angle ABC = 40°, \angle CBP = 140°, \angle CQP = 40°$

d $\angle ACT = \angle ABC$ (angle between tangent and chord equals angle in alternate segment)

$\angle ACT = 40°$

Chapter 7

Pre-test

1 a 0.89 **b** 9.51 **c** 0.27

d 3.25 **e** 2.37 **f** 7.75

2 a 1.1 **b** 3.6 **c** 22.3

d 5.4 **e** 7.7 **f** 2.8

3 a H = b, O = a, A = c **b** H = a, O = b, A = c

c H = a, O = c, A = b

4 a hypotenuse **b** adjacent **c** $\tan\theta$

5 a $\sin\theta = \frac{2}{3}$ **b** $\cos\theta = \frac{3}{4}$ **c** $\tan\theta = \frac{7}{8}$

6 a 44.4° **b** 7.1° **c** 53.1°

d 75.5° **e** 61.9° **f** 24.8°

7 a 180° **b** 270° **c** 45° **d** 225°

8 a $x = 50, y = 40$ **b** $x = 50, y = 40$

c $x = 40, y = 50$

Exercise 7A

1 a 0.799 **b** 0.951 **c** 1.192 **d** 0.931

e 0.274 **f** 11.664 **g** 0.196 **h** 0.999

2 a $\sin\theta$ **b** $\cos\theta$ **c** $\tan\theta$

3 a 1.80 **b** 2.94 **c** 3.42

d 2.38 **e** 22.33 **f** 12.47

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4 **a** 1.15 **b** 3.86 **c** 13.74 **d** 5.07
e 2.25 **f** 2.79 **g** 1.97 **h** 13.52
i 37.02 **j** 9.30 **k** 10.17 **l** 13.15

5 **a** 8.55 **b** 4.26 **c** 13.06 **d** 10.04
e 5.55 **f** 1.52 **g** 22.38 **h** 6.28
i 0.06 **j** 12.12 **k** 9.81 **l** 15.20

6 **a** $x = 2.5$ cm, $y = 4.33$ cm
b $x = 12.26$ cm, $y = 6.11$ cm
c $x = 0.20$ m, $y = 0.11$ m

7 **a** 125 m **b** 327 m

8 1.85 m

9 22.3 m

10 7.54 m

11 28.5 m

12 26.4 cm

13 **a** 4.5 cm **b** 8.5 mm

14 The student rounded $\tan 65°$ too early.

15 **a** 3.7 **b** 6.5 **c** 7.7

16 **a** **i** $a = c\sin\theta$ **ii** $b = c\cos\theta$ **iii** $\tan\theta = \dfrac{a}{b}$
iv $\tan\theta = \dfrac{c\sin\theta}{c\cos\theta} = \dfrac{\sin\theta}{\cos\theta}$ **v** Answers may vary.
b **i** $a = c\sin\theta$ **ii** $b = c\cos\theta$
iii $c^2 = a^2 + b^2$
iv $c^2 = (c\sin\theta)^2 + (c\cos\theta)^2$
$c^2 = c^2(\sin\theta)^2 + c^2(\cos\theta)^2$
$\therefore 1 = (\sin\theta)^2 + (\cos\theta)^2$
c **i** $\dfrac{b}{c}, \dfrac{b}{c}, \dfrac{a}{c}$ **ii** yes
iii $\cos\theta = \sin(90° - \theta)$

Exercise 7B

1 **a** 60° **b** 30° **c** 45°

2 **a** 23.58° **b** 60° **c** 11.31° **d** 5.74°
e 25.84° **f** 45° **g** 14.48° **h** 31.79°

3 **a** tangent **b** cosine **c** sine

4 **a** 60° **b** 45° **c** 48°35′
d 30° **e** 52°7′ **f** 32°44′

5 **a** $\alpha = 60°, \theta = 30°$ **b** $\alpha = 45°, \theta = 45°$
c $\alpha = 53.1°, \theta = 36.9°$ **d** $\alpha = 22.6°, \theta = 67.4°$
e $\alpha = 28.1°, \theta = 61.9°$ **f** $\alpha = 53.1°, \theta = 36.9°$

6 **a** 44.4°, 45.6° **b** 74.7°, 15.3° **c** 58.3°, 31.7°
d 23.9°, 66.1° **e** 82.9°, 7.1° **f** 42.4°, 47.6°

7 70.02°

8 31.1°

9 47.1°

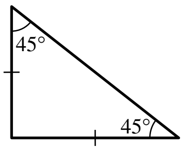
10 **a** 66.4° **b** 114.1° **c** 32.0°

11 **a** 1 **b** 30° **c** 45° **d** 63°26′

12 **a** **i** 45° **ii** 33.7°
b 11.3°

13 **a** Once one angle is known, the other can be determined by subtracting the known angle from 90°.
b $\alpha = 63.4°, \beta = 26.6°$

14 **a**



b $\tan 45 = \dfrac{x}{x} = 1$ **c** $\sqrt{2}x$
d $\sin 45° = \dfrac{x}{\sqrt{2}x} = \dfrac{1}{\sqrt{2}}$; $\cos 45°$ also equals $\dfrac{1}{\sqrt{2}}$.

15 **a** $\theta = 30°$ **b** $\alpha = 60°$ **c** $\sqrt{3}$
d **i** $\dfrac{1}{2}$ **ii** $\dfrac{1}{2}$ **iii** $\dfrac{\sqrt{3}}{2}$
iv $\dfrac{\sqrt{3}}{2}$ **v** $\dfrac{\sqrt{3}}{3}$ **vi** $\sqrt{3}$
e $AB = \dfrac{1}{2}x + \dfrac{\sqrt{3}}{2}x$

Exercise 7C

1 1866.03 m

2 39 m

3 28.31 m

4 4°16′

5 320 m

6 1509.53 m

7 32°

8 **a** 1.17 m **b** 1.50 m

9 8.69 cm

10 299 m

11 **a** 1.45° **b** 3.44° **c** 1.99°

12 yes

13 89.12 m

14 **a** **i** 8.7 cm **ii** 5 cm
b **i** 17.3 cm **ii** 20 cm
c Answers may vary.

15 321.1 km/h

16 **a** **i** 18° **ii** 72° **iii** 36° **iv** 54°
b **i** 0.77 m **ii** 2.38 m **iii** 2.02 m **iv** 1.47 m
c 3.85 m **d** 4.05 m **e** proof

Exercise 7D

1 **a** 0° **b** 45° **c** 90° **d** 135°
e 180° **f** 225° **g** 270° **h** 315°

2 **a** 050° **b** 060° **c** 139°
d 162° **e** 227° **f** 289°

3 **a** 200° **b** 082° **c** 335° **d** 164°

4 **b** **i** 022.5° **ii** 337.5°
iii 157.5° **iv** 247.5°

5 **a** 1.7 km **b** 3.6 km **c** 025°

6 **a** 121° **b** 301°

7 **a** 3.83 km **b** 6.21 km

8 **a** 14.77 cm **b** 2.6 cm

9 **a** 217° **b** 37°

10 **a** 1.414 km **b** 1.414 km **c** 2.914 km

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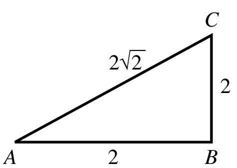
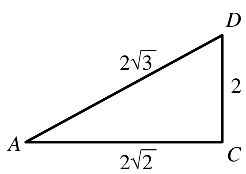
11 a 1.62 km **b** 5.92 km **c** 2.16 km
12 10.032 km
13 a i 045° **ii** 236.3° **iii** 26.6° **iv** 315°
b i 296.6° **ii** 116.6° **iii** 101.3° **iv** 246.8°
14 a i 2.5 km **ii** 2.82 km **iii** 5.32 km
b i 4.33 km **ii** 1.03 km **iii** 5.36 km
c i 45.2° **ii** 7.6 km
15 a 229.7°, 18.2 km **b** 55.1°, 12.3 km
16 a 212.98 m
b i 99.32 m **ii** 69.20 m
c 30.11 m
17 a 38.30 km **b** 57.86 km **c** 33.50°
18 a 4.34 km **b** 2.07 km **c** 4.81 km

Exercise 7E

1 a



b



c 35.3° **d** 45°
2 61.4°
3 a 37.609 m **b** 45.47°
4 a 57.409 m **b** 57.91°
5 a i 26.57° **ii** 11.18 cm
b 10.14°
6 a 7.31 m **b** 6.87 m
7 138.56 m
8 a i 2.25 m **ii** 2.59 m
b 40.98° **c** 3.43 m
9 a i 1.331 km **ii** 1.677 km
b 0.346 km
10 a camera C **b** 609.07 m
11 a 5.5 m **b** 34.5° **c** 34.7° **d** 0.2°
12 a 45° **b** 1.41 units
c 35.26° **d** 1.73 units
13 a i 1.55 **ii** 1.27 **iii** 2.82
b 34.34°
14 22°
15 a $\tan T = \frac{OA}{OT}$ **b** $OB = h\tan 48°$ **c** 61.4 m

Exercise 7F

1 b sin (135°) ≈ 0.7, cos (135°) ≈ –0.7
c sin (160°) ≈ 0.34, cos (160°) ≈ –0.94
2 a 149° **b** 128° **c** 135° **d** 93°
e 41° **f** 56° **g** 29° **h** 69°
3 a 0.34 **b** 0.98
c 0.63 **d** 0.77, –0.77
e 0.91, –0.91 **f** 0.42, –0.42
g 1.19, –1.19 **h** 2.75, –2.75
i 0.21, –0.21
4 a $\frac{1}{\sqrt{2}}$ **b** $\frac{1}{\sqrt{2}}$ **c** 1 **d** $\frac{\sqrt{3}}{2}$ **e** $\frac{1}{2}$
f $\frac{1}{\sqrt{3}}$ **g** $\sqrt{3}$ **h** $\frac{1}{2}$ **i** $\frac{\sqrt{3}}{2}$
5 a 140° **b** 160° **c** 115° **d** 155° **e** 138°
f 99° **g** 143° **h** 124° **i** 172°
6 a 30° **b** 55° **c** 86° **d** 70° **e** 45°
f 9° **g** 21° **h** 78° **i** 37°
7 a positive **b** positive **c** positive **d** negative
e negative **f** positive **g** negative **h** negative
8 a $\frac{\sqrt{3}}{2}$ **b** $\frac{\sqrt{2}}{2}$ **c** $\sqrt{3}$ **d** $\frac{\sqrt{2}}{2}$
e $-\frac{\sqrt{3}}{2}$ **f** $-\sqrt{3}$ **g** $\frac{\sqrt{2}}{2}$ **h** $-\frac{\sqrt{2}}{2}$
i $\frac{\sqrt{3}}{2}$ **j** $-\frac{\sqrt{3}}{3}$ **k** $-\frac{1}{2}$ **l** $\frac{1}{2}$
m –1 **n** 1 **o** 0 **p** undefined
9 a 30°, 150° **b** 45°, 135° **c** 60°, 120°
10 a 120° **b** 135° **c** 150°
d 120° **e** 135° **f** 150°
11 a $3\sqrt{2}$ **b** $3\sqrt{2}$ **c** $\frac{20\sqrt{3}}{3}$
d 14 **e** $5\sqrt{3}$ **f** 3
12 a 45° **b** 30° **c** 60°
13 a $\sqrt{3}$ **b** $\sqrt{2}$
14 a 13
b i $\frac{5}{13}$ **ii** $\frac{12}{13}$ **iii** $\frac{5}{12}$
15 a $\frac{1}{3}$ **b** $\frac{2\sqrt{7}}{7}$ **c** $\frac{2\sqrt{21}}{21}$
16 a $-\frac{5}{3}$ **b** $-\frac{\sqrt{5}}{2}$ **c** $-\frac{1}{7}$
17 a i 0.17 **ii** 0.17 **iii** 0.59 **iv** 0.59
v 0.99 **vi** 0.99 **vii** 0.37 **viii** 0.37
b $\sin a = \cos b$ when $a + b = 90$.
c i $90° - \theta$ **ii** $90° - \theta$
d i 70° **ii** 5° **iii** 19° **iv** 38°
e i $90° - \theta$ **ii** $\frac{b}{c}$ **iii** $\frac{b}{c}$
f $\frac{2\sqrt{5}}{5}$

Exercise 7G

1 a $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ **b** $\frac{12}{\sin \alpha} = \frac{6}{\sin \beta} = \frac{10}{\sin \theta}$
2 a 1.9 **b** 3.6 **c** 2.5
3 a 50.3° **b** 39.5° **c** 29.2°
4 a 7.9 **b** 16.5 **c** 19.1
d 9.2 **e** 8.4 **f** 22.7
5 a 38.0° **b** 51.5° **c** 28.8°
d 44.3° **e** 47.5° **f** 48.1°

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6 a 1.367 km b 74° c 2.089 km

7 27.0°

8 131.0 m

9 a $\angle ABC = 80°$, $\angle ACB = 40°$

b 122 km

10 a $\angle ABC = 80°$ b 61.3 km c 53.9 km

11 a 147.5° b 102.8° c 126.1°

d 100.5° e 123.9° f 137.7°

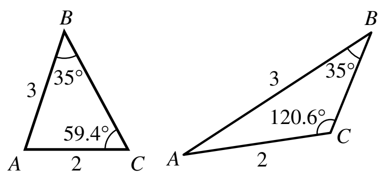
12 Impossible to find θ as such a triangle does not exist.

13 37.6° or 142.4°

14 a 6.1 b 104.0°

15 a 59.4° or 120.6°

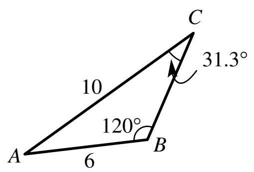
b



c 31.3°

d A triangle can have only one obtuse angle.

e



Exercise 7H

1 a $a^2 = 3^2 + 4^2 - 2 \times 3 \times 4 \times \cos 105°$

b $\cos\theta = \dfrac{5^2 + 9^2 - 7^2}{2 \times 5 \times 9}$

2 a 9.6 b 1.5 c 87.3° d 36.2°

3 a 16.07 cm b 8.85 m c 14.78 cm

d 4.56 m e 2.86 km f 8.14 m

4 a 81.79° b 104.48° c 64.62°

d 61.20° e 92.20° f 46.83°

5 310 m

6 32.2°, 49.6°, 98.2°

7 a 145.9° b 208.2°

8 383 km

9 7.76 m

10 a cosine rule 19.7 b sine rule 12.8

c sine rule 9.9 d cosine rule 1.4

e sine rule 3.0 f cosine rule 2.4

11 Obtuse, as cos of an obtuse angle gives a negative result.

12 a $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ b 121.9°

13 a, b 106°, 35°

14 a $CP = b - x$

b $c^2 = x^2 + h^2$

c $a^2 = h^2 + (b - x)^2$

d $a^2 = c^2 - x^2 + (b - x)^2 = c^2 + b^2 - 2bx$

e $\cos A = \dfrac{x}{c}$

f $x = c\cos A$ substitute into part **d**.

Exercise 7I

1 a 3.7 b 28.8 c 48.0

2 a α b θ c β

3 a 56.44° b 45.58° c 58.05°

4 a 4.4 cm^2 b 26.4 m^2 c 0.9 km^2

d 13.7 m^2 e 318.4 m^2 f 76.2 cm^2

5 a 11.9 cm^2 b 105.6 m^2 c 1.6 km^2

6 a 5.7 b 7.9 c 9.1

d 18.2 e 10.6 f 1.3

7 a 59.09 cm^2 b 1.56 mm^2 c 361.25 km^2

8 a 35.03 cm^2 b 51.68 m^2 c 6.37 km^2

9 a 965.88 m^2 b 214.66 m^2 c 0.72 km^2

10 a 17.3 m^2 b 48 cm^2 c 124.8 km

11 a Area $= ab\sin\theta$

b Area $= \dfrac{1}{2}a^2\sin 60° = \dfrac{\sqrt{3}}{4}a^2$

c Area $= \dfrac{1}{2}a^2\sin(180° - 2\theta) = \dfrac{1}{2}a^2\sin 2\theta$

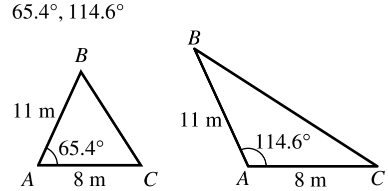
12 a i 129.9 cm^2 ii 129.9 cm^2

b They are equal because sin 60° and sin 120° are equal.

c For example, same side lengths with included angle 140°.

13 a 65.4°, 114.6°

b



14 a i 540° ii 108° iii 11.89 cm^2 iv 8.09 cm

v 72°, 36° vi 19.24 cm^2 vii 43.0 cm^2

b 65.0 cm^2 c Answers may vary.

Exercise 7J

1 a quadrant 1 b quadrant 3

c quadrant 4 d quadrant 2

2 a quadrants 1 and 2 b quadrants 2 and 4

c quadrants 2 and 3 d quadrants 1 and 4

e quadrants 1 and 3 f quadrants 3 and 4

3

θ	0°	90°	180°	270°	360°
$\sin\theta$	0	1	0	−1	0
$\cos\theta$	1	0	−1	0	1
$\tan\theta$	0	undefined	0	undefined	0

4 a 0.139 b 0.995 c −0.530 d −0.574

e −0.799 f −0.259 g 0.777 h −0.087

i 0.900 j −1.036 k 0.900 l −0.424

5 a quadrant 2, $\sin\theta$ positive, $\cos\theta$ negative, $\tan\theta$ negative

b quadrant 4, $\sin\theta$ negative, $\cos\theta$ positive, $\tan\theta$ negative

c quadrant 3, $\sin\theta$ negative, $\cos\theta$ negative, $\tan\theta$ positive

d quadrant 1, $\sin\theta$ positive, $\cos\theta$ positive, $\tan\theta$ positive

e quadrant 4, $\sin\theta$ negative, $\cos\theta$ positive, $\tan\theta$ negative

f quadrant 2, $\sin\theta$ positive, $\cos\theta$ negative, $\tan\theta$ negative

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g quadrant 3, sin θ negative, cos θ negative, tan θ positive
h quadrant 3, sin θ negative, cos θ negative, tan θ positive
i quadrant 3, sin θ negative, cos θ negative, tan θ positive
j quadrant 1, sin θ positive, cos θ positive, tan θ positive
k quadrant 4, sin θ negative, cos θ positive, tan θ negative
l quadrant 2, sin θ positive, cos θ negative, tan θ negative

6 a $-\sin 80°$ b $\cos 60°$ c $\tan 40°$ d $\sin 40°$
e $-\cos 55°$ f $-\tan 45°$ g $-\sin 15°$ h $-\cos 58°$
i $\tan 47°$ j $\sin 68°$ k $\cos 66°$ l $-\tan 57°$

7 a 30° b 60° c 24° d 40° e 71°
f 76° g 50° h 25° i 82°

8 a 42° b 47° c 34° d 9°
e 33° f 62° g 14° h 58°

9 a $0 < \theta < 90°$ b $90° < \theta < 180°$
c $270° < \theta < 360°$ d $180° < \theta < 270°$

10

θ_2	150°	315°	350°	195°	235°	140°	100°	35°	55°

11 a quadrant 4 b quadrant 1 c quadrant 2
d quadrant 2 e quadrant 1 f quadrant 3

12 a 45°
b i $-\frac{\sqrt{2}}{2}$ ii $-\frac{\sqrt{2}}{2}$ iii 1
c 30°
d i $-\frac{1}{2}$ ii $\frac{\sqrt{3}}{2}$ iii $-\frac{\sqrt{3}}{3}$
e 60°
f i $\frac{\sqrt{3}}{2}$ ii $-\frac{1}{2}$ iii $-\sqrt{3}$

13 a $\frac{\sqrt{2}}{2}$ b 0 c $-\frac{\sqrt{3}}{2}$ d $-\frac{\sqrt{3}}{2}$
e -1 f $-\frac{\sqrt{3}}{2}$ g $-\frac{1}{2}$ h $-\sqrt{3}$
i $-\frac{\sqrt{2}}{2}$ j -1 k $-\frac{\sqrt{3}}{3}$ l $\frac{1}{2}$
m 0 n undefined
o 1 p -1

14 As $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and both $\sin\theta$ and $\cos\theta$ are negative over this range, $\tan\theta$ is positive in the third quadrant.

15 As $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cos\theta = 0$ at 90° and 270°, the value of $\frac{\sin\theta}{\cos\theta}$ and, hence, $\tan\theta$ is undefined at these values.

16 a true b true c false d true
e true f false g true h false
i false j true k true l false

17 a i Check with your teacher.
ii True for these values.
b i $\sin 60° = \cos 30° = 0.866$,
$\sin 80° = \cos 10° = 0.985$,
$\sin 110° = \cos(-20°) = 0.940$,
$\sin 195° = \cos(-105°) = -0.259$
ii Their values are the same.
iii They add to 90°.
iv $\sin\theta = \cos(90° - \theta)$
v True for these values.

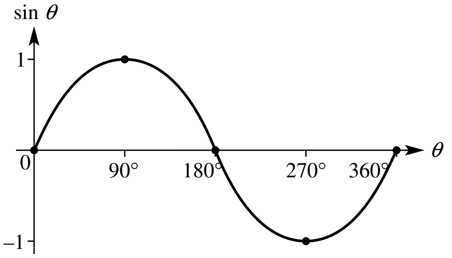
Exercise 7K

1 a

θ	0°	30°	60°	90°	120°	150°
$\sin\theta$	0	0.5	0.87	1	0.87	0.5

θ	180°	210°	240°	270°	300°	330°	360°
$\sin\theta$	0	−0.5	−0.87	−1	−0.87	−0.5	0

b

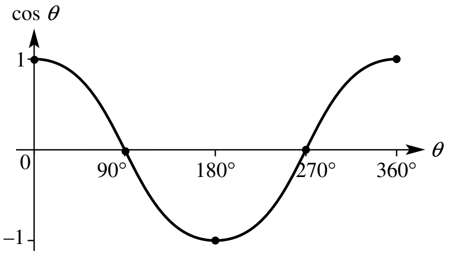


2 a

θ	0°	30°	60°	90°	120°	150°
$\cos\theta$	1	0.87	0.5	0	−0.5	−0.87

θ	180°	210°	240°	270°	300°	330°	360°
$\cos\theta$	−1	−0.87	−0.5	0	0.5	0.87	1

b



3 a i maximum = 1, minimum = −1
ii 0°, 180°, 360°
b i maximum = 1, minimum = −1
ii 90°, 270°
c i $90° < \theta < 270°$ ii $180° < \theta < 360°$

4 a i 0.82 ii −0.98 iii 0.87 iv −0.77
v −0.17 vi 0.26 vii −0.42 viii 0.57
b i 37°, 323° ii 53°, 307° iii 73°, 287°
iv 84°, 276° v 114°, 246° vi 102°, 258°
vii 143°, 217° viii 127°, 233°
c Graph not shown for this question

5 a i 0.42 ii 0.91 iii −0.64
iv −0.77 v 0.34 vi −0.82
vii −0.64 viii 0.94
b i 37°, 143° ii 12°, 168° iii 17°, 163°
iv 64°, 116° v 204°, 336° vi 233°, 307°
vii 224°, 316° viii 186°, 354°
c Graph not shown for this question

6 a true b false c false d true
e false f true g true h true
i true j false k true l true

7 a 110° b 60° c 350° d 260°
e 27° f 326° g 233° h 357°

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8 a 280° b 350° c 195° d 75°
e 136° f 213° g 24° h 161°

9 a 30° b 60° c 15° d 70°
e 55° f 80° g 55° h 25°
i 36° j 72° k 63° l 14°

10 a

θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

b i $\frac{1}{2}$ ii $-\frac{1}{2}$
iii $-\frac{\sqrt{2}}{2}$ iv 0
v $\frac{1}{2}$ vi $-\frac{\sqrt{3}}{2}$
vii 0 viii $\frac{\sqrt{2}}{2}$
ix $-\frac{\sqrt{3}}{2}$ x $-\frac{1}{2}$
xi $-\frac{\sqrt{2}}{2}$ xii $-\frac{1}{2}$
xiii $-\frac{\sqrt{2}}{2}$ xiv $\frac{\sqrt{3}}{2}$
xv $-\frac{\sqrt{3}}{2}$ xvi $\frac{\sqrt{3}}{2}$

11 a 45°, 315° b 60°, 120° c 30°, 150°
d 210°, 330° e 120°, 240° f 150°, 210°

12 a 17.5°, 162.5° b 44.4°, 135.6° c 53.1°, 306.9°
d 36.9°, 323.1° e 191.5°, 348.5° f 233.1°, 306.9°
g 113.6°, 246.4° h 49.5°, 310.5° i 28.7°, 151.3°

13 a 0, the maximum value of sin θ is 1.
b 0, the minimum value of cos θ is −1.

14 a Graph is reflected in the x-axis.
b Graph is reflected in the x-axis.
c Graph is dilated and constricted from the x-axis.
d Graph is dilated and constricted from the y-axis.
e Graph is translated up and down from the x-axis.
f Graph is translated left and right from the y-axis.

Puzzles and challenges

1 a 120°, 60° b 8.7 cm
2 225°
3 Use the cosine rule.
4 514 m
5 a 2 h 9 min b 308°
6 17.93°

Multiple-choice questions

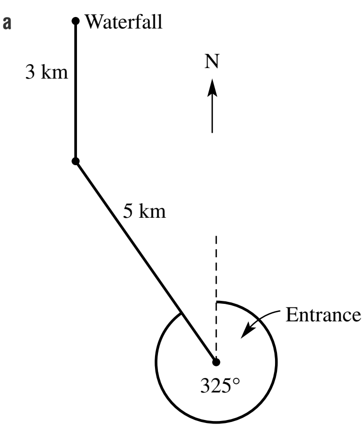
1 D 2 B 3 E 4 D 5 A
6 B 7 C 8 A 9 D 10 C

Short-answer questions

1 a 14.74 b 13.17
c $x = 11.55, y = 5.42$
2 a 45.6° b 64.8°
3 a 4 b $5\sqrt{3}$
4 6.1 m
5 $A = 115°, B = 315°, C = 250°, D = 30°$
6 a 98.3 km b 228.8 km c 336.8°
7 a 15.43 m b 52°
8 a i 15.5 cm ii 135.0 cm^2
b i 14.9 cm ii 111.3 cm^2
9 28.1 m
10 a 52.6° b 105.4°
11 a 12.5 b 42.8°
12 a i sin 60° ii −cos 30°
iii −tan 45° iv −sin 45°
b i $\frac{\sqrt{3}}{2}$ ii $-\frac{\sqrt{3}}{2}$
iii −1 iv $-\frac{\sqrt{2}}{2}$
c i negative ii positive
iii negative iv positive
13 a i 0.77 ii −0.97
b i 53°, 127 ii 197°, 343°
iii no value
c i true ii true
iii false

Extended-response questions

1 a



b 2.9 km west c 7.7 km
d i 21.9 m ii 38.0°
2 a 33.646° b 3177.54 m^2
c 41.00 m d 61.60 m
e i 65.66°, 114.34° ii 80.2 m, 43.1 m

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Chapter 8

Pre-test

1 a 3 b 2 c 5
2 a $4x + 2y$ b $2xy - x$ c $3x^2 - 3y^2$
3 a $2a$ b $-2m$ c $18a^2$
d $-6x^2y$ e $-\frac{1}{3}$ f $-\frac{6x}{y}$
4 a $4m + 4n$ b $-6x + 12$ c $6x^2 + 2x$ d $4a - 8a^2$
e $3x - 7$ f $-2x + 1$ g $7x + 10$ h $3x - 21$
5 a $x^2 + 2x - 3$ b $x^2 - 10x + 21$ c $6x^2 - x - 12$
d $35x^2 + 19x + 2$ e $16x^2 - 25$ f $100 - m^2$
g $x^2 + 14x + 49$ h $4x^2 - 12x + 9$
6 a $7(x + 1)$ b $-9x(1 + 3x)$ c $a(a + b + 3)$
d $(x + 3)(x - 3)$ e $(x + 5)(x + 4)$ f $(x - 2)(x + 7)$
g $(x - 1)(2x + 1)$ h $(x - 2)(5x - 1)$ i $(2x + 1)^2$
7 a $\frac{1}{2}$ b $\frac{x + 4}{2}$ c $\frac{3}{4}$
8 a 1 b -23 c 9 d 1
9 a $-\frac{1}{2}$ b 3 c $-\frac{5}{2}$
d $0, 5$ e $-2, 3$ f $\frac{1}{2}, -7$

Exercise 8A

1 a $x^2 + 2x$ b $x^2 + 4x + 3$ c $x^2 + 8x + 16$
2 a $a^2 + ab$ b $2x - 5$ c $a^2 - b^2$
d $(y + x)^2$ e $\frac{a - b}{2}$
3 a $6x$ b $-20x$ c $2x^2$
d $-4x^2$ e $\frac{x}{2}$ f $\frac{x}{3}$
g $-4x$ h $-3x$ i $-18x$
j $7x$ k $5x$ l $-13x$
4 a $2x + 10$ b $3x - 12$ c $-5x - 15$
d $-4x + 8$ e $6x - 3$ f $12x + 4$
g $-10x + 6$ h $-20x - 15$ i $2x^2 + 5x$
j $3x^2 - x$ k $2x - 2x^2$ l $6x - 3x^2$
m $-6x^2 - 4x$ n $-18x^2 + 6x$ o $-10x + 10x^2$
p $-4x + 16x^2$ q $4x + \frac{8}{5}$ r $6x - \frac{15}{4}$
s $-2x - \frac{1}{3}$ t $-2x + \frac{3}{2}$ u $-9x + \frac{3}{8}$
v $-2x - \frac{14}{9}$ w $\frac{9}{4}x^2 + 6x$ x $\frac{14}{5}x - \frac{6}{5}x^2$
5 a $2x^2 + 3x$ b $6x^2 - 3x$ c $2x^2 + 7x$
d $8x^2 + 7x$ e $2x^2 - 2x$ f $25x - 12x^2$
6 a $x^2 + 10x + 16$ b $x^2 + 7x + 12$ c $x^2 + 12x + 35$
d $x^2 + 5x - 24$ e $x^2 + x - 30$ f $x^2 + x - 6$
g $x^2 - 4x - 21$ h $x^2 - 10x + 24$ i $x^2 - 13x + 40$
j $x^2 + 10x + 25$ k $x^2 + 14x + 49$ l $x^2 + 12x + 36$
m $x^2 - 6x + 9$ n $x^2 - 16x + 64$ o $x^2 - 20x + 100$
p $x^2 - 16$ q $x^2 - 81$ r $4x^2 - 9$
s $9x^2 - 16$ t $16x^2 - 25$ u $64x^2 - 49$
7 a $6x^2 + 13x + 5$ b $12x^2 + 23x + 10$ c $10x^2 + 41x + 21$
d $9x^2 - 9x - 10$ e $20x^2 + 2x - 6$ f $6x^2 + 5x - 25$
g $16x^2 - 25$ h $4x^2 - 81$ i $25x^2 - 49$
j $14x^2 - 34x + 12$ k $25x^2 - 45x + 18$ l $56x^2 - 30x + 4$
m $4x^2 + 20x + 25$ n $25x^2 + 60x + 36$ o $49x^2 - 14x + 1$
8 a 3 b 3 c 3
d 8 e 1 f 2
9 a $2x^2 + 14x + 24$ b $3x^2 + 27x + 42$
c $-2x^2 - 20x - 32$ d $-4x^2 - 44x - 72$
e $5x^2 + 5x - 60$ f $3x^2 + 6x - 45$
g $-3a^2 + 15a + 42$ h $-5a^2 + 30a + 80$
i $4a^2 - 36a + 72$ j $3y^2 - 27y + 60$
k $-2y^2 + 22y - 48$ l $-6y^2 + 42y - 72$
m $12x^2 + 48x + 45$ n $18x^2 + 12x - 48$
o $-6x^2 - 10x + 56$ p $2x^2 + 12x + 18$
q $4m^2 + 40m + 100$ r $2a^2 - 28a + 98$
s $-3y^2 + 30y - 75$ t $12b^2 - 12b + 3$
u $-12y^2 + 72y - 108$
10 a $x^2 + 8x + 8$ b $x^2 + 9x - 10$ c $5x^2 - 5x + 2$
d $2x^2 + 10x + 11$ e $2x^2 + 20x + 44$ f $2y^2 - 4y + 5$
g $2y^2 - y - 43$ h $-9x^2 - 3x + 2$ i $2x^2 + 10x$
j $-2x^2 + 2x + 10$ k $-24a - 45$ l $b^2 + 54b + 5$
m $-8x + 14$ n $2x + 1$ o $x^2 + 10x + 18$
p $x^2 - 14x + 40$ q $-4x^2 + 36x - 78$ r $-25x^2 - 30x + 5$
11 a $x^2 - 12x + 36\text{ cm}^2$ b $x^2 + 10x - 200\text{ cm}^2$
12 a $(a + b)(a - b) = a^2 - ab + ba - b^2 = a^2 - b^2$
b $(a + b)^2 = (a + b)(a + b) = a^2 + ab + ba + b^2$
$= a^2 + 2ab + b^2$
c $(a - b)^2 = (a - b)(a - b) = a^2 - ab - ba + b^2$
$= a^2 - 2ab + b^2$
d $(a + b)^2 - (a - b)^2 = a^2 + ab + ba + b^2$
$- (a^2 - ab - ba + b^2) = 2ab + 2ab = 4ab$
13 a 618 b 220 c 567
d 1664 e 1386 f 891
g 3960 h 3480
14 a $-x^2 + 7x$ b $10a - 28$ c $4x^2 + 12x + 9$ d $4x + 8$
15 a $x^3 + 6x^2 + 11x + 6$
b $x^3 + 11x^2 + 38x + 40$
c $x^3 + 2x^2 - 15x - 36$
d $2x^3 - 13x^2 + 17x + 12$
e $2x^3 - x^2 - 63x + 90$
f $6x^3 - 35x^2 + 47x - 12$
16 a $2ab$ b $(a + b)^2 - c^2$
c $(a + b)^2 - c^2 = 2ab$
$c^2 = a^2 + 2ab + b^2 - 2ab$
$c^2 = a^2 + b^2$

Exercise 8B

1 a 7 b 6 c 8 d -5
e $2a$ f $3a$ g $-5a$ h $-3xy$
2 a $3(x - 6)$ b $4(x + 5)$ c $7(a + b)$
d $3(3a - 5)$ e $-5(x + 6)$ f $-2(2y + 1)$
g $-3(4a + 1)$ h $-b(2a + c)$ i $x(4x + 1)$

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j $x(5x-2)$ **k** $6b(b-3)$ **l** $7a(2a-3)$
m $5a(2-a)$ **n** $6x(2-5x)$ **o** $-x(2+x)$
p $-4y(1+2y)$ **q** $ab(b-a)$ **r** $2xy(xz-2)$
s $-12mn(m+n)$ **t** $3z^2(2xy-1)$

3 a $(x-1)(5-a)$ **b** $(x+2)(b+3)$ **c** $(x+5)(a-4)$
d $(x+2)(x+5)$ **e** $(x-4)(x-2)$ **f** $(x+1)(3-x)$
g $(x+3)(a+1)$ **h** $(x-2)(x-1)$ **i** $(x-6)(1-x)$

4 a $(x+3)(x-3)$ **b** $(x+5)(x-5)$
c $(y+7)(y-7)$ **d** $(y+1)(y-1)$
e $(2x-3)(2x+3)$ **f** $(6a-5)(6a+5)$
g $(1+9y)(1-9y)$ **h** $(10-3x)(10+3x)$
i $(5x-2y)(5x+2y)$ **j** $(8x-5y)(8x+5y)$
k $(3a+7b)(3a-7b)$ **l** $(12a-7b)(12a+7b)$

5 a $2(x+4)(x-4)$ **b** $5(x+3)(x-3)$
c $6(y+2)(y-2)$ **d** $3(y+4)(y-4)$
e $3(x+5y)(x-5y)$ **f** $3(a+10b)(a-10b)$
g $3(2x+3y)(2x-3y)$ **h** $7(3a+4b)(3a-4b)$
i $(x+9)(x+1)$ **j** $(x-7)(x-1)$
k $(a+5)(a-11)$ **l** $(a-8)(a-6)$
m $(4x+5)(2x+5)$ **n** $(y+7)(3y+7)$
o $(3x+11)(7x+11)$ **p** $3x(3x-10y)$

6 a $(x+\sqrt{7})(x-\sqrt{7})$ **b** $(x+\sqrt{5})(x-\sqrt{5})$
c $(x+\sqrt{19})(x-\sqrt{19})$ **d** $(x+\sqrt{21})(x-\sqrt{21})$
e $(x+\sqrt{14})(x-\sqrt{14})$ **f** $(x+\sqrt{30})(x-\sqrt{30})$
g $(x+\sqrt{15})(x-\sqrt{15})$ **h** $(x+\sqrt{11})(x-\sqrt{11})$
i $(x+2\sqrt{2})(x-2\sqrt{2})$ **j** $(x+3\sqrt{2})(x-3\sqrt{2})$
k $(x+3\sqrt{5})(x-3\sqrt{5})$ **l** $(x+2\sqrt{5})(x-2\sqrt{5})$
m $(x+4\sqrt{2})(x-4\sqrt{2})$ **n** $(x+4\sqrt{3})(x-4\sqrt{3})$
o $(x+5\sqrt{2})(x-5\sqrt{2})$ **p** $(x+10\sqrt{2})(x-10\sqrt{2})$
q $(x+2+\sqrt{6})(x+2-\sqrt{6})$
r $(x+5+\sqrt{10})(x+5-\sqrt{10})$
s $(x-3+\sqrt{11})(x-3-\sqrt{11})$
t $(x-1+\sqrt{7})(x-1-\sqrt{7})$
u $(x-6+\sqrt{15})(x-6-\sqrt{15})$
v $(x+4+\sqrt{21})(x+4-\sqrt{21})$
w $(x+1+\sqrt{19})(x+1-\sqrt{19})$
x $(x-7+\sqrt{26})(x-7-\sqrt{26})$

7 a $(x+4)(x+a)$ **b** $(x+7)(x+b)$ **c** $(x-3)(x+a)$
d $(x+2)(x-a)$ **e** $(x+5)(x-b)$ **f** $(x+3)(x-4a)$
g $(x-a)(x-4)$ **h** $(x-2b)(x-5)$ **i** $(x-2a)(3x-7)$

8 a $\left(x+\frac{\sqrt{2}}{3}\right)\left(x-\frac{\sqrt{2}}{3}\right)$ **b** $\left(x+\frac{\sqrt{3}}{2}\right)\left(x-\frac{\sqrt{3}}{2}\right)$
c $\left(x+\frac{\sqrt{7}}{4}\right)\left(x-\frac{\sqrt{7}}{4}\right)$ **d** $\left(x+\frac{\sqrt{5}}{6}\right)\left(x-\frac{\sqrt{5}}{6}\right)$
e $(x-2+2\sqrt{5})(x-2-2\sqrt{5})$
f $(x+4+3\sqrt{3})(x+4-3\sqrt{3})$
g $(x+1+5\sqrt{3})(x+1-5\sqrt{3})$
h $(x-7+2\sqrt{10})(x-7-2\sqrt{10})$
i $(\sqrt{3}x+2)(\sqrt{3}x-2)$
j $(\sqrt{5}x+3)(\sqrt{5}x-3)$
k $(\sqrt{7}x+\sqrt{5})(\sqrt{7}x-\sqrt{5})$
l $(\sqrt{6}x+\sqrt{11})(\sqrt{6}x-\sqrt{11})$
m $(\sqrt{2}x+3)(\sqrt{2}x-3)$
n $(\sqrt{5}x+4)(\sqrt{5}x-4)$
o $(\sqrt{3}x+\sqrt{10})(\sqrt{3}x-\sqrt{10})$
p $(\sqrt{13}x+\sqrt{7})(\sqrt{13}x-\sqrt{7})$

9 a $(x+2)(y-3)$ **b** $(a-4)(x+3)$
c $(a+5)(x-2)$ **d** $(y-4)(x-3)$
e $(a-3)(2x-1)$ **f** $(2a-5)(x+4)$

10 a $5(x+2\sqrt{6})(x-2\sqrt{6})$ **b** $3(x+3\sqrt{6})(x-3\sqrt{6})$
c $7(x+3\sqrt{2})(x-3\sqrt{2})$ **d** $2(x+4\sqrt{3})(x-4\sqrt{3})$
e $2(x+3+\sqrt{5})(x+3-\sqrt{5})$
f $3(x-1+\sqrt{7})(x-1-\sqrt{7})$
g $4(x-4+2\sqrt{3})(x-4-2\sqrt{3})$
h $5(x+6+3\sqrt{2})(x+6-3\sqrt{2})$

11 a 60 **b** 35 **c** 69 **d** 104
e 64 **f** 40 **g** 153 **h** 1260

12 a $4-(x+2)^2=(2-(x+2))(2+(x+2))=-x(x+4)$
b i $-x(x+6)$ **ii** $-x(x+8)$
iii $x(10-x)$ **iv** $(3-x)(7+x)$
v $(8-x)(6+x)$ **vi** $(6-x)(14+x)$

13 a $(x+a)^2=x^2+2ax+a^2\neq x^2+a^2$
b If $x=0$, then $(x+a)^2=x^2+a^2$; or if $a=0$, then $(x+a)^2=x^2+a^2$ is true for all real values of x.

14 $x^2-\frac{4}{9}=\frac{1}{9}(9x^2-4)=\frac{1}{9}(3x+2)(3x-2)$

or $x^2-\frac{4}{9}=\left(x+\frac{2}{3}\right)\left(x-\frac{2}{3}\right)$

$=\frac{1}{3}(3x+2)\frac{1}{3}(3x-2)$

$=\frac{1}{9}(3x+2)(3x-2)$

15 a $-(2x+5)$ **b** $-11(2y-3)$
c $16(a-1)$ **d** $20b$
e $-12s$ **f** $-28y$
g $(5w+7x)(-w-x)$ **h** $(4d+3e)(-2d+7e)$
i $12f(f+3j)$ **j** 0

16 a x^2+5y-y^2+5x
$=x^2-y^2+5x+5y$
$=(x-y)(x+y)+5(x+y)$
$=(x+y)(x-y+5)$
b i $(x+y)(x-y+7)$
ii $(x+y)(x-y-2)$
iii $(2x+3y)(2x-3y+2)$
iv $(5y+2x)(5y-2x+3)$

Exercise 8C

1 a 9, 2 **b** 10, 2 **c** 5, –3
d 4, –3 **e** –8, 3 **f** –10, 3
g –2, –5 **h** –12, –3

2 a Possible answer: $\frac{x-10}{x-10}=1$
b Possible answer: $\frac{3(x-7)}{x-7}=3$

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c Possible answer: $\dfrac{-5(x+3)}{x+3} = -5$

d Possible answer: $\dfrac{x+4}{3(x+4)} = \dfrac{1}{3}$

3 **a** $\frac{x}{2}$ **b** $\frac{x}{3}$ **c** 3 **d** $\frac{1}{5}$
e $\frac{1}{3}$ **f** $\frac{1}{4}$ **g** 5 **h** $\frac{2}{3}$
i $x+1$ **j** $x-2$ **k** $\frac{x-3}{2}$ **l** $1-2x$

4 **a** $(x+6)(x+1)$ **b** $(x+3)(x+2)$
c $(x+3)^2$ **d** $(x+5)(x+2)$
e $(x+4)(x+3)$ **f** $(x+9)(x+2)$
g $(x-1)(x+6)$ **h** $(x+3)(x-2)$
i $(x+4)(x-2)$ **j** $(x-1)(x+4)$
k $(x+10)(x-3)$ **l** $(x+11)(x-2)$
m $(x-2)(x-5)$ **n** $(x-4)(x-2)$
o $(x-4)(x-3)$ **p** $(x-1)^2$
q $(x-6)(x-3)$ **r** $(x-2)(x-9)$
s $(x-6)(x+2)$ **t** $(x-5)(x+4)$
u $(x-7)(x+2)$ **v** $(x-4)(x+3)$
w $(x+8)(x-4)$ **x** $(x-5)(x+2)$

5 **a** $2(x+5)(x+2)$ **b** $3(x+4)(x+3)$
c $2(x+9)(x+2)$ **d** $5(x-2)(x+1)$
e $4(x-5)(x+1)$ **f** $3(x-5)(x+2)$
g $-2(x+4)(x+3)$ **h** $-3(x-2)(x-1)$
i $-2(x-7)(x+2)$ **j** $-4(x-2)(x+1)$
k $-5(x+3)(x+1)$ **l** $-7(x-6)(x-1)$

6 **a** $(x-2)^2$ **b** $(x+3)^2$ **c** $(x+6)^2$
d $(x-7)^2$ **e** $(x-9)^2$ **f** $(x-10)^2$
g $2(x+11)^2$ **h** $3(x-4)^2$ **i** $5(x-5)^2$
j $-3(x-6)^2$ **k** $-2(x-7)^2$ **l** $-4(x+9)^2$

7 **a** $x+6$ **b** $x-3$ **c** $x-3$
d $\frac{1}{x+7}$ **e** $\frac{1}{x-5}$ **f** $\frac{1}{x-6}$
g $\frac{2}{x-8}$ **h** $\frac{x+4}{3}$ **i** $\frac{x-7}{5}$

8 **a** $\frac{5(x+2)(x-3)}{(x+3)(x+6)(x-2)}$ **b** $\frac{x-3}{3}$
c $\frac{2(x-1)}{x+5}$ **d** $\frac{4}{x+5}$
e $\frac{4}{x+7}$ **f** $\frac{6}{x-2}$
g $\frac{x+2}{x-1}$ **h** $\frac{x-4}{x+6}$

9 **a** $x-\sqrt{7}$ **b** $x+\sqrt{10}$ **c** $x-2\sqrt{3}$
d $\frac{1}{\sqrt{5}x-3}$ **e** $\frac{1}{\sqrt{3}x+4}$ **f** $\sqrt{7}x-\sqrt{5}$
g $x+1-\sqrt{2}$ **h** $x-3+\sqrt{5}$ **i** $x-6-\sqrt{6}$

10 **a** $\frac{2(x+3)}{3(x-5)}$ **b** $\frac{x-3}{4}$ **c** $\frac{3}{x-3}$
d $\frac{3}{2}$ **e** $\frac{x-2}{x+3}$ **f** $\frac{x+3}{x-1}$

11 $\dfrac{t^2-49}{5t-40} \times \dfrac{t^2-5t-24}{2t^2-8t-42} = \dfrac{\cancel{(t-7)}(t+7)}{5\cancel{(t-8)}} \times \dfrac{\cancel{(t-8)}\cancel{(t+3)}}{2\cancel{(t-7)}\cancel{(t+3)}}$
$= \dfrac{t+7}{10}$

12 **a** $x-3$ **b** $x+1$ **c** $x-8$
d $\frac{6}{x-2}$ **e** $\frac{4}{x+5}$ **f** $\frac{x-7}{5}$

13 **a** $\dfrac{a^2+2ab+b^2}{a^2+ab} \times \dfrac{a^2-ab}{a^2-b^2}$
$= \dfrac{\cancel{(a+b)^2}}{\cancel{a(a+b)}} \times \dfrac{\cancel{a(a-b)}}{\cancel{(a-b)(a+b)}}$
$= 1$

b Answer will vary.

14 **a** $\frac{a-b}{a}$ **b** 1
c $\frac{(a+b)^2}{(a-b)^2}$ **d** $\frac{(a+b)(a-b)}{a^2}$

15 **a** $\frac{3x-8}{(x+3)(x-4)}$ **b** $\frac{7x-36}{(x+2)(x-9)}$
c $\frac{x-12}{(x+4)(x-4)}$ **d** $\frac{3x-23}{(x+3)(x-3)(x-5)}$
e $\frac{x-14}{(x-3)(x+2)(x-6)}$ **f** $\frac{14x+9}{(x+3)(x+4)(x-8)}$
g $\frac{9-3x}{(x+5)(x-5)(x-1)}$ **h** $\frac{4x+11}{(x-1)^2(x+4)}$

Exercise 8D

1

ax^2+bx+c	$a \times c$	Two numbers that multiply to give $a \times c$ and add to give b	Factorisation
$6x^2+13x+6$	36	9 and 4	$(3x+2)(2x+3)$
$8x^2+18x+4$	32	16 and 2	$(4x+1)(2x+4)$
$12x^2+x-6$	−72	−8 and 9	$(4x+3)(3x-2)$
$10x^2-11x-6$	−60	−15 and 4	$(5x+2)(2x-3)$
$21x^2-20x+4$	84	−6 and −14	$(7x-2)(3x-2)$
$15x^2-13x+2$	30	−3 and −10	$(5x-1)(3x-2)$

2 **a** $(x+2)(x+5)$ **b** $(x+4)(x+6)$
c $(x+3)(x+7)$ **d** $(x-7)(x-2)$
e $(x-3)(x-4)$ **f** $(x-5)(x+3)$
g $(3x-4)(2x+1)$ **h** $(x-4)(3x+2)$
i $(2x-1)(4x+3)$ **j** $(x+4)(5x-2)$
k $(5x+6)(2x-3)$ **l** $(2x-1)(6x-5)$

3 **a** $(3x+1)(x+3)$ **b** $(2x+1)(x+1)$
c $(3x+2)(x+2)$ **d** $(3x-2)(x-1)$
e $(2x-1)(x-5)$ **f** $(5x-3)(x+1)$
g $(3x+1)(x-4)$ **h** $(3x+1)(x-1)$
i $(7x-5)(x+1)$ **j** $(2x-7)(x-1)$
k $(3x-4)(x+2)$ **l** $(2x-3)(x+4)$
m $(2x+1)(x-5)$ **n** $(13x+6)(x-1)$
o $(5x-2)(x-4)$ **p** $(4x-5)(2x-1)$
q $(3x-4)(2x+3)$ **r** $(5x-2)(2x+3)$
s $(3x+2)(2x+3)$ **t** $(4x-1)(x-1)$

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u $(4x-5)(2x-1)$ v $(2x-5)(4x-3)$
w $(3x-2)(2x-3)$ x $(3x-2)(3x+5)$

4 a $(6x+5)(3x+2)$ b $(4x+3)(5x+6)$
c $(7x-2)(3x+4)$ d $(5x-2)(6x+5)$
e $(8x+3)(5x-2)$ f $(7x+2)(4x-3)$
g $(6x-5)(4x-3)$ h $(9x-2)(5x-4)$
i $(5x-2)(5x-8)$

5 a $2(3x+4)(x+5)$ b $3(2x+3)(x-4)$
c $3(8x+1)(2x-1)$ d $4(4x-5)(2x-3)$
e $8(2x-1)(x-1)$ f $10(3x-2)(3x+5)$
g $-5(5x+4)(2x+3)$ h $3(2x-3)^2$
i $5(4x-1)(x-1)$

6 a $2x-5$ b $4x-1$ c $3x-2$
d $\frac{2}{3x+2}$ e $\frac{2}{7x-2}$ f $\frac{4}{2x-3}$
g $\frac{x+4}{3x+1}$ h $\frac{3x-1}{2x+3}$ i $\frac{5x+4}{7x-2}$
j $\frac{3x-2}{5x-2}$ k $\frac{2x+3}{7x+1}$ l $\frac{2x-3}{4x-5}$

7 a $3(x-5)(x-2)$
b -6 m. The cable is 6 m below the water.
c $x=2$ or $x=5$

8 a $\frac{3x+4}{x-3}$ b $\frac{3x+2}{4}$
c $\frac{1-x}{3}$ d $\frac{4x-3}{5x+1}$
e $\frac{5(2x+1)(2x-1)}{(x+5)(x-5)}$ f $\frac{x+2}{5}$
g 1 h $\frac{(4x-5)^2}{(x-3)^2}$

9 $-12x^2-5x+3$
$=-(12x^2+5x-3)$
$=-(3x-1)(4x+3)$
$=(1-3x)(4x+3)$
a $(3-2x)(4x+5)$ b $(5-2x)(3x+2)$
c $(4-3x)(4x+1)$ d $(3-4x)(2x-3)$
e $(2-7x)(2x-5)$ f $(3-5x)(3x+2)$

11 a $\frac{9x+2}{(2x-3)(4x+1)}$ b $\frac{5x+15}{(3x-1)(2x+5)}$
c $\frac{16x^2+5x}{(2x-5)(4x+1)}$ d $\frac{7x-12x^2}{(3x-2)(4x-1)}$
e $\frac{8x-5}{(2x+1)(2x-1)(3x-2)}$ f $\frac{11-3x}{(3x+5)(3x-5)(3x-2)}$
g $\frac{2}{(2x-5)(3x-2)}$ h $\frac{12x+3}{(5x-2)(2x-3)(2x+7)}$

Exercise 8E

1 a 9 b 36 c 1
d 4 e 16 f 25
g $\frac{25}{4}$ h $\frac{9}{4}$ i $\frac{81}{4}$

2 a $(x+2)^2$ b $(x+4)^2$ c $(x+5)^2$
d $(x-6)^2$ e $(x-3)^2$ f $(x-9)^2$

3 a $(x+1+\sqrt{5})(x+1-\sqrt{5})$
b $(x+2+\sqrt{7})(x+2-\sqrt{7})$
c $(x+4+\sqrt{10})(x+4-\sqrt{10})$
d $(x-3+\sqrt{11})(x-3-\sqrt{11})$
e $(x-6+\sqrt{22})(x-6-\sqrt{22})$
f $(x-5+\sqrt{3})(x-5-\sqrt{3})$

4 a $9, (x+3)^2$ b $36, (x+6)^2$
c $4, (x+2)^2$ d $16, (x+4)^2$
e $25, (x-5)^2$ f $1, (x-1)^2$
g $16, (x-4)^2$ h $36, (x-6)^2$
i $\frac{25}{4}, \left(x+\frac{5}{2}\right)^2$ j $\frac{81}{4}, \left(x+\frac{9}{2}\right)^2$
k $\frac{49}{4}, \left(x+\frac{7}{2}\right)^2$ l $\frac{121}{4}, \left(x+\frac{11}{2}\right)^2$
m $\frac{9}{4}, \left(x-\frac{3}{2}\right)^2$ n $\frac{49}{4}, \left(x-\frac{7}{2}\right)^2$
o $\frac{1}{4}, \left(x-\frac{1}{2}\right)^2$ p $\frac{81}{4}, \left(x-\frac{9}{2}\right)^2$

5 a $(x+2+\sqrt{3})(x+2-\sqrt{3})$
b $(x+3+\sqrt{7})(x+3-\sqrt{7})$
c $(x+1+\sqrt{5})(x+1-\sqrt{5})$
d $(x+5+\sqrt{29})(x+5-\sqrt{29})$
e $(x-4+\sqrt{3})(x-4-\sqrt{3})$
f $(x-6+\sqrt{26})(x-6-\sqrt{26})$
g $(x-2+\sqrt{7})(x-2-\sqrt{7})$
h $(x-4+\sqrt{21})(x-4-\sqrt{21})$

6 a not possible b not possible
c $(x+4+\sqrt{15})(x+4-\sqrt{15})$
d $(x+2+\sqrt{2})(x+2-\sqrt{2})$
e $(x+5+\sqrt{22})(x+5-\sqrt{22})$
f $(x+2+\sqrt{10})(x+2-\sqrt{10})$
g not possible
h $(x-3+\sqrt{3})(x-3-\sqrt{3})$
i $(x-6+\sqrt{34})(x-6-\sqrt{34})$
j not possible
k $(x-4+\sqrt{17})(x-4-\sqrt{17})$
l not possible

7 a $\left(x+\frac{3+\sqrt{5}}{2}\right)\left(x+\frac{3-\sqrt{5}}{2}\right)$
b $\left(x+\frac{7+\sqrt{41}}{2}\right)\left(x+\frac{7-\sqrt{41}}{2}\right)$
c $\left(x+\frac{5+\sqrt{33}}{2}\right)\left(x+\frac{5-\sqrt{33}}{2}\right)$
d $\left(x+\frac{9+\sqrt{93}}{2}\right)\left(x+\frac{9-\sqrt{93}}{2}\right)$

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e $\left(x - \frac{3 + \sqrt{7}}{2}\right)\left(x - \frac{3 - \sqrt{7}}{2}\right)$

f $\left(x - \frac{5 + \sqrt{23}}{2}\right)\left(x - \frac{5 - \sqrt{23}}{2}\right)$

g $\left(x - \frac{5 + \sqrt{31}}{2}\right)\left(x - \frac{5 - \sqrt{31}}{2}\right)$

h $\left(x - \frac{9 + \sqrt{91}}{2}\right)\left(x - \frac{9 - \sqrt{91}}{2}\right)$

8 a $2(x + 3 + \sqrt{5})(x + 3 - \sqrt{5})$

b $3(x + 2 + \sqrt{5})(x + 2 - \sqrt{5})$

c $4(x - 1 + \sqrt{5})(x - 1 - \sqrt{5})$

d $3(x - 4 + \sqrt{14})(x - 4 - \sqrt{14})$

e $-2(x + 1 + \sqrt{6})(x + 1 - \sqrt{6})$

f $-3(x + 5 + 2\sqrt{6})(x + 5 - 2\sqrt{6})$

g $-4(x + 2 + \sqrt{7})(x + 2 - \sqrt{7})$

h $-2(x - 4 + 3\sqrt{2})(x - 4 - 3\sqrt{2})$

i $-3(x - 4 + \sqrt{11})(x - 4 - \sqrt{11})$

9 a $3\left(x + \frac{3 + \sqrt{5}}{2}\right)\left(x + \frac{3 - \sqrt{5}}{2}\right)$

b $5\left(x + \frac{3 + \sqrt{37}}{2}\right)\left(x + \frac{3 - \sqrt{37}}{2}\right)$

c $2\left(x - \frac{5 + \sqrt{17}}{2}\right)\left(x - \frac{5 - \sqrt{17}}{2}\right)$

d $4\left(x - \frac{7 + \sqrt{37}}{2}\right)\left(x - \frac{7 - \sqrt{37}}{2}\right)$

e $-3\left(x + \frac{7 + \sqrt{57}}{2}\right)\left(x + \frac{7 - \sqrt{57}}{2}\right)$

f $-2\left(x + \frac{7 + \sqrt{65}}{2}\right)\left(x + \frac{7 - \sqrt{65}}{2}\right)$

g $-4\left(x - \frac{3 + \sqrt{29}}{2}\right)\left(x - \frac{3 - \sqrt{29}}{2}\right)$

h $-3\left(x - \frac{3 + \sqrt{17}}{2}\right)\left(x - \frac{3 - \sqrt{17}}{2}\right)$

i $-2\left(x - \frac{5 + \sqrt{41}}{2}\right)\left(x - \frac{5 - \sqrt{41}}{2}\right)$

10 a $x^2 - 2x - 24$
$= x^2 - 2x + (-1)^2 - (-1)^2 - 24$
$= (x - 1)^2 - 25$
$= (x - 1 + 5)(x - 1 - 5)$
$= (x + 4)(x - 6)$

b Using a quadratic trinomial and finding two numbers that multiply to -24 and add to -2.

11 a If the difference of two squares is taken, it involves the square root of a negative number.

b i yes ii yes iii no
iv no v no vi yes
vii yes viii no

c i $m \leqslant 4$ ii $m \leqslant 9$ iii $m \leqslant 25$

12 a $2(x + 4)\left(x - \frac{3}{2}\right)$

b $3\left(x + \frac{2 + \sqrt{13}}{3}\right)\left(x + \frac{2 - \sqrt{13}}{3}\right)$

c $4\left(x - \frac{7 + \sqrt{305}}{8}\right)\left(x - \frac{7 - \sqrt{305}}{8}\right)$

d Not able to be factorised.

e $-2\left(x + \frac{3 + \sqrt{41}}{4}\right)\left(x + \frac{3 - \sqrt{41}}{4}\right)$

f $-3\left(x + \frac{7 + \sqrt{13}}{6}\right)\left(x + \frac{7 - \sqrt{13}}{6}\right)$

g Not able to be factorised.

h $-2\left(x - \frac{3 + \sqrt{41}}{4}\right)\left(x - \frac{3 - \sqrt{41}}{4}\right)$

i $2(x - 1)\left(x + \frac{7}{2}\right)$

j $3\left(x + \frac{2 + \sqrt{19}}{3}\right)\left(x + \frac{2 - \sqrt{19}}{3}\right)$

k $-2\left(x + \frac{5}{2}\right)(x - 1)$

l $-3\left(x + \frac{4}{3}\right)(x + 1)$

Exercise 8F

1 a $0, -1$ b $0, 5$ c $0, 4$
d $3, -2$ e $-5, 4$ f $1, -1$
g $\sqrt{3}, -\sqrt{3}$ h $\sqrt{5}, -\sqrt{5}$ i $2\sqrt{2}, -2\sqrt{2}$
j $\frac{1}{2}, -\frac{7}{3}$ k $\frac{5}{4}, -\frac{2}{5}$ l $-\frac{3}{8}, -\frac{3}{4}$

2 a $x^2 + 2x - 3 = 0$ b $x^2 - 3x - 10 = 0$
c $x^2 - 5x + 6 = 0$ d $5x^2 - 2x - 7 = 0$
e $3x^2 - 14x + 8 = 0$ f $4x^2 + 4x - 3 = 0$
g $x^2 + x - 4 = 0$ h $2x^2 - 6x - 5 = 0$
i $x^2 - 4x + 12 = 0$ j $x^2 - 3x - 2 = 0$
k $3x^2 + 2x + 4 = 0$ l $x^2 + 3x - 6 = 0$

3 a 2 b 2 c 1 d 1 e 2
f 2 g 1 h 1 i 1

4 a $x = 0, 4$ b $x = 0, 3$ c $x = 0, -2$
d $x = 0, 4$ e $x = 0, 5$ f $x = 0, -2$
g $x = \sqrt{7}, -\sqrt{7}$ h $x = \sqrt{11}, -\sqrt{11}$ i $x = \sqrt{5}, -\sqrt{5}$
j $x = 0, 2$ k $x = 0, -5$ l $x = 0, -\frac{1}{7}$
m $x = 2, -2$ n $x = 3, -3$ o $x = 6, -6$

5 a $x = -2, -1$ b $x = -3, -2$ c $x = 2, 4$
d $x = 5, 2$ e $x = -6, 2$ f $x = -5, 3$
g $x = 5, -4$ h $x = 8, -3$ i $x = 4, 8$
j $x = -2$ k $x = -5$ l $x = 4$
m $x = 7$ n $x = 12$ o $x = -9$

6 a $x = -\frac{3}{2}, -4$ b $x = -\frac{1}{2}, -\frac{7}{2}$ c $x = 5, \frac{7}{2}$
d $x = \frac{1}{2}, 11$ e $x = -\frac{5}{3}, 3$ f $x = -\frac{3}{5}, 2$
g $x = \frac{4}{3}, -\frac{5}{2}$ h $x = \frac{3}{7}, -4$

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7 a $x = -2, -6$ b $x = -1, 11$ c $x = 3$
d $x = 2$ e $x = \frac{3}{2}, -2$ f $x = \frac{2}{3}, \frac{5}{2}$

8 a $x = 6, -4$ b $x = 8, -4$ c $x = 3$
d $x = -2, -5$ e $x = 5, 3$ f $x = 3, -3$
g $x = 4, -4$ h $x = -1, -9$ i $x = 5, -1$
j $x = -5$ k $x = 8$ l $x = 8, -8$
m $x = 3, -1$ n $x = -\frac{2}{3}, -4$ o $x = -\frac{1}{4}, -\frac{3}{2}$

9 a $x = 12, -7$ b $x = -5, 14$ c $x = -9, 2$
d $x = \frac{5}{2}, -4$ e $x = -\frac{4}{5}, 2$ f $x = 2, -\frac{5}{6}$
g $x = -3, 1$ h $x = 1, \frac{1}{2}$ i $x = 3, -2$

10 a $x = \pm 2, x = \pm 3$ b $x = \pm\sqrt{6}$
c no solutions

11 a i $x = 1, -2$ ii $x = 1, -2$
b no difference
c $3x^2 - 15x - 18 = 3(x^2 - 5x - 6)$ and, as seen in part **a**, the coefficient of 3 makes no difference when solving.

12 This is a perfect square $(x + 8)^2$, which has only 1 solution; i.e. $x = -8$.

13 The student has applied the null factor law incorrectly; i.e. when the product does not equal zero. Correct solution is:

$$x^2 - 2x - 8 = 7$$
$$x^2 - 2x - 15 = 0$$
$$(x - 5)(x + 3) = 0$$
$$x = 5 \text{ or } x = -3$$

14 a $x = -2, -1$ b $x = 1$ c $x = \frac{1}{2}, 5$
d $x = 8, -6$ e $x = -6, -2$ f $x = \frac{3}{2}, -4$
g $x = 8, -3$ h $x = 5, -3$ i $x = 2$
j $x = 4, -3$ k $x = 5, -2$ l $x = -5, 3$

Exercise 8G

1 b $x + 5$
c $x(x + 5) = 24$
d $x^2 + 5x - 24 = 0, x = -8, 3$
e breadth = 3 m, length = 8 m

2 a breadth = 6 m, length = 10 m
b breadth = 9 m, length = 7 m
c breadth = 14 mm, length = 11 mm

3 height = 8 cm, base = 6 cm

4 height = 2 m, base = 7 m

5 8 and 9 or −9 and −8

6 12 and 14

7 15 m

8 a 6 b 13 c 14

9 1 m

10 father 64, son 8

11 5 cm

12 a 55 b i 7 ii 13 iii 23

13 a 3.75 m b $t = 1$ second, 3 seconds
c The ball will reach this height both on the way up and on the way down.
d $t = 0$ seconds, 4 seconds
e $t = 2$ seconds
f The ball reaches a maximum height of 4 m.
g No, 4 metres is the maximum height. When $h = 5$, there is no solution.

14 a $x = 0, 100$
b The ball starts at the tee; i.e. at ground level, and hits the ground again 100 metres from the tee.
c $x = 2$ m or 98 m

15 5 m × 45 m

16 150 m × 200 m

Exercise 8H

1 a 1 b 9 c 100 d 625
e 4 f 25 g $\frac{25}{4}$ h $\frac{9}{4}$

2 a $(x + \sqrt{3})(x - \sqrt{3}) = 0$
b $(x + \sqrt{7})(x - \sqrt{7}) = 0$
c $(x + \sqrt{10})(x - \sqrt{10}) = 0$
d $(x + 1 + \sqrt{5})(x + 1 - \sqrt{5}) = 0$
e $(x + 3 + \sqrt{11})(x + 3 - \sqrt{11}) = 0$
f $(x - 1 + \sqrt{2})(x - 1 - \sqrt{2}) = 0$

3 a $x = \sqrt{2}, -\sqrt{2}$ b $x = \sqrt{7}, -\sqrt{7}$
c $x = \sqrt{10}, -\sqrt{10}$ d $x = 3 - \sqrt{5}, 3 + \sqrt{5}$
e $x = 4 - \sqrt{6}, 4 + \sqrt{6}$
f $x = -5 - \sqrt{14}, -5 + \sqrt{14}$

4 a $x = -3 - \sqrt{6}, -3 + \sqrt{6}$
b $x = -2 - \sqrt{2}, -2 + \sqrt{2}$
c $x = -5 - \sqrt{10}, -5 + \sqrt{10}$
d $x = -2 - \sqrt{6}, -2 + \sqrt{6}$
e $x = -4 - \sqrt{19}, -4 + \sqrt{19}$
f $x = -3 - \sqrt{14}, -3 + \sqrt{14}$
g $x = 4 - \sqrt{17}, 4 + \sqrt{17}$
h $x = 6 - \sqrt{39}, 6 + \sqrt{39}$
i $x = 1 - \sqrt{17}, 1 + \sqrt{17}$
j $x = 5 - \sqrt{7}, 5 + \sqrt{7}$
k $x = 3 - \sqrt{5}, 3 + \sqrt{5}$
l $x = 4 - \sqrt{7}, 4 + \sqrt{7}$
m $x = -3 - \sqrt{13}, -3 + \sqrt{13}$
n $x = -10 - \sqrt{87}, -10 + \sqrt{87}$
o $x = 7 - \sqrt{55}, 7 + \sqrt{55}$

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5 a $x = -4 - 2\sqrt{3}, -4 + 2\sqrt{3}$
b $x = -3 - 2\sqrt{2}, -3 + 2\sqrt{2}$
c $x = 5 - 2\sqrt{5}, 5 + 2\sqrt{5}$
d $x = 2 - 3\sqrt{2}, 2 + 3\sqrt{2}$
e $x = 5 - 2\sqrt{7}, 5 + 2\sqrt{7}$
f $x = -4 - 2\sqrt{6}, -4 + 2\sqrt{6}$
g $x = 1 - 4\sqrt{2}, 1 + 4\sqrt{2}$
h $x = -6 - 3\sqrt{6}, -6 + 3\sqrt{6}$
i $x = -3 - 5\sqrt{2}, -3 + 5\sqrt{2}$

6 a 2 b 2 c 0 d 0
e 0 f 2 g 2 h 0
i 0 j 2 k 2 l 0

7 a $x = \frac{-5 + \sqrt{17}}{2}, \frac{-5 - \sqrt{17}}{2}$
b $x = \frac{-3 + \sqrt{5}}{2}, \frac{-3 - \sqrt{5}}{2}$
c $x = \frac{-7 + \sqrt{29}}{2}, \frac{-7 - \sqrt{29}}{2}$
d $x = \frac{3 + \sqrt{17}}{2}, \frac{3 - \sqrt{17}}{2}$
e $x = \frac{1 + \sqrt{13}}{2}, \frac{1 - \sqrt{13}}{2}$
f $x = \frac{-5 + \sqrt{33}}{2}, \frac{-5 - \sqrt{33}}{2}$
g $x = \frac{7 + \sqrt{41}}{2}, \frac{7 - \sqrt{41}}{2}$
h $x = \frac{9 + \sqrt{61}}{2}, \frac{9 - \sqrt{61}}{2}$
i $x = \frac{-1 + \sqrt{17}}{2}, \frac{-1 - \sqrt{17}}{2}$
j $x = \frac{-9 + 3\sqrt{5}}{2}, \frac{-9 - 3\sqrt{5}}{2}$
k $x = \frac{3}{2} + \sqrt{3}, \frac{3}{2} - \sqrt{3}$
l $x = -\frac{5}{2} + \sqrt{5}, -\frac{5}{2} - \sqrt{5}$

8 a No real solution. b $x = \frac{-5 \pm \sqrt{17}}{2}$
c $x = \frac{5 \pm \sqrt{17}}{2}$ d $x = \frac{-9 \pm \sqrt{69}}{2}$
e $x = \frac{-5 \pm \sqrt{21}}{2}$ f $x = 3 \pm \sqrt{5}$

9 a $x = \frac{-3 \pm \sqrt{29}}{2}$ b $x = \frac{-5 \pm \sqrt{61}}{2}$
c No real solutions. d $x = 4 \pm \sqrt{5}$
e $x = -5 \pm 2\sqrt{5}$ f No real solutions.

10 breadth $= \frac{-3 + \sqrt{89}}{2}$ cm, length $= \frac{3 + \sqrt{89}}{2}$ cm

11 a i 1.5 km ii 1.5 km
b i 0 km or 400 km ii 200 km
c $200 \pm 100\sqrt{2}$ km

12 a $x^2 + 4x + 5 = 0$
$(x + 2)^2 + 1 = 0$, no real solutions
b $\left(x - \frac{3}{2}\right)^2 + \frac{3}{4} = 0$, no real solutions

13 Factorise by quadratic trinomial; i.e.
$(x + 6)(x-5) = 0$, $6 \times (-5) = -30$, and $6 + (-5) = -1$.
Therefore, $x = -6, 5$.

14 a $x = 3 \pm \sqrt{7}$ b $x = -4 \pm \sqrt{10}$ c $x = -2 \pm \sqrt{11}$
d $x = 1 \pm \sqrt{6}$ e $x = 4 \pm 2\sqrt{3}$ f $x = -5 \pm 2\sqrt{6}$

15 a Use the dimensions of rectangle *BCDE* and *ACDF* and the corresponding side lengths in similar rectangles.
b $a = \frac{1 \pm \sqrt{5}}{2}$

16 a $x = -1 \pm \frac{\sqrt{6}}{2}$ b $x = -1 \pm \sqrt{5}$
c $x = 4 \pm \sqrt{11}$ d $x = \frac{3 \pm \sqrt{5}}{2}$
e $x = \frac{-5 \pm \sqrt{17}}{2}$ f $x = \frac{-1 \pm \sqrt{13}}{2}$

Exercise 8I

1 a $a = 3, b = 2, c = 1$ b $a = 2, b = 1, c = 4$
c $a = 5, b = 3, c = -2$ d $a = 4, b = -3, c = 2$
e $a = 2, b = -1, c = -5$ f $a = -3, b = 4, c = -5$

2 a −8 b −31 c 49
d −23 e 41 f −44

3 a 1 b 0 c 2

4 a 2 b 0 c 1 d 2
e 2 f 2 g 0 h 0
i 2 j 1 k 0 l 2

5 a $x = \frac{-3 \pm \sqrt{17}}{2}$ b $x = \frac{-7 \pm \sqrt{65}}{2}$
c $x = \frac{7 \pm \sqrt{29}}{2}$ d $x = 4$
e $x = -1, -4$ f $x = -1, -7$
g $x = \frac{-7 \pm \sqrt{65}}{8}$ h $x = \frac{-5 \pm \sqrt{37}}{6}$
i $x = \frac{2 \pm \sqrt{22}}{3}$ j $x = \frac{5 \pm \sqrt{65}}{4}$
k $x = -\frac{4}{3}, 1$ l $x = \frac{-3 \pm \sqrt{19}}{5}$

6 a $x = -2 \pm \sqrt{3}$ b $x = 3 \pm \sqrt{5}$
c $x = -3 \pm \sqrt{11}$ d $x = \frac{-3 \pm 3\sqrt{5}}{2}$
e $x = 2 \pm 2\sqrt{2}$ f $x = \frac{4 \pm \sqrt{10}}{3}$
g $x = \frac{1 \pm \sqrt{7}}{2}$ h $x = \frac{3 \pm 2\sqrt{3}}{3}$
i $x = \frac{4 \pm \sqrt{31}}{5}$

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7 $\dfrac{-5 + \sqrt{105}}{2}$

8 a $x = \dfrac{3 \pm 2\sqrt{3}}{3}$ b $x = \dfrac{-2 \pm \sqrt{10}}{2}$ c $x = \dfrac{-5 \pm \sqrt{57}}{8}$

d $x = \dfrac{5 \pm \sqrt{17}}{4}$ e $x = \dfrac{-2 \pm \sqrt{13}}{3}$ f $x = 1 \pm \sqrt{6}$

g $x = \dfrac{1 \pm \sqrt{11}}{5}$ h $x = \dfrac{3 \pm \sqrt{41}}{4}$ i $x = \dfrac{5 \pm \sqrt{19}}{6}$

9 $\dfrac{3 + \sqrt{53}}{2}, \dfrac{-3 + \sqrt{53}}{2}$

10 a $x = 1 + 2\sqrt{2}$ b $10 + 6\sqrt{2}$ c $6 + 5\sqrt{2}$

11 63 cm

12 When $b^2 - 4ac = 0$, the solution reduces to $x = \dfrac{-b}{2a}$; i.e. a single solution.

13 Student's answers will vary.

14 $k = 6$ or -6

15 a i $k > 4$ ii $k = 4$ iii $k < 4$

b i $k > \dfrac{9}{8}$ ii $k = \dfrac{9}{8}$ iii $k < \dfrac{9}{8}$

c i $-2 < k < 2$ ii ± 2

iii $k > 2, k < -2$

d i no values ii no values

iii All values of k

Puzzles and challenges

1 $b = -4, c = 1$

2 47

3 a $\pm 2, \pm 1$ b ± 3

4 a $x = 0, 1$ b $x = 1, -2$

5 144 cm^2

6 25 km/h

7 1.6

8 $x^2 - 2x + 2 = (x - 1)^2 + 1$, as $(x - 1)^2 \geqslant 0$, $(x - 1)^2 + 1 > 0$

Multiple-choice questions

1 D	2 B	3 C
4 A	5 B	6 D
7 C	8 C	9 E
10 C	11 A	12 B

Short-answer questions

1 a $-2x + 26$ b $3x^2 + 11x - 20$

c $25x^2 - 4$ d $x^2 - 12x + 36$

e $7x + 22$ f $12x^2 - 23x + 10$

2 a $x^2 + 4x + 4$ b $4x^2 + 18x$ c $x^2 + 3x + 21$

3 a $(x + 7)(x - 7)$ b $(3x + 4)(3x - 4)$

c $(2x + 1)(2x - 1)$ d $3(x + 5)(x - 5)$

e $2(x + 3)(x - 3)$ f $(x + \sqrt{11})(x - \sqrt{11})$

g $-2(x + 2\sqrt{5})(x - 2\sqrt{5})$ h $(x + 5)(x - 3)$

i $(x - 3 + \sqrt{10})(x - 3 - \sqrt{10})$

4 a $(x - 6)(x - 2)$ b $(x + 12)(x - 2)$

c $-3(x - 6)(x - 1)$

5 a $(3x + 2)(x + 5)$ b $(2x - 3)(2x + 5)$

c $(6x + 1)(2x - 3)$ d $(3x - 2)(4x - 5)$

6 a $\dfrac{2x}{x + 3}$ b $\dfrac{x - 4}{4}$

7 a $(x + 4 + \sqrt{6})(x + 4 - \sqrt{6})$

b $(x + 5 + \sqrt{29})(x + 5 - \sqrt{29})$

c $(x - 3 + 2\sqrt{3})(x - 3 - 2\sqrt{3})$

d $\left(x + \dfrac{3 + \sqrt{17}}{2}\right)\left(x + \dfrac{3 - \sqrt{17}}{2}\right)$

e $\left(x + \dfrac{5 + \sqrt{13}}{2}\right)\left(x + \dfrac{5 - \sqrt{13}}{2}\right)$

f $\left(x + \dfrac{7 + \sqrt{31}}{2}\right)\left(x + \dfrac{7 - \sqrt{31}}{2}\right)$

8 a $x = 0, -4$ b $x = 0, 3$ c $x = 5, -5$

d $x = 3, 7$ e $x = 4$ f $x = -9, 4$

g $x = -2, \dfrac{1}{2}$ h $x = \dfrac{2}{3}, -\dfrac{5}{2}$ i $x = \dfrac{1}{9}, -\dfrac{3}{2}$

9 a $x = 3, -3$ b $x = 5, -1$

c $x = 4, -7$ d $x = -3, 6$

10 length = 8 m, width = 6 m

11 a $x = -2 \pm \sqrt{7}$ b $x = 3 \pm 2\sqrt{2}$

c $x = \dfrac{3 \pm \sqrt{17}}{2}$ d $x = \dfrac{-5 \pm 3\sqrt{5}}{2}$

12 a 1 solution b 2 solutions

c 0 solutions d 2 solutions

13 a $x = \dfrac{-3 \pm \sqrt{33}}{2}$ b $x = 1 \pm \sqrt{5}$

c $x = \dfrac{2 \pm \sqrt{14}}{2}$ d $x = \dfrac{1 \pm \sqrt{37}}{6}$

Extended-response questions

1 a i $15 + 2x$ m ii $12 + 2x$ m

b Area $= 4x^2 + 54x + 180$ m^2

c Trench $= 4x^2 + 54x$ m^2

d Minimum width is 1 m.

2 a $A = 63\pi$ m^2 b 0.46 m

c i $420 = 3\pi r^2 + 12\pi r$

ii $3\pi r^2 + 12\pi r - 420 = 0$

iii $r = 4.97$ m; i.e. $\pi r^2 + 4\pi r - 140 = 0$

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Chapter 9

Pre-test

1 **a** $x^2 + 2x - 3$ **b** $2x^2 - 9x - 5$ **c** $3x^2 + 2x - 8$

2

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9

3 **a** $(x - 3)(x + 3)$ **b** $(9x - 7)(9x + 7)$
c $(x + 4)(x + 1)$ **d** $(x - 5)(x - 4)$
e $(x + 8)(x - 2)$ **f** $3(x - 3)(x - 4)$

4 **a** $\left(x + 4 - \sqrt{15}\right)\left(x + 4 + \sqrt{15}\right)$
b $\left(x - 1 - \sqrt{10}\right)\left(x - 1 + \sqrt{10}\right)$

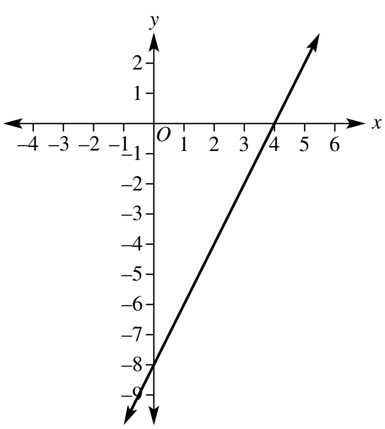
5 **a** $x = 1$ or -2 **b** $x = 2$ or -3 **c** $x = -5$ or -7
d $x = -\frac{3}{2}$ or 2 **e** $x = \frac{1}{3}$ or $-\frac{5}{2}$ **f** $x = -\frac{2}{7}$ or $\frac{4}{3}$

6 **a** $x = 2$ or -1 **b** $x = 3$ or -5 **c** $x = 3$ or -4
d $x = -1$ or -4 **e** $x = 1$ or 2 **f** $x = 4$ or 3

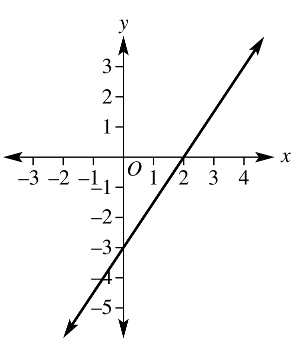
7 **a** $x = 0$ or 4 **b** $x = 4$
c $x = \sqrt{5}$ or $-\sqrt{5}$ **d** $x = -3$

8 **a** 0, 2 **b** 0 **c** $(1, -1)$

9 **a**



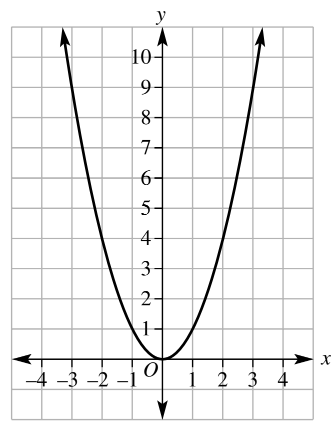
b



Exercise 9A

1

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9



2 **a** maximum
b $(-2, 4)$
c 2
d -5, 1
e $x = -2$

3 **a** **i** $(2, -5)$ **ii** $x = 2$
iii -1, 5 **iv** -3
b **i** $(2, 0)$ **ii** $x = 2$
iii 2 **iv** -1
c **i** $(2, 5)$ **ii** $x = 2$
iii no x-intercept **iv** 7
d **i** $(-3, 0)$ **ii** $x = -3$
iii -3 **iv** 4
e **i** $(2, -2)$ **ii** $x = 2$
iii 1, 3 **iv** 6
f **i** $(0, 3)$ **ii** $x = 0$
iii -3, 3 **iv** 3

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4

	Formula	Max or min	Reflected in the x-axis (yes/no)	Turning point	y value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = 3x^2$	min	no	(0, 0)	$y = 3$	narrower
b	$y = \frac{1}{2}x^2$	min	no	(0, 0)	$y = \frac{1}{2}$	wider
c	$y = 2x^2$	min	no	(0, 0)	$y = 2$	narrower
d	$y = -4x^2$	max	yes	(0, 0)	$y = -4$	narrower
e	$y = -\frac{1}{3}x^2$	max	yes	(0, 0)	$y = -\frac{1}{3}$	wider
f	$y = -2x^2$	max	yes	(0, 0)	$y = -2$	narrower

5

	Formula	Turning point	Axis of symmetry	y-intercept ($x = 0$)	x-intercept
a	$y = (x + 3)^2$	(−3, 0)	$x = -3$	9	−3
b	$y = (x - 1)^2$	(1, 0)	$x = 1$	1	1
c	$y = (x - 2)^2$	(2, 0)	$x = 2$	4	2
d	$y = (x + 4)^2$	(−4, 0)	$x = -4$	16	−4

6

	Formula	Turning point	y-intercept ($x = 0$)	y value when $x = 1$
a	$y = x^2 + 3$	(0, 3)	3	$y = 4$
b	$y = x^2 - 1$	(0, −1)	−1	$y = 0$
c	$y = x^2 + 2$	(0, 2)	2	$y = 3$
d	$y = x^2 - 4$	(0, −4)	−4	$y = -3$

7 a $x = 0$ **b** $x = 0$ **c** $x = 0$ **d** $x = 0$
e $x = 0$ **f** $x = 2$ **g** $x = -1$ **h** $x = -3$
i $x = 0$ **j** $x = 0$ **k** $x = 0$ **l** $x = -4$

8 a (0, 0) **b** (0, 7) **c** (0, 0) **d** (0, 0)
e (0, −4) **f** (2, 0) **g** (−1, 0) **h** (−3, 0)
i (0, −3) **j** (0, 2) **k** (0, −16) **l** (−4, 0)

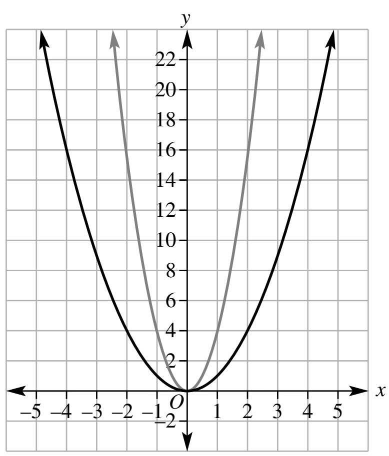
9 a 0 **b** 7 **c** 0 **d** 0
e −4 **f** 4 **g** 1 **h** −9
i −3 **j** 2 **k** −16 **l** −16

10 a H **b** C **c** G **d** D
e A **f** E **g** B **h** F

11 a $y = x^2 + 1$ **b** $y = x^2 - 2$
c $y = 2x^2$ **d** $y = 9 - x^2$

12 a $0, \pm 3, \pm 5$ **b** 5, 10, 12

13 a i



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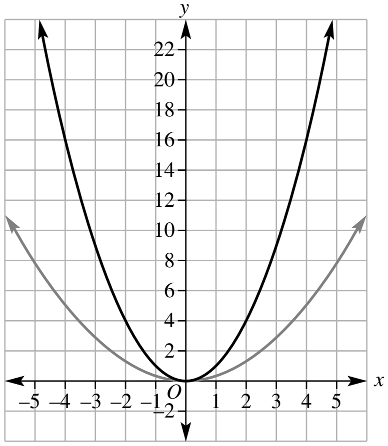

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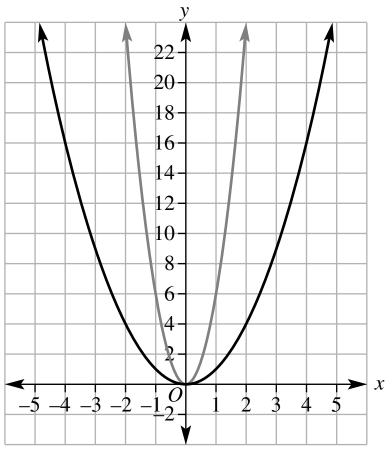

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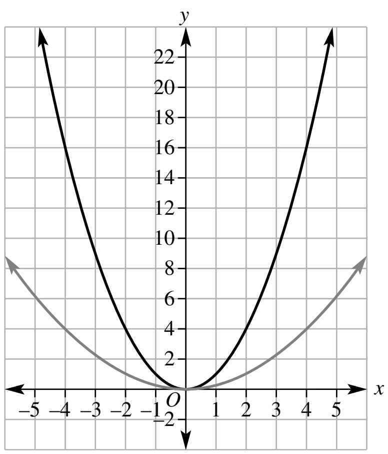

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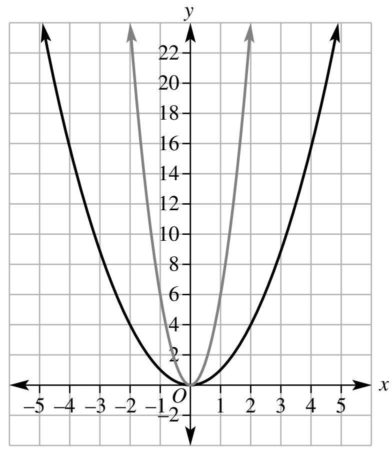

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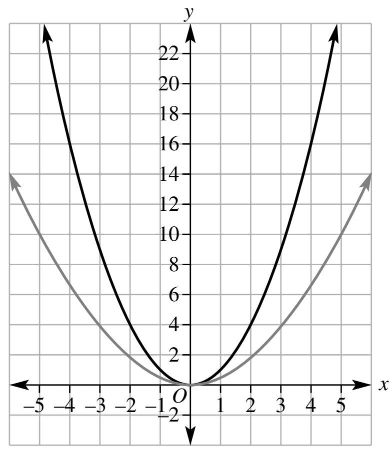
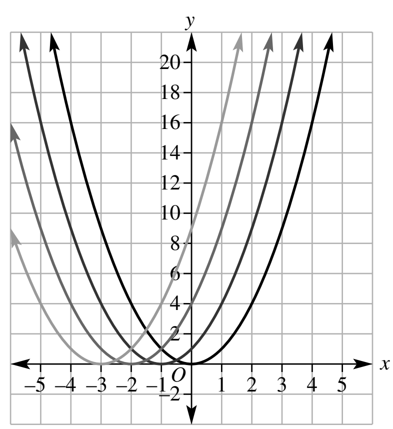


b The constant a determines the narrowness of the graph.

14 a i



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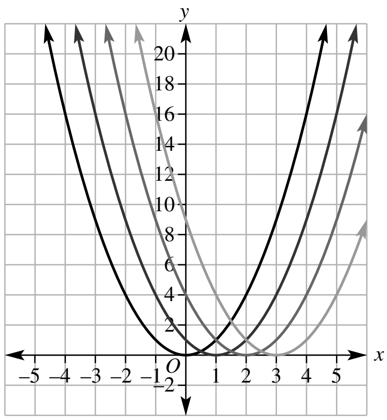
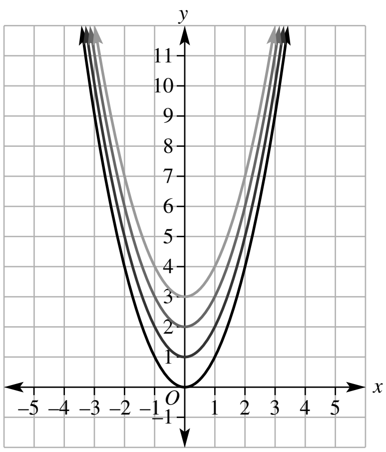


b The constant h determines whether the graph moves left or right from $y = x^2$.

15 a i



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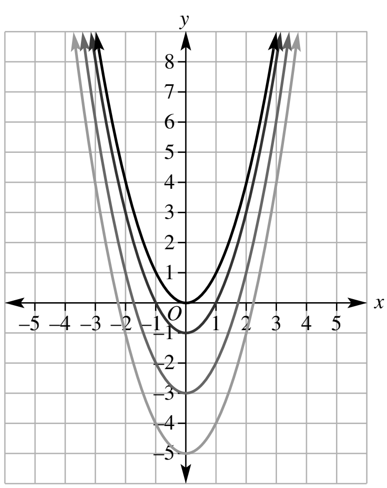


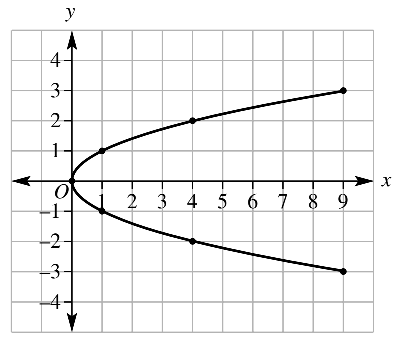
b The constant k determines whether the graph moves up or down from $y = x^2$.

16 Answers could be:

a $y = x^2 - 4$ **b** $y = (x - 5)^2$ **c** $y = x^2 + 3$

17 a $y = x^2 + 2$ **b** $y = -x^2 + 2$ **c** $y = (x + 1)^2$
d $y = (x - 2)^2$ **e** $y = 2x^2$ **f** $y = -3x^2$
g $y = (x + 1)^2 + 2$ **h** $y = \frac{1}{8}(x - 4)^2 - 2$

18 parabola on its side



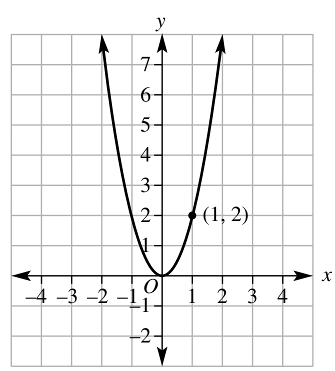
Exercise 9B

1 a (0, 0) **b** (0, 0) **c** (0, 3)
d (0, −3) **e** (0, 7) **f** (2, 0)
g (−5, 0) **h** (0, 0) **i** (0, 0)

2 a 3 **b** −3 **c** −3
d −1 **e** 1 **f** 4
g −16 **h** −25 **i** −2
j 2 **k** 6 **l** −63

3 a up **b** right **c** left
d down **e** down **f** left
g right **h** up

4 a



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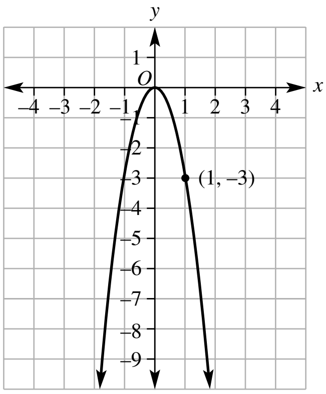
b



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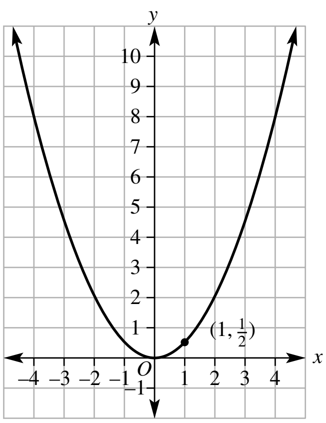

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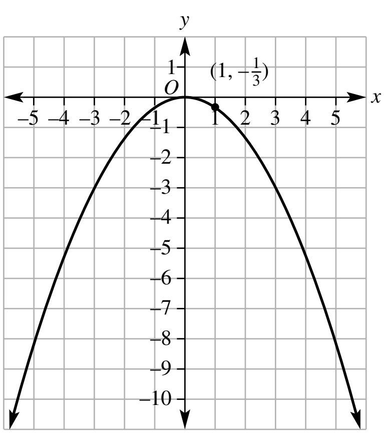

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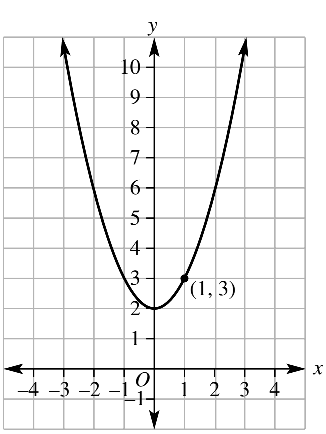

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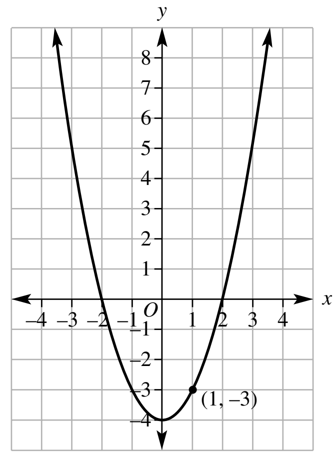

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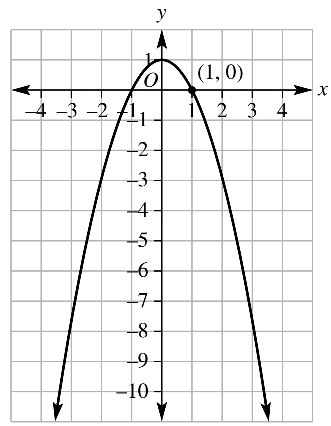

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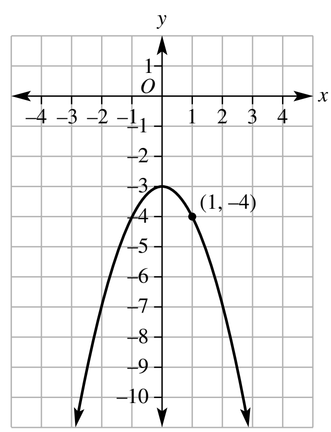

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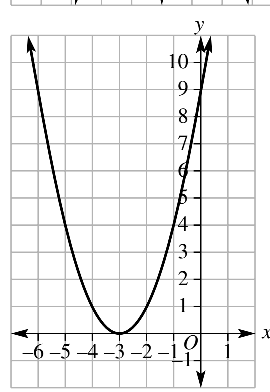

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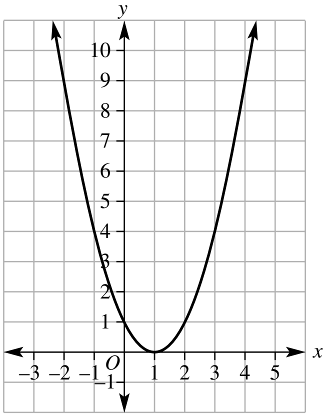

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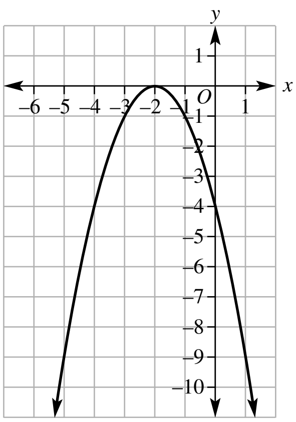

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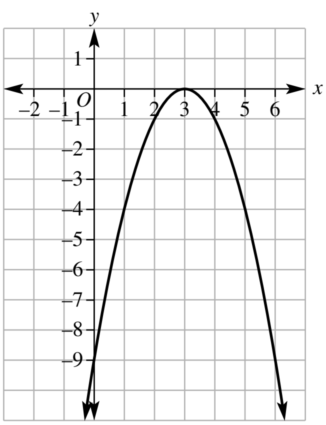
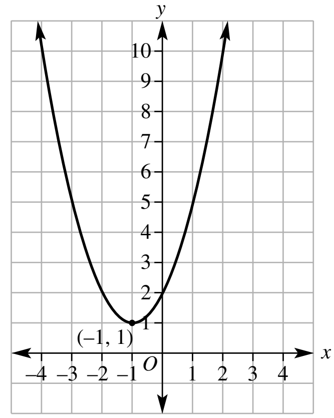


5 **a** (−3, 1) **b** (−2, −4) **c** (1, 3)
d (4, −2) **e** (3, −5) **f** (2, 2)
g (3, 3) **h** (2, 6) **i** (−1, 4)
j (2, −5) **k** (−1, −1) **l** (4, −10)

6 **a**



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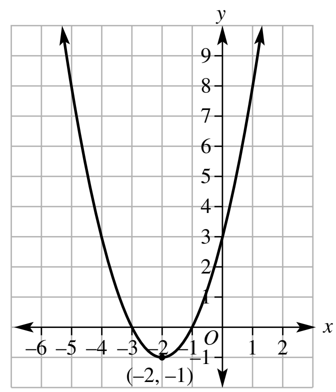

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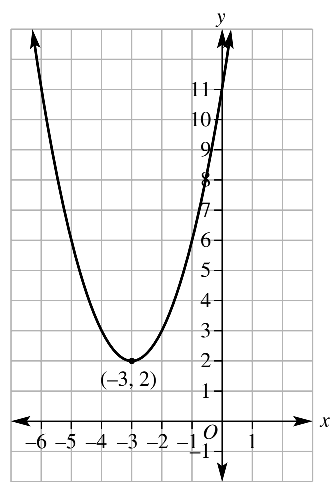

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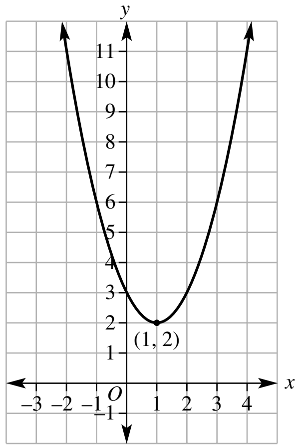

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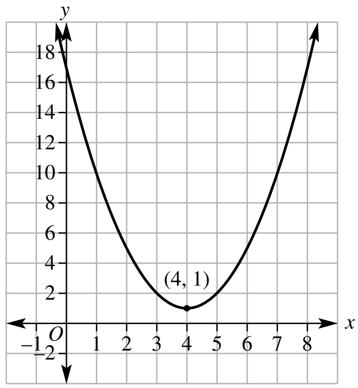

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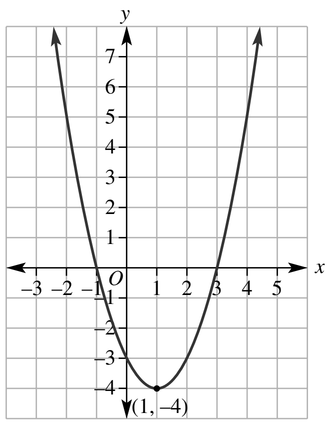

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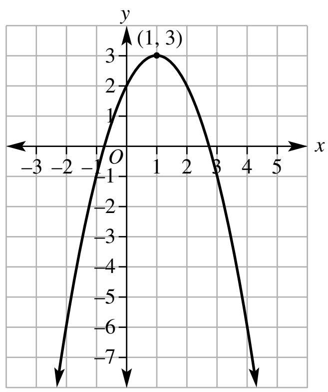

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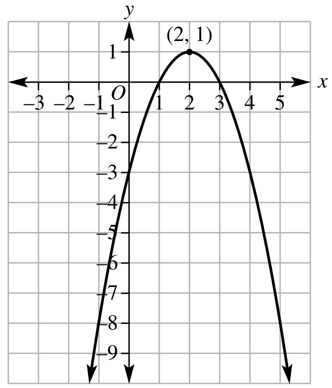

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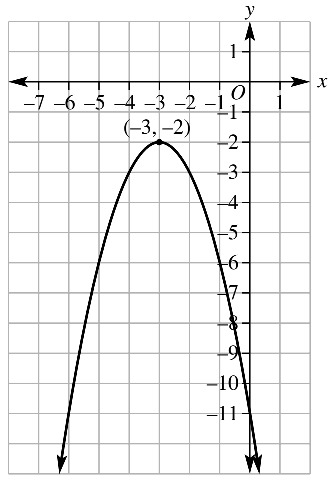

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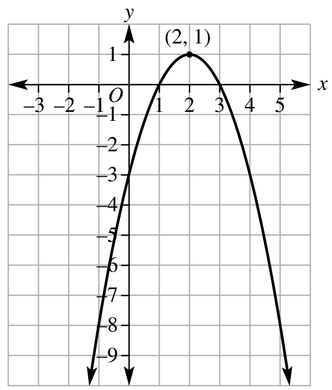

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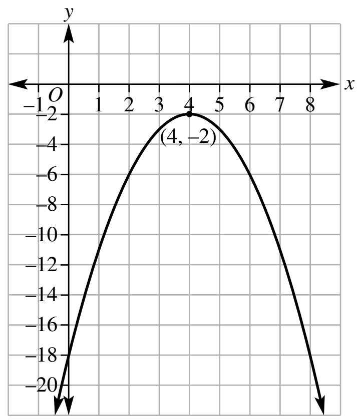

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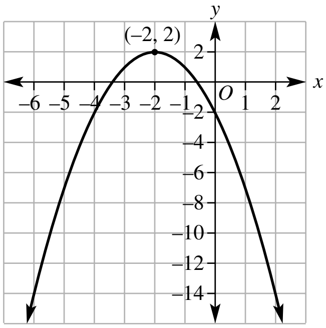


7 a $y = -x^2$ b $y = (x + 2)^2$ c $y = x^2 - 5$
d $y = x^2 + 4$ e $y = (x - 1)^2$ f $y = -x^2 + 2$
g $y = -(x + 3)^2$ h $y = (x + 5)^2 - 3$ i $y = (x - 6)^2 + 1$

8 a $y = 6x^2$ b $y = x^2 + 4$ c $y = (x - 3)^2$
d $y = -(x + 2)^2$ e $y = \frac{1}{2}x^2$ f $y = -x^2 + 2$
g $y = x^2 - 1$ h $y = (x - 1)^2$ i $y = -7x^2$

9 a maximum b (5, 25)
c 0 d 25 m
e i 21 m ii 21 m iii 0 m

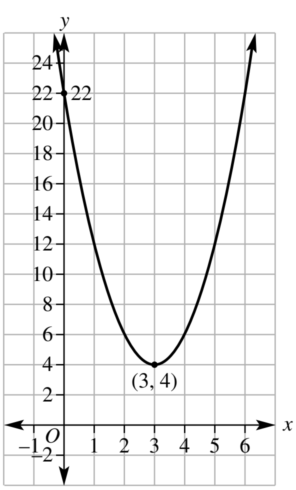
10 a $y = x^2 - 9$ b $y = (x - 2)^2$

11 (−3, 6), (3, 6)

12 a (1, 0) b (−2, 0) c (−3, 0) d (0, −4)
e (0, −2) f (0, 5) g (−4, −1) h (−2, 3)
i (5, 4) j (−2, 3) k (−3, −5) l (3, −3)

13 a translate 3 units right
b translate 2 units left
c translate 3 units down
d translate 7 units up
e reflect in x-axis
f translate 2 units left and 4 units down
g translate 5 units right and 8 units up
h reflect in x-axis, translate 3 units left
i reflect in x-axis, translate 6 units up

14 a (h, k) b $(0, ah^2 + k)$

15 a


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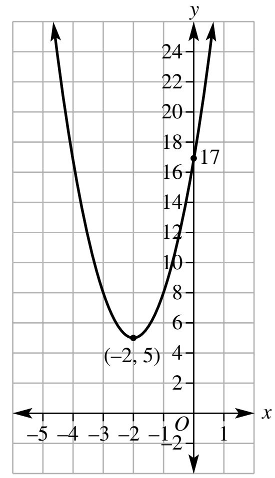

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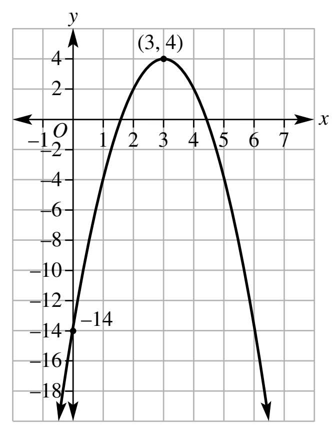

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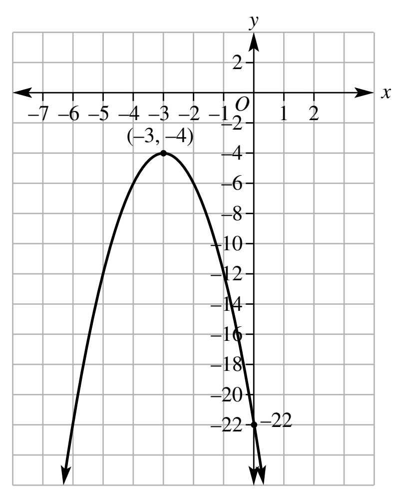

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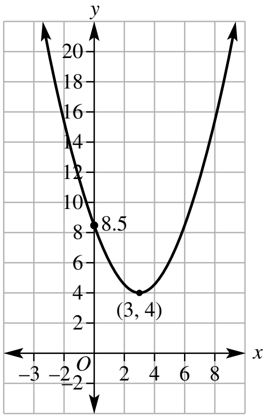

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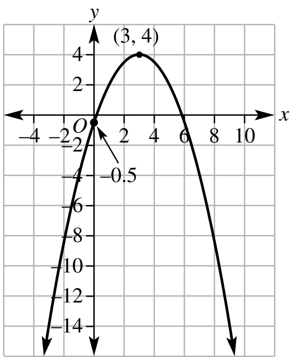

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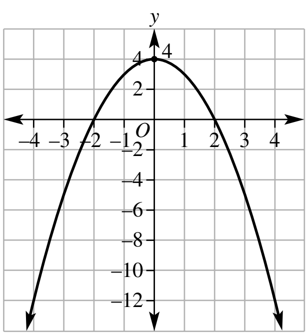

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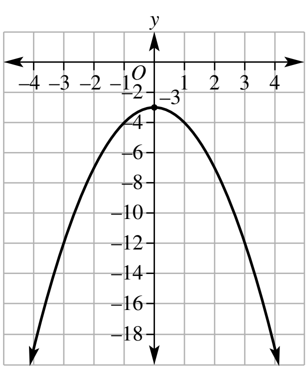

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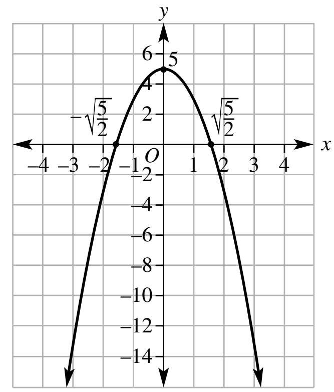

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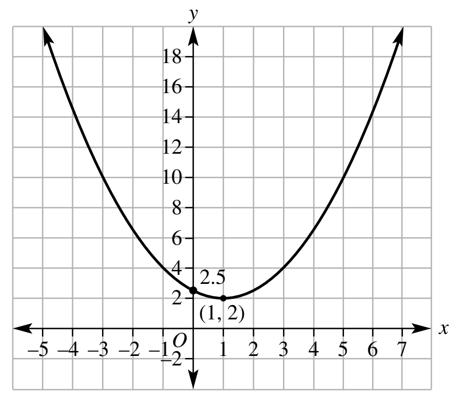

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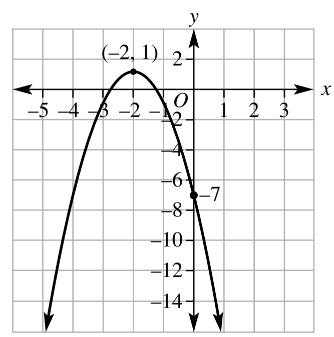

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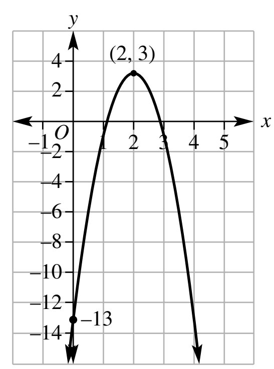

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Exercise 9C

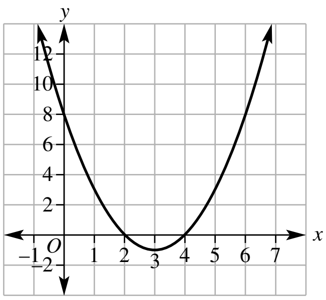
1 **a** $x = -1, x = 2$ **b** $x = 3, x = 4$ **c** $x = -1, x = -5$
d $x = 0, x = 3$ **e** $x = 0, x = 5$ **f** $x = 0, x = -2$
g $x = \pm\sqrt{5}$ **h** $x = \pm\sqrt{7}$ **i** $x = \pm 2\sqrt{2}$

2 **a** $x = -2, x = -1$ **b** $x = -4, x = 2$
c $x = 4$ **d** $x = 0, x = 4$
e $x = 0, x = 6$ **f** $x = 0, x = -5$
g $x = \pm 3$ **h** $x = \pm 5$
i $x = \pm\sqrt{10}$

3 **a** 2 **b** −8 **c** 16
d 0 **e** 0 **f** 0
g −9 **h** −25 **i** −10

4 **a** $(x + 6)(x - 8)$ **b** $x = -6, x = 8$
c $x = 1$ **d** $(1, -49)$

5 **a**



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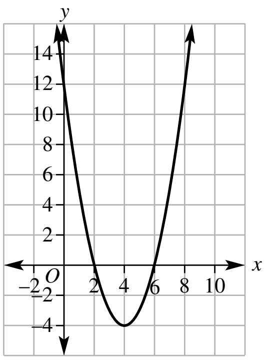

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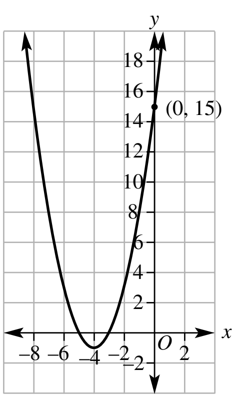

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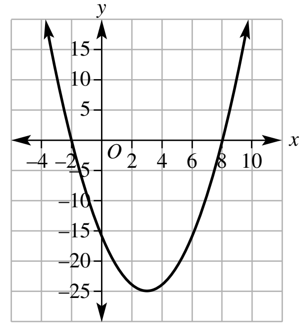

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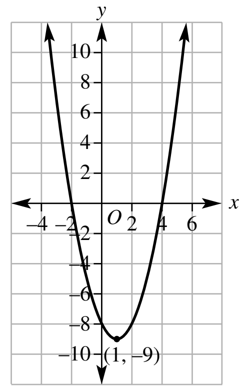

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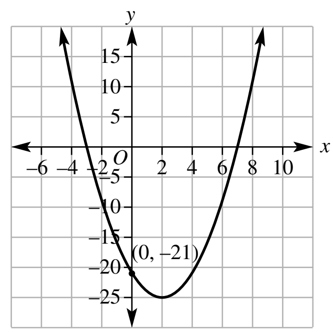

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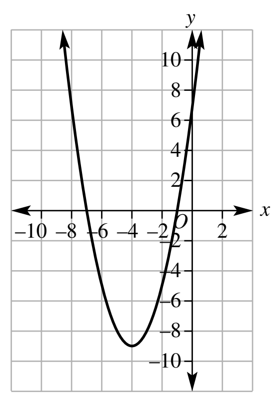

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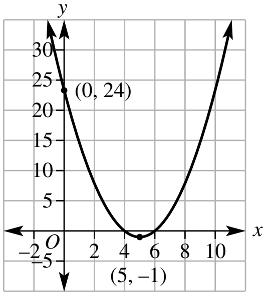

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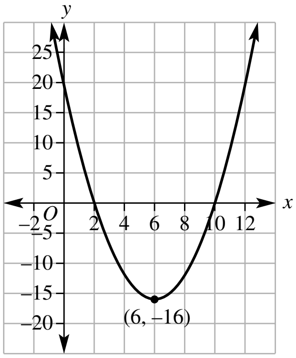

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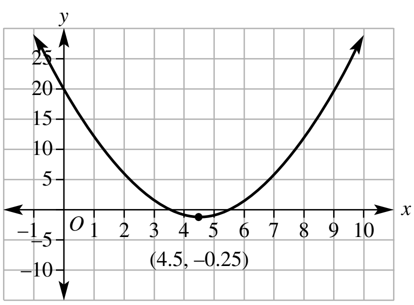

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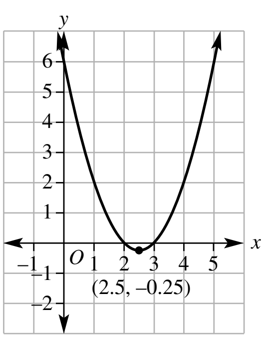

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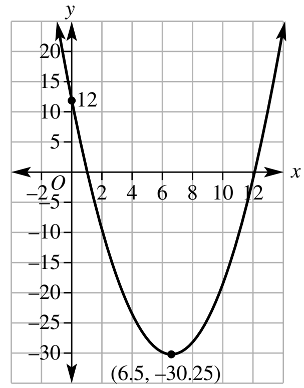

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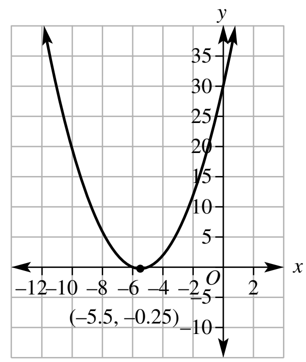

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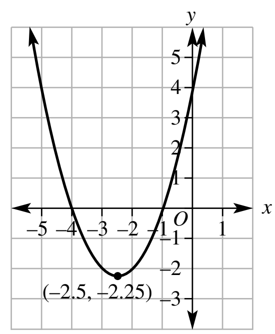

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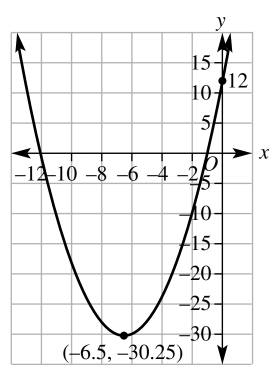

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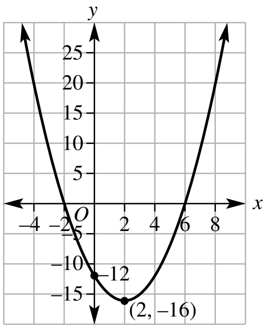

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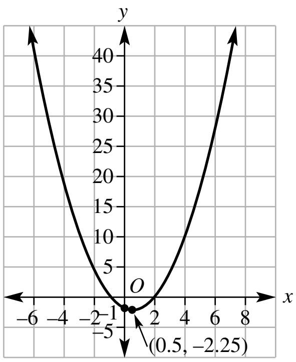

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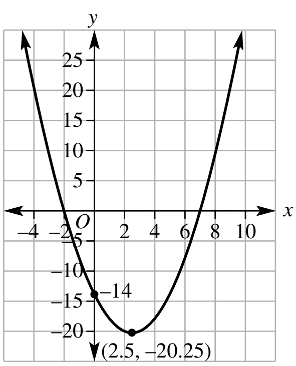

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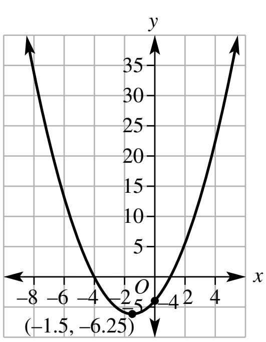

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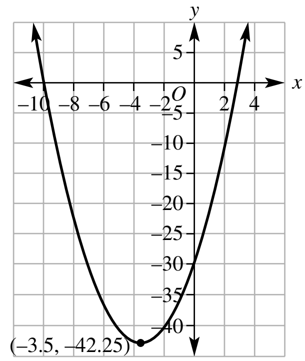

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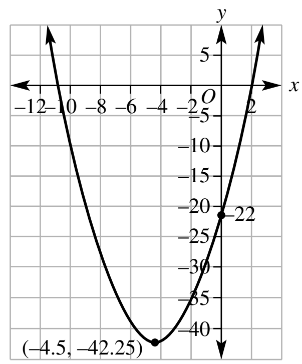

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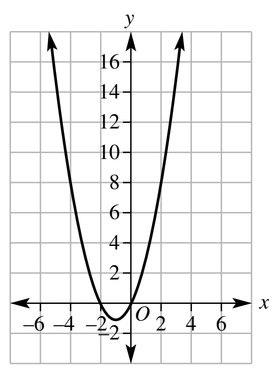

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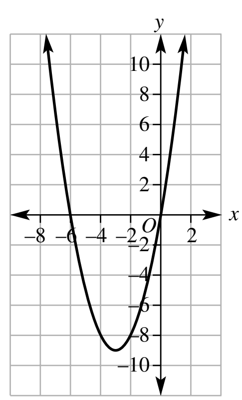

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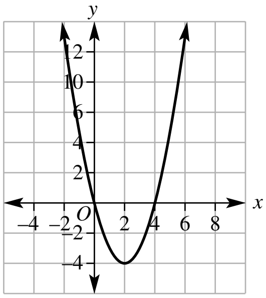

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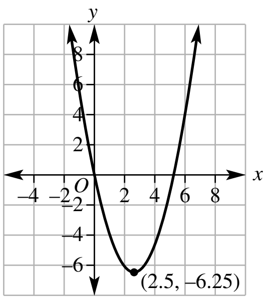

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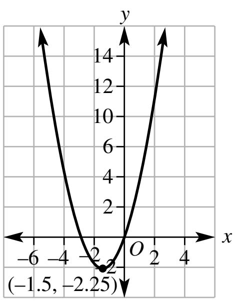

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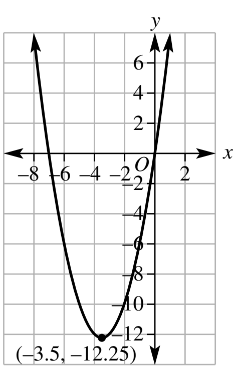

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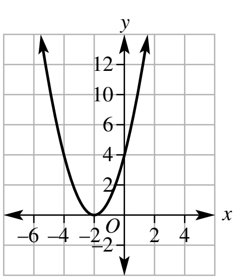

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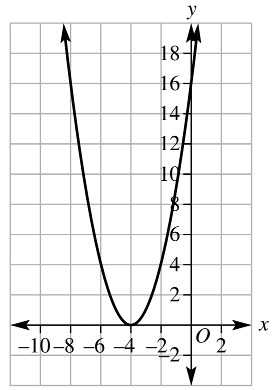

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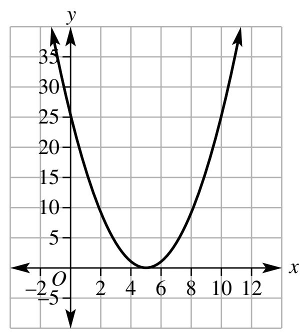

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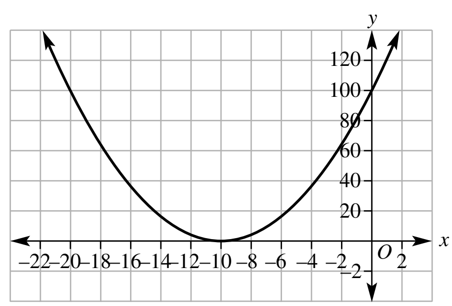

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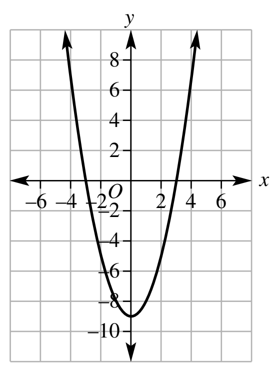

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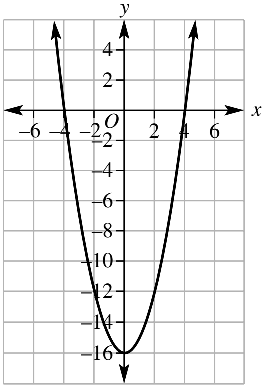

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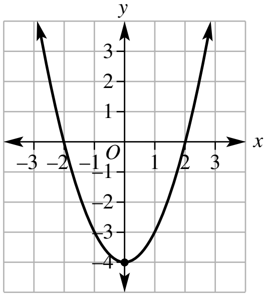
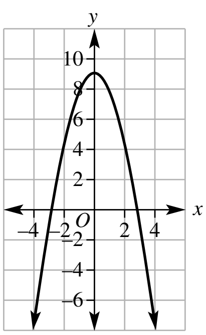


10 a (3.5, −4.5) **b** (3.5, −6.75) **c** (−3, −3)
d (−3, −4) **e** (0, −196) **f** (0, 196)
g (1, 0) **h** (1, 0)

11 a $a = -1, b = -3$, TP $(2, -1)$
b $a = 5, b = -1$, TP $(-2, -9)$
c $a = 2, b = -6$, TP $(2, -16)$

12 a x-intercepts: $\pm\sqrt{2}$, TP $(0, -2)$
b x-intercepts: $\pm\sqrt{11}$, TP $(0, -11)$
c x-intercepts: $\pm 5\sqrt{2}$, TP $(0, -50)$

13 a



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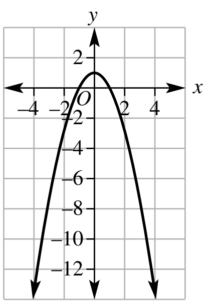

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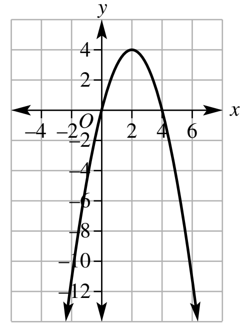

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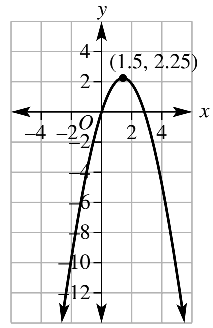

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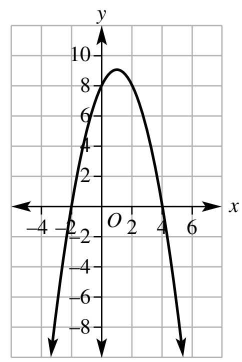

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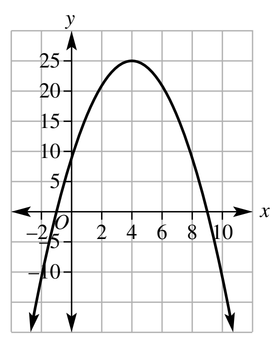


14 $a = -2$, TP $(1, 18)$

15 Coefficient does not change the x-intercept.

16 a $y = x^2 - 2x + 1 = (x - 1)^2$
Only one x-intercept, which is the turning point.

b Graph has a minimum (0, 2), therefore its lowest point is 2 units above the x-axis.

17 a $x = 4, x = -2$ **b** $(1, -9), (1, 9)$

c same x-coordinate; y-coordinate is reflected in the x-axis

18 a 0 **b** $0, -b$ **c** $\left(-\frac{b}{2}, -\frac{b^2}{4}\right)$

19 a $y = x(x - 4)$ **b** $y = x(x - 2)$

c $y = x(x + 6)$ **d** $y = (x + 3)(x - 3)$

e $y = (x + 2)(x - 2)$ **f** $y = (x + \sqrt{5})(x - \sqrt{5})$

g $y = (x + 4)(x - 2)$ **h** $y = (x - 1)(x - 5)$

i $y = (x + 1)(x - 3)$ **j** $y = -x(x - 4)$

k $y = -(x + 2)(x - 6)$ **l** $y = -(x - \sqrt{10})(x + \sqrt{10})$

Exercise 9D

1 a
$$\begin{aligned} y &= x^2 + 2x - 5 \\ &= x^2 + 2x + 1 - 1 - 5 \\ &= (x + 1)^2 - 6 \end{aligned}$$
TP is $(-1, -6)$.

b
$$\begin{aligned} y &= x^2 + 4x - 1 \\ &= x^2 + 4x + 4 - 4 - 1 \\ &= (x + 2)^2 - 5 \end{aligned}$$
TP is $(-2, -5)$.

c
$$\begin{aligned} y &= x^2 - 6x + 10 \\ &= x^2 - 6x + 9 - 9 + 10 \\ &= (x - 3)^2 + 1 \end{aligned}$$
TP is $(3, 1)$.

d
$$\begin{aligned} y &= x^2 - 3x - 7 \\ &= x^2 - 3x + \frac{9}{4} - \frac{9}{4} - 7 \\ &= \left(x - \frac{3}{2}\right)^2 - \frac{37}{4} \end{aligned}$$
TP is $\left(\frac{3}{2}, -\frac{37}{4}\right)$

2 a $x = \pm 3$ **b** $x = \pm\sqrt{3}$ **c** $x = 3, x = -1$

d $x = -5, x = 3$ **e** $x = \pm\sqrt{2} - 4$ **f** $x = 6 \pm \sqrt{5}$

3 a min $(3, 5)$ **b** max $(1, 3)$ **c** max $(-1, -2)$

d min $(-2, -5)$ **e** min $(-5, 10)$ **f** max $(7, 2)$

g max $(3, 8)$ **h** min $(3, -7)$

4 a 6 **b** -2 **c** 7

d 9 **e** -16 **f** -55

g 3 **h** 1 **i** -5

j -8 **k** 13 **l** -5

5 a $x = 5, x = 1$ **b** $x = -7, x = -1$ **c** $x = 9, x = -3$

d $x = -2 \pm \sqrt{5}$ **e** $x = 1 \pm \sqrt{10}$ **f** $x = 5 \pm \sqrt{3}$

g $x = 4$ **h** $x = -6$ **i** no x-intercept

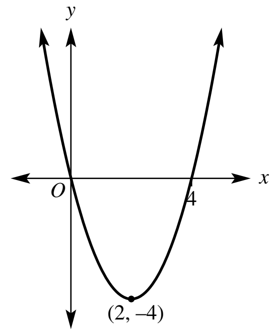
j no x-intercept **k** $x = 2 \pm \sqrt{5}$ **l** $x = 3 \pm \sqrt{10}$

6 a $x = -1, x = -5$ **b** $x = \pm\sqrt{7} - 3$

c $-4 \pm \sqrt{21}$ **d** $x = -1 \pm \sqrt{7}$

e no x-intercept **f** $x = 6 \pm \sqrt{41}$

7 a



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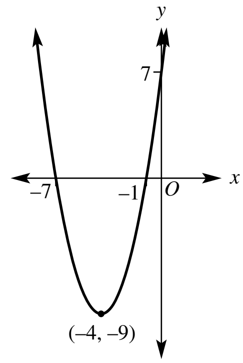

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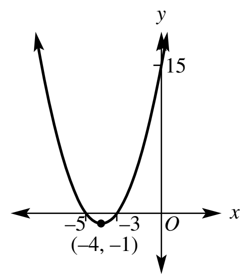

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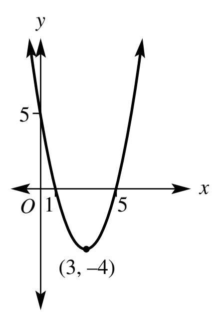

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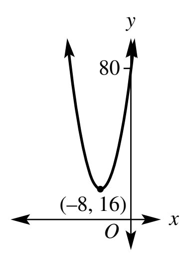

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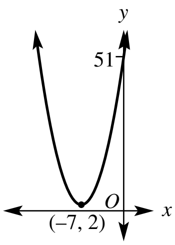

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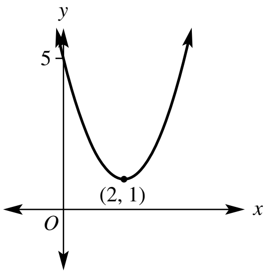

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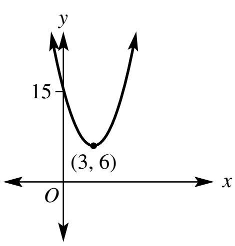

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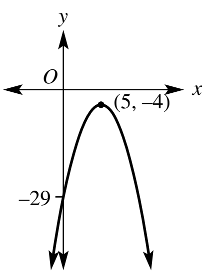

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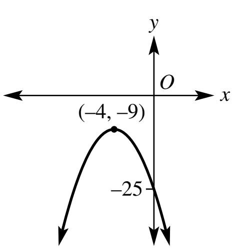

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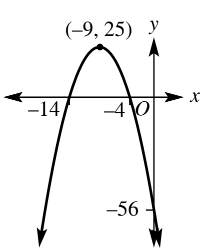

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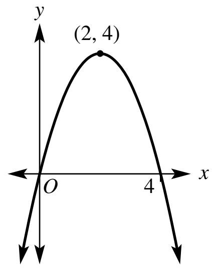

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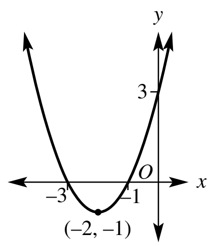

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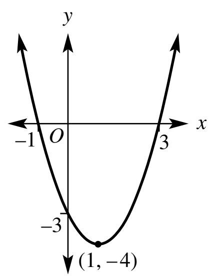

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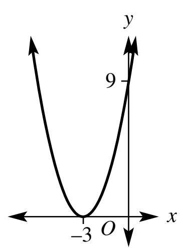

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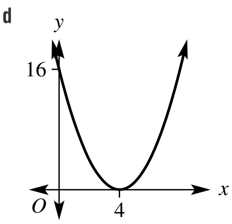

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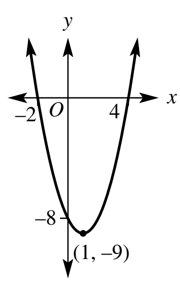

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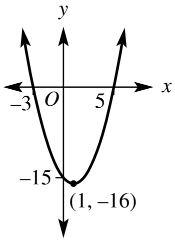

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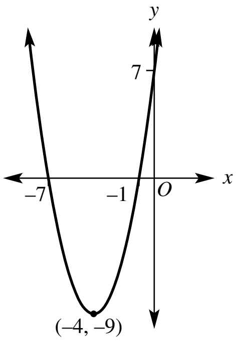

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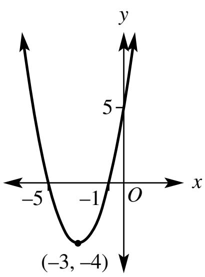

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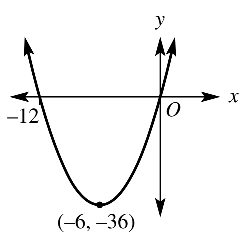

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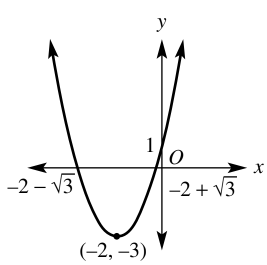

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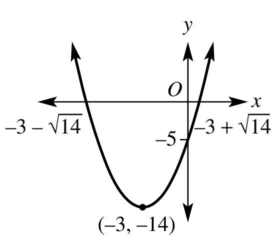

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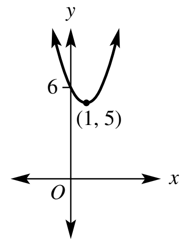

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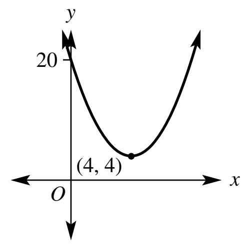

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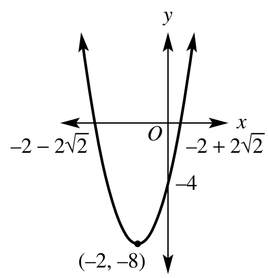

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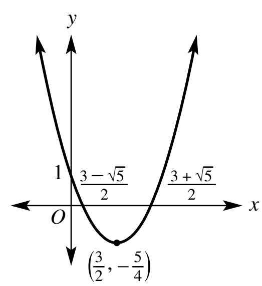

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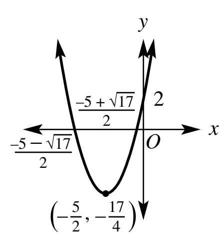

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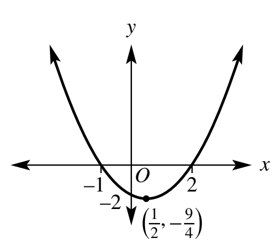

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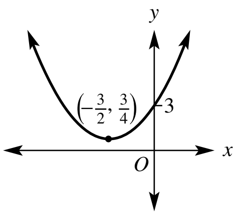
i



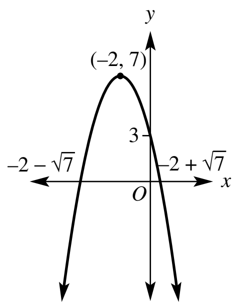
10 a 2 **b** 1 **c** 1 **d** 0 **e** 0 **f** 2

11 a $x = -1 \pm \sqrt{6}$

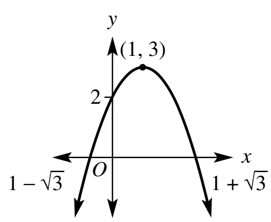
b $x = 3, 1$ **c** $x = 7, x = -1$ **d** $x = -2 \pm \sqrt{10}$

e $x = -1 \pm \sqrt{6}$ **f** $x = \dfrac{3 \pm \sqrt{5}}{2}$

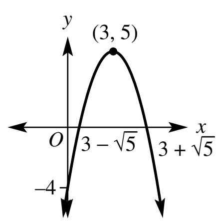
12 a $y = -(x + 2)^2 + 7$



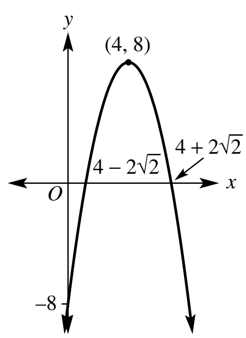
b $y = -(x - 1)^2 + 3$



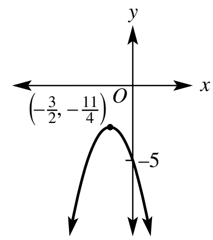
c $y = -(x - 3)^2 + 5$



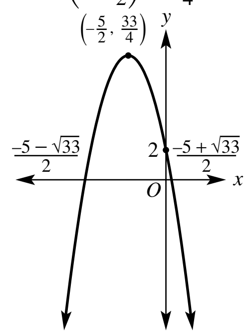
d $y = -(x - 4)^2 + 8$



e $y = -\left(x + \dfrac{3}{2}\right)^2 - \dfrac{11}{4}$



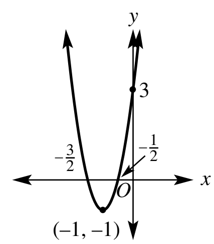
f $y = -\left(x + \dfrac{5}{2}\right)^2 + \dfrac{33}{4}$



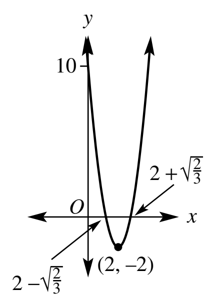
13 a $k > 0$ **b** $k = 0$ **c** $k < 0$

14
$$
\begin{aligned}
x^2 + bx + c &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\
&= \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + \frac{4c}{4} \\
&= \left(x + \frac{b}{2}\right)^2 - \frac{b^2 - 4c}{4}
\end{aligned}
$$

15 a $y = 4(x + 1)^2 - 1$



b $y = 3(x - 2)^2 - 2$



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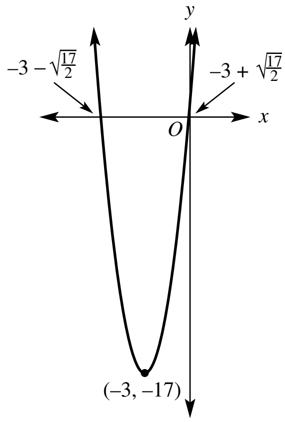
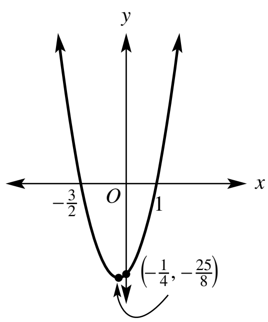
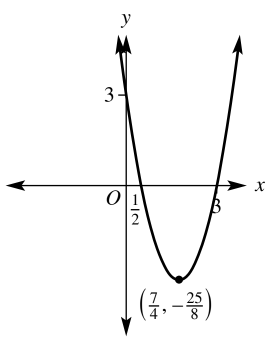
c $y = 2(x+3)^2 - 17$



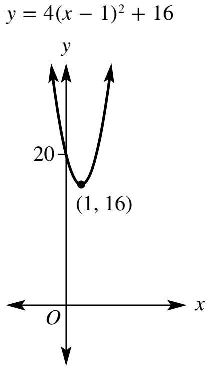
d $y = 2\left(x+\frac{1}{4}\right)^2 - \frac{25}{8}$



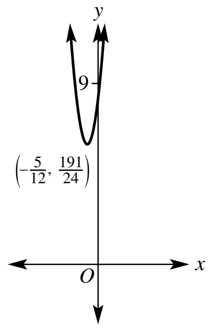
e $y = 2\left(x-\frac{7}{4}\right)^2 - \frac{25}{8}$



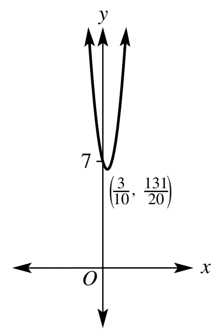
f $y = 4(x-1)^2 + 16$



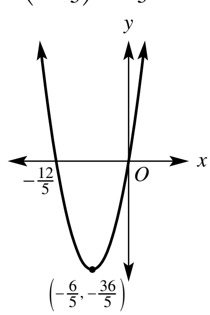
g $y = 6\left(x+\frac{5}{12}\right)^2 + \frac{191}{24}$



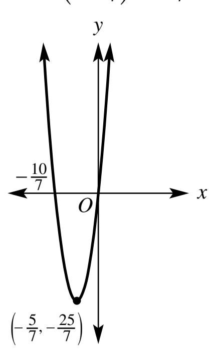
h $y = 5\left(x-\frac{3}{10}\right)^2 + \frac{131}{20}$



i $5\left(x+\frac{6}{5}\right)^2 - \frac{36}{5}$



j $y = 7\left(x+\frac{5}{7}\right)^2 - \frac{25}{7}$



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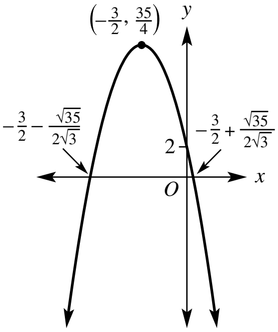
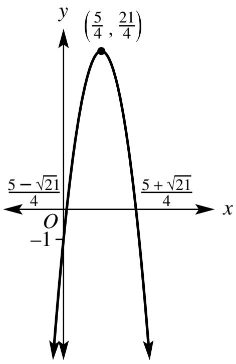
k $y = -3\left(x + \frac{3}{2}\right)^2 + \frac{35}{4}$



l $y = -4\left(x - \frac{5}{4}\right)^2 + \frac{21}{4}$



Exercise 9E

1 **a** 2 intercepts **b** 0 intercepts **c** 1 intercept

2 **a** $-1 \pm \sqrt{2}$ **b** 2.5, –1
c $\frac{-1 \pm \sqrt{17}}{4}$ **d** $\frac{-3 \pm \sqrt{15}}{3}$

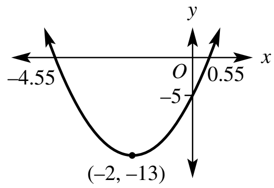
3 **a** zero **b** positive **c** negative

4 **a** 1 intercept **b** no intercepts **c** 2 intercepts
d 2 intercepts **e** no intercepts **f** 1 intercept
g 2 intercepts **h** 2 intercepts **i** 2 intercepts
j 0 intercepts **k** 2 intercepts **l** 2 intercepts

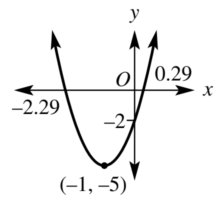
5 **a** 3 **b** 5 **c** –2
d –4 **e** 8 **f** –10
g 0 **h** 0 **i** –7

6 **a** (–1, 3) **b** (–2, –5) **c** (2, –1)
d (1, –5) **e** $\left(-\frac{3}{2}, 6\frac{1}{4}\right)$ **f** $\left(\frac{7}{2}, 5\frac{1}{4}\right)$
g $\left(-\frac{3}{4}, -5\frac{1}{8}\right)$ **h** $\left(\frac{3}{8}, -\frac{9}{16}\right)$ **i** (0, –9)
j $\left(\frac{1}{4}, -2\frac{3}{4}\right)$ **k** $\left(-\frac{1}{3}, \frac{1}{3}\right)$ **l** (0, 2)

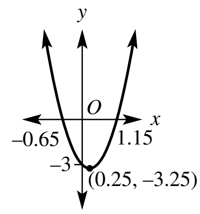
7 **a**



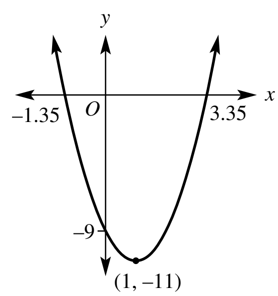
b



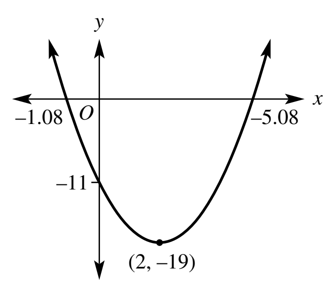
c



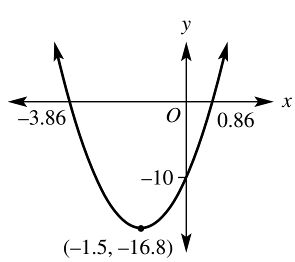
d



e



f



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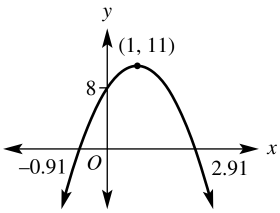
g



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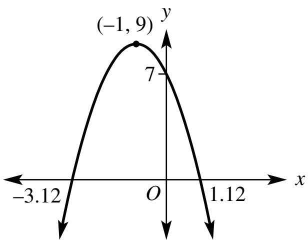

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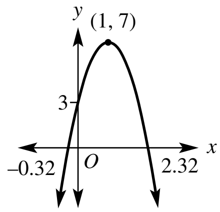

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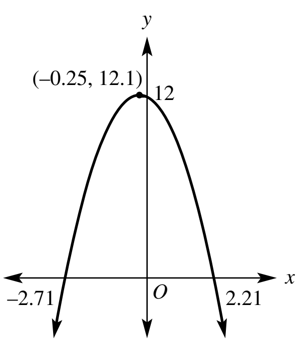
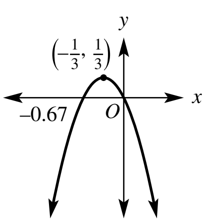
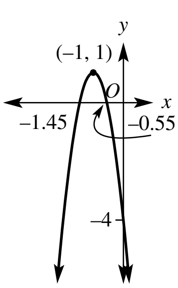


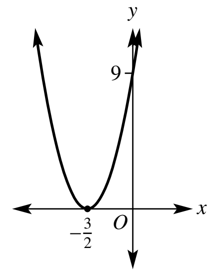
k



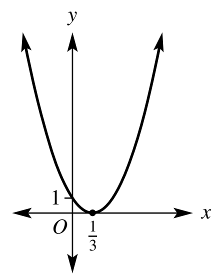
l



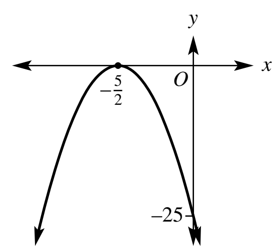
8 a



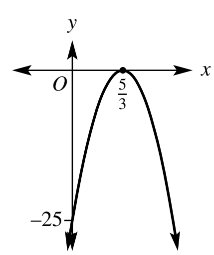
b



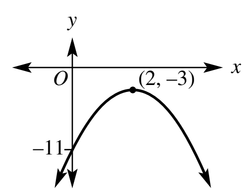
c



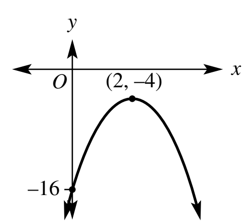
d



e



f



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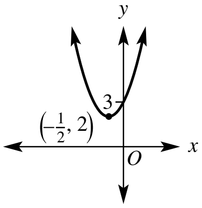
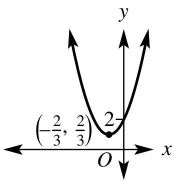
g



h



9 **a** $1 \pm \frac{2\sqrt{3}}{3}$ **b** $-1 \pm \frac{\sqrt{10}}{2}$ **c** $1 \pm \frac{\sqrt{10}}{2}$

d $\frac{-3 \pm \sqrt{15}}{2}$ **e** $2 \pm \frac{\sqrt{6}}{2}$ **f** $1 \pm \frac{\sqrt{30}}{5}$

10 $y = (x + 1)^2 - 6 = x^2 + 2x - 5$

11 **a** anything with $b^2 - 4ac > 0$

b anything with $b^2 - 4ac = 0$

c anything with $b^2 - 4ac < 0$

12 Number under square root = 0, therefore $x = \frac{-b}{2a}$ (one solution)

13 $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$

14 $y = -\frac{b^2}{4a} + c$

15

$$x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a} = 0$$

$$x^2 + \left(\frac{b}{a}\right)x = -\frac{c}{a}$$

$$x^2 + \left(\frac{b}{a}\right) + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} = -\frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

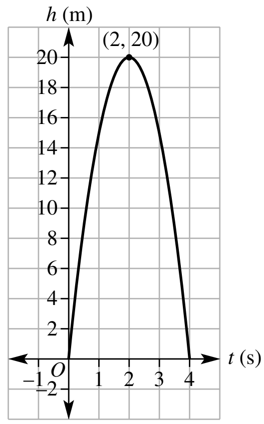
$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ as required}$$

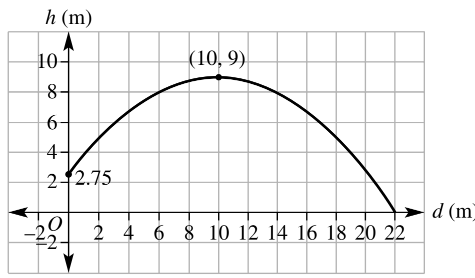
Exercise 9F

1 **a**



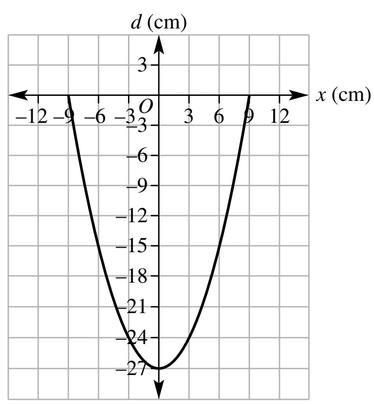
b $h = 20$ m **c** 4 s

2 **a**



b 9 m **c** 22 m

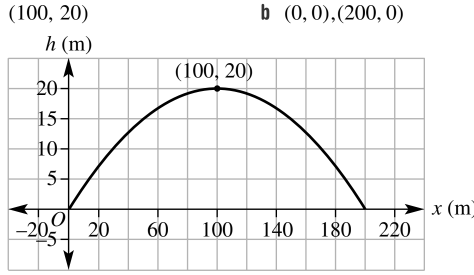
3 **a**



b 18 cm **c** 27 cm

4 **a** (100, 20) **b** (0, 0), (200, 0)

c



d 200 m **e** 20 m

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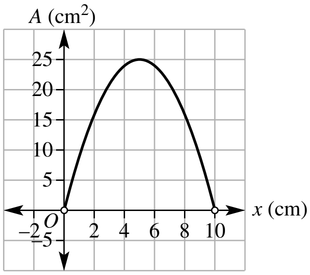
5 a $2 \times \text{length} = 20 - 2x$
$\text{length} = 10 - x$

b $A = x(10 - x)$ **c** $0 < x < 10$

d

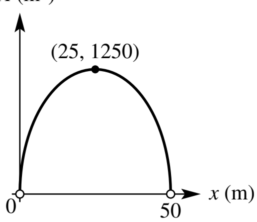


e 25 cm² **f** 5 cm by 5 cm

6 a $100 - 2x$

b $A = x(100 - 2x)$ **c** $0 < x < 50$

d

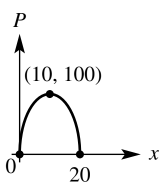


e 1250 m²

f breadth = 25 m, length = 50 m

7 a $20 - x$ **b** $P = x(20 - x)$

c

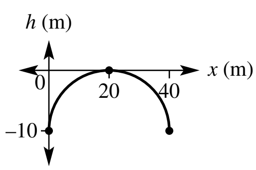


d i $x = 0$ or 20 **ii** $x = 10$

e 100

8 a (20, 0)

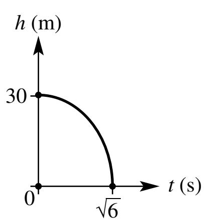
b



c 40 m **d** 10 m

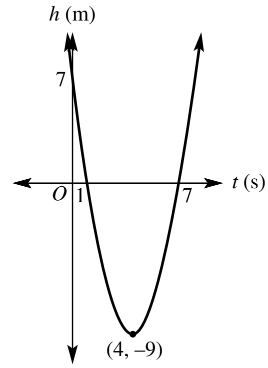
9 a $\sqrt{6}$ seconds

b



c $\sqrt{2}$ seconds

10 a



b i 1 second **ii** 7 seconds **iii** 4 seconds

c 9 m below sea level **d** at 3 and 5 seconds

11 a

P	0	1	2	3	4	5
H	0	0	1	3	6	10

b $H = \dfrac{P \times (P - 1)}{2}$

c 1225 handshakes

d 87 people

12 a 1 m

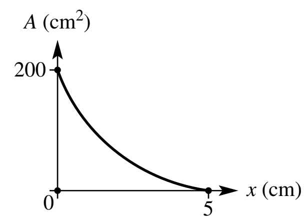
b No, 1 metre is the minimum height the kite falls to.

13 $P = x(64 - x)$ so maximum occurs at $x = 32$. Maximum product $= 32 \times (64 - 32) = 1024$.

14 a $A = (20 - 2x)(10 - 2x)$

b min: $x = 0$, max: $x = 5$

c

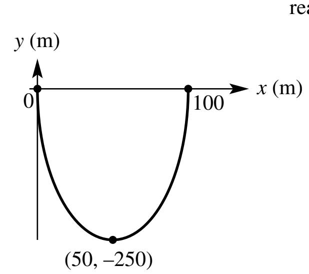


d Turning point occurs for an x value greater than 5.

e 1 cm

15 a 6 m **b** No, the maximum height reached is 4.5 m.

16 a



b i 2 **ii** none

c i (27.6, −200) and (72.4, −200)

ii (1.0, −10) and (99.0, −10)

d The highway meets the edge of the river (50 m along).

17 $5\frac{1}{24}$ m

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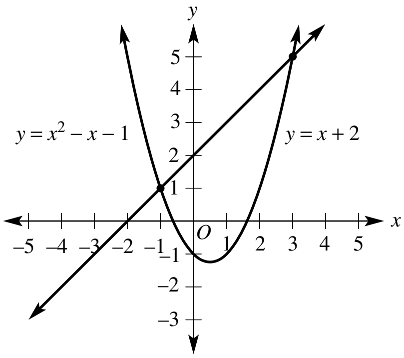
Exercise 9G

1 a one **b** zero, one or two

2 a (2, 12) **b** (−1, −3)

3 a



b (−1, 1) and (3, 5)

c
$$x^2 - x - 1 = x + 2$$
$$x^2 - 2x - 1 = 2$$
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$
$$x - 3 = 0 \text{ or } x + 1 = 0$$
$$x = 3 \text{ or } x = -1$$
When $x = 3, y = 3 + 2 = 5$
When $x = -1, y = -1 + 2 = 1$

4 a $x^2 + 3x + 6 = 0$ **b** $x^2 - 5x + 3 = 0$
c $x^2 + 3x - 12 = 0$ **d** $2x^2 + x - 1 = 0$
e $x^2 + 2x + 3 = 0$ **f** $x^2 - x + 4 = 0$

5 a $b^2 - 4ac < 0$ **b** $b^2 - 4ac > 0$ **c** $b^2 - 4ac = 0$

6 a (−3, 6) and (2, 6) **b** (−2, 12) and (6, 12)
c no solutions **d** (−3, −2) and $\left(-\frac{1}{2}, -2\right)$
e $\left(\frac{3}{2}, 0\right)$ **f** no solutions

7 a $x = 0, y = 0$ and $x = 3, y = 9$
b $x = 0, y = 0$ and $x = -2, y = 4$
c $x = -3, y = 9$ and $x = 6, y = 36$
d $x = 0, y = 5$ and $x = 3, y = 8$
e $x = -6, y = 34$ and $x = -2, y = 22$
f $x = -2, y = -3$ and $x = 3, y = 17$
g no solutions
h no solutions
i $x = -\frac{9}{2}, y = \frac{65}{2}$ and $x = -1, y = 8$
j $x = -\frac{5}{3}, y = -\frac{25}{3}$ and $x = 3, y = 1$
k $x = -3, y = 6$
l $x = -1, y = 2$

8 a $x = -4, y = 16$ and $x = 2, y = 4$
b $x = -1, y = 1$ and $x = 2, y = 4$
c $x = -1, y = 1$ and $x = \frac{1}{3}, y = \frac{1}{9}$
d $x = -2, y = 7$ and $x = -\frac{1}{2}, y = \frac{13}{4}$
e $x = -2, y = 0$ and $x = \frac{2}{3}, y = \frac{16}{9}$
f $x = -8, y = -55$ and $x = 2, y = 5$

9 a i no solutions
ii $x = -0.73, y = 1.54$ and $x = 2.73, y = 8.46$
iii $x = -1.37, y = -2.10$ and $x = 0.37, y = 3.10$
iv $x = -2.62, y = 8.24$ and $x = -0.38, y = 3.76$

b i $x = \frac{-1 \pm \sqrt{21}}{2}, y = \frac{-1 \pm \sqrt{21}}{2}$
ii $x = \frac{3 \pm \sqrt{5}}{2}, y = 3 \pm \sqrt{5}$
iii $x = \frac{-1 \pm \sqrt{13}}{2}, y = -1 \pm \sqrt{13}$
iv $x = \frac{-1 \pm \sqrt{17}}{2}, y = \pm\sqrt{17}$

10 a 2 **b** 0 **c** 2 **d** 0 **e** 1 **f** 2

11 Yes, the ball will hit the roof. This can be explained in a number of ways, using the discriminant we can see that the path of the ball intersects the equation of the roof $y = 10.6$.

12 a $x = -1, y = -2$ and $x = -\frac{1}{2}, y = -\frac{7}{4}$
b $x = \frac{5}{2}, y = -\frac{15}{4}$ and $x = 2, y = -4$
c $x = 1, y = 8$ and $x = 2, y = 7$
d $x = -6, y = -14$ and $x = 2, y = 2$

13 a (−1, 4) and $\left(\frac{1}{2}, 5\frac{1}{2}\right)$ **b** 212 m

14 a (3, −4)
b i $c > -4$ **ii** $c = -4$ **iii** $c < -4$

15 a $1 + 4k$
b i $k > -\frac{1}{4}$ **ii** $k = -\frac{1}{4}$ **iii** $k < -\frac{1}{4}$

16 a Discriminant from resulting equation is less than 0.
b $k \geqslant 2$

17 a $m = 2$ or $m = -6$
b The tangents are on different sides of the parabola, where one has a positive gradient and the other has a negative gradient.
c $m > 2$ or $m < -6$

Exercise 9H

1 a $f(x) = 8x$ **b** $f(x) = 9 - x^2$
c $f(x) = \frac{2}{x}$ **d** $f(x) = x(2x - 3)$
e $f(x) = 2^x + 1$

2 a true **b** true **c** false
d false **e** true

3 a $y \geqslant 0$ **b** $y > 0$ **c** $y > 9$
d $0 \leqslant y \leqslant 1$ **e** $y \geqslant 0$

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4 a function **b** function
c function **d** function
e not a function **f** function
g not a function **h** function
i not a function

5 a 4 **b** 10 **c** 28
d $5\frac{1}{2}$ **e** -2 **f** $3a + 4$

6 a 0 **b** 2 **c** -4
d 230 **e** 0.176 **f** $2k^3 - k^2 + k$

7 a $f(0) = 0, f(2) = 8, f(-4) = -16, f(a) = 4a,$
$f(a + 1) = 4a + 4$
b $f(0) = 1, f(2) = -3, f(-4) = -15, f(a) = 1 - a^2,$
$f(a + 1) = -a^2 - 2a$
c $f(0) = 1, f(2) = 4, f(-4) = \frac{1}{16}, f(a) = 2^a, f(a + 1) = 2^{a+1}$
d $f(0) = \text{undefined}, f(2) = 1, f(-4) = -\frac{1}{2}, f(a) = \frac{2}{a},$
$f(a + 1) = \frac{2}{(a + 1)}$
e $f(0) = -12, f(2) = 0, f(-4) = -12, f(a) = a^2 + 4a - 12,$
$f(a + 1) = a^2 + 6a - 7$
f $f(0) = 9, f(2) = 25, f(-4) = 73, f(a) = 4a^2 + 9,$
$f(a + 1) = 4a^2 + 8a + 13$

8 a all real x **b** all real x
c all real x **d** all real x
e all real x **f** all real x
g all real x **h** $x \neq 0$

9 a all real y **b** $y \geqslant 0$ **c** $y \geqslant 0$ **d** all real y
e $y > 0$ **f** $y > 0$ **g** $y \leqslant 2$ **h** $y \neq 0$

10 a i 5 **ii** -2 **iii** 3 **iv** -15 **v** 5
b $a = \frac{5}{3}$ represents the x value of the point where the line graphs intersect.

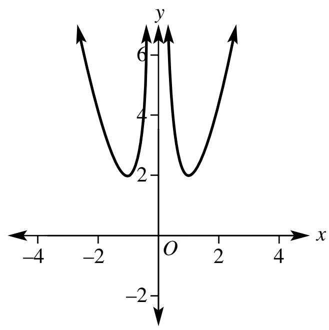
11 a i false **ii** false
b i false **ii** true
c i false **ii** false

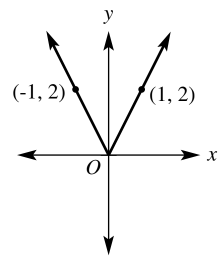
12 $4x + 2h - 3$

13 a They all pass the vertical line test, as each x value has only one y value.
b vertical lines in the form $x = a$
c The y value of the vertex is the maximum or minimum value of the parabola and therefore is essential when finding the range.
d i $y \geqslant -4$ **ii** $y \geqslant -12\frac{1}{4}$
iii $y \leqslant 1.125$ **iv** $y \geqslant 1$

14 a $x \neq 1$ **b** $x \neq -\frac{1}{2}$ **c** $x \neq 1$

15 a $x \geqslant 0$ **b** $x \geqslant 2$ **c** $x \geqslant -2$ **d** $x \leqslant 2$

16 a $f(a) = f(-a) = a^2 + \frac{1}{a^2}$
b

c The y-axis is the axis of symmetry for the function.

17 a i


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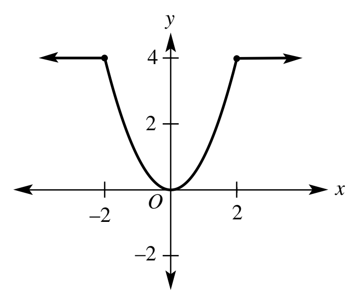
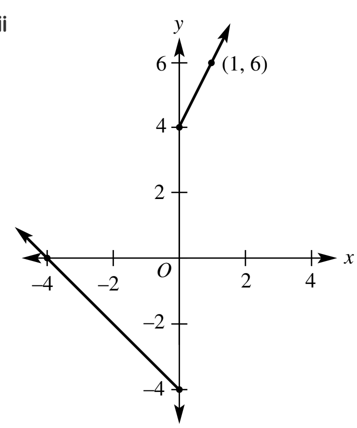
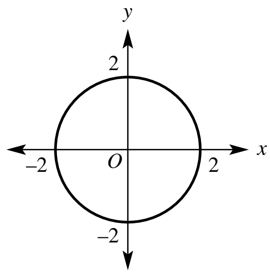

iii


b iii
c i $y \geqslant 0$ **ii** $0 \leqslant y \leqslant 4$ **iii** $y \geqslant -4$
d i 8, 8, 2 **ii** 34, 18, -2

Exercise 9I

1 $x^2 + y^2 = 4$



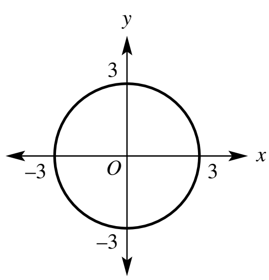
2 a $x = \pm\sqrt{5}$ **b** $x = \pm 4$ **c** $x = \pm\sqrt{3}$
d $y = \pm\sqrt{11}$ **e** $y = \pm\sqrt{57}$ **f** $y = \pm 2$

3 a (0, 0) or O **b** r

4 a $r = 6$ **b** $r = 9$ **c** $r = 12$
d $r = \sqrt{5}$ **e** $r = \sqrt{14}$ **f** $r = \sqrt{20} = 2\sqrt{5}$

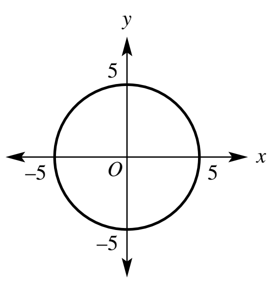
5 a $x^2 + y^2 = 4$ **b** $x^2 + y^2 = 49$ **c** $x^2 + y^2 = 10000$
d $x^2 + y^2 = 2601$ **e** $x^2 + y^2 = 6$ **f** $x^2 + y^2 = 10$
g $x^2 + y^2 = 1.21$ **h** $x^2 + y^2 = 0.25$

6 a (0, 0) **b** $r = 3$ **c** $y = \pm\sqrt{5}$
d $x = \pm\dfrac{\sqrt{27}}{2} = \pm\dfrac{3\sqrt{3}}{2}$
e



f $-3 \leqslant x \leqslant 3$ **g** $-3 \leqslant y \leqslant 3$

7 a (0, 0) **b** $r = 5$ **c** $y = \pm\dfrac{\sqrt{19}}{2}$ **d** $x = \pm 3$
e



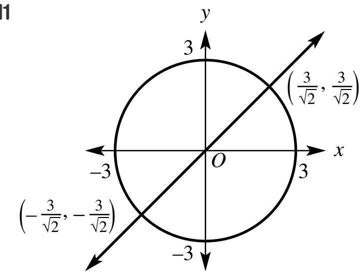
f $-5 \leqslant x \leqslant 5$ **g** $-5 \leqslant y \leqslant 5$

8 a $(1, \sqrt{3}), (1, -\sqrt{3})$
b $(-1, \sqrt{3}), (-1, -\sqrt{3})$
c $\left(\dfrac{1}{2}, \dfrac{\sqrt{15}}{2}\right), \left(\dfrac{1}{2}, -\dfrac{\sqrt{15}}{2}\right)$
d $\left(\dfrac{\sqrt{15}}{2}, -\dfrac{1}{2}\right), \left(-\dfrac{\sqrt{15}}{2}, -\dfrac{1}{2}\right)$
e $(0, -2)$
f $(2, 0), (-2, 0)$

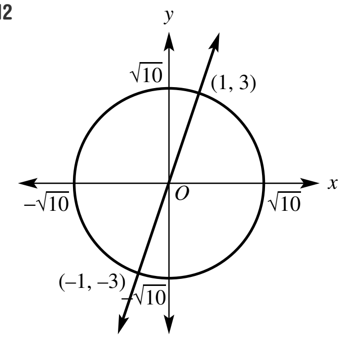
9 a ± 1 **b** ± 4
c $\pm\sqrt{3}$ **d** $\pm\sqrt{11}$

10 a $r = 2\sqrt{2}$ **b** $r = 2$ **c** $r = 3$
d $r = \sqrt{10}$ **e** $r = 2\sqrt{3}$ **f** $r = 2\sqrt{5}$

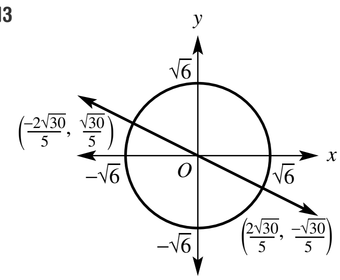
11



12



13



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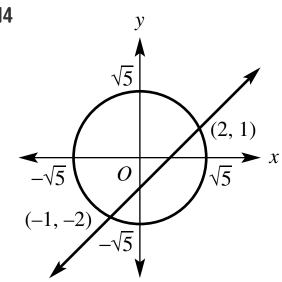

chord length $= 3\sqrt{2}$ units

15 a $m = \pm\sqrt{3}$ **b** $m > \sqrt{3}$ or $m < -\sqrt{3}$
c $-\sqrt{3} < m < \sqrt{3}$

16 a D **b** A **c** E **d** C **e** F **f** B

17 a $y = \pm\sqrt{16 - x^2} = \pm\sqrt{4^2 - x^2}$
b $x = \pm\sqrt{3 - y^2} = \pm\sqrt{(\sqrt{3})^2 - y^2}$

18 a Radius of graph is 2 so points are 2 units from (0, 0); i.e. <2.
b Radius of graph is 1 so points are 1 unit from (0, 0); i.e. -1 is the left-most point, which is not as far as -2.

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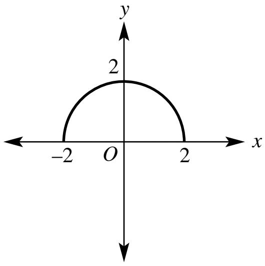
19 a i



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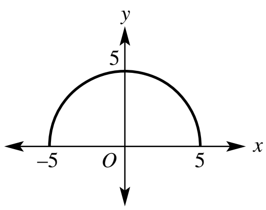

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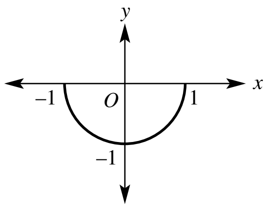

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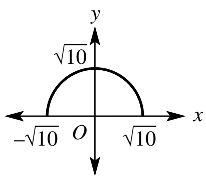

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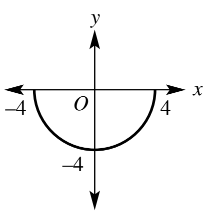

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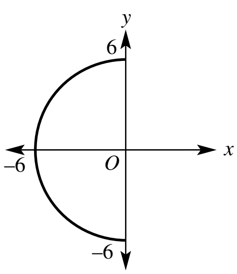
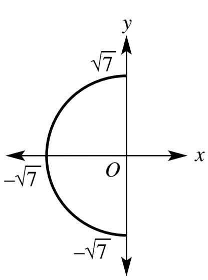


vii



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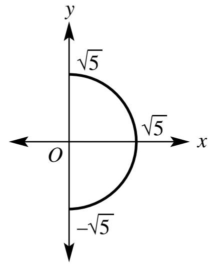
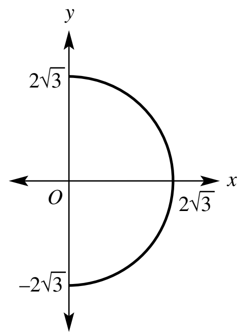


ix



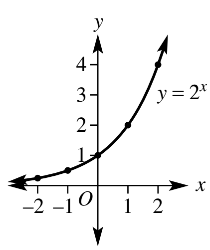
b **i** $y = \sqrt{25 - x^2}$ **ii** $y = -\sqrt{16 - x^2}$
iii $x = \sqrt{4 - y^2}$ **iv** $x = -\sqrt{1 - y^2}$
v $y = \sqrt{3 - x^2}$ **vi** $y = -\sqrt{5 - x^2}$
vii $x = \sqrt{10 - y^2}$ **viii** $x = -\sqrt{8 - y^2}$
ix $y = -\sqrt{18 - x^2}$

Exercise 9J

1 a

x	–2	–1	0	1	2
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

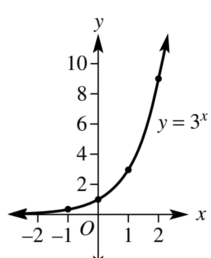
b



2 a

x	–2	–1	0	1	2
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

b



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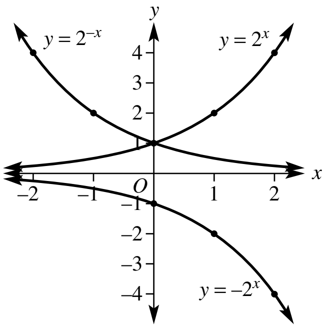
3 a

x	-2	-1	0	1	2
$y_1 = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$y_2 = -2^x$	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4
$y_3 = 2^{-x}$	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

b



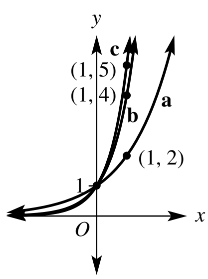
4 a $a^{-2} = \frac{1}{a^2} \neq -a^2$

b False since $3^{-2} = \frac{1}{3^2}$.

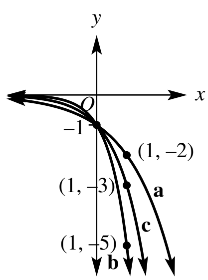
c $5^{-3}, 3^{-2}, 2^{-1}$

d $-9, -125, -\frac{1}{4}$

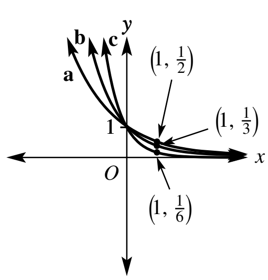
5



6



7



8 a i $(0, 1)$

ii $\left(-1, \frac{1}{3}\right)$

iii $(0, 1)$

iv $(2, 9)$

b i $(4, -16)$

ii $\left(-1, -\frac{1}{2}\right)$

iii $(0, -1)$

iv $(2, -4)$

c i $\left(1, \frac{1}{4}\right)$

ii $(-3, 64)$

iii $(0, 1)$

iv $\left(1, \frac{1}{4}\right)$

9 a $(2, 4)$

b $(2, 9)$

c $(1, -4)$

d $(-3, 8)$

10 a 1000

b i 2000

ii 8000

c i 2 years

ii 4 years

11 a $N = 2^t$

b $N = 2^{10} = 1024$

c 14 seconds

12 $x = 2.322$

13 a C

b A

c D

d E

e F

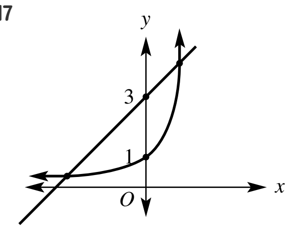
f B

14 Substitute $(2, 5)$ into the equation $y = 2^2 = 4 \neq 5$.

15 $y = 1$

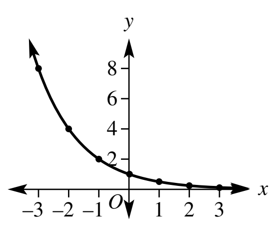
16 It is the asymptote.

17



Compared to $y = 2^x$, $y = x + 3$ meets the y-axis at 3; cuts the x-axis at -3; and maintains a constant gradient of 1. Therefore, the two graphs intersect twice.

18 a



They are the same graph.

b i $y = \left(\frac{1}{3}\right)^x$

ii $y = \left(\frac{1}{5}\right)^x$

iii $y = \left(\frac{1}{10}\right)^x$

c i $y = 4^{-x}$

ii $y = 7^{-x}$

iii $y = 11^{-x}$

d $\frac{1}{a} = a^{-1}$, thus $\left(\frac{1}{a}\right)^x = (a^{-1})^x = a^{-x}$ as required (or similar)

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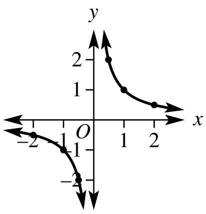
Exercise 9K

1 a

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y	$-\frac{1}{2}$	-1	-2	2	1	$\frac{1}{2}$

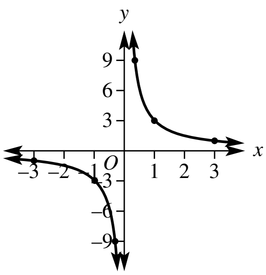
b



2 a

x	-3	-1	$-\frac{1}{3}$	$\frac{1}{3}$	1	3
y	-1	-3	-9	9	3	1

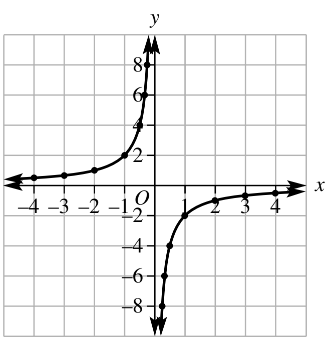
b



3 a

x	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y	$\frac{1}{2}$	1	2	4	8	-8	-4	-2	-1	$-\frac{1}{2}$

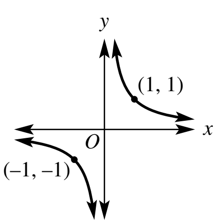
b



4 a $1 \div 0.1,\ 1 \div 0.01,\ 1 \div 0.001,\ 1 \div 0.00001$

b $x = \dfrac{1}{100}$ **c** 0.099 **d** 998

5 a



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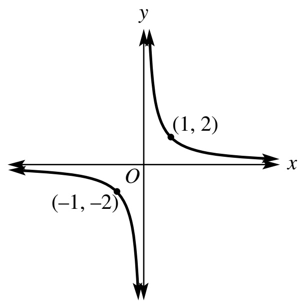

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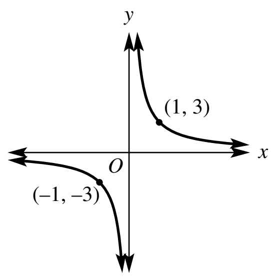

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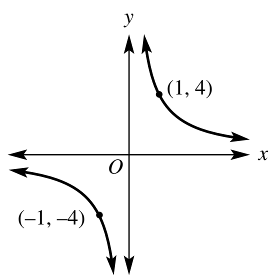

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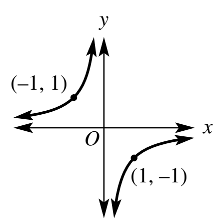

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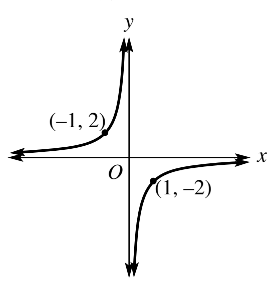

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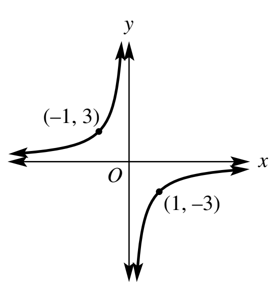

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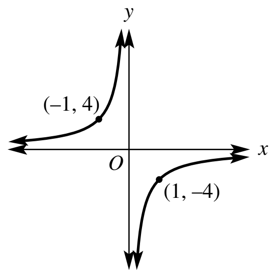
h



6 a (2, 1) **b** $\left(4, \frac{1}{2}\right)$

c (−1, −2) **d** $\left(-6, -\frac{1}{3}\right)$

7 a $\left(10, -\frac{1}{2}\right)$ **b** $\left(-4, \frac{5}{4}\right)$

c $\left(-7, \frac{5}{7}\right)$ **d** $\left(9, -\frac{5}{9}\right)$

8 a (1, 3) **b** (3, 1)

c $\left(-\frac{3}{2}, -2\right)$ **d** $\left(-\frac{1}{2}, -6\right)$

9 a yes **b** yes **c** no **d** no

10 a $\left(\frac{1}{2}, 2\right)$ **b** $\left(\frac{1}{6}, 6\right)$ **c** (−1, −1)

d $\left(-\frac{1}{10}, -10\right)$ **e** (1, 1), (−1, −1)

f $\left(-\frac{1}{2}, -2\right), \left(\frac{1}{2}, 2\right)$ **g** $\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right), \left(-\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$

h $\left(\frac{1}{\sqrt{5}}, \sqrt{5}\right), \left(-\frac{1}{\sqrt{5}}, -\sqrt{5}\right)$

11 a $\left(\frac{2}{3}, -3\right)$ **b** $\left(-\frac{1}{2}, 4\right)$

c $\left(4, -\frac{1}{2}\right)$ **d** $\left(-6, \frac{1}{3}\right)$

e (1, −2), (−1, 2) **f** $\left(\frac{1}{2}, -4\right), \left(-\frac{1}{2}, 4\right)$

g (2, −1), (−2, 1) **h** $(\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$

12 a E **b** C **c** D

d B **e** A **f** F

13 yes; $x = 0$ or $y = 0$

14 a zero **b** zero **c** infinity **d** infinity

15 The greater the coefficient, the closer the graph is to the asymptote.

16 a i $x = \frac{1 \pm \sqrt{5}}{2}, y = \frac{-1 \pm \sqrt{5}}{2}$

ii $x = 1 \pm \sqrt{2}, y = -1 \pm \sqrt{2}$

iii $x = -1 \pm \sqrt{2}, y = 1 \pm \sqrt{2}$

b no intersection, $\Delta < 0$

c $y = -x + 2, y = -x - 2$

Exercise 9L

1 a 27 **b** 125 **c** −64

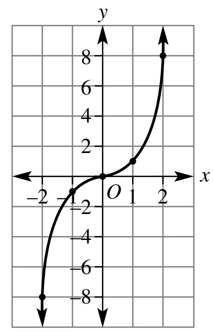
d −1000 **e** −216 **f** 2

g 1 **h** −3 **i** −5

2 a 0 **b** 54 **c** −2 **d** −128

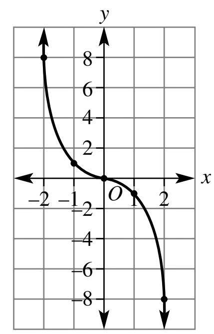
3 a

x	−2	−1	0	1	2
y	−8	−1	0	1	8



b

x	−2	−1	0	1	2
y	8	1	0	−1	−8



4 a (0, 2) **b** D

5 a $x = 4$ **b** $x = 5$ **c** $x = -3$

d $x = -2$ **e** $x = 10$ **f** $x = -3$

g $x = 6$ **h** $x = -3$ **i** $x = \frac{1}{2}$

j $x = \frac{1}{3}$ **k** $x = \frac{1}{5}$ **l** $x = \frac{1}{2}$

6 a $x = 1$ **b** $x = 3$ **c** $x = -1.4$

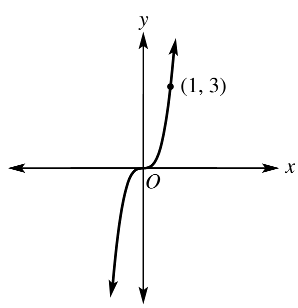
d $x = 3$ **e** $x = 4$ **f** $x = 6.1$

g $x = -2.9$ **h** $x = -4.3$

7 a $x = \sqrt[3]{21}$ **b** $x = \sqrt[3]{6}$

c $x = \sqrt[3]{-2}$ **d** $x = \sqrt[3]{18}$

8 a



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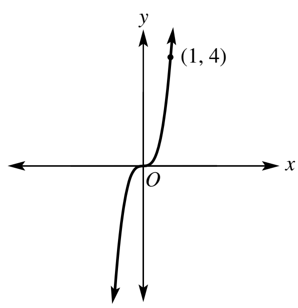

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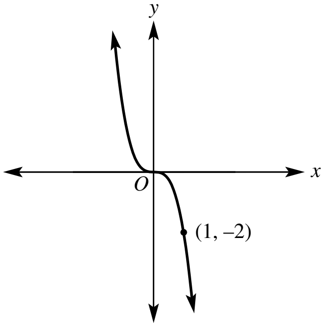

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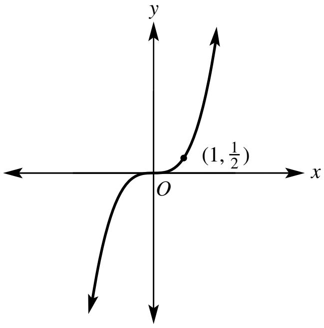

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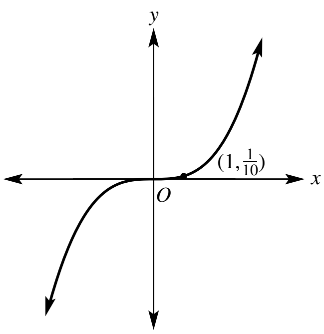

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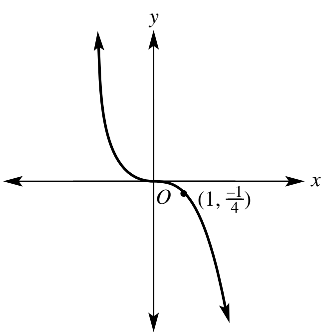
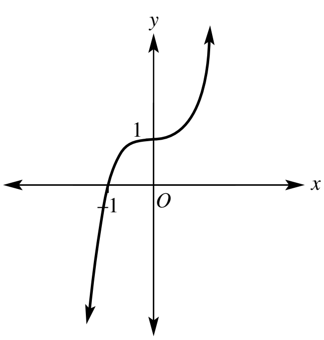


9 a



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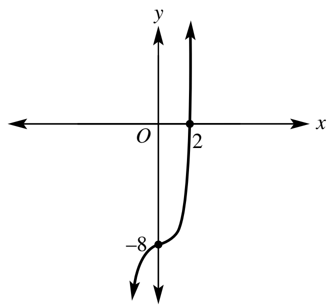

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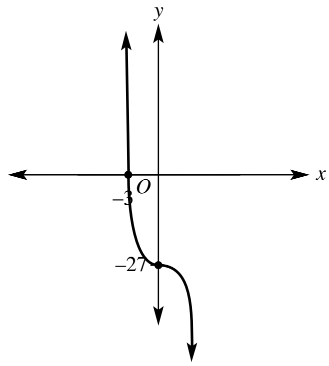

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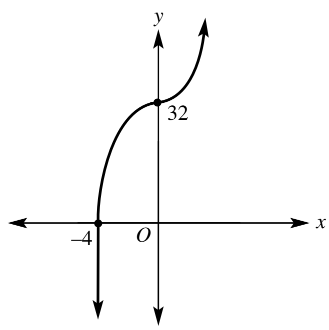

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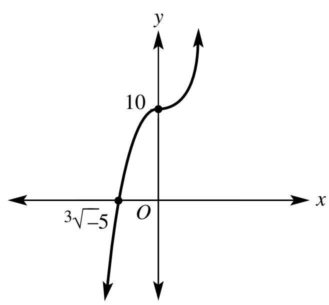

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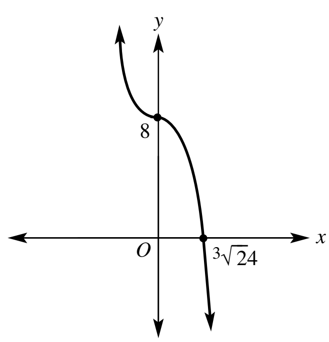

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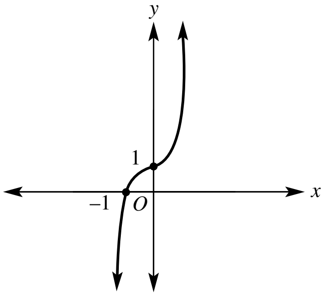
10 a



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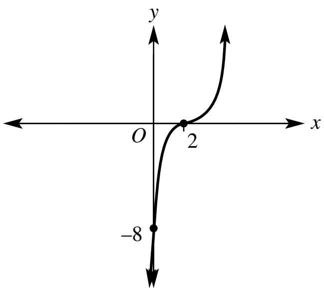

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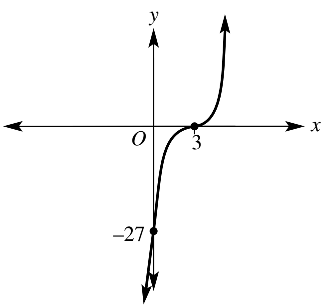

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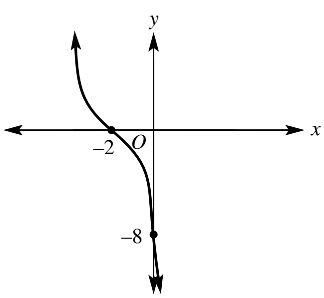

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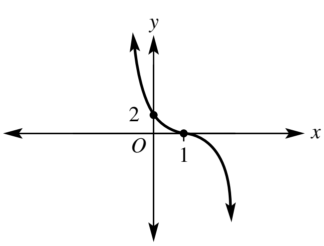

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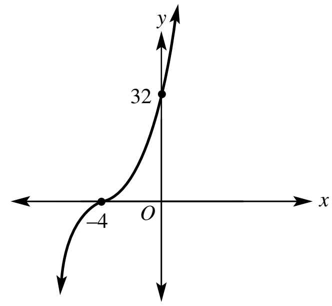
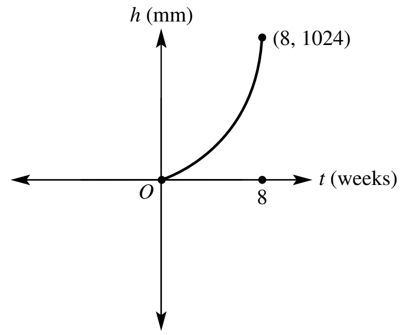


11 384 cm^2

12 a 54 mm **b** 5 weeks **c** after 8 weeks

d



13 a $x = 1$ **b** $x = 5$ **c** $x = -7$

d $x = 3$ **e** $x = -2$ **f** $x = -14$

14 6371 km

15 a $(-x)^2 = (-x) \times (-x) = x^2$

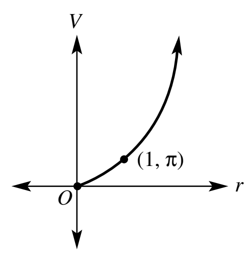
b $(-x)^3 = (-x) \times (-x) \times (-x) = -x^3$

c If n is odd, then $(-x)^n = -x^n$ for all values of x.

16 a $V = \pi r^3$

b $\sqrt{\dfrac{8000}{\pi}} = \dfrac{20}{\sqrt[3]{\pi}}$ units

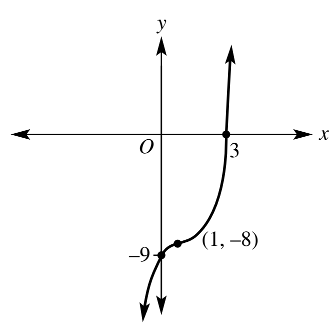
c Volume increases by a factor of 8; i.e. $(2)^3$.



17 a $a = -\dfrac{1}{2}, d = 0$ **b** $a = 2, d = 3$

c $a = 1, d = -2$ **d** $a = 1, d = 2$

18 a



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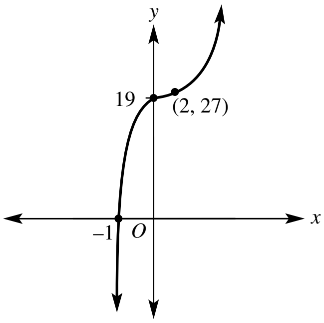

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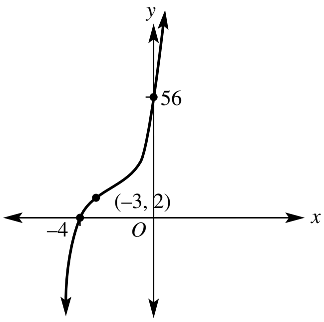

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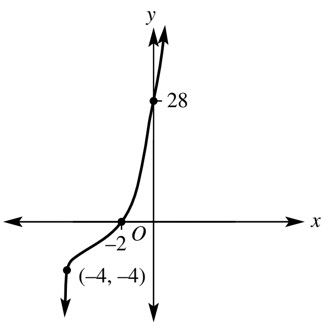

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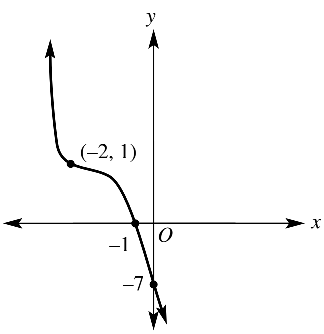

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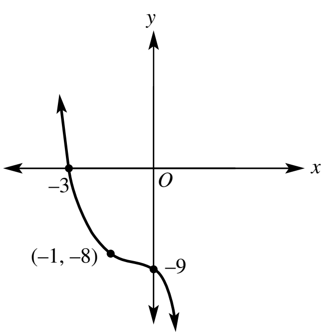

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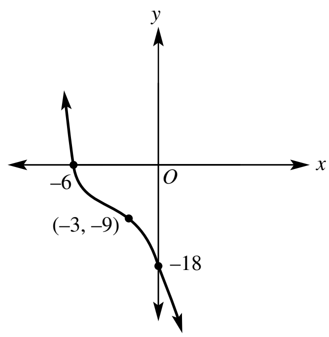

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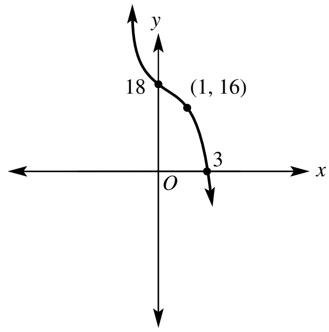
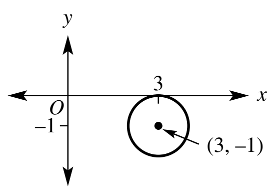


Exercise 9M

1 **a** up **b** down **c** right **d** left
e right **f** left **g** up **h** down
i right **j** left **k** down **l** up

2 **a**



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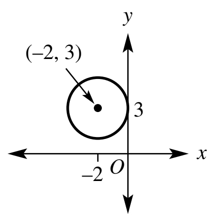

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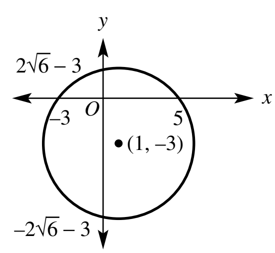

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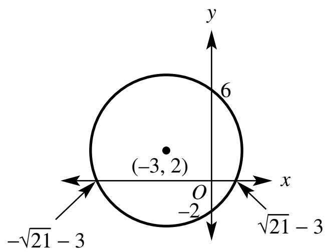

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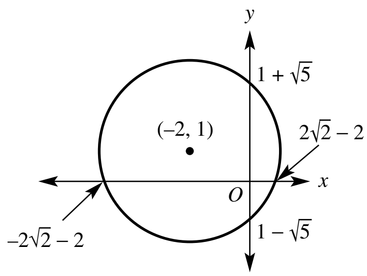

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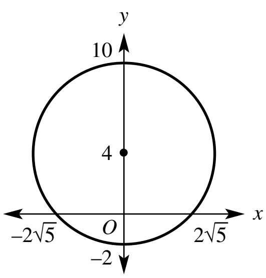

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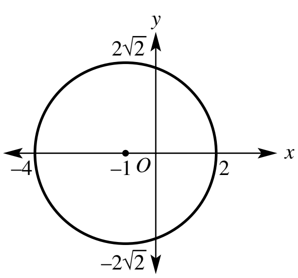

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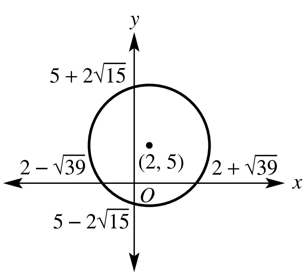

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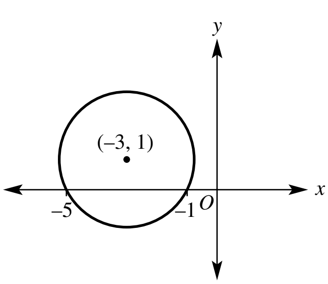

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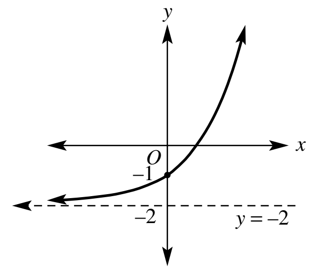

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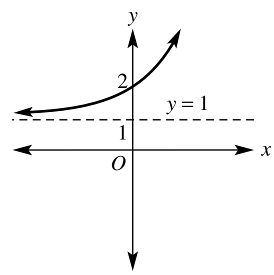

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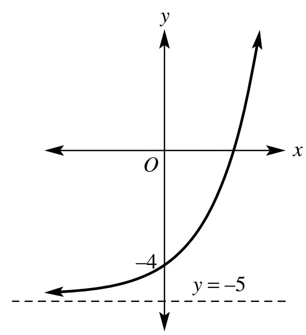

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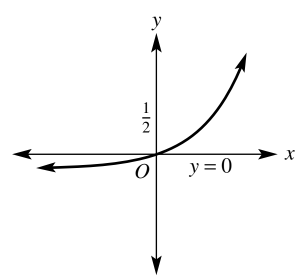

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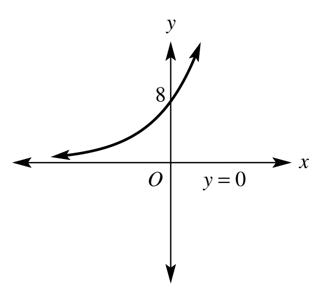

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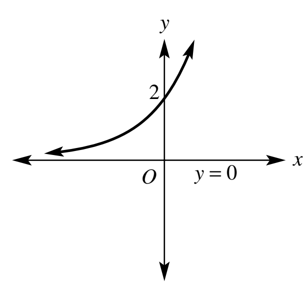

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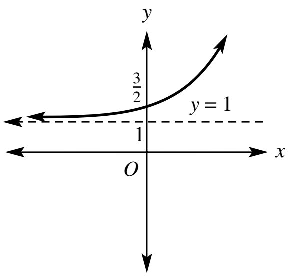
g



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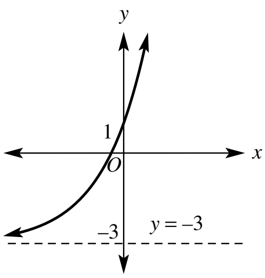

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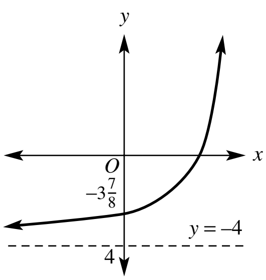
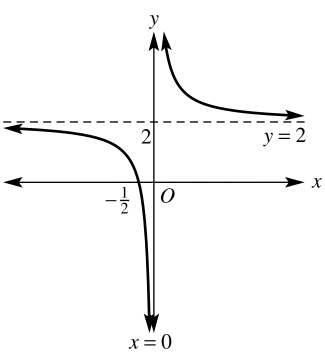


4 a



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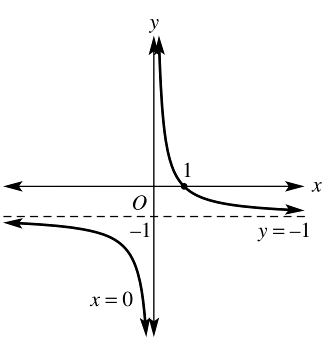

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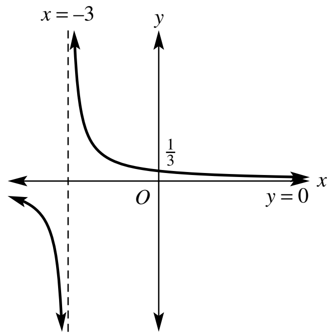

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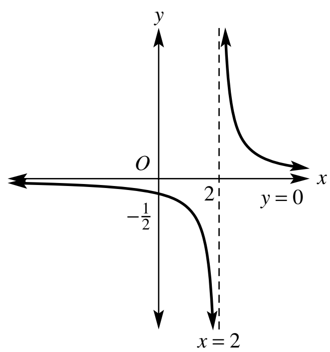

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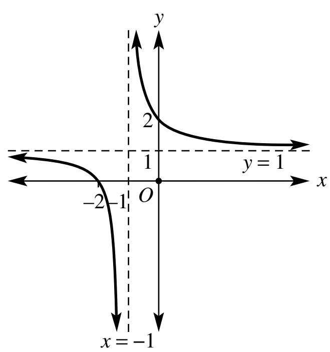

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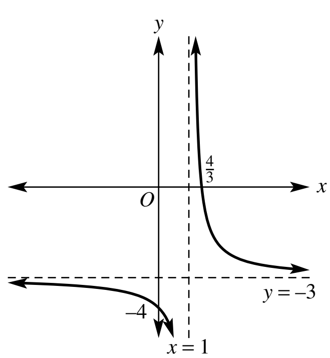

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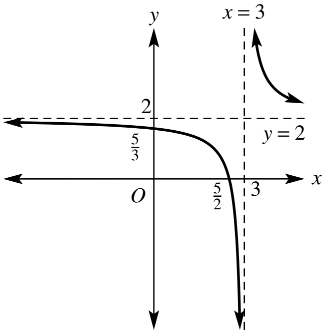

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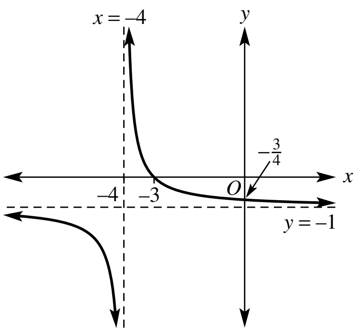

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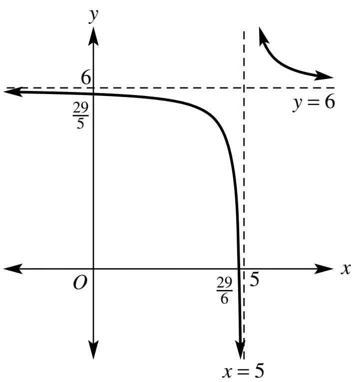

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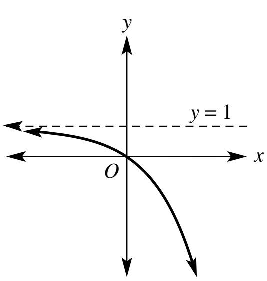

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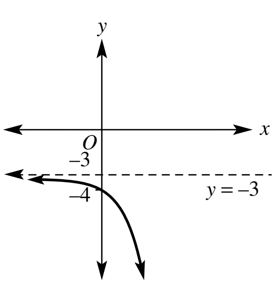

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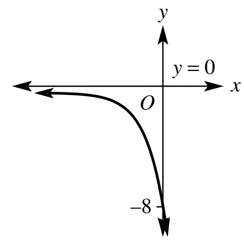

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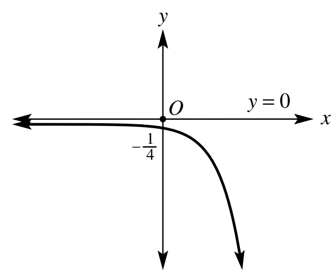

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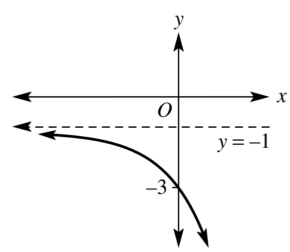

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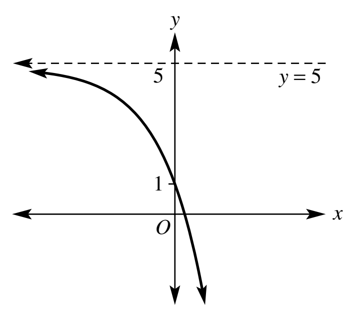

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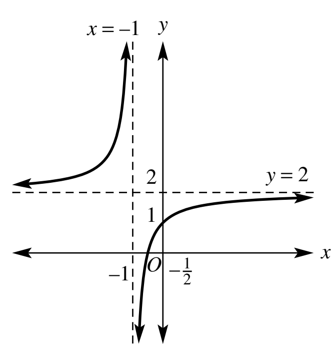

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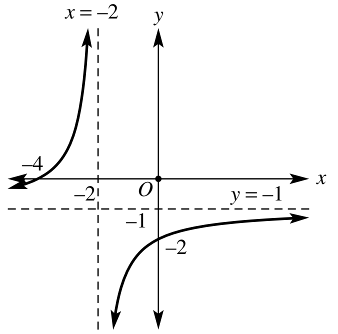

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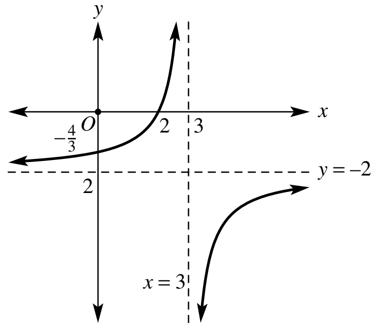

7 a $y = \dfrac{1}{x-2} - 1$ **b** $y = \dfrac{1}{x+1} + 3$ **c** $y = \dfrac{1}{x-1} + \dfrac{3}{2}$

8 a $\left(\dfrac{-3-\sqrt{5}}{2}, \dfrac{1-\sqrt{5}}{2}\right), \left(\dfrac{-3+\sqrt{5}}{2}, \dfrac{1+\sqrt{5}}{2}\right)$

b $\left(\sqrt{5}, 3+\sqrt{5}\right), \left(-\sqrt{5}, 3-\sqrt{5}\right)$

c $\left(\dfrac{-1-\sqrt{11}}{2}, \sqrt{11}\right), \left(\dfrac{-1+\sqrt{11}}{2}, -\sqrt{11}\right)$

d $(1, 2), \left(-\dfrac{3}{5}, -\dfrac{6}{5}\right)$

e $(-6, 3), (-2, -1)$

f $(3, 0), (-3, -2)$

9 a max $x = 5$, min $x = 1$

b max $y = 0$, min $y = -4$

10 a $(x-2)^2 + (y-1)^2 = 8$ **b** $(x+2)^2 + y^2 = 25$

c $(x+5)^2 + (y+3)^2 = 18$ **d** $y = \dfrac{1}{x-1} + 1$

e $y = \dfrac{1}{x+2} - 1$ **f** $y = \dfrac{-1}{x+3}$

11 a Solving $\dfrac{1}{x} = -x$ would require $x^2 = -1$, which is not possible.

b Circle has centre $(1, -2)$ and radius 2, so maximum 3 value on the circle is 0, which is less than 1.

c Exponential graph rises more quickly than the straight line and this line sits below the curve.

d Solving $\dfrac{2}{x+3} - 1 = \dfrac{1}{3x}$ gives a quadaratic with $\Delta < 0$, thus no points of intersection.

12 a i $(x+2)^2 + (y-1)^2 = 4, C(-2, 1), r = 2$

ii $(x+4)^2 + (y+5)^2 = 36, C(-4, -5), r = 6$

iii $(x-3)^2 + (y-2)^2 = 16, C(3, 2), r = 4$

iv $(x-1)^2 + (y+3)^2 = 15, C(1, -3), r = \sqrt{15}$

v $(x+5)^2 + (y+4)^2 = 24, C(-5, -4), r = 2\sqrt{6}$

vi $(x+3)^2 + (y+3)^2 = 18, C(-3, -3), r = 3\sqrt{2}$

vii $\left(x+\dfrac{3}{2}\right)^2 + (y-3)^2 = \dfrac{29}{4}, C\left(-\dfrac{3}{2}, 3\right), r = \dfrac{\sqrt{29}}{2}$

viii $\left(x+\dfrac{5}{2}\right)^2 + (y-2)^2 = \dfrac{49}{4}, C\left(-\dfrac{5}{2}, 2\right), r = \dfrac{7}{2}$

ix $\left(x-\dfrac{1}{2}\right)^2 + \left(y+\dfrac{3}{2}\right)^2 = \dfrac{3}{2}, C\left(\dfrac{1}{2}, -\dfrac{3}{2}\right), r = \sqrt{\dfrac{3}{2}}$

x $\left(x-\dfrac{3}{2}\right)^2 + \left(y-\dfrac{5}{2}\right)^2 = \dfrac{25}{2}, C\left(\dfrac{3}{2}, \dfrac{5}{2}\right), r = \dfrac{5}{\sqrt{2}}$

b $(x+2)^2 + (y-3)^2 = -2$, radius can't be negative

Exercise 9N

1 a direct proportion

b inverse proportion

c inverse proportion

d direct proportion

e neither

f inverse proportion

2 a Straight line with y-intercept; neither direct nor inverse (indirect) proportion.

b Straight line starting at (0, 0); direct proportion.

c Upward sloping curve so as x increases, y increases; neither direct nor inverse (indirect) proportion.

d Hyperbola shape so as x increases, y decreases; inverse (indirect) proportion.

3 a Fixed distance from home, zero gradient, stationary.

b Decreasing distance from home, negative constant gradient, lower constant speed.

c Increasing distance from home, positive constant gradient, higher constant speed.

d Increasing distance from home, positive varying gradient, increasing speed, accelerating.

e Increasing distance from home, positive varying gradient, decreasing speed, decelerating.

f Decreasing distance from home, negative varying gradient, decreasing speed, decelerating.

g Decreasing distance from home, negative varying gradient, increasing speed, accelerating.

4 a i $p = 4q$ **ii** $p = 60$ **iii** $q = 25$

b i $p = 50q$ **ii** $p = 750$ **iii** $q = 4$

5 a i $k = 72, y = \dfrac{72}{x}$ **ii** $y = 2$ **iii** $x = 24$

b i $k = 50, y = \dfrac{50}{x}$ **ii** $y = 0.5$ **iii** $x = 0.5$

6 A Positive variable rate of change, increasing speed, accelerating.

B Positive constant rate of change, constant speed.

C Positive varying rate of change, decreasing speed, decelerating.

D Zero rate of change, stationary.

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E Negative varying rate of change, increasing speed, accelerating.

F Negative constant rate of change, constant speed.

G Negative varying rate of change, decreasing speed, decelerating.

7 a y is increasing at an increasing rate.

b y is increasing at a decreasing rate.

c y is decreasing at an increasing rate.

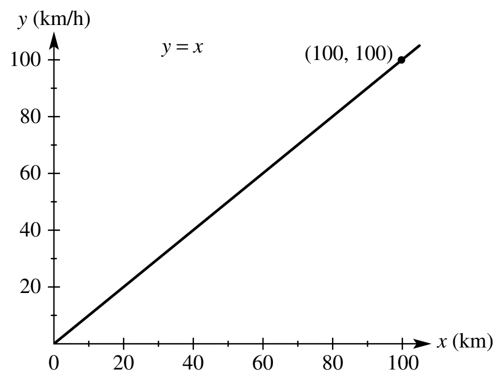
d y is decreasing at a decreasing rate.

8 a $k = \$244$/tonne **b** $P = 244n$

c $33 184 **d** 1175 tonnes

9 a $C = \frac{74}{s}$ **b** $4.93 **c** $2.47

10 a



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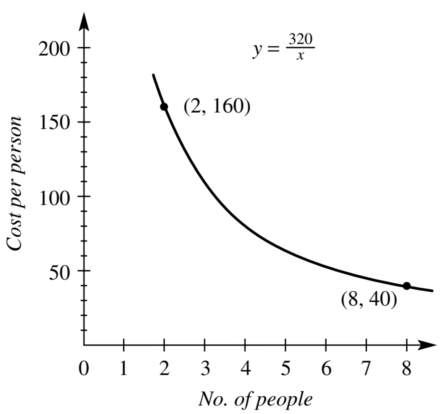

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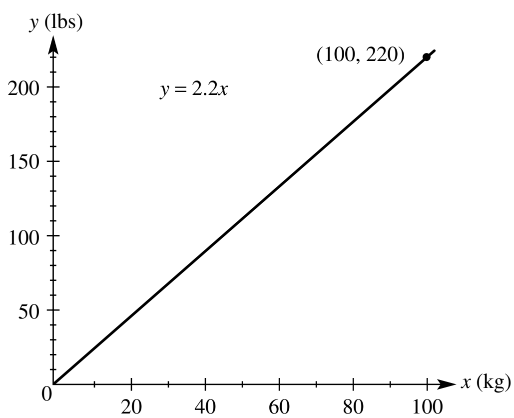

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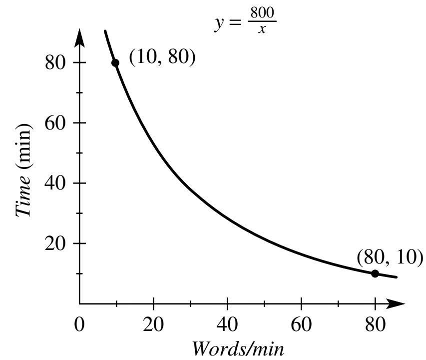


11 a decreasing at a decreasing rate

b increasing at an increasing rate

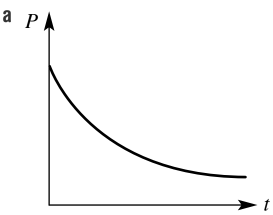
c increasing at a decreasing rate

d decreasing at an increasing rate

e decreasing at a constant rate

f increasing at a constant rate

12 a



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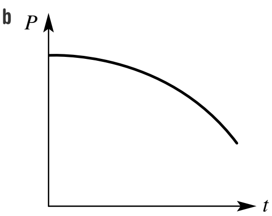

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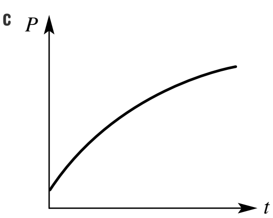

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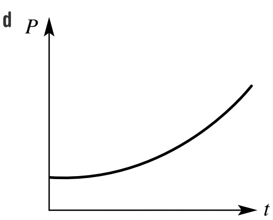
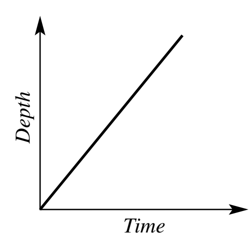


13 a



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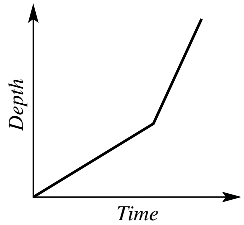

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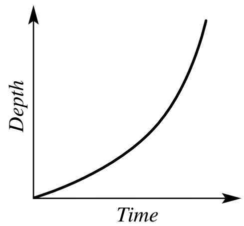

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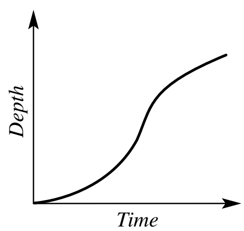

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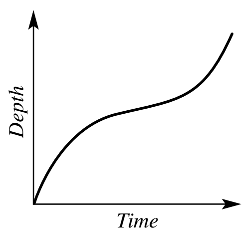

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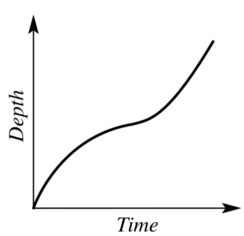
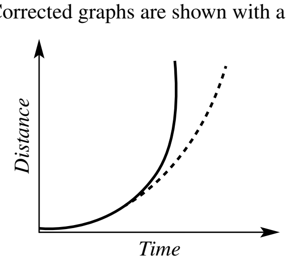


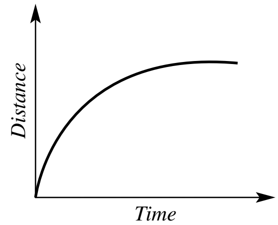
14 Corrected graphs are shown with a dashed line.

a



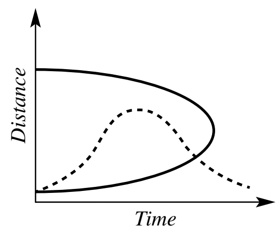
Vertical line incorrect. Can't change distance instantaneously.

b



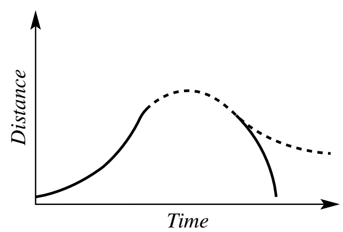
Graph correct.

c



Can't be in two places simultaneously. Curve must increase in gradient, turn, decrease in gradient.

d



No breaks in curve as continuous motion describes. Final deceleration segment needs a curve becoming flatter, showing a decreasing gradient.

15 **A** & **d**: School bus; distance increases at an increasing rate (acceleration), then a constant rate (steady speed) and then a decreasing rate (deceleration) becoming a zero rate (stopped).

B & **a**: Soccer player; distance increases at a constant rate (steady speed), then a zero rate (stopped) and then at an increasing rate (acceleration).

C & **c**: Motor bike; distance increases at a constant rate (steady speed), then at an increasing rate (acceleration), and then at a decreasing rate (deceleration) becoming a zero rate (stopped).

D & **b**: Rocket booster; distance increases at an increasing rate (upward acceleration), then a decreasing rate (deceleration when detached) becoming zero (fleetingly stopped). Distance then decreases at an increasing rate (acceleration towards Earth) and finally distance decreases at a constant rate (steady fall to Earth with parachute).

16 Various solutions; check with your teacher.

Exercise 90

1 **a** $A > 0, b > 0, h > 0$ **b** $b = \frac{2A}{h}$

c $h = \frac{2A}{b}$ **d** **i** 5 **ii** 5

e No, can't divide by 0.

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2 a i 0 **ii** 16 **iii** 1 **iv** 4 **v** 81

b all real numbers **c** $y \geqslant 0$

d $x = \pm\sqrt{y}$ **e** no

3 a i $-\frac{1}{3}$ **ii** $\frac{1}{2}$ **iii** undefined

b i 4 **ii** $\frac{1}{4}$ **iii** undefined

c $x \neq 0, y \neq 0$

4 a $a = \frac{P-b}{2}$ **b** $b = \frac{P-2\ell}{2}$ **c** $a = 2M - b$

d $c = \frac{b^2 - D}{4a}$ **e** $t = \frac{d}{s}$ **f** $r = \sqrt{\frac{A}{\pi}}$

g $a = \frac{2(s-ut)}{t^2}$ **h** $b = \pm\sqrt{c^2 - a^2}$ **i** $a = \frac{2A}{h} - b$

j $x = \sqrt{9 - y^2}$ **k** $h = \frac{Q^2}{2g}$ **l** $h = \frac{V}{\pi r^2}$

5 a $a = \pm\sqrt{b}$ **b** $a = \pm\sqrt{c^2 - b^2}$ **c** $a = \pm\sqrt{b+1}$

6 a no **b** $b = \frac{a}{c^2}$ **c** yes; $c \neq 0$.

d $c = \pm\sqrt{\frac{a}{b}}$ **e** yes; $b \neq 0$.

7 a $x \neq 0, y \neq 0$ **b** $x \neq 0, y \neq 2$

c $x \neq 0, y \neq -3$ **d** $x \neq 4, y \neq 0$

e $x \neq -2, y \neq 0$ **f** $x \neq 3, y \neq 2$

8 a $V \neq 0$ **b** $t \neq 0$

9 a $x = \frac{A}{y}$ **b** $x = \frac{2A}{h} - y$ **c** $x = 180 - y$

d $x = \sqrt{z^2 - y^2}$ **e** $x = \sqrt{\frac{4A}{\pi}}$ **f** $x = \sqrt{\frac{V}{h}}$

10 a i 5 **ii** −10 **b** 32

c $F = \frac{9}{5}C + 32$ **d** 212

e No; C and F can take any value.

11 a $x = \pm\sqrt{4 - y^2}$ **b** $y = \pm\sqrt{4 - x^2}$

c $-2 \leqslant x \leqslant 2, -2 \leqslant y \leqslant 2$

d i $-3 \leqslant x \leqslant 3, -3 \leqslant y \leqslant 3$

ii $-5 \leqslant x \leqslant 5, -5 \leqslant y \leqslant 5$

e Semicircle below x-axis with centre (0, 0) and radius 5.

12 a No; $2 + 3 = 5$ but $\frac{1}{2} + \frac{1}{3} \neq \frac{1}{5}$.

b $z = \frac{xy}{x+y}$

c $x = \frac{yz}{y-z}$

13 a i 2 **ii** −3 **iii** $\sqrt[3]{y}$

b i no **ii** no

c $r = \sqrt[3]{\frac{3V}{4\pi}}$

14 Yes; $x \geqslant 0$ in the first equation but $x > 0$ in the second equation.

15 a i 2.6 **ii** 3.1

b i $AB = \sqrt{2r^2(1 - \cos 0°)}$
$= \sqrt{2r^2(1-1)}$
$= 0$

ii $AB = \sqrt{2r^2(1 - \cos 180°)}$
$= \sqrt{2r^2(1-(-1))}$
$= \sqrt{4r^2}$
$= 2r$

c $r = \sqrt{\frac{AB^2}{2(1-\cos\theta)}}$ **d** 9

e $r > 0, AB > 0, 0° < \theta < 360°$.

Exercise 9P

1 a false **b** true **c** true **d** false

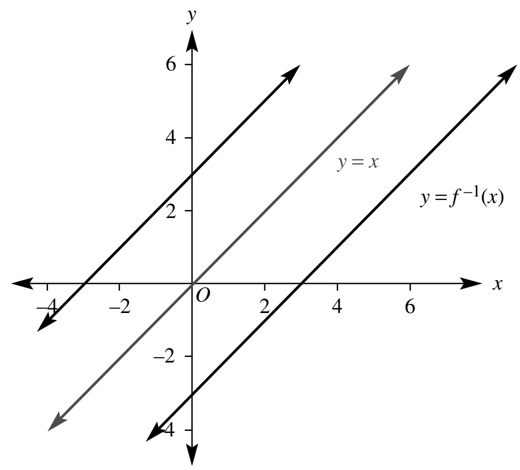
2 $C(2, 0), D(4, 1)$

3 a A horizontal line cuts the function at only one point.

b A horizontal line cuts the function at only one point.

c A horizontal line cuts the function at only one point.

4 a



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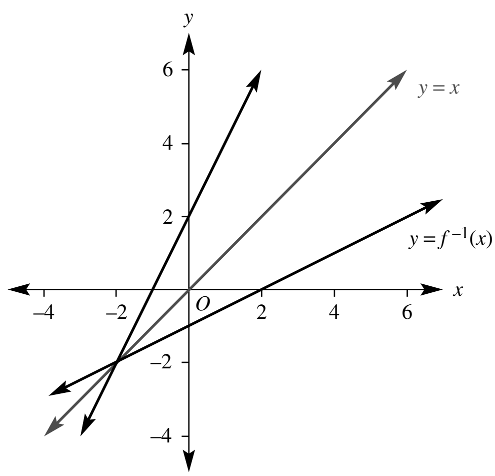

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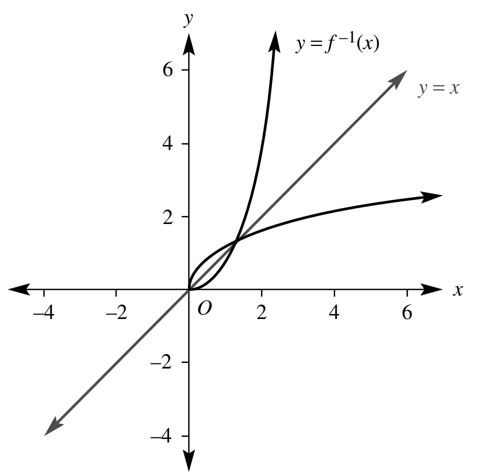

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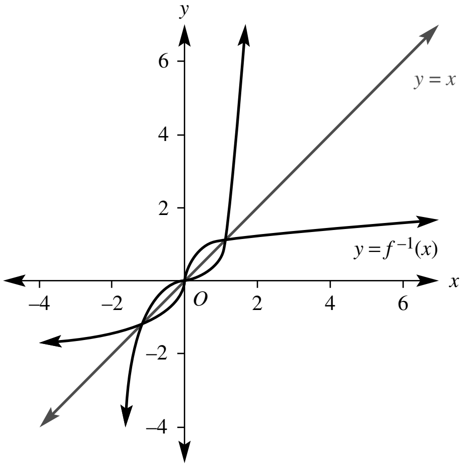


5 Note: Graphs not shown for this question.

a $f^{-1}(x) = x + 4$

b $f^{-1}(x) = \dfrac{x}{6}$

c $f^{-1}(x) = \dfrac{x - 4}{2}$

d $f^{-1}(x) = 4x$

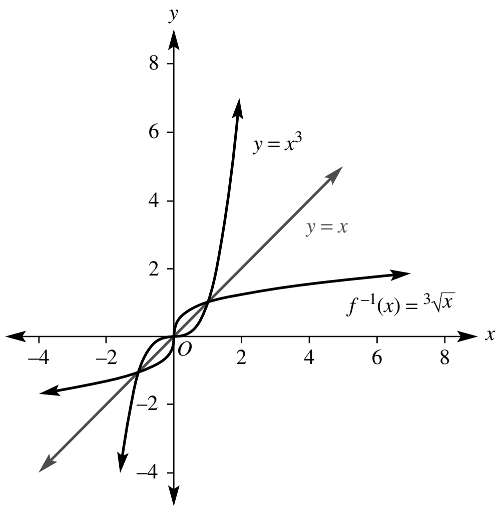
e $f^{-1}(x) = 2 - x$

f $f^{-1}(x) = \dfrac{6 - x}{3}$

g $f^{-1}(x) = 2x - 1$

h $f^{-1}(x) = \dfrac{1}{x}$

6 **a**



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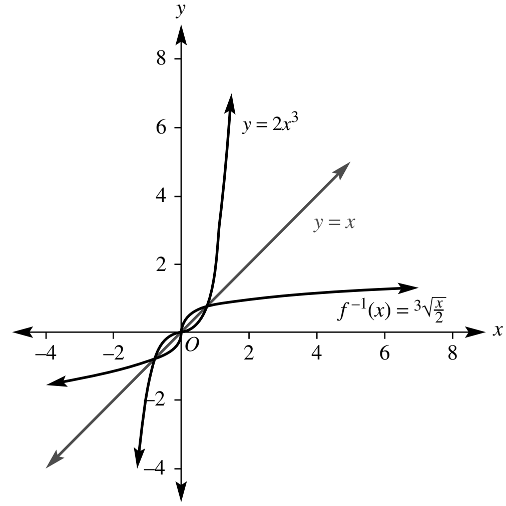

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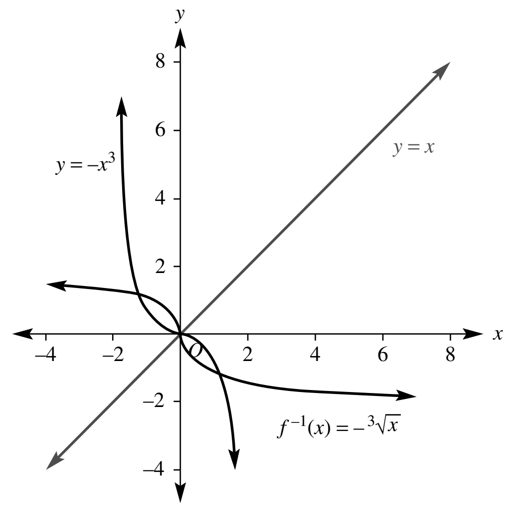

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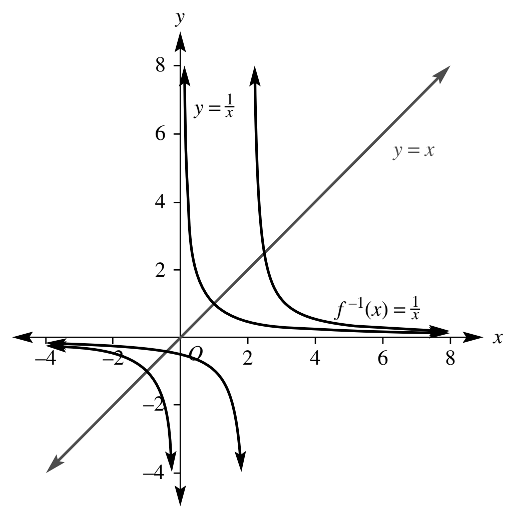

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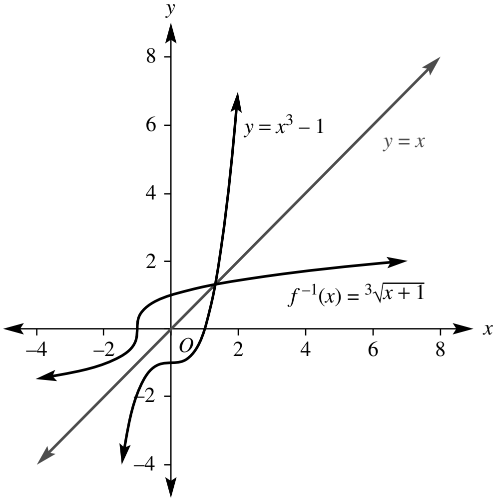

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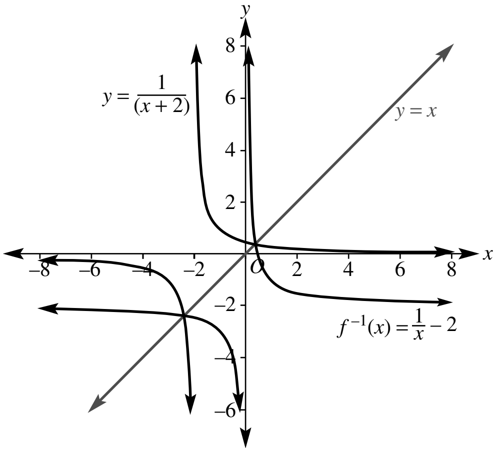

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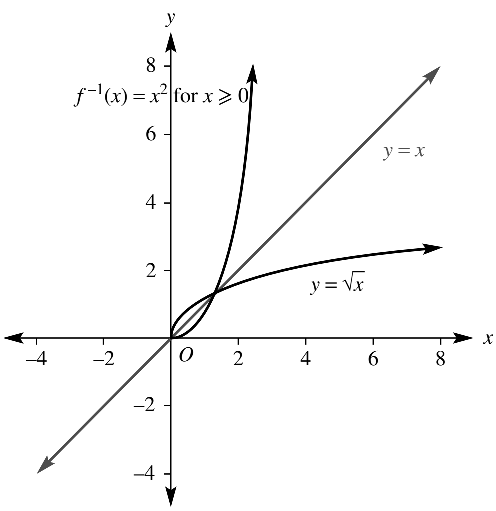

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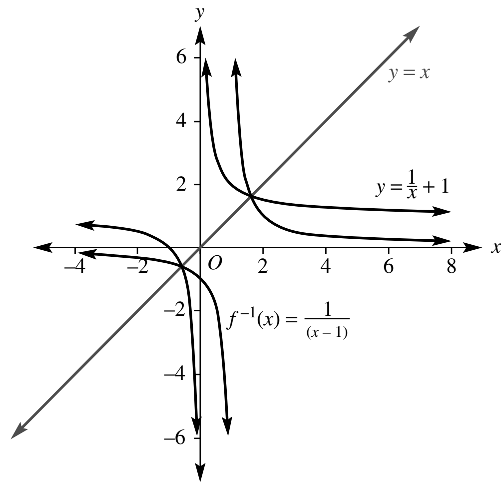


7 **a** $x \geqslant 1$ **b** $x \geqslant 2$ **c** $x \geqslant 0$

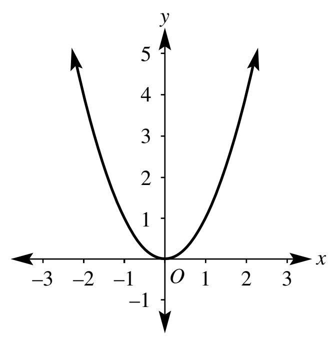
8 **a** Domain of $f^{-1}(x)$ is $2 \leqslant x \leqslant 4$; range of $f^{-1}(x)$ is $-1 \leqslant y \leqslant 1$.

b Domain of $f^{-1}(x)$ is $1 \leqslant x \leqslant 5$; range of $f^{-1}(x)$ is $-3 \leqslant y \leqslant -1$.

c Domain of $f^{-1}(x)$ is $0 \leqslant x \leqslant 5$; range of $f^{-1}(x)$ is $-2 \leqslant y \leqslant 0$.

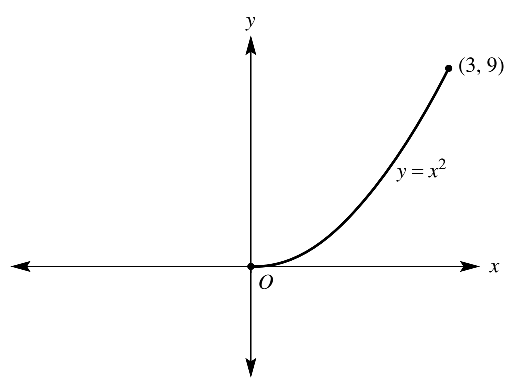
Note: Graphs not shown for this question.

9 **a**



b A horizontal line cuts the function at more than one point.

c



d $f^{-1}(x) = \sqrt{x}$; domain: $0 \leqslant x \leqslant 9$; range: $0 \leqslant y \leqslant 3$

10 **a** $x = 0$ and $x = 6$

b vertex $(3, -9)$

c A horizontal line cuts the function at more than one point.

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d $x \geqslant 3$ or $x \leqslant 3$

e Either the increasing section of the parabola or the decreasing section can be taken for an inverse.

11 As a parabola does not have an inverse unless the domain has been restricted, the vertex is important because it is from this point that the largest domain for an inverse can be found, as an inverse exists for the increasing or the decreasing part of the function.

12 a $x \geqslant -1$ **b** $x \geqslant -1$ **c** $x \geqslant -\frac{1}{2}$

d $x \leqslant \frac{1}{2}$ **e** $x \leqslant \frac{1}{2}$

13 a $f^{-1}(x) = \sqrt{(x-6)}$ **b** $f^{-1}(x) = \sqrt{(x+1)} + 1$

c $f^{-1}(x) = \sqrt{(x+4)} - 2$ **d** $f^{-1}(x) = \sqrt{\left(x + \frac{9}{4}\right)} + \frac{1}{2}$

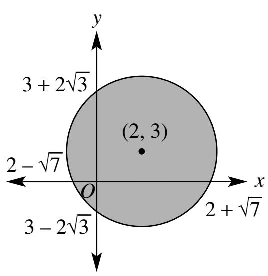
e $f^{-1}(x) = \sqrt{(x+16)} - 1$

Puzzles and challenges

1 a $-\frac{2}{3} \leqslant x \leqslant \frac{1}{2}$ **b** $x < -\frac{3}{4}$ or $x > \frac{1}{3}$

c $\frac{7-\sqrt{41}}{2} < x < \frac{7+\sqrt{41}}{2}$

2 $(x-2)^2 - (y-3)^2 \leqslant 16$



3 a $b^2 - 4ac < 0$ **b** $b^2 - 4ac = 0$ **c** $b^2 - 4ac > 0$

4 a $\left(-2-\sqrt{5}, -1-\sqrt{5}\right), \left(-2+\sqrt{5}, -1+\sqrt{5}\right)$

b $\left(\frac{-1-\sqrt{17}}{2}, \frac{1-\sqrt{17}}{2}\right), \left(\frac{-1+\sqrt{17}}{2}, \frac{1+\sqrt{17}}{2}\right)$

c $(-2, -1), (1, 2)$ **d** $(-4, -3), (-2, -1)$

5 $(x-2)^2 + (y+3)^2 = -15 + 9 + 4 = -2$, which is impossible.

6 a $k = \frac{1}{3}$ **b** $k < \frac{1}{3}$ **c** $k > \frac{1}{3}$

7 a $k = \pm\sqrt{20} = \pm 2\sqrt{5}$

b $k > 2\sqrt{5}$ or $k < -2\sqrt{5}$

c $-2\sqrt{5} < k < 2\sqrt{5}$

8 a $y = -(x+1)(x-3)$ **b** $y = \frac{3}{4}(x+2)^2 - 3$

c $y = x^2 - 2x - 3$

9 $\left(\frac{3}{4}, -\frac{73}{8}\right) a = 2, b = -3, c = -8$

10 20

11 a

D	1	2	3	4	5	6
M	1	3	7	15	31	63

b $M = 2^D - 1$

Multiple-choice questions

1 B **2** D **3** E **4** D **5** A
6 A **7** C **8** D **9** D **10** A
11 E **12** C **13** D **14** B **15** A
16 C **17** E **18** D

Short-answer questions

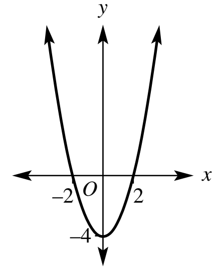
1 a minimum at $(1, -4)$ **b** $x = 1$

c $-1, 3$ **d** -3

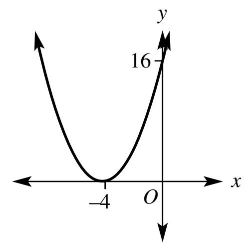
2 a minimum at $(2, 0)$ **b** maximum at $(0, 5)$

c maximum at $(-1, -2)$ **d** minimum at $(3, 4)$

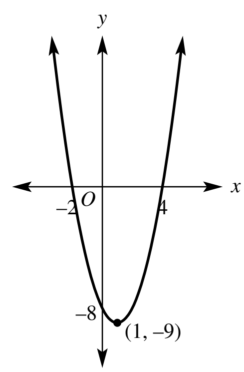
3 a



b



c

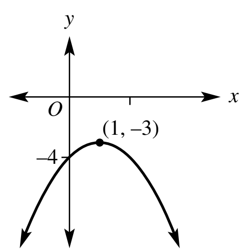


4 a i maximum at $(1, -3)$

ii -4

iii no x-intercepts

iv



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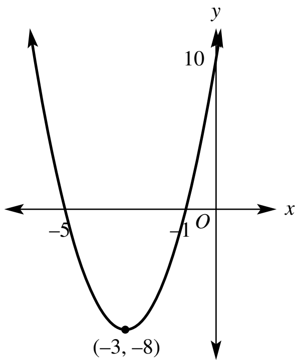
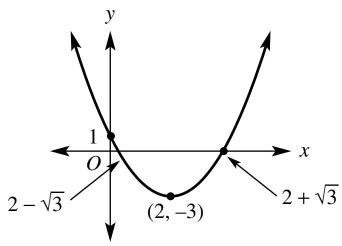
b **i** minimum at $(-3, -8)$

ii 10 **iii** -1 and -5

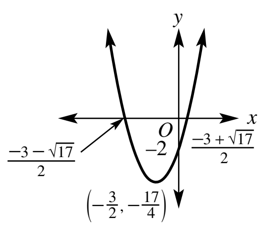
iv



5 **a**



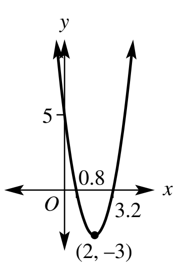
b



6 **a** 1 **b** 0 **c** 2 **d** 0

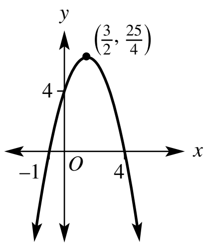
7 **a** **i** 5 **ii** $(2, -3)$ **iii** 0.8, 3.2

iv



b **i** 4 **ii** $\left(\frac{3}{2}, \frac{25}{4}\right)$ **iii** -1, 4

iv

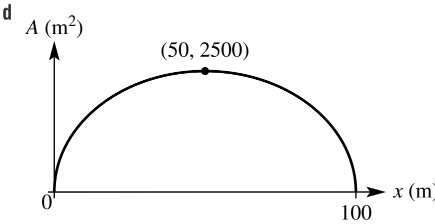


8 **a** $100 - x$

b $A = x(100 - x)$

c $0 < x < 100$

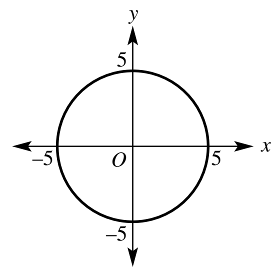
d



e 2500 m²

f 50 m by 50 m

9 **a**



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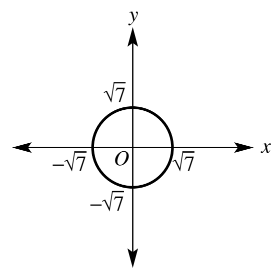

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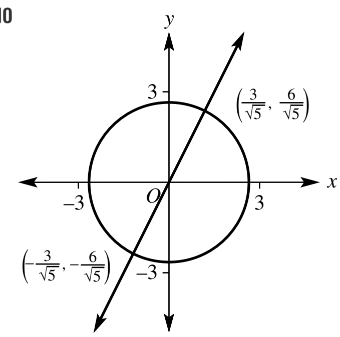
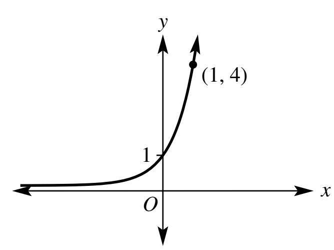


11 **a**



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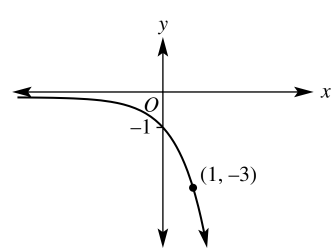

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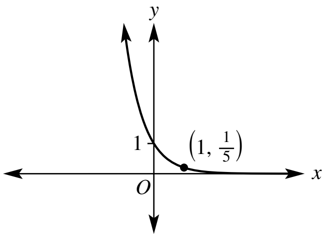
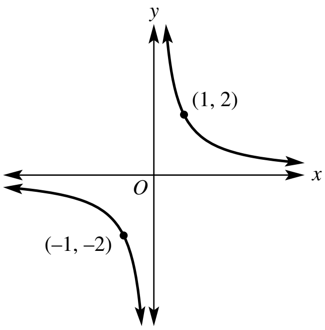
c



12 a



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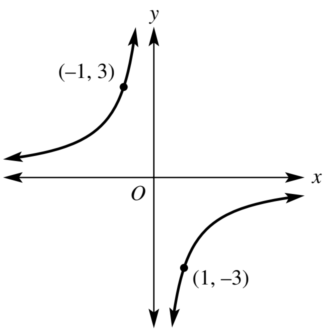
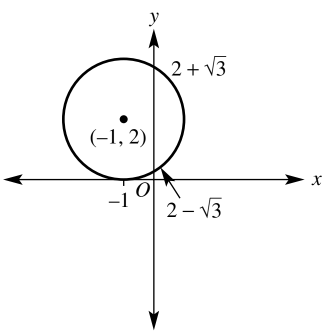


13 a $\left(\frac{4}{3}, 3\right)$

b $(\sqrt{2}, 2\sqrt{2})$ and $(-\sqrt{2}, -2\sqrt{2})$

14 a



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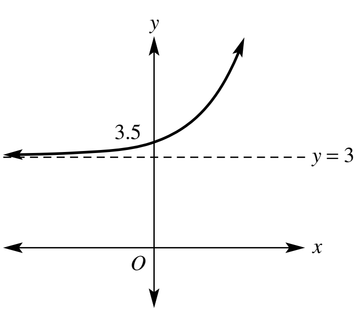

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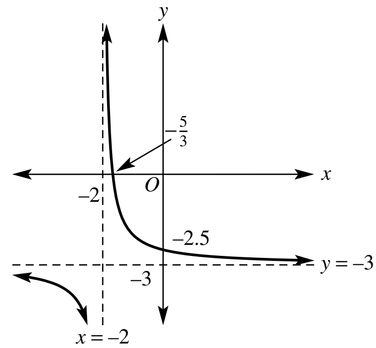


15 a $x = -6, y = 10$ and $x = 2, y = 10$

b no solutions

c $x = 0, y = 0$ and $x = 4, y = 8$

d $x = -1, y = 2$ and $x = \frac{1}{3}, y = \frac{10}{9}$

16 a show $b^2 - 4ac < 0$

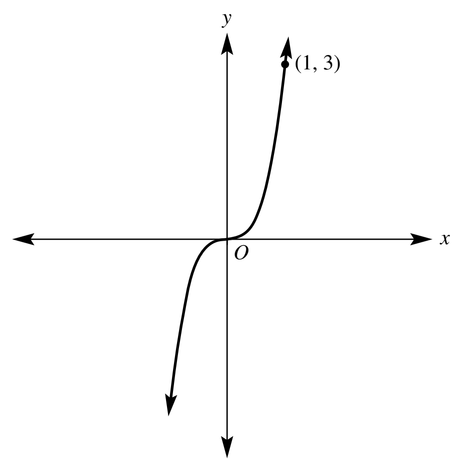
b show $b^2 - 4ac = 0$

c $k = -2$

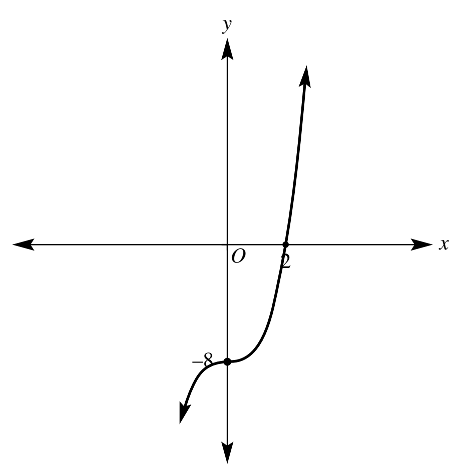
17 a $x = 3$ **b** $x = -2$ **c** $x = 1.8$ **d** $x = -1.7$

e $x = 4$ **f** $x = 3$ **g** $x = -2$ **h** $x = -2.5$

18 a



b



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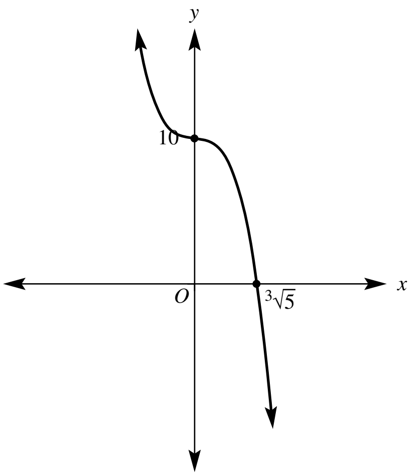
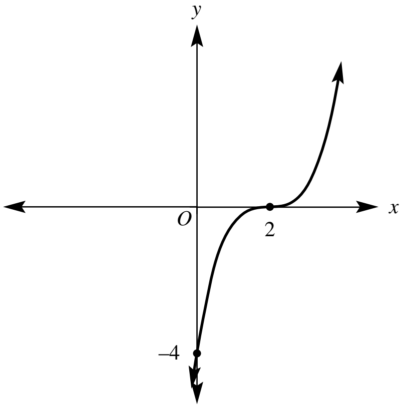
c



d



19 a -3 **b** -7 **c** -2.875
d 33 **e** $k^3 - 2k^2 + k - 3$

20 a domain: all real x values; range: all real y values
b domain: all real x values; range: $y = 4$ only
c domain: all real x values; range: $y \leqslant 4$
d domain: all real x, $x \neq 0$; range: all real y, $y \neq 0$
e domain: $x \geqslant 0$; range: $y \geqslant 0$

21 a $f^{-1}(x) = \frac{x+5}{4}$ **b** $f^{-1}(x) = \frac{x}{5}$
c $f^{-1}(x) = \sqrt[3]{\frac{x}{4}}$ **d** $f^{-1}(x) = \frac{3}{x}$
e $f^{-1}(x) = 1 - x$

22 $x < 0$

23 a i $y = \frac{72}{x}$ **ii** $y = 18$ **iii** $x = 102.86$
b CNY = 6.47 × AUD; AUD\$154 559.50
c i $k = 1200, y = \frac{1200}{x}$ or $xy = 1200, y = 15$
ii $k = 240, y = \frac{240}{x}$ or $xy = 240, x = 20$

24 a A: y is increasing at an increasing rate.
B: y is increasing at a decreasing rate.
C: y is decreasing at an increasing rate.
D: y is decreasing at a decreasing rate.
E: y is decreasing at a constant rate.

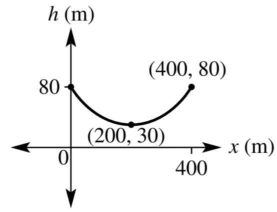
b A: i , v and viii
B: v, ix
C: ii and v
D: iii and ix
E: i, iv and vii
F: vii and ix
G: ii, vi and vii

25 a $b = \frac{P-a}{2}$ **b** $t = \frac{d}{s}$ **c** $r = \sqrt{\frac{A}{\pi}}$
d $b = \frac{2A}{h} - a$ **e** $a = \sqrt{c^2 - b^2}$ **f** $x - \pm\sqrt{\frac{y}{k}}$

26 a no **b** $a = \frac{y}{x^2}$
c yes, $x \neq 0$ **d** $x = \pm\sqrt{\frac{y}{a}}$
e $a \neq 0$; when $y \neq 0$ then a must have same sign as y.

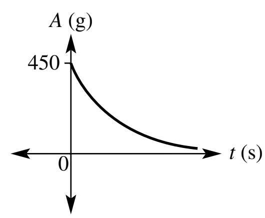
Extended-response questions

1 a (200, 30) **b** $0 \leqslant x \leqslant 400$ **c** $30 \leqslant h \leqslant 80$
d



e 400 m **f** 30 m **g** 80 m

2 a 450 g **b i** 150 g **ii** 5.6 g
c after 2 years
d



e 9.8 years

Chapter 10

Pre-test

1 a 8 **b** 2 **c** 1 **d** 9000
e $\frac{1}{27}$ **f** 32 **g** 64 **h** $\frac{1}{6}$

2 a x^7 **b** $12x^8$ **c** $2x$
d 2 **e** $12x^3$ **f** x^9y^3
g $\frac{8x^3}{y^3}$ **h** $\frac{y}{2}$ **i** $\frac{y}{24x}$

3 a 3 **b** 4 **c** 4
d -2 **e** -3 **f** $\frac{3}{2}$

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4 a $A = 10000 \times 1.05^n$ b $16289

5 a $x^2 + 4x + 3$ b $6x^2 + x - 2$ c $4x^2 - 25$
d $15x^2 + 2x - 1$ e 0 f $4x + 8$

6 a $5(1 - 2x)$ b $(x - 8)(x + 6)$ c $(x - 2)(x + 5)$
d $(x + 1)(2x - 3)$ e $3(x - 2)(5x + 4)$ f $-(x - 2)(x + 3)$

7 a 7 b 5 c 41

8 a 16 rem. 3 b 39 rem. 3 c 55 rem. 19

Exercise 10A

1

x	0	1	2	3	4	5
2^x	1	2	4	8	16	32
3^x	1	3	9	27	81	243
4^x	1	4	16	64	256	1024
5^x	1	5	25	125	625	3125
10^x	1	10	100	1000	10000	100000

2 a 4 b 4 c 3 d 4

3 a $\frac{1}{10000}$ b $\frac{1}{2}$ c $\frac{1}{4}$ d $\frac{1}{32}$
e $\frac{1}{27}$ f $\frac{1}{25}$ g $\frac{1}{64}$ h $\frac{1}{36}$

4 a $2^4 = 16$ b $10^2 = 100$ c $3^3 = 27$
d $2^{-2} = \frac{1}{4}$ e $10^{-1} = 0.1$ f $3^{-2} = \frac{1}{9}$

5 a $\log_2 8 = 3$ b $\log_3 81 = 4$ c $\log_2 32 = 5$
d $\log_4 16 = 2$ e $\log_{10}\frac{1}{10} = -1$ f $\log_5\frac{1}{125} = -3$

6 a 4 b 2 c 6 d 3
e 1 f 2 g 3 h 3
i 2 j 2 k 5 l 3
m 0 n 0 o 0 p 1

7 a −3 b −2 c −2 d −3
e −2 f −4 g −4 h −1
i −1 j −3 k −5 l −1
m −3 n −1 o −2 p −2

8 a 0.699 b 1.672 c 2.210
d −0.097 e −0.770 f −1.431

9 a 3 b 5 c 6 d 4
e 3 f 2 g 16 h 81
i 1000 j $\frac{1}{9}$ k $\frac{1}{4}$ l $\frac{1}{343}$
m 3 n 2 o 4 p 8
q 3 r 10 s 2 t −1

10 a

Time (min)	0	1	2	3	4	5
Population	1	2	4	8	16	32

b $P = 2^t$ c 256
d 14 min (round up) e $\log_2 10000$

11 Given that a is a positive number and $y = \log_a b$, then $a^y = b$, where y can be any negative number.
Example: $\log_2\frac{1}{4} = -2$.

12 No. A positive number to any power is always > 0.

13 a 16 b 26 c 6

14 $\log_a 1 = 0$ and dividing by 0 is not possible.

15 a $\frac{1}{4}$ b $\frac{1}{5}$ c $\frac{1}{2}$ d $\frac{1}{3}$
e $\frac{1}{2}$ f $\frac{1}{3}$ g $\frac{2}{3}$ h $\frac{4}{3}$
i $\frac{1}{2}$ j $\frac{1}{2}$ k $\frac{6}{5}$ l $\frac{4}{7}$

Exercise 10B

1 a x can't be negative or zero.
b $x > 0$
c y can be any real number.
d $-\infty < y < \infty$
e $y \to -\infty$
f $x = 0$
g x-intercept is 1.

2 1: $y = \log_2 x$ is the reflection of $y = 2^x$ in the line $y = x$.
2: The coordinates of each point on $y = \log_2 x$ are the coordinates in reverse order of a point on $y = 2^x$; e.g. (1, 0) and (0, 1).

3 a $x > 4$
b $x = 4$
c $y = \log_2 x$ is translated 4 units to the right.
d $y = \log_2 x$ is translated 5 units up.

4 a

Physical values	1	10	100	1000	10000	100000
log scale $\log_{10}$ (physical value)	0	1	2	3	4	5

b 1 unit added c factor of 100
d 4.2 e 2
f 4 g 0.903
h 0.301 added i 3.01

5 a

x	−3	−2	−1	0	1	2	3
$y = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$y = \log_2 x$	−3	−2	−1	0	1	2	3

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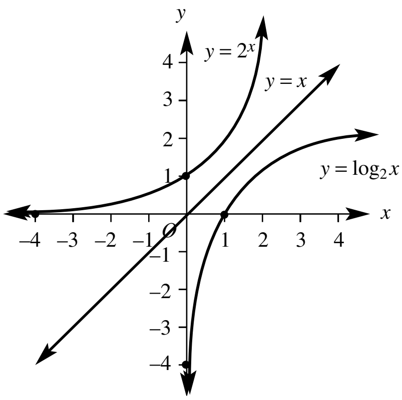
b



c **1** $y = \log_2 x$ is the mirror image (i.e. the reflection) of $y = 2^x$ in the line $y = x$.

2 The coordinates of each point on $y = 2^x$ are reversed to give the coordinates of a point on $y = \log_2 x$.

d $y = 2^x$ has y-intercept of 1; $y = \log_2 x$ has x-intercept of 1.

e $y = 0$ is the asymptote for $y = 2^x$; $x = 0$ is the asymptote for $y = \log_2 x$.

f $y = 2^x$ has no limitations on the values of x; $x > 0$ for $y = \log_2 x$.

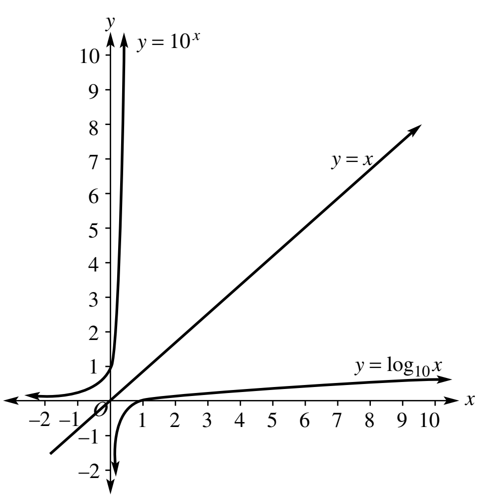
g $y > 0$ for $y = 2^x$; $y = \log_2 x$ has no limitations on the values of y.

6 **a**

x	−3	−2	−1	0	1	2	3
$y = 10^x$	0.001	0.01	0.1	1	10	100	1000

x	0.001	0.01	0.1	1	10	100	1000
$y = \log_2 x$	−3	−2	−1	0	1	2	3

b



c $y = 10^x$, y-intercept $= 1$; $y = \log_{10} x$, x-intercept $= 1$

d $(10, 1) y = \log_{10} 10 = 1$;, $(1, 0) \log_{10} 1 = 0$

e $y = 0$ is the asymptote for $y = 10^x$; $x = 0$ is the asymptote for $y = \log_{10} x$.

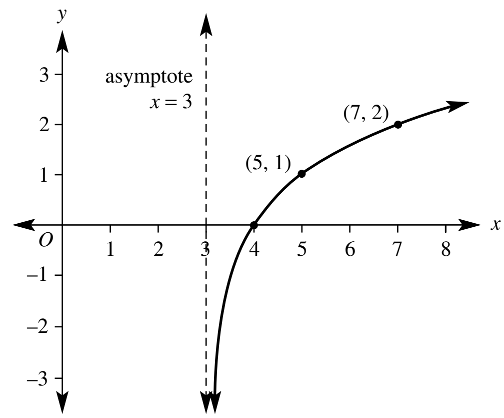
f $y = 10^x$ has no limitations on the values of x; $x > 0$ for $y = \log_{10} x$.

g $y > 0$ for $y = 10^x$; $y = \log_{10} x$ has no limitations on the values of y.

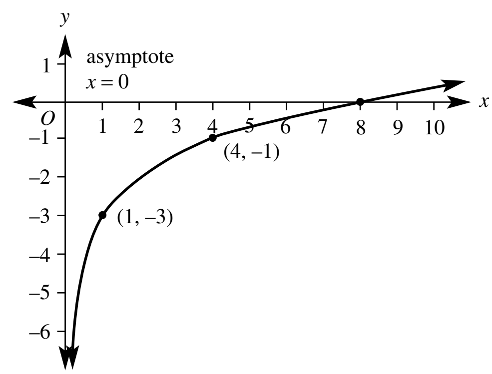
7

Graph	a	b	c	d
i	$x > 4$	$x = 4$	4 units right, 3 units up	(4.125, 0), no y-intercept
ii	$x > 3$	$x = 3$	3 units right, 2 units up	(3.25, 0), no y-intercept
iii	$x > -8$	$x = -8$	8 units left, 1.2 units down	(−5.7, 0), (0, 1.8)
iv	$x > -4$	$x = -4$	4 units left, 3.5 units down	(7.31, 0), (0, −1.5)

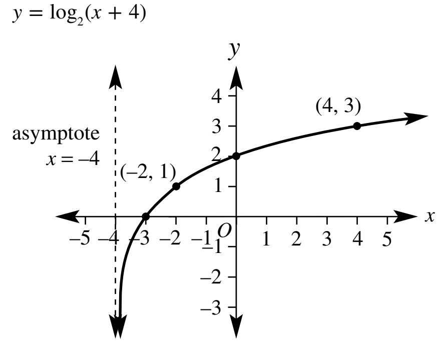
8 **a** $y = \log_2(x - 3)$



b $y = \log_2 x - 3$



c $y = \log_2(x + 4)$



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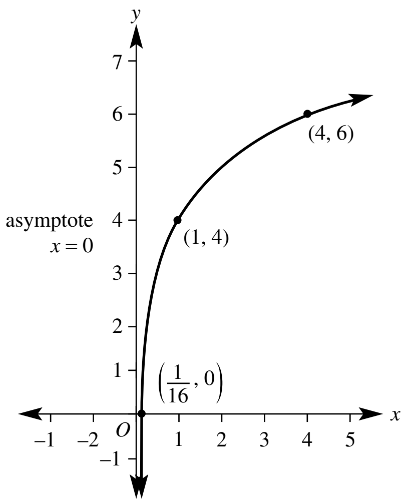
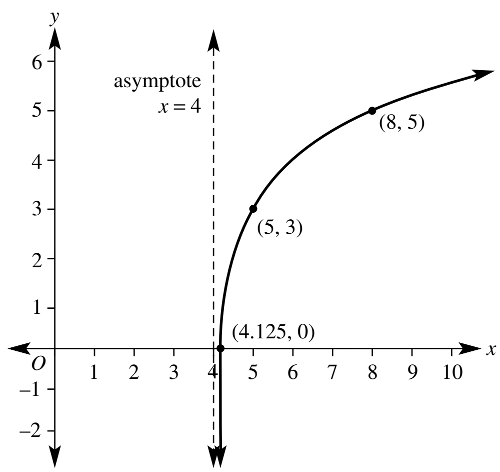
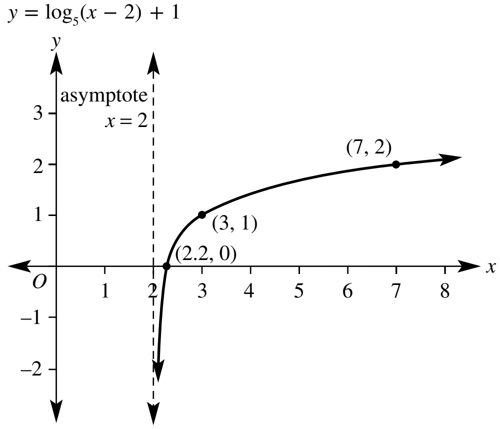
d $y = \log_2 x + 4$



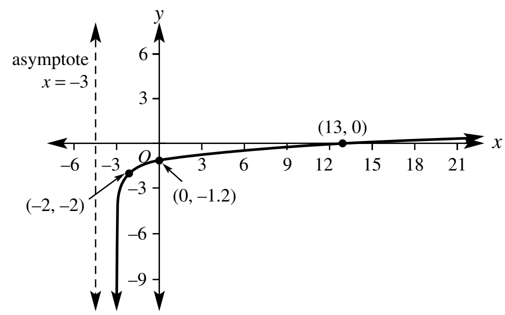
e $y = \log_2(x - 4) + 3$



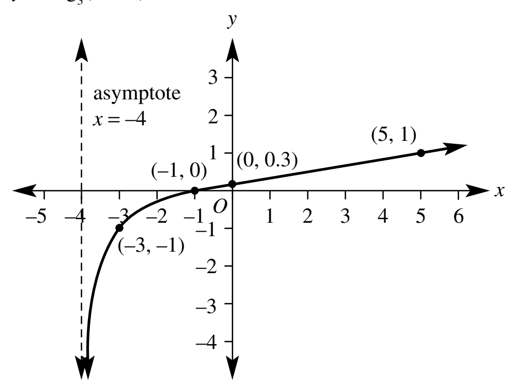
f $y = \log_5(x - 2) + 1$



g $y = \log_4(x + 3) - 2$



h $y = \log_3(x + 4) - 1$



9
$$\begin{aligned} \log_a a^x &= x \log_a a \\ &= x \times 1 \\ &= x \\ \therefore\ a^{\log_a x} &= x \\ \log_a a^{\log_a x} &= \log_a x \\ \log_a x \times \log_a a &= \log_a x \\ \log_a x \times 1 &= \log_a x \\ \log_a x &= \log_a x \\ \text{LHS} &= \text{RHS} \\ \therefore\ a^{\log_a x} &= x \end{aligned}$$

10 a $M = 4$

b $10^7 = 10\,000\,000$ times more intense

c $I = I_0 \times 10^M$

d Ratio of intensities $= \dfrac{I_0 \times 10^7}{I_0 \times 10^4} = 10^3 = 1000$

e $\dfrac{I_0 \times 10^{M1}}{I_0 \times 10^{M2}} = 2$

$10^{M1-M2} = 2$

$M1 - M2 = \log_{10} 2$

$M1 - M2 = 0.301$

f $10^{2.7} = 501.2 \approx 500$ times stronger

g $39.8 \approx 40$ times greater. Meeberrie is mostly an unpopulated area.

11 a 0 dB

b 60 dB

c 100 million times louder

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d 1000 times louder

e noise intensity doubles in loudness

f Band practice: 2 hours; belt sander: 1 hour; hand drill: 30 minutes; chainsaw: 15 min; rock concert: 1 min; iPod at peak volume: 30 s; jet engine at take-off: pain and immediate, permanent hearing damage.

12 a **i** pH = 2.4 acidic **ii** pH = 4 acidic

iii pH = 7.4 alkaline **iv** pH = 10 alkaline

b $[H^+] = 10^{-pH}$

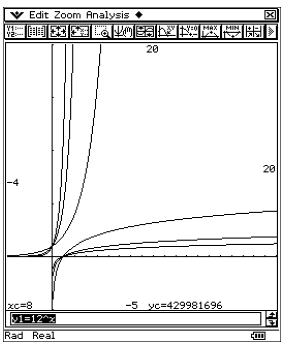
c 100 times more $[H^+]$

d $\frac{1}{10}$th of $[H^+]$ for pH increase of 1.

e $[H^+] = 1.49 \times 10^{-3}$ moles/litre, pH = 2.83

f $\left(\frac{10^{-7.9} - 10^{-8.1}}{10^{-8.1}}\right) \times 100 = 58\%$ increase in acidity

13 a



b Similarities: Both have an asymptote; each family of graphs intersect an axis at a common point; $x \to \infty, y \to \infty$ for both (but at different rates); and both curves have roughly a concave shape.

c Differences: Logarithmic graphs are the inverse of exponential graphs.

Exponential graphs are 'concave up' and logarithmic graphs are 'concave down'.

A logarithmic graph is the reflection of the exponential graph in the line $y = x$.

The coordinates of a point on an exponential graph are in the reverse order to the coordinates of the reflected point that is on the logarithmic graph.

Logarithmic graphs are valid for $x > 0$ and all real values of y, whereas exponential graphs are valid for all real values of x and $y > 0$.

14 a $y = \log_2(x + 3)$ **b** $y = \log_2(x - 2) + 3$

c $y = \log_5(x + 2) + 4$ **d** $y = \log_3(x + 3) - 1$

15 For $0 < x < 1$ and a given y value, the x-coordinate will decrease as the base a increases; hence, the log curve with the larger base turns more quickly towards the y-axis. For example, when $y = -1$, $x = 0.1$ on $y = \log_{10}x$ and $x = 0.5$ on $y = \log_2 x$, so points $y = \log_{10}x$ will be closer to the y-axis than $y = \log_2 x$.

For $x = 1$, $y = \log_a x$ will pass through $(1, 0)$ for all base values $(a > 1)$.

For $x > 1$ and fixed, the values of $\log_a x$ become smaller as the base a increases; hence, the log curve with the larger base is closer to the the x-axis. For example, when $x = 10$, $\log_{10}10 = 1$ and $\log_2 10 = 3.3$, so $y = \log_{10}x$ will be nearer to the x-axis than $y = \log_2 x$.

16 a $y = \log_5(x + 4) + 2$

b $y = \log_4(x - 1) + 3$

c $y = \log_2(x - 5) - 3$

d $y = \frac{1}{2}\log_3(x - 7) + 2$ or $y = \frac{1}{2}(\log_3(x - 7) + 4)$

e $y = 3^{x-4} + 1$

f $y = 2^{x-3} - 5$

g $y = 5^{x-2} + 8$

h $y = 4^{x-5} - 6$

i $y = \frac{1}{4}2^{x-3} - \frac{1}{2}$ or $y = \frac{1}{4}(2^{x-3} - 2)$

j $y = \frac{1}{2}\log_3\left(\frac{x-8}{5}\right) + 3.5$ or $y = \frac{1}{2}\left(\log_3\left(\frac{x-8}{5}\right) + 7\right)$

Exercise 10C

1 a $\log_b xy = \log_b x + \log_b y$ **b** $\log_b \frac{x}{y} = \log_b x - \log_b y$

c $\log_a b^m = m \times \log_a b$ **d** $\log_a a = 1$

e $\log_c 1 = 0$ **f** $\log_b \frac{1}{b} = -\log_a b$

2 a 2 **b** 1 **c** 2

d 4 **e** 12 **f** 3

g 4 **h** −1 **i** −1

3 a 2 **b** 5 **c** 3

d −4 **e** 12 **f** 0

4 a $\log_a 6$ **b** $\log_a 15$ **c** $\log_a 28$

d $\log_b 18$ **e** $\log_b 15$ **f** $\log_b 17$

5 a $\log_a 2$ **b** $\log_a 3$ **c** $\log_a 10$

d $\log_b 2$ **e** $\log_b\left(\frac{3}{2}\right)$ **f** $\log_b\left(\frac{7}{5}\right)$

6 a $\log_a 9$ **b** $\log_a 25$ **c** $\log_a 27$

d $\log_a 16$ **e** $\log_a 32$ **f** $\log_a 1000$

7 a 0 **b** 0 **c** 0 **d** 1

e 1 **f** 1 **g** 0 **h** 3

i $\frac{1}{3}$ **j** $\frac{2}{3}$ **k** 1 **l** $\frac{1}{2}$

8 a −2 **b** −3 **c** −3

d −1 **e** −2 **f** −5

9 a 1 **b** 1 **c** 3

d 2 **e** 2 **f** 2

10 a $\log_3 20$ **b** $\log_{10}48$ **c** $\log_{10}2$

d $\log_7 2$ **e** $\log_3 8$ **f** 0

g $\log_2\left(\frac{3}{4}\right)$ **h** $\log_5 6$

11 a $\frac{3}{2}$ **b** $\frac{5}{2}$ **c** $\frac{4}{3}$

d $\frac{3}{2}$ **e** $\frac{1}{3}$ **f** $\frac{4}{5}$

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12 a $\log_a \frac{1}{x} = \log_a 1 - \log_a x = 0 - \log_a x = -\log_a x$, as required.

b $\log_a \frac{1}{x} = \log_a x^{-1} = -\log_a x$, as required.

13 $\log_a \sqrt[n]{x} = \log_a x^{\frac{1}{n}} = \frac{1}{n}\log_a x = \frac{\log_a x}{n}$, as required.

14 a Recall the index law $a^m \times a^n = a^{m+n}$.

Now let $x = a^m$ and $y = a^n$ (1)

so $m = \log_a x$ and $n = \log_a y$ (2)

From (1): $xy = a^m \times a^n = a^{m+n}$

So $m + n = \log_a xy$

From (2): $m + n = \log_a x + \log_a y$

So $\log_a xy = \log_a x + \log_a y$, as required.

b Recall the index law $a^m \div a^n = a^{m-n}$.

Now let $x = a^m$ and $y = a^n$ (1)

so $m = \log_a x$ and $n = \log_a y$ (2)

From (1): $x \div y = \frac{x}{y} = a^m \div a^n = a^{m-n}$

So $m - n = \log_a \frac{x}{y}$

From (2): $m - n = \log_a x - \log_a y$

So $\log_a \frac{x}{y} = \log_a x - \log_a y$, as required.

c Recall the index law $(a^m)^n = a^{mn}$.

Let $x = a^m$

So $m = \log_a x$ (1)

$x^n = a^{mn}$

So $mn = \log_a x^n$

From (1): $n\log_a x = \log_a x^n$, as required.

Exercise 10D

1 a $\log_2 8 = 3$ **b** $\log_5 25 = 2$ **c** $\log_4 2 = \frac{1}{2}$
d $\log_3 10 = x$ **e** $\log_7 2 = x$ **f** $\log_{1.1} 7 = x$

2 a 3 **b** 4 **c** 2 **d** 8 **e** 2 **f** 0.1

3 a 0.845 **b** −0.222 **c** −0.125
d 1.277 **e** 0.780 **f** 0.897

4 a 1.465 **b** 3.459 **c** 1.594
d 6.871 **e** 1.177 **f** 2

5 a 1 **b** 1 **c** 3.969
d 1.727 **e** 6.579 **f** 1.528

6 a 2.585 **b** 1.893 **c** 1.209 **d** 1.129
e 1.559 **f** 6.579 **g** 3.322 **h** 1.262
i 0.356 **j** 3.969 **k** 3.106 **l** 1.137

7 a 2 days **b** 2.548 days **c** 3.322 days

8 a 14.21 years **b** 23.84 years **c** 47.19 years

9 a 10.48 years **b** 22.20 years **c** 91.17 years

10 a $A = 2000 \times 1.1^n$ **b** 8 years (rounded up)

11 a $F = 300\,000 \times 0.92^n$ **b** 8.4 years

12 a 69 years **b** 1386 years

13 a i $\frac{\log_{10} 7}{\log_{10} 2}$ **ii** $\frac{\log_{10} 16}{\log_{10} 3}$ **iii** $\frac{\log_{10} 1.3}{\log_{10} 5}$

b i $\frac{1}{\log_{10} 5}$ **ii** $\frac{3}{\log_{10} 2}$ **iii** $\frac{-1}{\log_{10} 3}$

c i 1.631 **ii** 1.167 **iii** −0.196

Exercise 10E

1 a 5
b i 3 **ii** −2 **iii** 1 **iv** −1
c 2

2 a linear **b** quadratic **c** quartic
d quadratic **e** constant **f** linear
g constant **h** quartic **i** cubic

3 a 4 **b** 3 **c** −2
d −2 **e** −9 **f** 2

4 **a, b, f** are polynomials.

5 a 14 **b** 92 **c** 8 **d** 4

6 a −5 **b** 11 **c** 1 **d** −45

7 a 0 **b** 92 **c** −4 **d** 42

8 a −2 **b** 25 **c** −22
d −17 **e** 17 **f** −351

9 a $-\frac{1}{2}$ **b** −1 **c** $\frac{1}{2}$

10 a 0 **b** 4 **c** −108

11 a i 30 m **ii** 24 m **iii** 0 m
b Yes, when $5 < x < 7$.

12 a 8 **b** $n + 1$ **c** 1 **d** 1

13 a $-\frac{9}{8}$ **b** $-\frac{20}{27}$ **c** $\frac{5}{8}$ **d** $\frac{27}{64}$
e $\frac{16}{27}$ **f** $-\frac{216}{125}$ **g** $-\frac{1}{2}$ **h** $-\frac{9}{8}$

14 a $2k^3 - k^2 - 5k - 1$ **b** $2b^3 - b^2 - 5b - 1$
c $16a^3 - 4a^2 - 10a - 1$ **d** $-2a^3 - a^2 + 5a - 1$
e $-16a^3 - 4a^2 + 10a - 1$ **f** $-54k^3 - 9k^2 + 15k - 1$
g $2a^3b^3 - a^2b^2 - 5ab - 1$ **h** $-2a^3b^3 - a^2b^2 + 5ab - 1$

15 a i 10 **ii** 2 **iii** 1
iv −13 **v** −9 **vi** −18
b i 3 **ii** −11 **iii** −22
c $a = 2$ and $b = -1$

Exercise 10F

1 a $x^2 + 2x$ **b** $x - 3x^2$
c $x^2 - 1$ **d** $x^2 + 6x - 55$
e $6x^2 - 13x - 5$ **f** $8x^2 - 26x + 15$

2 a $x^4 - 5x^3 + 4x^2 - 3$
b $-x^6 - 3x^4 + x^3 - x^2 + 13$
c $-3x^8 - x^6 - 6x + 3$

3 **a, b, c** are true.

4 a $x^3 - 3x^2$ **b** $x^4 - x^2$ **c** $2x^2 + 6x^3$
d $x^3 - x^4$ **e** $x^5 + 3x^4$ **f** $-3x^6 + 3x^3$
g $-2x^5 - 2x^4$ **h** $-x^7 + x^4$ **i** $-4x^7 + 8x^{10}$

5 a $x^5 + x^3 + 2x^2 + 2$ **b** $x^5 - x$
c $x^5 - x^4 - 3x^3 + 3x^2$ **d** $x^5 - x^3 - 2x^2 - 2x + 4$
e $x^5 + 2x^4 + 2x^3 - 2x^2 - 3x$ **f** $x^5 - 2x^4 + 5x^3 - 4x^2$
g $x^6 - x^5 + x^4 - 4x^3 + 2x^2 - x + 2$
h $x^6 - 5x^5 - x^4 + 8x^3 - 5x^2 - 2x + 2$
i $x^8 - x^6 + x^5 - 2x^4 - x^3 + 3x^2 + x - 3$

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6 **a** $x^5 - 2x^4 + 2x^3 - 3x^2 + 3x - 1$
b $x^6 + 2x^4 - 2x^3 + x^2 - 2x + 1$
c $x^4 - 4x^3 + 6x^2 - 4x + 1$

7 **a** $x^5 + 3x^4 - x^3 - 9x^2 - 2x + 8$
b $x^4 + 2x^3 - 3x^2 - 4x + 4$
c $x^6 + 4x^5 + 2x^4 - 12x^3 - 15x^2 + 8x + 16$

8 **a** $x^3 + x^2 - 4x + 1$ **b** $x^3 - x^2 + 6x - 1$
c $2x^3 + 5x^2 - 23x + 5$
d $-x^5 + 5x^4 - 2x^3 + 5x^2 - x + 1$
e $-x^6 - 2x^4 - x^2 + 4$
f $-x^6 - x^4 - 10x^3 + 26x^2 - 10x + 1$

9 $(x^2 + x - 1)^4 = x^8 + 4x^7 + 2x^6 - 8x^5 - 5x^4 + 8x^3 + 2x^2 - 4x + 1$

10 $(x^2 - x - 1)^2 - (x^2 - x + 1)^2 = x^4 - 2x^3 - x^2 + 2x + 1 - (x^4 - 2x^3 + 3x^2 - 2x + 1) = 4x - 4x^2$ as required (or could use difference of two squares)

11 Yes. Multiplicative axiom $ab = ba$.

12 **a** 3 **b** 5 **c** 7 **d** 12

13 **a** m **b** m **c** $m + n$
d $2m$ **e** $2m$ **f** $3n$

14 **a** $x^4 - x^3 + x^2 - x$ **b** $x^5 + 2x^4 - 3x^3$
c $x^3 + 4x^2 + x - 6$ **d** $6x^3 + 23x^2 - 5x - 4$
e $15x^3 - 11x^2 - 48x + 20$
f $x^5 + 3x^4 - x^3 - 3x^2 - 2x - 6$

Exercise 10G

1 **a** 1 **b** 3 **c** 0

2 **a** If $182 \div 3 = 60$ rem. 2, then $182 = 3 \times 60 + 2$.
b If $2184 \div 5 = 436$ rem. 4, then $2184 = 5 \times 436 + 4$.
c If $617 \div 7 = 88$ rem. 1, then $617 = 7 \times 88 + 1$.

3 $P(x) = (x - 1)(x^2 + 2x) + 3$

4 $P(x) = (x + 1)(3x^2 - 4x + 5) - 3$

5 **a** $2x^3 - x^2 + 3x - 2 = (x - 2)(2x^2 + 3x + 9) + 16$
b $2x^3 + 2x^2 - x - 3 = (x + 2)(2x^2 - 2x + 3) - 9$
c $5x^3 - 2x^2 + 7x - 1 = (x + 3)(5x^2 - 17x + 58) - 175$
d $-x^3 + x^2 - 10x + 4 = (x - 4)(-x^2 - 3x - 22) - 84$
e $-2x^3 - 2x^2 - 5x + 7 = (x + 4)(-2x^2 + 6x - 29) + 123$
f $-5x^3 + 11x^2 - 2x - 20 = (x - 3)(-5x^2 - 4x - 14) - 62$

6 **a** $6x^4 - x^3 + 2x^2 - x + 2$
$= (x - 3)(6x^3 + 17x^2 + 53x + 158) + 476$
b $8x^5 - 2x^4 + 3x^3 - x^2 - 4x - 6$
$= (x + 1)(8x^4 - 10x^3 + 13x^2 - 14x + 10) - 16$

7 **a** $x^2 - 2x + 3 - \dfrac{5}{x + 2}$ **b** $x^2 + 2x + 2 - \dfrac{1}{x - 1}$
c $x^3 - 3x^2 + 9x - 27 + \dfrac{79}{x + 3}$
d $x^3 + 4x^2 + 15x + 60 + \dfrac{240}{x - 4}$

8 $-1, 1, 2$

9

$$
\begin{array}{r@{}l}
 & \;6x^2 - 7x - 3 \\
x - 5 & \overline{)\,6x^3 - 37x^2 + 32x + 15} \\
 & \underline{6x^3 - 30x^2} \\
 & \quad -7x^2 + 32x \\
 & \quad \underline{-7x^2 + 35x} \\
 & \qquad\quad -3x + 15 \\
 & \qquad\quad \underline{-3x + 15} \\
 & \qquad\qquad\quad 0
\end{array}
$$

Remainder of 0, as required.

10 **a** 4 **b** $\dfrac{13}{8}$

11 **a** -8 **b** $-\dfrac{253}{16}$ **c** $-\dfrac{41}{27}$

12 **a** $x^3 - x^2 + 3x + 2 = (x^2 - 1)(x - 1) + 4x + 1$
b $2x^3 + x^2 - 5x - 1 = (x^2 + 3)(2x + 1) - 11x - 4$
c $5x^4 - x^2 + 2 = 5x(x^3 - 2) - x^2 + 10x + 2$

Exercise 10H

1 **a** -1 **b** 41 **c** -19 **d** -141

2 **a** 3 **b** -2

3 0

4 **a** 3 **b** 11 **c** 27 **d** 57
e -127 **f** -33 **g** -13 **h** -69

5 **a** 3 **b** 20 **c** 36 **d** 5

6 b, c and e are factors of $P(x)$.

7 b, d, f, g

8 **a** $x + 1$ **b** $x - 1, x + 1$ or $x + 2$
c $x + 2$ **d** $x - 2$

9 **a** $x - 2, x - 1$ and $x + 1$ **b** $x - 3, x - 1$ and $x + 2$
c $x - 3, x - 2$ and $x + 1$ **d** $x - 5, x - 1$ and $x + 4$

10 **a** -4 **b** -2 **c** -14 **d** 96

11 -38

12 **a** 5 **b** 1 **c** 5 **d** -3

13 **a** -2 **b** 23

14 **a** $a = -1$ and $b = 2$ **b** $a = 3$ and $b = -4$

Exercise 10I

1 $P(-1) = 0$

2 **a** $x = -3$ or 1 **b** $x = -2$ or 2 **c** $x = -\dfrac{1}{2}$ or 4
d $x = -3$ or 4 **e** $x = -3$ **f** $x = -4$ or -3

3 **a** $-3, 1, 2$ **b** $-7, -2, 1$ **c** $-4, 3, 4$
d $-\dfrac{1}{2}, -\dfrac{1}{3}, 3$ **e** $-\dfrac{2}{3}, -\dfrac{1}{2}, 3$ **f** $-\dfrac{2}{7}, \dfrac{1}{4}, \dfrac{2}{5}$
g $-\dfrac{12}{11}, -\dfrac{1}{2}, -\dfrac{1}{3}$ **h** $-\dfrac{3}{5}, -\dfrac{2}{19}, \dfrac{1}{2}$

4 **a** $(x - 3)(x - 2)(x + 1); -1, 2, 3$
b $(x + 1)(x + 2)(x + 3); -3, -2, -1$
c $(x - 3)(x - 2)(x - 1); 1, 2, 3$
d $(x - 4)(x - 3)(x - 1); 1, 3, 4$
e $(x - 6)(x + 1)(x + 2); -2, -1, 6$
f $(x - 2)(x + 3)(x + 5); -5, -3, 2$

5 **a** $x = 1$ or $1 + \sqrt{5}$ or $1 - \sqrt{5}$ **b** $x = -2$

6 **a** $x = -1, 3$ or 5 **b** $x = -3, -2$ or 1

7 **a** $x = -4, 1$ or 3 **b** $-2, -1$ or 3

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8 **a** 3 **b** 4 **c** n

9 **a** $x^2(x-1)$; 0, 1 **b** $x^2(x+1)$; -1, 0

c $x(x-4)(x+3)$; -3, 0, 4 **d** $2x^3(x+1)^2$; -1, 0

10 $0 = x^4 + x^2 = x^2(x^2+1)$

No solution to $x^2 + 1 = 0$, thus $x = 0$ is the only solution.

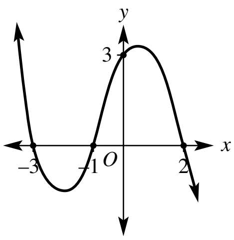
11 The discriminant of the quadratic is negative, implying solutions from the quadratic factor are not real; $x = 2$ is the only solution.

12 **a** $x = -4, -3, -2$ or 1 **b** $x = -2$ or 3

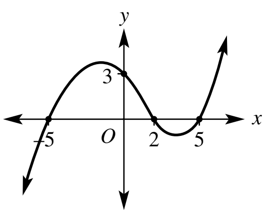
c $x = -3, -2, 1$ or 3 **d** $x = -2, \frac{1}{2}, 1$ or 2

Exercise 10J

1 **a**



b



2 **a** y-intercept = 12
x-intercepts = $-1, 3, 4$

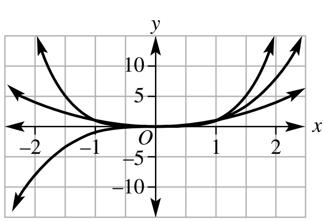
b y-intercept = -21
x-intercepts = $-3, -1, 7$

c y-intercept = 0
x-intercepts = $-2, 0, 4$

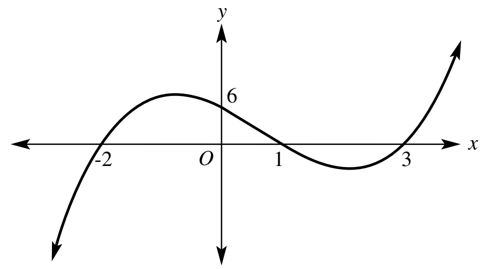
d y-intercept = 0
x-intercepts = $-7, 0, 5$

3 **a–c**

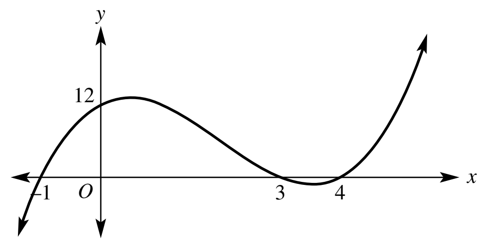
x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
$y = x^2$	4	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	4
$y = x^3$	-8	-1	$-\frac{1}{8}$	0	$\frac{1}{8}$	1	8
$y = x^4$	16	1	$\frac{1}{16}$	0	$\frac{1}{16}$	1	16



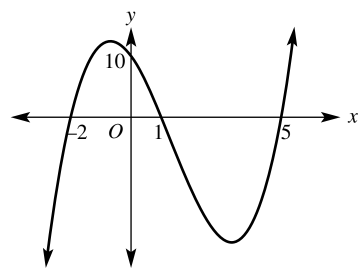
4 **a** y-intercept: 6
x-intercepts: $-2, 1, 3$



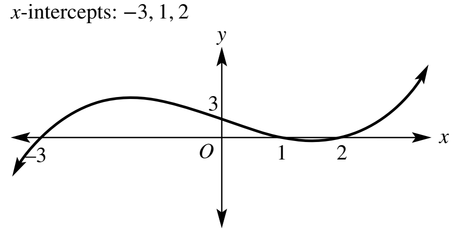
b y-intercept: 12
x-intercepts: $-1, 3, 4$



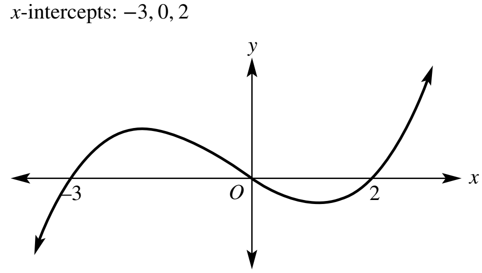
c y-intercept: 10
x-intercepts: $-2, 1, 5$



d y-intercept: 3
x-intercepts: $-3, 1, 2$



e y-intercept: 0
x-intercepts: $-3, 0, 2$



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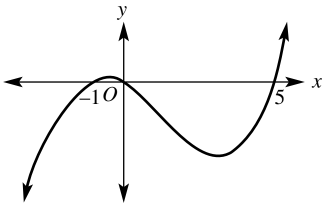
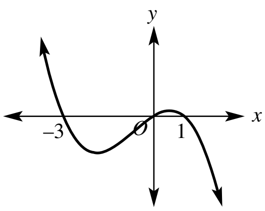
f y-intercept: 0

x-intercepts: $-1, 0, 5$



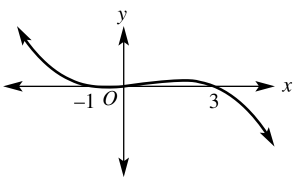
g y-intercept: 0

x-intercepts: $-3, 0, 1$



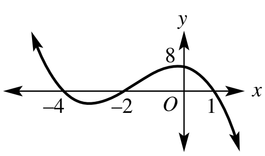
h y-intercept: 0

x-intercepts: $-1, 0, 3$



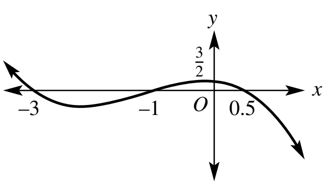
i y-intercept: 8

x-intercepts: $-4, -2, 1$

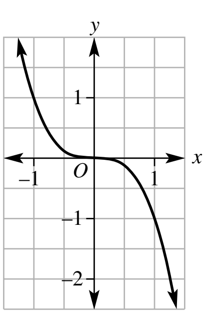


j y-intercept: $\frac{3}{2}$

x-intercepts: $-3, -1, \frac{1}{2}$



5 a



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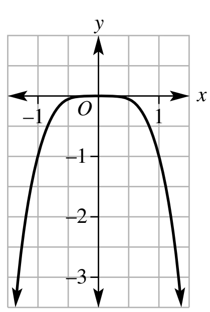


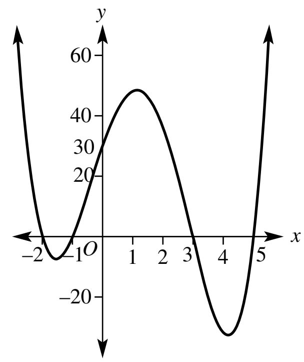
6 a $y = (x - 2)(x + 1)(x + 4)$

b $y = (x + 3)(x - 1)(x - 3)$

c $y = \frac{1}{2}x(x - 2)(x + 3)$

d $y = -\frac{1}{2}(x + 3)(x + 1)(x - 2)$

7 a



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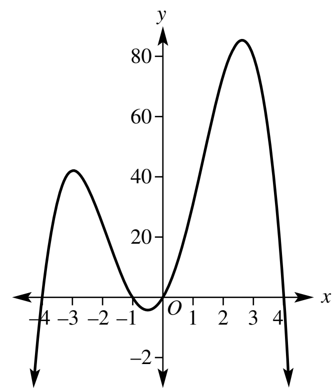
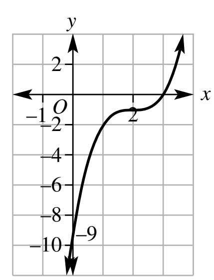


8 a



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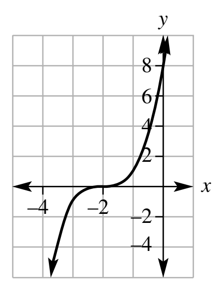

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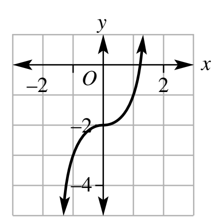
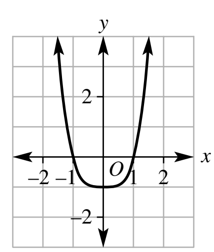


d



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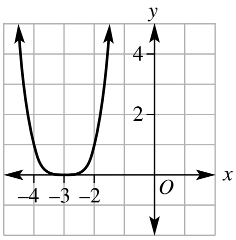
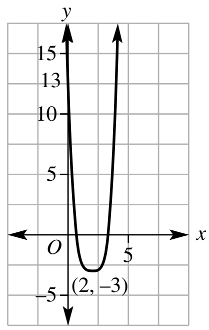
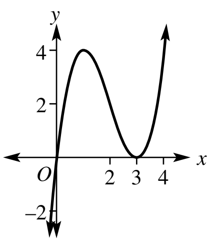
e



f



9 a



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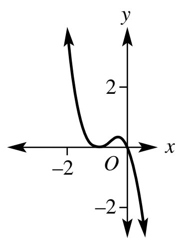

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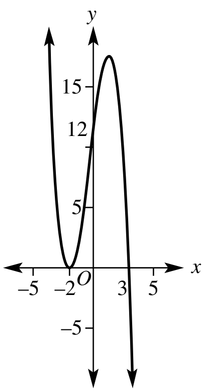

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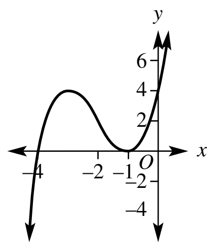

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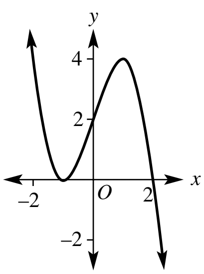

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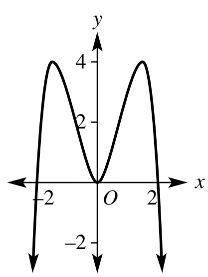
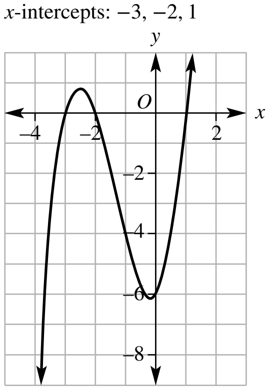


10 a i y-intercept $= -6$

ii $y = (x-1)(x+2)(x+3)$

iii x-intercepts: $-3, -2, 1$

iv

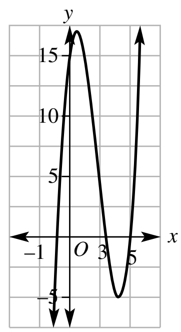


b i y-intercept $= 15$

ii $y = (x-5)(x-3)(x+1)$

iii x-intercepts: $-1, 3, 5$

iv

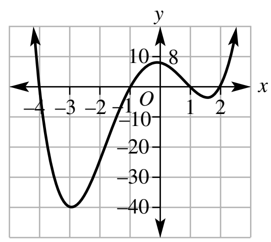


c i y-intercept $= 8$

ii $y = (x-2)(x-1)(x+1)(x+4)$

iii x-intercepts: $-4, -1, 1, 2$

iv

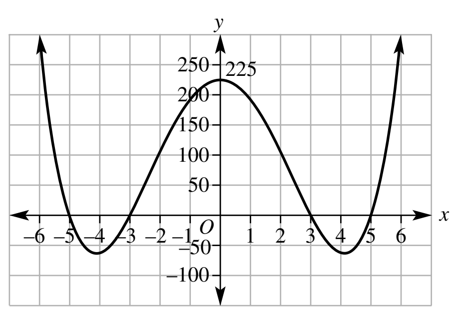


d i y-intercept $= 225$

ii $y = (x-5)(x-3)(x+3)(x+5)$

iii x-intercepts: $-5, -3, 3, 5$

iv



Puzzles and challenges

1 a 2 **b** 8 **c** $\frac{1}{2}$ **d** 3

2 a 1.43 **b** -1.43 **c** -2.71 **d** $x \geqslant -2.81$

3 $a = 2 \times 3^{\frac{1}{4}}, b = \frac{1}{4}\log_2 3$

4 $\dfrac{\log_{10} 2}{\log_{10} 1.1} = \log_{1.1} 2$

5 -2

6 $a = 5, b = -2$

7 Proof using long division required.

a $(x^3 - a^3) \div (x - a) = x^2 + ax + a^2$

b $(x^3 + a^3) \div (x + a) = x^2 - ax + a^2$

8 a $2 \leqslant x \leqslant 5$ or $x \leqslant -1$ **b** $-4 < x < 1$ or $x > 4$

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9 $y = \frac{1}{9}(x-3)^2(x+2)$

10 $y = -\frac{1}{10}x^2(x-3)(x+3)$

Multiple-choice questions

1 C	2 B	3 E	4 A
5 D	6 D	7 A	8 B
9 E	10 E	11 D	

Short-answer questions

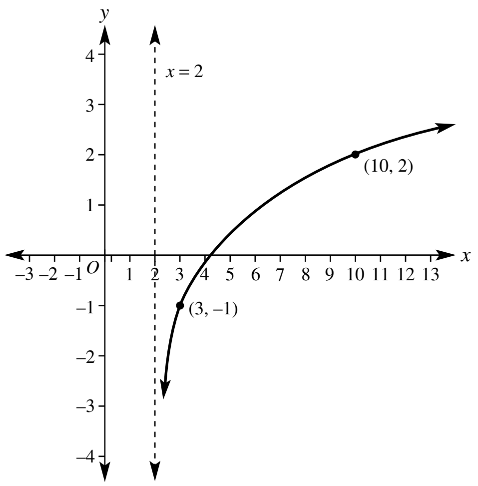
1 a $\log_2 16 = 4$ b $\log_{10} 1000 = 3$ c $\log_3 \frac{1}{9} = -2$

2 a $3^4 = 81$ b $4^{-2} = \frac{1}{16}$ c $10^{-1} = 0.1$

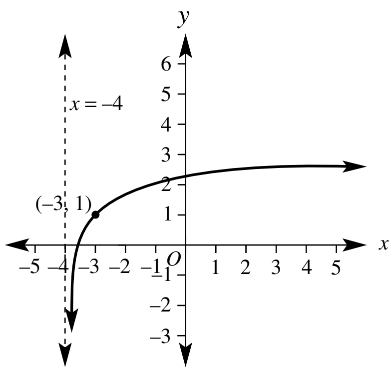
3 a 3 b 4 c 4
d 0 e −3 f −3
g −1 h −4 i −2

4 1 $y = \log_2 x$ is the reflection of $y = 2^x$ in the line $y = x$.
2 The coordinates of each point on $y = \log_2 x$ are the coordinates in reverse order of a point on $y = 2^x$

5 a $y = \log_2(x-2) - 1$
i $x > 2$
ii $x = 2$
iii moved right 2 units and down 1 unit
iv 4
v



b $y = \log_3(x+4) + 1$
i $x > -4$
ii $x = -4$
iii moved left 4 units and up 1 unit
iv −3.67, 2.26
v



6 $y = \log_5(x+2) + 3$

7 a $\text{pH} = \log_{10}\frac{1}{[\text{H}^+]}$ b 6.31×10^{-5} moles/L
c increase of 216%
d 794 times stronger intensity

8 a $\log_a 8$ b $\log_b 21$ c $\log_b 144$ d $\log_a 10$
e $\log_a 4$ f $\log_a 1000$ g 2 h 1
i $\frac{3}{2}$ j $4\log_a 2 = \log_a 16$

9 a $x = \log_3 6$ b $x = \log_{1.2} 2$

10 a $\frac{\log_{10} 13}{\log_{10} 2}$ b $\frac{\log_{10} 2}{\log_{10} 0.8}$

11 a −1 b 1 c −2 d −34

12 a $x^4 + 3x^2 + 2$ b $x^5 - x^4 - 3x^3$
c $x^5 + x^4 - 3x^3 - x^2 - x + 3$
d $x^6 + 2x^4 - 4x^3 + x^2 - 4x + 3$

13 a $x^3 + x^2 + 2x + 3 = (x-1)(x^2 + 2x + 4) + 7$
b $x^3 - 3x^2 - x + 1 = (x+1)(x^2 - 4x + 3) - 2$
c $2x^3 - x^2 + 4x - 7 = (x+2)(2x^2 - 5x + 14) - 35$
d $-2x^3 - x^2 - 3x - 4 = -(x-3)(2x^2 + 7x + 24) - 76$

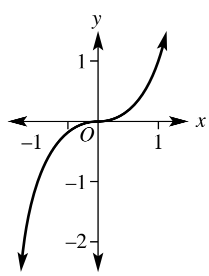
14 a −3 b −39 c −91 d 41

15 b, c and d are factors.

16 a $x = -2, 1$ or 3
b $x = -\frac{1}{3}, \frac{3}{2}$ or 5

17 a $(x-1)(x+2)(x+3) = 0$ $x = -3, -2$ or 1
b $(x+2)(x-5)(x-6) = 0$ $x = -2, 5$ or 6

18 a



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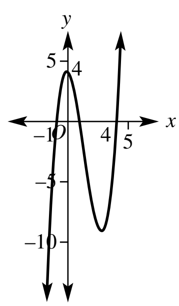

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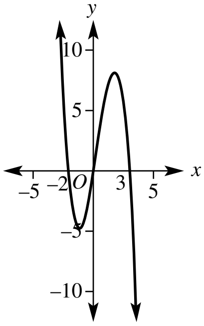
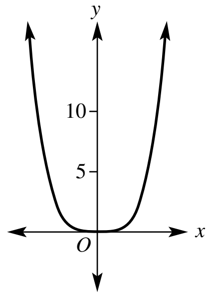
c



d



Extended-response questions

1 a B

b i \$121 000 **ii** \$115 369 **iii** \$272 034

c i 7.27 **ii** 6.17 **iii** 16.89

2 a i 32 **ii** 0

b There is no remainder; i.e. $P(1) = 0$.

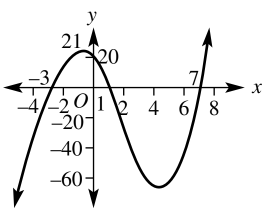
c $x^2 - 4x - 21$

d $(x - 7)(x - 1)(x + 3)$

e $x = 7, 1$ or -3

f $P(0) = 21$

g



Semester review 2

Chapter 6: Geometrical figures and circle geometry

Multiple-choice questions

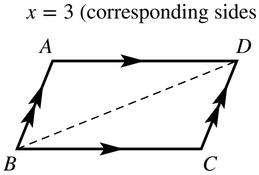
1 D **2** B **3** C **4** E **5** C

Short-answer questions

1 a $AB = DE$ (given)
$AC = DF$ (given)
$\angle BAC = 60° = \angle EDF$ (given)
$\therefore \triangle ABC \equiv \triangle DEF$ (SAS)
$a = 35$ (corresponding angles in congruent triangles)

b $BC = DC$ (given)
AC is common
$\angle ABC = 90° = \angle ADC$
$\therefore \triangle ABC \equiv \triangle ADC$ (RHS)
$x = 3$ (corresponding sides in congruent triangles)

2



$\angle DBC = \angle BDA$ (alternate angles in parallel lines)
$\angle BDC = \angle DBA$ (alternate angles in parallel lines)
BD is common
$\therefore \triangle BAD \equiv \triangle DCB$ (AAS)
Using congruence $BC = AD$ and $AB = DC$, corresponding sides in congruent triangles.

3 a $x = 6.75$ **b** $x = 2$

4 a $x = 8$ **b** $x = 5$ **c** $a = 32, b = 65$

d $x = 40$ **e** $a = 55$ **f** $a = 90, b = 60, c = 70$

5 a $x = 20$ **b** $x = 8$ **c** $a = 63, b = 55$

6 a $x = \frac{47}{5}$ **b** $x = \frac{29}{3}$ **c** $x = \frac{39}{5}$

Extended-response question

1 a $CD = 6$ cm (chords of equal length are equidistant from the centre)

b $OA = OD$ (radii of circle)
$OB = OC$ (radii of circle)
$AB = DC$ (chords of equal length are equidistant from the centre)
$\therefore \triangle OAB \equiv \triangle OCD$ (SSS)

c $OM = 4$ cm, Area $= 12$ cm^2

d 30.6%

e $\angle BOD = 106.2°$

Chapter 7: Trigonometry

Multiple-choice questions

1 E **2** B **3** A **4** D **5** C

Short-answer questions

1 a $x = 19.5$ **b** $\theta = 43.8, y = 9.4$

2 a i 150° **ii** 330° **b i** 310° **ii** 130°

3 a 32.174 m **b** 52.2°

4 a $x = 9.8$ **b** $\theta = 125.3$

5 95.1°

6 a $\tan\theta$

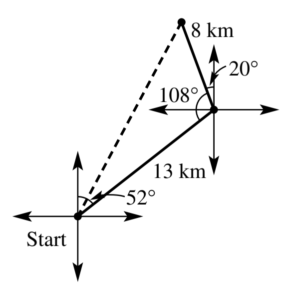
b i $\theta = 155$ **ii** $\theta = 35$ **iii** $\theta = 42$

c i $\frac{1}{2}$ **ii** $\frac{\sqrt{2}}{2}$ **iii** $-\frac{\sqrt{3}}{3}$

7 a ≈ 0.34 **b** $\theta \approx 233, 307$ **c** yes

Extended-response question

1 a 104.3 m **b**



c 17.242 km **d** 206°

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Chapter 8: Quadratic expressions and quadratic equations

Multiple-choice questions

1 C **2** B **3** D **4** B **5** D

Short-answer questions

1 **a** $9x^2 - 1$
b $4x^2 - 20x + 25$
c $-x^2 + 30x - 5$

2 **a** $(2x - y)(2x + y)$ **b** $(x + 2 + \sqrt{7})(x + 2 - \sqrt{7})$
c $3(x - 4)(x + 4)$ **d** $(x - 2)(x + 7)$
e $(x - 5)^2$ **f** $2(x - 6)(x - 2)$

3 **a** $(3x + 4)(x - 2)$ **b** $(3x - 1)(2x + 3)$
c $(5x - 4)(2x - 3)$

4 **a** $x = 0, 3$ **b** $x = -4, \frac{1}{2}$ **c** $x = 0, -5$
d $x = 4, -4$ **e** $x = \sqrt{7}, -\sqrt{7}$ **f** $x = 2$
g $x = 8, -3$ **h** $x = -2, \frac{1}{3}$

5 **a** $x = -8, 5$ **b** $x = 3, 7$ **c** $x = -4, 5$

6 **a** **i** $(x - 3 + \sqrt{5})(x - 3 - \sqrt{5})$
ii $(x + 2)^2 + 3$, does not factorise further
iii $\left(x + \frac{3}{2} - \frac{\sqrt{5}}{2}\right)\left(x + \frac{3}{2} + \frac{\sqrt{5}}{2}\right)$
b **i** $x = 3 \pm \sqrt{5}$
ii no solutions
iii $x = \frac{-3 \pm \sqrt{5}}{2}$

7 **a** $x = \frac{-3 \pm \sqrt{57}}{4}$ **b** $x = 2 \pm \sqrt{10}$

Extended-response question

1 **a** $4x^2 + 40x$ **b** $44\ \text{m}^2$
c $x = 3$ **d** $x = 2.2$

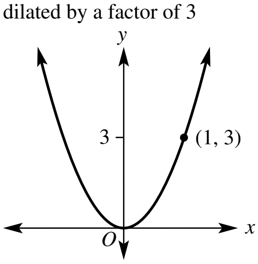
Chapter 9: Non-linear relationships, functions and their graphs

Multiple-choice questions

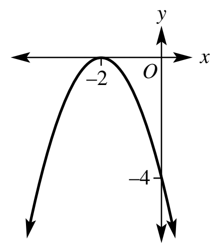
1 D **2** A **3** D
4 C **5** B **6** C

Short-answer questions

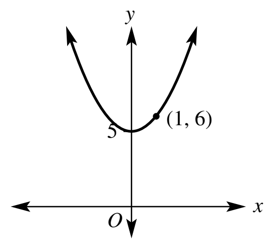
1 **a** dilated by a factor of 3



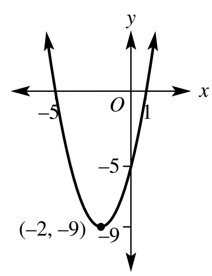
b reflected in x-axis and translated 2 units left



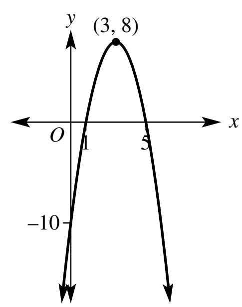
c translated 5 units up



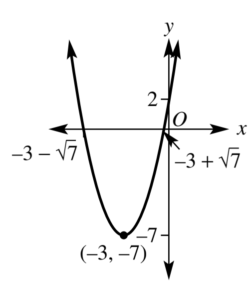
2 **a** -5 **b** $-5, 1$ **c** $(-2, -9)$
d



3 **a** maximum at $(3, 8)$ **b** -10 **c** $1, 5$
d



4 **a** $y = (x + 3)^2 - 7$



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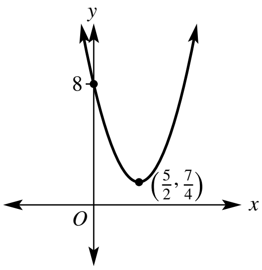
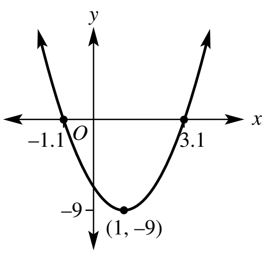
b $y = \left(x - \frac{5}{2}\right)^2 + \frac{7}{4}$

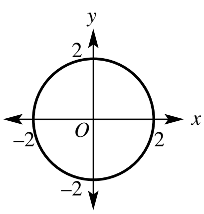


5 a Discriminant = 72, thus two x-intercepts.

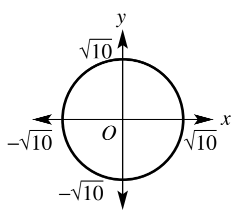
b



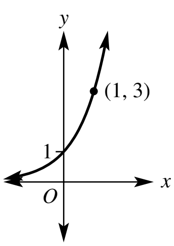
6 a



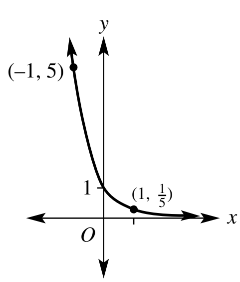
b



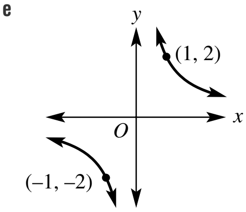
c



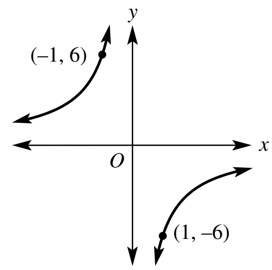
d



e



f



7 a -4 **b** -2 **c** 6

d 30 **e** $2k^2 + k - 7$

8 a $A > 0, a > 0, b > 0, h > 0$ **b** $h = \dfrac{2A}{a + b}$

c 4 **d** $a = \dfrac{2A}{h} - b$ **e** 3

9 a domain: all real x; range: all real y

b domain: all real x; range: all real y

c domain: all real x; range: $y \leqslant 4$

d domain: all real x, $x \neq 0$; range: all real y, $y \neq 0$

e domain: $x \geqslant -2$ range: $y \geqslant 0$

10 a $f^{-1}(x) = \dfrac{x}{4}$ **b** $f^{-1}(x) = 5 - x$

c $f^{-1}(x) = \sqrt[3]{4 - x}$ **d** $f^{-1}(x) = \dfrac{3}{2x}$

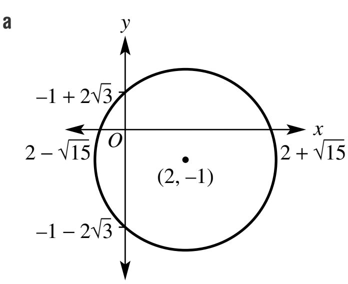
e $f^{-1}(x) = \dfrac{1}{1 - x}$

11 $x \geqslant 1$ or $x \leqslant 1$

12 a $(\sqrt{3}, 2\sqrt{3}), (-\sqrt{3}, -2\sqrt{3})$

b $(4, 16)$ **c** $\left(\frac{1}{2}, 4\right)\left(-\frac{1}{2}, -4\right)$

13 a



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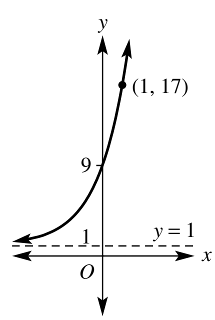

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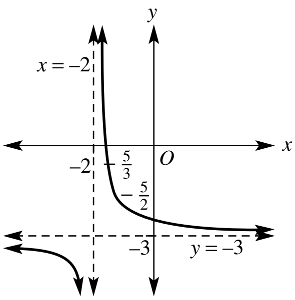
c

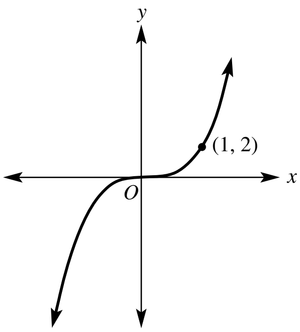


14 a $x = 3, y = 22$ and $x = -5, y = -2$

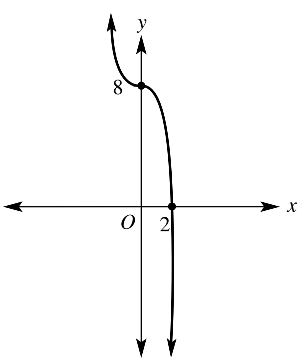
b no solutions

c $x = -1, y = 1$

15 a



b



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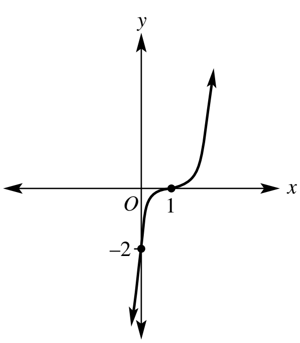
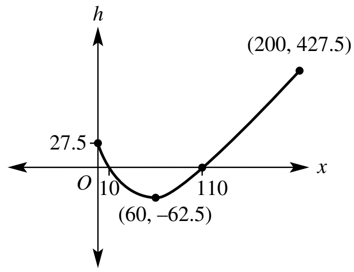


Extended-response questions

1 a



b 27.5 m **c** 10 m and 110 m from start

c 427.5 m **d** 62.5 m

2 a i Cost increases as the amount of hire time increases. If hire time doubles, then the cost will also double.

ii $C = 10n$

iii Cost = \$37

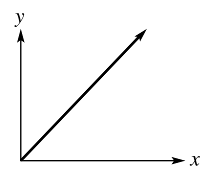
b i $y = 2$

ii $x = 24$

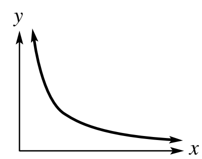
c i Speed is inversely proportional to the time taken to travel a certain distance. If the speed increases, then the time decreases.

ii 97.2 min = 1 h 37.2 min

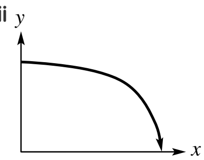
d i



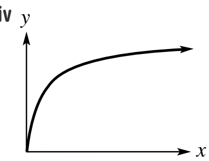
ii



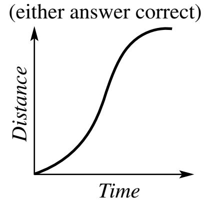
iii

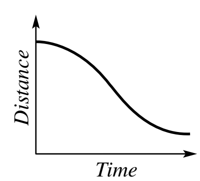


iv



v (either answer correct)





Chapter 10: Logarithms and polynomials

Multiple-choice questions

1 E **2** A **3** D **4** C **5** D

Short-answer questions

1 a 3 **b** –2 **c** –1 **d** 0

e 2 **f** 2 **g** 1 **h** 3

2 a $x = 3$ **b** $x = 3$ **c** $x = 81$

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3 a i $x = \log_3 30$ ii $x = \log_{2.4} 4$

b i $x = 2.460$ ii $x = 9.955$

4 a i 6 ii 0 iii -49 iv -5

b i $2x^6 + 6x^5 - 11x^4 - 25x^3 + 34x + 24$

ii $4x^6 - 12x^4 - 16x^3 + 9x^2 + 24x + 16$

5 $P(x) = (x - 3)(x^2 - x - 1) + 4$

6 a -24, not a factor b 0, a factor

c -40, not a factor

7 a $x = -1, 3, -6$ b $x = 0, \frac{5}{2}, -\frac{2}{3}$

c $x = -4, -2, 1$ d $x = -1, \frac{1}{2}, 2$

8 a $x > -4$

b $x = -4$

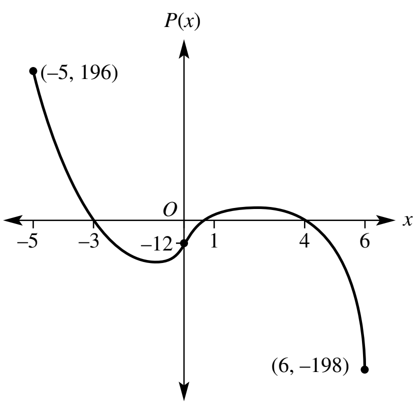
c $y = \log_3(x + 4) + 2$

d $-3.89, 3.26$

9 a $P(-3) = 0$

b $P(x) = -(x + 3)(2x - 1)(x - 4)$

c



Extended-response question

1 a 100 000 times louder

b 4.8 dB

c 355 times louder

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