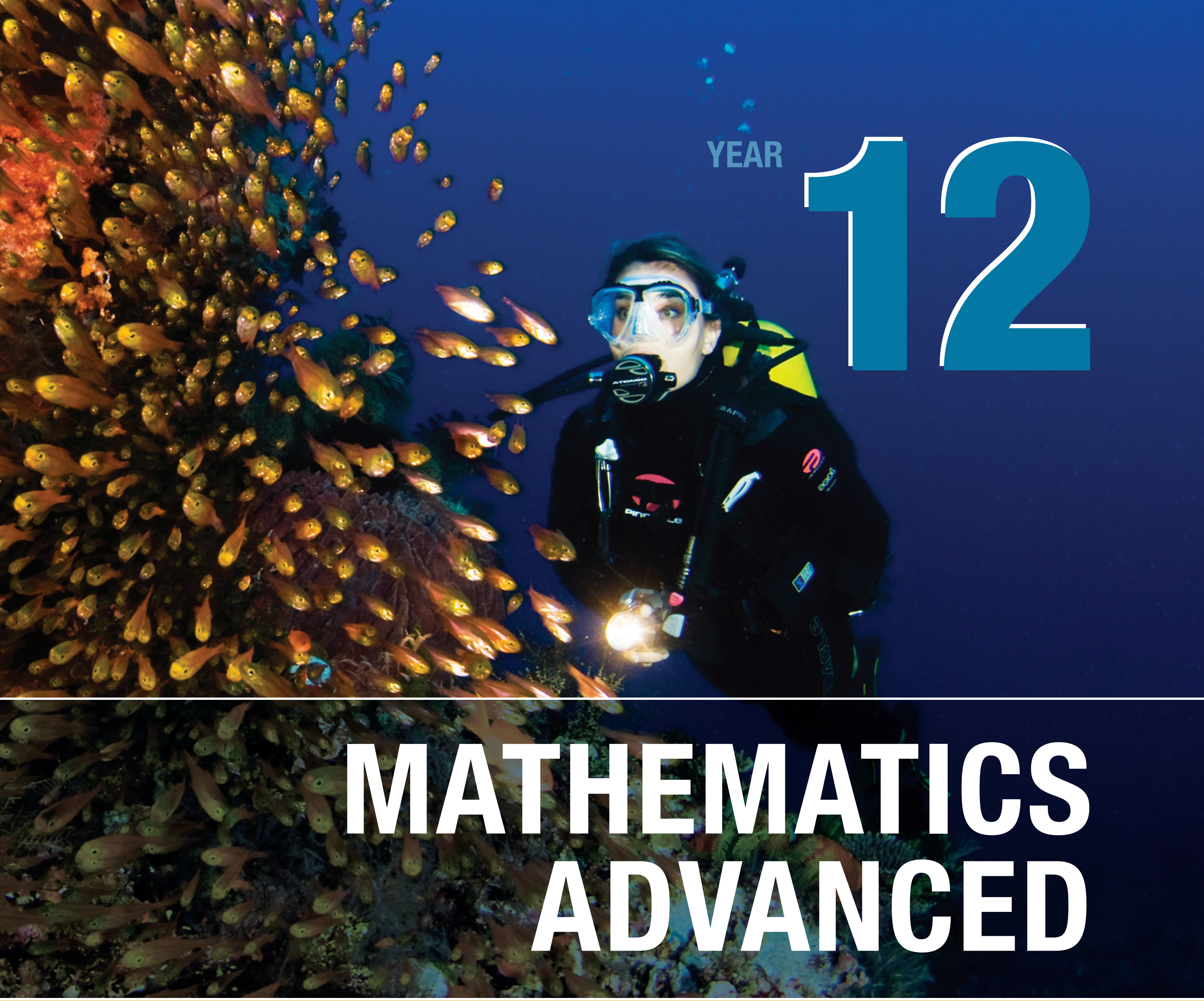

YEAR 12

MATHEMATICS ADVANCED

CambridgeMATHS

STAGE 6

BILL PENDER

DAVID SADLER | DEREK WARD | BRIAN DOROFAEFF | JULIA SHEA

INCLUDES INTERACTIVE TEXTBOOK POWERED BY CAMBRIDGE HOTMATHS

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Online appendices

See the interactive textbook for guides to spreadsheets, the Desmos graphing calculator, and links to scientific calculator guides.

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Rationale

CambridgeMATHS Advanced Year 12 covers all syllabus dotpoints for Year 12 of the Mathematics Advanced course being implemented in Term IV 2019. This rationale serves as a guide to how the book covers the dotpoints of the syllabus. Further documents are available in the teacher resources.

The exercises

No-one should try to do all the questions! We have written long exercises with a great variety of questions so that everyone will find enough questions of a suitable standard — they cater for differentiated teaching to a wide range of students. The division of all exercises into Foundation, Development and Challenge sections helps with this. Each student will need to tackle a selection of questions, and there should be plenty left for revision.

The **Foundation** section in each exercise provides a gentle start with many straightforward questions on each new skill and idea. Students need encouragement to assimilate comfortably the new ideas and methods presented in the text so that they are prepared and confident before tackling later problems.

The **Development** section is usually the longest, and is graded from reasonably straightforward questions to harder problems. The later questions may require the new content to be applied, they may require proof or explanation, or they may require content from previous sections to put the new ideas into a wider context.

The **Challenge** section is intended to match some of the more demanding questions that HSC Advanced examination papers often contain — we assume that this will continue with the HSC examinations on the new syllabuses. They may be algebraically or logically challenging, they may establish more difficult connections between topics, or they may require less obvious explanations or proofs. The section may be inappropriate at first reading.

The structure of calculus in the book

Calculus is the centre of this course, and the structure of calculus has determined the order in which the topics are presented in the book.

CHAPTER 1: Sequences and series are the discrete analogue of infinitesimal calculus. In particular, a definite integral consists of 'an infinite sum of infinitesimally thin strips', so that sums of series, which are conceptually much easier, should precede integration. Also, linear and exponential functions are the continuous analogues of APs and GPs.

CHAPTER 2: Systematic curve-sketching using differentiation is traditionally the first and most straightforward application of calculus. But before reaching for the derivative, it is important to give a systematic account of the various non-calculus approaches to curve-sketching.

CHAPTERS 3–4: Differentiation, curve-sketching with calculus, and integration, are three dramatic new ideas in school that change students' perceptions of the nature of mathematics, making it far more imaginative and speculative, and allowing easy solutions of problems that may otherwise seem impossible. The first encounter with these three ideas should involve only algebraic functions, because the complexities of the special functions associated with e^x and $\sin x$ are unnecessary here, and they cause confusion if introduced too early.

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Differentiation was introduced in Year 11, so curve-sketching and integration are the next two topics in the book. The derivative of e^x was established last year because of the interest in rates, but we have briefly left e^x aside in these two chapters so that the story is not confused.

CHAPTER 5: With the basic methods of calculus now established, the significance of e^x as a function that is its own derivative, and of $\log_e x$ as a primitive of x^{-1}, can now be explained, and the importance of the special number e made clear. This chapter is necessarily long — the ideas are unfamiliar and unsettling, their far-reaching significance is difficult to explain, and students need time to assimilate them.

CHAPTER 6: The other group of special functions are the trigonometric functions, with their special number π. The standard forms now multiply, but the basic methods of calculus remain the same, and by the end of Chapter 6 these methods will have been reviewed three times — for the exponential functions, for the logarithmic functions, and for the trigonometric functions.

CHAPTERS 7–8: These are application chapters. Chapter 7 applies calculus to motion and rates. Chapter 8 is the discrete analogue of Chapter 7, applying series to practical situations, particularly finance. The discussion of motion and rates in Chapter 7 allows a review of all the functions introduced in calculus. Motion, in particular, allows the derivatives to be perceived by the senses — the first derivative is velocity, which we see, and the second derivative is acceleration, which we feel.

Learning to handle applications of calculus is surprisingly difficult, and while a few questions have been included in exercises in previous chapters, motion and rates both need a sustained account if they are to be mastered. Motion requires confusing contrasts between displacement, distance and distance travelled, and between velocity and speed. Rates require further contrasting experiences of translating physical events into the abstractions of calculus.

CHAPTERS 9–10: Integration and the exponential function together make possible a coherent presentation of statistics and the normal distribution in these two chapters. Probability becomes an area, and most of the mysteries of $e^{-\frac{1}{2}x^2}$ can now be rigorously expounded.

Syllabus coverage of the chapters

Chapter 1: Sequences and series

Syllabus References: M1.2
M1.3

Chapter 1 presents the theory of arithmetic and geometric sequences and series. Its purpose at the start of the book is to give a wider mathematical context for linear and exponential functions in Chapters 2 and 5, for the derivative used in Chapter 3, and for the definite integral in Chapter 4. There are some practical examples throughout, but there are many more in Chapter 8, and it may be appropriate to bring forward some questions from Sections 8A–8C.

Arithmetic and geometric sequences are closely related, and are explained together as the theory progresses through sequences in Sections 1A–1C, problems in Section 1D, and the sums of series in Sections 1E–1G. The chapter concludes in Sections 1H–1I with limiting sums of GPs and the explanation — finally — of what a recurring decimal actually is.

Sigma notation is introduced for several reasons. It allows a more concise notation for series, it prepares the ground for the continuous sum $\int$ of integration, it makes more precise the rather vague use of $\sum$ in the statistics chapters, and the notation is needed in later mathematics courses. Nevertheless, neither the text nor the exercises rely on it, and the few questions that use it can easily be avoided or adapted.

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Chapter 2: Graphs and equations

Syllabus References: F1.2 dotpoint 2 (An alternative interval notation)

F1.2 dotpoint 7 (An alternative composite function notation)

F2

T3

The syllabus item F1 stresses 'any function within the scope of this syllabus'. As explained in the introduction, the methods used here will be extended to other functions as they are introduced in Chapters 5, 6 and 10, apart from the trigonometric graphs that are covered in Section 2I. It also stresses 'real-life contexts', which will be further developed in Chapter 7.

F1.2 dotpoints 2 & 7 and Section 2A: The chapter begins with two pieces of notation that were flagged last year as being a distraction at the start of Year 11.

- Bracket interval notation such as $[3, 6]$ or $(2, \infty)$ is an alternative for inequality interval notation such as $3 \le x \le 6$ or $x > 2$. It requires more sophistication than was appropriate at the start of Year 11, and in particular it requires the symbol $\cup$ for the union of sets, which was introduced for probability in Sections 10C–10D late in the Year 11 book.
- A composite function $g\big(f(x)\big)$ can also be written as $g \circ f(x)$, which was unnecessarily abstract for the start of Year 11.

F2 dotpoint 2, sub-dotpoints 1–2 & Sections 2A–2C: These sections consolidate and extend the curve-sketching methods from Year 11, particularly the sign of the function and asymptotes, and organise them into a curve-sketching menu, which gives a systematic approach to sketching an unknown function (this organisation of approaches to curve-sketching is only our suggestion).

F2 dotpoint 2, sub-dotpoints 3–4 & Sections 2D–2E: Graphing is closely related to the solving of equations and inequations. These sections formalise several important connections and methods.

F2 dotpoint 1 & Sections 2F–2H: After reviewing the translations, reflections and rotations introduced last year, these sections introduce dilations, then investigate which transformations do not commute with other transformations — a difficult question with a surprisingly simple answer. Replacement is our preferred method of relating transformations to the equations of the function or relation, but the formulae approach is also presented. The sections conclude with a complicated formula that transforms a function successively by four separate transformations.

T3 & Section 2I: Radians were introduced last year, so transformations of trigonometric graphs can be covered here. They lead to the four ideas of amplitude, period, phase and mean value, and they can be dealt with by the same complicated formula given in Section 2H.

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Chapter 3. Curve-sketching using the derivative

Syllabus References: C3.1

C3.2

C4.1 dotpoints 1, 2, 3, 11 (without the use of the integral sign)

The first and most straightforward application of the derivative is to assist in the sketch of a function by examining its gradient, stationary points, concavity and inflections. This chapter explains those procedures, which will be reviewed and extended progressively to exponential functions (Section 5E), logarithmic functions (Section 5H) and trigonometric functions (Section 6E), and used in various applications throughout the book, particularly in Chapter 7 on motion and rates.

C3.1, 3.2 dotpoints 1–2 & Sections 3A–3E: What is needed here for curve-sketching are *pointwise* definitions of increasing and decreasing, and of concave up and concave down. The *interval-wise* definitions are delayed until Section 7D, where they are needed for the discussion of rates in Sections 7D–7F. Section 3E concludes the discussion by extending the curve-sketching menu of Chapter 2 with two more steps involving the first and second derivatives.

C3.2 dotpoint 3 & Sections 3F–3G: The global maximum of a function is now introduced, followed by an exercise consisting of diverse examples of the use of these procedures to find the maxima or minima of functions in various practical situations.

C4.1 & Section 3H: Primitive functions have a strong graphical interpretation, and in preparation for the following integration chapter, it is useful to review and extend here the discussion of primitives in Section 8D of the Year 11 book.

Chapter 4. Integration

Syllabus References: C4.1

C4.2

Having briefly reviewed the primitive in Section 3H, this chapter now begins and develops integration in the traditional way by asking questions about areas of regions where some of the boundaries are neither lines nor arcs of circles. The procedures in this chapter will be reviewed and extended progressively to exponential functions (Sections 5D–5E), reciprocal (Sections 5I–5J) and trigonometric functions (Sections 6D–6E), and used in various applications throughout the book.

C4.2 dotpoints 1–3, 5–8 & Sections 4A–4D: These sections introduce the definite integral in terms of area, and first calculate integrals using area formulae. There is also some guided work on limiting sums of areas. The fundamental theorem is stated and used in Section 4B, but not proven until Section 4D, which is marked as Challenge. Using geometric arguments, Section 4C explores the behaviour of the definite integral when the curve is below the or the integral runs backwards, and identities are developed that can simplify the evaluation of a definite integral.

C4.2 dotpoints 4, 9–11 & Sections 4F–4H: Areas between curves are introduced, including areas between a curve and the . Then the trapezoidal rule is developed, first geometrically, then algebraically, then using spreadsheets to aid the computation. Some practical examples are given, and others occur throughout the book, particularly with motion in Chapter 7.

C4.1 & Sections 4E, 4I: The basic ideas of a primitive were discussed in Section 3H (*C4.1 dotpoints 1–3, 9–11*), and the remaining standard forms are dealt with in Chapters 5–6 as the new special functions are developed. The reverse chain rule, however, is introduced here with powers of functions (*C4.1 dotpoint 4*), in preparation for its further use with special functions.

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Chapter 5. The exponential and logarithmic functions

Syllabus References: F2, as relevant

C2.1 dotpoints 3–5

C2.2, as relevant

C3.1, as relevant

C3.2, as relevant

C4.1, as relevant

C4.2, as relevant

As stated in the introduction to these notes, the exponential and logarithmic functions are unfamiliar and difficult for students, and need their own sustained development rather than being interlocked with powers of x and trigonometric functions. All curve-sketching, equation-solving and calculus relevant to these functions is completed in these sections, in preparation for its further use, particularly in Chapter 7 on motion and rates and Chapter 10 on the normal distribution.

The topic was begun in Chapter 9 of the Year 11 book with the differentiation of the special number e and the exponential function e^x. The results, but not all the explanations, are reviewed here in Sections 5A and 5F. Some further review of that earlier chapter may be appropriate.

We draw attention to what are possibly unfamiliar standard forms in Section 5I (*C4.1 dotpoint 7*).

$$\int \frac{1}{x}\,dx = \log|x| + C \qquad \text{and} \qquad \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + C.$$

Students will now need to cope with absolute values in their integrals, and the further consequences of this standard form have made the topic more difficult than it used to be.

Section 5K on calculus with other bases is important because in ordinary language, half-lives and doubling times are so often used. The standard forms are to be learnt, but students should be aware of the other possibility — convert everything to base e before applying calculus.

Chapter 6. The trigonometric functions

Syllabus References: C2.1, as relevant

C2.2, as relevant

C3.1, as relevant

C3.2, as relevant

C4.1, as relevant

C4.2, as relevant

As in the previous chapter, the trigonometric functions are difficult for students, who learn best when these functions are developed separately before being combined with other functions. All curve-sketching, equation-solving and calculus relevant to these functions is completed in these sections, in preparation for its further use, particularly in Chapter 7 on motion and rates.

Radian measure, based on the special number π, was introduced in Chapter 9 of the Year 11 book, and the geometric formulae established there are important tools in this chapter. The graphs in radians were also discussed there at length, then subjected to four separate transformations in Section 2I.

The fact that the derivative of $\sin x$ is $\cos x$ should be informally clear by now, but the informal graphical proof is presented again. Using the geometric formulae, we are now able to prove in Section 6A using

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limits that $\sin x$ has gradient exactly 1 at the origin. The rest of the proof moves this result along the curve using compound angle formulae, which we do not have, so it has been removed to the Chapter Appendix.

The authors are concerned that the standard forms for integration on the current formulae sheet go significantly beyond the standard forms in C4.1, and seem rather sophisticated for Advanced students. There may be future adjustment here.

Chapter 7. Motion and rates

Syllabus References:

Motion: C1.2 dotpoint 3, C1,3 dotpoints 6–7, C3.1 dotpoint 2, C3.2 dotpoint 3

Rates: C1.2 dotpoints 1–2, C1.3 dotpoint 8, C1.4 dotpoint 4, C3.2 dotpoint 3,

M1.2 dotpoint 5, M1.4 dotpoint

Exponential function applications: E1.4 dotpoints 4, 6, 7

The syllabus references are scattered through the topics (the references above include both Year 11 and Year 12), but there is a constant concern that calculus be modelled by rates of various types, and in particular applied to motion and to exponential growth and decay. As explained in the introduction to this Rationale, some rates and motion questions have been asked already, but learning to apply calculus to practical situations is surprisingly difficult. What is needed is sustained and contrasting work to gain experience in reinterpreting the abstract objects of calculus. They appear in everyday things that we can see and feel, such as velocity, acceleration, the diminishing flow of water from a kettle, the rise and fall of the tides and the temperature. Then they appear in important contemporary concerns such as populations, radioactive decay and inflation.

Motion is particularly tricky and needs its own treatment in Section 7A–7C separate from other rates. Exponential growth and decay also need separate attention in Section 7F. The calculus of all the functions in the course so far reappears in these exercises.

There are several common ideas through the whole chapter. First, there are the correspondences,

gradient of a chord $\longleftrightarrow$ average rate $\longleftrightarrow$ average velocity,

gradient of a tangent $\longleftrightarrow$ instantaneous rate $\longleftrightarrow$ instantaneous velocity.

Secondly, concavity can be interpreted as indicating whether the rate of increase or decrease is increasing or decreasing, and in motion as the direction in which the particle is accelerating and whether the speed is increasing or decreasing. Thirdly, differentiation moves from the quantity to the rate (Section 7D), and integration from the rate to the quantity (Section 7E).

Section 7D finally gives a precise definition of *increasing in an interval* and *concave up in an interval*. Curve-sketching with calculus (Chapter 2) was not an appropriate time for these ideas, but they are part of the language of rates.

Chapter 8. Series and finance

Syllabus References: M1.1

M1.2 (review with applications)

M1.3 (review with applications)

M1.4

M1.2–M1.3: Sections 8A–8B: Arithmetic and geometric sequences and series were presented in Chapter 1 to give a wider context to the intense calculus of the next few chapters. Those formulae are reviewed here, but the emphasis is on applications of series to practical situations.

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Section 8B uses logarithms with problems involving GPs. These methods should be contrasted with trial-and-error, which is attractive for occasional use, but clumsy in comparison.

M1.1, M1.4 & Sections 8C–8E: These sections are still about APs and GPs, but specifically directed towards interest, annuities (or superannuation) and paying off loans. The algebra involved in these last two computations is difficult, but learning formulae should be discouraged, because the question only has to start or finish the transaction slightly differently and any formula will fail.

The basic approach presented here is to treat each payment as a separate investment. Another widely-used approach is recursion, where the monthly bank statement becomes the goal of the computations. Opportunity has been given to use recursion instead — later questions do not specify which approach is to be used, and earlier questions can easily be adapted.

Series are discrete objects, and their applications in Chapter 8 contrast with the applications of continuous functions in Chapter 7. This contrast has an interesting companion — the discrete probability distributions of Chapter 11 last year contrast with the continuous probability distributions of Chapter 10 this year. Datasets, on the other hand, are always discrete.

Chapter 9. Displaying and interpreting data

Syllabus References: S2.1–S2.2

This chapter is in sharp contrast to Chapters 10–11 in the Year 11 book on probability and discrete probability distributions. Those chapters were mostly about theoretical probabilities, with a little sampling and data-gathering at the end. This chapter, however, is all about data — its display in tables and charts, and the calculations of summary statistics — in order to gain a global view of a dataset that could otherwise be just a meaningless jumble. After this, we seek interpretation (requiring common sense and judgement) and prediction (risky), and we find a scientist to ask about causation.

Sections 9A–9C concern univariate data, and may be familiar from earlier years. Sections 9D–9F mostly concern bivariate data, which may be less familiar. Pearson's correlation coefficient and the least-squares line of best fit are a difficult part of this chapter. We dislike black boxes, so we have given the formulae for them in Section 9E and a short exercise to drill them, but what is intended is that they be calculated by technology, which can also be rather complicated. The rest of the chapter is quite independent of the formulae in Section 9E, which should be regarded as Challenge.

The technology used may be a calculator, or a spreadsheet, or special statistics software, and all these things may be online. Technology in schools is still very variable, with no agreed equipment, rules or procedures. We have given, here and elsewhere in the book, detailed instructions about using Excel, which is probably the most widely-used spreadsheet, but even this is a highly contentious decision because the Mac versions are different, the Windows versions keep changing, and many people prefer other technologies.

The final exercise is an investigation. It contains links to large datasets on the online interactive textbook, activities such as surveys to generate data for analysis, and investigations allowing the reader to search out raw data from the internet. Many things here could become projects.

One problem is a concern for the authors. No indication has been given about using a correction factor when calculating the variance of a sample that is not a population. We do not want to make difficult things more complicated, so we have used division by n, which is simpler, particularly when the formulae are developed for relative frequencies. We have warned about the technology issues that arise when computing variance, and we have written Challenge sections and questions, in this chapter and in Chapter 11 last year, where we have explained the correction as multiplying the variance by $\frac{n}{n-1}$. If a contrary ruling is ever given, adjustments will be needed.

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Chapter 10. Continuous probability distributions

Syllabus References: S3.1–S3.2

Chapter 10 moves via Section 10A from the data of Chapter 9 back to probability theory, but with the constant accompaniment of data, and this time the theory is about continuous random variables. Many things about data lead into the continuous theory — the relative frequencies, the histograms and their associated areas, the grouping that is so often necessary, the appearance of the frequency polygon — besides of course the nature of what is being measured.

Various examples of continuous distributions are given in Sections 10B–10C. We have included some distributions where the domain of the values is unbounded because the domain of the normal distribution is the whole real number line $(-\infty, \infty)$. We have done any calculations over unbounded domains without the usual complicated machinery, and the authors hope that such things will remain a side issue to the important statistical ideas going on. More machinery may be appropriate in some classes.

The standard normal distribution is discussed in Section 10D, then stretched and shifted in Section 10E to become the general normal distribution, where z-scores become a good example of the transformations studied in Chapter 2.

We have used a short table of values of the normal in our calculations so that everything is accessible to students whose only technology is a standard non-statistics scientific calculator. Some practice doing this is useful for everyone, but as technology develops, such tables will fall out of use just as log tables have done. Reading a printed table backwards is a particular nuisance because interpolation is clumsy and time-consuming.

The inflections and variance of the standard normal are difficult, but should be accessible to most students. On the other hand, the proof that $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx = \sqrt{2\pi}$ will unfortunately have to wait for far more sophisticated methods of calculus at some later time.

As in Chapter 9, Exercise 10G concludes the chapter with an investigation exercise using large datasets, and projects can easily be developed from these questions.

Bill Pender, June 2019

The Book of Nature is written in the language of Mathematics.

— The seventeenth-century Italian scientist Galileo

It is more important to have beauty in one's equations than to have them fit experiment.

— The twentieth-century English physicist Dirac

Even if there is only one possible unified theory, it is just a set of rules and equations. What is it that breathes fire into the equations and makes a universe for them to describe? The usual approach of science of constructing a mathematical model cannot answer the questions of why there should be a universe for the model to describe.

— Steven Hawking, A Brief History of Time

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Overview

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Features of the print textbook

1 Refer to the *Rationale* for details of question categories in the exercises and syllabus coverage.
2 Each section begins at the top of the page to make them easy to find and access.
3 Plenty of numbered worked examples are provided, with video versions for most of them.
4 Important concepts are formatted in numbered boxes for easy reference.
5 Investigation exercises and suggestions for projects are included.
6 Proofs for important results are provided in certain chapters.
7 Chapter review exercises assess learning in the chapter.

Downloadable PDF textbook

8 The convenience of a downloadable PDF textbook has been retained for times when users cannot go online. PDF search and commenting tools are enabled.

Digital resources in the Interactive Textbook powered by the HOTmaths platform *(shown on the page opposite)*

The Interactive Textbook is an online HTML version of the print textbook powered by the HOTmaths platform, completely designed and reformatted for on-screen use, with easy navigation. It is included with the print book, or available as digital-only purchase. Its features include:

9 Video versions of the examples to encourage independent learning.
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12 The full suite of the HOTmaths learning management system and communication tools are included in the platform, with similar interfaces and controls.
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17 Definitions pop up for key terms in the text, and are also provided in a dictionary.
18 Spreadsheet files are provided to support questions and examples based on the use of such technology.
19 Online guides are provided to spreadsheets and the Desmos graphing calculator, while links to scientific calculator guides on the internet are provided.
20 Users who had Year 11 digital accounts may access Year 11 material for revision of prior knowledge.
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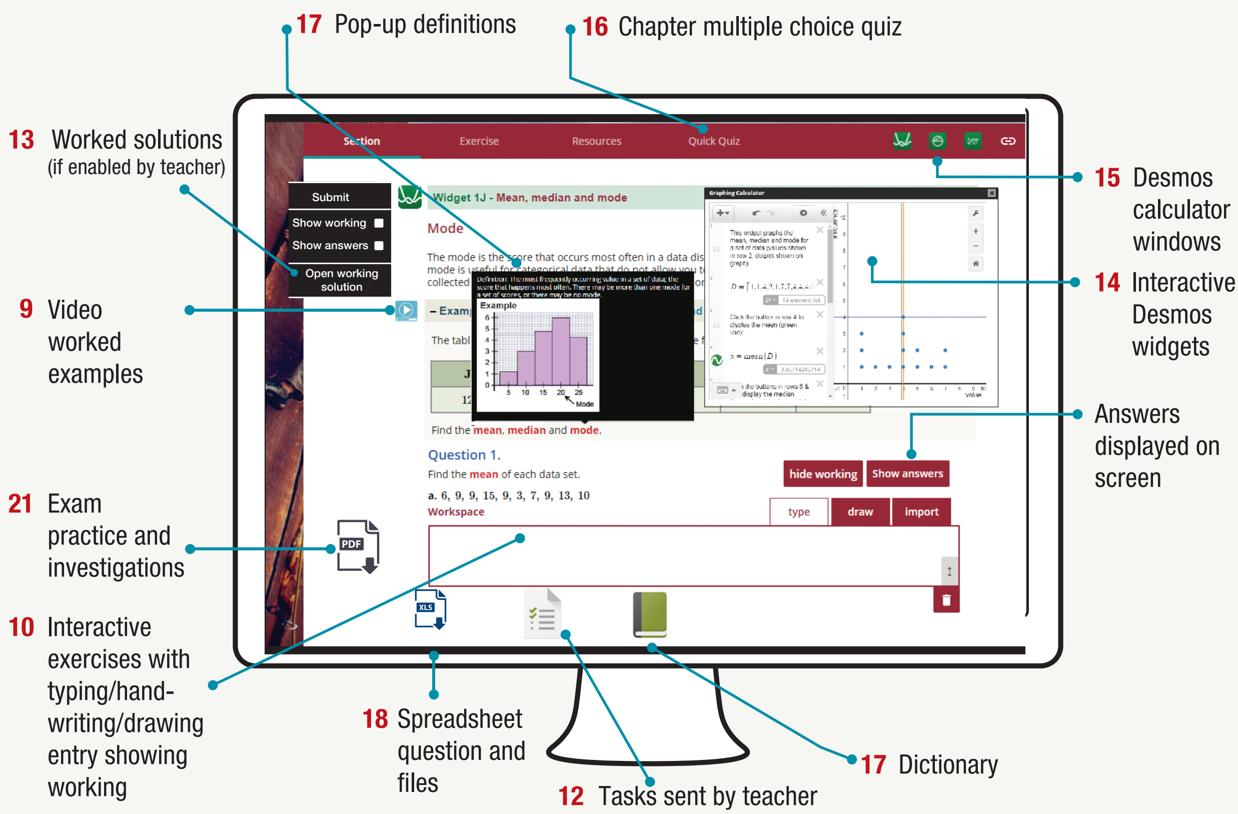

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22 The Online Teaching Suite is automatically enabled with a teacher account and appears in the teacher's copy of the Interactive Textbook. All the assets and resources are in the Teacher Resources panel for easy access.

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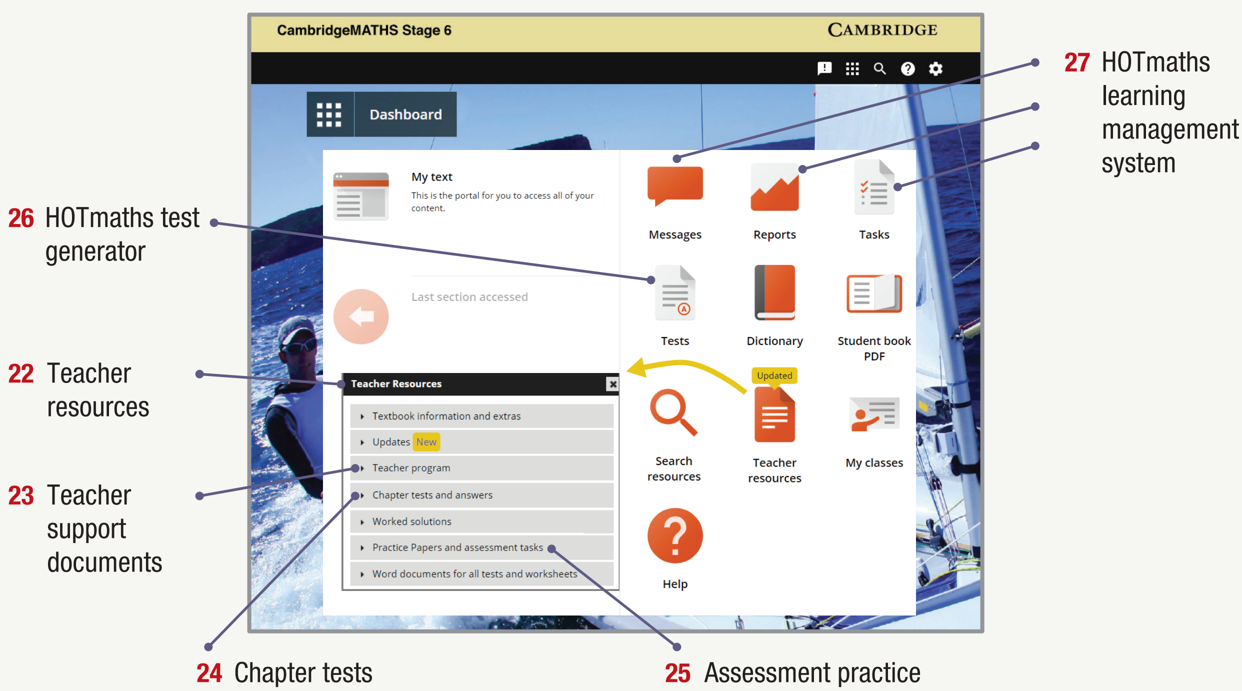

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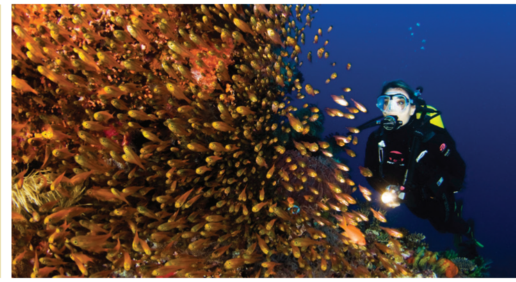

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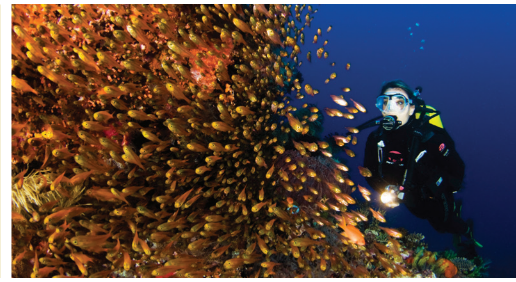

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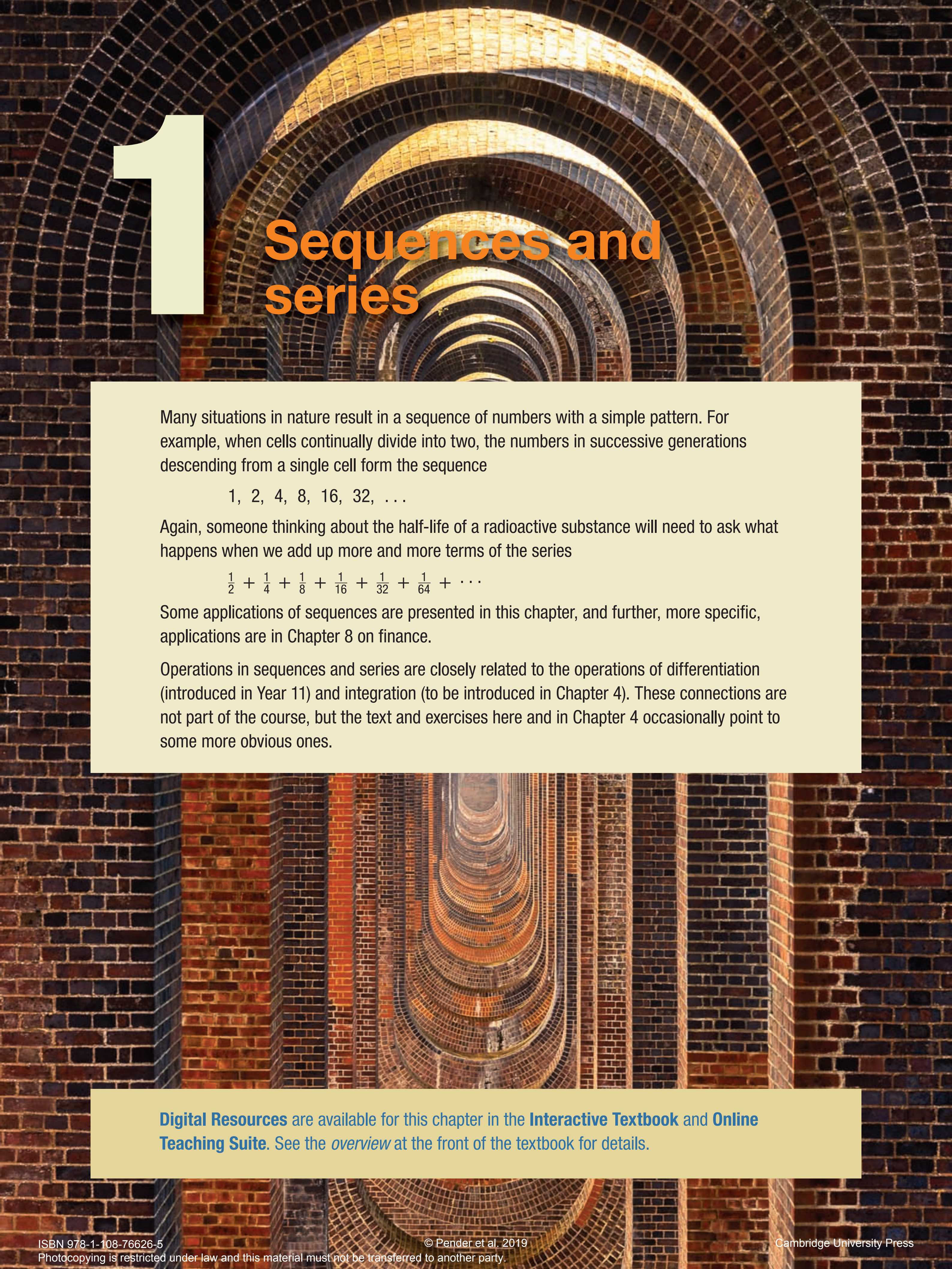

1 Sequences and series

Many situations in nature result in a sequence of numbers with a simple pattern. For example, when cells continually divide into two, the numbers in successive generations descending from a single cell form the sequence

$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ \ldots$$

Again, someone thinking about the half-life of a radioactive substance will need to ask what happens when we add up more and more terms of the series

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{64} + \cdots$$

Some applications of sequences are presented in this chapter, and further, more specific, applications are in Chapter 8 on finance.

Operations in sequences and series are closely related to the operations of differentiation (introduced in Year 11) and integration (to be introduced in Chapter 4). These connections are not part of the course, but the text and exercises here and in Chapter 4 occasionally point to some more obvious ones.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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1A Sequences and how to specify them

A typical *infinite sequence* is formed by arranging the positive odd integers in increasing order:

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, \ldots$$

The three dots . . . indicate that the sequence goes on forever, with no last term. The sequence starts with the first term 1, then has second term 3, third term 5, and so on. The symbol T_n will usually be used to stand for the nth term, thus

$$T_1 = 1, \quad T_2 = 3, \quad T_3 = 5, \quad T_4 = 7, \quad T_5 = 9, \quad \ldots$$

The two-digit odd numbers less than 100 form a *finite sequence*:

$$1,\ 3,\ 5,\ 7,\ \ldots,\ 99$$

where the dots . . . stand for the 45 terms that have been omitted.

There are three different ways to specify a sequence, and it is important to be able to display a given sequence in each of these different ways.

Write out the first few terms

The easiest way is to write out the first few terms until the pattern is clear to the reader. Continuing with our example of the positive odd integers, we could write the sequence as

$$1,\ 3,\ 5,\ 7,\ 9,\ \ldots$$

This sequence clearly continues as $11,\ 13,\ 15,\ 17,\ 19,\ \ldots$, and with a few more calculations, it becomes clear that $T_{11} = 21$, $T_{14} = 27$, and $T_{16} = 31$.

Give a formula for the *n*th term

The formula for the nth term of this sequence is

$$T_n = 2n - 1,$$

because the nth term is always 1 less than $2n$. Giving the formula does not rely on the reader recognising a pattern, and any particular term of the sequence can now be calculated quickly:

$$\begin{aligned} T_{30} &= 60 - 1 \\ &= 59 \end{aligned} \qquad \begin{aligned} T_{100} &= 200 - 1 \\ &= 199 \end{aligned} \qquad \begin{aligned} T_{244} &= 488 - 1 \\ &= 487 \end{aligned}$$

Say where to start and how to proceed

The sequence of odd positive integers starts with 1, then each term is 2 more than the previous one. Thus the sequence is completely specified by writing down these two statements:

$$\begin{aligned} &T_1 = 1, && \text{(start the sequence with 1)} \\ &T_n = T_{n-1} + 2, \quad \text{for } n \geq 2. && \text{(every term is 2 more than the previous term)} \end{aligned}$$

Such a specification is called a *recursive* formula of a sequence. Most of the sequences studied in this chapter are based on this idea.

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Example 1 1A

a Write down the first five terms of the sequence given by $T_n = 7n - 3$.
b Describe how each term T_n can be obtained from the previous term T_{n-1}.

SOLUTION

a $T_1 = 7 - 3 = 4$ $\quad T_2 = 14 - 3 = 11$ $\quad T_3 = 21 - 3 = 18$ $\quad T_4 = 28 - 3 = 25$ $\quad T_5 = 35 - 3 = 32$

b Each term is 7 more than the previous term. That is, $T_n = T_{n-1} + 7$.

Example 2 1A

a Find the first five terms of the sequence given by $T_1 = 14$ and $T_n = T_{n-1} + 10$.
b Write down a formula for the nth term T_n.

SOLUTION

a $T_1 = 14$ $\quad T_2 = T_1 + 10 = 24$ $\quad T_3 = T_2 + 10 = 34$ $\quad T_4 = T_3 + 10 = 44$ $\quad T_5 = T_4 + 10 = 54$

b From this pattern, the formula for the nth term is clearly $T_n = 10n + 4$.

1 THREE WAYS TO SPECIFY A SEQUENCE

- Write out the first few terms until the pattern is clear to the reader.
- Give a formula for the nth term T_n.
- Say where to start and how to proceed. That is:
  - Say what the value of T_1 is.
  - Then for $n \geq 2$, give a formula for T_n in terms of the preceding terms.

Using the formula for T_n to solve problems

Many problems about sequences can be solved by forming an equation using the formula for T_n.

Example 3 1A

Find whether 300 and 400 are terms of the sequence $T_n = 7n + 20$.

SOLUTION

Put $T_n = 300$.
Then $7n + 20 = 300$
$7n = 280$
$n = 40$.
Hence 300 is the 40th term.

Put $T_n = 400$.
Then $7n + 20 = 400$
$7n = 380$
$n = 54\frac{2}{7}$.
Hence 400 is not a term of the sequence.

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Example 4 1A

a Find how many negative terms there are in the sequence $T_n = 12n - 100$.
b Find the first positive term of the sequence $T_n = 7n - 60$.

SOLUTION

a Put $T_n < 0$.
Then $12n - 100 < 0$
$n < 8\frac{1}{3}$,
so there are eight negative terms.

b Put $T_n > 0$.
Then $7n - 60 > 0$
$7n > 60$
$n > 8\frac{4}{7}$.
Thus the first positive term is $T_9 = 3$.

Note: The question, 'Find the first positive term' requires two answers:

- Which number term is it?
- What is its value?

Thus the correct answer is, 'The first positive term is $T_9 = 3$'.

Exercise 1A FOUNDATION

1 Alex collects stamps. He found a collection of 700 stamps in the attic a few years ago, and every month since then he has been buying 150 interesting stamps to add to his collection. Thus the numbers of stamps at the end of each month after his discovery form a sequence

$850, \ 1000, \ \ldots$

a Copy and continue the sequence to at least 12 terms followed by dots . . . .
b After how many months did his collection first exceed 2000 stamps?

2 Write down the next four terms of each sequence.

a $5, 10, 15, \ldots$ **b** $6, 16, 26, \ldots$ **c** $2, 4, 8, \ldots$ **d** $3, 6, 12, \ldots$
e $38, 34, 30, \ldots$ **f** $39, 30, 21, \ldots$ **g** $24, 12, 6, \ldots$ **h** $81, 27, 9, \ldots$
i $-1, 1, -1, \ldots$ **j** $1, 4, 9, \ldots$ **k** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$ **l** $16, -8, 4, \ldots$

3 Find the first four terms of each sequence. You will need to substitute $n = 1$, $n = 2$, $n = 3$ and $n = 4$ into the formula for the nth term T_n.

a $T_n = 6n$ **b** $T_n = 5n - 2$ **c** $T_n = 2^n$ **d** $T_n = 5^n$
e $T_n = 20 - n$ **f** $T_n = 6 - 2n$ **g** $T_n = 3 \times 2^n$ **h** $T_n = 7 \times 10^n$
i $T_n = n^3$ **j** $T_n = n(n + 1)$ **k** $T_n = (-1)^n$ **l** $T_n = (-3)^n$

4 Write down the first four terms of each sequence described below.

a The first term is 6, and every term after that is 2 more than the previous term.
b The first term is 11, and every term after that is 50 more than the previous term.
c The first term is 15, and every term after that is 3 less than the previous term.
d The first term is 12, and every term after that is 8 less than the previous term.
e The first term is 5, and every term after that is twice the previous term.
f The first term is $\frac{1}{3}$, and every term after that is three times the previous term.
g The first term is 18, and every term after that is half the previous term.
h The first term is -100, and every term after that is one fifth of the previous term.

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5 Write out the first twelve terms of the sequence 7, 12, 17, 22, . . .

- **a** How many terms are less than 30?
- **b** How many terms are less than 60?
- **c** How many terms lie between 20 and 40?
- **d** How many terms lie between 10 and 50?
- **e** What is the 10th term?
- **f** What number term is 37?
- **g** Is 87 a term in the sequence?
- **h** Is 201 a term in the sequence?
- **i** Find the first term greater than 45.
- **j** Find the last term less than 43.

6 Write out the first twelve terms of the sequence $\frac{3}{4}$, $1\frac{1}{2}$, 3, 6, . . .

- **a** How many terms are less than 30?
- **b** How many terms are less than 400?
- **c** How many terms lie between 20 and 100?
- **d** How many terms lie between 1 and 1000?
- **e** What is the 10th term?
- **f** What number term is 192?
- **g** Is 96 a term in the sequence?
- **h** Is 100 a term in the sequence?
- **i** Find the first term greater than 200.
- **j** Find the last term less than 50.

DEVELOPMENT

7 For each sequence, write out the first five terms. Then explain how each term is obtained from the previous term.

- **a** $T_n = 12 + n$
- **b** $T_n = 4 + 5n$
- **c** $T_n = 15 - 5n$
- **d** $T_n = 3 \times 2^n$
- **e** $T_n = 7 \times (-1)^n$
- **f** $T_n = 80 \times \left(\frac{1}{2}\right)^n$

8 The nth term of a sequence is given by $T_n = 3n + 1$.

- **a** Put $T_n = 40$, and hence show that 40 is the 13th term of the sequence.
- **b** Put $T_n = 30$, and hence show that 30 is not a term of the sequence.
- **c** Similarly, find whether 100, 200 and 1000 are terms of the sequence.

9 Answer each question by forming an equation and solving it.

- **a** Find whether 16, 35 and 111 are terms of the sequence $T_n = 2n - 5$.
- **b** Find whether 44, 200 and 306 are terms of the sequence $T_n = 10n - 6$.
- **c** Find whether 40, 72 and 200 are terms of the sequence $T_n = 2n^2$.
- **d** Find whether 8, 96 and 128 are terms of the sequence $T_n = 2^n$.

10 The nth term of a sequence is given by $T_n = 10n + 4$.

- **a** Put $T_n < 100$, and hence show that the nine terms T_1 to T_9 are less than 100.
- **b** Put $T_n > 56$, and hence show that the first term greater than 56 is $T_6 = 64$.
- **c** Similarly, find how many terms are less than 500.
- **d** Find the first term greater than 203, giving its number and its value.

11 Answer each question by forming an inequation and solving it.

- **a** How many terms of the sequence $T_n = 2n - 5$ are less than 100?
- **b** How many terms of the sequence $T_n = 4n + 6$ are less than 300?
- **c** What is the first term of the sequence $T_n = 3n + 5$ greater than 127?
- **d** What is the first term of the sequence $T_n = 7n - 44$ greater than 100?

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12 In each part, the two lines define a sequence T_n. The first line gives the first term T_1. The second line defines how each subsequent term T_n is obtained from the previous term T_{n-1}. Write down the first four terms of each sequence.

a $T_1 = 3,$
$T_n = T_{n-1} + 2, \quad \text{for } n \ge 2.$

b $T_1 = 5,$
$T_n = T_{n-1} + 12, \quad \text{for } n \ge 2.$

c $T_1 = 6,$
$T_n = T_{n-1} - 3, \quad \text{for } n \ge 2.$

d $T_1 = 12,$
$T_n = T_{n-1} - 10, \quad \text{for } n \ge 2.$

e $T_1 = 5,$
$T_n = 2T_{n-1}, \quad \text{for } n \ge 2.$

f $T_1 = 4,$
$T_n = 5T_{n-1}, \quad \text{for } n \ge 2.$

g $T_1 = 20,$
$T_n = \frac{1}{2}T_{n-1}, \quad \text{for } n \ge 2.$

h $T_1 = 1,$
$T_n = -T_{n-1}, \quad \text{for } n \ge 2.$

CHALLENGE

13 Write down the first four terms of each sequence. Then state which terms of the whole sequence are zero.

a $T_n = \sin 90n°$ **b** $T_n = \cos 90n°$ **c** $T_n = \cos 180n°$ **d** $T_n = \sin 180n°$

14 The *Fibonacci sequence* is defined by

$$F_1 = 1, \qquad F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \ge 3.$$

Write out the first 12 terms of the sequence. Explain why every third term of the sequence is even and the rest are odd.

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1B Arithmetic sequences

A simple type of sequence is an *arithmetic sequence*. This is a sequence such as

$$3,\ 13,\ 23,\ 33,\ 43,\ 53,\ 63,\ 73,\ 83,\ 93,\ \ldots,$$

in which the difference between successive terms is constant — in this example each term is 10 more than the previous term. Notice that all the terms can be generated from the *first term* 3 by repeated addition of this *common difference* 10.

In the context of successive terms of sequences, the word *difference* will always mean some term minus the previous term.

Definition of an arithmetic sequence

Arithmetic sequences are called APs for short. The initials stand for 'arithmetic progression' — an old name for the same thing.

2 ARITHMETIC SEQUENCES

- The *difference* between successive terms in a sequence T_n always means some term minus the previous term, that is,

  $$\text{difference} = T_n - T_{n-1}, \quad \text{where } n \ge 2.$$

- A sequence T_n is called an *arithmetic sequence* or AP if

  $$T_n - T_{n-1} = d, \quad \text{for all } n \ge 2,$$

  where d is a constant, called the *common difference*.

- The terms of an arithmetic sequence can be generated from the first term by repeated addition of this common difference,

  $$T_n = T_{n-1} + d, \quad \text{for all } n \ge 2.$$

Example 5 1B

Test whether each sequence is an AP. If the sequence is an AP, find its first term a and its common difference d.

a $46,\ 43,\ 40,\ 37,\ \ldots$

b $1,\ 4,\ 9,\ 16,\ \ldots$

SOLUTION

a $T_2 - T_1 = 43 - 46 = -3$ $\quad T_3 - T_2 = 40 - 43 = -3$ $\quad T_4 - T_3 = 37 - 40 = -3$

Hence the sequence is an AP with $a = 46$ and $d = -3$.

b $T_2 - T_1 = 4 - 1 = 3$ $\quad T_3 - T_2 = 9 - 4 = 5$ $\quad T_4 - T_3 = 16 - 9 = 7$

The differences are not all the same, so the sequence is not an AP.

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A formula for the nth term of an AP

Let the first term of an AP be a and the common difference be d. Then the first few terms of the sequence are

$$T_1 = a, \qquad T_2 = a + d, \qquad T_3 = a + 2d, \qquad T_4 = a + 3d, \qquad \dots$$

From this pattern, the general formula for the nth term is clear:

3 THE nTH TERM OF AN AP

$$T_n = a + (n - 1)d$$

where a is the first term and d is the common difference.

Example 6 1B

Write out the first five terms, and calculate the 20th term, of the AP with:

a $a = 2$ and $d = 5$,

b $a = 20$ and $d = -3$.

SOLUTION

a $2,\ 7,\ 12,\ 17,\ 22,\ \dots$

$$\begin{aligned} T_{20} &= a + 19d \\ &= 2 + 19 \times 5 \\ &= 97 \end{aligned}$$

b $20,\ 17,\ 14,\ 11,\ 8,\ \dots$

$$\begin{aligned} T_{20} &= a + 19d \\ &= 20 + 19 \times (-3) \\ &= -37 \end{aligned}$$

Example 7 1B

a Find a formula for the nth term of the sequence 26, 35, 44, 53, ...

b How many terms are there in the sequence 26, 35, 44, 53, . . . , 917?

SOLUTION

a The sequence is an AP with $a = 26$ and $d = 9$.

$$\begin{aligned} \text{Hence } T_n &= a + (n - 1)d \\ &= 26 + 9(n - 1) \\ &= 26 + 9n - 9 \\ &= 17 + 9n. \end{aligned}$$

b
$$\begin{aligned} \text{Put} \qquad T_n &= 917. \\ \text{Then } 17 + 9n &= 917 \\ 9n &= 900 \\ n &= 100, \end{aligned}$$

so there are 100 terms in the sequence.

Solving problems involving APs

Now that we have the formula for the nth term T_n, many problems can be solved by forming an equation and solving it.

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Example 8 — 1B

a Show that the sequence 200, 193, 186, . . . is an AP.
b Find a formula for the nth term.
c Find the first negative term.

SOLUTION

a Because $T_2 - T_1 = -7$
and $T_3 - T_2 = -7$,
it is an AP with $a = 200$ and $d = -7$.

b Hence
$$\begin{aligned} T_n &= 200 - 7(n - 1) \\ &= 200 - 7n + 7 \\ &= 207 - 7n. \end{aligned}$$

c Put $T_n < 0$.
$$\begin{aligned} \text{Then } 207 - 7n &< 0 \\ 207 &< 7n \\ n &> 29\tfrac{4}{7}, \end{aligned}$$
so the first negative term is $T_{30} = -3$.

Example 9 — 1B

The first term of an AP is 105 and the 10th term is 6. Find the common difference and write out the first five terms.

SOLUTION

First, we know that $T_1 = 105$,
that is, $a = 105$. (1)
Secondly, we know that $T_{10} = 6$,
so using the formula for the 10th term, $a + 9d = 6$. (2)
Substituting (1) into (2),
$$\begin{aligned} 105 + 9d &= 6 \\ 9d &= -99 \\ d &= -11, \end{aligned}$$
so the common difference is $d = -11$ and the sequence is 105, 94, 83, 72, 61, . . .

Arithmetic sequences and linear functions

Take a linear function such as $f(x) = 30 - 8x$, and substitute the positive integers. The result is an arithmetic sequence

$$22,\ 14,\ 6,\ -2,\ -10,\ \ldots.$$

x	1	2	3	4	5
$f(x)$	22	14	6	-2	-10

The formula for the nth term of this AP is $T_n = 22 - 8(n - 1) = 30 - 8n$. This is a function whose domain is the set of positive integers, and its equation is the same as the linear function above, with only a change of pronumeral from x to n.

Every arithmetic sequence can be generated in this way.

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Exercise 1B

FOUNDATION

1 Write out the next three terms of these sequences. They are all APs.

a $2,\ 6,\ 10,\ \ldots$ b $3,\ 8,\ 13,\ \ldots$ c $35,\ 25,\ 15,\ \ldots$

d $11,\ 5,\ -1,\ \ldots$ e $4\frac{1}{2},\ 6,\ 7\frac{1}{2},\ \ldots$ f $8,\ 7\frac{1}{2},\ 7,\ \ldots$

2 Write out the first four terms of the APs whose first terms and common differences are:

a $a = 3$ and $d = 2$ b $a = 7$ and $d = 2$ c $a = 7$ and $d = -4$

d $a = 17$ and $d = 11$ e $a = 30$ and $d = -11$ f $a = -9$ and $d = 4$

g $a = 4\frac{1}{2}$ and $d = -\frac{1}{2}$ h $a = 3\frac{1}{2}$ and $d = -2$ i $a = 0.9$ and $d = 0.7$

3 Find the differences $T_2 - T_1$ and $T_3 - T_2$ for each sequence to test whether it is an AP. If the sequence is an AP, state the values of the first term a and the common difference d.

a $3,\ 7,\ 11,\ \ldots$ b $11,\ 7,\ 3,\ \ldots$ c $10,\ 17,\ 24,\ \ldots$

d $10,\ 20,\ 40,\ \ldots$ e $50,\ 35,\ 20,\ \ldots$ f $23,\ 34,\ 45,\ \ldots$

g $-12,\ -7,\ -2,\ \ldots$ h $-40,\ 20,\ -10,\ \ldots$ i $1,\ 11,\ 111,\ \ldots$

j $8,\ -2,\ -12,\ \ldots$ k $-17,\ 0,\ 17,\ \ldots$ l $10,\ 7\frac{1}{2},\ 5,\ \ldots$

4 Use the formula $T_n = a + (n - 1)d$ to find the 11th term T_{11} of the APs in which:

a $a = 7$ and $d = 6$ b $a = 15$ and $d = -7$ c $a = 10\frac{1}{2}$ and $d = 4$

5 Use the formula $T_n = a + (n - 1)d$ to find the eighth term T_8 of the APs in which:

a $a = 1$ and $d = 4$ b $a = 100$ and $d = -7$ c $a = -13$ and $d = 6$

6 a Find the first term a and the common difference d of the AP $6,\ 16,\ 26,\ \ldots$

b Find the ninth term T_9, the 21st term T_{21} and the 100th term T_{100}.

c Use the formula $T_n = a + (n - 1)d$ to find a formula for the nth term T_n.

7 a Find the first term a and the common difference d of the AP $-20,\ -9,\ 2,\ \ldots$

b Find the eighth term T_8, the 31st term T_{31} and the 200th term T_{200}.

c Use the formula $T_n = a + (n - 1)d$ to find a formula for the nth term T_n.

8 a Find the first term a and the common difference d of the AP $300,\ 260,\ 220,\ \ldots$

b Find the seventh term T_7, the 51st term T_{51} and the 1000th term T_{1000}.

c Use the formula $T_n = a + (n - 1)d$ to find a formula for the nth term T_n.

DEVELOPMENT

9 Find $T_3 - T_2$ and $T_2 - T_1$ to test whether each sequence is an AP. If the sequence is an AP, use the formula $T_n = a + (n - 1)d$ to find a formula for the nth term T_n.

a $8,\ 11,\ 14,\ \ldots$ b $21,\ 15,\ 9,\ \ldots$ c $8,\ 4,\ 2,\ \ldots$

d $-3,\ 1,\ 5,\ \ldots$ e $1\frac{3}{4},\ 3,\ 4\frac{1}{4},\ \ldots$ f $12,\ -5,\ -22,\ \ldots$

g $\sqrt{2},\ 2\sqrt{2},\ 3\sqrt{2},\ \ldots$ h $1,\ 4,\ 9,\ 16,\ \ldots$ i $-2\frac{1}{2},\ 1,\ 4\frac{1}{2},\ \ldots$

10 a Use the formula $T_n = a + (n - 1)d$ to find the nth term T_n of $165,\ 160,\ 155,\ \ldots$

b Solve $T_n = 40$ to find the number of terms in the finite sequence $165,\ 160,\ 155,\ \ldots,\ 40$.

c Solve $T_n < 0$ to find the first negative term of the sequence $165,\ 160,\ 155,\ \ldots$

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11 Use the formula $T_n = a + (n - 1)d$ to find the number of terms in each finite sequence.

- **a** $10,\ 12,\ 14,\ \dots,\ 30$
- **b** $1,\ 4,\ 7,\ \dots,\ 100$
- **c** $105,\ 100,\ 95,\ \dots,\ 30$
- **d** $100,\ 92,\ 84,\ \dots,\ 4$
- **e** $-12,\ -10\frac{1}{2},\ -9,\ \dots,\ 0$
- **f** $2,\ 5,\ 8,\ \dots,\ 2000$

12 Find T_n for each AP. Then solve $T_n < 0$ to find the first negative term.

- **a** $20,\ 17,\ 14,\ \dots$
- **b** $50,\ 45,\ 40,\ \dots$
- **c** $67,\ 60,\ 53,\ \dots$
- **d** $82,\ 79,\ 76,\ \dots$
- **e** $345,\ 337,\ 329,\ \dots$
- **f** $24\frac{1}{2},\ 24,\ 23\frac{1}{2},\ \dots$

13 The nth term of an arithmetic sequence is $T_n = 7 + 4n$.

- **a** Write out the first four terms, and hence find the values of a and d.
- **b** Find the sum and the difference of the 50th and the 25th terms.
- **c** Prove that $5T_1 + 4T_2 = T_{27}$.
- **d** Which term of the sequence is 815?
- **e** Find the last term less than 1000 and the first term greater than 1000.
- **f** Find which terms are between 200 and 300, and how many of them there are.

14
- **a** Let T_n be the sequence $8,\ 16,\ 24,\ \dots$ of positive multiples of 8.
  - **i** Show that the sequence is an AP, and find a formula for T_n.
  - **ii** Find the first term of the sequence greater than 500 and the last term less than 850.
  - **iii** Hence find the number of multiples of 8 between 500 and 850.
- **b** Use the same steps to find the number of multiples of 11 between 1000 and 2000.
- **c** Use the same steps to find the number of multiples of 7 between 800 and 2000.

15
- **a** The first term of an AP is $a = 7$ and the fourth term is $T_4 = 16$. Use the formula $T_n = a + (n - 1)d$ to find the common difference d. Then write down the first four terms.
- **b** The first term of an AP is $a = 100$ and the sixth term is $T_6 = 10$. Find the common difference d using the formula $T_n = a + (n - 1)d$. Then write down the first five terms.
- **c** Find the 20th term of an AP with first term 28 and 11th term 108.
- **d** Find the 100th term of an AP with first term 32 and 20th term -6.

16 Ionian Windows charges \$500 for the first window, then \$300 each additional window.

- **a** Write down the cost of 1 window, 2 windows, 3 windows, 4 windows, . . .
- **b** Show that this is an AP, and write down the first term a and common difference d.
- **c** Use the formula $T_n = a + (n - 1)d$ to find the cost of 15 windows.
- **d** Use the formula $T_n = a + (n - 1)d$ to find a formula for the cost of n windows.
- **e** Form an inequation and solve it to find the maximum number of windows whose total cost is less than \$10 000.

17 Many years ago, 160 km of a railway line from Nevermore to Gindarinda was built. On 1st January 2001, work was resumed, with 20 km of new track completed each month.

- **a** Write down the lengths of track 1 month later, 2 months later, 3 months later, ...
- **b** Show that this is an AP, and write down the first term a and common difference d.
- **c** Use the formula $T_n = a + (n - 1)d$ to find how much track there was after 12 months.
- **d** Use the formula $T_n = a + (n - 1)d$ to find a formula for the length after n months.
- **e** The distance from Nevermore to Gindarinda is 540 km. Form an equation and solve it to find how many months it took to complete the track.

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18 **a** Write down the first few terms of the AP generated by substituting the positive integers into the linear function $f(x) = 12 - 3x$. Then write down a formula for the nth term.

b **i** Find the formula of the nth term T_n of the AP $-3,\ -1,\ 1,\ 3,\ 5,\ \ldots$ Then write down the linear function $f(x)$ that generates this AP when the positive integers are substituted into it.

ii Graph the function and mark the points $(1, -3)$, $(2, -1)$, $(3, 1)$, $(4, 3)$, $(5, 5)$.

CHALLENGE

19 Find the common difference of each AP. Then find x if $T_{11} = 36$.

a $5x - 9,\ 5x - 5,\ 5x - 1,\ \ldots$

b $16,\ 16 + 6x,\ 16 + 12x,\ \ldots$

20 Find the common difference of each AP. Then find a formula for the nth term T_n.

a $\log_3 2,\ \log_3 4,\ \log_3 8,\ \ldots$

b $\log_a 54,\ \log_a 18,\ \log_a 6,\ \ldots$

c $x - 3y,\ 2x + y,\ 3x + 5y,\ \ldots$

d $5 - 6\sqrt{5},\ 1 + \sqrt{5},\ -3 + 8\sqrt{5},\ \ldots$

e $1.36,\ -0.52,\ -2.4,\ \ldots$

f $\log_a 3x^2,\ \log_a 3x,\ \log_a 3,\ \ldots$

21 **a** What are the first term and difference of the AP generated by substituting the positive integers into the linear function with gradient m and y-intercept b?

b What are the gradient and y-intercept of the linear function that generates an AP with first term a and difference d when the positive integers are substituted into it?

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1C Geometric sequences

A *geometric sequence* is a sequence like this:

$$2,\ 6,\ 18,\ 54,\ 162,\ 486,\ 1458,\ \dots$$

in which the *ratio* of successive terms is constant — in this example, each term is 3 times the previous term. Because the ratio is constant, all the terms can be generated from the *first term* 2 by repeated multiplication by this *common ratio* 3.

In the context of successive terms of sequences, the word *ratio* will always mean some term divided by the previous term.

Definition of a geometric sequence

The old name was 'geometric progression' and geometric sequences are called GPs for short.

4 GEOMETRIC SEQUENCES

- The *ratio* of successive terms in a sequence T_n always means some term divided by the previous term, that is,
$$\text{ratio} = \frac{T_n}{T_{n-1}},\ \text{where } n \ge 2.$$
- A sequence T_n is called a *geometric sequence* if
$$\frac{T_n}{T_{n-1}} = r,\ \text{for all } n \ge 2,$$
where r is a non-zero constant, called the *common ratio*.
- The terms of a geometric sequence can be generated from the first term by repeated multiplication by this common ratio,
$$T_n = T_{n-1} \times r,\ \text{for all } n \ge 2.$$

Thus arithmetic sequences have a common difference and geometric sequences have a common ratio, so the methods of dealing with them are quite similar.

Example 10 1C

Test whether each sequence is a GP. If the sequence is a GP, find its first term a and its ratio r.

a $40,\ 20,\ 10,\ 5,\ \dots$ **b** $5,\ 10,\ 100,\ 200,\ \dots$

SOLUTION

a Here $\dfrac{T_2}{T_1} = \dfrac{20}{40} = \tfrac{1}{2}$ and $\dfrac{T_3}{T_2} = \dfrac{10}{20} = \tfrac{1}{2}$ and $\dfrac{T_4}{T_3} = \dfrac{5}{10} = \tfrac{1}{2}$,

so the sequence is a GP with $a = 40$ and $r = \frac{1}{2}$.

b Here $\dfrac{T_2}{T_1} = \dfrac{10}{5} = 2$ and $\dfrac{T_3}{T_2} = \dfrac{100}{10} = 10$ and $\dfrac{T_4}{T_3} = \dfrac{200}{100} = 2$.

The ratios are not all the same, so the sequence is not a GP.

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A formula for the *n*th term of a GP

Let the first term of a GP be a and the common ratio be r. Then the first few terms of the sequence are

$$T_1 = a, \quad T_2 = ar, \quad T_3 = ar^2, \quad T_4 = ar^3, \quad T_5 = ar^4, \quad \dots$$

From this pattern, the general formula for the nth term is clear:

5 THE *n*TH TERM OF A GP

$$T_n = ar^{n-1}$$

where a is the first term and r is the common ratio.

Example 11 **1C**

Write out the first five terms, and calculate the 10th term, of the GP with:

a $a = 3$ and $r = 2$,

b $a = 7$ and $r = 10$.

SOLUTION

a 3, 6, 12, 24, 48, ...

$$\begin{aligned} T_{10} &= ar^9 \\ &= 3 \times 2^9 \\ &= 1536 \end{aligned}$$

b 7, 70, 700, 7000, 70 000, ...

$$\begin{aligned} T_{10} &= a \times r^9 \\ &= 7 \times 10^9 \\ &= 7\,000\,000\,000 \end{aligned}$$

Zeroes and GPs don't mix

No term of a GP can be zero. For example, if $T_2 = 0$, then $\dfrac{T_3}{T_2}$ would be undefined, contradicting the definition that $\dfrac{T_3}{T_2} = r$.

Similarly, the ratio of a GP cannot be zero. Otherwise $T_2 = ar$ would be zero, which is impossible, as we have explained above.

Negative ratios and alternating signs

The sequence

$$2, \ -6, \ 18, \ -54, \ \dots$$

is an important type of GP. Its ratio is $r = -3$, which is negative, so the terms are alternately positive and negative.

Example 12 **1C**

a Show that $2, \ -6, \ 18, \ -54, \ \dots$ is a GP and find its first term a and ratio r.

b Find a formula for the nth term, and hence find T_6 and T_{15}.

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SOLUTION

a Here $\dfrac{T_2}{T_1} = \dfrac{-6}{2}$ $= -3$ and $\dfrac{T_3}{T_2} = \dfrac{18}{-6}$ $= -3$ and $\dfrac{T_4}{T_3} = \dfrac{-54}{18}$ $= -3$,

so the sequence is a GP with $a = 2$ and $r = -3$.

b Using the formula for the nth term,

$$\begin{aligned} T_n &= ar^{n-1} \\ &= 2 \times (-3)^{n-1}. \end{aligned}$$

Hence $T_6 = 2 \times (-3)^5 = -486$ and $T_{15} = 2 \times (-3)^{14} = 2 \times 3^{14}$, because 14 is even.

Using a switch to alternate the sign

Here are two classic GPs with ratio -1:

$$-1,\ 1,\ -1,\ 1,\ -1,\ 1,\ \dots \qquad \text{and} \qquad 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ \dots$$

The first has formula $T_n = (-1)^n$, and the second has formula $T_n = (-1)^{n-1}$.

These sequences provide a way of writing any GP that alternates in sign using a *switch*. For example, the sequence $2,\ -6,\ 18,\ -54,\ \dots$ in the previous worked example has formula $T_n = 2 \times (-3)^{n-1}$, which can also be written as

$$T_n = 2 \times 3^{n-1} \times (-1)^{n-1}$$

to emphasise the alternating sign, and $-2,\ 6,\ -18,\ 54,\ \dots$ can be written as

$$T_n = 2 \times 3^{n-1} \times (-1)^n.$$

Solving problems involving GPs

As with APs, the formula for the nth term allows many problems to be solved by forming an equation and solving it.

Example 13 — 1C

a Find a formula for the nth term of the geometric sequence $5,\ 10,\ 20,\ \dots$.

b Hence find whether 320 and 720 are terms of this sequence.

SOLUTION

a The sequence is a GP with $a = 5$ and $r = 2$.

Hence
$$\begin{aligned} T_n &= ar^{n-1} \\ &= 5 \times 2^{n-1}. \end{aligned}$$

b Put $T_n = 320$.

Then
$$\begin{aligned} 5 \times 2^{n-1} &= 320 \\ 2^{n-1} &= 64 \\ n - 1 &= 6 \\ n &= 7, \end{aligned}$$

so 320 is the seventh term T_7.

c Put $T_n = 720$.

Then
$$\begin{aligned} 5 \times 2^{n-1} &= 720 \\ 2^{n-1} &= 144. \end{aligned}$$

But 144 is not a power of 2, so 720 is not a term of the sequence.

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Example 14 1C

The first term of a GP is 448 and the seventh term is 7. Find the common ratio and write out the first seven terms.

SOLUTION

First, we know that $T_1 = 448$
that is, $a = 448.$ (1)
Secondly, we know that $T_7 = 7$
so using the formula for the 7th term, $ar^6 = 7.$ (2)
substituting (1) into (2), $448r^6 = 7$

$$r^6 = \tfrac{1}{64}$$
$$r = \tfrac{1}{2} \text{ or } -\tfrac{1}{2}.$$

Thus either the ratio is $r = \frac{1}{2}$, and the sequence is

$$448,\ 224,\ 112,\ 56,\ 28,\ 14,\ 7,\ \ldots$$

or the ratio is $r = -\frac{1}{2}$, and the sequence is

$$448,\ -224,\ 112,\ -56,\ 28,\ -14, 7,\ \ldots$$

Geometric sequences and exponential functions

Take the exponential function $f(x) = 54 \times 3^{-x}$, and substitute the positive integers. The result is a geometric sequence

$$18,\ 6,\ 2,\ \tfrac{2}{3},\ \tfrac{2}{9},\ \ldots$$

x	1	2	3	4	5
$f(x)$	18	6	2	$\frac{2}{3}$	$\frac{2}{9}$

The formula for the nth term of this GP is $T_n = 18 \times \left(\frac{1}{3}\right)^{n-1} = 54 \times 3^{-n}$. This is a function whose domain is the set of positive integers, and its equation is the same as the exponential function, with only a change of pronumeral from x to n.

Thus the graph of an arithmetic sequence is the positive integer points on the graph of a linear function, and the graph of a geometric sequence is the positive integer points on the graph of an exponential function.

Exercise 1C

FOUNDATION

1 Write out the next three terms of each sequence. They are all GPs.

a $1,\ 2,\ 4,\ \ldots$
b $81,\ 27,\ 9,\ \ldots$
c $-7,\ -14,\ -28$
d $-2500,\ -500,\ -100,\ \ldots$
e $3,\ -6,\ 12,\ \ldots$
f $-25,\ 50,\ -100,\ \ldots$
g $5,\ -5,\ 5,\ \ldots$
h $-1000,\ 100,\ -10,\ \ldots$
i $0.04,\ 0.4,\ 4,\ \ldots$

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2 Write out the first four terms of the GPs whose first terms and common ratios are:

a $a = 1$ and $r = 3$
b $a = 12$ and $r = 2$
c $a = 5$ and $r = -2$
d $a = 18$ and $r = \frac{1}{3}$
e $a = 18$ and $r = -\frac{1}{3}$
f $a = 50$ and $r = \frac{1}{5}$
g $a = 6$ and $r = -\frac{1}{2}$
h $a = -13$ and $r = 2$
i $a = -7$ and $r = -1$

3 Find the ratios $\dfrac{T_3}{T_2}$ and $\dfrac{T_2}{T_1}$ for each sequence to test whether it is a GP. If the sequence is a GP, write down the first term a and the common ratio r.

a $4, 8, 16, \ldots$
b $16, 8, 4, \ldots$
c $7, 21, 63, \ldots$
d $-4, -20, -100, \ldots$
e $2, 4, 6, \ldots$
f $-1000, -100, -10, \ldots$
g $-80, 40, -20, \ldots$
h $29, 29, 29, \ldots$
i $1, 4, 9, \ldots$
j $-14, 14, -14, \ldots$
k $6, 1, \frac{1}{6}, \ldots$
l $-\frac{1}{3}, 1, -3, \ldots$

4 Use the formula $T_n = ar^{n-1}$ to find the fourth term of the GP with:

a $a = 5$ and $r = 2$
b $a = 300$ and $r = \frac{1}{10}$
c $a = -7$ and $r = 2$
d $a = -64$ and $r = \frac{1}{2}$
e $a = 11$ and $r = -2$
f $a = -15$ and $r = -2$

5 Use the formula $T_n = ar^{n-1}$ to find an expression for the 70th term of the GP with:

a $a = 1$ and $r = 3$
b $a = 5$ and $r = 7$
c $a = 8$ and $r = -3$

6 a Find the first term a and the common ratio r of the GP $7, 14, 28, \ldots$
b Find the sixth term T_6 and an expression for the 50th term T_{50}.
c Find a formula for the nth term T_n.

7 a Find the first term a and the common ratio r of the GP $10, -30, 90, \ldots$
b Find the sixth term T_6 and an expression for the 25th term T_{25}.
c Find a formula for the nth term T_n.

8 a Find the first term a and the common ratio r of the GP $-80, -40, -20, \ldots$
b Find the 10th term T_{10} and an expression for the 100th term T_{100}.
c Find a formula for the nth term T_n.

DEVELOPMENT

9 Find $\dfrac{T_3}{T_2}$ and $\dfrac{T_2}{T_1}$ to test whether each sequence is a GP. If the sequence is a GP, use the formula $T_n = ar^{n-1}$ to find a formula for the nth term, then find T_6.

a $10, 20, 40, \ldots$
b $180, 60, 20, \ldots$
c $64, 81, 100, \ldots$
d $35, 50, 65, \ldots$
e $\frac{3}{4}, 3, 12, \ldots$
f $-48, -24, -12, \ldots$

10 Find the common ratio of each GP, find a formula for T_n, and find T_6.

a $1, -1, 1, \ldots$
b $-2, 4, -8, \ldots$
c $-8, 24, -72, \ldots$
d $60, -30, 15, \ldots$
e $-1024, 512, -256, \ldots$
f $\frac{1}{16}, -\frac{3}{8}, \frac{9}{4}, \ldots$

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11 Use the formula $T_n = ar^{n-1}$ to find how many terms there are in each finite sequence.

a $1,\ 2,\ 4,\ \dots,\ 64$ **b** $-1,\ -3,\ -9,\ \dots,\ -81$ **c** $8,\ 40,\ 200,\ \dots,\ 125\,000$

d $7,\ 14,\ 28,\ \dots,\ 224$ **e** $2,\ 14,\ 98,\ \dots,\ 4802$ **f** $\frac{1}{25},\ \frac{1}{5},\ 1,\ \dots,\ 625$

12 a The first term of a GP is $a = 25$ and the fourth term is $T_4 = 200$. Use the formula $T_n = ar^{n-1}$ to find the common ratio r, then write down the first five terms.

b The first term of a GP is $a = 3$ and the sixth term is $T_6 = 96$. Use the formula $T_n = ar^{n-1}$ to find the common ratio r, then write down the first six terms.

c The first term of a GP is $a = 1$ and the fifth term is $T_5 = 81$. Use the formula $T_n = ar^{n-1}$ to find the common ratio r, then write down the first five terms.

13 Use the formula $T_n = ar^{n-1}$ to find the common ratio r of a GP for which:

a $a = 486$ and $T_5 = \frac{2}{27}$ **b** $a = 1000$ and $T_7 = 0.001$

c $a = 32$ and $T_6 = -243$ **d** $a = 5$ and $T_7 = 40$

14 The nth term of a geometric sequence is $T_n = 25 \times 2^n$.

a Write out the first six terms and hence find the values of a and r.

b Which term of the sequence is 6400?

c Find in factored form $T_{50} \times T_{25}$ and $T_{50} \div T_{25}$.

d Prove that $T_9 \times T_{11} = 25 \times T_{20}$.

e Write out the terms between 1000 and 100 000. How many of them are there?

f Verify by calculations that $T_{11} = 51\,200$ is the last term less than 100 000 and that $T_{12} = 102\,400$ is the first term greater than 100 000.

15 A piece of paper 0.1 mm thick is folded successively 100 times. How thick is it now?

16 a Write down the first few terms of the GP generated by substituting the positive integers into the exponential function $f(x) = \frac{4}{25} \times 5^x$. Then write down a formula for the nth term.

b i Find the formula of the nth term T_n of the GP $5,\ 10,\ 20,\ 40,\ 80\ \dots$ Then write down the exponential function $f(x)$ that generates this GP when the positive integers are substituted into it.

ii Graph the function (without the same scale on both axes) and mark the points $(1, 5)$, $(2, 10)$, $(3, 20)$, $(4, 40)$, $(5, 80)$.

CHALLENGE

17 Find the nth term of each GP.

a $\sqrt{6},\ 2\sqrt{3},\ 2\sqrt{6},\ \dots$ **b** $ax,\ a^2x^3,\ a^3x^5,\ \dots$ **c** $-\frac{x}{y}, -1, -\frac{y}{x}, \dots$

18 a Find a formula for T_n in $2x,\ 2x^2,\ 2x^3,\ \dots$ Then find x if $T_6 = 2$.

b Find a formula for T_n in $x^4,\ x^2,\ 1,\ \dots$ Then find x if $T_6 = 3^6$.

c Find a formula for T_n in $2^{-16}x,\ 2^{-12}x,\ 2^{-8}x,\ \dots$ Then find x if $T_6 = 96$.

19 a What are the first term and common ratio of the GP generated by substituting the positive integers into the exponential function $f(x) = cb^x$?

b What is the equation of the exponential function that generates a GP with first term a and ratio r when the positive integers are substituted into it?

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1D Solving problems involving APs and GPs

This section deals with APs and GPs together and presents some further approaches to problems about the terms of APs and GPs.

A condition for three numbers to be in AP or GP

The three numbers $10,\ 25,\ 40$ form an AP because the differences $25 - 10 = 15$ and $40 - 25 = 15$ are equal.

Similarly, $10,\ 20,\ 40$ form a GP because the ratios $\frac{20}{10} = 2$ and $\frac{40}{20} = 2$ are equal.

These situations occur quite often and a formal statement is worthwhile:

6 THREE NUMBERS IN AP OR GP

- Three numbers a, b and c form an AP if
  $$b - a = c - b$$
- Three numbers a, b and c form a GP if
  $$\frac{b}{a} = \frac{c}{b}$$

Example 15 1D

a Find the value of x if $3,\ x,\ 12$ form an AP.

b Find the value of x if $3,\ x,\ 12$ form a GP.

SOLUTION

a Because $3,\ x,\ 12$ form an AP,

$$\begin{aligned} x - 3 &= 12 - x \\ 2x &= 15 \\ x &= 7\tfrac{1}{2}. \end{aligned}$$

b Because $3,\ x,\ 12$ form a GP,

$$\begin{aligned} \frac{x}{3} &= \frac{12}{x} \\ x^2 &= 36 \\ x &= 6 \text{ or } -6. \end{aligned}$$

Solving problems leading to simultaneous equations

Many problems about APs and GPs lead to simultaneous equations. These are best solved by elimination.

7 PROBLEMS ON APS AND GPS LEADING TO SIMULTANEOUS EQUATIONS

- With APs, eliminate a by subtracting one equation from the other.
- With GPs, eliminate a by dividing one equation by the other.

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Example 16 1D

The third term of an AP is 16 and the 12th term is 79. Find the 41st term.

SOLUTION

Let the first term be a and the common difference be d.

Because $T_3 = 16$, $\quad a + 2d = 16, \qquad (1)$

and because $T_{12} = 79$, $\quad a + 11d = 79. \qquad (2)$

Subtracting (1) from (2), $\quad 9d = 63$ (the key step that eliminates a)

$d = 7.$

Subtracting into (1), $\quad a + 14 = 16$

$a = 2.$

Hence $\quad T_{41} = a + 40d$

$= 282.$

Example 17 1D

Find the first term a and the common ratio r of a GP in which the fourth term is 6 and the seventh term is 162.

SOLUTION

Because $T_4 = 6$, $\quad ar^3 = 6, \qquad (1)$

and because $T_7 = 162$, $\quad ar^6 = 162. \qquad (2)$

Dividing (2) by (1), $\quad r^3 = 27$ (the key step that eliminates a)

$r = 3.$

Substituting into (1), $\quad a \times 27 = 6$

$a = \frac{2}{9}.$

Solving GP problems involving trial-and-error or logarithms

Equations and inequations involving the terms of a GP are index equations, so logarithms are needed for a systematic approach.

Trial-and-error, however, is quite satisfactory for simpler problems, and the reader may prefer to leave the application of logarithms until Chapter 8.

Example 18 1D

a Find a formula for the nth term of the geometric sequence 2, 6, 18, ...

b Use trial-and-error to find the first term greater than 1 000 000.

c Use logarithms to find the first term greater than 1 000 000.

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SOLUTION

a This is a GP with $a = 2$ and $r = 3$,
so $T_n = ar^{n-1}$
$= 2 \times 3^{n-1}$.

b Put $T_n > 1000000$.
Using the calculator, $T_{12} = 354294$
and $T_{13} = 1062882$.
Hence the first term over 1000000 is $T_{13} = 1062882$.

c Put $T_n > 1000000$.

$$\begin{aligned}
\text{Then } 2 \times 3^{n-1} &> 1000000 \\
3^{n-1} &= 500000 \\
n - 1 &> \log_3 500000 \quad \text{(remembering that } 2^3 = \text{ means } 3 = \log_2 8) \\
n - 1 &> \frac{\log_{10} 500000}{\log_{10} 3} \quad \text{(the change-of-base formula)} \\
n - 1 &> 11.94\ldots \\
n &> 12.94\ldots
\end{aligned}$$

Hence the first term over 1000000 is $T_{13} = 1062882$.

Exercise 1D

FOUNDATION

1 Find the value of x if each set of numbers below forms an arithmetic sequence.
(Hint: Form an equation using the identity $T_2 - T_1 = T_3 - T_2$, then solve it to find x.)

a $5,\ x,\ 17$ **b** $32,\ x,\ 14$ **c** $-12,\ x,\ -50$
d $-23,\ x,\ 7$ **e** $x,\ 22,\ 32$ **f** $-20,\ -5,\ x$

2 Each triple of number forms a geometric sequence. Find the value of x. (Hint: Form an equation using the identity $\dfrac{T_2}{T_1} = \dfrac{T_3}{T_2}$, then solve it to find x.)

a $2,\ x,\ 18$ **b** $48,\ x,\ 3$ **c** $-10,\ x,\ -90$
d $-98,\ x,\ -2$ **e** $x,\ 20,\ 80$ **f** $-1,\ 4,\ x$

3 Find x if each triple of three numbers forms: **i** an AP, **ii** a GP.

a $4,\ x,\ 16$ **b** $1,\ x,\ 49$ **c** $16,\ x,\ 25$ **d** $-5,\ x,\ -20$
e $x,\ 10,\ 50$ **f** $x,\ 12,\ 24$ **g** $x,\ -1,\ 1$ **h** $x,\ 6,\ -12$
i $20,\ 30,\ x$ **j** $-36,\ 24,\ x$ **k** $-\frac{1}{4},\ -3,\ x$ **l** $7,\ -7,\ x$

DEVELOPMENT

4 In these questions, substitute the last term into $T_n = a + (n - 1)d$ or $T_n = ar^{n-1}$.

a Find the first six terms of the AP with first term $a = 7$ and sixth term $T_6 = 42$.
b Find the first four terms of the GP with first term $a = 27$ and fourth term $T_4 = 8$.
c Find the first eleven terms of the AP with $a = 40$ and $T_{11} = 5$.
d Find the first seven terms of the GP with $a = 1$ and $T_7 = 1000000$.
e Find the first five terms of the AP with $a = 3$ and $T_5 = 48$.
f Find the first five terms of the GP with $a = 3$ and $T_5 = 48$.

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5 Use simultaneous equations and the formula $T_n = a + (n - 1)d$ to solve these problems.
  a Find the first term and common difference of the AP with $T_{10} = 18$ and $T_{20} = 48$.
  b Find the first term and common difference of the AP with $T_2 = 3$ and $T_{10} = 35$.
  c Find the first term and common difference of the AP with $T_5 = 24$ and $T_9 = -12$.
  d Find the first term and common difference of the AP with $T_4 = 6$ and $T_{12} = 34$.

6 Use simultaneous equations and the formula $T_n = ar^{n-1}$ to solve these problems.
  a Find the first term and common ratio of the GP with $T_3 = 16$ and $T_6 = 128$.
  b Find the first term and common ratio of the GP with $T_3 = 1$ and $T_6 = 64$.
  c Find the first term and common ratio of the GP with $T_2 = \frac{1}{3}$ and $T_6 = 27$.
  d Find the first term and common ratio of the GP with $T_5 = 6$ and $T_9 = 24$.

7 a The third term of an AP is 7 and the seventh term is 31. Find the eighth term.
  b The common difference of an AP is -7 and the 10th term is 3. Find the second term.
  c The common ratio of a GP is 2 and the sixth term is 6. Find the second term.

8 Use either trial-and-error or logarithms to solve these problems.
  a Find the smallest value of n such that $3^n > 1\,000\,000$.
  b Find the largest value of n such that $5^n < 1\,000\,000$.
  c Find the smallest value of n such that $7^n > 1\,000\,000\,000$.
  d Find the largest value of n such that $12^n < 1\,000\,000\,000$.

9 Let T_n be the sequence $2,\ 4,\ 8,\ \dots$ of powers of 2.
  a Show that the sequence is a GP, and show that the nth term is $T_n = 2^n$.
  b Find how many terms are less than 1 000 000. (You will need to solve the inequation $T_n < 1\,000\,000$ using trial-and-error or logarithms.)
  c Use the same method to find how many terms are less than 1 000 000 000.
  d Use the same method to find how many terms are less than 10^{20}.
  e How many terms are between 1 000 000 and 1 000 000 000?
  f How many terms are between 1 000 000 000 and 10^{20}?

10 Find a formula for T_n for these GPs. Then find how many terms exceed 10^{-6}. (You will need to solve the inequation $T_n > 10^{-6}$ using trial-and-error or logarithms.)
  a $98,\ 14,\ 2,\ \dots$ b $25,\ 5,\ 1,\ \dots$ c $1,\ 0.9,\ 0.81,\ \dots$

11 When light passes through one sheet of very thin glass, its intensity is reduced by 3%. (Hint: 97% of the light gets though each sheet.)
  a If the light passes through 50 sheets of this glass, find by what percentage (correct to the nearest 1%) the intensity will be reduced.
  b What is the minimum number of sheets that will reduce the intensity below 1%?

12 a Find a and d for the AP in which $T_6 + T_8 = 44$ and $T_{10} + T_{13} = 35$.
  b Find a and r for the GP in which $T_2 + T_3 = 4$ and $T_4 + T_5 = 36$.
  c The fourth, sixth and eighth terms of an AP add to -6. Find the sixth term.

13 Each set of three numbers forms an AP. Find x and write out the numbers.
  a $x - 1,\ 17,\ x + 15$ b $2x + 2,\ x - 4,\ 5x$
  c $x - 3,\ 5,\ 2x + 7$ d $3x - 2,\ x,\ x + 10$

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14 Each set of three numbers forms a GP. Find x and write out the numbers.

a $x,\ x+1,\ x$ **b** $2-x,\ 2,\ 5-x$

15 Find x and write out the three numbers if they form: **i** an AP, **ii** a GP.

a $x,\ 24,\ 96$ **b** $0.2,\ x,\ 0.00002.$ **c** $x,\ 0.2,\ 0.002.$

d $x-4,\ x+1,\ x+11$ **e** $x-2,\ x+2,\ 5x-2$ **f** $\sqrt{5}+1,\ x,\ \sqrt{5}-1$

g $\sqrt{2},\ x,\ \sqrt{8}$ **h** $2^4,\ x,\ 2^6$ **i** $7,\ x,\ -7$

16 [The relationship between APs and GPs]

a The AP $1,\ 3,\ 5,\ 7,\ 9,\ \ldots$ has first term 1 and difference 2. Show that the sequence

$$2^1,\ 2^3,\ 2^5,\ 2^7,\ 2^9,\ \ldots$$

is a GP, and find its first term and common ratio.

b The GP $3,\ 6,\ 12,\ 24,\ 48\ \ldots$ has first term 3 and ratio 2. Show that the sequence

$$\log_2 3,\ \log_2 6,\ \log_2 12,\ \log_2 24,\ \log_2 48$$

is an AP, and find its first term and common difference.

CHALLENGE

17 **a** Find a and b if $a,\ b,\ 1$ forms a GP, and $b,\ a,\ 10$ forms an AP.

b Find a and b if $a,\ 1,\ a+b$ forms a GP, and $b,\ \frac{1}{2},\ a-b$ forms an AP.

c The first, second and fourth terms of an AP $T_n = a + (n-1)d$ form a geometric sequence. Show that either $d = 0$ or $d = a$.

d The first, second and fifth terms of an AP $T_n = a + (n-1)d$ form a geometric sequence. Show that either $d = 0$ or $d = 2a$.

18 [The relationship between APs and GPs]

a Show that $2^5,\ 2^2,\ 2^{-1},\ 2^{-4},\ \ldots$ is a GP. Then find its nth term.

b Show that $\log_2 96,\ \log_2 24,\ \log_2 6, \ldots$ is an AP. Then show that $T_n = 7 - 2n + \log_2 3$.

c Show that for any positive base $b \neq 1$, if T_n is an AP with first term a and difference d, then b^{T_n} is a GP with first term b^a and ratio b^d.

d Show that for any positive base $b \neq 1$, if T_n is a GP with first term $a > 0$ and ratio $r > 0$, then $\log_b T_n$ is an AP with first term $\log_b a$ and difference $\log_b r$.

19 [Extension — geometric sequences in musical instruments] The pipe lengths in a rank of organ pipes decrease from left to right. The lengths form a GP, and the 13th pipe along is exactly half the length of the first pipe (making an interval called an *octave*).

a Show that the ratio of the GP is $r = \left(\frac{1}{2}\right)^{\frac{1}{12}}$.

b Show that the eighth pipe along is just over two-thirds the length of the first pipe (this interval is called a *perfect fifth*).

c Show that the fifth pipe along is just under four-fifths the length of the first pipe (a *major third*).

d Find which pipes are about three-quarters (a *perfect fourth*) and five-sixths (a *minor third*) the length of the first pipe.

e What simple fractions are closest to the relative lengths of the third pipe (a *major second*) and the second pipe (a *minor second*)?

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1E Adding up the terms of a sequence

Adding the terms of a sequence is often important. For example, a boulder falling from the top of a high cliff falls 5 metres in the first second, 15 metres in the second second, 25 metres in the third second, and so on. The distance that it falls in the first 10 seconds is the sum of the 10 numbers

$$5 + 15 + 25 + 35 + \cdots + 95 = 500.$$

A notation for the sums of terms of a sequence

For any sequence $T_1, T_2, T_3, \ldots$, define S_n to be the sum of the first n terms of the sequence.

8 THE SUM OF THE FIRST n TERMS OF A SEQUENCE

Given a sequence $T_1, T_2, T_3, \ldots$, define

$$S_n = T_1 + T_2 + T_3 + \cdots + T_n.$$

The sum S_n is variously called:

- the *sum of the first n terms* of the sequence,
- the *sum to n terms* of the sequence,
- the *nth partial sum* of the sequence ('partial' meaning 'part of the sequence').

For example, the sum of the first 10 terms of the sequence $5, 15, 25, 35, \ldots$ is

$$\begin{aligned} S_{10} &= 5 + 15 + 25 + 35 + 45 + 55 + 65 + 75 + 85 + 95 \\ &= 500, \end{aligned}$$

which is also called the 10th partial sum of the sequence.

The sequence $S_1, S_2, S_3, S_4, \ldots$ of sums

The partial sums $S_1, S_2, S_3, S_4, \ldots$ form another sequence. For example, with the sequence $5, 15, 25, 35, \ldots$,

$$S_1 = 5 \qquad \begin{aligned} S_2 &= 5 + 15 \\ &= 20 \end{aligned} \qquad \begin{aligned} S_3 &= 5 + 15 + 25 \\ &= 45 \end{aligned} \qquad \begin{aligned} S_4 &= 5 + 15 + 25 + 35 \\ &= 80 \end{aligned}$$

Example 19 1E

Copy and complete this table for the successive sums of a sequence.

n	1	2	3	4	5	6	7	8	9	10
T_n	5	15	25	35	45	55	65	75	85	95
S_n										

SOLUTION

Each entry for S_n is the sum of all the terms T_n up to that point.

n	1	2	3	4	5	6	7	8	9	10
T_n	5	15	25	35	45	55	65	75	85	95
S_n	5	20	45	80	125	180	245	320	405	500

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Recovering the sequence from the partial sums

Suppose we know that the partial sums S_n of a sequence are the successive squares,

S_n: 1, 4, 9, 16, 25, 36, 49, 64, ...

and we want to recover the terms T_n. The first term is $T_1 = S_1 = 1$, and then we can take successive differences, giving the sequence

T_n: 1, 3, 5, 7, 9, 11, 13, 15, ...

9 RECOVERING THE TERMS FROM THE PARTIAL SUMS

The original sequence T_n can be recovered from the sequence S_n of partial sums by taking successive differences,

$$T_1 = S_1$$
$$T_n = S_n - S_{n-1}, \text{ for } n \geq 2.$$

Example 20 — 1E

By taking successive differences, list the terms of the original sequence.

n	1	2	3	4	5	6	7	8	9	10
T_n										
S_n	1	5	12	22	35	51	70	92	117	145

SOLUTION

Each entry for T_n is the difference between two successive sums S_n.

n	1	2	3	4	5	6	7	8	9	10
T_n	1	4	7	10	13	16	19	22	25	28
S_n	1	5	12	22	35	51	70	92	117	145

Example 21 — 1E

Confirm the example given above by proving algebraically that if the partial sums S_n of a sequence are the successive squares, then the sequence T_n is the sequence of odd numbers.

SOLUTION

We are given that $S_n = n^2$.

Hence $T_1 = S_1$

$= 1$, which is the first odd number,

and for $n \geq 2$, $T_n = S_n - S_{n-1}$

$= n^2 - (n-1)^2$

$= 2n - 1$, which is the nth odd number.

Note: Taking successive differences in a sequence is analogous to differentiation in calculus, and the results have many similarities to differentiation. For example, in the worked example above, taking finite differences of a quadratic function yields a linear function. The last question in the Challenge has further analogies, which are not pursued in this course.

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Sigma notation

This is a concise notation for the sums of a sequence. For example:

$$\begin{aligned}\sum_{n=2}^{5} n^2 &= 2^2 + 3^2 + 4^2 + 5^2 \\ &= 4 + 9 + 16 + 25 \\ &= 54\end{aligned}$$

$$\begin{aligned}\sum_{n=6}^{10} n^2 &= 6^2 + 7^2 + 8^2 + 9^2 + 10^2 \\ &= 36 + 49 + 64 + 81 + 100 \\ &= 330\end{aligned}$$

The first sum says 'evaluate the function n^2 for all the integers from $n = 2$ to $n = 5$, then add up the resulting values'. There are 4 terms, and their sum is 54.

10 SIGMA NOTATION

Suppose that $T_1,\ T_2,\ T_3,\ \dots$ is a sequence. Then

$$\sum_{n=5}^{20} T_n = T_5 + T_6 + T_7 + T_8 + \cdots + T_{20}$$

(Any two integers can obviously be substituted for the numbers 5 and 20.)

We used the symbol $\sum$ before in Chapter 11 of the Year 11 book. It stands for the word 'sum', and is a large version of the Greek capital letter Σ called 'sigma' and pronounced 's'. The superscripts and subscripts on the sigma sign, however, are used for the first time in this chapter.

Example 22 1E

Evaluate these sums.

a $\displaystyle\sum_{n=4}^{7} (5n + 1)$

b $\displaystyle\sum_{n=1}^{5} (-2)^n$

SOLUTION

a $$\begin{aligned}\sum_{n=4}^{7} (5n + 1) &= 21 + 26 + 31 + 36 \\ &= 114\end{aligned}$$

b $$\begin{aligned}\sum_{n=1}^{5} (-2)^n &= -2 + 4 - 8 + 16 - 32 \\ &= -22\end{aligned}$$

Series

The word *series* is often used imprecisely, but it always refers to the activity of adding up terms of a sequence. For example, the phrase

'the series $1 + 4 + 9 + \cdots$'

means that one is considering the successive partial sums $S_1 = 1,\ S_2 = 1 + 4,\ S_3 = 1 + 4 + 9,\ \dots$ of the sequence of positive squares.

The precise definition is that a *series* is the sequence of partial sums of a sequence. That is, given a sequence $T_1,\ T_2,\ T_3,\ \dots$, the corresponding series is the sequence

$$S_1 = T_1,\ S_2 = T_1 + T_2,\ S_3 = T_1 + T_2 + T_3,\ S_4 = T_1 + T_2 + T_3 + T_4,\ \dots$$

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Exercise 1E

FOUNDATION

1 Find the sum S_4 of the first four terms of each sequence.

a $3,\ 5,\ 7,\ 9,\ 11,\ 13,\ \ldots$

b $2,\ 6,\ 18,\ 54,\ 162,\ 486,\ \ldots$

c $6,\ 2,\ -2,\ -6,\ -10,\ -14,\ \ldots$

d $2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \ldots$

2 Find the sum S_3 of the first three terms of each series.

a $200 + 150 + 100 + 50 + 0 + \cdots$

b $32 - 16 + 8 - 4 + 2 - 1 + \cdots$

c $-24 - 18 - 12 - 6 + 0 + 6 + \cdots$

d $5.1 + 5.2 + 5.3 + 5.4 + 5.5 + 5.6 + \cdots$

3 Find the sums S_1, S_2, S_3, S_4 and S_5 for each sequence.

a $10,\ 20,\ 30,\ 40,\ 50,\ 60,\ \ldots$

b $1,\ -3,\ 9,\ -27,\ 81,\ -243,\ \ldots$

c $1,\ 4,\ 9,\ 16,\ 25,\ 36,\ \ldots$

d $3,\ 4\frac{1}{2},\ 6,\ 7\frac{1}{2},\ 9,\ 10\frac{1}{2},\ 12,\ \ldots$

4 Find the sums S_4, S_5 and S_6 for each series. (You will need to continue each series first.)

a $1 - 2 + 3 - 4 + \cdots$

b $81 + 27 + 9 + 3 + \cdots$

c $30 + 20 + 10 + \cdots$

d $0.1 + 0.01 + 0.001 + 0.0001 + \cdots$

5 Copy and complete these tables of a sequence and its partial sums.

T_n	2	5	8	11	14	17	20
S_n							

T_n	2	−4	6	−8	10	−12	14
S_n							

T_n	40	38	36	34	32	30	28
S_n							

T_n	7	−7	7	−7	7	−7	7
S_n							

DEVELOPMENT

6 Each table below gives the successive sums S_1, S_2, S_3, ... of a sequence. By taking successive differences, write out the terms of the original sequence.

T_n							
S_n	1	4	9	16	25	36	49

T_n							
S_n	−3	−8	−15	−24	−35	−48	−63

T_n							
S_n	2	6	14	30	62	126	254

T_n							
S_n	8	0	8	0	8	0	8

7 [The Fibonacci and Lucas sequences] Each table below gives the successive sums S_n of a sequence. By taking successive differences, write out the terms of the original sequence.

T_n								
S_n	1	2	3	5	8	13	21	34

T_n								
S_n	3	4	7	11	18	29	47	76

8 Rewrite each partial sum without sigma notation, then evaluate it.

a $\displaystyle\sum_{n=1}^{6} 2n$

b $\displaystyle\sum_{n=1}^{6} (3n + 2)$

c $\displaystyle\sum_{k=3}^{7} (18 - 3n)$

d $\displaystyle\sum_{n=5}^{8} n^2$

e $\displaystyle\sum_{n=1}^{4} n^3$

f $\displaystyle\sum_{n=0}^{5} 2^n$

g $\displaystyle\sum_{n=2}^{4} 3^n$

h $\displaystyle\sum_{\ell=1}^{31} (-1)^{\ell}$

i $\displaystyle\sum_{\ell=1}^{40} (-1)^{\ell-1}$

j $\displaystyle\sum_{n=5}^{105} 4$

k $\displaystyle\sum_{n=0}^{4} (-1)^n(n + 5)$

l $\displaystyle\sum_{n=0}^{4} (-1)^{n+1}(n + 5)$

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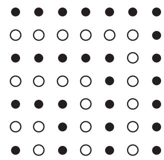
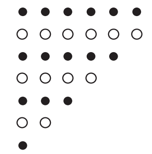
9 **a** Use the dot diagram on the right to explain why the sum of the first n odd positive integers is n^2.



b Use the dot diagram on the right to explain why the sum of the first n positive integers is $\frac{1}{2}n(n + 1)$.



CHALLENGE

10 Rewrite each sum in sigma notation, starting each sum at $n = 1$. Do not evaluate it.

a $1^3 + 2^3 + 3^3 + \cdots + 40^3$

b $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{40}$

c $3 + 4 + 5 + \cdots + 22$

d $2 + 2^2 + 2^3 + \cdots + 2^{12}$

e $-1 + 2 - 3 + \cdots + 10$

f $1 - 2 + 3 - \cdots - 10$

11 **a** The partial sums of a sequence T_n are given by $S_n = 2^n$. Use the formula in Box 9 to find a formula for T_n.

b Confirm your answer by writing out the calculation in table form, as in Question 6.

c In Chapter 9 of the Year 11 book, you differentiated $y = e^x$. What is the analogy to these results?

12 **a** Prove that $n^3 - (n - 1)^3 = 3n^2 - 3n + 1$.

b The partial sums of a sequence T_n are given by $S_n = n^3$. Use the formula in Box 9 to find a formula for T_n.

c The terms of the sequence T_n are the partial sums of a third sequence U_n. Use the formula in Box 9 to find a formula for T_n.

d Confirm your answer by writing out in table form the successive taking of differences in **b** and **c**.

e In Chapter 8 of the Year 11 volume, you differentiated powers of x. What is the analogy to these result?

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1F Summing an arithmetic series

There are two formulae for adding up the first n terms of an AP.

Adding the terms of an AP

Consider adding the first six terms of the AP

$5 + 15 + 25 + 35 + 45 + 55 + \cdots$

$$\begin{aligned}
&\text{Writing out the sum,} & S_6 &= 5 + 15 + 25 + 35 + 45 + 55.\\
&\text{Reversing the sum,} & S_6 &= 55 + 45 + 35 + 25 + 15 + 5,\\
&\text{and adding the two,} & 2S_6 &= 60 + 60 + 60 + 60 + 60 + 60\\
& & &= 6 \times 60, \text{ because there are 6 terms in the series.}\\
&\text{Dividing by 2,} & S_6 &= \tfrac{1}{2} \times 6 \times 60\\
& & &= 180.
\end{aligned}$$

Notice that 60 is the sum of the first term $T_1 = 5$ and the last term $T_6 = 55$.

In general, let $\ell = T_n$ be the last term of an AP with first term a and difference d.

$$\begin{aligned}
&\text{Then} & S_n &= a + (a + d) + (a + 2d) + \cdots + (\ell - 2d) + (\ell - d) + \ell.\\
&\text{Reversing the sum,} & S_n &= \ell + (\ell - d) + (\ell - 2d) + \cdots + (a + 2d) + (a + d) + a,\\
&\text{and adding,} & 2S_n &= (a + \ell) + (a + \ell) + \cdots + (a + \ell) + (a + \ell) + (a + \ell)\\
& & &= n \times (a + \ell), \text{ because there are } n \text{ terms in the series.}\\
&\text{Dividing by 2,} & S_n &= \tfrac{1}{2}n(a + \ell).
\end{aligned}$$

Example 23 1F

Add up all the integers from 100 to 200 inclusive.

SOLUTION

The sum $100 + 101 + \cdots + 200$ is an AP with 101 terms.

The first term is $a = 100$ and the last term is $\ell = 200$.

$$\begin{aligned}
\text{Using}\quad S_n &= \tfrac{1}{2}n(a + \ell),\\
S_{101} &= \tfrac{1}{2} \times 101 \times (100 + 200)\\
&= \tfrac{1}{2} \times 101 \times 300\\
&= 15\,150.
\end{aligned}$$

An alternative formula for summing an AP

This alternative form is equally important.

$$\begin{aligned}
&\text{The previous formula is} & S_n &= \tfrac{1}{2}n(a + \ell), \text{ where } \ell = T_n = a + (n - 1)d.\\
&\text{Substituting } \ell = a + (n - 1)d, & S_n &= \tfrac{1}{2}n\big(a + a(n - 1)d\big)\\
&\text{so} & S_n &= \tfrac{1}{2}n\big(2a + (n - 1)d\big).
\end{aligned}$$

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11 TWO FORMULAE FOR SUMMING AN AP

Suppose that the first term a of an AP, and the number n of terms, are known.

- When the last term $\ell = T_n$ is known, use $S_n = \frac{1}{2}n(a + \ell)$.
- When the difference d is known, use $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$.

If you have a choice, use the first because it is simpler.

Example 24 1F

Consider the arithmetic series $100 + 94 + 88 + 82 + \cdots$.

a Find S_{10}.

b Find S_{41}.

SOLUTION

The series is an AP with $a = 100$ and $d = -6$.

a Using $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$,

$$\begin{aligned} S_{10} &= \tfrac{1}{2} \times 10 \times (2a + 9d) \\ &= 5 \times (200 - 54) \\ &= 730. \end{aligned}$$

b Similarly,

$$\begin{aligned} S_{41} &= \tfrac{1}{2} \times 41 \times (2a + 40d) \\ &= \tfrac{1}{2} \times 41 \times (200 - 240) \\ &= \tfrac{1}{2} \times 41 \times (-40) \\ &= -820. \end{aligned}$$

Example 25 1F

a Find how many terms are in the sum $41 + 45 + 49 + \cdots + 401$.

b Hence evaluate the sum $41 + 45 + 49 + \cdots + 401$.

SOLUTION

a The series is an AP with first term $a = 41$ and difference $d = 4$.

To find the numbers of terms, put $T_n = 401$

$$\begin{aligned} a + (n - 1)d &= 401 \\ 41 + 4(n - 1) &= 401 \\ 4(n - 1) &= 360 \\ n - 1 &= 90 \\ n &= 91. \end{aligned}$$

Thus there are 91 terms in the series.

b Because we now know both the difference d and the last term $\ell = T_{91}$, either formula can be used. It's always easier to use $S_n = \frac{1}{2}n(a + \ell)$ if you can.

Using $S_n = \frac{1}{2}n(a + \ell)$,

$$\begin{aligned} S_{91} &= \tfrac{1}{2} \times 91 \times (41 + 401) \\ &= \tfrac{1}{2} \times 91 \times 442 \\ &= 20\,111. \end{aligned}$$

OR

Using $S_n = \frac{1}{2}n(2a + (n - 1)d)$,

$$\begin{aligned} S_{91} &= \tfrac{1}{2} \times 91 \times (2a + 90d) \\ &= \tfrac{1}{2} \times 91 \times (82 + 360) \\ &= 20\,111. \end{aligned}$$

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Solving problems involving the sums of APs

Problems involving sums of APs are solved using the formulae developed for the nth term T_n and the sum S_n of the first n terms.

Example 26 — 1F

a Find an expression for the sum S_n of n terms of the series $40 + 37 + 34 + \cdots$.

b Hence find the least value of n for which the partial sum S_n is negative.

SOLUTION

The sequence is an AP with $a = 40$ and $d = -3$.

a
$$\begin{aligned} S_n &= \tfrac{1}{2}n(2a + (n-1)d) \\ &= \tfrac{1}{2} \times n \times \big(80 - 3(n-1)\big) \\ &= \tfrac{1}{2} \times n \times (80 - 3n + 3) \\ &= \frac{n(83 - 3n)}{2} \end{aligned}$$

b
$$\begin{aligned} \text{Put} \qquad S_n &< 0. \\ \text{Then} \qquad \frac{n(83-3n)}{2} &< 0 \\ \boxed{\times 2} \qquad n(83 - 3n) &< 0 \\ \boxed{\div n} \qquad 83 - 3n &< 0, \quad \text{because } n \text{ is positive,} \\ 83 &< 3n \\ n &> 27\tfrac{2}{3}. \end{aligned}$$

Hence S_{28} is the first sum that is negative.

Example 27 — 1F

The sum of the first 10 terms of an AP is zero, and the sum of the first and second terms is 24. Find the first three terms.

SOLUTION

$$\begin{aligned} \text{The first piece of information given is} \qquad S_{10} &= 0 \\ 5(2a + 9d) &= 0 \\ \boxed{\div 5} \qquad 2a + 9d &= 0. \qquad (1) \\ \text{The second piece of information given is} \qquad T_1 + T_2 &= 24 \\ a + (a + d) &= 24 \\ 2a + d &= 24. \qquad (2) \\ \text{Subtracting (2) from (1),} \qquad 8d &= -24 \\ d &= -3, \\ \text{and substituting this into (2),} \qquad 2a - 3 &= 24 \\ a &= 13\tfrac{1}{2}. \end{aligned}$$

Hence the AP is $13\frac{1}{2} + 10\frac{1}{2} + 7\frac{1}{2} + \cdots$.

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Exercise 1F

FOUNDATION

1 Let $S_7 = 2 + 5 + 8 + 11 + 14 + 17 + 20$. By reversing the sum and adding in columns, evaluate S_7.

2 State how many terms each sum has, then find the sum using $S_n = \frac{1}{2}n(a + \ell)$.

a $1 + 2 + 3 + 4 + \cdots + 100$
b $1 + 3 + 5 + 7 + \cdots + 99$
c $2 + 4 + 6 + \cdots + 100$
d $3 + 6 + 9 + 12 + \cdots + 300$
e $101 + 103 + 105 + \cdots + 199$
f $1001 + 1002 + 1003 + \cdots + 10\,000$

3 Use $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$ to find the sum S_6 of the first 6 terms of the series with:

a $a = 5$ and $d = 10$
b $a = 8$ and $d = 2$
c $a = -3$ and $d = -9$
d $a = -7$ and $d = -12$

4 State the first term a and the difference d for each series below. Then use the formula $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$ to find the sum S_{21} of the first 21 terms of each series.

a $2 + 6 + 10 + \cdots$
b $3 + 10 + 17 + \cdots$
c $-6 - 1 + 4 + \cdots$
d $10 + 5 + 0 - \cdots$
e $-7 - 10 - 13 - \cdots$
f $1\frac{1}{2} + 3\frac{1}{2} + 5\frac{1}{2} + \cdots$

5 Use the formula $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$ to find the sum of the stated number of terms.

a $2 + 5 + 8 + \cdots$ (12 terms)
b $40 + 33 + 26 + \cdots$ (21 terms)
c $-6 - 2 + 2 + \cdots$ (200 terms)
d $33 + 30 + 27 + \cdots$ (23 terms)
e $-10 - 7\frac{1}{2} - 5 + \cdots$ (13 terms)
f $10\frac{1}{2} + 10 + 9\frac{1}{2} + \cdots$ (40 terms)

6 First use the formula $T_n = a + (n - 1)d$ to find how many terms there are in each sum. Then use the formula $S_n = \frac{1}{2}n(a + \ell)$ to find the sum, where ℓ is the last term T_n.

a $50 + 51 + 52 + \cdots + 150$
b $8 + 15 + 22 + \cdots + 92$
c $-10 - 3 + 4 + \cdots + 60$
d $4 + 7 + 10 + \cdots + 301$
e $6\frac{1}{2} + 11 + 15\frac{1}{2} + \cdots + 51\frac{1}{2}$
f $-1\frac{1}{3} + \frac{1}{3} + 2 + \cdots + 13\frac{2}{3}$

7 Find these sums by any appropriate method.

a $2 + 4 + 6 + \cdots + 1000$
b $1000 + 1001 + \cdots + 3000$
c $1 + 5 + 9 + \cdots$ (40 terms)
d $10 + 30 + 50 + \cdots$ (12 terms)

DEVELOPMENT

8 Use $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$ to find and simplify the sum of the first n terms of each series.

a $5 + 10 + 15 + \cdots$
b $10 + 13 + 16 + \cdots$
c $3 + 7 + 11 + \cdots$
d $-9 - 4 + 1 + \cdots$
e $5 + 4\frac{1}{2} + 4 + \cdots$
f $\big(1 - \sqrt{2}\big) + 1 + \big(1 + \sqrt{2}\big) + \cdots$

9 Use either standard formula for S_n to find a formula for the sum of the first n:

a positive integers,
b odd positive integers,
c positive integers divisible by 3,
d odd positive multiples of 100.

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10 **a** How many legs are there on 15 fish, 15 ducks, 15 dogs, 15 beetles, 15 spiders, and 15 ten-legged grubs? How many of these creatures have the mean number of legs?

b Matthew Flinders High School has 1200 pupils, with equal numbers of each age from 6 to 17 years inclusive. It also has 100 teachers and ancillary staff, all aged 30 years, and one Principal aged 60 years. What is the total of the ages of everyone in the school?

c An advertising graduate earns \$28 000 per annum in her first year, then each successive year her salary rises by \$1600. What are her total earnings over 10 years?

11 By substituting appropriate values of k, find the first term a and last term ℓ of each sum. Then evaluate the sum using $S_n = \frac{1}{2}n(a + \ell)$. (Note that all four series are APs.)

a $\sum_{k=1}^{200} (600 - 2k)$

b $\sum_{k=1}^{61} (93 - 3k)$

c $\sum_{k=1}^{40} (3k - 50)$

d $\sum_{k=10}^{30} (5k + 3)$

12 Solve these questions using the formula $S_n = \frac{1}{2}n(a + \ell)$ whenever possible — otherwise use the formula $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$.

a Find the last term if a series with 10 terms and first term -23 has sum -5.

b Find the first term if a series with 40 terms and last term $8\frac{1}{2}$ has sum 28.

c Find the common difference if a series with 8 terms and first term 5 has sum 348.

d Find the first term if a series with 15 terms and difference $\frac{2}{7}$ has sum -15.

13 Beware! These questions require quadratic equations to find solutions for n.

a Show that the sum to n terms of the AP $60 + 52 + 44 + 36 + \cdots$ is $S_n = 4n(16 - n)$.

b Hence find how many terms must be taken to make the sum: **i** zero, **ii** negative.

c Find the two values of n for which the sum S_n is 220.

d Show that $S_n = -144$ has two integer solutions, but that only one has meaning.

e For what values of n does the sum S_n exceed 156?

f Prove that no sum S_n can exceed 256.

g Write out the first 16 terms and sums, and check your results.

14 First use the formula $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$ to find the sum S_n for each arithmetic series. Then use quadratic equations to find the number of terms if S_n has the given value.

a $42 + 40 + 38 + \cdots$ where $S_n = 0$

b $60 + 57 + 54 + \cdots$ where $S_n = 0$

c $45 + 51 + 57 + \cdots$ where $S_n = 153$

d $2\frac{1}{2} + 3 + 3\frac{1}{2} + \cdots$ where $S_n = 22\frac{1}{2}$

CHALLENGE

15 **a** Logs of wood are stacked with 10 on the top row, 11 on the next, and so on. If there are 390 logs, find the number of rows, and the number of logs on the bottom row.

b A stone dropped from the top of a 245-metre cliff falls 5 metres in the first second, 15 metres in the second second, and so on in arithmetic sequence. Find a formula for the distance after n seconds, and find how long the stone takes to fall to the ground.

c A truck spends several days depositing truckloads of gravel from a quarry at equally spaced intervals along a straight road. The first load is deposited 20 km from the quarry, the last is 10 km further along the road. If the truck travels 550 km during these deliveries, including its return to the quarry after the last delivery, how many trips does it make, and how far apart are the deposits?

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16 **a** The sum of the first and fourth terms of an AP is 16, and the sum of the third and eighth terms is 4. Find the sum of the first 10 terms.

b The sum of the first 10 terms of an AP is zero, and the 10th term is -9. Find the first and second terms.

c The sum to 16 terms of an AP is 96, and the sum of the second and fourth terms is 45. Find the fourth term, and show that the sum to four terms is also 96.

17 Find the sums of these APs, whose terms are logarithms.

a $\log_a 2 + \log_a 4 + \log_a 8 + \cdots + \log_a 1024$

b $\log_5 243 + \log_5 81 + \log_5 27 + \cdots + \log_5 \frac{1}{243}$

c $\log_b 36 + \log_b 18 + \log_b 9 + \cdots + \log_b \frac{9}{8}$

d $\log_x \frac{27}{8} + \log_x \frac{9}{4} + \log_x \frac{3}{2} + \cdots$ (10 terms)

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1G Summing a geometric series

There is also a simple formula for finding the sum of the first n terms of a GP. The approach, however, is quite different from the approach used for APs.

Adding up the terms of a GP

This method is easier to understand with a general GP. Let us find the sum S_n of the first n terms of the GP $a + ar + ar^2 + \cdots$

Writing out the sum, $\quad S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}. \quad (1)$

Multiplying both sides by r, $\quad rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \quad (2)$

Subtracting (1) from (2), $(r - 1)\, S_n = ar^n - a.$

Then provided that $r \neq 1$, $\quad S_n = \dfrac{a(r^n - 1)}{r - 1}.$

If $r < 1$, there is a more convenient form. Taking opposites of top and bottom,

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

Method for summing a GP

Thus again there are two forms to remember.

12 TWO FORMULAE FOR SUMMING A GP

Suppose that the first term a, the ratio r, and the number n of terms are known.

- When $r > 1$, use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$.
- When $r < 1$, use the formula $S_n = \dfrac{a(1 - r^n)}{1 - r}$.

Example 28 **1G**

a Find the sum of all the powers of 5 from 5^0 to 5^7.

b Find the sum of the first six terms of the geometric series $2 - 6 + 18 - \cdots$.

SOLUTION

a The sum $5^0 + 5^1 + \cdots + 5^7$ is a GP with $a = 1$ and $r = 5$.

Using $S_n = \dfrac{a(r^n - 1)}{r - 1}$, $\quad$ (in this case $r > 1$)

$$\begin{aligned} S_8 &= \frac{a(r^8 - 1)}{r - 1} \quad \text{(there are 8 terms)} \\ &= \frac{1 \times (5^8 - 1)}{5 - 1} \\ &= 97\,656. \end{aligned}$$

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b To find the sum of the first six terms:
The series $2 - 6 + 18 - \cdots$ is a GP with $a = 2$ and $r = -3$.

$$\text{Using } S_n = \frac{a(1 - r^n)}{1 - r}, \qquad \text{(in this case } r < 1)$$

$$\begin{aligned} S_6 &= \frac{a(1 - r^6)}{1 - r} \\ &= \frac{2 \times (1 - (-3)^6)}{1 + 3} \\ &= -364. \end{aligned}$$

Solving problems about the sums of GPs

As always, read the question very carefully and write down all the information in symbolic form.

Example 29 1G

The sum of the first four terms of a GP with ratio 3 is 200. Find the four terms.

SOLUTION

It is known that $S_4 = 200.$

$$\text{Using the formula, } \frac{a(3^4 - 1)}{3 - 1} = 200$$

$$\begin{aligned} \frac{80a}{2} &= 200 \\ 40a &= 200 \\ a &= 5. \end{aligned}$$

So the series is $5 + 15 + 45 + 135 + \cdots$

Solving problems involving trial-and-error or logarithms

As remarked already in Section 1D, logarithms are needed for solving GP problems systematically, but trial-and-error is quite satisfactory for simpler problems.

Example 30 1G

a Find a formula for the sum of the first n terms of the GP $2 + 6 + 18 + \cdots$.
b How many terms of this GP must be taken for the sum to exceed one billion?

SOLUTION

a The sequence is a GP with $a = 2$ and $r = 3$,

$$\begin{aligned} \text{so } S_n &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{2(3^n - 1)}{3 - 1} \\ &= 3^n - 1. \end{aligned}$$

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b Put $S_n > 1\,000\,000\,000.$
Then $3^n - 1 > 1\,000\,000\,000$
$3^n > 1\,000\,000\,001.$
Using trial-and-error on the calculator,
$3^{18} = 387\,420\,489$
and $3^{19} = 1\,162\,261\,467,$
so S_{19} is the first sum over one billion.

OR

Put $S_n > 1\,000\,000\,000.$
Then $3^n - 1 > 1\,000\,000\,000$
$3^n > 1\,000\,000\,001$
$$n > \frac{\log_{10} 1\,000\,000\,001}{\log_{10} 3}$$
$n > 18.86\ldots,$
so S_{19} is the first sum over one billion.

An exceptional case

If the ratio of a GP is 1, then the formula for S_n doesn't work, because the denominator $r - 1$ would be zero. All the terms, however, are equal to the first term a, so the formula for the partial sum S_n is just

$$S_n = an.$$

This series is also an AP with first term a and difference 0. The last term is a, so

$$S_n = \tfrac{1}{2}n(a + \ell) = \tfrac{1}{2}n(a + a) = an.$$

Exercise 1G

FOUNDATION

1 Let $S_6 = 2 + 6 + 18 + 54 + 162 + 486$. By taking $3S_6$ and subtracting S_6 in columns, evaluate S_6.

2 'As I was going to St Ives, I met a man with seven wives. Each wife had seven sacks, each sack had seven cats, each cat had seven kits. Kits, cats, sacks and wives, how many were going to St Ives?'

Only the speaker was going *to* St Ives, but how many were going the other way?

3 **a** Use the formula $S_7 = \dfrac{a(r^7 - 1)}{r - 1}$ to find $1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 + 3^6$.

b Use the formula $S_7 = \dfrac{a(1 - r^7)}{1 - r}$ to find $1 - 3 + 3^2 - 3^3 + 3^4 - 3^5 + 3^6$.

4 Find these sums using $S_n = \dfrac{a(r^n - 1)}{r - 1}$ when $r > 1$, or $S_n = \dfrac{a(1 - r^n)}{1 - r}$ when $r < 1$. Then find a formula for the sum S_n of the first n terms of each series.

a $1 + 2 + 4 + 8 + \cdots$ (10 terms)
b $2 + 6 + 18 + \cdots$ (5 terms)
c $-1 - 10 - 100 - \cdots$ (5 terms)
d $-1 - 5 - 25 - \cdots$ (5 terms)
e $1 - 2 + 4 - 8 + \cdots$ (10 terms)
f $2 - 6 + 18 - \cdots$ (5 terms)
g $-1 + 10 - 100 + \cdots$ (5 terms)
h $-1 + 5 - 25 + \cdots$ (5 terms)

5 Find these sums. Then find a formula for the sum S_n of the first n terms of each series. Be careful when dividing by $1 - r$, because $1 - r$ is a fraction in each case.

a $8 + 4 + 2 + \cdots$ (10 terms)
b $9 + 3 + 1 + \cdots$ (6 terms)
c $45 + 15 + 5 + \cdots$ (5 terms)
d $\frac{2}{3} + 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8}$
e $8 - 4 + 2 - \cdots$ (10 terms)
f $9 - 3 + 1 - \cdots$ (6 terms)
g $-45 + 15 - 5 + \cdots$ (5 terms)
h $\frac{2}{3} - 1 + \frac{3}{2} - \frac{9}{4} + \frac{27}{8}$

DEVELOPMENT

6 Find an expression for S_n. Hence approximate S_{10} correct to four significant figures.

a $1 + 1.2 + (1.2)^2 + \cdots$
b $1 + 0.95 + (0.95)^2 + \cdots$
c $1 + 1.01 + (1.01)^2 + \cdots$
d $1 + 0.99 + (0.99)^2 + \cdots$

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7 The King takes a chessboard of 64 squares, and places 1 grain of wheat on the first square, 2 grains on the next square, 4 grains on the next square, and so on.

a How many grains are on: **i** the last square, **ii** the whole chessboard?

b Given that 1 litre of wheat contains about 30 000 grains, how many cubic kilometres of wheat are there on the chessboard?

8 Find S_n and S_{10} for each series, rationalising the denominators in your answers.

a $1 + \sqrt{2} + 2 + \cdots$ **b** $2 - 2\sqrt{5} + 10 - \cdots$

9 Find these sums. First write out some terms and identify a and r.

a $\displaystyle\sum_{n=1}^{7} 3 \times 2^n$ **b** $\displaystyle\sum_{n=3}^{8} 3^{n-1}$ **c** $\displaystyle\sum_{n=1}^{8} 3 \times 2^{3-n}$

10 **a** The first term of a GP is $\frac{1}{8}$ and the fifth term is 162. Find the first five terms of the GP, then find their sum.

b The first term of a GP is $-\frac{3}{4}$ and the fourth term is 6. Find the sum of the first six terms.

c The second term of GP is 0.08 and the third term is 0.4. Find the sum to eight terms.

d The ratio of a GP is $r = 2$ and the sum to eight terms is 1785. Find the first term.

e A GP has ratio $r = -\frac{1}{2}$ and the sum to eight terms is 425. Find the first term.

11 **a** Each year when the sunflower paddock is weeded, only half the previous weight of weed is dug out. In the first year, 6 tonnes of weed is dug out.

i How much is dug out in the 10th year?

ii What is the total dug out over 10 years (correct to four significant figures)?

b Every two hours, half of a particular medical isotope decays. If there was originally 20 grams, how much remains after a day (correct to two significant figures)?

c The price of Victoria shoes is increasing over a 10-year period by 10% per annum, so that the price in each of those 10 years is P, $1.1 \times P$, $(1.1)^2 \times P$, ... I buy one pair of these shoes each year.

i Find an expression for the total paid over 10 years (correct to the nearest cent).

ii Hence find the initial price P if the total paid is \$900.

12 Find a formula for S_n, and hence find n for the given value of S_n.

a $5 + 10 + 20 + \cdots$ where $S_n = 315$ **b** $5 - 10 + 20 - \cdots$ where $S_n = -425$

c $18 + 6 + 2 + \cdots$ where $S_n = 26\frac{8}{9}$ **d** $48 - 24 + 12 - \cdots$ where $S_n = 32\frac{1}{4}$

CHALLENGE

13 **a** Show that the sum S_n of the first n terms of $7 + 14 + 28 + \cdots$ is $S_n = 7(2^n - 1)$.

b For what value of n is S_n equal to 1785?

c Show that $T_n = 7 \times 2^{n-1}$, and find how many terms are less than 70 000.

d Use trial-and-error to find the first sum S_n that is greater than 70 000.

e Prove that the sum S_n of the first n terms is always 7 less than the $(n + 1)$th term.

14 The powers of 3 that are greater than 1 form a GP 3, 9, 27, ...

a Find how many powers of 3 there are between 2 and 10^{20}.

b Show that $S_n = \frac{3}{2}(3^n - 1)$, and find the smallest value of n for which $S_n > 10^{20}$.

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1H The limiting sum of a geometric series

There is a sad story of a perishing frog, dying of thirst only 8 metres from the edge of a waterhole. He first jumps 4 metres towards it, his second jump is 2 metres, then each successive jump is half the previous jump. Does the frog perish?

The jumps form a GP, whose terms T_n and sums S_n are as follows:

T_n	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	...
S_n	4	6	7	$7\frac{1}{2}$	$7\frac{3}{4}$	$7\frac{7}{8}$	$7\frac{15}{16}$	...

The successive jumps $4,\ 2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \ldots$ have limit zero, because they are halving each time. It seems too that the successive sums S_n have limit 8, meaning that the frog's total distance gets 'as close as we like' to 8 metres. So provided that the frog can stick his tongue out even the merest fraction of a millimetre, eventually he will get some water to drink and be saved.

The limiting sum of a GP

We can describe all this more precisely by looking at the sum S_n of the first n terms and examining what happens as $n \to \infty$.

The series $4 + 2 + 1 + \frac{1}{2} + \cdots$ is a GP with $a = 4$ and $r = \frac{1}{2}$.
Using the formula for the sum to n terms of the series,

$$
\begin{aligned}
S_n &= \frac{a(1 - r^n)}{1 - r} \qquad \text{(using this formula because } r < 1) \\
&= \frac{4\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} \\
&= 4 \times \left(1 - \left(\tfrac{1}{2}\right)^n\right) \div \tfrac{1}{2} \\
&= 8\left(1 - \left(\tfrac{1}{2}\right)^n\right).
\end{aligned}
$$

As n increases, the term $\left(\frac{1}{2}\right)^n$ gets progressively closer to zero:

$$\left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4}, \quad \left(\tfrac{1}{2}\right)^3 = \tfrac{1}{8}, \quad \left(\tfrac{1}{2}\right)^4 = \tfrac{1}{16}, \quad \left(\tfrac{1}{2}\right)^5 = \tfrac{1}{32}, \quad \left(\tfrac{1}{2}\right)^6 = \tfrac{1}{64}, \quad \ldots$$

so that $\left(\frac{1}{2}\right)^n$ has limit zero as $n \to \infty$.

Hence S_n does indeed have limit $8(1 - 0) = 8$, as the table of values suggested.

There are several different common notations and words for this situation:

13 NOTATIONS FOR THE LIMITING SUM

Take as an example the series $4 + 2 + 1 + \frac{1}{2} + \cdots$.

- $S_n \to 8$ as $n \to \infty$. ('S_n has limit 8 as n increases without bound.')
- $\lim\limits_{n \to \infty} S_n = 8$. ('The limit of S_n, as n increases without bound, is 8.')
- The series $4 + 2 + 1 + \frac{1}{2} + \cdots$ has *limiting sum* $S_\infty = 8$.
- The series $4 + 2 + 1 + \frac{1}{2} + \cdots$ *converges to the limit* $S_\infty = 8$.
- $4 + 2 + 1 + \frac{1}{2} + \cdots = 8$.

The symbols S_∞ and S are both commonly used for the limiting sum.

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The general case

Suppose now that T_n is a GP with first term a and ratio r, so that

$$T_n = ar^{n-1} \qquad \text{and} \qquad S_n = \frac{a(1-r^n)}{1-r}.$$

Suppose also that the ratio r lies in the interval $-1 < r < 1$.

Then as $n \to \infty$, the successive powers $r^1, r^2, r^3, r^4, \ldots$ get smaller and smaller,

that is, $\quad r^n \to 0 \quad$ and $\quad 1 - r^n \to 1$.

Thus both the nth term T_n and the sum S_n *converge* to a limit,

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty} ar^{n-1} = 0, \qquad \text{and} \qquad \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}.$$

14 THE LIMITING SUM OF A GEOMETRIC SERIES

- Suppose that $|r| < 1$, that is, $-1 < r < 1$.
  Then $\quad r^n \to 0$ as $n \to \infty$,
  so the terms and the partial sums of the GP both *converge to a limit*,
  $$\lim_{n\to\infty} T_n = 0 \qquad \text{and} \qquad S_\infty = \lim_{n\to\infty} S_n = \frac{a}{1-r}.$$
- If $|r| \geq 1$, then the partial sums S_n do not converge to a limit.

Example 31 — 1H

Explain why these series have limiting sums, and find them.

a $18 + 6 + 2 + \cdots$

b $18 - 6 + 2 - \cdots$

SOLUTION

a Here $a = 18$ and $r = \frac{1}{3}$.
Because $-1 < r < 1$, the series converges.

$$S_\infty = \frac{18}{1 - \frac{1}{3}} = 18 \times \tfrac{3}{2} = 27$$

b Here $a = 18$ and $r = -\frac{1}{3}$.
Because $-1 < r < 1$, the series converges.

$$S_\infty = \frac{18}{1 + \frac{1}{3}} = 18 \times \tfrac{3}{4} = 13\tfrac{1}{2}$$

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Example 32 1H

a For what values of x does the series $1 + (x - 2) + (x - 2)^2 + \cdots$ converge?
b When the series does converge, what is its limiting sum?

SOLUTION

The sequence is a GP with first term $a = 1$ and ratio $r = x - 2$.

a The GP converges when
$$\begin{aligned} -1 &< r < 1 \\ -1 &< x - 2 < 1 \\ \boxed{+2} \qquad 1 &< x < 3. \end{aligned}$$

b The limiting sum is then
$$\begin{aligned} S_\infty &= \frac{1}{1 - (x - 2)} \\ &= \frac{1}{3 - x}. \end{aligned}$$

Solving problems involving limiting sums

As always, the first step is to write down in symbolic form everything that is given in the question.

Example 33 1H

Find the ratio of a GP whose first term is 10 and whose limiting sum is 40.

SOLUTION

We know that $S_\infty = 40.$

Using the formula, $\dfrac{a}{1 - r} = 40$

and substituting $a = 10$ gives $\dfrac{10}{1 - r} = 40$
$$\begin{aligned} 10 &= 40(1 - r) \\ 1 &= 4 - 4r \\ 4r &= 3 \\ r &= \tfrac{3}{4}. \end{aligned}$$

Sigma notation for infinite sums

When $-1 < r < 1$ and the GP converges, the limiting sum S_∞ can also be written as an infinite sum, either using sigma notation or using dots, so that
$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \qquad \text{or} \qquad a + ar + ar^2 + \cdots = \frac{a}{1 - r},$$
and we say that 'the series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$ converges to $\dfrac{a}{1 - r}$'.

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Exercise 1H

FOUNDATION

1 **a** Copy and complete the table of values opposite for the GP with $a = 18$ and $r = \frac{1}{3}$.

n	1	2	3	4	5	6
T_n	18	6	2	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{2}{27}$
S_n						

b Find the limiting sum using $S_\infty = \dfrac{a}{1 - r}$.

c Find the difference $S_\infty - S_6$.

2 **a** Copy and complete the table of values opposite for the GP with $a = 24$ and $r = -\frac{1}{2}$.

n	1	2	3	4	5	6
T_n	24	-12	6	-3	$1\frac{1}{2}$	$-\frac{3}{4}$
S_n						

b Find the limiting sum using $S_\infty = \dfrac{a}{1 - r}$.

c Find the difference $S_\infty - S_6$.

3 Each GP below has ratio $r = \frac{1}{2}$. Identify the first term a and hence find S_∞.

a $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ **b** $8 + 4 + 2 + \cdots$ **c** $-4 - 2 - 1 - \cdots$

4 Each GP below has ratio $r = -\frac{1}{3}$. Identify the first term a and hence find S_∞.

a $1 - \frac{1}{3} + \frac{1}{9} - \cdots$ **b** $36 - 12 + 4 - \cdots$ **c** $-60 + 20 - 6\frac{2}{3} + \cdots$

5 Each GP below has first term $a = 60$. Identify the ratio r and hence find S_∞.

a $60 + 15 + 3\frac{3}{4} + \cdots$ **b** $60 + 15 + 3\frac{3}{4} + \cdots$ **c** $60 - 12 + 2\frac{2}{5} - \cdots$

6 Find each ratio r to test whether there is a limiting sum. Find the limiting sum if it exists.

a $1 - \frac{1}{2} + \frac{1}{4} - \cdots$ **b** $1 + \frac{1}{3} + \frac{1}{9} + \cdots$ **c** $1 - \frac{2}{3} + \frac{4}{9} - \cdots$

d $1 + \frac{3}{5} + \frac{9}{25} + \cdots$ **e** $4 - 6 + 9 - \cdots$ **f** $12 + 4 + \frac{4}{3} + \cdots$

g $1000 + 100 + 10 + \cdots$ **h** $1000 - 100 + 10 - \cdots$ **i** $1 - 1 + 1 - \cdots$

j $100 + 90 + 81 + \cdots$ **k** $-2 + \frac{2}{5} - \frac{2}{25} + \cdots$ **l** $-\frac{2}{3} - \frac{2}{15} - \frac{2}{75} - \cdots$

7 Bevin dropped the Nelson Bros Bouncy Ball from a height of 8 metres. It bounced continually, each successive height being half of the previous height.

a Show that the first distance travelled down-and-up is 12 metres, and explain why the successive down-and-up distances form a GP with $r = \frac{1}{2}$.

b Through what distance did the ball 'eventually' travel?

DEVELOPMENT

8 These examples will show that a GP does not have a limiting sum when $r \geq 1$ or $r \leq -1$. Copy and complete the tables for these GPs, then describe the behaviour of S_n as $n \to \infty$.

a $r = 1$ and $a = 10$

n	1	2	3	4	5	6
T_n						
S_n						

b $r = -1$ and $a = 10$

n	1	2	3	4	5	6
T_n						
S_n						

c $r = 2$ and $a = 10$

n	1	2	3	4	5	6
T_n						
S_n						

d $r = -2$ and $a = 10$

n	1	2	3	4	5	6
T_n						
S_n						

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9 For each series, find S_∞ and S_4, then find the difference $S_\infty - S_4$.

a $80 + 40 + 20 + \cdots$ **b** $100 + 10 + 1 + \cdots$ **c** $100 - 80 + 64 - \cdots$

10 When Brownleigh Council began offering free reflective house numbers to its 10 000 home owners, 20% installed them in the first month. The number installing them in the second month was only 20% of those in the first month, and so on.

a Show that the numbers installing them each month form a GP.

b How many home owners will 'eventually' install them? ('Eventually' means take S_∞.)

c How many eventual installations were not done in the first four months?

11 The Wellington Widget Factory has been advertising its unbreakable widgets every month. The first advertisement brought in 1000 sales, but every successive advertisement is only bringing in 90% of the previous month's sales.

a How many widget sales will the advertisements 'eventually' bring in?

b About how many eventual sales were not brought in by the first 10 advertisements?

12 Find, in terms of x, an expression for the limiting sum of each series on the left. Then solve the given equation to find x.

a $5 + 5x + 5x^2 + \cdots = 10$ **b** $5 + 5x + 5x^2 + \cdots = 3$

c $5 - 5x + 5x^2 - \cdots = 15$ **d** $x + \frac{x}{3} + \frac{x}{9} + \cdots = 2$

e $x - \frac{x}{3} + \frac{x}{9} - \cdots = 2$ **f** $x + \frac{2x}{3} + \frac{4x}{9} + \cdots = 2$

13 Find the condition for each GP to have a limiting sum, then find that limiting sum.

a $7 + 7x + 7x^2 + \cdots$ **b** $2x + 6x^2 + 18x^3 + \cdots$

c $1 + (x - 1) + (x - 1)^2 + \cdots$ **d** $1 + (1 + x) + (1 + x)^2 + \cdots$

14 Find the limiting sum of each series, if it exists.

a $1 + (1.01) + (1.01)^2 + \cdots$ **b** $1 - 0.99 + (0.99)^2 - \cdots$

c $1 + (1.01)^{-1} + (1.01)^{-2} + \cdots$ **d** $0.72 - 0.12 + 0.02 - \cdots$

15 Find the limiting sums if they exist, rationalising denominators if necessary.

a $16\sqrt{5} + 4\sqrt{5} + \sqrt{5} + \cdots$ **b** $108\sqrt{7} - 36\sqrt{7} + 12\sqrt{7} - \cdots$

c $7 + \sqrt{7} + 1 + \cdots$ **d** $4 - 2\sqrt{2} + 2 - \cdots$

e $5 - 2\sqrt{5} + 4 - \cdots$ **f** $9 + 3\sqrt{10} + 10 + \cdots$

g $1 + \left(1 - \sqrt{3}\right) + \left(1 - \sqrt{3}\right)^2 + \cdots$ **h** $1 + \left(2 - \sqrt{3}\right) + \left(2 - \sqrt{3}\right)^2 + \cdots$

16 Expand each series for a few terms. Then write down a and r, and find the limiting sum.

a $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ **b** $\sum_{n=1}^{\infty} 7 \times \left(\frac{1}{2}\right)^n$ **c** $\sum_{n=1}^{\infty} 40 \times \left(-\frac{3}{5}\right)^n$

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CHALLENGE

17 Suppose that $T_n = ar^{n-1}$ is a GP with a limiting sum.

a Find the ratio r if the limiting sum equals 5 times the first term.

b Find the first three terms if the second term is 6 and the limiting sum is 27.

c Find the ratio if the sum of all terms except the first equals 5 times the first term.

d Show that the sum S of all terms from the third on is $\dfrac{ar^2}{1 - r}$.

i Hence find r if S equals the first term.

ii Find r if S equals the second term.

iii Find r if S equals the sum of the first and second terms.

18 Find the condition for each GP to have a limiting sum, then find that limiting sum.

a $1 + (x^2 - 1) + (x^2 - 1)^2 + \cdots$

b $1 + \dfrac{1}{1 + x^2} + \dfrac{1}{(1 + x^2)^2} + \cdots$

19 Suppose that the series $v + v^2 + v^3 + \cdots$ has a limiting sum w.

a Write w in terms of v.

b Find v in terms of w.

c Hence find the limiting sum of the series $w - w^2 + w^3 - \cdots$, assuming that $|w| < 1$.

d Test your results with $v = \frac{1}{3}$.

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1I Recurring decimals and geometric series

It would not have been easy in Chapter 2 of the Year 11 book to convert a recurring decimal back to a fraction. Now, however, we can express a recurring decimal as an infinite GP — its value is the limiting sum of that GP, which is easily expressed as a fraction.

Example 34 1I

Express these recurring decimals as infinite GPs. Then use the formula for the limiting sum to find their values as fractions reduced to lowest terms.

a $0.\dot{2}\dot{7}$ **b** $2.6\dot{4}\dot{5}$

SOLUTION

a Expanding the decimal, $0.\dot{2}\dot{7} = 0.272727\ldots$

$$= 0.27 + 0.0027 + 0.000027 + \cdots$$

This is an infinite GP with first term $a = 0.27$ and ratio $r = 0.01$.

Hence
$$\begin{aligned} 0.\dot{2}\dot{7} &= \frac{a}{1-r} \\ &= \tfrac{0.27}{0.99} \\ &= \tfrac{27}{99} \\ &= \tfrac{3}{11}. \end{aligned}$$

b This example is a little more complicated, because the first part is not recurring.

Expanding the decimal, $2.6\dot{4}\dot{5} = 2.645454545\ldots$

$$= 2.6 + (0.045 + 0.00045 + \cdots)$$

This is 2.6 plus an infinite GP with first term $a = 0.045$ and ratio $r = 0.01$.

Hence
$$\begin{aligned} 2.6\dot{4}\dot{5} &= 2.6 + \tfrac{0.045}{0.99} \\ &= \tfrac{26}{10} + \tfrac{45}{990} \\ &= \tfrac{286}{110} + \tfrac{5}{110} \\ &= \tfrac{291}{110}. \end{aligned}$$

15 EXPRESSING A RECURRING DECIMAL AS A FRACTION

- To convert a recurring decimal as a fraction, write the recurring part as a GP.
- The ratio will be between 0 and 1, so the series will have an infinite sum.

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Exercise 1I

FOUNDATION

Note: These prime factorisations will be useful in this exercise:

$$9 = 3^2 \qquad 999 = 3^3 \times 37 \qquad 99999 = 3^2 \times 41 \times 271$$
$$99 = 3^2 \times 11 \qquad 9999 = 3^2 \times 11 \times 101 \qquad 999999 = 3^3 \times 7 \times 11 \times 13 \times 37$$

1 Write each recurring decimal as an infinite GP. Then use the formula for the limiting sum of a GP to express it as a rational number in lowest terms.

a $0.\dot{3}$ **b** $0.\dot{1}$ **c** $0.\dot{7}$ **d** $0.\dot{6}$

2 Write each recurring decimal as an infinite GP. Then use the formula for the limiting sum of a GP to express it as a rational number in lowest terms.

a $0.\dot{2}\dot{7}$ **b** $0.\dot{8}\dot{1}$ **c** $0.\dot{0}\dot{9}$ **d** $0.\dot{1}\dot{2}$
e $0.\dot{7}\dot{8}$ **f** $0.\dot{0}2\dot{7}$ **g** $0.\dot{1}3\dot{5}$ **h** $0.\dot{1}8\dot{5}$

DEVELOPMENT

3 Write each recurring decimal as the sum of an integer or terminating decimal and an infinite GP. Then express it as a fraction in lowest terms.

a $12.\dot{4}$ **b** $7.\dot{8}\dot{1}$ **c** $8.4\dot{6}$ **d** $0.2\dot{3}\dot{6}$

4 **a** Express the repeating decimal $0.\dot{9}$ as an infinite GP, and hence show that it equals 1.
b Express $2.7\dot{9}$ as 2.7 plus an infinite GP, and hence show that it equals 2.8.

CHALLENGE

5 Use GPs to express these as fractions in lowest terms.

a $0.\dot{0}95\dot{7}$ **b** $0.\dot{2}47\dot{5}$ **c** $0.\dot{2}3076\dot{9}$ **d** $0.\dot{4}2857\dot{1}$
e $0.25\dot{5}\dot{7}$ **f** $1.1\dot{0}3\dot{7}$ **g** $0.0\dot{0}027\dot{1}$ **h** $7.7\dot{7}1428\dot{5}$

6 Last year we proved in Section 2B that $\sqrt{2}$ is irrational. Why can we now conclude that when $\sqrt{2}$ is written as a decimal, it is not a recurring decimal?

Two techniques in mental arithmetic: You would have used quite a bit of mental arithmetic in this chapter. Here are some techniques that are well worth knowing and practising to make life easier (and this course does not require them).

7 **Doubling and halving** are easy. This means that when multiplying and dividing with even numbers, we can break down the calculation into smaller pieces that can be done mentally.

a To multiply by an even number, take out the factors of 2, then multiply the resulting odd numbers together, then use doubling to get the final answer.

$$14 \times 24 = 2^4 \times (7 \times 3) = 2^4 \times 21 = 2^3 \times 42 = 2^2 \times 84 = 2 \times 168 = 336$$

b To multiply by a multiple of 5, combine each 5 with a 2 using doubling and halving

$$15 \times 26 = 30 \times 13 = 390$$
$$125 \times 108 = 250 \times 54 = 500 \times 27 = 1000 \times 13\tfrac{1}{2} = 13500$$

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c To divide by 5 or a multiple of 5, double top and bottom.

$$\frac{62}{5} = \frac{124}{10} = 12.4 \qquad \frac{48}{15} = \frac{96}{30} = 3.2$$

d Some practice — make up your own, and use a calculator to check.

$$11 \times 44,\ 12 \times 77,\ 18 \times 14,\ 14 \times 35,\ 15 \times 21,\ 75 \times 16,\ \frac{85}{5},\ \frac{42}{5},\ \frac{36}{15}$$

8 **The difference of squares** makes multiplying two odd numbers straightforward as long as you know your squares.

$$13 \times 17 = (15 - 2)(15 + 2) = 15^2 - 2^2 = 225 - 4 = 221$$

a To find a square, add and subtract to give a product that can be done easily, then use the difference of squares in reverse. In this example, multiplying by 20 is simple.

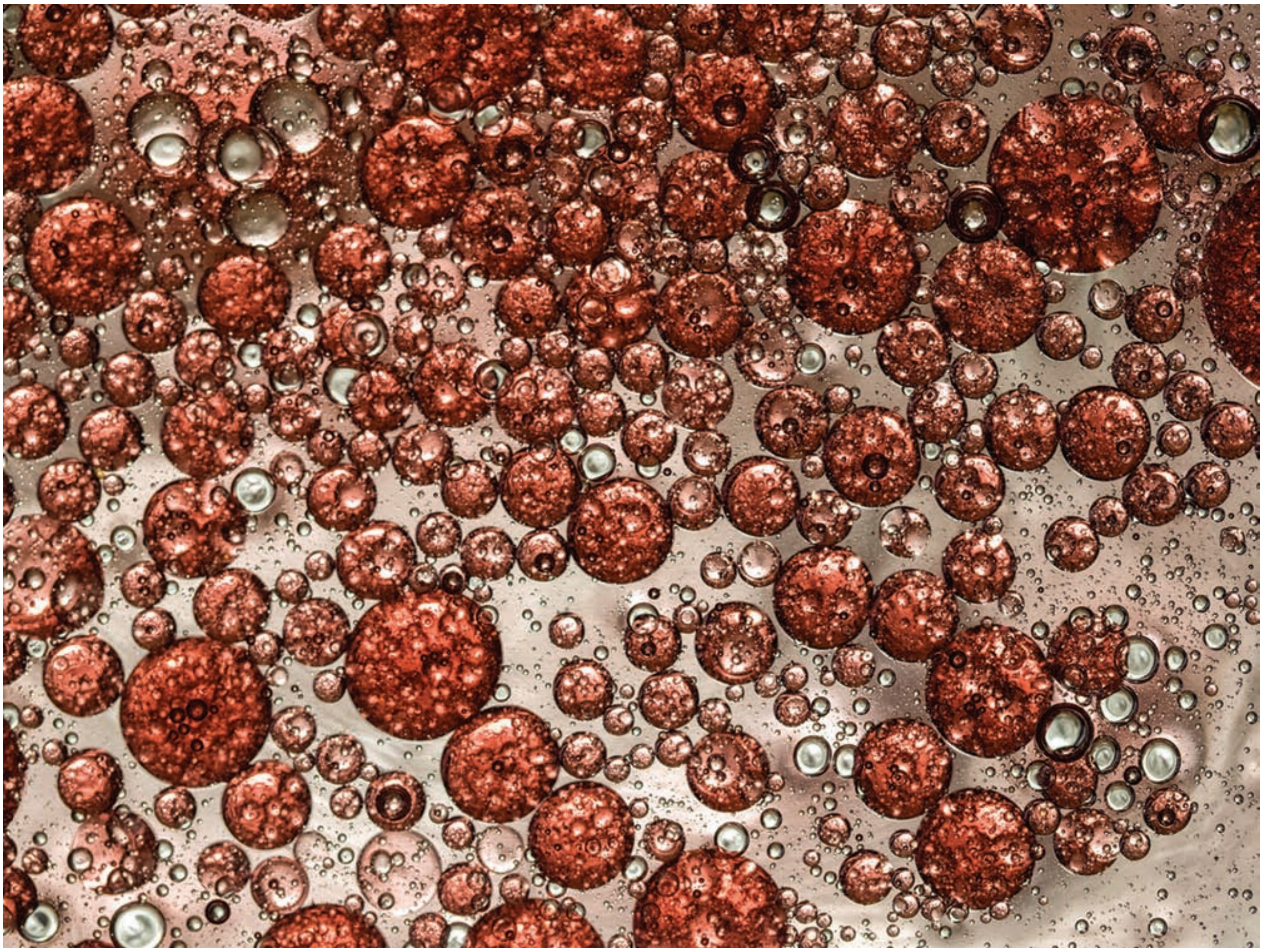
$$23^2 = (20 \times 26) + 3^2 = (10 \times 52) + 3^2 = 520 + 9 = 529$$

b Half-integers can easily be squared in this way.

$$\left(8\tfrac{1}{2}\right)^2 = (8 \times 9) + \left(\tfrac{1}{2}\right)^2 = 72\tfrac{1}{4}$$

c Some practice — it is worth learning by heart the squares up to 20^2.

$$9 \times 13,\ 17 \times 23,\ 23 \times 37,\ 17^2,\ 13 \times 21,\ 18^2,\ 17 \times 19,\ 19^2,\ 17 \times 21,\ 41^2,\ 28^2$$



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Chapter 1 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 1 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 Write out the first 12 terms of the sequence $50,\ 41,\ 32,\ 23,\ \ldots$

a How many positive terms are there?
b How many terms lie between 0 and 40?
c What is the 10th term?
d What number term is -13?
e Is -100 a term in the sequence?
f What is the first term less than -35?

2 The nth term of a sequence is given by $T_n = 58 - 6n$.

a Find the first, 20th, 100th and the 1 000 000th terms.
b Find whether 20, 10, -56 and -100 are terms of the sequence.
c Find the first term less than -200, giving its number and its value.
d Find the last term greater than -600, giving its number and its value.

3 Find the original sequence T_n if its partial sums S_n are:

a the sequence $4,\ 11,\ 18,\ 25,\ 32,\ 39,\ \ldots,$
b the sequence $0,\ 1,\ 3,\ 6,\ 10,\ 15,\ 21,\ \ldots,$
c given by $S_n = n^2 + 5,$
d given by $S_n = 3^n,$

4 Evaluate these expressions:

a $\displaystyle\sum_{n=3}^{6} (n^2 - 1)$
b $\displaystyle\sum_{n=-2}^{2} (5n - 3)$
c $\displaystyle\sum_{n=0}^{6} (-1)^n$
d $\displaystyle\sum_{n=1}^{6} \left(\tfrac{1}{2}\right)^n$

5 **a** Write out the first eight terms of the sequence $T_n = 5 \times (-1)^n$.

b Find the sum of the first seven terms and the sum of the first eight terms.
c How is each term obtained from the previous term?
d What are the 20th, 75th and 111th terms?

6 Test each sequence to see whether it is an AP, a GP or neither. State the common difference of any AP and the common ratio of any GP.

a $76,\ 83,\ 90,\ \ldots$
b $100,\ -21,\ -142,\ \ldots$
c $1,\ 4,\ 9,\ \ldots$
d $6,\ 18,\ 54,\ \ldots$
e $6,\ 10,\ 15,\ \ldots$
f $48,\ -24,\ 12,\ \ldots$

7 **a** State the first term and common difference of the AP $23,\ 35,\ 47,\ \ldots$

b Use the formula $T_n = a + (n - 1)d$ to find the 20th term and the 600th term.
c Show that the formula for the nth term is $T_n = 11 + 12n$.

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d Hence find whether 143 and 173 are terms of the sequence.

e Hence find the first term greater than 1000 and the last term less than 2000.

f Hence find how many terms there are between 1000 and 2000.

8 A shop charges \$20 for one case of soft drink and \$16 for every subsequent case.

a Show that the costs of 1 case, 2 cases, 3 cases, . . . form an AP and state its first term and common difference.

b Hence find a formula for the cost of n cases.

c What is the largest number of cases that I can buy with \$200, and what is my change?

d My neighbour paid \$292 for some cases. How many did he buy?

9 **a** Find the first term and common ratio of the GP $50,\ 100,\ 200,\ \ldots$

b Use the formula $T_n = ar^{n-1}$ to find a formula for the nth term.

c Hence find the eighth term and the twelfth term.

d Find whether 1600 and 4800 are terms of the sequence.

e Find the product of the fourth and fifth terms.

f Use logarithms, or trial-and-error on the calculator, to find how many terms are less than 10 000 000.

10 On the first day that Barry exhibited his paintings, there were 486 visitors. On each subsequent day, there were only a third as many visitors as on the previous day.

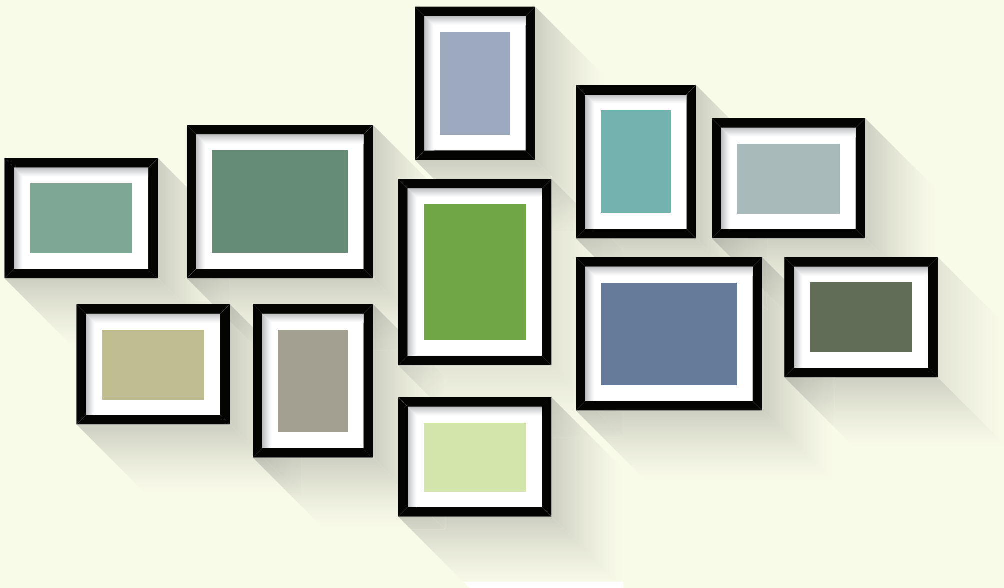
a Show that the number of visitors on successive days forms a GP and state the first term and common ratio.

b Write out the terms of the GP until the numbers become absurd.

c For how many days were there at least 10 visitors?

d What was the total number of visitors while the formula was still valid?

e Use the formula $S_\infty = \dfrac{a}{1 - r}$ to find the 'eventual' number of visitors if the absurdity of fractional numbers of people were ignored.



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Review

11 Find the second term x of the sequence $15,\ x,\ 135$:

a if the sequence is an AP, **b** if the sequence is a GP.

12 Use the formula $S_n = \frac{1}{2}n\big(2a + (n-1)d\big)$ to find the sum of the first 41 terms of each AP.

a $51 + 62 + 73 + \cdots$ **b** $100 + 75 + 50 + \cdots$ **c** $-35 - 32 - 29 - \cdots$

13 Use the formula $T_n = a + (n-1)d$ to find the number of terms in each AP, then use the formula $S_n = \frac{1}{2}n(a + \ell)$ to find the sum of the series.

a $23 + 27 + 31 + \cdots + 199$

b $200 + 197 + 194 + \cdots - 100$

c $12 + 12\frac{1}{2} + 13 + \cdots + 50$

14 Use $S_n = \dfrac{a(r^n - 1)}{r - 1}$ or $S_n = \dfrac{a(1 - r^n)}{1 - r}$ to find the sum of the first 6 terms of each GP.

a $3 + 6 + 12 + \cdots$ **b** $6 - 18 + 54 - \cdots$ **c** $-80 - 40 - 20 - \cdots$

15 Find the limiting sum of each GP, if it exists.

a $240 + 48 + 9\frac{3}{5} + \cdots$ **b** $-6 + 9 - 13\frac{1}{2} + \cdots$ **c** $-405 + 135 - 45 + \cdots$

16 a For what values of x does the GP $(2 + x) + (2 + x)^2 + (2 + x)^3 + \cdots$ have a limiting sum?

b Find a formula for the value of this limiting sum when it does exist.

17 Use the formula for the limiting sum of a GP to express as a fraction:

a $0.\dot{3}\dot{9}$ **b** $0.\dot{4}6\dot{8}$ **c** $12.30\dot{4}\dot{5}$

18 a The second term of an AP is 21 and the ninth term is 56. Find the 100th term.

b Find the sum of the first 20 terms of an AP with third term 10 and 12th term -89.

c The third term of a GP is 3 and the eighth term is -96. Find the sixth term.

d Find the difference of the AP with first term 1 if the sum of the first 10 terms is -215.

e Find how many terms there are in an AP with first term $4\frac{1}{2}$ and difference -1 if the sum of all the terms is 8.

f Find the common ratio of a GP with first term 60 and limiting sum 45.

g The sum of the first 10 terms of a GP with ratio -2 is 682. Find the fourth term.

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2 Graphs and equations

This chapter completes non-calculus curve-sketching and the use of graphs to solve equations and inequations.

Sections 2A–2C discuss four approaches to curve-sketching — domain, odd and even symmetry, zeroes and sign, and asymptotes — and combine then into an informal menu for sketching any curve given its equation. This is in preparation for Chapter 3, where the methods of calculus will be added to gain more information about the graph.

Section 2A also emphasises the role of the graph in solving inequations. Sections 2D–2E develop graphical methods further to help solve various equations and inequations.

Sections 2F–2I review transformations, and add stretching (or dilation) to the list of available transformations. The problem of combining two or more transformations is addressed, including a crucial and difficult question, 'Does it matter in which order the transformations are applied?'

Trigonometric graphs, in particular, benefit very greatly from a systematic approach using translations and dilations, because the amplitude, period, phase and mean value all depend on them. The final Section 2I deals with these graphs.

Whether explicitly suggested in an exercise or not, graphing software in any form is always very useful in confirming results and improving approximations. It is particularly suited to investigating what happens when changes are made to a function's equation.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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2A The sign of a function

When we are sketching a curve, we usually want to know very early where the curve is above the x-axis and where it is below the x-axis. That means knowing where the function is positive and where it is negative. This section provides an approach to this question using a table of signs.

The other main concern of this chapter is solving equations and inequations graphically. We can always put all the terms on the left. For example,

$$x^3 + 1 \geq x^2 + x \qquad \text{can be written as} \qquad x^3 - x^2 - x + 1 \geq 0.$$

This procedure reduces every inequation to one of the four forms

$$f(x) < 0 \qquad \text{or} \qquad f(x) \leq 0 \qquad \text{or} \qquad f(x) \geq 0 \qquad \text{or} \qquad f(x) > 0,$$

so that solving an inequation is the same as finding the sign of the function $f(x)$.

Before all this, however, two extra pieces of notation need to be introduced.

- Bracket interval notation is an alternative to inequality interval notation.
- Composition of functions has an alternative notation $f \circ g$.

Bracket interval notation

There is an alternative notation for intervals that will make the notation in this section and later a little more concise. The notation encloses the endpoints of the interval in brackets, using a square bracket if the endpoint is included and a round bracket if the endpoint is not included.

Here are the five examples from the Year 11 book written in both notations:

Diagram	Using inequalities	Using brackets
	$\frac{1}{3} \leq x \leq 3$	$\left[\frac{1}{3}, 3\right]$
Read this as, 'The closed interval from $\frac{1}{3}$ to 3'.		
	$-1 < x < 5$	$(-1, 5)$
Read this as, 'The open interval from -1 to 5'.		
	$-2 \leq x < 3$	$[-2, 3)$
Read this as, 'The interval from -2 to 3, including -2 but excluding 3'.		
	$x \geq -5$	$[-5, \infty)$
Read this as, 'The closed ray from -5 to the right'.		
	$x < 2$	$(-\infty, 2)$
Read this as, 'The open ray from 2 to the left'.		

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The first interval $\left[\frac{1}{3}, 3\right]$ is *closed*, meaning that it contains all its endpoints.

The second interval $(-1, 5)$ is *open*, meaning that it does not contain any of its endpoints.

The third interval $[-2, 3)$ is neither open nor closed — it contains one of its endpoints, but does not contain the other endpoint.

The fourth interval $[-5, \infty)$ is *unbounded on the right*, meaning that it continues towards infinity. It only has one endpoint -5, which it contains, so it is closed.

The fifth interval $(-\infty, 2)$ is *unbounded on the left*, meaning that it continues towards negative infinity. It only has one endpoint 2, which it does not contain, so it is open.

'Infinity' and 'negative infinity', with their symbols ∞ and $-\infty$, are not numbers. They are ideas used in specific situations and phrases to make language and notation more concise. Here, they indicate that an interval is unbounded on the left or right, and the symbol $(-\infty, 2)$ means 'all real numbers less than 2'.

Bracket interval notation has some details that need attention.

- The variable x or y or whatever is missing. This can be confusing when we are talking about domain and range, or solving an inequation for some variable. When, however, we are just thinking about 'all real numbers greater than 100', no variable is involved, so the notation $(100, \infty)$ is more satisfactory than $x > 100$.
- The notation can be dangerously ambiguous. For example, the open interval $(-1, 5)$ can easily be confused with the point $(-1, 5)$ in the coordinate plane.
- Infinity and negative infinity are not numbers, as remarked above.
- The set $\mathbb{R}$ of all real numbers can be written as $(-\infty, \infty)$.
- The notation $[4, 4]$ is the one-member set $\{4\}$, called a *degenerate interval* because it has length zero.
- Notations such as $(4, 4)$, $(4, 4]$, $[7, 3]$ and $[7, 3)$ all suggest the empty set, if they mean anything at all, and should be avoided in this course.

1 BRACKET INTERVAL NOTATION

- A square bracket means that the endpoint is included, and a round bracket means that the endpoint is not included.
- For $a < b$, we can form the four *bounded intervals* below. The first is closed, the last is open, and the other two are neither open nor closed.
$$[a, b] \quad \text{and} \quad [a, b) \quad \text{and} \quad (a, b] \quad \text{and} \quad (a, b).$$
- For any real number a, we can form the four *unbounded intervals* below. The first two are closed, and the last two are open.
$$[a, \infty) \quad \text{and} \quad (-\infty, a] \quad \text{and} \quad (a, \infty) \quad \text{and} \quad (-\infty, a).$$
- The notation $(-\infty, \infty)$ means the whole real number line $\mathbb{R}$.
- The notation $[a, a]$ is the one-member set $\{a\}$, called a *degenerate interval*.
- An interval is called *closed* if it contains all its endpoints, and *open* if it doesn't contain any of its endpoints.

For those who enjoy precision, the interval $(-\infty, \infty)$ is both open and closed (it has no endpoints), and a degenerate interval $[a, a]$ is closed.

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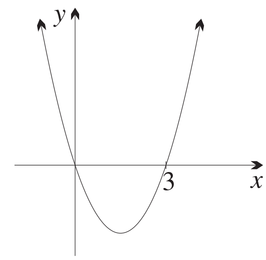
The union of intervals



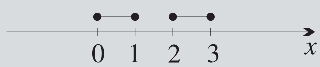
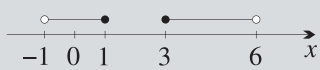
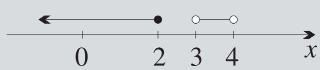
The graph to the right is the quadratic $y = x(x - 3)$. From the graph, we can see that the inequation

$$x(x - 3) \geq 0 \qquad \text{has solution} \qquad x \leq 0 \ \text{ or } \ x \geq 3.$$

This set is the *union* of the two intervals $(-\infty, 0]$ and $[3, \infty)$, so when using bracket interval notation, we write the set as

$$(-\infty, 0] \cup [3, \infty).$$

Here are some further examples using both types of interval notation. The close association between the word 'or' and the union of sets was discussed in Sections 10C and 10D of the Year 11 book in the context of probability.

Diagram	Using inequalities	Using brackets
	$0 \leq x \leq 1$ or $2 \leq x \leq 3$	$[0, 1] \cup [2, 3]$
	$-1 < x \leq 1$ or $3 \leq x < 6$	$(-1, 1] \cup [3, 6)$
	$x \leq 2$ or $3 < x < 4$	$(-\infty, 2] \cup (3, 4)$

Some alternative notation for composite functions

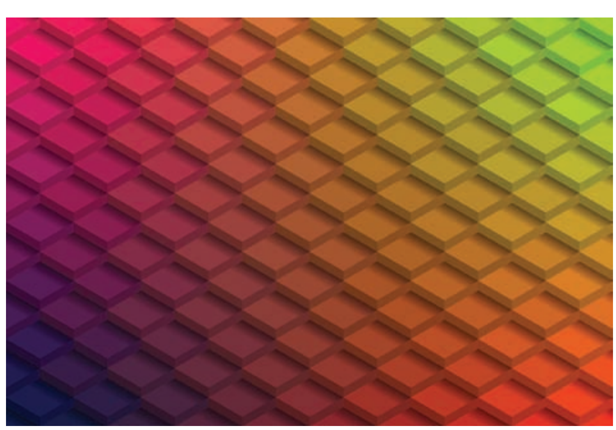
If $f(x)$ and $g(x)$ are two functions, the composite $g\big(f(x)\big)$ of two function $f(x)$ and $g(x)$ can also be written as $g \circ f(x)$. Thus

$$g \circ f(x) = g\big(f(x)\big), \quad \text{for all } x \text{ for which } f(x) \text{ and } g\big(f(x)\big) \text{ are defined.}$$

The advantage of this notation is that the composite function $g\big(f(x)\big)$ has a clear symbol $g \circ f$ that displays the composition of functions as a binary operator $\circ$ on the set of functions, with notation analogous to addition $a + b$, which is a binary operator on the set of numbers.

Be careful, however, when calculating $g \circ f(2)$, to apply the function f before the function g, because $g \circ f(x)$ means $g\big(f(x)\big)$. Section 4E of the Year 11 book developed composite functions in some detail, and Exercise 2A contains only a few mostly computational questions as practice of the new notation.

The composite $g \circ f(x)$ is often written with extra brackets as $(g \circ f)(x)$, and readers may prefer to add these extra brackets.



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Example 1 2A

If $f(x) = x + 3$ and $g(x) = x^2$, find:

a **i** $g \circ f(5)$ **ii** $f \circ g(5)$ **iii** $g \circ g(5)$ **iv** $f \circ f(5)$

b **i** $g \circ f(x)$ **ii** $f \circ g(x)$ **iii** $g \circ g(x)$ **iv** $f \circ f(x)$

SOLUTION

a **i** $g \circ f(5) = g(8)$
$= 64$

ii $f \circ g(5) = f(25)$
$= 28$

iii $g \circ g(5) = g(25)$
$= 625$

iv $f \circ f(5) = f(8)$
$= 11$

b **i** $g \circ f(x) = g(x + 3)$
$= (x + 3)^2$

ii $f \circ g(x) = f(x^2)$
$= x^2 + 3$

iii $g \circ g(x) = g(x^2)$
$= x^4$

iv $f \circ f(x) = f(x + 3)$
$= x + 6$

Where can a function change sign?

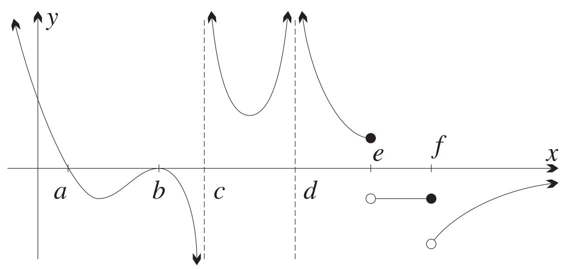
Now we can begin the main topic of this section.

Sketching polynomials in Section 3G of the Year 11 book required a table of test values dodging around the zeroes to see where the function changed signs. The functions in this chapter, however, may also have breaks in the curve, called *discontinuities*, and we need to dodge around them as well.

2 WHERE CAN A FUNCTION CHANGE SIGN?

The only places where a function may possibly change sign are zeroes and discontinuities.

Informally, a function is called *continuous at* $x = a$ if $f(x)$ is defined at $x = a$ and the curve $y = f(x)$ can be drawn through the point $(a, f(a))$ without lifting the pen off the paper. Otherwise the value $x = a$ is called a *discontinuity of* $f(x)$. Continuity was discussed in Section 8K of the Year 11 book.



The graph above has discontinuities at $x = c$, $x = d$, $x = e$ and $x = f$, and has zeroes at $x = a$ and $x = b$. The function changes sign at the zero $x = a$ and at the discontinuities $x = c$ and $x = e$, and nowhere else. Notice that it does not change sign at the zero $x = b$ or at the discontinuities $x = d$ and $x = f$.

The statement in Box 2 goes to the heart of what the real numbers are and what continuity means. In this course, the sketch above is sufficient justification of it.

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A table of signs

As a consequence, we can examine the sign of a function using a table of test values dodging around any zeroes and discontinuities. We add a third row for the sign, so that the table becomes a *table of signs*.

3 EXAMINING THE SIGN OF A FUNCTION

To examine the sign of a function, draw up a *table of signs*. This is a table of test values that dodge around any zeroes and discontinuities.

Finding the zeroes of a function has been a constant topic. To find discontinuities, assume that the function in the course are continuous at every value in their domains, except where there is an obvious problem.

Solving polynomial inequations

A simpler form of this table of signs was introduced in Section 3G of Year 11 to sketch polynomials. Polynomials do not have any discontinuities, so the test values only needed to dodge around the zeroes.

The attention in Section 3G was on the sketch of the function, whereas in this section our attention is also on solving inequations, as can be seen in the remaining worked examples. The graphs are not strictly necessary for this particular purpose, but they are very useful because they allow us to see the whole situation.

Example 2 2A

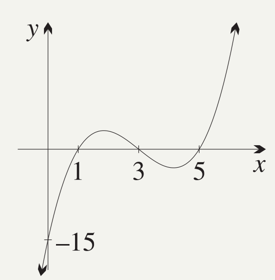
a Draw up a table of signs of the function $y = (x - 1)(x - 3)(x - 5)$.
b Sketch the graph of the function.
c State, using both interval notations, where the function is positive and where it is negative.
d From the graph, or from the table of signs, write down the solutions of $(x - 1)(x - 3)(x - 5) \le 0$ using both interval notations.

SOLUTION

a There are zeroes at 1, 3 and 5, and no discontinuities.

x	0	1	2	3	4	5	6
y	-15	0	3	0	-3	0	15
sign	$-$	0	$+$	0	$-$	0	$+$

b



c Hence y is positive for $1 < x < 3$ or $x > 5$, or alternatively $(1, 3) \cup (5, \infty)$, and negative for $x < 1$ or $3 < x < 5$, or alternatively $(-\infty, 1) \cup (3, 5)$.

d $x \le 1$ or $3 \le x \le 5$, or alternatively $(-\infty, 1] \cup [3, 5]$.

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Example 3 2A

a Rearrange the inequation $x^3 + 1 \le x^2 + x$ by moving all terms to the left.
b Factor $y = x^3 - x^2 - x + 1$ by grouping.
c Draw up a table of signs to establish where the function is positive and where it is negative.
d Hence sketch the graph of $y = x^3 - x^2 - x + 1$.
e From the graph, or from the table of signs, find the solution of $x^3 + 1 \le x^2 + x$.

SOLUTION

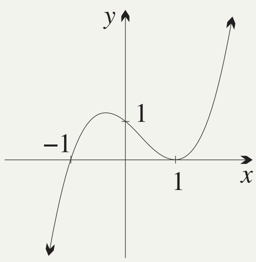
a
$$x^3 + 1 \le x^2 + x$$
$\boxed{-x^2 - x}$
$$x^3 - x^2 - x + 1 \le 0$$

b
$$\begin{aligned} y &= x^3 - x^2 - x + 1 \\ &= x^2(x-1) - (x-1) \\ &= (x^2 - 1)(x - 1) \\ &= (x+1)(x-1)^2 \end{aligned}$$

c The LHS has zeroes at 1 and -1, and no discontinuities.

x	-2	-1	0	1	2
y	-9	0	1	0	3
sign	$-$	0	$+$	0	$+$

d



e Rewrite the inequation as $x^3 - x^2 - x + 1 \le 0$. Then from the graph, the solution is $x \le -1$ or $x = 1$, or alternatively $(-\infty, -1] \cup [1, 1]$.

Solving inequations involving discontinuities

When the function has discontinuities, the method is the same, except that the test values now need to dodge around discontinuities as well as zeroes.

As always, a sketch is very useful because it shows the whole situation.

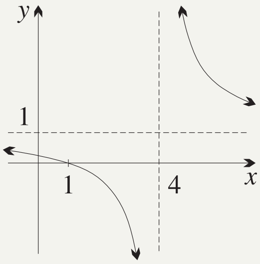
Example 4 2A

Examine the sign of $y = \dfrac{x-1}{x-4}$.

SOLUTION

There is a zero at $x = 1$, and a discontinuity at $x = 4$.

x	0	1	2	4	5
y	$\frac{1}{4}$	0	$-\frac{1}{2}$	$*$	4
sign	$+$	0	$-$	$*$	$+$



Hence y is positive for $x < 1$ or $x > 4$, and negative for $1 < x < 4$.
(This graph has a horizontal and a vertical asymptote, which will be discussed in Section 2B.)

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Example 5 2A

Examine the sign of $y = \dfrac{1}{1 + x^2}$.

SOLUTION

The function is always positive because $1 + x^2$ is always at least 1. There is thus no need to use a table of signs.

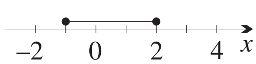
A table of signs can still be used, however. The function has no zeroes because the numerator is never zero, and has no discontinuities because the denominator is never zero. Hence one test value $f(0) = 1$ establishes that the function is always positive.

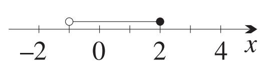
x	0
y	1
sign	$+$

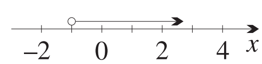
Exercise 2A FOUNDATION

1 For each number line, write the graphed interval using:

i inequality interval notation, **ii** bracket interval notation.

a 

b 

c 

2 For each interval given by inequality interval notation:

i draw the interval on a number line, **ii** write it using bracket interval notation.

a $-1 \le x < 2$ **b** $x \le 2$ **c** $x < 2$

3 For each interval given by bracket interval notation:

i draw the interval on a number line, **ii** write it using inequality interval notation.

a $[-1, \infty)$ **b** $(-1, 2)$ **c** $(-\infty, \infty)$

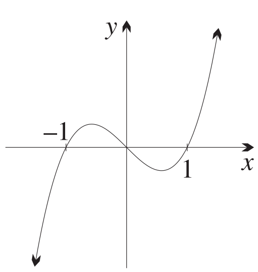
4 If $f(x) = 5x$ and $g(x) = 2^x$, find:

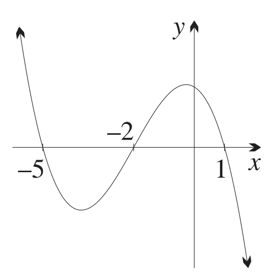
a **i** $g \circ f(3)$ **ii** $f \circ g(3)$ **iii** $g \circ g(3)$ **iv** $f \circ f(3)$

b **i** $g \circ f(x)$ **ii** $f \circ g(x)$ **iii** $g \circ g(x)$ **iv** $f \circ f(x)$

5 Write down the values of x for which each curve is:

i below the x-axis, **ii** above the x-axis.

a 

b 

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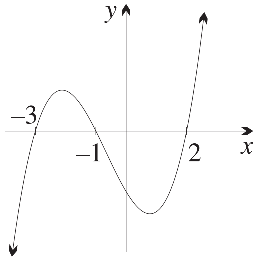
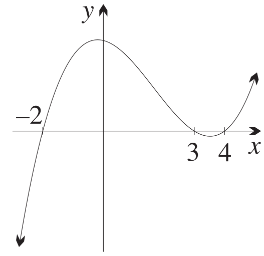
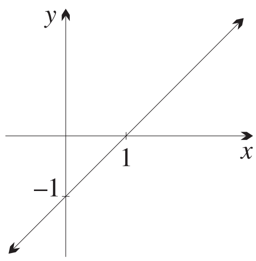
6 For each curve, write down the values of x for which:

i $y = 0$, ii y is positive, iii y is negative.

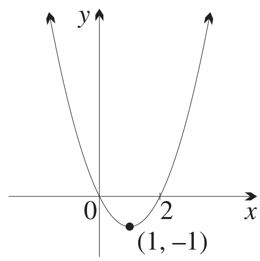
a


b


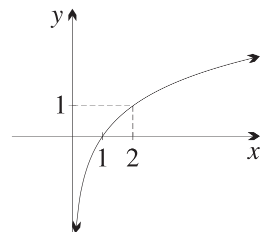
7 For each function graphed, use bracket interval notation to state where the function is negative. Also state whether each function is one-to-one or many-to-one.

a


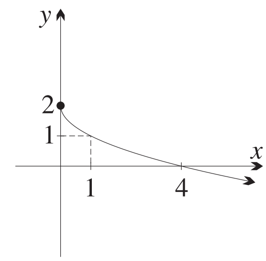
$f(x) = x - 1$

b


$f(x) = x^2 - 2x$

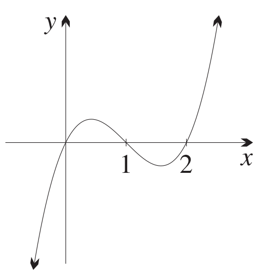
c


$f(x) = \log_2 x$

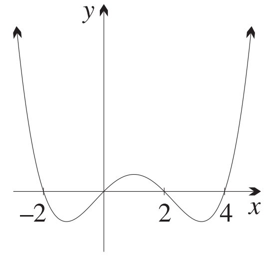
d


$f(x) = 2 - \sqrt{x}$

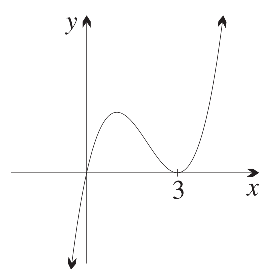
8 Use the given graph of the LHS to help solve each inequation (inequality interval notation).

a


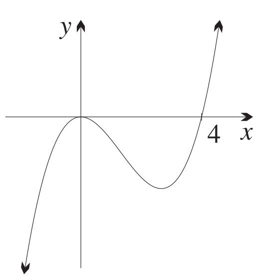
$x(x - 1)(x - 2) \le 0$

b


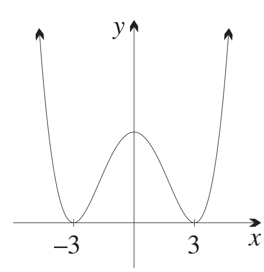
$x(x + 2)(x - 2)(x - 4) < 0$

c


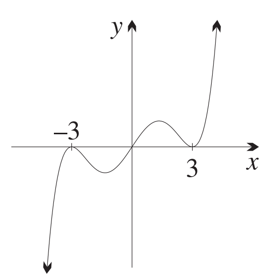
$x(x - 3)^2 > 0$

d


$x^2(x - 4) \ge 0$

e


$(x - 3)^2(x + 3)^2 \le 0$

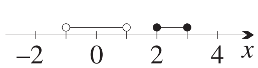
f


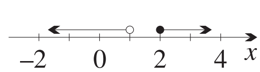
$x(x - 3)^2(x + 3)^2 \ge 0$

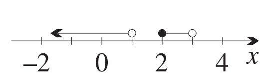
DEVELOPMENT

9 For each number line, write the graphed compound interval using:

i inequality interval notation, ii bracket interval notation.

a


b


c


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10 For each compound interval given by inequality interval notation:

i draw the number line graph, **ii** write it using bracket interval notation.

a $x = -1$ or $x \geq 2$ **b** $x \leq -1$ or $2 < x \leq 3$ **c** $-1 < x \leq 1$ or $x > 2$

11 For each compound interval given by bracket interval notation:

i draw the number line graph, **ii** write it using inequality interval notation.

a $[-1, 1] \cup [2, \infty)$ **b** $[-1, 1) \cup (2, 3]$ **c** $(-1, 1] \cup [3, 3]$

12 Re-write the solutions to Question 8 using bracket interval notation.

13 Show that the zeroes of $y = (x + 1)(x - 1)$ are $x = 1$ and $x = -1$. Then copy and complete the table of test values to examine the sign, and sketch the graph.

x	-2	-1	0	1	2
y					
sign					

14 Apply the methods used in the previous question to sketch these quadratics, cubics and quartics. Mark all x- and y-intercepts.

a $y = (x + 1)(x + 3)$ **b** $y = (x - 1)(3 - x)$ **c** $y = (1 - x)(x + 2)^2$

d $y = x(x - 2)(x + 2)$ **e** $y = (2 - x)x(x + 2)(x + 4)$ **f** $y = (x - 1)^2(x - 3)^2$

15 Use the methods of the Questions 13 and 14 to sketch each polynomial. You will first need to factor each polynomial completely. Begin by finding any common factors.

a $f(x) = x^3 - 4x$ **b** $f(x) = x^3 - 5x^2$ **c** $f(x) = x^3 - 4x^2 + 4x$

16 Solve these inequations from the graphs in the previous question, or from the tables of values used to construct them. Begin by getting all terms onto the one side.

a $x^3 > 4x$ **b** $x^3 < 5x^2$ **c** $x^3 + 4x \leq 4x^2$

17 Factor each equation completely, and hence find the x-intercepts of the graph. Factor parts **b** and **c** by grouping in pairs.

a $y = x^3 - x$ **b** $y = x^3 - 2x^2 - x + 2$ **c** $y = x^3 + 2x^2 - 4x - 8$

18 For each function in the previous question, examine the sign of the function around each zero and hence draw a graph of the function.

19 Find all zeroes of these functions, and any values of x where the function is discontinuous. Then analyse the sign of the function by taking test values around these zeroes and discontinuities.

a $f(x) = \dfrac{x}{x - 3}$ **b** $f(x) = \dfrac{x - 4}{x + 2}$ **c** $f(x) = \dfrac{x + 3}{x + 1}$

d $f(x) = \dfrac{1}{x^2 - 1}$ **e** $f(x) = \dfrac{x^2 - 4}{x}$ **f** $f(x) = \dfrac{x^2 - 4}{x^2 - 16}$

20 Prove that composition of functions is associative, that is,

$\big((f \circ g) \circ h\big)(x) = \big(f \circ (g \circ h)\big)(x)$, for all x where both sides are defined.

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CHALLENGE

21 If necessary, collect all terms on the LHS and factor. Then solve the inequation by finding any zeroes and discontinuities and drawing up a table of test values around them.

a $(x-1)(x-3)(x-5) < 0$

b $(x+3)(x-1)(x-4) \geq 0$

c $(x-1)^2(x-3)^2 > 0$

d $(x+2)x(x-2)(x-4) > 0$

e $x^3 > 9x$

f $x^4 \geq 5x^3$

22 a Consider the function $y = \dfrac{|x|}{x-1}$.

i Determine the natural domain,

ii Find any intercepts with the x- and y-axes,

iii Determine where the function is zero, positive or negative, using a table of signs, if necessary.

iv Use appropriate graphing software or applications to confirm your answers.

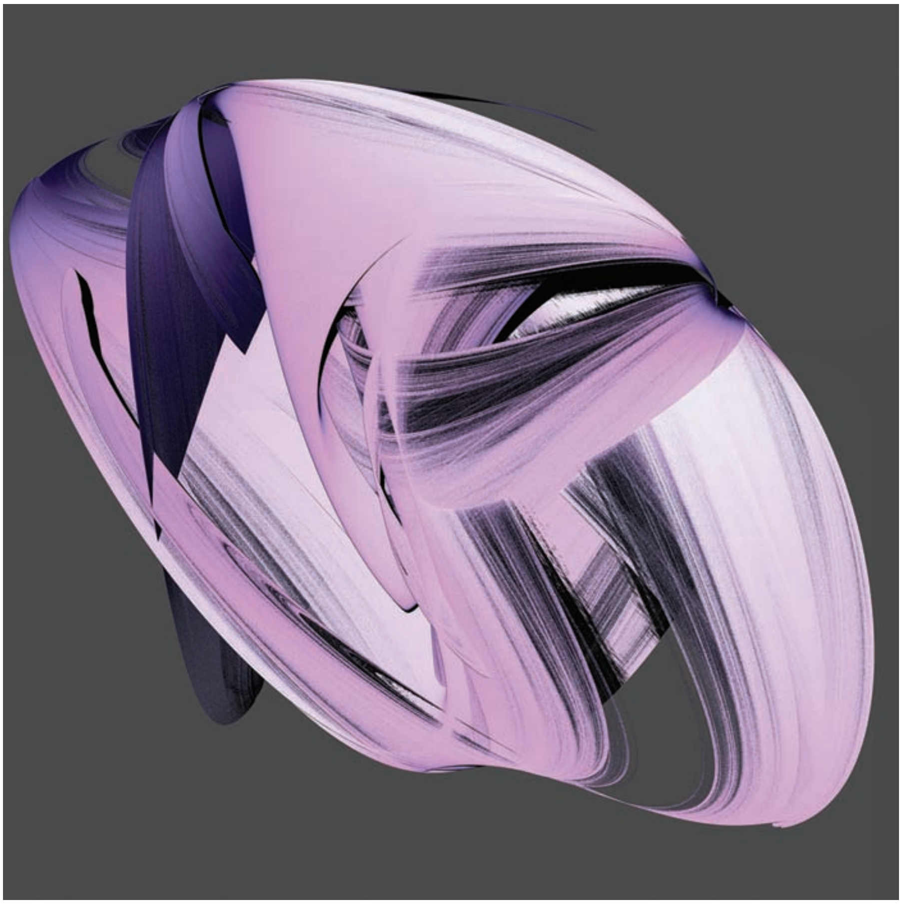
b Repeat the steps of part **a** for $y = \dfrac{|x|}{x^2-1}$.

23 An interval is called *closed* if it contains all its endpoints, and *open* if it doesn't contain any of its endpoints.

a Explain why the degenerate interval $[5]$ is closed.

b Explain why the interval $(-\infty, \infty)$ is closed.

c Explain why the same interval $(-\infty, \infty)$ is also open.



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2B Vertical and horizontal asymptotes

So far in this course we have discussed three steps in sketching an unknown function (leaving transformations aside for the moment). After factoring:

1. Identify the domain.
2. Test whether the function has even or odd symmetry or neither.
3. Identify the zeroes and discontinuities and draw up a table of signs.

This section introduces a fourth step:

4. Identify a curve's vertical and horizontal asymptotes.

This may also involve describing the curve's behaviour near them.

Asymptotes

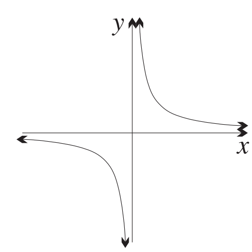
Asymptotes occur naturally with algebraic fractions such as

$$y = \frac{2}{3 - x} \qquad \text{or} \qquad y = \frac{1}{4 - x^2} \qquad \text{or} \qquad y = \frac{x - 1}{x - 4}.$$

In the functions above, the denominator is very small when x is near 3 for the first function, or near 2 or -2 for the second function, or near 4 for the third function. In all three functions, the denominator is very large when x is very large. There are two straightforward principles to keep in mind:

- The reciprocal of a very small number is a very large number.
  The reciprocal of a very large number is a very small number.
- The reciprocal of a positive number is positive.
  The reciprocal of a negative number is negative.

All this is well demonstrated by the rectangular hyperbola $y = \dfrac{1}{x}$, which has a vertical and a horizontal asymptote, as discussed in Section 3H of the Year 11 book.



Horizontally, when x is a very large number, positive or negative, y is a very small number with the same sign, so the curve approaches the x-axis on the left and on the right. We described this situation earlier by the statement:

'As $x \to \infty$, $y \to 0$, and as $x \to -\infty$, $y \to 0$.'

Vertically, when x is a very small number, positive or negative, y is a very large number with the same sign, so the curve flies off to ∞ or to $-\infty$ near $x = 0$. We described this situation earlier by the statement:

x	-1	0	1
y	-1	*	1
sign	$-$	*	$+$

'As $x \to 0^-$, $y \to -\infty$, and as $x \to 0^+$, $y = +\infty$.'

'As x approaches 0 from the left, . . . , and as x approaches 0 from the right, . . .'.

In more difficult situations than this, use a table of signs, as introduced in the previous section, to check the sign and distinguish between $y \to \infty$ and $y \to -\infty$.

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4 TESTING FOR VERTICAL ASYMPTOTES

Always factor the function first as far as possible.

- If the denominator has a zero at $x = a$, and the numerator is *not* zero at $x = a$, then the vertical line $x = a$ is an asymptote.
- The choice between $y \to \infty$ and $y \to -\infty$ can be made by looking at a table of signs.

Once the vertical asymptote has been identified, the behaviour of the curve near it can then be described using the notation $x \to a^+$ and $x \to a^-$.

Example 6 2B

a Find the vertical asymptote of the function $y = \dfrac{2}{3 - x}$.

b Construct a table of signs, and describe the behaviour of the curve near it.

c Name the horizontal asymptote, and describe the behaviour of y for large x.

d Sketch the curve.

SOLUTION

a When $x = 3$, the denominator vanishes, but the numerator does not, so $x = 3$ is an asymptote.

b There are no zeroes, and there is a discontinuity at $x = 3$.

x	0	3	4
y	$\frac{2}{3}$	*	-2
sign	$+$	*	$-$

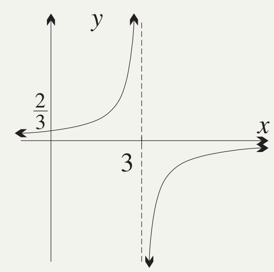
From the table of signs:

As $x \to 3^+$, $y \to -\infty$,
and as $x \to 3^-$, $y \to +\infty$.

c The x-axis is a horizontal asymptote:

As $x \to \infty$, $y \to 0$, and as $x \to -\infty$, $y \to 0$.

d



Example 7 2B

a Test whether the function $f(x) = \dfrac{1}{4 - x^2}$ is even or odd.

b Find any vertical asymptotes of the function $y = \dfrac{1}{4 - x^2}$.

c Construct a table of signs, and describe the behaviour of the curve near it.

d Name the horizontal asymptote, and describe the behaviour of y for large x.

e Sketch the curve.

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SOLUTION

a $f(-x) = \dfrac{1}{4 - (-x)^2}$
$= f(x),$

so the function is even.

b Factoring, $y = \dfrac{1}{(2 - x)(2 + x)}$.

When $x = 2$ or $x = -2$, the denominator vanishes, but the numerator does not, so $x = 2$ and $x = -2$ are asymptotes.

c There are no zeroes, and there are discontinuities at $x = 2$ and $x = -2$.

From the table of signs:

As $x \to (-2)^-$, $y \to -\infty$, and as $x \to (-2)^+$, $y \to +\infty$.

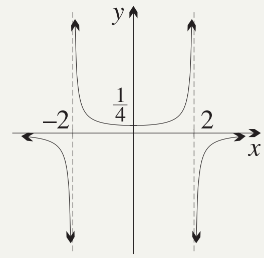
As $x \to 2^-$, $y \to +\infty$, and as $x \to 2^+$, $y \to -\infty$.

x	-3	-2	0	2	3
y	$-\frac{1}{5}$	$*$	$\frac{1}{4}$	$*$	$-\frac{1}{5}$
sign	$-$	$*$	$+$	$*$	$-$

d The x-axis is a horizontal asymptote:

As $x \to \infty$, $y \to 0$, and as $x \to -\infty$, $y \to 0$.

e



Horizontal asymptotes, and the behaviour as $x \to \infty$ and as $x \to -\infty$

It was very straightforward in the examples above to see that the x-axis is an asymptote to each curve. But it is not so straightforward to find the horizontal asymptotes, if indeed they exist, for curves such as

$$y = \frac{x - 1}{x - 4} \quad \text{or} \quad y = \frac{x - 1}{x^2 - 4}.$$

Such curves are called *rational functions* because they are the ratio of two polynomials. For rational functions, *dividing top and bottom by the highest power of x in the denominator* makes the situation clear.

5 BEHAVIOUR FOR LARGE x

- Divide top and bottom by the highest power of x in the denominator.
- Then use the fact that $\dfrac{1}{x} \to 0$ as $x \to \infty$ and as $x \to -\infty$.
- If $f(x)$ tends to a definite limit b as $x \to \infty$ or as $x \to -\infty$, then the horizontal line $y = b$ is a horizontal asymptote on the right or on the left.

The next worked example uses these methods to find the asymptotes of the function sketched in Section 2A (worked Example 4).

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Example 8 2B

a Examine the behaviour of the function $y = \dfrac{x-1}{x-4}$ as $x \to \infty$ and as $x \to -\infty$, noting any horizontal asymptotes.

b Find any vertical asymptotes of the function $y = \dfrac{x-1}{x-4}$. Then use a table of signs to discuss the behaviour of the curve near them.

c Sketch the curve.

SOLUTION

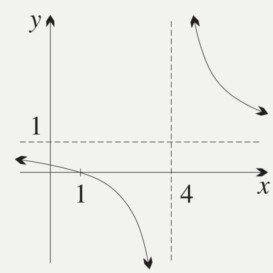
a Dividing top and bottom by x gives

$$y = \frac{1 - \frac{1}{x}}{1 - \frac{4}{x}},$$

so $y \to \dfrac{1-0}{1-0} = 1$ as $x \to \infty$ and as $x \to -\infty$.

Hence $y = 1$ is a horizontal asymptote.

c



b When $x = 4$, the denominator vanishes, but the numerator does not, so $x = 4$ is an asymptote.

From the table of signs to the right, dodging around the zero at $x = 1$ and the discontinuity at $x = 4$:

As $x \to 4^-$, $y \to -\infty$, and as $x \to 4^+$, $y \to +\infty$,

x	0	1	2	4	5
y	$\frac{1}{4}$	0	$-\frac{1}{2}$	*	4
sign	+	0	−	*	+

Example 9 2B

Examine the behaviour of these functions as $x \to \infty$ and as $x \to -\infty$, noting any horizontal asymptotes.

a $y = \dfrac{x-1}{x^2-4}$

b $y = \dfrac{x^2-1}{x-4}$

c $y = \dfrac{3-5x-4x^2}{4-5x-3x^2}$

d $y = 2^x + \dfrac{1}{x^2+1} + 3$

SOLUTION

a Dividing top and bottom by x^2, $y = \dfrac{\frac{1}{x} - \frac{1}{x^2}}{1 - \frac{4}{x^2}}$.

Hence as $x \to \infty$, $y \to 0$, and as $x \to -\infty$, $y \to 0$, and the x-axis $y = 0$ is a horizontal asymptote.

b Dividing top and bottom by x, $y = \dfrac{x - \frac{1}{x}}{1 - \frac{4}{x}}$.

Hence as $x \to \infty$, $y \to \infty$, and as $x \to -\infty$, $y \to -\infty$, and there are no horizontal asymptotes.

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c $y = \dfrac{3 - 5x - 4x^2}{4 - 5x - 3x^2}$

Dividing top and bottom by x^2, $y = \dfrac{\frac{3}{x^2} - \frac{5}{x} - 4}{\frac{4}{x^2} - \frac{5}{x} - 3}$.

Hence as $x \to \infty$, $y \to \frac{4}{3}$, and as $x \to -\infty$, $y \to \frac{4}{3}$, and $y = \frac{4}{3}$ is a horizontal asymptote.

d $y = 2^x + \dfrac{1}{x^2 + 1} + 3$

As $x \to \infty$, $y \to \infty$, and as $x \to -\infty$, $y \to 3$.

Hence $y = 3$ is a horizontal asymptote on the left.

Exercise 2B

FOUNDATION

Note: Later parts of some questions here and in Exercise 2C use calculus to identify details such as stationary points. These are not essential for the basic sketch of the curve, and readers may like to leave those parts until after a review of calculus in Chapter 3.

1 **a** Sketch $y = \dfrac{1}{x - 1}$ after carrying out the following steps:

- **i** State the natural domain.
- **ii** Find the y-intercept.
- **iii** Explain why $y = 0$ is a horizontal asymptote.
- **iv** Draw up a table of test values to examine the sign.
- **v** Identify any vertical asymptotes, and use the table of signs to write down its behaviour near any vertical asymptotes.

b Repeat the steps of part **a** to sketch $y = \dfrac{2}{3 - x}$.

c Likewise sketch $y = -\dfrac{2}{x + 2}$.

d Now sketch $y = \dfrac{5}{2x + 5}$.

2 Follow the steps of Question 1 to investigate the function $y = \dfrac{2}{(x - 1)^2}$ and hence sketch it.

3 Investigate the domain, intercepts, sign and asymptotes of the function $y = -\dfrac{1}{(x - 2)^2}$ and hence sketch its graph.

4 Find the horizontal asymptotes of these functions by dividing through by the highest power of x in the denominator and taking the limit as $x \to \infty$ and as $x \to -\infty$.

a $f(x) = \dfrac{1}{x - 2}$ **b** $f(x) = \dfrac{x - 3}{x + 4}$ **c** $f(x) = \dfrac{2x + 1}{3 - x}$

d $f(x) = \dfrac{5 - x}{4 - 2x}$ **e** $\dfrac{1}{x^2 + 1}$ **f** $\dfrac{x}{x^2 + 4}$

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DEVELOPMENT

5 Let $y = \dfrac{3}{(x + 1)(x - 3)}$.

a State the natural domain.
b Find the y-intercept.
c Show that $y = 0$ is a horizontal asymptote.
d Draw up a table of test values to examine the sign.
e Identify the vertical asymptotes, and use the table of signs to write down its behaviour near them.
f Sketch the graph of the function and state its range.

6 **a** Follow steps similar to those in Question 5 in order to sketch $y = \dfrac{4}{4 - x^2}$.
b What is the range of $y = \dfrac{4}{4 - x^2}$?

7 Let $y = \dfrac{2}{x^2 + 1}$.

a Determine the horizontal asymptote.
b Explain why there are no vertical asymptotes.
c Show that the tangent to the curve is horizontal at the y-intercept.
d Sketch $y = \dfrac{2}{x^2 + 1}$. Use a table of signs if needed.
e What is the range of the function?
f Is this function one-to-one or many-to-one?

8 Sketch the function $y = \dfrac{x - 1}{x + 1}$ by carrying out the following steps.

a State the natural domain.
b Find the y-intercept.
c Determine the horizontal asymptote.
d Investigate the vertical asymptote.
e Sketch the function.
f What is its range?
g Is this function one-to-one or many-to-one?

9 Follow the steps of Question 8 to sketch these graphs.

a $y = \dfrac{x}{x + 2}$ **b** $y = \dfrac{x + 1}{x - 2}$ **c** $y = \dfrac{2x - 1}{x + 1}$

10 Consider the function $y = \dfrac{3x}{x^2 + 1}$.

a Show that it is an odd function.
b Show that it has only one intercept with the axes at the origin.
c Show that the x-axis is a horizontal asymptote.
d Hence sketch the curve.

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11 **a** Show that $y = \dfrac{4 - x^2}{4 + x^2}$ is even.

b Find its three intercepts with the axes.

c Determine the equation of the horizontal asymptote.

d Sketch the curve.

12 **a** Consider the function $y = \dfrac{x^2 + 5x + 6}{x^2 - 4x + 3}$.

i Determine the horizontal asymptote.

ii Factor the numerator and denominator.

iii Hence determine any vertical asymptotes.

b Follow similar steps to find the asymptotes of these rational functions.

i $y = \dfrac{x^2 - 2x + 1}{x^2 + 5x + 4}$ **ii** $y = \dfrac{x - 5}{x^2 + 3x - 10}$ **iii** $y = \dfrac{1 - 4x^2}{1 - 9x^2}$

(Computer sketches of these curves may be useful to put these features in context.)

13 **a** **i** Sketch $y = \cos x$ for $0 \le x \le 2\pi$, showing the points where $y = -1, 0,$ or 1.

ii Where are the vertical asymptotes of $\sec x = \dfrac{1}{\cos x}$ in this domain?

iii Use the graph of $y = \cos x$ to determine the behaviour of $y = \sec x$ at each of these asymptotes.

iv When $\cos x$ is positive, $0 < \cos x \le 1$. Use this result to show that $\sec x \ge 1$ for these values of x. Where is $\sec x = 1$?

v What happens when $\cos x$ is negative?

vi Hence sketch $y = \sec x$ on the same number plane.

b Likewise sketch $y = \sin x$ and $y = \operatorname{cosec} x$ for $0 \le x \le 2\pi$.

14 Consider the function $f(x) = \dfrac{x}{x^2 - 4}$.

a Show that the function is odd. What symmetry does its graph have?

b State the domain of the function and the equations of any vertical asymptotes.

c Use a table of test values of $f(x)$ to analyse the sign of the function.

d What value does $f(x)$ approach as x becomes large? Hence write down the equation of the horizontal asymptote.

e Use the quotient rule to show that the derivative of $f(x)$ is $f'(x) = -\dfrac{x^2 + 4}{(x^2 - 4)^2}$.

f Explain why the curve $y = f(x)$ has no tangent that is horizontal, and why the curve is always decreasing.

g Sketch the graph of $y = f(x)$, showing all important features.

h Use the graph to state the range of the function.

15 Consider the function $y = x + \dfrac{1}{x}$.

a Show that the function is odd. What symmetry does its graph have?

b State the domain of the function and the equation of the vertical asymptote.

c Use a table of test values of y to analyse the sign of the function.

d Show that the derivative is $y' = \dfrac{x^2 - 1}{x^2}$.

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e Find any points where the tangent is horizontal.

f Sketch the graph of the function. (You may assume that the diagonal line $y = x$ is an asymptote to the curve. This is because for large x, the term $\frac{1}{x}$ becomes very small.)

g Write down the range of the function.

16 **a** Factor the denominator of $y = \frac{x - 1}{x^2 - 1}$ and hence explain why there is no vertical asymptote at $x = 1$.

b Sketch the function, carefully noting the domain.

CHALLENGE

17 **a** Notice that $(x - 1) = (x + 1) - 2$. Use this result to show that the function $y = \frac{x - 1}{x + 1}$ in Question 8 can be written as $y = 1 - \frac{2}{x + 1}$.

b Hence confirm the horizontal asymptote found in Question 8.

c Likewise, find the horizontal asymptotes of the functions in Question 9 by re-writing each numerator.

18 In each question above, the horizontal asymptote was the same on the left as $x \to -\infty$, and on the right as $x \to \infty$. This is not always the case. Consider the function $y = \frac{1 - e^x}{1 + e^x}$.

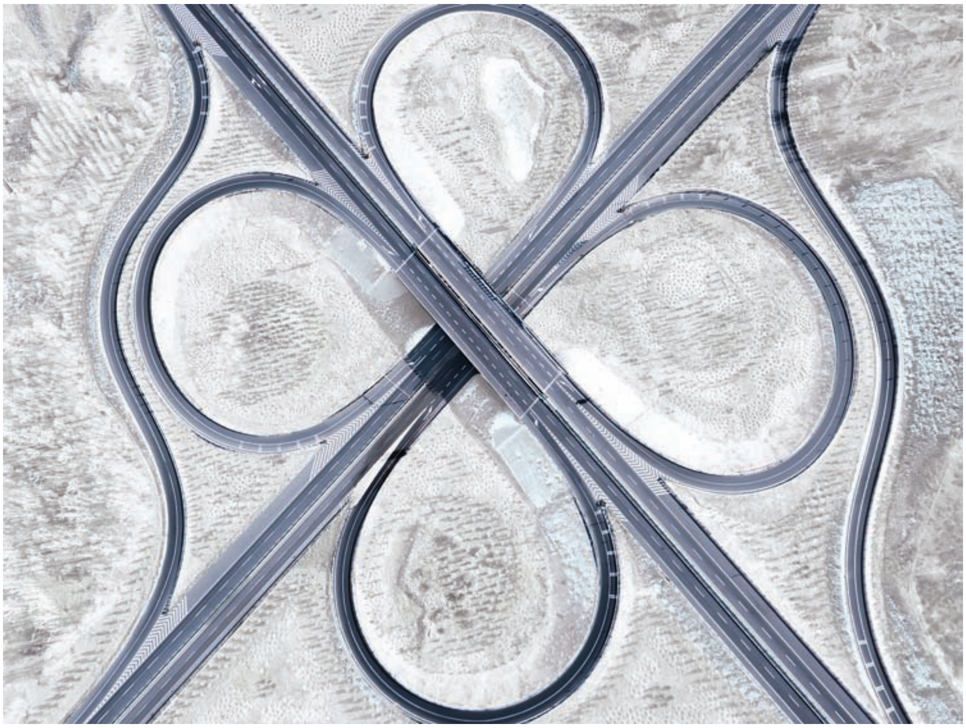
a Find $\lim\limits_{x \to -\infty} y$.

b Multiply the numerator and denominator by e^{-x}, then find $\lim\limits_{x \to \infty} y$.

c Determine any intercepts with the axes.

d Using no other information, sketch this curve.

e Test algebraically whether the curve is even, odd or neither.



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2C A curve-sketching menu

We can now combine four approaches to curve-sketching into an informal four-step approach for sketching an unknown graph. This simple menu cannot possibly deal with every possible graph. Nevertheless, it will allow the main features of a surprising number of functions to be found. Two further steps involving calculus will be added in Chapter 3.

A 'sketch' of a graph is not an accurate plot. It is a neat diagram showing the main features of the curve.

6 SKETCHES

- A sketch should show any x- and y-intercepts if they are accessible, any vertical or horizontal asymptotes, and any other significant points on the curve.
- There should always be some indication of scale on both axes, and both axes should be labelled.

A curve-sketching menu

Here is our informal four-step approach to sketching an unknown function.

7 A CURVE SKETCHING MENU

0 **Preparation:** Combine any fractions using a common denominator, then factor top and bottom as far as possible.
1 **Domain:** What is the domain? (*Always* do this first.)
2 **Symmetry:** Is the function odd, or even, or neither?
3 **A** **Intercepts:** What are the y-intercept and the x-intercepts (zeroes)?
B **Sign:** Where is the function positive, and where is it negative?
4 **A** **Vertical asymptotes:** Examine the behaviour near any discontinuities, noting any vertical asymptotes.
B **Horizontal asymptotes:** Examine the behaviour of $f(x)$ as $x \to \infty$ and as $x \to -\infty$, noting any horizontal asymptotes.

Finding the domain and finding the zeroes may both require factoring, which is the reason why the preparatory Step 0 is useful. Factoring, however, may not always be possible, even with the formula for the roots of a quadratic, and in such cases approximation methods may be useful.

Questions will normally give guidance as to what is required. Our menu is not an explicit part of the course, but rather a suggested way to organise the approaches presented in the course.

Putting it all together — the first example

All that remains is to give two examples of the whole process.

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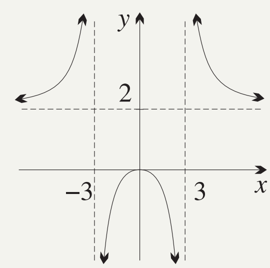
Example 10 2C

Apply the steps in Box 7 to sketch $f(x) = \dfrac{2x^2}{x^2 - 9}$.

SOLUTION



0 Preparation: $f(x) = \dfrac{2x^2}{(x - 3)(x + 3)}$.

1 Domain: $x \neq 3$ and $x \neq -3$.

2 Symmetry:
$$\begin{aligned} f(-x) &= \frac{2(-x)^2}{(-x)^2 - 9} \\ &= \frac{2x^2}{x^2 - 9} \\ &= f(x). \end{aligned}$$

so $f(x)$ is even, with line symmetry in the y-axis.

3 Intercepts and sign: When $x = 0$, $y = 0$.
There is a zero at $x = 0$, and discontinuities at $x = 3$ and $x = -3$.

x	-4	-3	-1	0	1	3	4
$f(x)$	$\frac{32}{7}$	$*$	$-\frac{1}{4}$	0	$-\frac{1}{4}$	$*$	$\frac{32}{7}$
sign	$+$	$*$	$-$	0	$-$	$*$	$+$

4 Vertical asymptotes: At $x = 3$ and $x = -3$, the denominator vanishes, but the numerator does not, so $x = 3$ and $x = -3$ are vertical asymptotes. To make this more precise, it follows from the table of signs that

$$\begin{aligned} f(x) &\to \infty \text{ as } x \to 3^+ \quad \text{and} \quad f(x) \to -\infty \text{ as } x \to 3^-, \\ f(x) &\to -\infty \text{ as } x \to (-3)^+ \quad \text{and} \quad f(x) \to \infty \text{ as } x \to (-3)^-. \end{aligned}$$

Horizontal asymptotes: Dividing through by x^2, $f(x) = \dfrac{2}{1 - \frac{9}{x^2}}$,

so $f(x) \to 2$ as $x \to \infty$ and as $x \to -\infty$,
and $y = 2$ is a horizontal asymptote.

Putting it all together — the second example

The second example requires a common denominator. The calculations involving intercepts and sign have been done with an alternative approach using signs rather than numbers.

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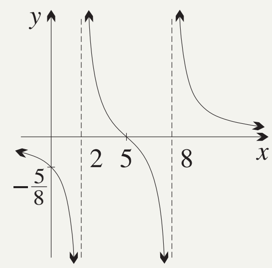
Example 11 2C

Apply the steps in Box 7 above to sketch $f(x) = \dfrac{1}{x-2} + \dfrac{1}{x-8}$.

SOLUTION



0 $f(x) = \dfrac{(x-8)+(x-2)}{(x-2)(x-8)}$

$= \dfrac{2x-10}{(x-2)(x-8)}$

$= \dfrac{2(x-5)}{(x-2)(x-8)}$.

1 The domain is $x \neq 2$ and $x \neq 8$.

2 $f(x)$ is neither even nor odd.

3 When $x = 0$, $y = -\frac{5}{8}$.
There is a zero at $x = 5$, and discontinuities at $x = 2$ and $x = 8$.

x	0	2	3	5	7	8	10
$x-2$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-5$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-8$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f(x)$	$-$	*	$+$	0	$-$	*	$+$

If only the signs are calculated, at least these three lines of working should be shown.

4 At $x = 2$ and $x = 8$ the denominator vanishes, but the numerator does not, so $x = 2$ and $x = 8$ are vertical asymptotes.
From the original form of the given equation, $f(x) \to 0$ as $x \to \infty$ and as $x \to -\infty$, so $y = 0$ is a horizontal asymptote.

Exercise 2C FOUNDATION

Note: As remarked in the previous exercise, later parts of some questions use calculus, but these details are not essential for the basic sketch of the curve. Readers may like to leave those parts until after a review of calculus in Chapter 3.

1 Complete the following steps in order to sketch the graph of $y = \dfrac{9}{x^2 - 9}$.

- **a** Factor the function as far as possible.
- **b** State the domain using bracket interval notation.
- **c** Show that the function is even. What symmetry does the graph have?
- **d** Write down the coordinates of any intercepts with the axes.
- **e** Investigate the sign of the function using a table of test values. Where is $y \leq 0$?
- **f** Write down the equation of any vertical asymptote.

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g What value does y approach as $x \to \infty$ and as $x \to -\infty$? Hence write down the equation of the horizontal asymptote.

h Sketch the graph of the function showing these features.

i The graph appears to be horizontal at its y-intercept. Use the fact that $y' = \dfrac{-18x}{(x^2 - 9)^2}$ (which you may want to prove) to confirm that the graph is horizontal there.

2 Complete the following steps in order to sketch the graph of $y = \dfrac{x}{4 - x^2}$.

a Factor the function as far as possible.

b State the domain using bracket interval notation.

c Show that the function is odd. What symmetry does the graph have?

d Write down the coordinates of any intercepts with the axes.

e Investigate the sign of the function using a table of test values. Where is $y \geq 0$?

f Write down the equation of any vertical asymptote.

g What value does y approach as $x \to \infty$ and as $x \to -\infty$? Hence write down the equation of the horizontal asymptote.

h Sketch the graph of the function showing these features.

i Use the fact that $y' = \dfrac{(x^2 + 4)}{(4 - x^2)^2}$ (which you may want to prove using the quotient rule) to explain why the graph always has a positive gradient.

3 Let $y = x^3 - 4x$.

a Factor this function.

b State the domain using bracket interval notation.

c Write down the coordinates of any intercepts with the axes.

d Show that the function is odd. What symmetry does the graph have?

e Does this function have any asymptotes?

f Use this information and a table of signs to sketch the curve.

g The graph seems to have a peak somewhere in the interval $-2 < x < 0$ and a trough in the interval $0 < x < 2$. These are stationary points, where $y' = 0$. Show that $y' = 3x^2 - 4$, then solve $y' = 0$ to find the x-coordinates of these points (they will involve surds) and add them to the diagram.

DEVELOPMENT

4 Follow these steps to graph $y = f(x)$, where $f(x) = \dfrac{1}{x - 1} + \dfrac{1}{x - 4}$.

a Combine the two fractions using a common denominator, then factor the numerator and denominator as far as possible.

b State the domain.

c Explain why the function is neither even nor odd (the answers to part **a** may help).

d Write down the coordinates of any intercepts with the axes.

e Investigate the sign of the function using a table of test values. Where is $y > 0$?

f Write down the equation of any vertical asymptote.

g What value does $f(x)$ approach as $x \to \infty$ and as $x \to -\infty$? Hence write down the equation of the horizontal asymptote.

h Sketch the graph of the function showing these features.

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5 **a** Factor the numerator and denominator of $y = \dfrac{3x - 3}{x^2 - 2x - 3}$.

b State the domain and any intercepts with the axes.

c Explain why the function is neither even nor odd (the answers to part **a** may help).

d Write down the equations of the asymptotes.

e Sketch the graph of this curve.

6 **a** Factor the right-hand side of $y = \dfrac{x^2 + 2x + 1}{x^2 + 2x - 3}$.

b State the domain and any intercepts with the axes.

c Explain why the function is neither even nor odd.

d Use a table of signs to explain why the x-axis is tangent to the graph at the x-intercept.

e Write down the equations of the asymptotes.

f Sketch the graph of this curve.

g What is the range of this function?

7 Let $f(x) = \dfrac{x^2 - 4}{x^2 - 4x}$. You may assume that $f(x)$ is neither even nor odd.

a Factor $f(x)$.

b State the domain of $f(x)$ and write down the intercepts of $y = f(x)$.

c Write down the equations of the asymptotes.

d Sketch the graph of $y = f(x)$.

e What is the range of this function?

f The graph crosses its horizontal asymptote in the interval $0 < x < 4$. Find the coordinates of this point and add it to the graph.

8 Follow the curve-sketching menu to sketch the graph of $y = \dfrac{2}{x + 1} - \dfrac{1}{x}$. Include on your sketch the x-coordinates of any points where the tangent to the curve is horizontal.

9 Let $y = x^3 - 6x^2 + 8x$.

a State the domain using inequality interval notation.

b Write down the coordinates of any intercepts with the axes.

c Does this function have any asymptotes?

d Use this information and a table of signs to sketch the curve.

e The graph seems to be horizontal somewhere in the interval $0 < x < 2$, and again in the interval $2 < x < 4$. Find y' and solve $y' = 0$ using the quadratic formula to find the x-coordinates of these stationary points, then add them to the diagram.

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10 The curve $y = e^{-\frac{1}{2}x^2}$ is essential in statistics because it is related to the normal distribution, studied later in this course.

a What is the domain of this function?

b Determine whether the function is even or odd (or neither).

c Find any intercepts with the axes.

d Investigate any asymptotes.

e By considering the maximum value of $-x^2$, find the highest point on this curve.

f Sketch the curve, and hence state its range.

CHALLENGE

11 We can apply calculus to the function $y = e^{-\frac{1}{2}x^2}$ drawn in Question 10.

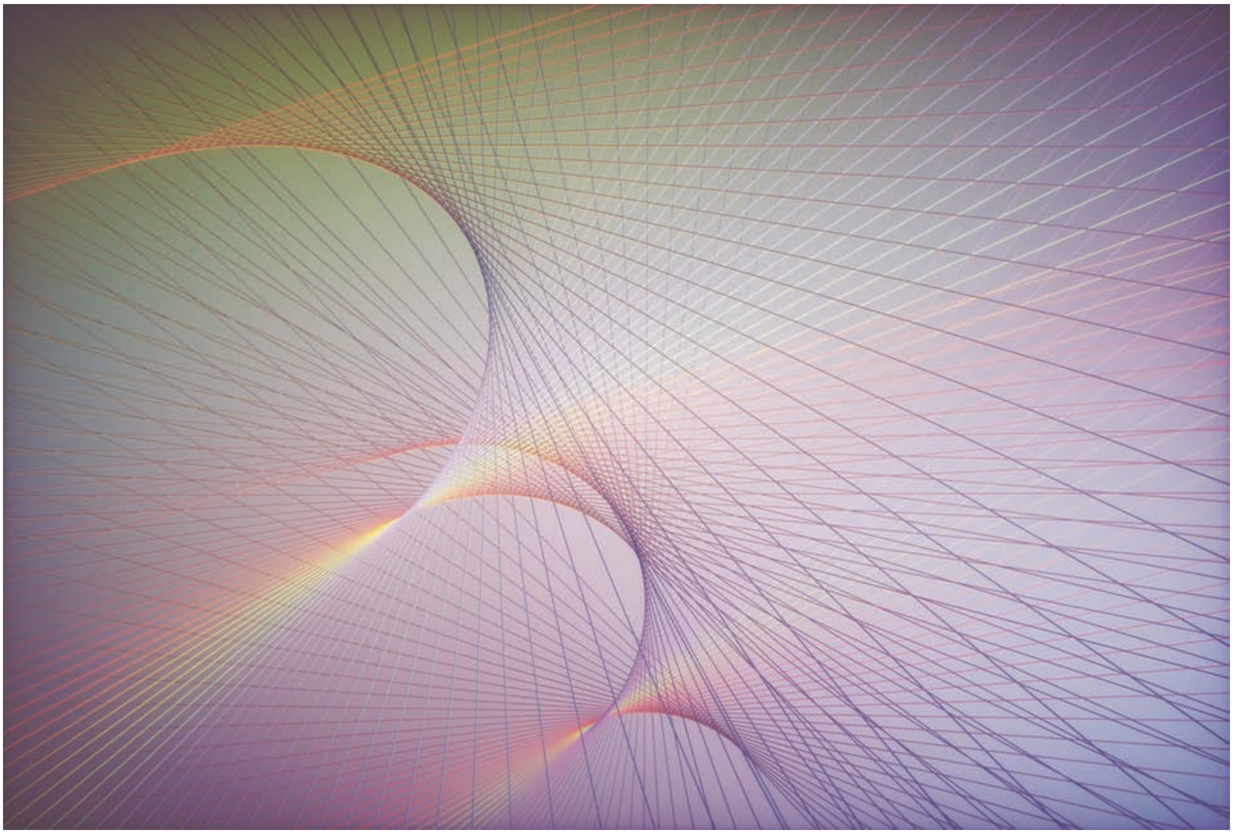
a Use the chain rule to prove that $y' = -xe^{-\frac{1}{2}x^2}$.

b Hence confirm that the tangent is horizontal at its y-intercept.

c Which is higher, $y = e^{-\frac{1}{2}x^2}$ or $y = 2^{-\frac{1}{2}x^2}$?

12 The graph of $y = e^{-\frac{1}{2}x^2}$ drawn in Question 10 appears to be steepest at $x = -1$ where it has positive gradient, and at $x = 1$ where it has negative gradient. In order to confirm this, the range of $y' = -xe^{-\frac{1}{2}x^2}$ is needed.

Use the curve sketching menu to sketch the graph of $f(x) = -xe^{-\frac{1}{2}x^2}$. Include on the sketch the x-coordinates of any points where the tangent is horizontal. Hence prove the result.



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2D Solving inequations

The equation $3x = 12$ has solution $x = 4$. When the equals sign is replaced by <, or by ≤, or by >, or by ≥, the result is an *inequation*, and we say,

'The inequation $3x > 12$ has solution $x > 4$',

because substituting any number greater than 4 makes the statement true, and substituting any other number makes the statement false. In this section and the next, various inequations are solved using algebraic and graphical methods.

An *inequality* is an inequation such as $x^2 \ge 0$ that is true for all values of x. Inequalities are thus similar to *identities*, which are equations that are true for all values of x, such as $(x + 3)^2 = x^2 + 6x + 9$.

There are, however, different conventions about the words 'inequation' and 'inequality'. Often the word 'inequation' is not used at all, and the word 'inequality' is used for both objects. Don't be alarmed if you are asked to 'solve an inequality'.

The meaning of 'less than'

There are a geometric and an algebraic interpretation of the phrase 'less than'. Suppose that a and b are real numbers.

8 THE MEANING OF $a < b$

The geometric interpretation
We say that $a < b$ if a is to the left of b on the number line:

The algebraic interpretation
We say that $a < b$ if $b - a$ is positive.

The first interpretation is geometric, relying on the idea of a 'line' and of one point being 'on the left-hand side of' another. The second interpretation requires that the term 'positive number' be already understood. Both interpretations are useful when solving inequations.

Solving linear inequations

As reviewed in Chapter 1 of the Year 11 book, the rules for adding and subtracting from both sides, and for multiplying or dividing both sides, are exactly the same as for equations, with one qualification — the inequality symbol reverses when multiplying or dividing by a negative.

9 SOLVING A LINEAR INEQUATION

Use the same methods as for linear equations, except that:

- When multiplying or dividing both sides by a negative number, the inequality symbol is reversed.
- As with equations, never multiply or divide by 0.

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Example 12 2D

Solve these inequations:

a $3x - 7 \le 8x + 18$ **b** $20 > 2 - 3x \ge 8$

SOLUTION

a
$$3x - 7 \le 8x + 18$$
$\boxed{+(-8x + 7)}$ $-5x \le 25$

$\boxed{\div(-5)}$ $x \ge -5$

b
$$20 > 2 - 3x \ge 8$$
$\boxed{-2}$ $18 > -3x \ge 6$

$\boxed{\div(-3)}$ $-6 < x \le -2$

Solving quadratic inequations

The most straightforward way to solve a quadratic inequation is to sketch the graph of the associated parabola.

10 SOLVING A QUADRATIC INEQUATION

- Move everything to the left-hand side.
- Sketch the graph of $y =$ LHS, showing the x-intercepts.
- Read the solution off the graph.

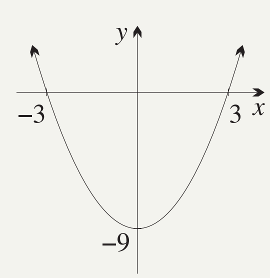
Example 13 2D

Solve each inequation by constructing a function and sketching it.

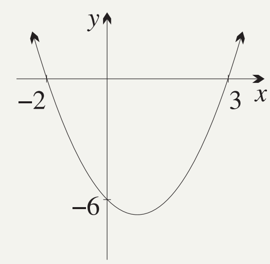
a $x^2 > 9$ **b** $x + 6 \ge x^2$

SOLUTION

a Moving everything onto the left, $x^2 - 9 > 0$,
then factoring, $(x - 3)(x + 3) > 0$.
We now sketch the graph of $y = (x - 3)(x + 3)$,
and examine where the graph is above the x-axis.
Thus the values of x for which $y > 0$ are
$x > 3$ or $x < -3$.
(This example $x^2 > 9$ is easy, and can be done without working.)



b Moving everything onto the left, $x^2 - x - 6 \le 0$,
then factoring, $(x - 3)(x + 2) \le 0$.
We now sketch the graph of $y = (x - 3)(x + 2)$,
and examine where the graph is below or on the x-axis.
Thus the values of x for which $y \le 0$ are
$-2 \le x \le 3$.



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Solving absolute value equations and inequations on the number line

Most equations and inequations involving absolute values in this course are simple enough to be solved using distances on the number line.

11 SOLVING SIMPLE ABSOLUTE VALUE EQUATIONS AND INEQUATIONS

1 Force the equation or inequation into one of the following forms:

$$|x - b| = a, \quad \text{or} \quad |x - b| < a, \quad \text{or} \quad |x - b| \geq a, \quad \text{or} \ldots.$$

2 Find the solution using distance on a number line.

Example 14 2D

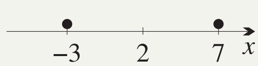
Solve these equations and inequations on the number line:

a $|x - 2| = 5$ **b** $|x + 3| = 4$ **c** $|3x + 7| < 3$ **d** $|7 - \frac{1}{4}x| \geq 3$

SOLUTION

a

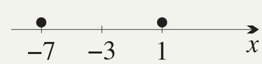
$$|x - 2| = 5$$
$$(\text{distance from } x \text{ to } 2) = 5$$



so $x = -3$ or $x = 7$.

b

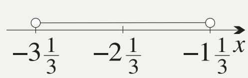
$$|x + 3| = 4$$
$$(\text{distance from } x \text{ to } -3) = 4$$



so $x = -7$ or $x = 1$.

c

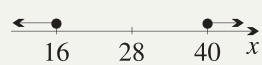
$$|3x + 7| < 3$$
$$\boxed{\div 3} \qquad |x + 2\tfrac{1}{3}| < 1$$
$$\left(\text{distance from } x \text{ to } -2\tfrac{1}{3}\right) < 1$$



so $-3\frac{1}{3} < x < -1\frac{1}{3}$.

d

$$|7 - \tfrac{1}{4}x| \geq 3$$
$$\boxed{\times 4} \qquad |28 - x| \geq 12$$
$$(\text{distance from } x \text{ to } 28) \geq 12$$



so $x \leq 16$ or $x \geq 40$.

Solving absolute value equations and inequations algebraically

We saw in Section 4D of the Year 11 book how an absolute value equation of the form $|f(x)| = a$ can be solved algebraically by rewriting the equation.

Rewrite an equation $|f(x)| = a$ as $f(x) = a$ or $f(x) = -a$.

We can take a similar approach to solving an inequation $|f(x)| < a$ or $|f(x)| > a$.

Rewrite an inequation $|f(x)| < a$ as $-a < f(x) < a$.

Rewrite an inequation $|f(x)| > a$ as $f(x) < -a$ or $f(x) > a$.

The absolute value $|f(x)|$ cannot be negative. Thus if a is negative:

- $|f(x)| = a$ and $|f(x)| < a$ have no solutions, and
- $|f(x)| > a$ is true for all values of x in the domain of $f(x)$.

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Example 15 2D

a Solve $|9 - 2x| = 5$. **b** Solve $|9 - 2x| < 5$. **c** Solve $|9 - 2x| > 5$.

SOLUTION

a

$$\begin{aligned} |9 - 2x| &= 5 \\ 9 - 2x = 5 \text{ or } 9 - 2x &= -5 \\ \boxed{-9} \qquad -2x = -4 \text{ or } -2x &= -14 \\ \boxed{\div(-2)} \qquad x = 2 \text{ or } x &= 7 \end{aligned}$$

b

$$\begin{aligned} |9 - 2x| &< 5 \\ -5 < 9 - 2x &< 5 \\ \boxed{-9} \qquad -14 < -2x &< 4 \\ \boxed{\div(-2)} \qquad 7 > x &> 2 \\ \text{that is,} \qquad 2 < x &< 7. \end{aligned}$$

c

$$\begin{aligned} |9 - 2x| &> 5 \\ 9 - 2x < -5 \text{ or } 9 - 2x &> 5 \\ \boxed{-9} \qquad -2x < -14 \text{ or } -2x &> -4 \\ \boxed{\div(-2)} \qquad x > 7 \text{ or } x &< 2 \\ \text{that is,} \qquad x < 2 \text{ or } x &> 7. \end{aligned}$$

12 SOLVING AN ABSOLUTE VALUE EQUATION OR INEQUATION

- Rewrite an equation $|f(x)| = a$ as $f(x) = a$ or $f(x) = -a$.
- Rewrite an inequation $|f(x)| < a$ as $-a < f(x) < a$.
- Rewrite an inequation $|f(x)| > a$ as $f(x) < -a$ or $f(x) > a$.

Exercise 2D

FOUNDATION

1 Solve each inequation, and graph your solution on the number line.

a $x - 2 < 3$ **b** $3x \ge -6$ **c** $4x - 3 \le -7$ **d** $6x - 5 < 3x - 17$

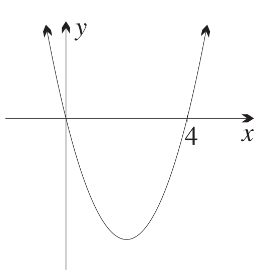
2 Solve each inequation, then write your answer using bracket interval notation.

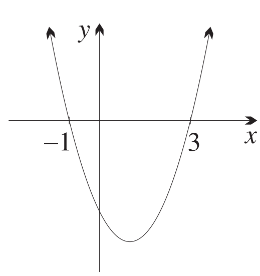
a $-2x < 6$ **b** $-5x \ge -50$ **c** $3 - 2x > 7$

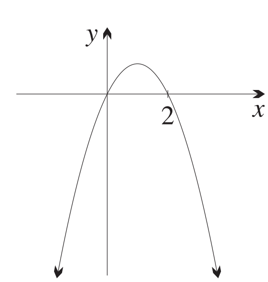
d $-4 - x \le 1$ **e** $3 - 3x \le 19 + x$ **f** $12 - 7x > -2x - 18$

3 Use the given graph of the LHS to help solve each inequation.

a $x(x - 4) < 0$ **b** $(x - 3)(x + 1) \ge 0$ **c** $x(2 - x) \le 0$







4 Sketch the associated parabola (vertices are unnecessary) and hence solve:

a $(x + 2)(x - 4) < 0$ **b** $(x - 3)(x + 1) > 0$ **c** $(2 - x)(x - 5) \ge 0$

d $(x + 1)(x + 3) \ge 0$ **e** $(2x - 1)(x - 5) > 0$ **f** $(3x + 5)(x + 4) \le 0$

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5 Solve each equation or inequation using distances. Graph the solution on a number line.

a $|x - 4| = 1$ **b** $|x - 3| = 7$ **c** $|x + 2| = 2$ **d** $|x + 1| = 6$

e $|x - 2| < 3$ **f** $|x - 7| \geq 2$ **g** $|x + 3| > 4$ **h** $|x + 10| \leq 6$

DEVELOPMENT

6 Solve each double inequation, and graph your solution on the number line.

a $3 < x + 2 < 6$ **b** $-5 < x - 3 \leq 4$ **c** $-1 \leq 2x \leq 3$ **d** $-7 < 5x + 3 \leq 3$

7 Solve each double inequation, and write your answer in bracket interval notation.

a $-4 < -2x < 8$ **b** $-2 \leq -x \leq 1$ **c** $-7 \leq 5 - 3x < 4$ **d** $-4 < 1 - \frac{1}{3}x \leq 3$

8 Solve each inequation.

a $\dfrac{x}{5} - \dfrac{x}{2} < 3$ **b** $\frac{1}{4}x + 1 \geq \frac{1}{2}x$

c $\dfrac{x + 1}{4} - \dfrac{2x - 1}{3} \leq 1$ **d** $\frac{1}{6}(2 - x) - \frac{1}{3}(2 + x) > 2$

9 Factor the LHS, then sketch an appropriate parabola in order to solve:

a $x^2 + 2x - 3 < 0$ **b** $x^2 - 5x + 4 \geq 0$ **c** $x^2 + 6x + 8 > 0$

d $x^2 - x - 6 \leq 0$ **e** $2x^2 - x - 3 \leq 0$ **f** $4 + 3x - x^2 > 0$

10 Solve:

a $x^2 \leq 1$ **b** $x^2 > 3x$ **c** $x^2 \geq 144$

d $x^2 > 0$ **e** $x^2 + 9 \leq 6x$ **f** $4x - 3 \geq x^2$

11 Write down and solve a suitable inequation to find the values of x for which the line $y = 5x - 4$ is below the line $y = 7 - \frac{1}{2}x$.

12 **a** Sketch the lines $y = 1 - x$, $y = 2$ and $y = -1$ on a number plane and find the points of intersection.

b Solve the inequation $-1 < 1 - x \leq 2$ and relate the answer to the graph.

13 Solve these equations and inequations using distances.

a $|7x| = 35$ **b** $|2x - 1| = 11$ **c** $|7x - 3| = 11$ **d** $|3x + 2| = 8$

e $|3x - 5| \leq 4$ **f** $|6x - 7| > 5$ **g** $|2x + 1| < 3$ **h** $|5x + 4| \geq 6$

14 Solve these equations and inequations algebraically (some were in Question 13).

a $|2x| = 10$ **b** $|x - 2| = 6$ **c** $|3x + 2| = 8$ **d** $|5x + 2| = 9$

e $|x - 2| < 3$ **f** $|3x - 5| \leq 4$ **g** $|5x + 4| \geq 6$ **h** $|6x - 7| > 5$

CHALLENGE

15 **a** **i** What value of x makes $|x - 4|$ zero?

ii Simplify $|x - 4| + x + 1$ when $x \geq 4$.

iii Simplify $|x - 4| + x + 1$ when $x < 4$.

b **i** What value of x makes $|x + 3|$ zero?

ii Simplify $|x + 3| + 1 - x$ when $x \geq -3$.

iii Simplify $|x + 3| + 1 - x$ when $x < -3$.

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c **i** What value of x makes $|2x + 4|$ zero?

ii Simplify $|2x + 4| - x + 5$ when $x \geq -2$.

iii Simplify $|2x + 4| - x + 5$ when $x < -2$.

d **i** What value of x makes $|3x - 3|$ zero?

ii Simplify $|3x - 3| + x - 1$ when $x \geq 1$.

iii Simplify $|3x - 3| + x - 1$ when $x < 1$.

16 **a** Consider the equation $|2x| + x - 1 = 0$.

i Simplify the equation when $x \geq 0$ and hence find any solutions for which $x \geq 0$.

ii Simplify the equation when $x < 0$ and hence find any solutions for which $x < 0$.

b Consider the equation $|3x - 6| + x = 4$.

i Simplify the equation when $x \geq 2$ and hence find any solutions for which $x \geq 2$.

ii Simplify the equation when $x < 2$ and hence find any solutions for which $x < 2$.

c Consider the equation $|x + 1| - \frac{1}{2}x = 3$.

i Simplify the equation when $x \geq -1$ and hence find any solutions for which $x \geq -1$.

ii Simplify the equation when $x < -1$ and hence find any solutions for which $x < -1$.

d Consider the equation $|3x - 2| = x + 6$.

i Simplify the equation when $x \geq \frac{2}{3}$ and hence find any solutions for which $x \geq \frac{2}{3}$.

ii Simplify the equation when $x < \frac{2}{3}$ and hence find any solutions for which $x < \frac{2}{3}$.

17 **a** Draw a sketch of the curve $y = x^2 - 4$ and use it to solve the inequation $x^2 - 4 \geq 0$. Hence write down the natural domain of $\sqrt{x^2 - 4}$.

b Solve the inequation $x^2 - 4 > 0$ and hence write down the natural domain of $\dfrac{5}{\sqrt{x^2 - 4}}$.

18 Use the methods of the previous question to write down the domain of:

a $\sqrt{4 - x^2}$ **b** $\dfrac{1}{\sqrt{4 - x^2}}$ **c** $\sqrt{x^2 - 4}$ **d** $\dfrac{5}{\sqrt{x^2 - 4}}$

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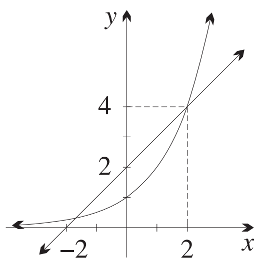
2E Using graphs to solve equations and inequations

In this section, graphs are used to solve more general equations and inequations. The advantage of such an approach is that once the graphs are drawn, it is usually obvious from the picture how many solutions there are, and indeed if there are any solutions at all, as well as their approximate values. Exact solutions can sometimes then be calculated once the situation has been sorted out from the picture.

Constructing two functions from a given equation

Here is an equation that cannot be solved algebraically, so that a graphical approach is appropriate:

$$2^x = x + 2.$$



To the right, $y =$ LHS and $y =$ RHS are graphed together. (In other situations, some rearrangement of the equation first may be appropriate.)

The first thing to notice is that there are two solutions, because the graphs intersect twice.

The second thing is to examine what the values of the two solutions are. One solution is exactly $x = 2$, because

$$2^2 = 4 = 2 + 2, \qquad \text{so } (2, 4) \text{ lies on both graphs.}$$

The other solution is just to the right of $x = -2$. From the graph, we might guess $x \doteqdot -1.7$, and if necessary we can refine this solution in several ways:

- Plot the graphs carefully on graph paper (an old method that works).
- Use trial and error on a calculator (see Question 11 in Exercise 2E).
- Use graphing software that can generate approximations (if you have it).

Counting the number of solutions of an equation

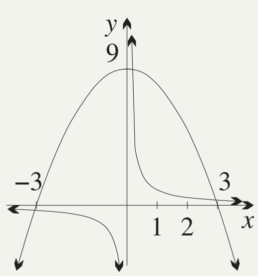
Often, however, we only want to know how many solutions an equation has, and roughly where they are.

Example 16 — 2E

a Graph $y = 9 - x^2$ and $y = \dfrac{1}{x}$ on the one set of axes.

b Use your graph to investigate the equation $9 - x^2 = \dfrac{1}{x}$. How many solutions does the equation have, and approximately where are they?

SOLUTION



a The two functions are sketched to the right.

b There are three points of intersection of the two graphs.
Thus there are three solutions:
– one just to the left of $x = -3$,
– one just to the right of $x = 0$,
– and one just to the left of $x = 3$.

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13 GRAPHICAL SOLUTION OF AN EQUATION

- Sketch the graphs of $y = \text{LHS}$ and $y = \text{RHS}$ on one pair of axes.
  - It may be appropriate to rearrange the original equation first.
- The solutions are the x-coordinates of any points of intersection.
- You may be interested only in the number of solutions.
- The graph will give a rough idea where any solution lies.

Solving an inequation using graphs

Now consider the inequation

$$2^x < x + 2.$$

From the sketch at the start of the section the curve $y = 2^x$ is only below the curve $y = x + 2$ between the two points of intersection. Hence the solution of the inequation is approximately $-1.7 < x < 2$.

14 GRAPHICAL SOLUTION OF AN INEQUATION

- Sketch the graphs of $y = \text{LHS}$ and $y = \text{RHS}$ on one pair of axes.
- Then examine which curve lies above the other at each value of x.

Absolute value equations and inequations — graphical solutions

The most straightforward approach to absolute value equations and inequations is to draw a sketch to sort out the situation. Then the exact values can usually be found algebraically.

The next worked example benefits greatly from the diagram that makes the situation so clear.

Example 17 2E

a Draw the graph of $y = |2x - 5|$.
b Write down the equations of the right-hand and left-hand branches.
c One the same diagram, draw the graph of $y = x + 2$.
d Hence find the points P and Q of intersection.
e Solve $|2x - 5| = x + 2$.
f Solve $|2x - 5| \ge x + 2$.

SOLUTION

a To find the x-intercept, put $x = 0$, then $2x - 5 = 0$
$$x = 2\tfrac{1}{2}.$$
To find the y-intercept, put $x = 0$, then
$$\begin{aligned} y &= |0 - 5| \\ &= 5. \end{aligned}$$
We can now sketch the right-hand branch by symmetry (or use a table of values).

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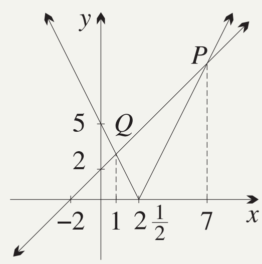
b For $x \ge 2\frac{1}{2}$, $y = 2x - 5$.
For $x < 2\frac{1}{2}$, $y = -2x + 5$.

c The two graphs intersect at two points P and Q as shown.



d The points P and at Q can be now found algebraically.
P is the intersection of $y = x + 2$ with $y = 2x - 5$,

$$x + 2 = 2x - 5$$
$$x = 7, \quad \text{so} \quad P = (7, 9),$$

and Q is the intersection of $y = x + 2$ with $y = -2x + 5$,

$$x + 2 = -2x + 5$$
$$x = 1, \quad \text{so} \quad Q = (1, 3).$$

e Hence the solutions are $x = 7$ or $x = 1$.

f Look at where $y = |2x - 5|$ is on or above $y = x + 2$.
From the graph, $x \le 1$ or $x \ge 7$.

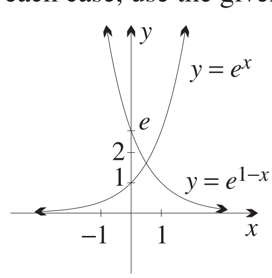
Exercise 2E

FOUNDATION

Note: Graphing software would be particularly useful in this exercise.

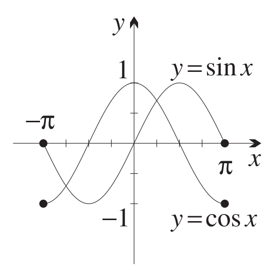
1 In each case, use the given graph to determine the number of solutions of the equation.

a



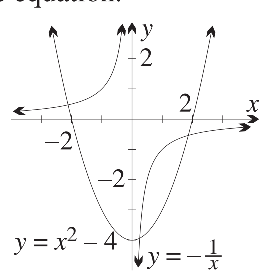
$e^x = e^{1-x}$

b



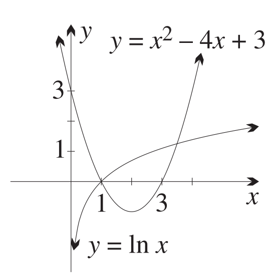
$\cos x = \sin x$, for $-\pi \le x \le \pi$

c



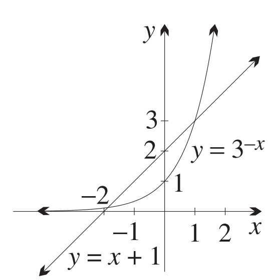
$x^2 - 4 = -\frac{1}{x}$

d



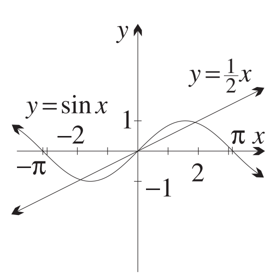
$\ln x = x^2 - 4x + 3$

e



$3^x = x + 2$

f



$\sin x = \frac{1}{2}x$

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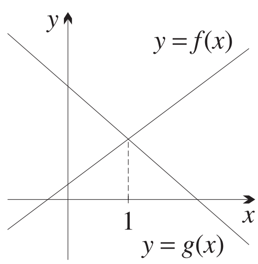
2 For each equation in Question 1, read the solutions from the graph, approximated correct to one decimal place where appropriate.

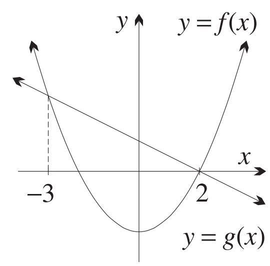
3 Write down the values of x for which:

i $y = f(x)$ is above $y = g(x)$, **ii** $y = f(x)$ is below $y = g(x)$.

a

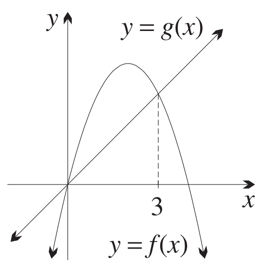


b

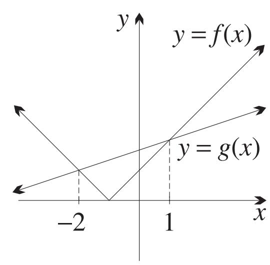


4 State the values of x for which: **i** $f(x) = g(x)$, **ii** $f(x) > g(x)$, **iii** $f(x) < g(x)$.

a

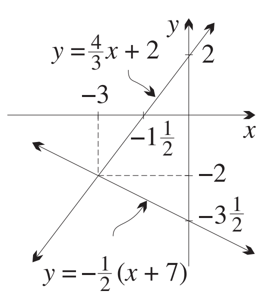


b



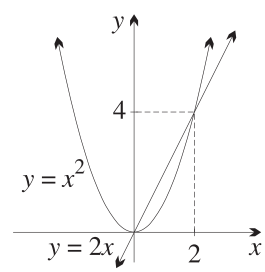
5 Use the given graphs to help solve each inequation.

a



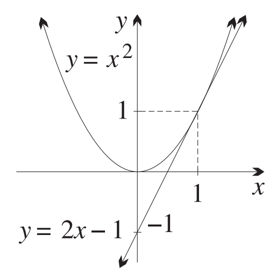
$\frac{4}{3}x + 2 \le -\frac{1}{2}(x + 7)$

b



$x^2 \le 2x$

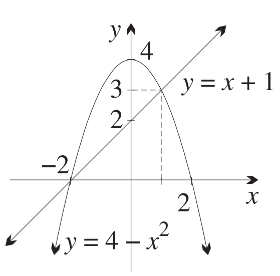
c



$x^2 \le 2x - 1$

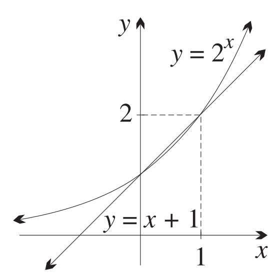
6 Solve these inequations using the given graphs.

a



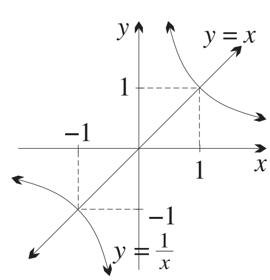
$4 - x^2 < x + 2$

b



$2^x \le x + 1$

c



$\frac{1}{x} < x$

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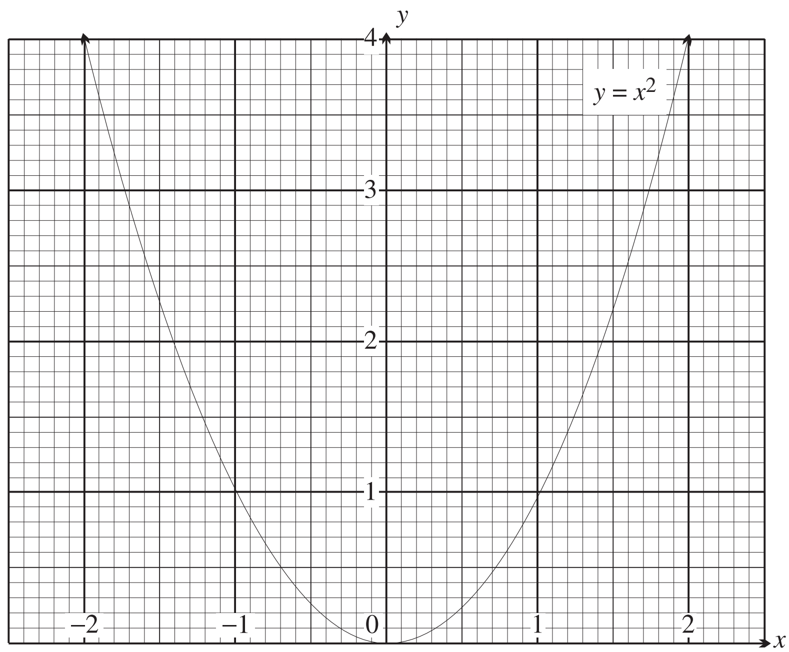
7



In preparation for the following questions, photocopy the sketch above, which shows the graph of the function

$$y = x^2, \quad \text{for } -2 \le x \le 2.$$

a Read $\sqrt{2}$ and $\sqrt{3}$ off the graph, correct to one decimal place, by locating 2 and 3 on the y-axis and reading the answer off the x-axis. Check your approximations using a calculator.

b Draw on the graph the line $y = x + 2$, and hence read off the graph the solutions of $x^2 = x + 2$. Then check your solution by solving $x^2 = x + 2$ algebraically.

c From the graph, write down the solution of $x^2 > x + 2$.

d Draw a suitable line to solve $x^2 = 2 - x$ and $x^2 \le 2 - x$. Check your results by solving $x^2 = 2 - x$ algebraically.

e Draw $y = x + 1$, and hence solve $x^2 = x + 1$ approximately. Check your result algebraically.

f Find approximate solutions for these quadratic equations by rearranging each with x^2 as subject, and drawing a suitable line on the graph.

i $x^2 + x = 0$ **ii** $x^2 - x - \frac{1}{2} = 0$ **iii** $2x^2 - x - 1 = 0$

DEVELOPMENT

8 In each part:

i Carefully sketch each pair of equations.

ii Read off the points of intersection.

iii Write down the equation satisfied by the x-coordinates of the points of intersection.

a $y = x - 2$ and $y = 3 - \frac{1}{4}x$ **b** $y = x$ and $y = 2x - x^2$

c $y = \dfrac{2}{x}$ and $y = x - 1$ **d** $y = x^3$ and $y = x$

9 Use your graphs from the previous question to solve these inequations.

a $x - 2 \ge 3 - \frac{1}{4}x$ **b** $x < 2x - x^2$

c $\dfrac{2}{x} > x - 1$ **d** $x^3 > x$

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10 Explain how the graphs of Question 1 parts **a**, **b** and **c** could be used to solve:

a $e^{2x} = e$ **b** $\tan x = 1$ **c** $x^3 - 4x + 1 = 0$.

11 [Solving an equation by trial and error]

a At the start of Section 2E we sketched $y = 2^x$ and $y = x + 2$ on one set of axes, and saw that there is a solution a little to the right of $x = -2$. Fill in the table of values below, and hence find the negative solution of $2^x = x + 2$ correct to three decimal places.

x	-2	-1.7	-1.6	-1.68	-1.69	-1.691	-1.6905
2^x							
$x + 2$							

b For parts **c** and **e** of Question 1, use trial and error on the calculator to find the negative solution of the equation, correct to three decimal places.

12 a Sketch on the same number plane the functions $y = |x + 1|$ and $y = \frac{1}{2}x - 1$.

b Hence explain why all real numbers are solutions of the inequation $|x + 1| > \frac{1}{2}x - 1$.

13 a Draw a sketch of the curve $y = 2^x$ and the line $y = -1$. Hence explain why the inequation $2^x \le -1$ has no solutions.

b Draw $y = 2x - 1$ and $y = 2x + 3$ on the same number plane, and hence explain why the inequation $2x - 1 \le 2x + 3$ is true for all real values of x.

14 Sketch each pair of equations, and hence find the points of intersection.

a $y = |x + 1|$ and $y = 3$ **b** $y = |x - 2|$ and $y = x$

c $y = |2x|$ and $2x - 3y + 8 = 0$ **d** $y = |x| - 1$ and $y = 2x + 2$

15 Use your answers to the previous question to help solve:

a $|x + 1| \le 3$ **b** $|x - 2| > x$

c $|2x| \ge \dfrac{2x + 8}{3}$ **d** $|x| > 2x + 3$

16 [Break-even point]

A certain business has fixed costs of \$900 plus costs of \$30 per item sold. The sale price of each item is \$50. If enough items are sold then the company is just able to pay its total costs. That point is called the *break-even point*. Companies may use several different methods to graph this information. Here are two such methods. In each case, let n be the number of items sold.

a i The gross profit per item is $\$50 - \$30 = \$20$. Sketch the graph of the gross profit for n items $y = \$20 \times n$.

ii On the same graph, sketch the fixed costs $y = \$900$.

iii The point where these two lines cross is the break-even point. How many items need to be sold to break even?

b i On a new graph, draw $y = \$50 \times n$, the total sales for n items.

ii On the same graph, draw $y = \$900 + \$30 \times n$, the total cost for n items.

iii Does the break-even point for this graph agree with the result of part **a**?

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CHALLENGE

17 **a** Sketch $y = x^2 - 2$, $y = x$ and $y = -x$ on the same number plane, and find all points of intersection of the three functions.

b Hence find the solutions of $x^2 - 2 = |x|$.

c Hence solve $x^2 - 2 > |x|$.

18 Sketch graphs of the LHS and RHS of each equation on the same number plane in order to find the number of solutions (tables of values may help). Do not attempt to solve them.

a $1 - \frac{1}{2}x = x^2 - 2x$

b $|2x| = 2^x$

c $x^3 - x = \frac{1}{2}(x + 1)$

d $4x - x^2 = \dfrac{1}{x}$

e $2^x = 2x - x^2$

f $2^{-x} - 1 = \dfrac{1}{x}$

19 Draw appropriate graphs, using a computer or graphics calculator, in order to find the solutions of these equations correct to one decimal place.

a $x^3 = 2(x - 2)^2$

b $x^3 = \sqrt{4 - x^2}$

c $2^x = -x(x + 2)$

d $2^{-x} = 2x - x^2$

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2F Review of translations and reflections

This short section reviews translations and reflections in preparation for dilations.

Translations and reflections

Here are the rules from Chapter 4 of the Year 11 book.

15 A SUMMARY OF SHIFTING AND REFLECTING

Transformation	By replacement	By function rule
Shift horizontally h right	Replace x by $x - h$	$y = f(x) \to y = f(x - h)$
Shift vertically k up	Replace y by $y - k$	$y = f(x) \to y = f(x) + k$
Reflect in the y-axis	Replace x by $-x$	$y = f(x) \to y = f(-x)$
Reflect in the x-axis	Replace y by $-y$	$y = f(x) \to y = -f(x)$
Rotate 180° about O	Replace x by $-x$, y by $-y$	$y = f(x) \to y = -f(-x)$

The equation of a transformed relation can always be obtained *by replacement*, whether or not the relation is a function. The second method, *by function rule*, can only be used when the relation is a function.

Example 18 2F

Write down the equation of the resulting graph when each transformation below is applied to the circle $(x - 1)^2 + (y + 2)^2 = 1$.

a shift left 3 units **b** reflect in the x-axis **c** rotate 180° about O

Then sketch all four circles on one set of axes.

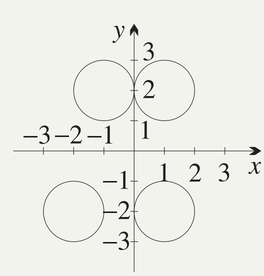
SOLUTION

The original circle is $(x - 1)^2 + (y + 2)^2 = 1.$

a Shifting left 3 units gives $((x + 3) - 1)^2 + (y + 2)^2 = 1$
$(x + 2)^2 + (y + 2)^2 = 1.$

b Reflecting in the x-axis gives $(x - 1)^2 + (-y + 2)^2 = 1$
$(x - 1)^2 + (y - 2)^2 = 1.$

c Rotating 180° about O gives $(-x - 1)^2 + (-y + 2)^2 = 1$
$(x + 1)^2 + (y - 2)^2 = 1.$



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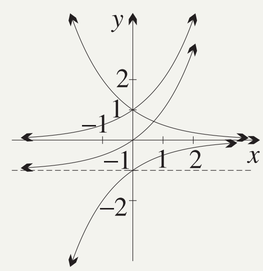
Example 19 2F

Write down the equation of the resulting graph when each transformation below is applied the exponential curve $y = 2^x$:

a shift down 1 unit, **b** reflect in the y-axis, **c** rotate 180° about O.

Then sketch all four curves on one set of axes.

SOLUTION



The original curve is $y = 2^x$.

a Shifting down 1 unit gives $y = 2^x - 1$.

b Reflecting in the y-axis gives $y = 2^{-x}$.

c Rotating 180° about O gives $y = -2^{-x}$.

Exercise 2F

FOUNDATION

1 Write down the new equation for each function or relation after the given shift has been applied. Draw a graph of the image after the shift.

a $y = x^2$: right 2 units
b $y = 2^x$: down 1 unit
c $y = x^2 - 1$: down 3 units
d $y = \frac{1}{x}$: right 3 units
e $x^2 + y^2 = 4$: up 1 unit
f $y = \log_2 x$: left 1 units
g $y = \sin x$: left $\frac{\pi}{2}$ units
h $y = \sqrt{x}$: up 2 units

2 Repeat Question 1 for the reflection in the given axis, or rotation about the origin.

a x-axis **b** y-axis **c** x-axis **d** x-axis
e y-axis **f** rotate 180° **g** rotate 180° **h** y-axis

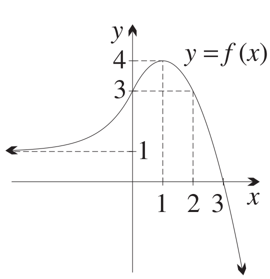
3 In which parts of Question 2 was the result the same as the original function? In each case, explain geometrically why that happened.

4 Use your understanding of translations, and completion of the square where necessary, to determine the centre and radius of each circle.

a $(x + 1)^2 + y^2 = 4$
b $(x - 1)^2 + (y - 2)^2 = 1$
c $x^2 - 4x + y^2 = 0$
d $x^2 + y^2 - 6y = 16$

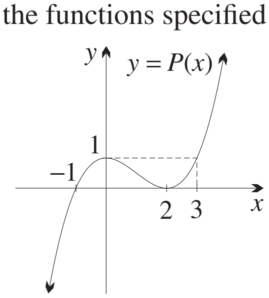
5 In each case an unknown function has been drawn. Draw the functions specified below it.

a



i $y = f(x - 2)$ **ii** $y = f(x) - 2$

b



i $y = P(x + 1)$ **ii** $y = P(x) + 1$

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DEVELOPMENT

6 Write down the equation for each function after the given translation has been applied.

a $y = x^2$: left 1 unit, up 2 units

b $y = \frac{1}{x}$: right 2 units, up 3 units

c $y = \cos x$: right $\frac{\pi}{3}$ units, down 2 units

d $y = e^x$: left 2 units, down 1 unit

7 The composition of functions can sometimes result in translations.

a Let $h(x) = x - 3$. Draw the following using the graph of $f(x)$ given in Question **5a**.

i $y = f \circ h(x)$

ii $y = h \circ f(x)$

b Let $k(x) = x + 2$. Draw the following using the graph of $P(x)$ given in Question **5b**.

i $y = P \circ k(x)$

ii $y = k \circ P(x)$

8 In each part explain how the graph of each subsequent equation is a transformation of the first graph (there may be more than one answer), then sketch each curve.

a From $y = -x$: **i** $y = -x + 2$ **ii** $y = -x - 2$ **iii** $y = x + 4$

b From $y = x^2$: **i** $y = (x + 1)^2$ **ii** $y = -(x + 1)^2$ **iii** $y = (x + 1)^2 - 1$

c From $y = \sqrt{x}$: **i** $y = \sqrt{x + 4}$ **ii** $y = -\sqrt{x}$ **iii** $y = -\sqrt{x + 4}$

d From $y = \dfrac{2}{x}$: **i** $y = \dfrac{2}{x} - 1$ **ii** $y = \dfrac{2}{x + 2} - 1$ **iii** $y = -\dfrac{2}{x}$

e From $y = \sin x$: **i** $y = \sin x - 1$ **ii** $y = \sin\left(x - \frac{\pi}{4}\right) - 1$ **iii** $y = -\sin x$

9 Answer these questions about the cubic $y = x^3 - 3x$.

a Find the coordinates of the two points where the tangent is horizontal.

b The cubic is shifted 1 unit up.

i Write down the equation of this new cubic.

ii Show that the x-coordinates where the tangent is horizontal have not changed.

c The original cubic is shifted 1 unit left.

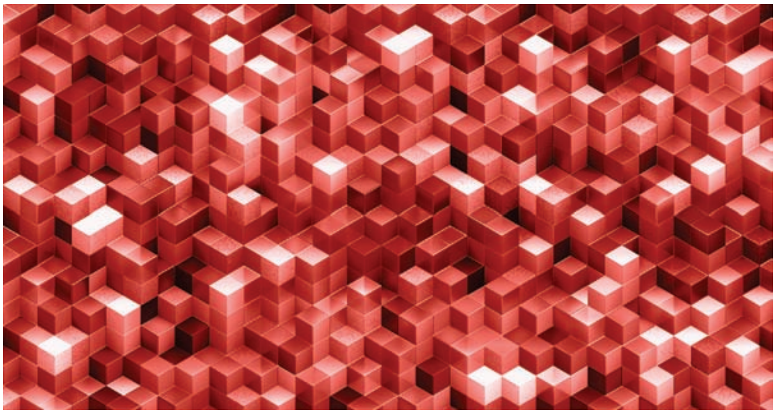
i Write down the equation of this third cubic, expanding any brackets.

ii Show that the y-coordinates where the tangent is horizontal have not changed.

10 Complete the square then sketch each circle, stating the centre and radius. Find any intercepts with the axes by substituting $x = 0$ and $y = 0$.

a $x^2 + 4x + y^2 - 8y = 0$

b $x^2 - 2x + y^2 + 4y = -1$



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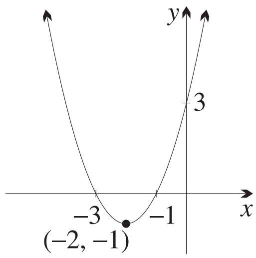
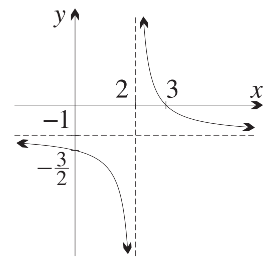
11 Describe each graph below as a standard curve transformed either by two shifts, or by a reflection followed by a shift. Hence write down its equation.

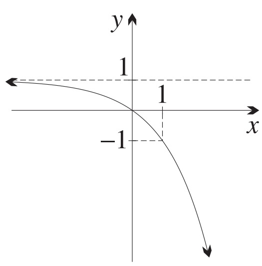
a



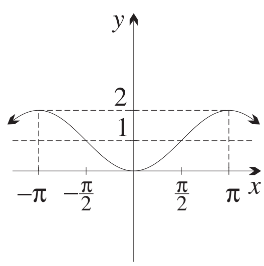
b



c



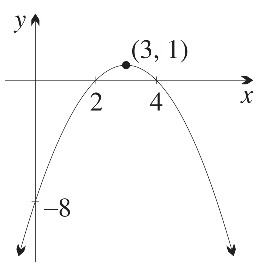
d



CHALLENGE

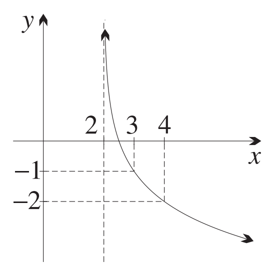
12 Describe each graph below as the given standard curve transformed by a reflection followed by two shifts, and hence write down its equation.

a



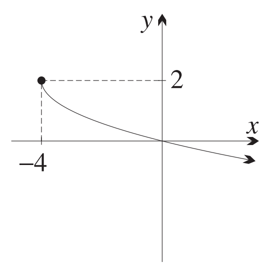
Start with $y = x^2$.

b



Start with $y = \log_2 x$.

c



Start with $y = \sqrt{x}$.

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2G Dilations

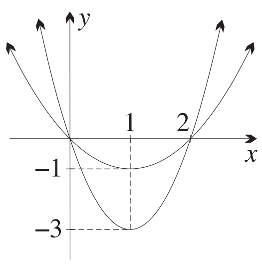
A dilation is a stretch of a curve in one direction. For example, a dilation distorts a circle into an ellipse. Dilations are another kind of transformation of curves, and they belong naturally with the translations and reflections that were reviewed in the previous section. Most of the functions in the course can be reduced to very simple functions using a combination of translations, reflections and dilations.

Stretching a graph vertically

Compare the graphs of

$$y = x(x - 2) \qquad \text{and} \qquad y = 3x(x - 2).$$

Each value in the table below for $y = 3x(x - 2)$ is three times the corresponding value in the table for $y = x(x - 2)$. This means that the graph of $y = 3x(x - 2)$ is obtained from the graph of $y = x(x - 2)$ by *stretching away from the x-axis in the vertical direction* by a factor of 3:



x	-2	-1	0	1	2	3	4
$x(x-2)$	8	3	0	-1	0	3	8
$3x(x-2)$	24	9	0	-3	0	9	24

We can rewrite the equation $y = 3x(x - 2)$ as $\frac{y}{3} = x(x - 2)$. This makes it clear that the stretching has been obtained by replacing y by $\frac{y}{3}$.

The x-axis is the *axis of dilation*. Points on the x-axis do not move, and all other points on the graph triple their distance from the x-axis.

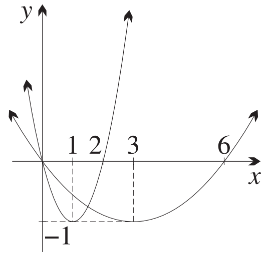
16 VERTICAL DILATIONS — STRETCHING A GRAPH VERTICALLY

- To stretch a graph in a vertical direction by a factor of a, replace y by $\dfrac{y}{a}$.
- Alternatively, if the graph is a function, the new function rule is $y = a\,f(x)$.
- The *axis of dilation* for these transformations is the x-axis.

Stretching a graph horizontally

By analogy with the previous example, the graph of $y = x(x - 2)$ can be *stretched horizontally away from the y-axis* by a factor of 3 by replacing x by $\frac{x}{3}$, giving the new function

$$y = \frac{x}{3}\left(\frac{x}{3} - 2\right) = \frac{1}{9}x(x - 6).$$



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Two table of values should make this clear. The first table is the original graph, the second is the new function.

x	-2	-1	0	1	2	3	4
$x(x-2)$	8	3	0	-1	0	3	8

x	-6	-3	0	3	6	9	12
$\frac{x}{3}\left(\frac{x}{3}-2\right)$	8	3	0	-1	0	3	8

The y-coordinates in each table are the same, but we needed to treble the x-coordinates to produce those same y-coordinates. Thus the y-axis is the *axis of dilation*, and the *dilation factor* is 3, because the point $(0, 0)$ on the y-axis does not move, and all other points on the graph triple their distance from the y-axis.

17 HORIZONTAL DILATIONS — STRETCHING A GRAPH HORIZONTALLY

- To stretch the graph in a horizontal direction by a factor of a, replace x by $\frac{x}{a}$.
- Alternatively, if the graph is a function, the new function rule is $y = f\left(\frac{x}{a}\right)$.
- The *axis of dilation* for these transformations is the y-axis.

Example 20 2G

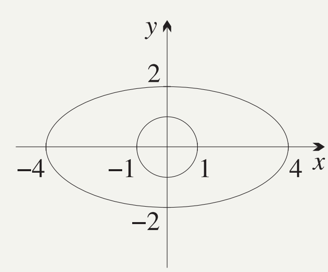
Obtain the graph of $\frac{x^2}{16} + \frac{y^2}{4} = 1$ from the graph of the circle $x^2 + y^2 = 1$.

SOLUTION

The equation can be rewritten as

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1,$$

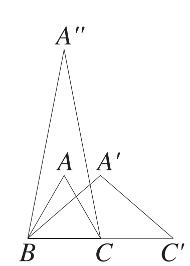
which is the unit circle stretched vertically by a factor of 2 and horizontally by a factor of 4.



Note: Any curve of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is called an *ellipse*. It can be obtained from the unit circle $x^2 + y^2 = 1$ by stretching horizontally by a factor of a and vertically by a factor of b, so that its x-intercepts are a and $-a$ and its y-intercepts are b and $-b$.

Enlargements

The dilation of a figure is usually not similar to the original. For example, the equilateral triangle ABC in the figure to the right with its base on the x-axis is stretched to the squat isosceles triangle $A'BC'$ by a horizontal dilation with factor 2, and it is stretched to the skinny isosceles triangle $A''BC$ by a vertical dilation with factor 3.



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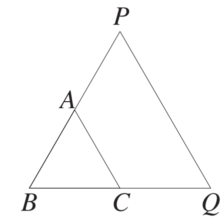
But if two dilations with the same factor 2, one horizontal and the other vertical, are applied in order to the equilateral triangle ABC — the order does not matter — the result is the similar equilateral triangle PBQ. Such a combined transformation is called an *enlargement*, and the factor 2 is called the *enlargement factor* or *similarity factor*.



In the coordinate plane, the *centre* of an enlargement is normally taken as the origin.

18 ENLARGMENTS

- An enlargement of a figure is similar to the original. In particular, matching angles are equal, and the ratios of matching lengths are equal.
- The composition of two dilations with the same factor, one horizontal and one vertical, is an *enlargement* with centre the origin.
- To apply an enlargement with factor a, replace x by $\dfrac{x}{a}$ and y by $\dfrac{y}{a}$.
- Alternatively, if the graph is a function, the new function rule is $y = af\left(\dfrac{x}{a}\right)$.

Example 21 — 2G

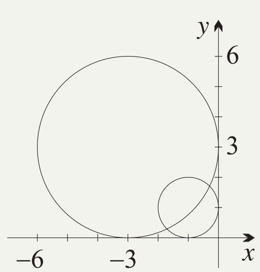
Apply an enlargement with centre the origin and factor 3 to the circle $(x + 1)^2 + (y - 1)^2 = 1$. Write down the new function, then sketch both curves.

SOLUTION

The new function is $\left(\dfrac{x}{3} + 1\right)^2 + \left(\dfrac{y}{3} - 1\right)^2 = 1$

$\boxed{\times\, 3^2 = 9}$ $\qquad (x + 3)^2 + (y - 3)^2 = 9.$

The two circles are sketched to the right.



Stretching with a fractional or negative factor

In the upper diagram to the right, a vertical dilation with factor $\frac{1}{2}$ has been applied to the parabola $y = x^2 + 2$ to yield the parabola

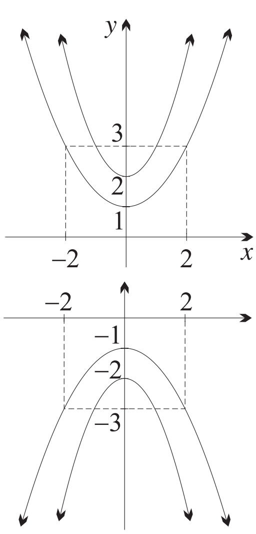
$$\frac{y}{\frac{1}{2}} = x^2 + 2, \qquad \text{that is,} \qquad y = \tfrac{1}{2}x^2 + 1.$$

The result is a compression, but we still call it a dilation.

In the lower diagram to the right, vertical dilations with factors -1 and $-\frac{1}{2}$ have been applied to the same parabola $y = x^2 + 2$. The results are the parabolas

$$y = -x^2 - 2 \qquad \text{and} \qquad y = -\tfrac{1}{2}x^2 - 1.$$

The first parabola is the reflection of the original in the x-axis. The second parabola is the reflection in the x-axis of the compressed image.



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When the dilation factor is negative, the dilation can be thought of as a dilation with positive factor followed by a reflection.

In particular, a reflection is a dilation with factor -1.

19 DILATIONS WITH A FRACTIONAL OR NEGATIVE FACTOR

- If the dilation factor is between 0 and 1, the graph is compressed.
- If the dilation factor is negative, the dilation is the composition of a dilation with positive factor and a reflection — the order does not matter.
- In particular:
  - A reflection is a dilation with factor -1.
  - A rotation of 180° about the origin is an enlargement with factor -1, and is often called a *reflection in the origin*.

Example 22 2G

Write down the new functions when each dilation is applied to the parabola $y = (x-3)(x-5)$. Then sketch the four curves on one set of axes.

a A horizontal dilation with factor -2.

b A horizontal dilation with factor $-\frac{1}{2}$.

c A vertical dilation with factor -1.

SOLUTION

a Replacing x by $\dfrac{x}{-2}$,

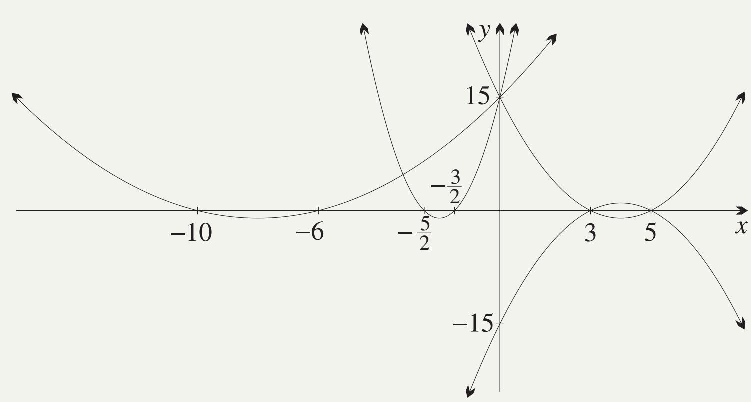
$$y = \left(-\tfrac{x}{2} - 3\right)\left(-\tfrac{x}{2} - 5\right)$$
$$y = \tfrac{1}{4}(x+6)(x+10)$$

b Replacing x by $\dfrac{x}{-1/2} = -2x$,

$$y = (-2x-3)(-2x-5)$$
$$y = 4\left(x+1\tfrac{1}{2}\right)\left(x+2\tfrac{1}{2}\right)$$

c Replacing y by $\dfrac{y}{-1} = -y$,

$$-y = (x-3)(x-5)$$
$$y = -(x-3)(x-5)$$



Note: The word 'dilation' is often used to mean 'enlargement', but in this course, it means a stretching in just one direction. Be careful when looking at other sources.

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Exercise 2G

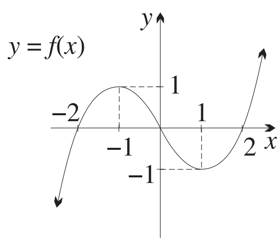
FOUNDATION

1 Write down the new equation for each function or relation after the given dilation has been applied. Draw a graph of the image after the shift.

a $y = x^2$: horizontally by $\frac{1}{2}$
b $y = 2^x$: vertically by 2
c $y = x^2 - 1$: vertically by -1
d $y = \frac{1}{x}$: horizontally by 2
e $x^2 + y^2 = 4$: vertically by $\frac{1}{3}$
f $y = \log_2 x$: horizontally by -1
g $y = \sin x$: horizontally by $\frac{1}{2}$
h $y = \sqrt{x}$: vertically by -2

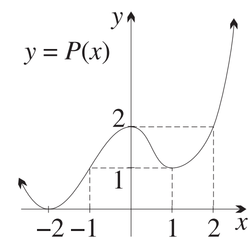
2 In each case an unknown function has been drawn. Use dilations to draw the new functions indicated below it.

a $y = f(x)$



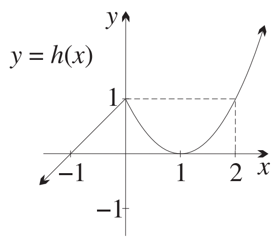
i $y = f(2x)$ **ii** $y = 2f(x)$

b $y = P(x)$



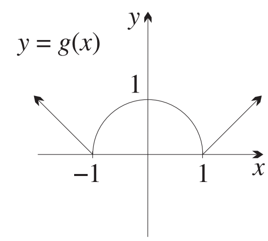
i $y = P\left(\frac{x}{2}\right)$ **ii** $y = \frac{1}{2}P(x)$

c $y = h(x)$



i $\frac{y}{2} = h(x)$ **ii** $y = h\left(\frac{x}{2}\right)$

d $y = g(x)$



i $2y = g(x)$ **ii** $y = g(2x)$

3 Sketch $x + y = 1$. Then explain how each graph below may be obtained by dilations of the first graph (there may be more than one answer), and sketch it.

a $\frac{x}{2} + y = 1$
b $\frac{x}{2} + \frac{y}{4} = 1$
c $2x + y = 1$

4 **a** The circle $(x - 3)^2 + y^2 = 4$ is enlarged by factor $\frac{1}{3}$ with centre the origin. Write down the new equation and draw both circles on the one set of axes.

b The hyperbola $y = \dfrac{1}{x}$ is enlarged by factor $\sqrt{3}$ with centre the origin. Write down the new equation and draw both hyperbolas on the one set of axes.

DEVELOPMENT

5 In each case, graph the three given equations on one set of axes by using dilations.

a $y = x(4 + x)$, $y = 2x(4 + x)$, and $y = \frac{x}{2}\left(4 + \frac{x}{2}\right)$.

b $x^2 + y^2 = 36$, $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 36$, and $(2x)^2 + (3y)^2 = 36$.

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6 The composition of functions can sometimes result in dilations.

a Let $k(x) = 3x$. Draw the following using the graph of $h(x)$ given in Question 2(c).

i $y = h \circ k(x)$ **ii** $y = k \circ h(x)$

b Let $\ell(x) = \frac{1}{3}x$. Draw the following using the graph of $g(x)$ given in Question 2(d).

i $y = g \circ \ell(x)$ **ii** $y = \ell \circ g(x)$

7 Sketch each group of three trigonometric functions on the one set of axes.

a $y = \sin x, y = 3 \sin x, y = 3 \sin 2x$ **b** $y = \cos x, y = \cos \frac{x}{2}, y = 2 \cos \frac{x}{2}$

8 Answer these questions about the cubic $y = x^3 - 3x$.

a Find the coordinates of the two points where the tangent is horizontal.

b The cubic is dilated vertically by factor 2.

i Write down the equation of this new cubic.

ii Show that the x-coordinates where the tangent is horizontal have not changed.

c The original cubic is dilated horizontally by factor 3.

i Write down the equation of this third cubic.

ii Show that the y-coordinates where the tangent is horizontal have not changed.

9 In each case identify how the graph of the second equation can be obtained from the graph of the first by a suitable dilation.

a $y = x^2 - 2x$ and $y = 3x^2 - 6x$ **b** $y = \dfrac{1}{x - 4}$ and $y = \dfrac{1}{2x - 4}$

c $y = \cos x$ and $y = \cos \frac{x}{4}$ **d** $y = \dfrac{1}{x + 1}$ and $y = \dfrac{2}{x + 1}$

10 Consider the hyperbola $y = \dfrac{1}{x}$.

a The hyperbola is stretched horizontally by factor 2. Write down its equation.

b The original hyperbola is stretched vertically by factor 2. Write down its equation.

c What do you notice about the answers to parts **a** and **b**?

d Can the hyperbolas in parts **a** or **b** be obtained by an enlargement?

e Investigate whether there are any other functions that exhibit similar behaviour.

11 Consider the parabola $y = x^2$.

a The parabola is dilated horizontally by factor $\frac{1}{2}$. Write down its equation.

b The original parabola is dilated vertically by factor 4. Write down its equation.

c What do you notice about the answers to parts **a** and **b**?

d Can the parabolas in parts **a** or **b** be obtained by an enlargement?

e Investigate whether there are any other functions that exhibit similar behaviour.

12 The mass M grams of a certain radioactive substance after t years is modelled by the formula $M = 3 \times 2^{-\frac{1}{53}t}$.

a Find the initial mass.

b Find the time taken for the mass to halve, called the *half-life*.

c Suppose now that the initial mass is doubled.

i Explain this in terms of a dilation and hence write down the new equation for M.

ii Show that the dilation does not change the value of the half-life.

13 Show that the equation $y = mx$ of a straight line through the origin is unchanged by any enlargement with centre the origin.

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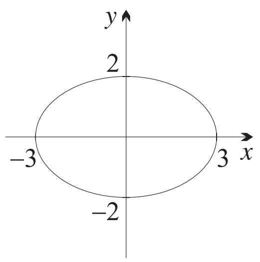
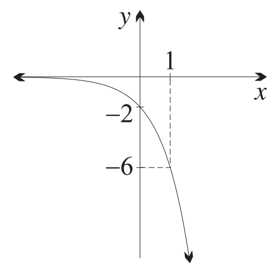
CHALLENGE

14 Describe each graph below as a standard curve transformed by dilations, and hence write down its equation.

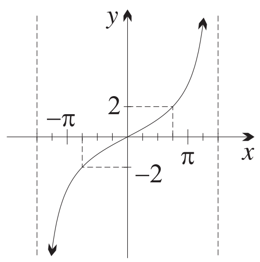
a



b



c



15 **a** For each pair of curves, suggest two simple and distinct transformations by which the second equation may be obtained from the first.

i $y = 2^x, y = 2^{x+1}$ **ii** $y = \dfrac{1}{x}, y = \dfrac{k^2}{x}$ **iii** $y = 3^x, y = 3^{-x}$

b Investigate other combinations of curves and transformations with similar ambiguity.

16 The parabola $y = x^2$ is stretched horizontally by factor a. Clearly a horizontal stretch by factor $\frac{1}{a}$ will restore the original parabola. What other stretch will produce a new parabola that appears identical to the original parabola $y = x^2$?

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2H Combinations of transformations

We will now apply several transformations to a graph one after the other.

In this section we will mostly regard reflections in the axes as dilations with factor -1, and of course a rotation of 180° about the origin is the composition of two reflections. This reduces the types of transformations to just four — two translations and two dilations:

20 A SUMMARY OF TRANSFORMATIONS

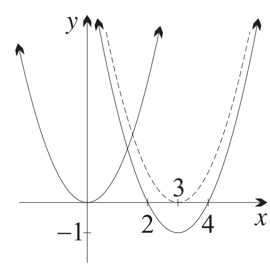
Transformation	By replacement	By function rule
Shift horizontally right h	Replace x by $x - h$	$y = f(x) \to y = f(x - h)$
Shift vertically up k	Replace y by $y - k$	$y = f(x) \to y = f(x) + k$
Stretch horizontally factor a	Replace x by $\frac{x}{a}$	$y = f(x) \to y = f\left(\frac{x}{a}\right)$
Stretch vertically factor b	Replace y by $\frac{y}{b}$	$y = f(x) \to y = bf(x)$

We shall see that it sometimes matters in which the order the two transformations are applied. Two transformations are said to *commute* if the order in which they are applied does not matter, whatever graph they are applied to.

Two translations always commute

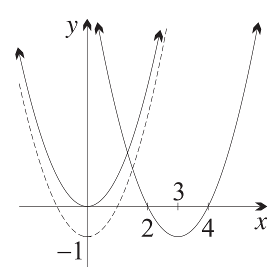
Suppose that the parabolic graph $y = x^2$ is shifted right 3 and then down 1.

$$\begin{aligned} &\text{Shifting right 3,} & y &= (x-3)^2, \\ &\text{and shifting down 1,} & y + 1 &= (x-3)^2 \\ & & y &= (x-3)^2 - 1. \end{aligned}$$



The result is exactly the same if the graph $y = x^2$ is shifted down 1 and then right 3.

$$\begin{aligned} &\text{Shifting down 1,} & y + 1 &= x^2 \\ & & y &= x^2 - 1, \\ &\text{and shifting right 3,} & y &= (x-3)^2 - 1. \end{aligned}$$



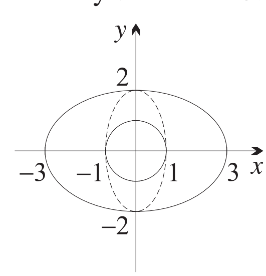
Thus the two translations commute.

In general, any two translations commute.

Two dilations always commute

Suppose that the circle graph $x^2 + y^2 = 1$ is stretched vertically with factor 2 and then horizontally with factor 3.

$$\begin{aligned} &\text{Stretching vertically factor 2,} & x^2 + \left(\frac{y}{2}\right)^2 &= 1 \\ & & x^2 + \frac{y^2}{4} &= 1, \\ &\text{and stretching horizontally factor 3,} & \left(\frac{x}{3}\right)^2 + \frac{y^2}{4} &= 1 \\ & & \frac{x^2}{9} + \frac{y^2}{4} &= 1. \end{aligned}$$



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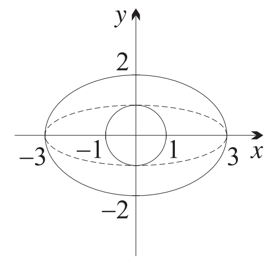
The result is the same if the graph is stretched horizontally with factor 3 and then vertically with factor 2.



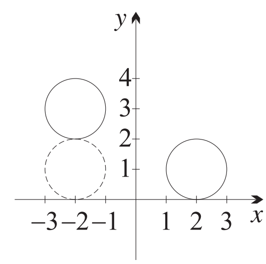
Stretching horizontally factor 3, $\dfrac{x^2}{9} + y^2 = 1$,

and stretching vertically factor 2, $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$.

Thus the two dilations commute, and in general, any two dilations commute.

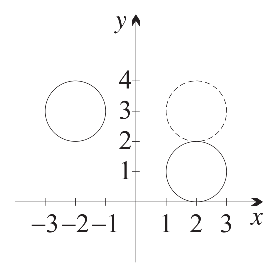
A horizontal dilation and a vertical translation commute

Apply a reflection in the y-axis (horizontal dilation with factor -1), then shift up 2, successively to the circle $(x - 2)^2 + (y - 1)^2 = 1$.



$$\begin{aligned} &\text{Reflecting in the } y\text{-axis,} && (-x - 2)^2 + (y - 1)^2 = 1 \\ & && (x + 2)^2 + (y - 1)^2 = 1, \\ &\text{and shifting up 2,} && (x + 2)^2 + (y - 2 - 1)^2 = 1 \\ & && (x + 2)^2 + (y - 3)^2 = 1. \end{aligned}$$

The resulting circle $(x + 2)^2 + (y - 3)^2 = 1$ is the same if the transformations are done in the reverse order.



$$\begin{aligned} &\text{Shifting up 2,} && (x - 2)^2 + (y - 3)^2 = 1, \\ &\text{and reflecting in the } y\text{-axis,} && (-x - 2)^2 + (y - 3)^2 = 1 \\ & && (x + 2)^2 + (y - 3)^2 = 1. \end{aligned}$$

Thus the two transformations commute. In general, any horizontal dilation and any vertical translation commute. Similarly, any vertical dilation and any horizontal translation commute.

21 COMMUTING TRANSFORMATIONS

- Any two translations commute.
- Any two dilations commute (including reflections).
- A translation and a dilation commute if one is vertical and the other horizontal.

Transformations that do not commute

In the remaining case, the transformations do not commute. That is, a translation and a dilation do not commute when they are both horizontal or both vertical. The next worked examples give two examples of this.

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Example 23 2H

a The reflection in the y-axis (horizontal dilation with factor -1) and the translation left 2 are applied successively to the circle $(x+3)^2+y^2=1$. Find the equation of the resulting graph, and sketch it.

b Repeat when the transformations are done in the reverse order.

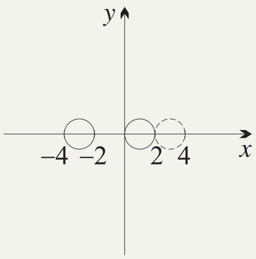
SOLUTION

a Applying the reflection, $(-x+3)^2+y^2=1$

$(x-3)^2+y^2=1,$

and applying the translation, $(x+2-3)^2+y^2=1$

$(x-1)^2+y^2=1.$

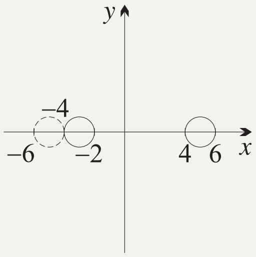


b Applying the translation, $(x+2+3)^2+y^2=1$

$(x+5)^2+y^2=1,$

and applying the reflection, $(-x+5)^2+y^2=1$

$(x-5)^2+y^2=1.$



Example 24 2H

a The vertical dilation with factor $\frac{1}{2}$ and the translation down 3 units are applied successively to the parabola $y=x^2+4$. Find the equation of the resulting graph, and sketch it.

b Repeat when the transformations are done in the reverse order.

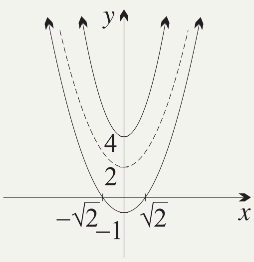
SOLUTION

a Applying the dilation, $y=\frac{1}{2}(x^2+4)$

$y=\frac{1}{2}x^2+2,$

and applying the translation, $y=\frac{1}{2}x^2+2-3$

$y=\frac{1}{2}x^2-1.$

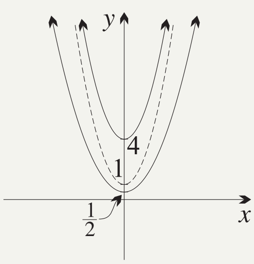


b Applying the translation, $y=x^2+4-3$

$y=x^2+1,$

and applying the dilation, $y=\frac{1}{2}(x^2+1)$

$y=\frac{1}{2}x^2+\frac{1}{2}.$



22 TRANSFORMATIONS THAT DO NOT COMMUTE

- A vertical translation and a vertical dilation do not commute.
- A horizontal translation and a horizontal dilation do not commute.

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A universal formula involving all four transformations

When the graph is a function, there is a universal formula that allows the four transformations to be applied to any function $y = f(x)$. The formula is:

$$y = kf(a(x + b)) + c.$$

This formula is useful because it applies all four transformations to any function $y = f(x)$. It is useful for trigonometric functions, and for computer programs. The formula is tricky to use, however, and although readers must know the formula, most problems should be done using the methods already presented.

Here is how to analyse the successive transformations involved in this formula.

Start with	$y = f(x)$
Stretching horizontally with factor $\frac{1}{a}$ gives	$y = f(ax)$.
Shifting left b gives	$y = f\big(a(x + b)\big)$
Stretching vertically with factor k gives	$y = kf\big(a(x + b)\big)$
Shifting up c gives	$y = kf\big(a(x + b)\big) + c$

Another way to analyse this formula is to rewrite it progressively so that the four successive transformations can be seen:

$$y = kf\big(a(x + b)\big) + c$$

$$\boxed{-\,c} \qquad y - c = kf\big(a(x + b)\big)$$

$$\boxed{\div\, k} \qquad \frac{y - c}{k} = f\big(a(x + b)\big)$$

$$\frac{y - c}{k} = f\left(\frac{x + b}{1/a}\right)$$

$$\frac{y - c}{k} = f\left(\frac{x - (-b)}{1/a}\right)$$

23 A UNIVERSAL FORMULA INVOLVING ALL FOUR TRANSFORMATIONS

The following sequence of transformations transforms the function $y = f(x)$ to

$$y = kf\big(a(x + b)\big) + c.$$

1 Stretch horizontally with factor $\frac{1}{a}$.

2 Shift left b.

3 Stretch vertically with factor k.

4 Shift up c.

Alternatively, the vertical dilation and translation (step 3 then step 4) could be done before the horizontal dilation and translation (step 1 then 2).

Exercise 2H

FOUNDATION

1 Let $y = x^2 - 2x$. Sketch the graph of this function showing the intercepts and vertex.

a **i** Sketch the parabola after shifting right 1 unit. Find its equation, expanding any brackets.

ii The parabola in part **i** is then shifted up 2 units. Sketch the new graph and find its equation.

b **i** The original parabola $y = x^2 - 2x$ is translated up 2 units. Sketch the result and find its equation.

ii Sketch the parabola in part **i** after translating right 1 unit. Find its equation, expanding any brackets.

c Parts **a** and **b** used the same two translations, right 1 unit and up 2 units, but in a different order. Do these transformations commute?

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2 As in Question 1, start with the parabola $y = x^2 - 2x$.

a **i** The parabola is stretched horizontally with factor 2. Sketch the situation and find its equation.

ii The parabola in part **i** is then stretched vertically with factor 3. Sketch the new graph and find its equation.

b **i** The original parabola $y = x^2 - 2x$ is stretched vertically with factor 3. Sketch the result and find its equation.

ii The parabola in part **i** is then stretched horizontally with factor 2. Sketch the situation and find its equation.

c Parts **a** and **b** used the same two dilations, horizontally with factor 2 and vertically with factor 3, but in a different order. Do these transformations commute?

3 Once again, start with the parabola $y = x^2 - 2x$.

a **i** The parabola is dilated horizontally with factor 2. Sketch the situation and find its equation.

ii The parabola in part **i** is then translated up 1 unit. Sketch the new graph and find its equation.

b **i** The original parabola $y = x^2 - 2x$ is shifted up 1 unit. Sketch the result and find its equation.

ii The parabola in part **i** is then dilated horizontally with factor 2. Sketch the situation and find its equation.

c Parts **a** and **b** used the same two transformations, stretched horizontally with factor 2 and shifted up 1 unit, but in a different order. Do these transformations commute?

4 Let $y = x^2 - 2x$. Sketch the graph of this function showing the intercepts and vertex.

a **i** The parabola is shifted right by 1 unit. Sketch the situation and find its equation, expanding any brackets.

ii The shifted parabola is then reflected in the y-axis. Sketch the new graph and find its equation.

b **i** The original parabola $y = x^2 - 2x$ is reflected in the y-axis. Sketch the result and find its equation.

ii The reflected parabola is then shifted right by 1 unit. Sketch the situation and find its equation, expanding any brackets.

c Parts **a** and **b** each used a shift right 1 unit and reflection in the y-axis, but in a different order. Do these two transformations commute?

DEVELOPMENT

5 Which of these pairs of transformations commute?

a reflection in the y-axis and horizontal translation,

b vertical dilation and vertical translation,

c vertical dilation and reflection in the x-axis,

d horizontal translation and vertical translation,

e horizontal dilation and horizontal translation,

f reflection in the x-axis and horizontal translation.

6 Write down the new equation for each function or relation after the given transformations have been applied. Draw a graph of the image.

a $y = x^2$: right 2 units, then dilate by factor $\frac{1}{2}$ horizontally

b $y = 2^x$: down 1 unit then reflect in the y-axis

c $y = x^2 - 1$: down 3 units, then dilate by factor -1 vertically

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d $y = \frac{1}{x}$: right 3 units then dilate by factor 2 vertically

e $x^2 + y^2 = 4$: up 2 units then dilate by factor $\frac{1}{2}$ vertically

f $y = \log_2 x$: left 1 units then dilate by factor 2 horizontally

g $y = \sin x$: left π units then reflect in the x-axis

h $y = \sqrt{x}$: up 2 units then dilate by factor -1 horizontally

7 Identify the various transformations to help graph these trigonometric functions. Make sure the transformations are applied in the correct order when they do not commute.

a $y = \sin 2x + 1$ **b** $y = 2\sin x + 1$ **c** $y = 2\sin\left(x + \frac{\pi}{4}\right)$ **d** $y = \sin\left(2x + \frac{\pi}{4}\right)$

8 Determine the equation of the curve after the given transformations have been applied in the order stated.

a $y = x^2$: left 1, down 4, dilate horizontally by 2

b $y = x^2$: down 4, dilate horizontally by 2, left 1

c $y = 2^x$: down 1, right 1, dilate vertically by -2

d $y = \dfrac{1}{x}$: right 2, dilate by 2 vertically, up 1

CHALLENGE

9 Identify the transformations of these trigonometric functions and hence sketch them.

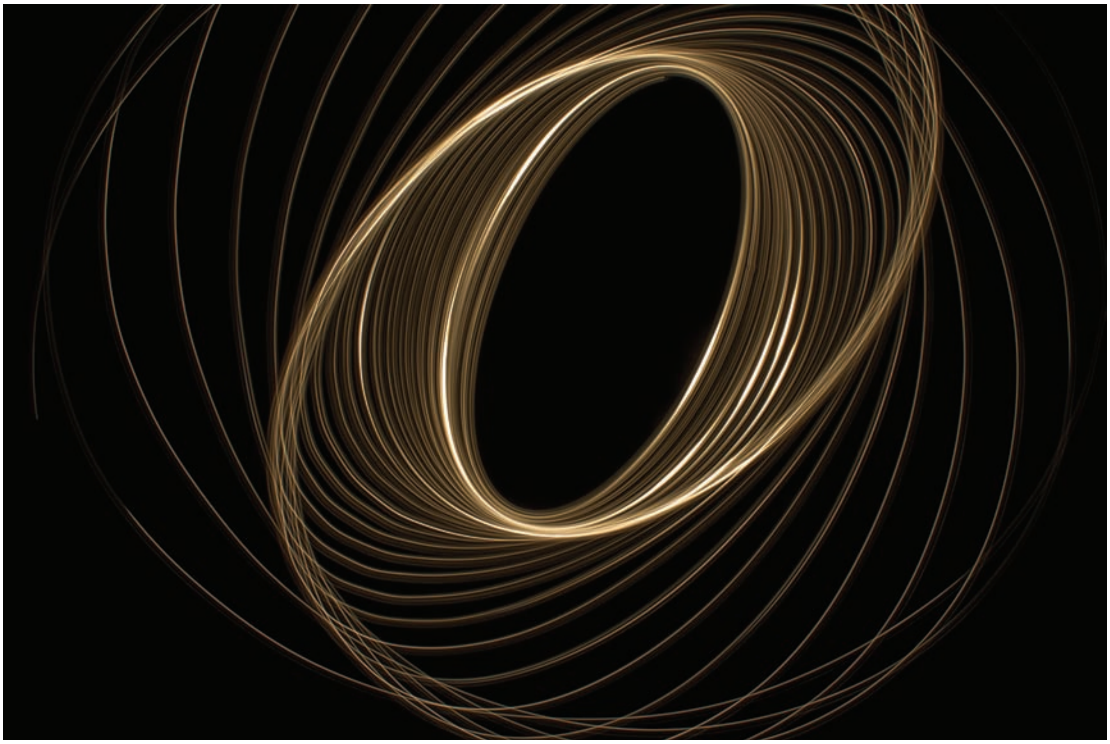
a $y = 3\cos 2x + 1$ **b** $y = 2\cos\left(x - \frac{\pi}{3}\right) + 2$

c $y = \cos\left(\frac{1}{2}x - \frac{\pi}{3}\right) + 1$ **d** $y = \cos\frac{1}{2}\left(x - \frac{\pi}{3}\right) + 1$

10 The parabola $y = (x - 1)^2$ is shifted 2 left and then reflected in the y-axis.

a Show that the new parabola has the same equation.

b Investigate why this has happened.



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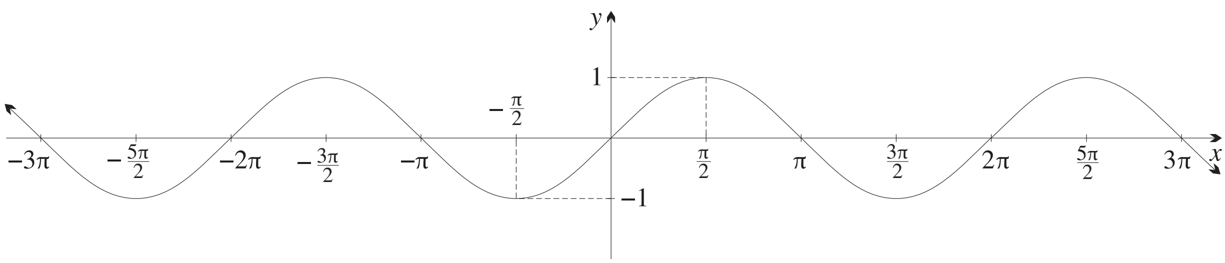
2I Trigonometric graphs

In Sections 9G–9J of the Year 11 book, we developed *radians* as a way of measuring angles. An angle measured in radians is a pure number, without units, and the important conversions between radians and the old familiar degrees are:

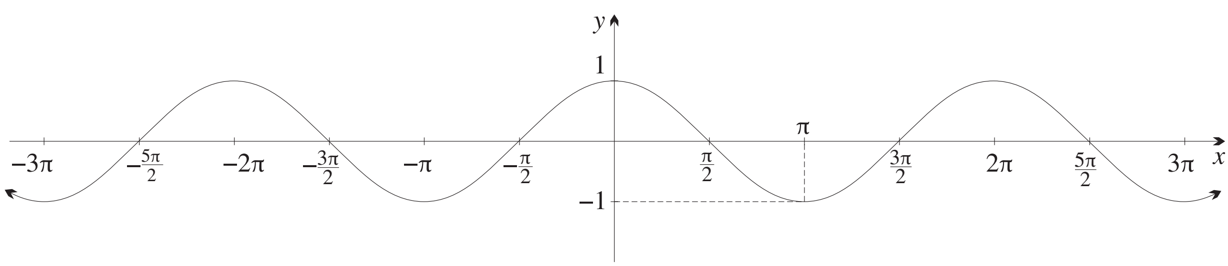
$$2\pi = 360°, \quad \pi = 180°, \quad \tfrac{\pi}{2} = 90°, \quad \tfrac{\pi}{3} = 60°, \quad \tfrac{\pi}{4} = 45°, \quad \tfrac{\pi}{6} = 30°.$$

In the final Section 9J we drew all six trigonometric graphs in radians. This present section deals only with $\sin x$, $\cos x$ and $\tan x$ — these three graphs are:

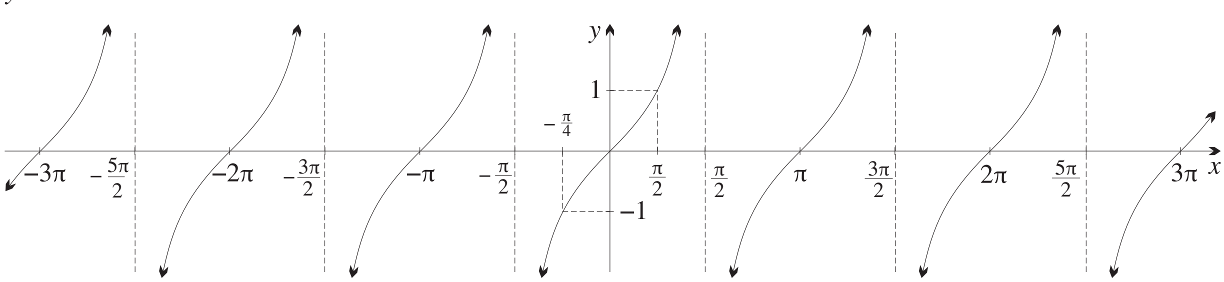
$y = \sin x$



$y = \cos x$



$y = \tan x$



We remarked in Section 9J that the trigonometric graphs were now drawn in the form appropriate for calculus. In particular, we shall prove in Chapter 6 that all three curves above have gradient 1 or -1 at all their x-intercepts.

Transformations are our concern here, however. The investigation Exercise 9J in the Year 11 book dealt thoroughly with the symmetries of these three graphs under translations, reflections in the axes, and rotations about the origin. We now have dilations, and this section shows how to generate any basic wave graph by combinations of translations and dilations.

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Reflections and rotations do not need review, so all the dilations in this section have positive factors. We will consider separately, then in combination:

- Vertical dilations with positive factors — this leads to the *amplitude*.
- Horizontal dilations with positive factors — this leads to the *period*.
- Translations left and right — this leads to the *phase*.
- Translations up and down — this leads to the *mean value*.

Vertical dilations and amplitude

The *amplitude* of a wave is the maximum height of the wave above its mean position. The graphs on the previous page both show that $y = \sin x$ and $y = \cos x$ have a maximum value of 1, a minimum value of -1 and a mean value of 0 (the average of 1 and -1). Thus both have amplitude 1.

Now let us apply a vertical dilation with factor a to $y = \sin x$.

Replacing y by $\dfrac{y}{a}$ gives $\qquad \dfrac{y}{a} = \sin x,$

and multiplying by a, $\qquad y = a \sin x.$

This function is also a wave, but its amplitude is now a, because it has maximum value $y = a$, minimum value $y = -a$, and mean value $y = 0$.

Exactly the same argument applies to $y = \cos x$.

24 VERTICAL DILATIONS AND AMPLITUDE

- The *amplitude* of a wave is the maximum height of the wave above its mean position.
- $y = \sin x$ and $y = \cos x$ both have amplitude 1.
- $y = a \sin x$ and $y = a \cos x$ both have amplitude a.
- $y = a \sin x$ and $y = a \cos x$ are the results of stretching $y = \sin x$ or $y = \cos x$ vertically with factor a.

We can stretch the function $y = \tan x$ vertically to $y = a \tan x$ in the usual way. But the function increases without bound near its asymptotes, so the idea of amplitude makes no sense. Instead, we can conveniently tie down the vertical scale of $y = a \tan x$ by using the fact that $\tan \frac{\pi}{4} = 1$, so when $x = \frac{\pi}{4}$, $y = a$.

Horizontal dilations and period

The trigonometric functions are called *periodic functions* because each graph repeats itself exactly over and over again. The *period* of such a function is the length of the smallest repeating unit.

The graphs of $y = \sin x$ and $y = \cos x$ on the previous page are waves, with a pattern that repeats every revolution. Thus they both have period 2π.

The graph of $y = \tan x$, on the other hand, has a pattern that repeats every half-revolution. Thus it has period π.

25 THE PERIODS OF THE TRIGONOMETRIC FUNCTIONS

- The *period* of a function that repeats is the length of the smallest repeating unit.
- $y = \sin x$ and $y = \cos x$ have period 2π (that is, a full revolution).
- $y = \tan x$ has period π (that is, half a revolution).

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Now consider the function $y = \sin nx$.

We can write this as $\qquad y = \sin \dfrac{x}{1/n}$,

which shows that it is a horizontal dilation of $y = \sin x$ with factor $\dfrac{1}{n}$.

Because $y = \sin x$ has period 2π, the dilation $y = \sin nx$ therefore has period $\dfrac{2\pi}{n}$.

The same arguments apply to $y = \cos nx$ and $y = \tan nx$.

26 HORIZONTAL DILATIONS AND PERIOD

- $y = \sin nx$ and $y = \cos nx$ have period $\dfrac{2\pi}{n}$.
- $y = \tan nx$ has period $\dfrac{\pi}{n}$.
- All three functions are the results of stretching $y = \sin x$, $y = \cos x$ or $y = \tan x$ horizontally with factor $\dfrac{1}{n}$.

The next worked example examines the amplitude and period together.

Example 25 2I

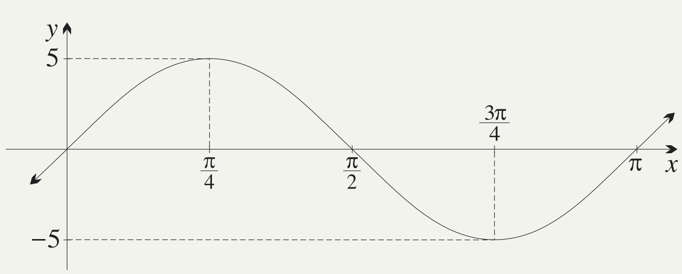
Find the period and amplitude of:

a $y = 5 \sin 2x$ **b** $y = 2 \tan \frac{1}{3}x$

Then sketch one period of the function, showing all intercepts, turning points and asymptotes.

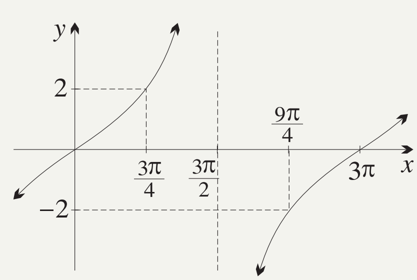
SOLUTION

a $y = 5 \sin 2x$ has an amplitude of 5, and a period of $\dfrac{2\pi}{2} = \pi$.



b $y = 2 \tan \dfrac{1}{3}x$ has period $\dfrac{\pi}{1/3} = 3\pi$.

It has no amplitude, but when $x = \frac{3\pi}{4}$, $y = 2 \tan \frac{\pi}{4} = 2$.



Horizontal translations and phase

The *initial phase angle*, or simply *phase*, of a trigonometric function is the angle when $x = 0$. Thus a function such as $y = \sin\left(x + \frac{\pi}{3}\right)$ has phase $\frac{\pi}{3}$, and $y = \sin x$ itself has phase 0.

Let us apply a translation left by α to the function $\quad y = \sin x$.

Replacing x by $x - (-\alpha) = x + a$ gives $\quad y = \sin(x + \alpha)$,

which is a sine wave with phase α, because when $x = 0$, the angle is $0 + \alpha = \alpha$.

The same argument applies to $y = \cos x$ and $y = \tan x$.

27 HORIZONTAL TRANSLATIONS AND PHASE

- The phase of a trigonometric function is the angle when $x = 0$.
- $y = \sin(x + \alpha)$, $y = \cos(x + \alpha)$ and $y = \tan(x + \alpha)$ all have phase α.
- All three functions are the result of shifting $y = \sin x$, $y = \cos x$ or $y = \tan x$ left by α.

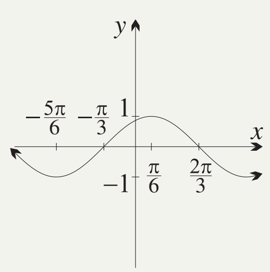
Example 26 2I

Use horizontal translations to sketch these functions, and state their phase.

a $y = \sin\left(x + \frac{\pi}{3}\right)$ **b** $y = \cos\left(x - \frac{5\pi}{2}\right)$ **c** $y = \tan\left(x - \frac{\pi}{4}\right)$

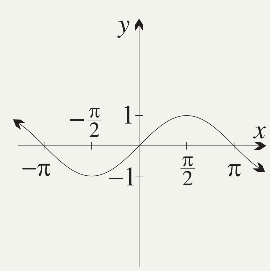
SOLUTION

a



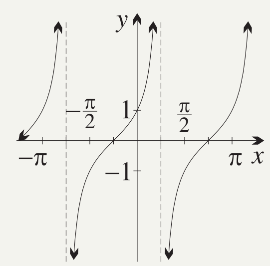
Shift left $\frac{\pi}{3}$, phase $\frac{\pi}{3}$

b



Shift right $\frac{5\pi}{2}$, phase $-\frac{5\pi}{2}$

c



Shift right $\frac{\pi}{4}$, phase $-\frac{\pi}{4}$

Note: The phase is not uniquely defined, because we can add and subtract multiples of the period. For example:

- In part **b**, it may be more convenient to write the phase as $-\frac{5\pi}{2} + 2\pi = -\frac{\pi}{2}$, or $-\frac{5\pi}{2} + 4\pi = \frac{3\pi}{2}$.
- In part **a**, we could also say that the phase is $\frac{\pi}{3} - 2\pi = -\frac{5\pi}{3}$.
- In part **c**, we could also say that the phase is $-\frac{\pi}{4} + \pi = \frac{3\pi}{4}$.

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Combining period and phase

The two dilations and one translation that we have introduced into trigonometry so far all commute, except only that a horizontal dilation and a horizontal translation do not commute — this needs attention. Consider the function

$$y = \sin\left(2x + \tfrac{\pi}{3}\right) \quad \text{or equivalently} \quad y = \sin 2\left(x + \tfrac{\pi}{6}\right).$$

The period is $\dfrac{2\pi}{2} = \pi$. The phase is $0 + \frac{\pi}{3} = \frac{\pi}{3}$, or equivalently $2\left(0 + \frac{\pi}{6}\right) = \frac{\pi}{3}$.

The first form $y = \sin\left(2x + \frac{\pi}{3}\right)$ of the equation regards the function as $y = \sin x$,

- shifted left $\frac{\pi}{3}$, giving $y = \sin\left(x + \frac{\pi}{3}\right)$,
- then stretched horizontally with factor $\frac{1}{2}$, giving $y = \sin\left(2x + \frac{\pi}{3}\right)$.

The second form $y = \sin 2\left(x + \frac{\pi}{6}\right)$ of the equation regards it as $y = \sin x$,

- stretched horizontally with factor $\frac{1}{2}$, giving $y = \sin 2x$,
- then shifted left $\frac{\pi}{6}$, giving $y = \sin 2\left(x + \frac{\pi}{6}\right)$.

The second way is what is suggested in the formula at the end of Section 2H, but either approach gets the result. In both cases, find the phase by putting $x = 0$.

Here is the general statement, but it is mostly better to work with transformations of each example individually than remember complicated formulae.

28 COMBINING PERIOD AND PHASE

- The function $y = \sin(nx + \alpha)$ has period $\dfrac{2\pi}{n}$ and phase α.
- Written as $y = \sin(nx + \alpha)$ it suggests transforming $y = \sin x$ by
  - a shift left α, followed by a horizontal stretch with factor $\dfrac{1}{n}$.
- Written as $y = \sin n\left(x + \dfrac{\alpha}{n}\right)$ it suggests transforming $y = \sin x$ by
  - a horizontal stretch with factor $\dfrac{1}{n}$, followed by a shift left by $\dfrac{\alpha}{n}$.

Vertical translations and the mean value

If there is a vertical translation, do it last so that it does not get confused with the vertical stretch associated with the amplitude (a vertical dilation and a vertical translation do not commute). A vertical translation shifts the mean value of the wave from 0 to some other value.

29 VERTICAL TRANSLATIONS AND THE MEAN VALUE

- The *mean value* of a wave is the mean of its maximum and minimum values.
- $y = \sin x + c$ and $y = \cos x + c$ both have mean value c.
- $y = \sin x + c$ is the result of shifting $y = \sin x$ up c.
- In a combination of transformations, do any vertical translation last.

The function $y = \tan x + c$ is not a wave, and has no mean value.

The next worked example below puts all four transformations together.

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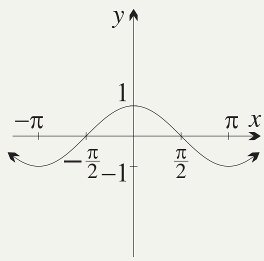
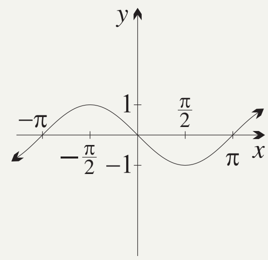
Example 27 2I

Use four successive transformations to sketch $y = 3\cos\left(2x + \frac{\pi}{2}\right) - 2$, and specify the amplitude, period, phase and mean value (and ignore x-intercepts this time).

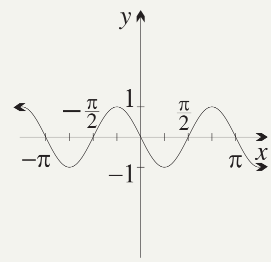
SOLUTION



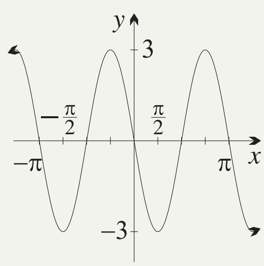
1 Start with $y = \cos x$.



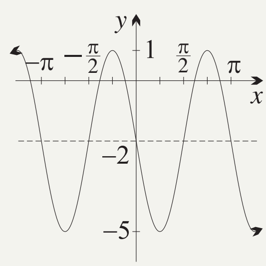
2 Shift $y = \cos x$ left $\frac{\pi}{2}$,
giving $y = \cos\left(x + \frac{\pi}{2}\right)$.
Phase is now $0 + \frac{\pi}{2} = \frac{\pi}{2}$.



3 Stretch horizontally factor $\frac{1}{2}$,
giving $y = \cos\left(2x + \frac{\pi}{2}\right)$.
The period is now $\frac{2\pi}{2} = \pi$.



4 Then stretch vertically with factor 3,
giving $y = 3\cos\left(2x + \frac{\pi}{2}\right)$.
The amplitude is now 3.



5 Shift the whole thing down 2 units,
giving $y = 3\cos\left(2x + \frac{\pi}{2}\right) - 2$.
The mean value is now -2.

Alternatively, rewrite the function as $y = 3\cos 2\left(x + \frac{\pi}{4}\right) - 2$. This suggests that the first two transformations are now:

- Stretch horizontally with factor $\frac{1}{2}$.
- Then shift left $\frac{\pi}{4}$.

Oddness and evenness of the trigonometric functions

Box 30 quickly reviews the oddness and evenness of the sine, cosine and tangent functions. These are crucially important properties of the three functions.

30 ODDNESS AND EVENNESS OF THE TRIGONOMETRIC FUNCTIONS

- The functions $y = \sin x$ and $y = \tan x$ are odd functions. Thus:
  $$\sin(-x) = -\sin x \quad \text{and} \quad \tan(-x) = -\tan x.$$
  and they both have point symmetry in the origin.
- The function $y = \cos x$ is an even function. Thus:
  $$\cos(-x) = \cos x.$$
  and it has line symmetry in the y-axis.

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Graphical solutions of trigonometric equations

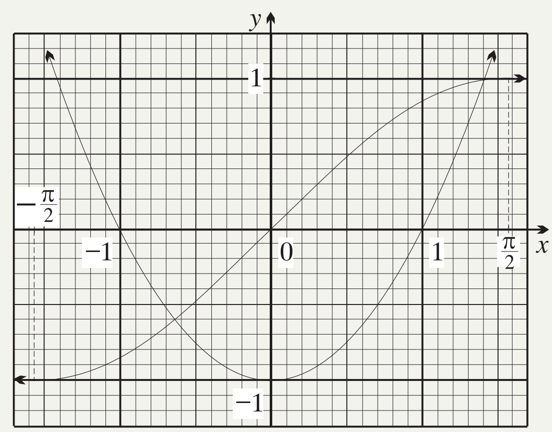
Many trigonometric equations cannot be solved by algebraic methods. Approximation methods using the graphs can usually be used instead and a graph-paper sketch will show:

- how many solutions there are, and
- the approximate values of the solutions.

Example 28 2I

a Find, by drawing a graph, the number of solutions of $\sin x = x^2 - 1$.

b Then use the graph to find approximations correct to one decimal place.



SOLUTION

a Here are $y = \sin x$ and $y = x^2 - 1$. Clearly the equation has two solutions.

b The positive solution is $x \doteqdot 1.4$, and the negative solution is $x \doteqdot -0.6$.

Note: Technology is particularly useful here. It allows sketches to be drawn quickly, and many programs will give the approximate coordinates of the intersections.

Exercise 2I FOUNDATION

Technology: Computer sketching can provide experience of a large number of graphs similar to the ones listed in this exercise. In particular, it is very useful in making clear the importance of period and amplitude and in reinforcing the formulae for them.

1 **a** Sketch the graph of each function for $0 \le x \le 2\pi$, stating the amplitude in each case.

  i $y = \frac{1}{2}\sin x$ **ii** $y = 2\sin x$ **iii** $y = 3\sin x$

b Describe the transformation from $y = \sin x$ to $y = k\sin x$. (Assume that k is positive.)

c How does the graph of $y = k\sin x$ change as k increases?

2 **a** Sketch the graph of each function for $0 \le x \le 2\pi$, and state the period in each case.

  i $y = \cos\frac{1}{2}x$ **ii** $y = \cos 2x$ **iii** $y = \cos 3x$

b Describe the transformation from $y = \cos x$ to $y = \cos nx$. (Assume that n is positive.)

c How does the graph of $y = \cos nx$ change as n increases?

3 **a** Sketch the graph of each function for $0 \le x \le 2\pi$, and state the period in each case.

  i $y = \tan x$ **ii** $y = \tan\frac{1}{2}x$ **iii** $y = \tan 2x$

b Describe the transformation from $y = \tan x$ to $y = \tan ax$. (Assume that a is positive.)

c How does the graph of $y = \tan ax$ change as a increases?

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4 **a** Sketch the graph of each function for $0 \le x \le 2\pi$, and state the phase in each case.

i $y = \sin\left(x + \frac{\pi}{2}\right)$ **ii** $y = \sin(x + \pi)$ **iii** $y = \sin(x + 2\pi)$

b Describe the transformation from $y = \sin x$ to $y = \sin(x + \alpha)$. (Assume that α is positive.)

c Describe the transformation when α is a multiple of 2π.

5 **a** Sketch the graph of each function for $0 \le x \le 2\pi$, and state the mean value and the range in each case.

i $y = \cos x + 1$ **ii** $y = \cos x + 2$ **iii** $y = \cos x + \frac{1}{2}$

b Describe the transformation from $y = \cos x$ to $y = \cos x + c$. (Assume that c is positive.)

c How does the graph of $y = \cos x + c$ change as c increases?

DEVELOPMENT

6 State the amplitude and period of each function, then sketch its graph for $-\pi \le x \le \pi$.

a $y = 3\cos 2x$ **b** $y = 2\sin\frac{1}{2}x$ **c** $y = \tan\frac{3x}{2}$ **d** $y = 2\cos 3x$

7 Write down a sequence of transformations that will transform $y = \sin x$ to the given function, and hence sketch the given function for $0 \le x \le 2\pi$.

a $y = 3\sin 3x$ **b** $y = -2\sin\frac{x}{2}$ **c** $y = 3\sin\left(x - \frac{\pi}{2}\right) + 2$

8 Write down a sequence of transformations that will transform $y = \cos x$ to the given function, and hence sketch the given function for $-\pi \le x \le \pi$.

a $y = 5\cos\frac{1}{2}x$ **b** $y = -2\cos 2x - 2$ **c** $y = \cos\left(2\left(x - \frac{\pi}{2}\right)\right)$

9 Write down a sequence of transformations that will transform $y = \sin x$ to the given function.

a $y = \sin\left(3x + \frac{\pi}{2}\right)$ **b** $y = \frac{1}{4}\sin(4x - \pi) - 4$ **c** $y = -6\sin\left(\frac{x}{2} + \frac{\pi}{4}\right)$

10 **a** What is the period and phase of each function in Question 9?

b What are the period and phase of these functions?

i $y = 3\sin 2\left(x - \frac{\pi}{3}\right)$ **ii** $y = \frac{5}{2}\cos\frac{1}{3}(x + \pi)$ **iii** $y = 2\tan 3\left(x + \frac{\pi}{8}\right)$

11 Solve each equation, for $0 \le x \le 2\pi$. Then indicate the solutions on a diagram showing sketches of the functions on the LHS and RHS of the equation.

a $2\sin\left(x - \frac{\pi}{3}\right) = 1$ **b** $2\cos 2x = -1$

12 Solve each equation, for $0 \le x \le \pi$, giving solutions correct to 3 decimal places.

a $\cos(x + 0.2) = -0.3$ **b** $\tan 2x = 0.5$

13 **a** Find the vertex of the parabola $y = x^2 - 2x + 4$.

b Hence show graphically that $x^2 - 2x + 4 > 3\sin x$ for all real values of x.

CHALLENGE

14 **a** Sketch the graph of $y = 2\cos x$ for $-2\pi \le x \le 2\pi$.

b On the same diagram, carefully sketch the line $y = 1 - \frac{1}{2}x$, showing its x- and y-intercepts.

c How many solutions does the equation $2\cos x = 1 - \frac{1}{2}x$ have?

d Mark with the letter P the point on the diagram from which the negative solution of the equation in part **c** is obtained.

e Prove algebraically that if n is a solution of the equation in part **c**, then $-2 \le n \le 6$.

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15 **a** What is the period of the function $y = \sin \frac{\pi}{2}x$?

b Sketch the curve $y = 1 + \sin \frac{\pi}{2}x$, for $0 \le x \le 4$.

c Through what fixed point does the line $y = mx$ always pass for varying values of m?

d By considering possible points of intersection of the graphs of $y = 1 + \sin \frac{\pi}{2}x$ and $y = mx$, find the values of m for which the equation $\sin \frac{\pi}{2}x = mx - 1$ has exactly one real solution in the domain $0 \le x \le 4$.

16 The depth of water in Dolphin Bay varies according to the tides. The depth is modelled by the equation $x = 2\cos\left(\frac{\pi}{7}t\right) + 8$, where x metres is the depth and t hours is the time since the last high tide. Last Saturday, it was high tide at 7 am.

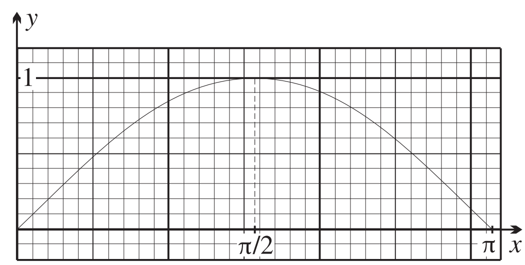
a How deep is the bay at high tide?

b How deep is the bay at low tide?

c When did the first low tide after 7 am occur?

d At what time last Saturday morning was the depth 9 metres?

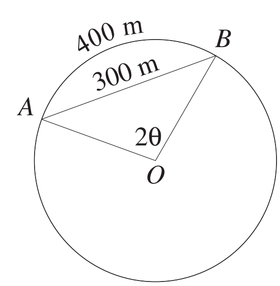
17 **a** **i** Photocopy this graph of $y = \sin x$, for $0 \le x \le \pi$, and on it graph the line $y = \frac{3}{4}x$.



ii Measure the gradient of $y = \sin x$ at the origin.

iii For what values of k does $\sin x = kx$ have a solution, for $0 < x < \pi$?

b The diagram shows points A and B on a circle with centre O, where $\angle AOB = 2\theta$, chord AB has length 300 metres, and the minor arc AB has length 400 metres.

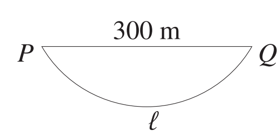


i Show that $\sin \theta = \frac{3}{4}\theta$.

ii Use the graph from part **a i** to determine θ, correct to one decimal place.

iii Hence find $\angle AOB$ in radians, correct to one decimal place, and show that the radius of the circle is about 154 metres.

c P and Q are two points 300 metres apart. The circular arc PQ has length ℓ metres.



i If C is the centre of the arc and $\angle PCQ = 2\alpha$, show that $\sin \alpha = \dfrac{300\alpha}{\ell}$.

ii Use your answer to part **a iii** to find the possible range of values of ℓ.

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Chapter 2 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

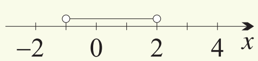
Chapter 2 Multiple-choice quiz

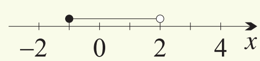
- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

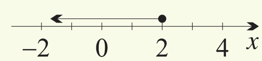
Chapter review exercise

1 For each number line, write the graphed interval using:

i inequality interval notation, **ii** bracket interval notation.

a 

b 

c 

2 If $f(x) = x^2 - 1$ and $g(x) = x + 1$, find:

a **i** $f \circ g(-2)$ **ii** $g \circ f(-2)$ **iii** $f \circ f(-2)$ **iv** $g \circ g(-2)$

b **i** $f \circ g(x)$ **ii** $g \circ f(x)$ **iii** $f \circ f(x)$ **iv** $g \circ g(x)$

3 Find the horizontal asymptotes of these functions by dividing through by the highest power of x in the denominator, and taking the limit as $x \to \infty$ and as $x \to -\infty$.

a $f(x) = \dfrac{1}{x + 2}$ **b** $f(x) = \dfrac{x - 3}{2x + 5}$ **c** $\dfrac{x}{x^2 + 1}$

4 Let $y = x^3 - 9x^2 + 18x$.

a State the domain using inequality interval notation.

b Write down the coordinates of any intercepts with the axes.

c Does this function have any asymptotes?

d Use this information and a table of values to sketch the curve.

e The graph seems to be horizontal somewhere in the interval $0 < x < 3$, and again in the interval $3 < x < 6$. Use calculus to find the x-coordinates of these points, and add them to the diagram.

5 Solve each double inequation, then write your answer in bracket interval notation.

a $-6 < -3x \le 12$ **b** $-2 < 2x + 1 < 1$

c $-7 \le 5 + 4x < 7$ **d** $-4 \le 1 - \frac{1}{2}x \le 3$

6 Carefully draw the graphs of the LHS and RHS of each equation on the same number plane in order to find the number of solutions. Do not attempt to solve them.

a $x - 2 = \log_2 x$

b $\cos x = 1 - x^2$

c $x(x - 2)(x + 2) = 2 - |x|$

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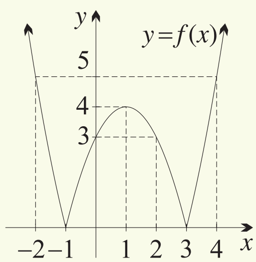
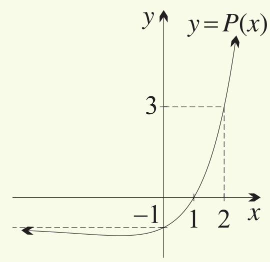
7 In each case an unknown function has been drawn. Draw the functions specified below it.

a



i $y = f(x - 1)$ **ii** $y = f(x) + 1$

b



i $y = P(x + 1)$ **ii** $y = P(x) - 1$

8 In each case apply the indicated dilation to the corresponding function in Question 7 and draw the resulting graph.

a **i** $y = f\left(\frac{1}{2}x\right)$ **ii** $y = \frac{1}{2}f(x)$ **b** **i** $y = P(2x)$ **ii** $y = 2P(x)$

9 In each case, completely factor the given polynomial where necessary and hence sketch its graph. A table of values may also help. Then use the graph to solve $f(x) \le 0$.

a $f(x) = (x + 1)(x - 3)$ **b** $f(x) = x(x - 2)(x + 1)$ **c** $f(x) = x^2 - 4x - 5$
d $f(x) = 3 - 2x - x^2$ **e** $f(x) = 2x - x^2 - x^3$ **f** $f(x) = x^3 + 4x^2 + 4x$

10 Let $y = \dfrac{4}{(x + 2)(2 - x)}$.

a State the natural domain.
b Find the y-intercept.
c Show that $y = 0$ is a horizontal asymptote.
d Draw up a table of signs.
e Identify the vertical asymptotes, and use the table of signs to describe its behaviour near them.
f Sketch the graph of the function and state its range using bracket interval notation

11 **a** Factor the right-hand side of $y = \dfrac{3x + 3}{x^2 + 2x - 3}$.
b State the domain and any intercepts with the axes.
c Explain why the function is neither even nor odd.
[HINT: The answers to **a** may help.]
d Find the equations of the asymptotes.
e Sketch the graph of this curve.

12 Solve these equations and inequations algebraically.

a $|2x| = 7$ **b** $|3x - 2| = 1$ **c** $|3x + 5| \le 4$ **d** $|6x + 7| > 5$

13 Carefully sketch the functions on the LHS and RHS of each inequation on the same number plane. Then use the graph to solve the inequations.

a $x - 1 \ge 1 + \frac{1}{2}x$ **b** $\dfrac{1}{1 - x} > 1 - 2x$

c $|2x| \le x + 3$ **d** $\left|\frac{1}{2}x + 1\right| > \frac{1}{4}(x + 5)$

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14 Write down the equation for each function after the given translations have been applied.

a $y = x^2$: right 2 units, up 1 unit
b $y = \frac{1}{x}$: left 2 units, down 3 units
c $y = \sin x$: left $\frac{\pi}{6}$ units, down 1 unit
d $y = e^x$: right 2 units, up 1 unit

15 In each case identify how the graph of the second equation can be obtained from the graph of the first by a suitable dilation.

a $y = x^2 - 2x$ and $y = \frac{1}{4}x^2 - x$
b $y = \dfrac{1}{x-4}$ and $y = \dfrac{1}{2x-8}$
c $y = \cos x$ and $y = \frac{1}{3}\cos x$
d $y = \dfrac{1}{x+1}$ and $y = \dfrac{2}{x+2}$

16 Which of these pairs of transformations commute?

a reflection in the y-axis and reflection in the x-axis,
b vertical reflection and vertical translation,
c horizontal translation and horizontal dilation,
d vertical translation and horizontal dilation.

17 Identify the various transformations of the standard functions and hence graph each. Make sure the transformations are applied in the correct order when they do not commute.

a $y = 4 - 2^x$
b $y = \frac{1}{2}(x-2)^2 - 1$
c $y = 2\sin\left(x + \frac{\pi}{6}\right) + 1$

18 Write down the amplitude and period, then sketch the graph for $-\pi \le x \le \pi$.

a $y = 4\sin 2x$
b $y = \frac{3}{2}\cos\frac{1}{2}x$

19 **a** Explain how the graph of $y = \tan x$ can be transformed into the graph of $y = 1 - \tan x$
b Hence sketch $y = 1 - \tan x$ for $-\pi \le x \le \pi$.

20 Write down a sequence of transformations that will transform $y = \cos x$ into:

a $y = 3\cos(-x) - 2$
b $y = 4\cos\left(4\left(x + \frac{\pi}{2}\right)\right)$
c $y = \cos\left(2x - \frac{\pi}{3}\right)$

21 What is the phase of the three functions in Question 19?

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3 Curve-sketching using the derivative

This chapter will use the derivative to extend the systematic approach to sketching curves developed in Chapter 2 by asking two further questions:

1 Where is the curve sloping upwards, where is it sloping downwards, and where does it have any maximum or minimum values?

2 Where is the curve concave up, where is it concave down, and are there points of inflection where the curve changes from one concavity to the other?

These are standard procedures for investigating unfamiliar curves. In particular, the algorithm for finding the maximum and minimum values of a function can be applied to all sorts of practical and theoretical questions.

The chapter concludes with a fuller account of primitives than was appropriate in Year 11, in preparation for integration in Chapter 4.

Curve-sketching software is very useful when studying this chapter, because it can easily show the effect on the graph of changing the equation of the curve.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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3A Increasing, decreasing and stationary at a point

We have used the terms *increasing* and *decreasing* freely so far without precise definitions. This section uses tangents to formalise the ideas of increasing and decreasing at a point. Later, in Chapter 7, we will use chords to formalise the ideas of increasing and decreasing over an interval.

Tangents and the behaviour of a curve at a point

At a point where a curve is sloping upwards, the tangent has positive gradient, and y is increasing as x increases. At a point where it is sloping downwards, the tangent has negative gradient, and y is decreasing as x increases.

Let $f(x)$ be a function that can be differentiated at $x = a$.

1 INCREASING, DECREASING AND STATIONARY AT A POINT

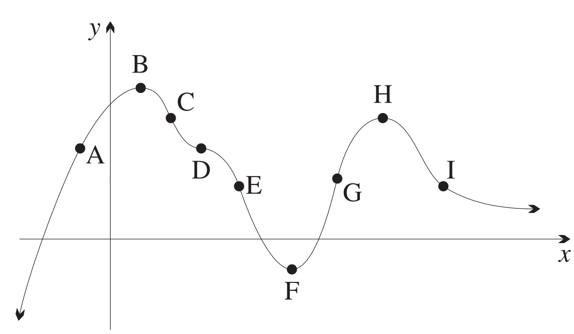
Let $f(x)$ be a function that can be differentiated at $x = a$.

- If $f'(a) > 0$, then $f(x)$ is called *increasing* at $x = a$.
- If $f'(a) < 0$, then $f(x)$ is called *decreasing* at $x = a$.
- If $f'(a) = 0$, then $f(x)$ is called *stationary* at $x = a$.

For example, the curve in the diagram to the right is:

- increasing at A and G,
- decreasing at C, E and I,
- stationary at B, D, F and H.

Think about the tangents to the curve at each of the nine points.



Example 1 3A

Differentiate $f(x) = x^3 - 12x$. Hence find whether the curve $y = f(x)$ is increasing, decreasing or stationary at the point where:

a $x = 5$ **b** $x = 2$ **c** $x = 0$

SOLUTION

Differentiating, $f'(x) = 3x^2 - 12$.

a $f'(5) = 75 - 12 > 0$, so the curve is increasing at $x = 5$.

b $f'(2) = 12 - 12 = 0$, so the curve is stationary at $x = 2$.

c $f'(0) = 0 - 12 < 0$, so the curve is decreasing at $x = 0$.

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Example 2 3A

For what value(s) of x is the curve $y = x^4 - 4x$ stationary?

SOLUTION

Differentiating, $y' = 4x^3 - 4$
$= 4(x^3 - 1)$.

Put $y' = 0$ to find where the curve is stationary.

Then $x^3 = 1$
$x = 1$.

Example 3 3A

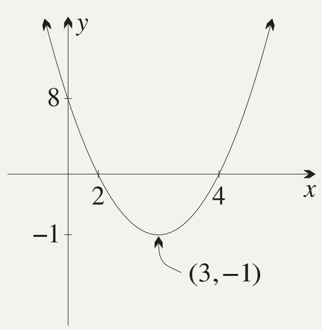
a Differentiate $y = (x - 2)(x - 4)$.

b Hence find the values of x where the curve is stationary, and where it is decreasing. Then sketch the curve.

SOLUTION

a Expanding, $y = x^2 - 6x + 8$,
and differentiating, $y' = 2x - 6$
$= 2(x - 3)$.

b When $x = 3$, $y' = 0$, so the curve is stationary at $x = 3$.
When $x < 3$, $y' < 0$, so the curve is decreasing for $x < 3$.



Example 4 3A

a Show that $f(x) = x^3 + x - 1$ is always increasing.

b Find $f(0)$ and $f(1)$, and hence explain why the curve has exactly one x-intercept.

SOLUTION

a Differentiating, $f'(x) = 3x^2 + 1$.
Because squares can never be negative, $f'(x)$ can never be less than 1, so the function is increasing for every value of x.

b Substituting, $f(0) = -1$ and $f(1) = 1$.
Because $f(0)$ is negative and $f(1)$ is positive, and the curve is continuous, the curve crosses the x-axis somewhere between 0 and 1.
Because the function is increasing for every value of x, it can never go back and cross the x-axis at a second point.

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Exercise 3A

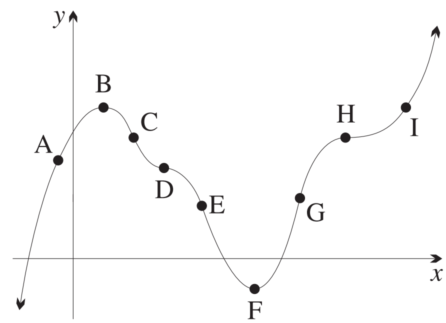
FOUNDATION

1 In the diagram to the right, name the points where:

a $f'(x) > 0$

b $f'(x) < 0$

c $f'(x) = 0$



2 Find the derivative of each function. By substituting $x = 1$ into the derivative, determine whether the function is increasing, decreasing or stationary at $x = 1$.
(A function is increasing at some point on the curve when $\frac{dy}{dx} > 0$ there, it is decreasing when $\frac{dy}{dx} < 0$ there, and it is stationary when $\frac{dy}{dx} = 0$ there.)

a $y = x^2$ **b** $y = x^2 - 2x$ **c** $y = 3x^2 - 8x$

d $y = x^2 - 3x + 7$ **e** $y = x^3 + 5x$ **f** $y = 4x^3 - 3x^4$

3 **a** Find the derivative $f'(x)$ of $f(x) = x^2 - 6x + 11$.

b Hence find whether the curve $y = f(x)$ is increasing, decreasing or stationary at:

i $x = 0$ **ii** $x = 1$ **iii** $x = 3$ **iv** $x = 4$ **v** $x = -1$

4 **a** Find the derivative $f'(x)$ of $f(x) = x^3 - 6x^2 + 9x$.

b Hence find whether the curve $y = f(x)$ is increasing, decreasing or stationary at:

i $x = 0$ **ii** $x = 1$ **iii** $x = 2$ **iv** $x = 3$ **v** $x = -1$

5 By finding where the derivative is zero, find the x-coordinates of the stationary points of each function.

a $y = x^2 - 2x$ **b** $y = x^2 - 4x + 3$ **c** $y = x^2 + 6x + 9$

d $y = 2x^2 - 16x$ **e** $y = x^3 - 3x^2$ **f** $y = x^3 - 12x$

6 **a** Explain why $y = -5x + 2$ is decreasing for all x.

b Explain why $y = x + 7$ is increasing for all x.

c Explain why $f(x) = x^3$ is increasing for all values of x, apart from $x = 0$, where it is stationary.

d Explain why $f(x) = x^2$ is increasing for $x > 0$ and decreasing for $x < 0$. What happens at $x = 0$?

DEVELOPMENT

7 Differentiate each function using the chain rule. Then evaluate $f'(0)$ to establish whether the curve is increasing, decreasing or stationary at $x = 0$.

a $f(x) = (x - 1)^3$ **b** $f(x) = (2x - 1)^4$ **c** $f(x) = (x^2 + 3)^2$

8 Differentiate each function using the product rule. Then evaluate $f'(1)$ to establish whether the curve is increasing, decreasing or stationary at $x = 1$.

a $f(x) = (x - 5)(x + 3)$ **b** $f(x) = (x - 2)(x^2 + 5)$ **c** $f(x) = (x^4 + 2)(1 - x^3)$

9 Differentiate each function using the quotient rule. Then evaluate $f'(2)$ to establish whether the curve is increasing, decreasing or stationary at $x = 2$.

a $f(x) = \dfrac{x}{x + 1}$ **b** $f(x) = \dfrac{x + 1}{x - 1}$ **c** $f(x) = \dfrac{x^2}{x + 2}$

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10 Differentiate each function by first writing it in index form. Then evaluate $f'(1)$ to establish whether the curve is increasing, decreasing or stationary at $x = 1$.

a $f(x) = \sqrt{x}$ **b** $f(x) = \dfrac{1}{x}$ **c** $f(x) = -\dfrac{1}{x^2}$

11 **a** Find $f'(x)$ for the function $f(x) = 4x - x^2$.

b For what values of x is:

i $f'(x) > 0$, **ii** $f'(x) < 0$, **iii** $f'(x) = 0$?

c Find $f(2)$. Then, by interpreting these results geometrically, sketch $y = f(x)$.

12 **a** Find $f'(x)$ for the function $f(x) = x^2 - 4x + 3$.

b For what values of x is:

i $f'(x) > 0$, **ii** $f'(x) < 0$, **iii** $f'(x) = 0$?

c Evaluate $f(2)$. Then, by interpreting these results geometrically, sketch $y = f(x)$.

13 **a** Let $f(x) = x^3 - 3x^2 - 9x - 2$. Show that $f'(x) = 3(x - 3)(x + 1)$.

b By sketching a graph of $y = f'(x)$, show that $f(x)$ is increasing when $x > 3$ or $x < -1$.

14 **a** Find the derivative $f'(x)$ of $f(x) = x^3 + 2x^2 + x + 7$.

b Use factoring or the quadratic formula to find the zeroes of $f'(x)$.

c Sketch the graph of $y = f'(x)$.

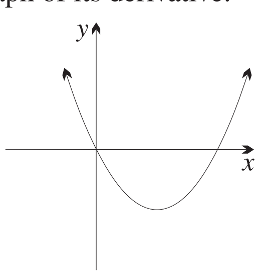
d Hence find the values of x for which $f(x)$ is decreasing.

15 Find the derivative of each function. By solving $y' > 0$, find the values of x for which the function is increasing.

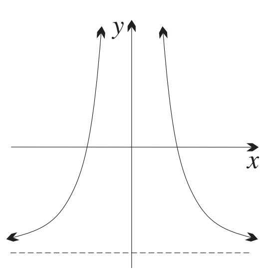
a $y = x^2 - 4x + 1$ **b** $y = 7 - 6x - x^2$ **c** $y = 2x^3 - 6x$ **d** $y = x^3 - 3x^2 + 7$

16 The graphs of four functions a, b, c and d are shown below. The graphs of the derivatives of these functions, in scrambled order, are shown in I, II, III and IV. Match the graph of each function with the graph of its derivative.

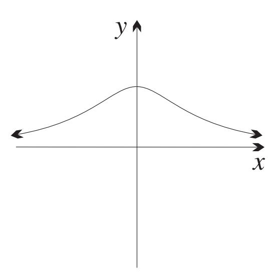
a



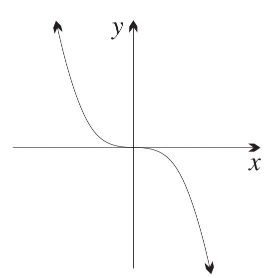
b



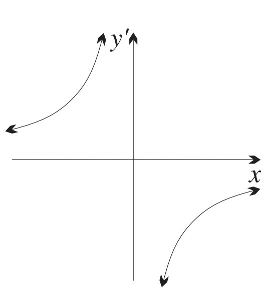
c



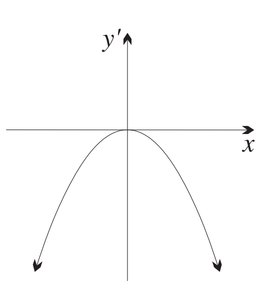
d



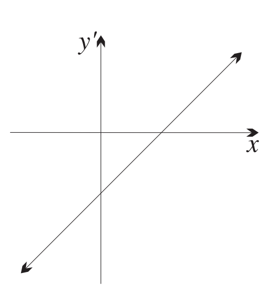
I



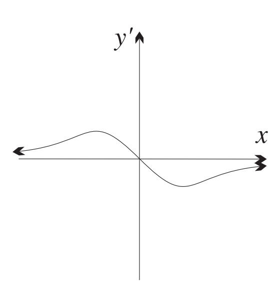
II



III



IV



17 **a** Differentiate $f(x) = -\dfrac{1}{x}$, and hence prove that $f(x)$ increases for all x in its domain.

b Sketch a graph of $f(x) = -\dfrac{1}{x}$, and explain why $f(-1) > f(2)$ despite this fact.

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18 **a** Use the quotient rule to find the derivative $f'(x)$ of $f(x) = \dfrac{2x}{x - 3}$.

b Explain why $f(x)$ is decreasing for all $x \neq 3$.

19 **a** Use the quotient rule to find the derivative $f'(x)$ of $f(x) = \dfrac{x^3}{x^2 + 1}$.

b Explain why $f(x)$ is increasing for all x, apart from $x = 0$ where it is stationary.

20 **a** Find $f'(x)$ for the function $f(x) = \frac{1}{3}x^3 + x^2 + 5x + 7$.

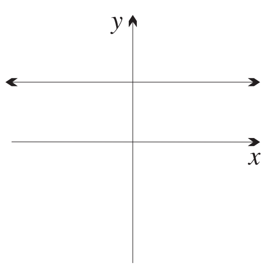
b By completing the square, show that $f'(x) = (x + 1)^2 + 4$, and hence explain why $f(x)$ is increasing for all x.

c Evaluate $f(-3)$ and $f(0)$, and hence explain why the curve $y = f(x)$ has exactly one x-intercept.

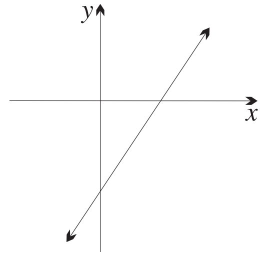
CHALLENGE

21 Look carefully at each function graphed below to establish where it is increasing, decreasing and stationary. Hence sketch the graph of the derivative of the function.

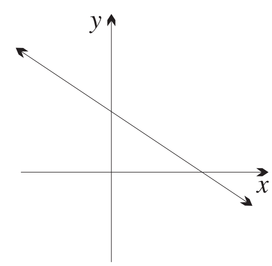
a



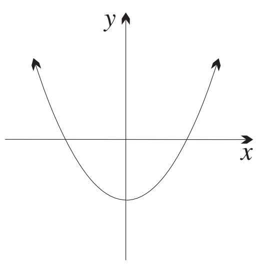
b



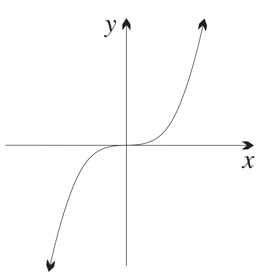
c



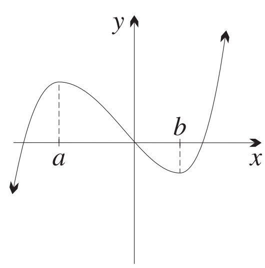
d



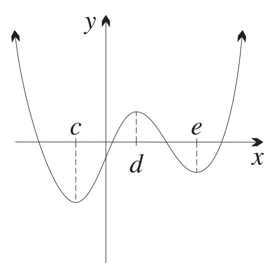
e



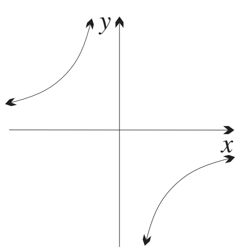
f



g



h



22 **a** If $f(x) = -x^3 + 2x^2 - 5x + 3$, find $f'(x)$.

b By evaluating the discriminant $\triangle$, show that $f'(x) < 0$ for all values of x.

c Hence deduce the number of solutions of the equation $3 - 5x + 2x^2 - x^3 = 0$.

23 Sketch possible graphs of continuous curves that have the properties below.

a $f(1) = f(-3) = 0$,
$f'(-1) = 0$,
$f'(x) > 0$ when $x < -1$,
$f'(x) < 0$ when $x > -1$.

b $f(2) = f'(2) = 0$,
$f'(x) > 0$ for all $x \neq 2$.

c $f(x)$ is odd,
$f(3) = 0$ and $f'(1) = 0$,
$f'(x) > 0$ for $x > 1$,
$f'(x) < 0$ for $0 \leq x < 1$.

d $f(x) > 0$ for all x,
$f'(0) = 0$,
$f'(x) < 0$ for $x < 0$,
$f'(x) > 0$ for $x > 0$.

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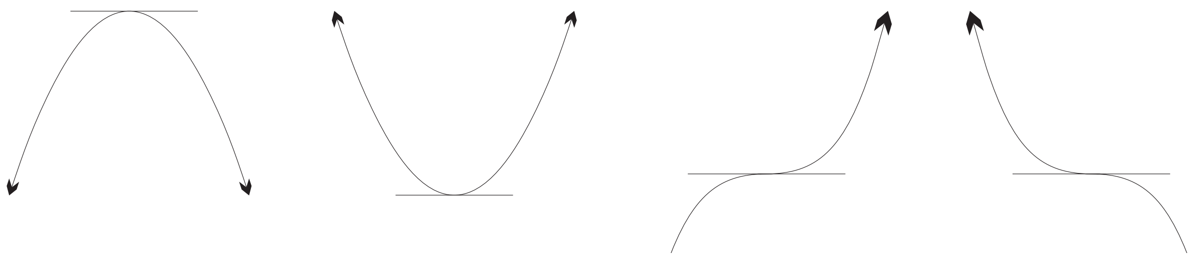
3B Stationary points and turning points

Stationary points on a curve that is not a constant function near that point can be classified into four different types:



Maximum turning point | Minimum turning point | Stationary point of inflection | Stationary point of inflection

Turning points

The first stationary point is a *maximum turning point* — the curve turns smoothly from increasing to decreasing, with a maximum value at the point.

The second stationary point is a *minimum turning point* — the curve turns smoothly from decreasing to increasing, with a minimum value at the point.

2 TURNING POINTS

A stationary point is called a *turning point* if the derivative changes sign around the point.
- At a *maximum turning point*, the curve changes from increasing to decreasing.
- At a *minimum turning point*, the curve changes from decreasing to increasing.

Stationary points of inflection

In the third and fourth diagrams above, there is no turning point. In the third diagram, the curve is increasing on both sides of the stationary point, and in the fourth, the curve is decreasing on both sides.

Instead, the curve *flexes* around the stationary point, changing *concavity* from downwards to upwards, or from upwards to downwards. The surprising effect is that *the tangent at this type of stationary point actually crosses the curve.*

3 POINTS OF INFLECTION

- A *point of inflection* is a point on the curve where the tangent crosses the curve. This means that the concavity changes from upwards to downwards, or from downwards to upwards, around the point.
- A *stationary point of inflection* is a point of inflection where the tangent is horizontal. Thus it is both a point of inflection and a stationary point.

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Local or relative maximum and minimum

A *local* or *relative maximum* is a point where the curve reaches a maximum in its immediate neighbourhood. Sometimes there is no tangent at the point — look at points C and I in the diagram below.

4 LOCAL MAXIMA AND MINIMA

Let $A\big(a, f(a)\big)$ be a point on $y = f(x)$. There may or may not be a tangent at A.

- The point A is called a *local* or *relative maximum* if

  $f(x) \le f(a)$, for all x in some small interval around a.
- Similarly, A is called a *local* or *relative minimum* if

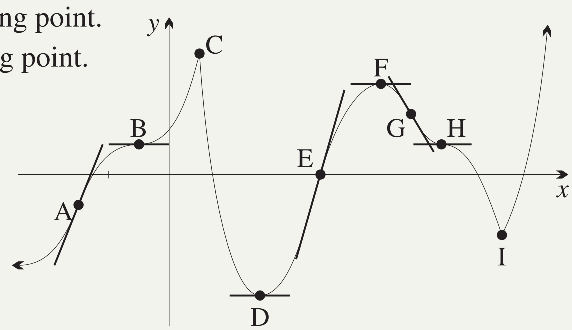
  $f(x) \ge f(a)$, for all x in some small interval around a.

Example 5 3B

Classify the points labelled A–I in the diagram below.

SOLUTION

C and F are local maxima, but only F is a maximum turning point.
D and I are local minima, but only D is a minimum turning point.
B and H are stationary points of inflection.
A, E and G are also points of inflection, but are not stationary points.



Note: The point F is called a *maximum turning point* rather than a 'local maximum turning point'. This is because when we are classifying turning points, we are only ever interested in the immediate neighbourhood of the point.

Analysing stationary points with a table of slopes

Section 2A explained how a function can only change sign at a zero or a discontinuity. Similarly, the derivative $f'(x)$ can only change sign at a zero or discontinuity of $f'(x)$, meaning a stationary point of $f(x)$ or a point where $f(x)$ is not differentiable.

This gives a straightforward method for analysing the stationary points. The method also gives an overall picture of the shape of the function.

5 USING THE DERIVATIVE $f'(x)$ TO ANALYSE STATIONARY POINTS AND SLOPE

1. Find the zeroes and discontinuities of the derivative $f'(x)$.
2. Then draw up a table of test values of the derivative $f'(x)$ dodging around its zeroes and discontinuities, with the slopes underneath, to see where the gradient changes sign.

The resulting *table of slopes* shows not only the nature of each stationary point, but also where the function is increasing and decreasing across its whole domain. This gives an outline of the shape of the curve, in preparation for a proper sketch.

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Example 6 3B

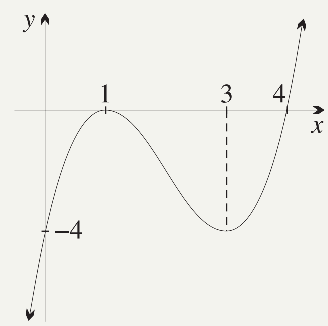
Find the stationary points of the cubic $y = x^3 - 6x^2 + 9x - 4$, use a table of slopes to determine their nature, and sketch the curve.

SOLUTION

$$\frac{dy}{dx} = 3x^2 - 12x + 9$$
$$= 3(x^2 - 4x + 3)$$
$$= 3(x - 1)(x - 3),$$

so y' has zeroes at $x = 1$ and 3, and no discontinuities.

x	0	1	2	3	4
y'	9	0	-3	0	9
slope	/	—	\	—	/



When $x = 1$, $y = 1 - 6 + 9 - 4$
$= 0$,
and when $x = 3$, $y = 27 - 54 + 27 - 4$
$= -4$.

Hence $(1, 0)$ is a maximum turning point, and $(3, -4)$ is a minimum turning point.

Note: Only the signs of y' are relevant, but if the actual values of y' are not calculated, some other argument should be given as to how the signs were obtained.

Example 7 3B

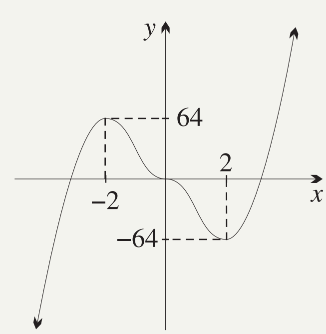
Find the stationary points of the quintic $f(x) = 3x^5 - 20x^3$, use a table of slopes to determine their nature, and sketch the curve.

SOLUTION

$$f'(x) = 15x^4 - 60x^2$$
$$= 15x^2(x^2 - 4)$$
$$= 15x^2(x - 2)(x + 2),$$

so $f'(x)$ has zeroes at $x = -2$, $x = 0$ and $x = 2$, and has no discontinuities.

x	-3	-2	-1	0	1	2	3
$f'(x)$	675	0	-45	0	-45	0	675
slope	/	—	\	—	\	—	/



When $x = 0$, $y = 0 - 0 = 0$,
when $x = 2$ $y = 96 - 160 = -64$
and when $x = -2$, $y = -96 + 160 = 64$.

Hence $(-2, 64)$ is a maximum turning point, $(2, -64)$ is a minimum turning point, and $(0, 0)$ is a stationary point of inflection.

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Note: This function $f(x) = 3x^5 - 20x^3$ is odd, and it has as its derivative $f'(x) = 15x^4 - 60x^2$, which is even. In general, the derivative of an even function is odd, and the derivative of an odd function is even (see question 18 in Exercise 3B). This provides a useful check.

Finding pronumerals in a function

In this worked example, the pronumerals in a function are found using information about a stationary point of the curve.

Example 8 3B

The graph of the cubic $f(x) = x^3 + ax^2 + bx$ has a stationary point at $A(2, 2)$. Find a and b.

SOLUTION

To find the two unknown constants, we need two independent equations.

Because $f(2) = 2$, $\quad 2 = 8 + 4a + 2b$

$2a + b = -3. \quad (1)$

Differentiating, $\quad f'(x) = 3x^2 + 2ax + b,$

and because $f'(2) = 0$, $\quad 0 = 12 + 4a + b$

$4a + b = -12. \quad (2)$

Subtracting (1) from (2), $\quad 2a = -9$

$a = -4\tfrac{1}{2},$

and substituting into (1), $-9 + b = -3$

$b = 6.$

Exercise 3B

FOUNDATION

1 By finding where the derivative equals zero, determine the x-coordinates of any stationary points of each function.

a $y = x^2 - 6x + 8$ b $y = x^2 + 4x + 3$ c $y = x^3 - 3x$

2 By finding where the derivative equals zero, determine the coordinates of any stationary points of each function. (Remember that you find the y-coordinate by substituting the x-coordinate into the original function.)

a $y = x^2 - 4x + 7$ b $y = x^2 - 8x + 16$ c $y = 3x^2 - 6x + 1$

d $y = -x^2 + 2x - 1$ e $y = x^3 - 3x^2$ f $y = x^4 - 4x + 1$

3 Find the derivative of each function and complete the given table to determine the nature of the stationary point. Sketch each graph, indicating all important features.

a $y = x^2 - 4x + 3$:

x	1	2	3
y'			
slope			

b $y = 12 + 4x - x^2$:

x	1	2	3
y'			
slope			

c $y = x^2 + 6x + 8$:

x	-4	-3	-2
y'			
slope			

d $y = 15 - 2x - x^2$:

x	-2	-1	0
y'			
slope			

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4 Differentiate each function and show that there is a stationary point at $x = 1$. Then use a table of test values of $f'(x)$ to determine the nature of the stationary point at $x = 1$.

a $f(x) = x^2 - 2x - 3$

b $f(x) = 15 + 2x - x^2$

c $f(x) = x^3 + 3x^2 - 9x + 2$

d $f(x) = x^3 - 3x^2 + 3x + 1$

5 Find the stationary point of each function and use a table of test values of $\dfrac{dy}{dx}$ to determine its nature. Sketch each graph, indicating all intercepts with the axes.

a $y = x^2 + 4x - 12$

b $y = 5 - 4x - x^2$

DEVELOPMENT

6 a Show that the derivative of $y = x^3 - 3x^2$ is $\dfrac{dy}{dx} = 3x(x - 2)$.

b Use a table of slopes to show that there is a maximum turning point at $(0, 0)$ and a minimum turning point at $(2, -4)$.

c Sketch the graph of the function, showing all important features.

7 a Show that the derivative of $y = 12x - x^3$ is $y' = 3(2 - x)(2 + x)$.

b Use a table of test values of y' to show that there is a maximum turning point at $(2, 16)$ and a minimum turning point at $(-2, -16)$.

c Sketch the graph of the function, showing all important features.

8 Find the stationary points of each function, then determine their nature using a table of slopes. Sketch each graph. (You need not find the x-intercepts.)

a $y = 2x^3 + 3x^2 - 36x + 15$

b $y = x^3 + 4x^2 + 4x$

c $y = 16 + 4x^3 - x^4$

d $y = 3x^4 - 16x^3 + 24x^2 + 11$

9 a Use the product rule to show that if $y = x(x - 2)^3$, then $y' = 2(2x - 1)(x - 2)^2$.

b Find any stationary points and use a table of gradients to classify them.

c Sketch the graph of the function, indicating all important features.

10 a Use the product rule to show that if $y = x^2(x - 4)^2$, then $\dfrac{dy}{dx} = 4x(x - 4)(x - 2)$.

b Find any stationary points and use a table of gradients to classify them.

c Sketch the graph of the function, indicating all important features.

11 a Use the product rule to show that if $y = (x - 5)^2(2x + 1)$, then $y' = 2(x - 5)(3x - 4)$.

b Find any stationary points and use a table of slopes to classify them.

c Sketch the graph of the function, indicating all important features.

12 a The tangent to the curve $y = x^2 + ax - 15$ is horizontal at the point where $x = 4$. Find the value of a.

b The curve $y = x^2 + ax + 7$ has a turning point at $x = -1$. Find the value of a.

13 a The curve $f(x) = ax^2 + 4x + c$ has a turning point at $(-1, 1)$. Find a and c.

b Find b and c if $y = x^3 + bx^2 + cx + 5$ has stationary points at $x = -2$ and $x = 4$.

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14 The curve $y = ax^2 + bx + c$ passes through the points $(1, 4)$ and $(-1, 6)$, and there is a maximum turning point at $x = -\frac{1}{2}$.

a Show that $a + b + c = 4$, $a - b + c = 6$ and $-a + b = 0$.

b Hence find the values of a, b and c.

15 The line $y = 2x$ is the tangent to the curve $y = ax^2 + bx + c$ at the origin, and there is a maximum turning point at $x = 1$.

a Explain why $c = 0$.

b Explain why $\frac{dy}{dx} = 2$ when $x = 0$ and use this fact to deduce that $b = 2$.

c Show that $2a + b = 0$ and hence find the value of a.

CHALLENGE

16 a If $f(x) = \dfrac{3x}{x^2 + 1}$, show that $f'(x) = \dfrac{3(1 - x)(1 + x)}{(x^2 + 1)^2}$.

b Hence find any stationary points and determine their nature.

c Sketch the graph of $y = f(x)$, indicating all important features.

d Hence state how many roots the equation $\dfrac{3x}{x^2 + 1} = c$ has for:

i $c > \frac{3}{2}$ **ii** $c = \frac{3}{2}$ **iii** $0 < c < \frac{3}{2}$ **iv** $c = 0$

(Hint: Sketch the horizontal line $y = c$ on the same number plane and see how many times the graphs intersect.)

17 The function $y = ax^3 + bx^2 + cx + d$ has a relative maximum at $(-2, 27)$ and a relative minimum at $(1, 0)$. Find the values of a, b, c and d using the following steps.

a Find $\frac{dy}{dx}$ and show that $3a + 2b + c = 0$ and $12a - 4b + c = 0$.

b Using the fact that $(1, 0)$ and $(-2, 27)$ lie on the curve, show that $a + b + c + d = 0$ and $-8a + 4b - 2c + d = 27$. By subtracting, eliminate d from these two equations.

c Solve the simultaneous equations $3a + 2b + c = 0$, $12a - 4b + c = 0$ and $9a - 3b + 3c = -27$.

d Find the value of d.

18 a It was claimed just after worked Example 7 that the derivative of an even function is odd. Draw graphs of some even functions to explain why this is so.

b Similarly, draw graphs to explain why the derivative of an odd function is even.

c Explain how this works when differentiating powers of x.

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3C Second and higher derivatives

The derivative of the derivative of a function is called the *second derivative* of the function. There are various notations, including

$$\frac{d^2y}{dx^2} \quad \text{and} \quad f''(x) \quad \text{and} \quad f^{(2)}(x) \quad \text{and} \quad y'' \quad \text{and} \quad y^{(2)}.$$

This section reviews the algebra of higher derivatives from Section 8D of the Year 11 book in preparation for the geometric implications of the second derivative in the next section.

Example 9 3C

Find the successive derivatives of $y = x^4 + x^3 + x^2 + x + 1$.

SOLUTION

$$y = x^4 + x^3 + x^2 + x + 1 \qquad \frac{d^2y}{dx^2} = 12x^2 + 6x + 2 \qquad \frac{d^4y}{dx^4} = 24$$

$$\frac{dy}{dx} = 4x^3 + 3x^2 + 2x + 1 \qquad \frac{d^3y}{dx^3} = 24x + 6 \qquad \frac{d^5y}{dx^5} = 0$$

Because the fifth derivative is zero, all the higher derivatives are also zero.

Example 10 3C

Find the first four derivatives of $f(x) = x^{-1}$, giving each answer as a fraction.

SOLUTION

$$\begin{aligned} f'(x) &= -x^{-2} \\ &= -\frac{1}{x^2} \end{aligned} \qquad \begin{aligned} f''(x) &= 2x^{-3} \\ &= \frac{2}{x^3} \end{aligned} \qquad \begin{aligned} f^{(3)}(x) &= -6x^{-4} \\ &= -\frac{6}{x^4} \end{aligned} \qquad \begin{aligned} f^{(4)}(x) &= 24x^{-5} \\ &= \frac{24}{x^5} \end{aligned}$$

Exercise 3C FOUNDATION

1 Find the first, second and third derivatives of each function.

a $y = x^3$ **b** $y = x^{10}$ **c** $y = x^7$ **d** $y = x^2$ **e** $y = 2x^4$
f $y = 3x^5$ **g** $y = 4 - 3x$ **h** $y = x^2 - 3x$ **i** $y = 4x^3 - x^2$ **j** $y = 4x^5 + 2x^3$

2 Expand each product, then find the first and second derivatives.

a $y = x(x + 3)$ **b** $y = x^2(x - 4)$ **c** $y = (x - 2)(x + 1)$
d $y = (3x + 2)(x - 5)$ **e** $y = 3x^2(2x^3 - 3x^2)$ **f** $y = 4x^3(x^5 + 2x^2)$

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DEVELOPMENT

3 Find the first, second and third derivatives of each function.
a $y = x^{0.3}$ **b** $y = x^{-1}$ **c** $y = x^{-2}$ **d** $y = 5x^{-3}$ **e** $y = x^2 + x^{-1}$

4 By writing each function with a negative index, find its first and second derivatives.
a $f(x) = \dfrac{1}{x^3}$ **b** $f(x) = \dfrac{1}{x^4}$ **c** $f(x) = \dfrac{3}{x^2}$ **d** $f(x) = \dfrac{2}{x^3}$

5 Use the chain rule to find the first and second derivatives of each function.
a $y = (x + 1)^2$ **b** $y = (3x - 5)^3$ **c** $y = (1 - 4x)^2$ **d** $y = (8 - x)^{11}$

6 By writing each function with a negative index, find its first and second derivatives.
a $y = \dfrac{1}{x + 2}$ **b** $y = \dfrac{1}{(3 - x)^2}$ **c** $y = \dfrac{1}{(5x + 4)^3}$ **d** $y = \dfrac{2}{(4 - 3x)^2}$

7 By writing each function with fractional indices, find its first and second derivatives.
a $f(x) = \sqrt{x}$ **b** $f(x) = \sqrt[3]{x}$ **c** $f(x) = x\sqrt{x}$
d $f(x) = \dfrac{1}{\sqrt{x}}$ **e** $f(x) = \sqrt{x + 2}$ **f** $f(x) = \sqrt{1 - 4x}$

8 **a** Find $f'(x)$ and $f''(x)$ for the function $f(x) = x^3 + 3x^2 + 5x - 6$.
b Hence evaluate:
i $f'(0)$ **ii** $f'(1)$ **iii** $f''(0)$ **iv** $f''(1)$

9 **a** If $f(x) = 3x + x^3$, find:
i $f'(2)$ **ii** $f''(2)$ **iii** $f'''(2)$ **iv** $f''''(2)$
b If $f(x) = (2x - 3)^4$, find:
i $f'(1)$ **ii** $f''(1)$ **iii** $f'''(1)$ **iv** $f''''(1)$

10 Use the quotient rule to find the first derivative of each function. Then use the chain rule to find the second derivative.
a $y = \dfrac{x}{x + 1}$ **b** $y = \dfrac{x - 1}{2x + 5}$

11 If $f(x) = x(x - 1)^4$, use the product rule to find $f'(x)$ and $f''(x)$.

12 Find the values of x for which $y'' = 0$ if:
a $y = x^4 - 6x^2 + 11$ **b** $y = x^3 + x^2 - 5x + 7$

CHALLENGE

13 **a** Find the first, second and third derivatives of x^n.
b Find the nth and $(n + 1)$th derivatives of x^n.

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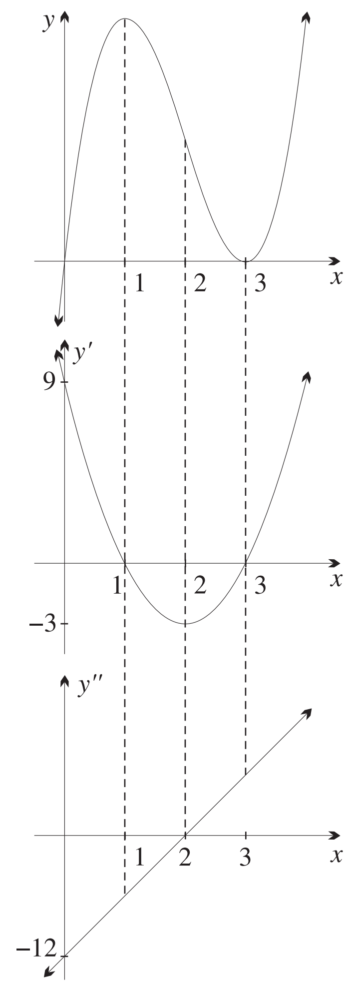
3D Concavity and points of inflection

Sketched to the right are a cubic function and its first and second derivatives. These sketches will show how the *concavity* of the original graph can be determined from the sign of the second derivative.

$$\begin{aligned} y &= x^3 - 6x^2 + 9x = x(x-3)^2 \\ y' &= 3x^2 - 12x + 9 = 3(x-1)(x-3) \\ y'' &= 6x - 12 \qquad\quad = 6(x-2) \end{aligned}$$



The sign of each derivative tells us whether the function above it is increasing or decreasing. Thus the second graph describes the gradient of the first, and the third graph describes the gradient of the second.

To the right of $x = 2$, the top graph is concave up. This means that as one moves along the curve to the right from $x = 2$, the tangent gets steeper, with its gradient steadily increasing. Thus for $x > 2$, the gradient function y' is increasing as x increases, as can be seen in the middle graph. The bottom graph is the gradient of the middle graph, and accordingly y'' is positive for $x > 2$.

Similarly, to the left of $x = 2$ the top graph is concave down. This means that its gradient function y' is steadily decreasing as x increases. The bottom graph is the derivative of the middle graph, so y'' is negative for $x < 2$.

This example demonstrates that the concavity of a graph $y = f(x)$ at any value $x = a$ is determined by the sign of its second derivative at $x = a$.

6 CONCAVITY AND THE SECOND DERIVATIVE

- If $f''(a)$ is negative, the curve is concave down at $x = a$.
- If $f''(a)$ is positive, the curve is concave up at $x = a$.

Points of inflection

As foreshadowed in Section 3B, a *point of inflection* is a point where the tangent crosses the curve. This means that the curve curls away from the tangent on opposite sides of the tangent, and this in turn means that the concavity changes sign around the point.

The three diagrams above show how the point of inflection at $x = 2$ results in a minimum turning point at $x = 2$ in the middle graph of y'. Hence the bottom graph of y'' has a zero at $x = 2$ and changes sign around $x = 2$.

This discussion gives us a method of analysing concavity and finding points of inflection. Once again, we use the fact that y'' can only change sign at a zero or a discontinuity of y''.

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7 USING $f''(x)$ TO ANALYSE CONCAVITY AND FIND POINTS OF INFLECTION

A *point of inflection* is a point where the tangent crosses the curve.

1. Find the zeroes and discontinuities of the second derivative $f''(x)$.
2. Then use a table of test values of the second derivative $f''(x)$ dodging around its zeroes and discontinuities, with the concavities underneath, to see where the concavity changes sign.

The table of concavities will show not only any points of inflection, but also the concavity of the graph across its whole domain.

Before drawing the sketch, it is often useful to find the gradient of the tangent at each point of inflection. Such tangents are called *inflectional tangents*.

Example 11 — 3D

a Find any turning points of $f(x) = x^5 - 5x^4$.

b Draw up a table of concavities. Find any points of inflection and the gradients of the inflectional tangents, and describe the concavity. Then sketch.

SOLUTION

Here
$$\begin{aligned} f(x) &= x^5 - 5x^4 &&= x^4(x-5) \\ f'(x) &= 5x^4 - 20x^3 &&= 5x^3(x-4) \\ f''(x) &= 20x^3 - 60x^2 &&= 20x^2(x-3). \end{aligned}$$

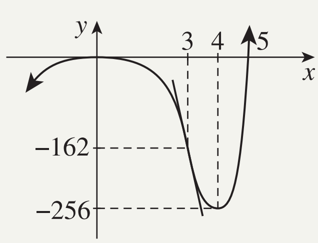
a $f'(x)$ has zeroes at $x = 0$ and $x = 4$, and no discontinuities:

x	-1	0	1	4	5
$f'(x)$	25	0	-15	0	625
slope	/	—	\	—	/

So $(0, 0)$ is a maximum turning point, and $(4, -256)$ is a minimum turning point.

b $f''(x)$ has zeroes at $x = 0$ and $x = 3$, and no discontinuities:

x	-1	0	1	3	4
$f''(x)$	-80	0	-40	0	320
concavity	⌢	.	⌢	.	⌣



So $(3, -162)$ is a point of inflection, but $(0, 0)$ is not.
Because $f'(3) = -135$, the inflectional tangent has gradient -135.
The graph is concave down for $x < 0$ and $0 < x < 3$, and concave up for $x > 3$.

Note: The example given above is intended to show that $f''(x) = 0$ is NOT a sufficient condition for a point of inflection. *The sign of $f''(x)$ must also change around the point* — this happened at $x = 3$, but not at $x = 0$.

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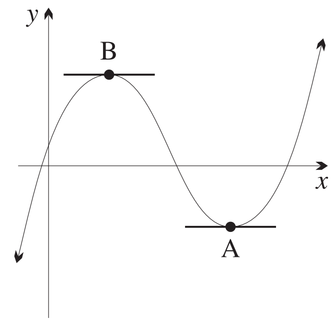
Using the second derivative to test stationary points

If a curve is concave up at a stationary point, then the point is a minimum turning point, as at the point A.

Similarly, if a curve is concave down at a stationary point, then the point is a maximum turning point, as at the point B. This gives an alternative test of a stationary point.



8 USING THE SECOND DERIVATIVE TO TEST A STATIONARY POINT

Suppose that the curve $y = f(x)$ has a stationary point at $x = a$.

- If $f''(a) > 0$, the curve is concave up at $x = a$, and there is a minimum turning point there.
- If $f''(a) < 0$, the curve is concave down at $x = a$, and there is a maximum turning point there.
- If $f''(a) = 0$, *more work is needed. Go back to the table of values of* $f'(x)$.

The third dotpoint is most important — all four cases shown on page 124 are possible for the shape of the curve at $x = a$ when the second derivative is zero there.

The previous example of the point $(0, 0)$ on $y = x^5 - 5x^4$ shows that a stationary point where $f''(x) = 0$ can be a turning point. The next worked example is an example where a stationary point turns out to be a point of inflection.

Example 12 3D

Use the second derivative, if possible, to determine the nature of the stationary points of the graph of $f(x) = x^4 - 4x^3$. Find also any points of inflection, examine the concavity over the whole domain, and sketch the curve.

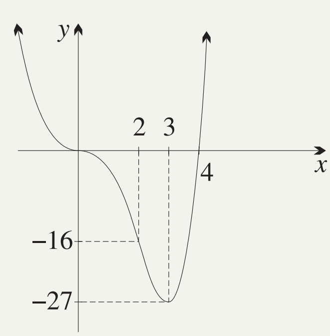
SOLUTION

Here,
$$\begin{aligned} f(x) &= x^4 - 4x^3 &&= x^3(x - 4) \\ f'(x) &= 4x^3 - 12x^2 &&= 4x^2(x - 3) \\ f''(x) &= 12x^2 - 24x &&= 12x(x - 2), \end{aligned}$$

so $f'(x)$ has zeroes at $x = 0$ and $x = 3$, and no discontinuities.

Because $f''(3) = 36$ is positive, $(3, -27)$ is a minimum turning point, but $f''(0) = 0$, *so no conclusion can be drawn about* $x = 0$.

x	-1	0	1	3	4
$f'(x)$	-16	0	-8	0	64
slope	\	—	\	—	/



So $(0, 0)$ is a stationary point of inflection.

$f''(x)$ has zeroes at $x = 0$ and $x = 2$, and no discontinuities,

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x	-1	0	1	2	3
$f''(x)$	36	0	-12	0	36
concavity	$\smile$	$\cdot$	$\frown$	$\cdot$	$\smile$

So, besides the horizontal inflection at $(0, 0)$,
there is a non-stationary inflection at $(2, -16)$,
and the inflectional tangent at $(2, -16)$ has gradient -16.
The graph is concave down for $0 < x < 2$, and concave up for $x < 0$ and for $x > 2$.

Finding pronumerals in a function

In this worked example, a pronumeral in a function is found using information about the concavity of the graph.

Example 13 3D

For what values of b is $y = x^4 - bx^3 + 5x^2 + 6x - 8$ concave down when $x = 2$?

SOLUTION

Differentiating, $y' = 4x^3 - 3bx^2 + 10x + 6$,
and differentiating again, $y'' = 12x^2 - 6bx + 10$,
so when $x = 2$, $y'' = 48 - 12b + 10$
$= 58 - 12b$.

In order for the curve to be concave down at $x = 2$,

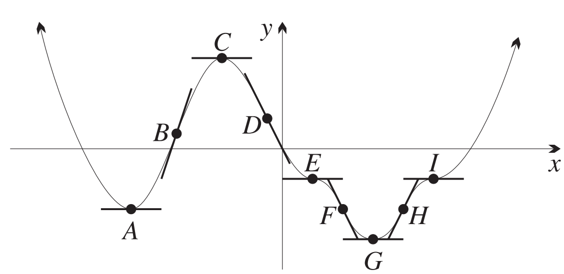
$$58 - 12b < 0$$
$$12b > 58$$
$$b > 4\tfrac{5}{6}.$$

Exercise 3D

FOUNDATION

1 Complete the table below for the function to the right. At each point, state whether the first and second derivatives are positive, negative or zero.

Point	A	B	C	D	E	F	G	H	I
y'									
y''									



2 Find $f''(x)$ for each function. By evaluating $f''(0)$, state whether the curve is concave up $(f''(x) > 0)$ or concave down $(f''(x) < 0)$ at $x = 0$.

a $f(x) = x^3 - 3x^2$

b $f(x) = x^3 + 4x^2 - 5x + 7$

c $f(x) = x^4 + 2x^2 - 3$

d $f(x) = 6x - 7x^2 - 8x^4$

3 By showing that $f'(2) = 0$, prove that each curve has a stationary point at $x = 2$. Then evaluate $f''(2)$ to determine the nature of the stationary point.

a $f(x) = x^2 - 4x + 4$

b $f(x) = 5 + 4x - x^2$

c $f(x) = x^3 - 12x$

d $f(x) = 2x^3 - 3x^2 - 12x + 5$

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4 A curve is concave up when $\dfrac{d^2y}{dx^2} > 0$ and concave down when $\dfrac{d^2y}{dx^2} < 0$.

a Explain why $y = x^2 - 3x + 7$ is concave up for all values of x.

b Explain why $y = -3x^2 + 2x - 4$ is concave down for all values of x.

5 **a** Find the second derivative $\dfrac{d^2y}{dx^2}$ of $y = x^3 - 3x^2 - 5x + 2$.

b Hence find the values of x for which the curve is:

i concave up,

ii concave down.

6 **a** Find the second derivative $\dfrac{d^2y}{dx^2}$ of $y = x^3 - x^2 - 5x + 1$.

b Hence find the values of x for which the curve is:

i concave up,

ii concave down.

DEVELOPMENT

7 A function has second derivative $y'' = 3x^3(x + 3)^2(x - 2)$. Determine the x-coordinates of the points of inflection on the graph of the function.

8 **a** If $f(x) = x^3 - 3x$, show that $f'(x) = 3(x - 1)(x + 1)$ and $f''(x) = 6x$.

b By solving $f'(x) = 0$, find the coordinates of any stationary points.

c Examine the sign of $f''(1)$ and $f''(-1)$ to determine their nature.

d Find the coordinates of the point of inflection. Remember that you must show that the sign of $f''(x)$ changes about this point.

e Sketch the graph of $f(x)$, indicating all important features.

9 **a** If $f(x) = x^3 - 6x^2 - 15x + 1$, show that $f'(x) = 3(x - 5)(x + 1)$ and $f''(x) = 6(x - 2)$.

b Find any stationary points and use the sign of $f''(x)$ to determine their nature.

c Find the coordinates of any points of inflection, testing them with a table of concavities.

d Sketch the graph of $f(x)$, indicating all important features.

10 **a** If $y = x^3 - 3x^2 - 9x + 11$, show that $y' = 3(x - 3)(x + 1)$ and $y'' = 6(x - 1)$.

b Find any stationary points and use the sign of y'' to determine their nature.

c Find the coordinates of any points of inflection, testing them with a table of concavities.

d Sketch the graph of the function, indicating all important features.

11 **a** If $y = 3 + 4x^3 - x^4$, show that $y' = 4x^2(3 - x)$ and $y'' = 12x(2 - x)$.

b Find any stationary points and use a table of test values of y' to determine their nature.

c Find the coordinates of any points of inflection.

d Sketch the graph of the function, indicating all important features.

12 Find the range of values of x for which the curve $y = 2x^3 - 3x^2 - 12x + 8$ is:

a increasing, that is $y' > 0$,

b decreasing, that is $y' < 0$,

c concave up, that is $y'' > 0$,

d concave down, that is $y'' < 0$.

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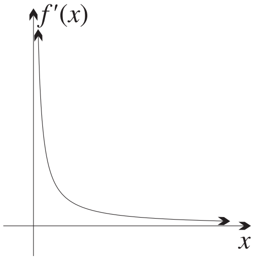
13 **a** If $y = x^3 + 3x^2 - 72x + 14$, find y' and y''.
b Show that the curve has a point of inflection at $(-1, 88)$.
c Show that the gradient of the tangent at the point of inflection is -75.
d Hence find the equation of the tangent at the point of inflection.

14 **a** If $f(x) = x^3$ and $g(x) = x^4$, find $f'(x)$, $f''(x)$, $g'(x)$ and $g''(x)$.
b Both $f(x)$ and $g(x)$ have a stationary point at $(0, 0)$. Evaluate $f''(x)$ and $g''(x)$ when $x = 0$. Can you determine the nature of the stationary points from this calculation?
c Use tables of values of $f'(x)$ and $g'(x)$ to determine the nature of the stationary points.

15 **a** Find a if the curve $y = x^3 - ax^2 + 3x - 4$ has an inflection at the point where $x = 2$.
b For what values of a is $y = x^3 + 2ax^2 + 3x - 4$ concave up at the point where $x = -1$?
c Find a and b if the curve $y = x^4 + ax^3 + bx^2$ has an inflection at $(2, 0)$.
d For what values of a is $y = x^4 + ax^3 - x^2$ concave up and increasing when $x = 1$?

16 The diagram to the right shows the graph of the derivative $y = f'(x)$ of the function $y = f(x)$, with domain $x > 0$.



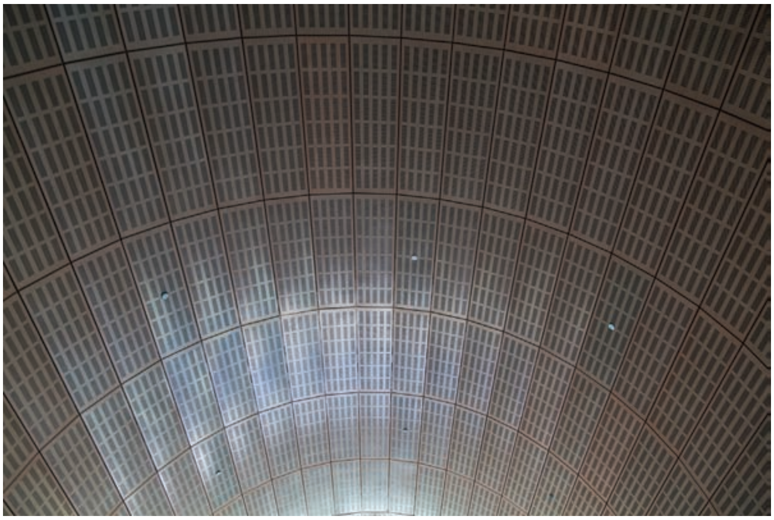
a State whether the graph of $y = f(x)$ is increasing or decreasing throughout its domain.
b State whether the graph of $y = f(x)$ is concave up or concave down throughout its domain.

CHALLENGE

17 Sketch a small section of the graph of the continuous function $f(x)$ about $x = a$ if:
a $f'(a) > 0$ and $f''(a) > 0$,
b $f'(a) > 0$ and $f''(a) < 0$,
c $f'(a) < 0$ and $f''(a) > 0$,
d $f'(a) < 0$ and $f''(a) < 0$.

18 A function has equation $y = \frac{1}{3}x^3 - 3x^2 + 11x - 9$.
a Show that the function has no stationary points.
b Show that there is a point of inflection.
c How many x-intercepts does the graph of the function have? Justify your answer.

19 A curve has equation $y = ax^3 + bx^2 + cx + d$ and crosses the x-axis at $x = -1$. It has a turning point at $(0, 5)$ and a point of inflection at $x = \frac{1}{2}$. Find the values of a, b, c and d.



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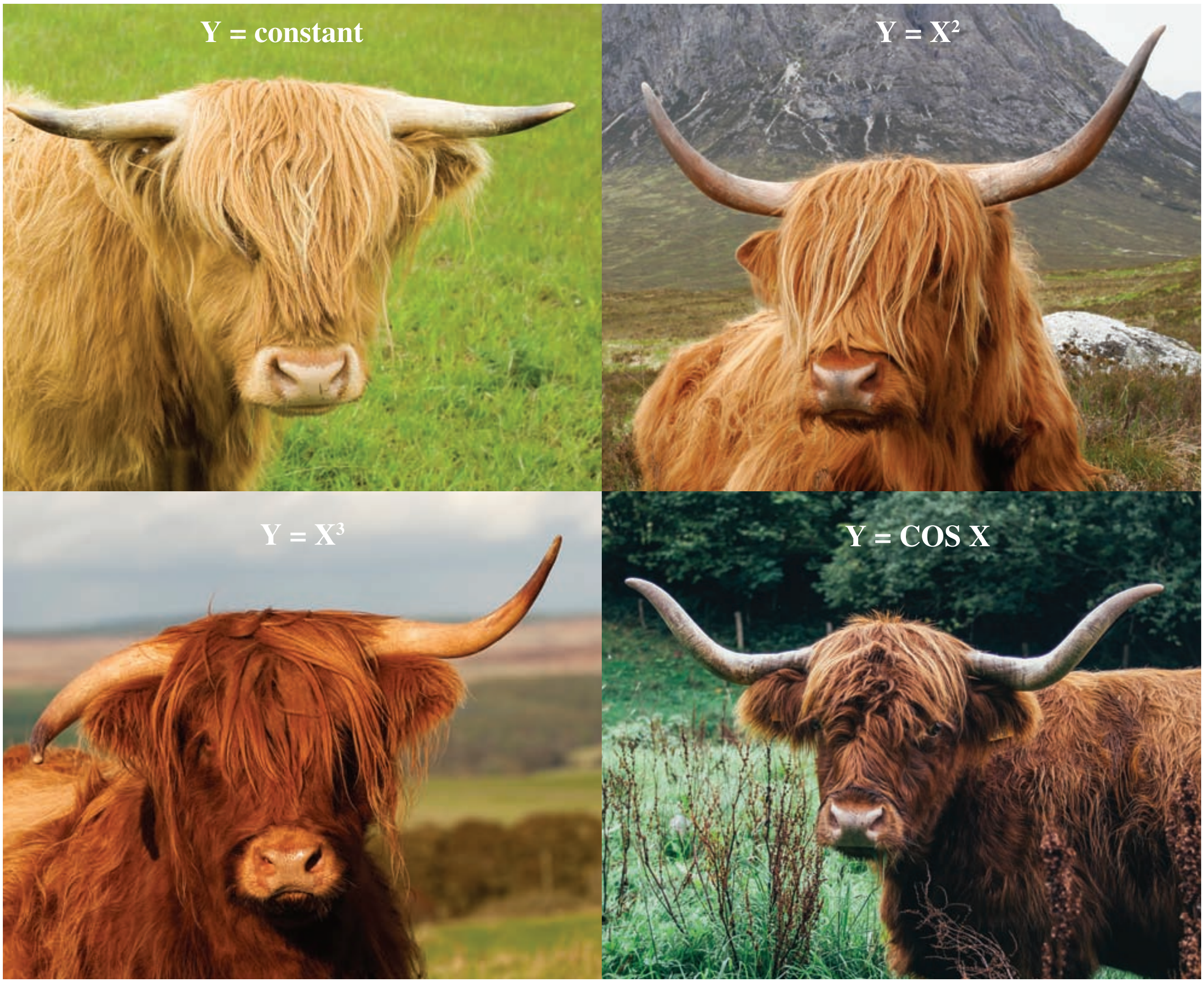

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3E Systematic curve sketching with the derivative

In Section 2C of the last chapter, we developed a systematic four-step approach to sketching an unfamiliar curve:

1 domain, **2** symmetry, **3** intercepts and sign, **4** asymptotes

This chapter has used the derivative to examine the gradient and concavity of curves and to find their turning points and inflections. We can now add these methods to the menu as steps 5 and 6.

Few curves in this course would require consideration of all the points in the summary below. Questions almost always give some guidance as to which methods to use for any particular function.

9 A SUMMARY OF CURVE-SKETCHING METHODS

1 Domain: Find the domain of $f(x)$. (*Always* do this first.)

2 Symmetry: Find whether the function is even or odd, or neither.

3 A Intercepts: Find the y-intercept and all x-intercepts (zeroes).

B Sign: Use a table of test values of $f(x)$, that is, a table of signs, to find where the function is positive, and where it is negative.

4 A Vertical asymptotes: Examine any discontinuities to see whether there are vertical asymptotes there.

B Horizontal asymptotes: Examine the behaviour of $f(x)$ as $x \to \infty$ and as $x \to -\infty$, noting any horizontal asymptotes.

5 The first derivative:

A Find the zeroes and discontinuities of $f'(x)$.

B Use a table of test values of $f'(x)$, that is, a table of slopes, to determine the nature of the stationary points and the slope of the function throughout its domain.

6 The second derivative:

A Find the zeroes and discontinuities of $f''(x)$.

B Use a table of test values of $f''(x)$, that is, a table of concavities, to find any points of inflection and the concavity of the function throughout its domain.

7 Any other features:

A routine warning of incompleteness.

The final Step 7 is a routine warning that many important features of functions will not be picked up using this menu. For example, every parabola has an axis of symmetry, but the even-and-odd test only picks up that axis of symmetry when it is the y-axis. Even more importantly, the trigonometric functions repeat periodically, and tests for periodicity are not mentioned.

An example of a curve with turning points and asymptotes

The curve in the worked example below has three asymptotes and a turning point. Such curves are never easy to analyse, but it is worth having one such example that combines the calculus approaches of the present chapter with the previous non-calculus approaches.

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Example 14 3E

Consider the curve $y = \dfrac{1}{x(x - 4)}$.

a Write down the domain of the function.

b Use a table of test values to analyse the sign of the function.

c Find any vertical and horizontal asymptotes.

d Show that the derivative is $y' = \dfrac{2(2 - x)}{x^2(x - 4)^2}$.

e Find all the zeroes and discontinuities of $f'(x)$. Then use a table of test values of $f'(x)$ to analyse stationary points and find where the function is increasing and decreasing.

f Sketch the curve and hence write down the range of the function.

SOLUTION

a The domain of the function is $x \neq 0$ and $x \neq 4$.

b The function is never zero, and it has discontinuities at $x = 0$ and $x = 4$.

x	-1	0	2	4	5
y	$\frac{1}{5}$	*	$-\frac{1}{4}$	*	$\frac{1}{5}$

Hence y is positive for $x < 0$ or $x > 4$, and y is negative for $0 < x < 4$.

c The lines $x = 0$ and $x = 4$ are vertical asymptotes.
Also, $y \to 0$ as $x \to \infty$ and as $x \to -\infty$, so the x-axis is a horizontal asymptote.

d Differentiating using the chain rule,

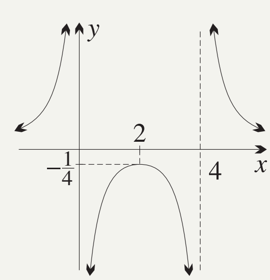
$$\begin{aligned} y' &= \frac{-1}{x^2(x - 4)^2} \times (2x - 4) \\ &= \frac{2(2 - x)}{x^2(x - 4)^2}. \end{aligned}$$

Let $u = x^2 - 4x$.
The $y = \dfrac{1}{u}$.
Hence $\dfrac{du}{dx} = 2x - 4$
and $\dfrac{dy}{du} = -\dfrac{1}{u^2}$.

e Hence y' has a zero at $x = 2$ and discontinuities at $x = 0$ and $x = 4$.

x	-1	0	1	2	3	4	5
y'	$\frac{6}{25}$	*	$\frac{2}{9}$	0	$-\frac{2}{9}$	*	$-\frac{6}{25}$
slope	/	*	/	—	\	*	\

Thus there is a maximum turning point at $\left(2, -\frac{1}{4}\right)$, the curve is increasing for $x < 2$ (except at $x = 0$), and it is decreasing for $x > 2$ (except at $x = 4$).



f The graph is sketched to the right.
From the graph, the range is $y > 0$ or $y \leq -\frac{1}{4}$.

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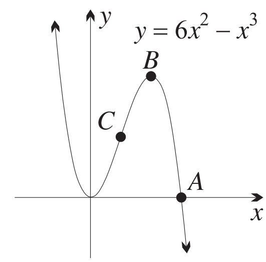
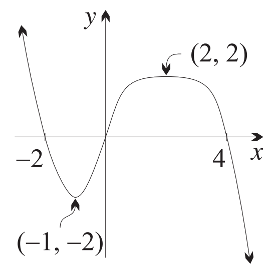
Exercise 3E

FOUNDATION



1 The diagram to the right shows a sketch of $y = 6x^2 - x^3$. The curve cuts the x-axis at A, and it has a maximum turning point at B and a point of inflection at C.
  a Find the coordinates of A.
  b Find the coordinates of B.
  c Find the coordinates of C.



2 The diagram to the right shows a curve $y = f(x)$. From the sketch, find the values of x for which:
  a $f'(x) = 0$,
  b $f''(x) = 0$,
  c $f(x)$ is increasing,
  d $f''(x) > 0$.

3 a Find the x-intercepts of the parabola $y = x^2 - 5x - 14$. (You may use either factoring or the quadratic formula.)
  b By putting $x = 0$, find the y-intercept.
  c Solve $\dfrac{dy}{dx} = 0$ and hence find the coordinates of the stationary point.
  d By examining the sign of $\dfrac{d^2y}{dx^2}$, establish the nature of the stationary point.
  e Sketch the graph of the function, indicating all important features.

4 Using the steps outlined in the previous question, sketch the graphs of:
  a $y = x^2 - 8x$
  b $y = 6 - x - x^2$

DEVELOPMENT

5 a Show that $y = 27x - x^3$ is an odd function. What symmetry does its graph display?
  b Show that $y' = 3(9 - x^2)$ and $y'' = -6x$.
  c Find the coordinates of the stationary points. Then determine their nature, either by examining the sign of $f''(3)$ and $f''(-3)$, or by means of a table of test values of y'.
  d Show, using a table of test values of y'', that $x = 0$ is a point of inflection.
  e By substituting into the gradient function y', find the gradient at the inflection.
  f Sketch the graph of the function, indicating all important features.

6 a If $f(x) = 2x^3 - 3x^2 + 5$, show that $f'(x) = 6x(x - 1)$ and $f''(x) = 6(2x - 1)$.
  b Find the coordinates of the stationary points. Then determine their nature, either by examining the sign of $f''(0)$ and $f''(1)$, or by means of a table of test values of y'.
  c Explain why there is a point of inflection at $x = \frac{1}{2}$, and find the gradient there.
  d Sketch the graph of $f(x)$, indicating all important features.

7 Find the first and second derivatives of each function below. Hence find the coordinates of any stationary points and determine their nature. Then find any points of inflection. Sketch the graph of each function. You do not need to find the x-intercepts in part **b**.
  a $y = x(x - 6)^2$
  b $y = x^3 - 3x^2 - 24x + 5$

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8 **a** If $y = 12x^3 - 3x^4 + 11$, show that $y' = 12x^2(3 - x)$ and $y'' = 36x(2 - x)$.

b By solving $y' = 0$, find the coordinates of any stationary points.

c By examining the sign of y'', establish the nature of the stationary point at $x = 3$. Why does this method fail for the stationary point at $x = 0$?

d Use a table of test values of y' to show that there is a stationary point of inflection at $x = 0$.

e Show that there is a change in concavity at $x = 2$.

f Sketch the graph of the function, showing all important features.

9 Using the method outlined in the previous question, sketch $y = x^4 - 16x^3 + 72x^2 + 10$.

CHALLENGE

Note: The next three questions involve functions that may have both turning points and asymptotes. As a consequence, the analysis of each function is quite long and complicated.

10 **a** Show that the derivative of $f(x) = \dfrac{1}{x^2 - 4}$ is $f'(x) = -\dfrac{2x}{(x^2 - 4)^2}$.

b Show that $y = f(x)$ has a stationary point at $x = 0$. Then determine its nature, using a table of test values of $f'(x)$.

c Show that the function is even. What sort of symmetry does its graph have?

d State the domain of the function and the equations of any vertical asymptotes.

e What value does $f(x)$ approach as $x \to \infty$ and as $x \to -\infty$? Hence write down the equation of the horizontal asymptote.

f Sketch the graph of $y = f(x)$, showing all important features.

g Use the graph to state the range of the function.

11 **a** Show that the derivative of $f(x) = \dfrac{x}{x^2 - 4}$ is $f'(x) = -\dfrac{x^2 + 4}{(x^2 - 4)^2}$.

b Explain why the curve $y = f(x)$ has no stationary points, and why the curve is always decreasing.

c Given that $f''(x) = \dfrac{2x^3 + 24x}{(x^2 - 4)^3}$, show that $(0, 0)$ is a point of inflection. Then find the gradient of the tangent at this point.

d State the domain of the function and the equations of any vertical asymptotes.

e What value does $f(x)$ approach as x becomes large? Hence write down the equation of the horizontal asymptote.

f Show that the function is odd. What symmetry does its graph have?

g Use a table of test values of y to analyse the sign of the function.

h Sketch the graph of $y = f(x)$, showing all important features.

i Use the graph to state the range of the function.

12 **a** Show that the derivative of $y = x + \dfrac{1}{x}$ is $y' = \dfrac{x^2 - 1}{x^2}$.

b Find the stationary points and determine their nature.

c Show that the function is odd. What symmetry does its graph have?

d State the domain of the function and the equation of the vertical asymptote.

e Use a table of test values of y to analyse the sign of the function.

f Sketch the graph of the function. (You may assume that the diagonal line $y = x$ is an asymptote to the curve. This is because for large x, the term $1/x$ is very close to zero.)

g Write down the range of the function.

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3F Global maximum and minimum

Australia has many high mountain peaks, each of which is a *local* or *relative maximum*, because each is the highest point relative to other peaks in its immediate locality. Mount Kosciuszko is the highest of these, but it is still not a *global* or *absolute maximum*, because there are higher peaks on other continents of the globe. Mount Everest in Asia is the global maximum over the whole world.

10 GLOBAL MAXIMUM AND MINIMUM

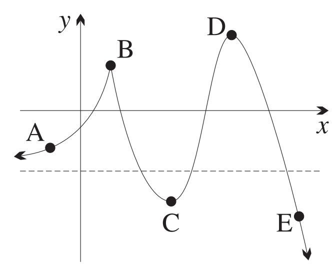
Let $A\left(a, f(a)\right)$ be a point on a curve $y = f(x)$.

- The point A is a *global* or *absolute maximum* if
  $f(x) \le f(a)$, for all x in the domain.
- Similarly, A is a *global* or *absolute minimum* if
  $f(x) \ge f(a)$, for all x in the domain.

The following diagrams illustrate what has to be considered in the general case.

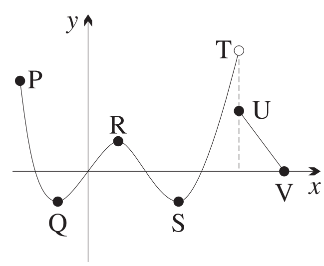
In the upper diagram, the domain is all real numbers.

- There are local maxima at the point B, where $f'(x)$ is undefined, and at the turning point D. This point D is also the global maximum.
- There is a local minimum at the turning point C, which is lower than all points on the curve to the left past A. There is no global minimum, however, because the curve goes infinitely far downwards to the right of E.



In the lower diagram, the domain is the closed interval on the x-axis from P to V.

- There are local maxima at the turning point R and at the endpoint P. There is no global maximum, however, because the point T has been omitted from the curve.
- There are local minima at the two turning points Q and S, and at the endpoint V. These points Q and S have equal heights and are thus both global minima.



Testing for global maximum and minimum

These examples show that there are three types of points that must be considered and compared when finding the global maximum and minimum of a function $f(x)$ defined on some domain.

11 TESTING FOR GLOBAL MAXIMUM AND MINIMUM

Examine and compare:

- turning points,
- boundaries of the domain (or the behaviour for large x),
- discontinuities of $f'(x)$ (to pick up sharp corners or discontinuities).

More simply, examine and compare turning points and boundary points — and discontinuities if there are any.

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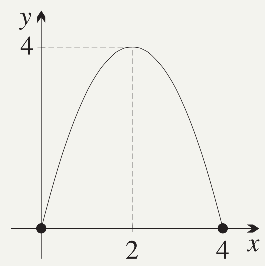
Example 15 3F

Find the absolute maximum and minimum of $f(x) = 4x - x^2$ over the domain $0 \le x \le 4$.
Note: Calculus is not needed here because the function is a quadratic.

SOLUTION

The graph is a concave-down quadratic.
Factoring, $f(x) = x(4 - x)$, so the x-intercepts are $x = 0$ and $x = 4$.
Taking their average, the axis of symmetry is $x = 2$,
and substituting, the vertex is $(2, 4)$.
Hence, from the sketch, the absolute maximum is 4 at $x = 2$,
and the absolute minimum is 0 at the endpoints where $x = 0$ or 4.



Example 16 3F

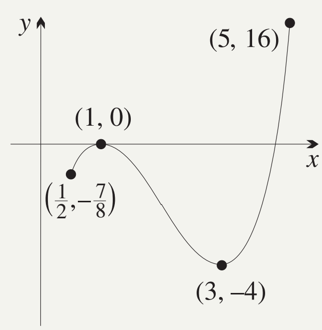
Find the global maximum and minimum of the function $f(x) = x^3 - 6x^2 + 9x - 4$, where $\frac{1}{2} \le x \le 5$.

SOLUTION

The unrestricted curve was sketched in Section 3B (worked Example 6), and substituting the boundaries,

$$f\left(\tfrac{1}{2}\right) = -\tfrac{7}{8} \qquad \text{and} \qquad f(5) = 16.$$

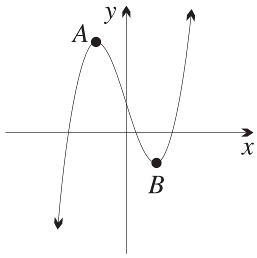
Hence the global maximum is 16, at $x = 5$,
and the global minimum is -4, at $x = 3$.



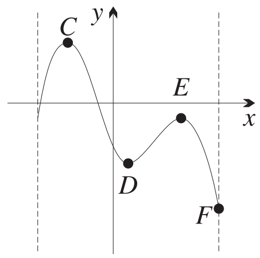
Exercise 3F FOUNDATION

1 In the diagrams below, classify each labelled point as one of the following: (i) global maximum, (ii) global minimum, (iii) local maximum, (iv) local minimum, (v) horizontal point of inflection.

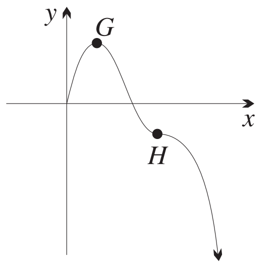
a



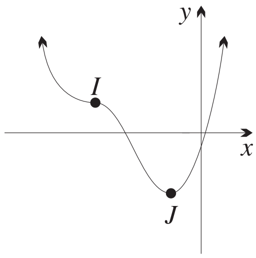
b



c



d



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2 Sketch each function and state its global minimum and maximum in the specified domain.

a $y = x^2, -2 \le x \le 2$

b $y = 5 - x, 0 \le x \le 3$

c $y = \sqrt{16 - x^2}, -4 \le x \le 4$

d $y = |x|, -5 \le x \le 1$

e $y = \sqrt{x}, 0 \le x \le 8$

f $y = 1/x, -4 \le x \le -1$

g $y = \begin{cases} -1, & \text{for } x < -2, \\ x + 1, & \text{for } -2 \le x < 1, \\ 2, & \text{for } x \ge 1. \end{cases}$

DEVELOPMENT

3 Sketch the graph of each function, clearly indicating any stationary points. Determine the absolute minimum and maximum of the function in the specified domain.

a $y = x^2 - 4x + 3, 0 \le x \le 5$

b $y = x^3 - 3x^2 + 5, -3 \le x \le 2$

c $y = 3x^3 - x + 2, -1 \le x \le 1$

d $y = x^3 - 6x^2 + 12x, 0 \le x \le 3$

CHALLENGE

4 Find (i) any local maxima or minima, and (ii) the global maximum and minimum of the function $y = x^4 - 8x^2 + 11$ for each domain.

a $1 \le x \le 3$

b $-4 \le x \le 1$

c $-1 \le x \le 0$

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3G Applications of maximisation and minimisation

Here are some of many practical applications of maximisation and minimisation.

- Maximise the volume of a box built from a rectangular sheet of cardboard.
- Minimise the fuel used in a flight.
- Maximise the profits from manufacturing and selling an article.
- Minimise the amount of metal used in a can of soft drink.

Such problems can be solved using calculus, provided that a clear functional relationship can first be established.

12 MAXIMISATION AND MINIMISATION PROBLEMS

Usually a diagram should be drawn. Then:

1 Introduce the two variables from which the function is to be formed.
 'Let y (or whatever) be the quantity that is to be maximised,
 and let x (or whatever) be the quantity that can be varied.'

2 Form an equation in the two variables, noting any restrictions.

3 Find the global maximum or minimum.

4 Write a careful conclusion.

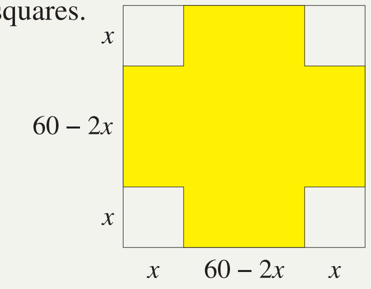
Note: A claim that a stationary point is a maximum or minimum must always be justified by a proper analysis of the nature of the stationary point.

Example 17 3G

An open rectangular box is to be made by cutting square corners out of a square piece of cardboard measuring 60 cm $\times$ 60 cm, and folding up the sides. What is the maximum volume of the box, and what are its dimensions then?

SOLUTION

Let V be the volume of the box, and let x be the side lengths of the cut-out squares. Then the box is x cm high, with base a square of side length $60 - 2x$,



so $V = x(60 - 2x)^2$

$= 3600x - 240x^2 + 4x^3$, where $0 \le x \le 30$.

Differentiating, $V' = 3600 - 480x + 12x^2$

$= 12(x - 30)(x - 10)$,

so V' has zeroes at $x = 10$ and $x = 30$, and no discontinuities.

Also, $V'' = -480 + 24x$,

so $V''(10) = -240 < 0$ and $V''(30) = 240 > 0$.

Hence $(10, 16000)$ is the global maximum in the domain $0 \le x \le 30$,

and the maximum volume is 16000 cm^3 when the box is 10 cm $\times$ 40 cm $\times$ 40 cm.

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Example 18 3G

A certain cylindrical soft drink can is required to have a volume of 250 cm^3.

a Show that the height of the can is $\dfrac{250}{\pi r^2}$, where r is the base radius.

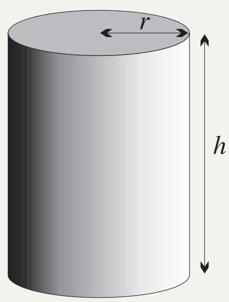
b Show that the total surface area is $S = 2\pi r^2 + \dfrac{500}{r}$.

c Show that $r = \dfrac{5}{\pi^{\frac{1}{3}}}$ gives a global minimum of S in the domain $r > 0$.

d Show that to minimise the surface area of the can, the diameter of its base should equal its height.

SOLUTION

a Let the height of the can be h cm.



$$\begin{aligned} \text{Then volume} &= \pi r^2 h \\ 250 &= \pi r^2 h \\ h &= \frac{250}{\pi r^2}. \end{aligned}$$

b Each end has area πr^2 and the curved side has area $2\pi rh$, so

$$\begin{aligned} S &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r^2 + 2\pi r \times \frac{250}{\pi r^2} \\ &= 2\pi r^2 + \frac{500}{r}, \quad \text{where } r > 0. \end{aligned}$$

c Differentiating,

$$\begin{aligned} \frac{dS}{dr} &= 4\pi r - \frac{500}{r^2} \\ &= \frac{4\pi r^3 - 500}{r^2}. \end{aligned}$$

To find stationary points, put

$$\begin{aligned} \frac{dS}{dr} &= 0 \\ 4\pi r^3 &= 500 \\ r^3 &= \frac{125}{\pi} \\ r &= \frac{5}{\pi^{\frac{1}{3}}}. \end{aligned}$$

Differentiating again,

$$\frac{d^2S}{dr^2} = 4\pi + \frac{1000}{r^3},$$

which is positive for all $r > 0$.

Hence the stationary point is a global minimum in the domain $r > 0$.

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d When $r = \dfrac{5}{\pi^{\frac{1}{3}}}$, $\quad h = \dfrac{250}{\pi r^2}$

$$= \frac{250}{\pi} \times \frac{\pi^{\frac{2}{3}}}{25}$$

$$= \frac{10}{\pi^{\frac{1}{3}}}$$

$$= 2r.$$

Hence the minimum surface area occurs when the diameter equals the height.

Cost and time problems

There is often an optimum speed at which the costs of running a boat or truck are minimised.

- At slow speeds, wages and fixed costs rise.
- At high speeds, the costs of fuel and wear rise.

If some formula for these costs can be found, calculus can find the best speed.

Example 19 3G

The cost C (in dollars per hour) of running a boat depends on the speed v km/h of the boat according to the formula $C = 500 + 40v + 5v^2$.

a Show that the total cost for a trip of 100 km is $T = \dfrac{50000}{v} + 4000 + 500v$.

b What speed will minimise the total cost of the trip?

SOLUTION

a Because time $= \dfrac{\text{distance}}{\text{speed}}$, the time for the trip is $\dfrac{100}{v}$ hours.

Hence the total cost is $\quad T = (\text{cost per hour}) \times (\text{time for the trip})$

$$= (500 + 40v + 5v^2) \times \frac{100}{v}$$

$$= \frac{50000}{v} + 4000 + 500v, \quad \text{where } v > 0.$$

b Differentiating, $\quad \dfrac{dT}{dv} = -\dfrac{50000}{v^2} + 500$

$$= \frac{500(-100 + v^2)}{v^2}$$

$$= \frac{500(v - 10)(v + 10)}{v^2},$$

so $\dfrac{dT}{dv}$ has a single zero at $v = 10$ in the domain $v > 0$, and no discontinuities.

Differentiating again, $\quad \dfrac{d^2T}{dv^2} = \dfrac{100000}{v^3}$, which is positive for all $v > 0$,

so $v = 10$ gives a global minimum in the domain $v > 0$.

Thus a speed of 10 km/h will minimise the cost of the trip.

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Exercise 3G

FOUNDATION

Note: You must always prove that any stationary point is a maximum or minimum, either by creating a table of test values of the derivative, or by substituting into the second derivative. It is never acceptable to assume this from the wording of a question.

1 At time t seconds, a particle has height $h = 3 + t - 2t^2$ metres.

  a Find $\frac{dh}{dt}$ and show that the maximum height occurs after 0.25 seconds.

  b Find the maximum height.

2 a Given that $P = xy$ and $2x + y = 12$, show that $P = 12x - 2x^2$.

  b Find $\frac{dP}{dx}$ and hence determine the value of x that maximises P.

  c Hence find the maximum value of P.

3 a Given that $Q = x^2 + y^2$ and $x + y = 8$, show that $Q = 2x^2 - 16x + 64$.

  b Find $\frac{dQ}{dx}$ and hence determine the value of x that minimises Q.

  c Hence find the minimum value of P.

4 A rectangle has a fixed perimeter of 20 cm. Let the length of the rectangle be x.

  a Show that the width of the rectangle is $10 - x$.

  b Hence show that the area of the rectangle is given by $A = 10x - x^2$.

  c Find $\frac{dA}{dx}$ and hence find the value of x that maximises A.

  d Hence find the maximum possible area of the rectangle.

5 A landscaper is constructing a rectangular garden bed. Three of the sides are to be fenced using 40 metres of fencing, while an existing wall will form the fourth side of the rectangle.

  a Let x be the length of each of the two sides perpendicular to the wall. Show that the side parallel to the wall has length $40 - 2x$.

  b Show that the area of the garden bed is given by $A = 40x - 2x^2$.

  c Find $\frac{dA}{dx}$ and hence find the value of x that maximises A.

  d Find the maximum possible area of the garden bed.

6 The quantity V of vitamins present in a patient's bloodstream t hours after taking the vitamin tablets is given by $V = 4t^2 - t^3$, for $0 \le t \le 3$. Find $\frac{dV}{dt}$ and hence determine when the quantity of vitamins in the patient's bloodstream is at its maximum.

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DEVELOPMENT

7 A rectangle has a constant area of 36 cm^2.

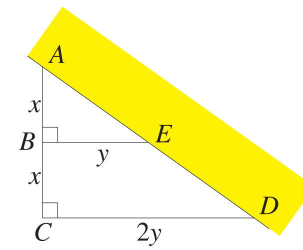
a If x is the length of the rectangle, show that the width is $\dfrac{36}{x}$.

b Show that the perimeter of the rectangle is given by $P = 2x + \dfrac{72}{x}$.

c Show that $\dfrac{dP}{dx} = 2 - \dfrac{72}{x^2}$ and hence that the minimum value of P occurs at $x = 6$.

d Find the minimum possible perimeter of the rectangle.

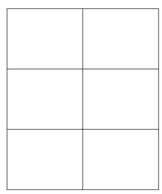
8 A farmer has a field of total area 1200 m^2. To keep his animals separate, he sets up his field with fences at AC, CD and BE, as shown in the diagram. The side AD is beside a river, so no fence is needed there. The point B is the midpoint of AC, and CD is twice the length of BE. Let $AB = x$ and $BE = y$.



a Show that the total length of fencing is $L = 2x + \dfrac{1800}{x}$.

b Hence find the values of x and y that allow the farmer to use the least possible length of fencing.

9 A window frame consisting of six equal rectangles is illustrated to the right. Only 12 metres of frame is available for its construction.



a If the entire frame has height h metres and width w metres, show that $w = \frac{1}{4}(12 - 3h)$.

b Show that the area of the window is $A = 3h - \frac{3}{4}h^2$.

c Find $\dfrac{dA}{dh}$ and hence find the dimensions of the frame for which the area of the window is maximised.

10 A 10 cm length of wire is cut into two pieces from which two squares are formed.

a If one piece has length x, find the side length of each square.

b Show that the combined area of the two squares is $A = \frac{1}{8}(x^2 - 10x + 50)$.

c Find $\dfrac{dA}{dx}$ and hence find the value of x that minimises A.

d Find the least possible combined area.

11 The total cost of producing x telescopes per day is given by $C = \left(\frac{1}{5}x^2 + 15x + 10\right)$ dollars, and each telescope is sold for a price of $\left(47 - \frac{1}{3}x\right)$ dollars.

a Find an expression for the revenue R raised from the sale of x telescopes per day.

b Find an expression for the daily profit $P = R - C$ made if x telescopes are sold.

c How many telescopes should be made daily in order to maximise the profit?

12 The sum of the height h of a cylinder and the circumference of its base is 10 metres.

a Show that $h = 10 - 2\pi r$, where r is the radius of the cylinder.

b Show that the volume of the cylinder is $V = \pi r^2(10 - 2\pi r)$.

c Find $\dfrac{dV}{dr}$ and hence find the value of r at which the volume is a maximum.

d Hence find the maximum possible volume of the cylinder.

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13 A closed cylindrical can is to have a surface area of 60π cm^2.

a Let the cylinder have height h and radius r. Show that $h = \dfrac{30 - r^2}{r}$.

b Show that the volume of the can is $V = \pi r(30 - r^2)$.

c Find $\dfrac{dV}{dr}$ and hence find the maximum possible volume of the can.

14 A box with volume 32 cm^3 has a square base and no lid. Let the square base have length x and the box have height h.

a Show that the surface area of the box is $S = x^2 + 4xh$.

b Show that $h = \dfrac{32}{x^2}$ and hence that $S = x^2 + \dfrac{128}{x}$.

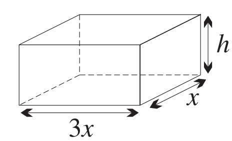
c Find the dimensions of the box that minimise its surface area.

15 **a** An open rectangular box is to be formed by cutting squares of side length x cm from the corners of a rectangular sheet of metal that has length 40 cm and width 15 cm.

b Show that the volume of the box is given by $V = 600x - 110x^2 + 4x^3$.

c Find $\dfrac{dV}{dx}$ and hence find the value of x that maximises the volume of the box.

16 The steel frame of a rectangular prism, as illustrated in the diagram, is three times as long as it is wide.

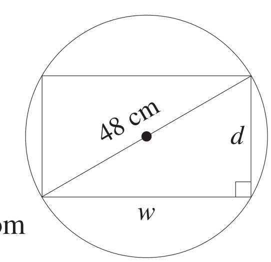


a Find an expression, in terms of x and y, for the length of steel S required to construct the frame.

b The prism has a volume of 4374 m^3. Show that $y = \dfrac{1458}{x^2}$ and hence show that $S = 16x + \dfrac{5832}{x^2}$.

c Show that $\dfrac{dS}{dx} = \dfrac{16(x^3 - 729)}{x^3}$ and hence find the dimensions of the frame so that the minimum amount of steel is used.

17 Engineers have determined that the strength s of a rectangular beam varies as the product of the width w and the square of the depth d of the beam; that is, $s = kwd^2$ for some constant k.



a A particular cylindrical log has a diameter of 48 cm. Use Pythagoras' theorem to show that $s = kw(2304 - w^2)$.

b Hence find the dimensions of the strongest rectangular beam that can be cut from the log.

CHALLENGE

18 A closed rectangular box has length x cm, width y cm and height h cm. It is to be made from 300 cm^2 of thin sheet metal, and the perimeter of the base is to be 40 cm.

a Show that the volume of the box is given by $V = 150h - 20h^2$.

b Hence find the dimensions of the box that meets all the requirements and has the maximum possible volume.

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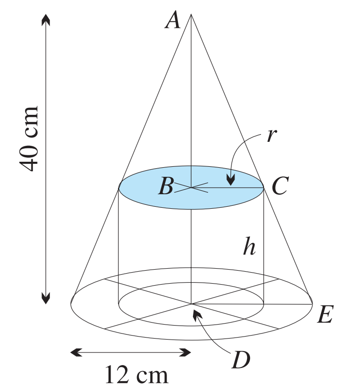
19 A cylinder of height h cm and radius r cm is enclosed in a cone of height 40 cm and radius 12 cm.

a Explain why $\triangle ABC \,|||\, \triangle ADE$.

b By using ratios of corresponding sides, show that $h = 40 - \frac{10}{3}r$.

c Show that the volume of the cylinder is given by $V = 40\pi r^2 - \frac{10}{3}\pi r^3$.

d Find $\dfrac{dV}{dr}$ and hence find the value of r for which the volume of the cylinder is maximised.

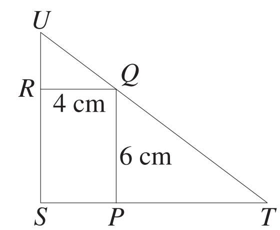


20 In the diagram to the right, $PQRS$ is a rectangle with sides $PQ = 6$ cm and $QR = 4$ cm. The side SP is extended to T, and the side SR is extended to U, so that T, Q and U are collinear. Let $PT = x$ cm and $RU = y$ cm.

a Show that $xy = 24$.

b Show that the area of $\triangle TSU$ is given by $A = 24 + 3x + \dfrac{48}{x}$.

c Hence find the minimum possible area of $\triangle TSU$.



21 A page of a book is to have 80 cm^2 of printed material. There is to be a 2 cm margin at the top and bottom and a 1 cm margin on each side of the page. Let the page have width x and height y.

a Show that $(y - 4)(x - 2) = 80$ and hence that $y = 4 + \dfrac{80}{x - 2}$.

b Show that the area of the page is $A = \dfrac{4x^2 + 72x}{x - 2}$.

c Use the quotient rule to show that $\dfrac{dA}{dx} = \dfrac{4(x^2 - 4x - 36)}{(x - 2)^2}$.

d What should be the dimensions of the page in order to use the least amount of paper?

22 A transport company runs a truck from Hobart to Launceston, a distance of 250 km, at a constant speed of v km/h. For a given speed v, the cost per hour is $6400 + v^2$ cents.

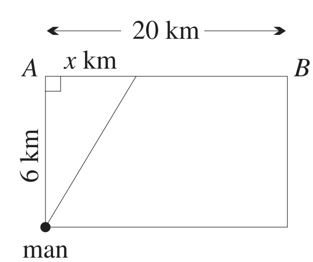
a Show that the cost of the trip, in cents, is $C = 250\left(\dfrac{6400}{v} + v\right)$.

b Find the speed at which the cost of the journey is minimised.

c Find the minimum cost of the journey.

23 A man in a rowing boat is presently 6 km from the nearest point A on the shore. He wants to reach, as soon as possible, a point B that is a further 20 km along the shore from A.

a He can row at 8 km/h and he can run at 10 km/h. He rows to a point on the shore x km from A, and then he runs to B. Show that the time taken for the journey is $T = \frac{1}{8}\sqrt{36 + x^2} + \frac{1}{10}(20 - x)$.
(Hint: Recall that time = distance/speed.)



b The boundaries of the domain in this situation are $x = 0$ (in which case he rows directly to A), and $x = 20$ (in which case he rows all the way to B). Find the values of T, correct to two decimal places where necessary, corresponding to these boundary conditions.

c Use calculus to show that T has a local minimum at $x = 8$.

d Hence find the minimum possible time for the journey.

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3H Primitive functions

This section reverses the process of differentiation and asks, 'What can we say about a function if we know its derivative?' The results of this section will be needed when integration is introduced in Chapter 4.

This topic was briefly mentioned in Section 8D of the Year 11 book, but without any terminology or formulae, and we begin the discussion again here.

Functions with the same derivative

Many different functions may all have the same derivative. For example, all these functions have the same derivative $2x$:

$$x^2, \qquad x^2 + 3, \qquad x^2 - 2, \qquad x^2 + 4\tfrac{1}{2}, \qquad x^2 - \pi.$$

These functions are all the same apart from a constant term. This is true generally — *any two functions with the same derivative differ only by a constant.*

13 FUNCTIONS WITH THE SAME DERIVATIVE

A If a function $f(x)$ has derivative zero in an interval $a < x < b$, then $f(x)$ is a constant function in $a < x < b$.

B If $f'(x) = g'(x)$ for all x in an interval $a < x < b$, then $f(x)$ and $g(x)$ differ by a constant in $a < x < b$.

Proof

A Because the derivative is zero, the gradient of the curve is zero throughout the interval. The curve is therefore a horizontal straight line, and $f(x)$ is a constant function.

B Take the difference between $f(x)$ and $g(x)$ and apply part **A**.

$$\begin{aligned}
\text{Let} \qquad h(x) &= f(x) - g(x). \\
\text{Then} \qquad h'(x) &= f'(x) - g'(x) \\
&= 0, \quad \text{for all } x \text{ in the interval } a < x < b. \\
\text{Hence by part } \mathbf{a}, \qquad h(x) &= C, \quad \text{where } C \text{ is a constant,} \\
\text{so} \qquad f(x) - g(x) &= C, \quad \text{as required.}
\end{aligned}$$

The family of curves with the same derivative

Continuing with the very first example, the various functions whose derivatives are $2x$ are all of the form

$$f(x) = x^2 + C, \qquad \text{where } C \text{ is a constant.}$$

By taking different values of the constant C, these functions form an infinite family of curves, each consisting of the parabola $y = x^2$ translated upwards or downwards.

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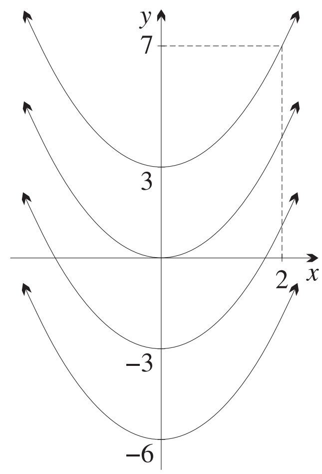
Initial or boundary conditions

If we know also that the graph of the function passes through a particular point, say $(2, 7)$, then we can evaluate the constant C by substituting the point into $f(x) = x^2 + C$,

$$7 = 4 + C.$$

Thus $C = 3$ and hence $f(x) = x^2 + 3$ — in place of the infinite family of functions, there is now a single function.

Such an extra condition is called an *initial condition* or *boundary condition*.



Primitives

We need a suitable name for the result of this reverse process. The words *primitive* and *anti-derivative* are both used.

14 A PRIMITIVE OF A FUNCTION

- A function $F(x)$ is called *a primitive* or *an anti-derivative* of $f(x)$ if the derivative of $F(x)$ is $f(x)$,
  $F'(x) = f(x)$.
- If $F(x)$ is any primitive of $f(x)$ then *the general primitive* of $f(x)$ is
  $F(x) + C$, where C is a constant.

The general primitive of $f(x)$ is also called just *the primitive* of $f(x)$. For example, each these functions is *a primitive* of $x^2 + 1$,

$$\tfrac{1}{3}x^3 + x, \qquad \tfrac{1}{3}x^3 + x + 7, \qquad \tfrac{1}{3}x^3 + x - 13, \qquad \tfrac{1}{3}x^3 + x + 4\pi,$$

whereas *the primitive* of $x^2 + 1$ is $\tfrac{1}{3}x^3 + x + C$, where C is a constant.

A rule for finding primitives

We have seen that a primitive of x is $\tfrac{1}{2}x^2$, and a primitive of x^2 is $\tfrac{1}{3}x^3$. Reversing the formula $\dfrac{d}{dx}(x^{n+1}) = (n + 1)x^n$ gives the general rule:

15 FINDING PRIMITIVES

If $\dfrac{dy}{dx} = x^n$, where $n \neq -1$,

then $y = \dfrac{x^{n+1}}{n + 1} + C$, for some constant C.

'Increase the index by 1 and divide by the new index.'

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Example 20 3H

Find the primitives (or anti-derivatives) of:

a $x^3 + x^2 + x + 1$

b $5x^3 + 7$

SOLUTION

a Let $\dfrac{dy}{dx} = x^3 + x^2 + x + 1.$

Then $y = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C,$

for some constant C.

b Let $f'(x) = 5x^3 + 7.$

Then $f(x) = \frac{5}{4}x^4 + 7x + C,$

for some constant C.

Example 21 3H

Rewrite each function with negative or fractional indices, and find the primitive.

a $\dfrac{1}{x^2}$

b $\sqrt{x}$

SOLUTION

a Let
$$f'(x) = \frac{1}{x^2} = x^{-2}.$$

Then
$$f(x) = -x^{-1} + C, \quad \text{where } C \text{ is a constant,}$$
$$= -\frac{1}{x} + C.$$

b Let
$$\frac{dy}{dx} = \sqrt{x} = x^{\frac{1}{2}}.$$

Then $y = \frac{2}{3}x^{\frac{3}{2}} + C,$ where C is a constant.

Linear extension

Reversing the formula $\dfrac{d}{dx}(ax + b)^{n+1} = a(n + 1)(ax + b)^n$ gives:

16 EXTENSION TO POWERS OF LINEAR FUNCTIONS

If $\dfrac{dy}{dx} = (ax + b)^n,$ where $n \neq -1,$

then $y = \dfrac{(ax + b)^{n+1}}{a(n + 1)} + C,$ for some constant C.

'Increase the index by 1, then divide by the new index and by the coefficient of x.'

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Example 22 3H

Find the primitives of:

a $(3x + 1)^4$ **b** $(1 - 3x)^6$ **c** $\dfrac{1}{(x + 1)^2}$ **d** $\sqrt{x + 1}$

SOLUTION

a Let $\dfrac{dy}{dx} = (3x + 1)^4$.

Then $y = \dfrac{(3x + 1)^5}{5 \times 3} + C$

where C is a constant,

$= \dfrac{(3x + 1)^5}{15} + C.$

b Let $\dfrac{dy}{dx} = (1 - 3x)^6$.

Then $y = \dfrac{(1 - 3x)^7}{7 \times (-3)} + C,$

where C is a constant,

$y = -\dfrac{(1 - 3x)^7}{21} + C.$

c Let $\dfrac{dy}{dx} = \dfrac{1}{(x + 1)^2}$

$= (x + 1)^{-2}$.

Then $y = \dfrac{(x + 1)^{-1}}{-1} + C,$

where C is a constant,

$= -\dfrac{1}{x + 1} + C.$

d Let $\dfrac{dy}{dx} = \sqrt{x + 1}$

$= (x + 1)^{\frac{1}{2}}$.

Then $y = \dfrac{(x + 1)^{\frac{3}{2}}}{\frac{3}{2}} + C,$

where C is a constant,

$y = \frac{2}{3}(x + 1)^{\frac{3}{2}} + C.$

Finding the primitive, given an initial condition

If the derivative of a function is known, plus an initial condition (or boundary condition), then substitute the condition into the general primitive to find the constant, and hence the original function.

17 FINDING A FUNCTION, GIVEN ITS DERIVATIVE AND AN INITIAL CONDITION

- First find the primitive, taking care to include the constant of integration.
- Then substitute the known value of the function to work out the constant.

Example 23 3H

Given that $\dfrac{dy}{dx} = 6x^2 + 1$, and $y = 12$ when $x = 2$, find y as a function of x.

SOLUTION

So $\dfrac{dy}{dx} = 6x^2 + 1,$

$y = 2x^3 + x + C,$ for some constant C.

substituting $x = 2$ and $y = 12$, $12 = 16 + 2 + C,$

so $C = -6$, and hence $y = 2x^3 + x - 6.$

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Example 24 3H

Given that $f''(x) = 12(x - 1)^2$, and $f(0) = f(1) = 0$, find $f(4)$.

SOLUTION

We know that	$f''(x) = 12(x - 1)^2.$
Taking the primitive,	$f'(x) = 4(x - 1)^3 + C,$ for some constant C,
and taking the primitive again,	$f(x) = (x - 1)^4 + Cx + D,$ for some constant D.
Because $f(0) = 0$,	$0 = 1 + 0 + D$
	$D = -1.$
Because $f(1) = 0$,	$0 = 0 + C - 1.$
Hence $C = 1$, so	$f(4) = 81 + 4 - 1$
	$= 84.$

Exercise 3H

FOUNDATION

1 Find the primitive of each function.

a x^6 **b** x^3 **c** x^{10} **d** $3x$ **e** 5 **f** $5x^9$ **g** $21x^6$ **h** 0

2 Find the primitive of each function.

a $x^2 + x^4$ **b** $4x^3 - 5x^4$ **c** $2x^2 + 5x^7$
d $x^2 - x + 1$ **e** $3 - 4x + 16x^7$ **f** $3x^2 - 4x^3 - 5x^4$

3 Find the primitive of each function, after first expanding the product.

a $x(x - 3)$ **b** $(x + 1)(x - 2)$ **c** $(3x - 1)(x + 4)$
d $x^2(5x^3 - 4x)$ **e** $2x^3(4x^4 + 1)$ **f** $(x - 3)(1 + x^2)$

4 Find y as a function of x if:

a $y' = 2x + 3$ and: **i** $y = 3$ when $x = 0$, **ii** $y = 8$ when $x = 1$.
b $y' = 9x^2 + 4$ and: **i** $y = 1$ when $x = 0$, **ii** $y = 5$ when $x = 1$.
c $y' = 3x^2 - 4x + 7$ and: **i** $y = 0$ when $x = 0$, **ii** $y = -1$ when $x = 1$.

DEVELOPMENT

5 Write each function using a negative power of x. Then find the primitive function, writing it as a fraction without a negative index.

a $\dfrac{1}{x^2}$ **b** $\dfrac{1}{x^3}$ **c** $-\dfrac{2}{x^3}$ **d** $-\dfrac{3}{x^4}$ **e** $\dfrac{1}{x^2} - \dfrac{1}{x^3}$

6 Write each function using a fractional index, and hence find the primitive.

a $\sqrt{x}$ **b** $\dfrac{1}{\sqrt{x}}$ **c** $\sqrt[3]{x}$ **d** $\dfrac{2}{\sqrt{x}}$ **e** $\sqrt[5]{x^3}$

7 Find y as a function of x if $\dfrac{dy}{dx} = \sqrt{x}$ and:

a $y = 1$ when $x = 0$, **b** $y = 2$ when $x = 9$.

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8 Find each family of curves whose gradient function is given below. Then sketch the family, and find the member of the family passing through $A(1, 2)$.

a $\dfrac{dy}{dx} = -4x$ **b** $\dfrac{dy}{dx} = 3$ **c** $\dfrac{dy}{dx} = 3x^2$ **d** $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$

9 Recall that if $\dfrac{dy}{dx} = (ax + b)^n$, then $y = \dfrac{(ax + b)^{n+1}}{a(n + 1)} + C$, for some constant C. Use this result to find the primitive of each function.

a $(x + 1)^3$ **b** $(x - 2)^5$ **c** $(x + 5)^2$ **d** $(2x + 3)^4$ **e** $(3x - 4)^6$

f $(5x - 1)^3$ **g** $(1 - x)^3$ **h** $(1 - 7x)^3$ **i** $\dfrac{1}{(x - 2)^4}$ **j** $\dfrac{1}{(1 - x)^{10}}$

10 Find the primitive of each function. Use the rule given in the previous question and the fact that $\sqrt{u} = u^{\frac{1}{2}}$.

a $\sqrt{x + 1}$ **b** $\sqrt{x - 5}$ **c** $\sqrt{1 - x}$ **d** $\sqrt{2x - 7}$ **e** $\sqrt{3x - 4}$

11 **a** Find y if $y' = (x - 1)^4$, given that $y = 0$ when $x = 1$.

b Find y if $y' = (2x + 1)^3$, given that $y = -1$ when $x = 0$.

c Find y if $y' = \sqrt{2x + 1}$, given that $y = \frac{1}{3}$ when $x = 0$.

12 **a** Find the equation of the curve through the origin whose gradient is $\dfrac{dy}{dx} = 3x^4 - x^3 + 1$.

b Find the curve passing through $(2, 6)$ with gradient function $\dfrac{dy}{dx} = 2 + 3x^2 - x^3$.

c Find the curve through the point $\left(\frac{1}{5}, 1\right)$ with gradient function $y' = (2 - 5x)^3$.

13 Find y if $\dfrac{dy}{dt} = 8t^3 - 6t^2 + 5$, and $y = 4$ when $t = 0$. Hence find y when $t = 2$.

CHALLENGE

14 The primitive of x^n is $\dfrac{x^{n+1}}{n + 1}$, provided that $n \neq -1$. Why can't this rule be used when $n = -1$?

15 Find y if $y'' = 6x + 4$, given that when $x = 1$, $y' = 2$ and $y = 4$.
(Hint: Find y' and use the condition $y' = 2$ when $x = 1$ to find the constant of integration. Then find y and use the condition $y = 4$ when $x = 1$ to find the second constant.)

16 A function $f(x)$ has second derivative $f''(x) = 2x - 10$. Its graph passes through the point $(3, -34)$, and at this point the tangent has a gradient of 20.

a Show that $f'(x) = x^2 - 10x + 41$.

b Hence find $f(x)$, and show that its graph cuts the y-axis at $(0, -121)$.

17 If $y'' = 8 - 6x$, show that $y = 4x^2 - x^3 + Cx + D$, for some constants C and D. Hence find the equation of the curve given that it passes through the points $(1, 6)$ and $(-1, 8)$.

18 In Question **8d** above, you showed that the curve with derivative $f'(x) = -\dfrac{1}{x^2}$ passing through $A(1, 2)$ is $y = \dfrac{1}{x} + 1$. This is not strictly true, because the asymptote at $x = 0$ allows the two branches of the curve to move independently so that each branch has its own arbitrary constant.

a Prove that the function $g(x) = \begin{cases} \dfrac{1}{x} + 1, & \text{for } x > 0, \\ \dfrac{1}{x} + 7, & \text{for } x < 0, \end{cases}$ also satisfies the two conditions of Question **8d**.

b Find the piecewise-defined equation of the function $h(x)$ with the same derivative $h'(x) = -\dfrac{1}{x^2}$ passing through the points $A(2, 5)$ and $B(-1, 0)$.

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Chapter 3 Review

Review activity

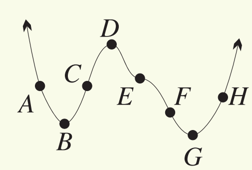
- Create your own summary of this chapter on paper or in a digital document.

Chapter 3 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 In the diagram to the right, name the points where:



a $f'(x) > 0$ **b** $f'(x) < 0$ **c** $f'(x) = 0$
d $f''(x) > 0$ **e** $f''(x) < 0$ **f** $f''(x) = 0$

2 **a** Find the derivative $f'(x)$ of the function $f(x) = x^3 - x^2 - x - 7$.
b Hence find whether $f(x)$ is increasing, decreasing or stationary at:
i $x = 0$ **ii** $x = 1$ **iii** $x = -1$ **iv** $x = 3$

3 **a** Find the derivative $f'(x)$ of the function $f(x) = x^2 - 4x + 3$.
b Find the values of x for which $f(x)$ is:
i increasing, **ii** decreasing, **iii** stationary.

4 Differentiate each function, then evaluate $f'(1)$ to determine whether the function is increasing, decreasing or stationary at $x = 1$.
a $f(x) = x^3$ **b** $f(x) = (x + 2)(x - 3)$
c $f(x) = (x - 1)^5$ **d** $f(x) = \dfrac{x + 1}{x - 3}$

5 Find the first and second derivatives of:
a $y = x^7$ **b** $y = x^3 - 4x^2$ **c** $y = (x - 2)^5$ **d** $y = \dfrac{1}{x}$

6 Find $f''(x)$ for each function. By evaluating $f''(1)$, state whether the curve is concave up or concave down at $x = 1$.
a $f(x) = x^3 - 2x^2 + 4x - 5$ **b** $f(x) = 6 - 2x^3 - x^4$

7 **a** Find the second derivative $f''(x)$ of the function $f(x) = 2x^3 - 3x^2 + 6x - 1$.
b Find the values of x for which $f(x)$ is:
i concave up, **ii** concave down.

8 Find the values of x for which the curve $y = x^3 - 6x^2 + 9x - 11$ is:
a increasing, **b** decreasing, **c** concave up, **d** concave down.

Review

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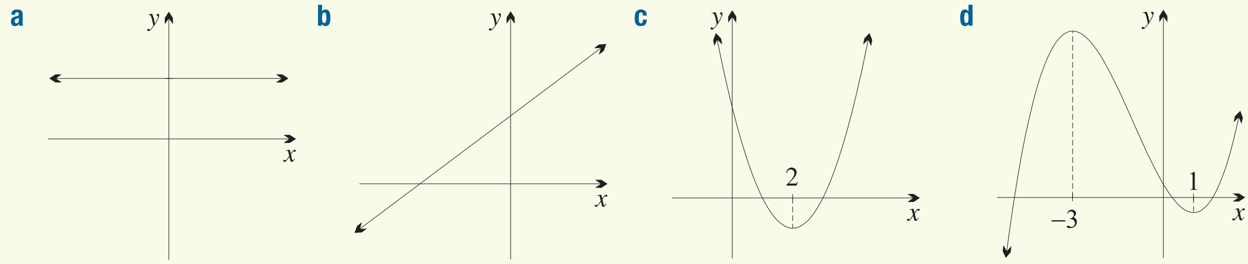
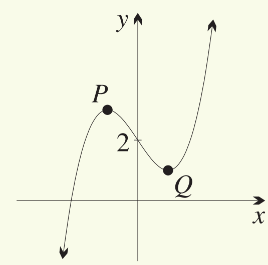
9 Look carefully at each function graphed below to establish where it is increasing, decreasing and stationary. Hence sketch the graph of the derivative of each function.



10 The curve $y = x^3 + x^2 - x + 2$ is graphed to the right. The points P and Q are stationary points.



a Find the coordinates of P and Q.

b For what values of x is the curve concave up?

c For what values of k are there three distinct solutions of the equation $x^2 + x^2 - x + 2 = k$?

11 Sketch the graph of each function, indicating all stationary points and points of inflection.

a $y = x^2 - 6x - 7$

b $y = x^3 - 6x^2 + 8$

c $y = 2x^3 - 3x^2 - 12x + 1$

12 a Sketch the graph of the function $y = x^3 - 3x^2 - 9x + 11$, indicating all stationary points.

b Hence determine the global maximum and minimum values of the function in the domain $-2 \le x \le 6$.

13 a The tangent to $y = x^2 - ax + 9$ is horizontal at $x = -1$. Find the value of a.

b The curve $y = ax^2 + bx + 3$ has a turning point at $(-1, 0)$. Find the values of a and b.

14 a Show that the curve $y = x^4 - 4x^3 + 7$ has a point of inflection at $(2, -9)$.

b Find the gradient of the curve at this point of inflection.

c Hence show that the tangent at the point of inflection is $16x + y - 23 = 0$.

15 The number S of students logged onto a particular website over a five-hour period is given by the formula $S = 175 + 18t^2 - t^4$, for $0 \le t \le 5$.

a What is the initial number of students that are logged on?

b How many students are logged on at the end of the five hours?

c What was the maximum number of students logged onto the website during the five-hour period?

16 A rectangular sheet of cardboard measures 16 cm by 6 cm. Equal squares of side length x cm are cut out of the corners and the sides are turned up to form an open rectangular box.

a Show that the volume V of the box is given by $V = 4x^3 - 44x^2 + 96x$.

b Find, in exact form, the maximum volume of the box.

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Review

17 A coal chute is built in the shape of an upturned cone. The sum of the base radius r and the height h is 12 metres.

a Show that the volume V of the coal chute is given by $V = 4\pi r^2 - \frac{1}{3}\pi r^3$. (Recall that the volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$.)

b Find the radius of the cone that yields the maximum volume.

18 Find the primitive of each function.

a x^7 **b** $2x$ **c** 4 **d** $10x^4$ **e** $8x + 3x^2 - 4x^3$

19 Find the primitive of each function after first expanding the brackets.

a $3x(x - 2)$ **b** $(x + 1)(x - 5)$ **c** $(2x - 3)^2$

20 Find the primitive of each function.

a $(x + 1)^5$ **b** $(x - 4)^7$ **c** $(2x - 1)^3$

21 Find the primitive of each function after writing the function as a power of x.

a $\dfrac{1}{x^2}$ **b** $\sqrt{x}$

22 Find the equation of the curve passing through the point $(2, 5)$ with gradient function $f'(x) = 3x^2 - 4x + 1$.

23 If $f'(x) = 4x - 3$ and $f(2) = 7$, find $f(4)$.

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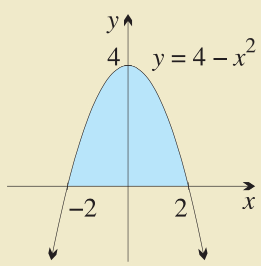
4 Integration

The calculation of areas has so far been restricted to regions bounded by straight lines or parts of circles. This chapter will extend the study of areas to regions bounded by more general curves.

For example, it will be possible to calculate the area of the shaded region in the diagram to the right, bounded by the parabola $y = 4 - x^2$ and the x-axis.



The method developed in this chapter is called *integration*. We will soon show that finding tangents and finding areas are inverse processes, so that integration is the inverse process of differentiation. This surprising result is called the *fundamental theorem of calculus* — the word 'fundamental' is well chosen because the theorem is the basis of the way in which calculus is used throughout mathematics and science.

Graphing software that can also estimate selected areas is useful in the chapter to illustrate how answers change as the curves and boundaries are varied.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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4A Areas and the definite integral

All area formulae and calculations of area are based on two principles:

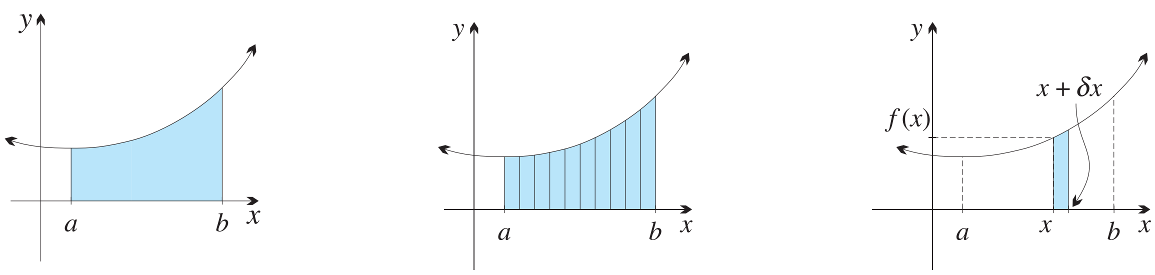
1. Area of a rectangle $=$ length $\times$ breadth.
2. When a region is dissected, the area is unchanged.

A region bounded by straight lines, such as a triangle or a trapezium, can be cut up and rearranged into rectangles with a few well-chosen cuts. Dissecting a curved region into rectangles, however, requires an infinite number of rectangles, and so must be a limiting process, just as differentiation is.

A new symbol — the definite integral

Some new notation is needed to reflect this process of infinite dissection as it applies to functions and their graphs.

The diagram on the left below shows the region contained between a given curve $y = f(x)$ and the x-axis, from $x = a$ to $x = b$, where $a \le b$. The curve must be continuous and, for the moment, never go below the x-axis.



In the middle diagram, the region has been dissected into a number of thin strips. Each strip is approximately a rectangle, but only roughly so, because the upper boundary is curved. The area of the region is the sum of the areas of all the strips.

The third diagram shows just one of the strips, above the value x on the x-axis. Its height at the left-hand end is $f(x)$, and provided the strip is very thin, the height is still about $f(x)$ at the right-hand end. Let the width of the strip be δx, where δx is, as usual in calculus, thought of as being very small. Then, roughly,

$$\begin{aligned}\text{area of strip} &\doteqdot \text{height} \times \text{width}\\ &\doteqdot f(x)\,\delta x.\end{aligned}$$

Adding up all the strips, using sigma notation for the sum,

$$\begin{aligned}\text{area of shaded region} &\doteqdot \sum_{x=a}^{b} \text{area of each strip}\\ &\doteqdot \sum_{x=a}^{b} f(x)\,\delta x.\end{aligned}$$

Now imagine that there are infinitely many of these strips, each infinitesimally thin, so that the inaccuracy disappears. This involves taking the limit, and we might expect to see something like this

$$\text{area of shaded region} = \lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\,\delta x,$$

but instead, we use the brilliant and flexible notation introduced by Leibnitz.

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The width δx is replaced by the symbol dx, which suggests an infinitesimal width, and an old form $\int$ of the letter S is used to suggest an infinite sum under a smooth curve. The result is the strange-looking symbol $\int_a^b f(x)\,dx$. We now *define* this symbol to be the shaded area,

$$\int_a^b f(x)\,dx = \text{area of shaded region.}$$

The definite integral

This new object $\int_a^b f(x)\,dx$ is called a *definite integral*. The rest of the chapter is concerned with evaluating definite integrals and applying them.

1 THE DEFINITE INTEGRAL

Let $f(x)$ be a function that is continuous in a closed interval $[a, b]$, where $a \le b$.
For the moment, suppose that $f(x)$ is never negative in the interval.

- The *definite integral* $\int_a^b f(x)\,dx$ is defined to be the area of the region between the curve and the x-axis, from $x = a$ to $x = b$.
- The function $f(x)$ is called the *integrand*, and the values $x = a$ and $x = b$ are called the *lower* and *upper limits* (or *bounds*) of the integral.

The name 'integration' suggests putting many parts together to make a whole. The notation arises from building up the region from an infinitely large number of infinitesimally thin strips. Integration is 'making a whole' from these thin slices.

Evaluating definite integrals using area formulae

When the function is linear or circular, the definite integral can be calculated from the graph using well-known area formulae, although a quicker method will be developed later for linear functions.

Here are the relevant area formulae:

2 AREA FORMULAE FOR TRIANGLE, TRAPEZIUM AND CIRCLE

Triangle:	Area $= \frac{1}{2}bh$	$= \frac{1}{2} \times$ base $\times$ height
Trapezium:	Area $= \frac{1}{2}(a+b)h$	$=$ average of parallel sides $\times$ width
Circle:	Area $= \pi r^2$	$= \pi \times$ square of the radius

For a trapezium, h is the perpendicular distance between the parallel sides. Depending on the orientation, the word 'height' or 'width' may be more appropriate. Similarly, any side of a triangle may be taken as its 'base'.

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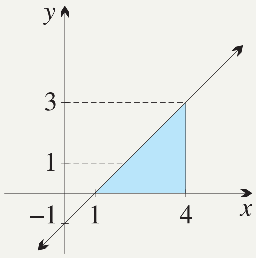
Example 1 4A

Evaluate using a graph and area formulae:

a $\displaystyle\int_1^4 (x - 1)\,dx$ **b** $\displaystyle\int_2^4 (x - 1)\,dx$

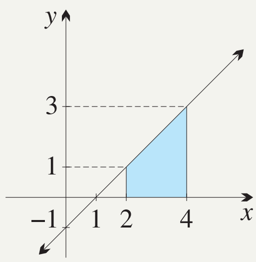
SOLUTION

a The graph of $y = x - 1$ has gradient 1 and y-intercept -1.
The area represented by the integral is the shaded triangle, with base $4 - 1 = 3$ and height 3.



$$\text{Hence} \int_1^4 (x - 1)\,dx = \tfrac{1}{2} \times \text{base} \times \text{height}$$
$$= \tfrac{1}{2} \times 3 \times 3$$
$$= 4\tfrac{1}{2}.$$

b The function $y = x - 1$ is the same as before.
The area represented by the integral is the shaded trapezium, with width $4 - 2 = 2$ and parallel sides of length 1 and 3.



$$\text{Hence} \int_2^4 (x - 1)\,dx = \text{average of parallel sides} \times \text{width}$$
$$= \frac{1 + 3}{2} \times 2$$
$$= 4.$$

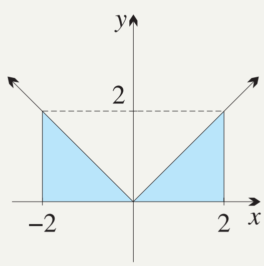
Example 2 4A

Evaluate using a graph and area formulae:

a $\displaystyle\int_{-2}^{2} |x|\,dx$ **b** $\displaystyle\int_{-5}^{5} \sqrt{25 - x^2}\,dx$

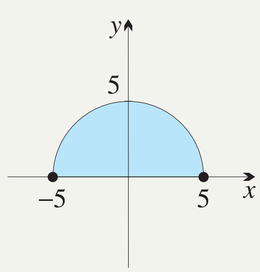
SOLUTION

a The function $y = |x|$ is a V-shape with vertex at the origin.
Each shaded triangle has base 2 and height 2.



$$\text{Hence} \int_{-2}^{2} |x|\,dx = 2 \times \left(\tfrac{1}{2} \times \text{base} \times \text{height}\right)$$
$$= 2 \times \left(\tfrac{1}{2} \times 2 \times 2\right)$$
$$= 4.$$

b The shaded region under $y = \sqrt{25 - x^2}$ is a semicircle, with centre at the origin and radius 5.



$$\text{Hence} \int_{-5}^{5} \sqrt{25 - x^2}\,dx = \tfrac{1}{2} \times \pi r^2$$
$$= \tfrac{1}{2} \times 5^2 \times \pi$$
$$= \tfrac{25\pi}{2}.$$

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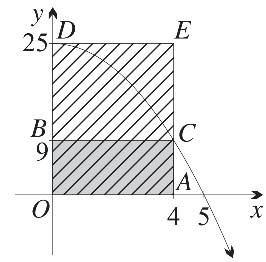
Using upper and lower rectangles to trap an integral



The diagram to the right illustrates how the integral

$$\int_0^4 (25 - x^2)\,dx$$

can easily be trapped between two rectangles,

- the *lower rectangle OABC* (or inner rectangle), and
- the *upper rectangle OADE* (or outer rectangle).

This allows us to trap the integral between two values, as calculated in part **a** of the worked example below.

Using more and more rectangles allows the integral to be trapped between closer and closer bounds, as in parts **b** and **c**. In Section 4D, we will prove the fundamental theorem of calculus using upper and lower rectangles.

Example 3 — 4A

a Evaluate the bounds on $\int_0^4 (25 - x^2)\,dx$ indicated in the diagram above.

b Subdivide the interval $[0, 4]$ as $[0, 2] \cup [2, 4]$ to tighten the bounds.

c Subdivide $[0, 4]$ into four subintervals to tighten the bounds further.

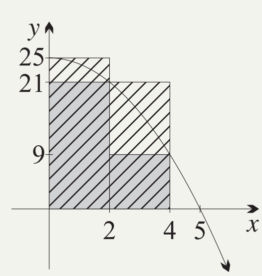
SOLUTION

a In the diagram above,

$$\begin{aligned}\text{area of lower (or inner) rectangle } OACB &= 9 \times 4 \\ &= 36, \\ \text{area of upper (or outer) rectangle } OAED &= 25 \times 4 \\ &= 100,\end{aligned}$$

so $36 < \int_0^4 (25 - x^2)\,dx < 100.$

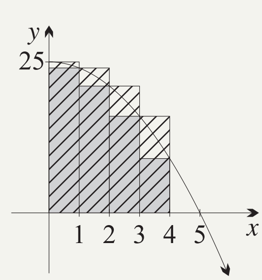
b Subdividing the interval $[0, 4]$ as $[0, 2] \cup [2, 4]$,



$$\begin{aligned}\text{area of 2 lower rectangles} &= 21 \times 2 + 9 \times 2 \\ &= 60, \\ \text{area of 2 upper rectangles} &= 25 \times 2 + 21 \times 2 \\ &= 92,\end{aligned}$$

so $60 < \int_0^4 (25 - x^2)\,dx < 92.$

c Subdividing $[0, 4]$ into four subintervals,



$$\begin{aligned}\text{area of 4 lower rectangles} &= 24 + 21 + 16 + 9 \\ &= 70, \\ \text{area of 4 upper rectangles} &= 25 + 24 + 21 + 16 \\ &= 86,\end{aligned}$$

so $70 < \int_0^4 (25 - x^2)\,dx < 86.$

The exact value of the integral is $78\frac{2}{3}$, as calculated later in worked Example 5(c).

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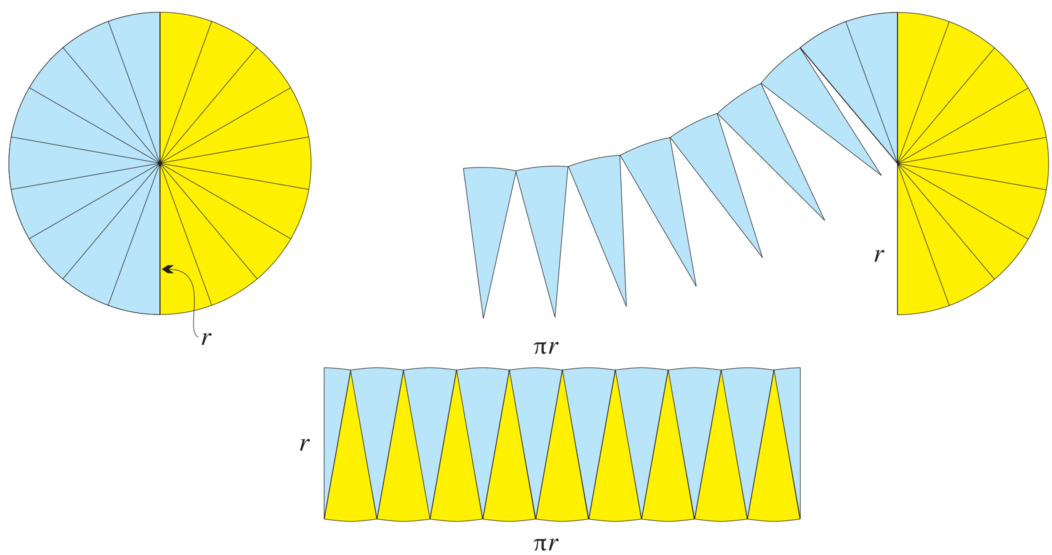
The area of a circle

In earlier years, the formula $A = \pi r^2$ for the area of a circle was developed. Because the boundary is a curve, some limiting process had to be used in its proof. For comparison with the notation for the definite integral explained at the start of this section, here is the most common version of that argument — a little rough in its logic, but very quick. It involves dissecting the circle into infinitesimally thin sectors and then rearranging them into a rectangle.

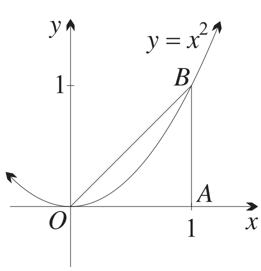


The height of the rectangle in the lower diagram is r. Because the circumference $2\pi r$ is divided equally between the top and bottom sides, the length of the rectangle is πr. Hence the rectangle has area πr^2, which is therefore the area of the circle.

Exercise 4A

FOUNDATION

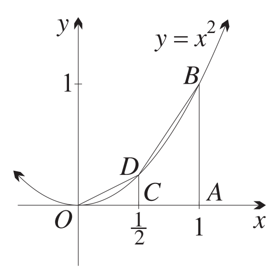
1



a Find the area of $\triangle OAB$ in the diagram above.

b Hence explain why $\int_0^1 x^2\,dx < \frac{1}{2}$.

2

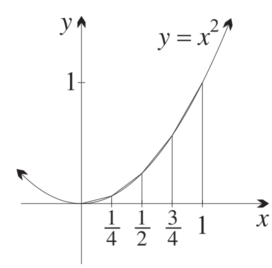


a Find the area of $\triangle OCD$ in the diagram above.

b Find the area of the trapezium $CABD$.

c Hence explain why $\int_0^1 x^2\,dx < \frac{3}{8}$.

3



a Use the diagram above to show that $\int_0^1 x^2\,dx < \frac{11}{32}$.

b Explain why $\frac{11}{32}$ is a better approximation to $\int_0^1 x^2\,dx$ than $\frac{3}{8}$ is.

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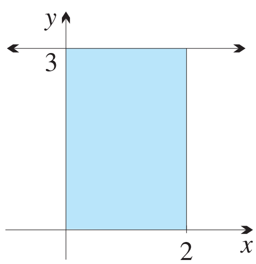
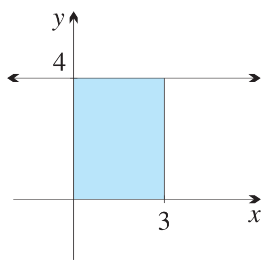
4 Use area formulae to calculate the definite integrals in these sketches.

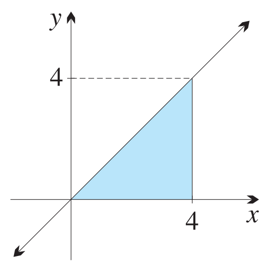
a $\int_0^2 3\,dx$



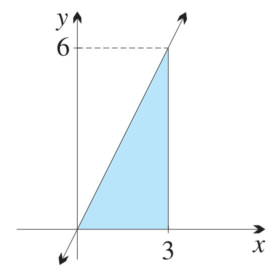
b $\int_0^3 4\,dx$



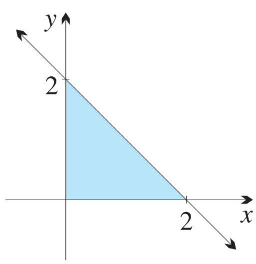
c $\int_0^4 x\,dx$



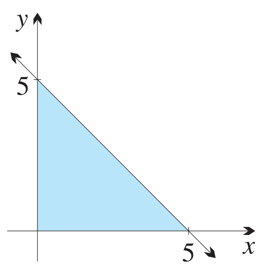
d $\int_0^3 2x\,dx$



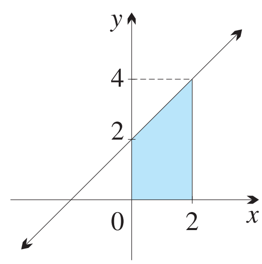
e $\int_0^2 (2 - x)\,dx$



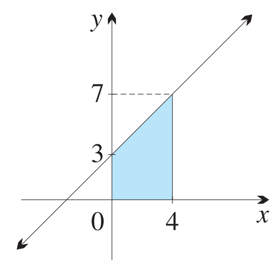
f $\int_0^5 (5 - x)\,dx$



g $\int_0^2 (x + 2)\,dx$

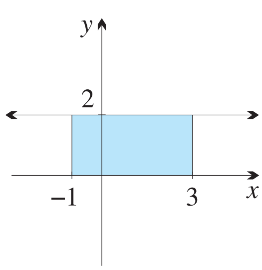


h $\int_0^4 (x + 3)\,dx$

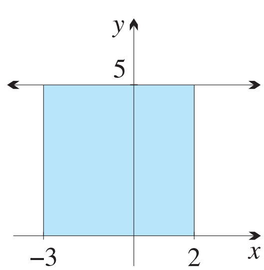


5 Use area formulae to calculate the sketched definite integrals.

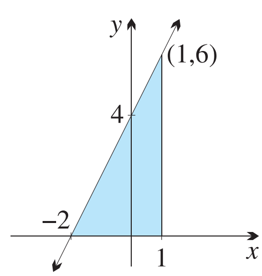
a $\int_{-1}^3 2\,dx$



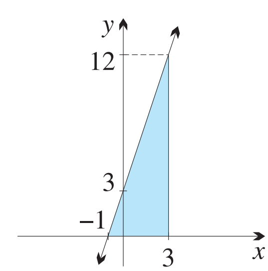
b $\int_{-3}^2 5\,dx$



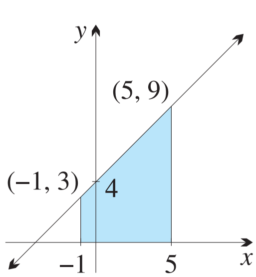
c $\int_{-2}^1 (2x + 4)\,dx$



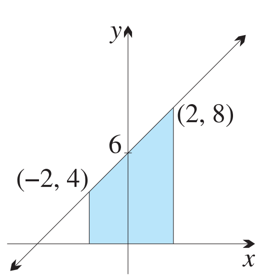
d $\int_{-1}^3 (3x + 3)\,dx$



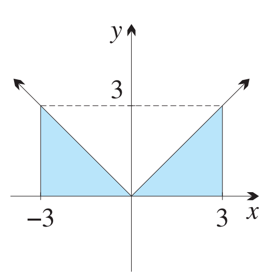
e $\int_{-1}^5 (x + 4)\,dx$



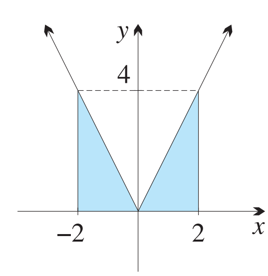
f $\int_{-2}^2 (x + 6)\,dx$



g $\int_{-3}^3 |x|\,dx$



h $\int_{-2}^2 |2x|\,dx$



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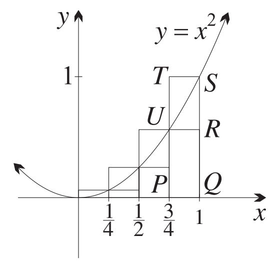
DEVELOPMENT

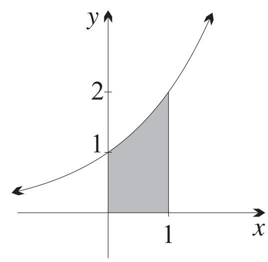
6 a In the diagram to the right, add the areas of the lower rectangles. (For example, $PQRU$ is a lower rectangle.)

b Add the areas of the upper rectangles. (For example, $PQST$ is an upper rectangle.)

c Hence explain why $\frac{7}{32} < \int_0^1 x^2\,dx < \frac{15}{32}$.



7 The area of the region in the diagram to the right is given by $\int_0^1 2^x\,dx$.



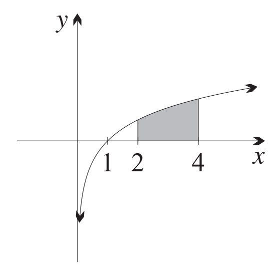
a Use one lower and one upper rectangle to show that $1 < \int_0^1 2^x\,dx < 2$.

b Use 2 lower and 2 upper rectangles of equal width to show that (with decimals rounded to one place) $1.2 < \int_0^1 2^x\,dx < 1.7$.

c Use 4 lower and 4 upper rectangles of equal width to show that $1.3 < \int_0^1 2^x\,dx < 1.6$.

d What trend can be identified in the parts above?

8 The area of the region in the diagram to the right is given by $\int_2^4 \ln x\,dx$.



a Use 2 lower and 2 upper rectangles to show that (with decimals rounded to 2 places) $1.79 < \int_2^4 \ln x\,dx < 2.48$.

b Use 4 lower and 4 upper rectangles of equal width to show that $1.98 < \int_2^4 \ln x\,dx < 2.33$.

c Use 8 lower and 8 upper rectangles of equal width to show that $2.07 < \int_2^4 \ln x\,dx < 2.24$.

d What trend can be identified in the parts above?

9 Sketch a graph of each definite integral, then use an area formula to calculate it.

a $\int_0^3 5\,dx$ b $\int_{-3}^0 5\,dx$ c $\int_{-1}^4 5\,dx$ d $\int_{-2}^6 5\,dx$

e $\int_{-5}^0 (x+5)\,dx$ f $\int_0^2 (x+5)\,dx$ g $\int_2^4 (x+5)\,dx$ h $\int_{-1}^3 (x+5)\,dx$

i $\int_4^8 (x-4)\,dx$ j $\int_4^{10} (x-4)\,dx$ k $\int_5^7 (x-4)\,dx$ l $\int_6^{10} (x-4)\,dx$

m $\int_{-2}^2 |x|\,dx$ n $\int_{-4}^4 |x|\,dx$ o $\int_0^5 |x-5|\,dx$ p $\int_5^{10} |x-5|\,dx$

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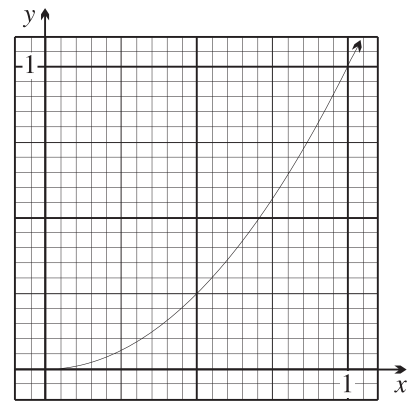
10 Sketch a graph of each definite integral, then use an area formula to calculate it.

a $\int_{-4}^{4} \sqrt{16 - x^2}\, dx$ **b** $\int_{-5}^{0} \sqrt{25 - x^2}\, dx$



11 The diagram to the right shows the graph of $y = x^2$ from $x = 0$ to $x = 1$, drawn on graph paper.

The scale is 20 little divisions to 1 unit. This means that 400 little squares make up 1 square unit.

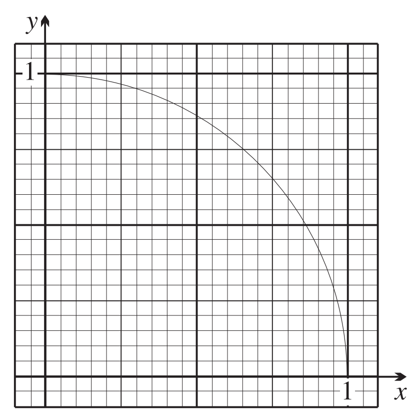
a Count how many little squares there are under the graph from $x = 0$ to $x = 1$ (keeping reasonable track of fragments of squares), then divide by 400 to approximate $\int_0^1 x^2\, dx$.

Write your answer correct to 2 decimal places.

b By counting the appropriate squares, approximate:

i $\int_0^{\frac{1}{2}} x^2\, dx$ **ii** $\int_{\frac{1}{2}}^{1} x^2\, dx$

Confirm that the sum of the answers to parts **i** and **ii** is the answer to part **a**.



12 The diagram to the right shows the quadrant

$$y = \sqrt{1 - x^2}, \quad \text{from } x = 0 \text{ to } x = 1.$$

As before, the scale is 20 little divisions to 1 unit.

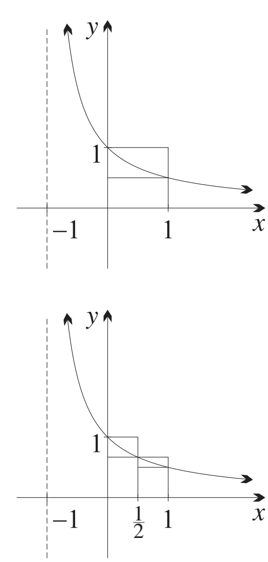
a Count how many little squares there are under the graph from $x = 0$ to $x = 1$.

b Divide by 400 to approximate $\int_0^1 \sqrt{1 - x^2}\, dx$.

Write your answer correct to 2 decimal places.

c Hence, using the fact that a quadrant has area $\frac{1}{4}\pi r^2$, find an approximation for π. Give your answer correct to 2 decimal places.

CHALLENGE



13 Let $A = \int_0^1 \frac{1}{x + 1}\, dx$.

a Use the areas of the lower and upper rectangles in the top diagram to show that $\frac{1}{2} < A < 1$.

b Use the areas of the 2 lower and 2 upper rectangles in the bottom diagram to show that $\frac{7}{12} < A < \frac{5}{6}$. (That is, $0.58 < A < 0.83$, correct to 2 decimal places.)

c Use 3 lower and 3 upper rectangles of equal width to show that $\frac{37}{60} < A < \frac{47}{60}$. (That is, $0.62 < A < 0.78$, correct to 2 decimal places.)

d Finally, use 4 lower and 4 upper rectangles of equal width to show that $\frac{533}{840} < A < \frac{319}{420}$. (That is, correct to 2 decimal places, $0.63 < A < 0.76$.)

e As the number of rectangles increases, is the interval within which A lies getting bigger or smaller?

f The exact value of A is $\ln 2 = 0.693147\ldots$. Do the lower and upper limits of the intervals in parts **a** to **d** seem to be approaching the exact value?

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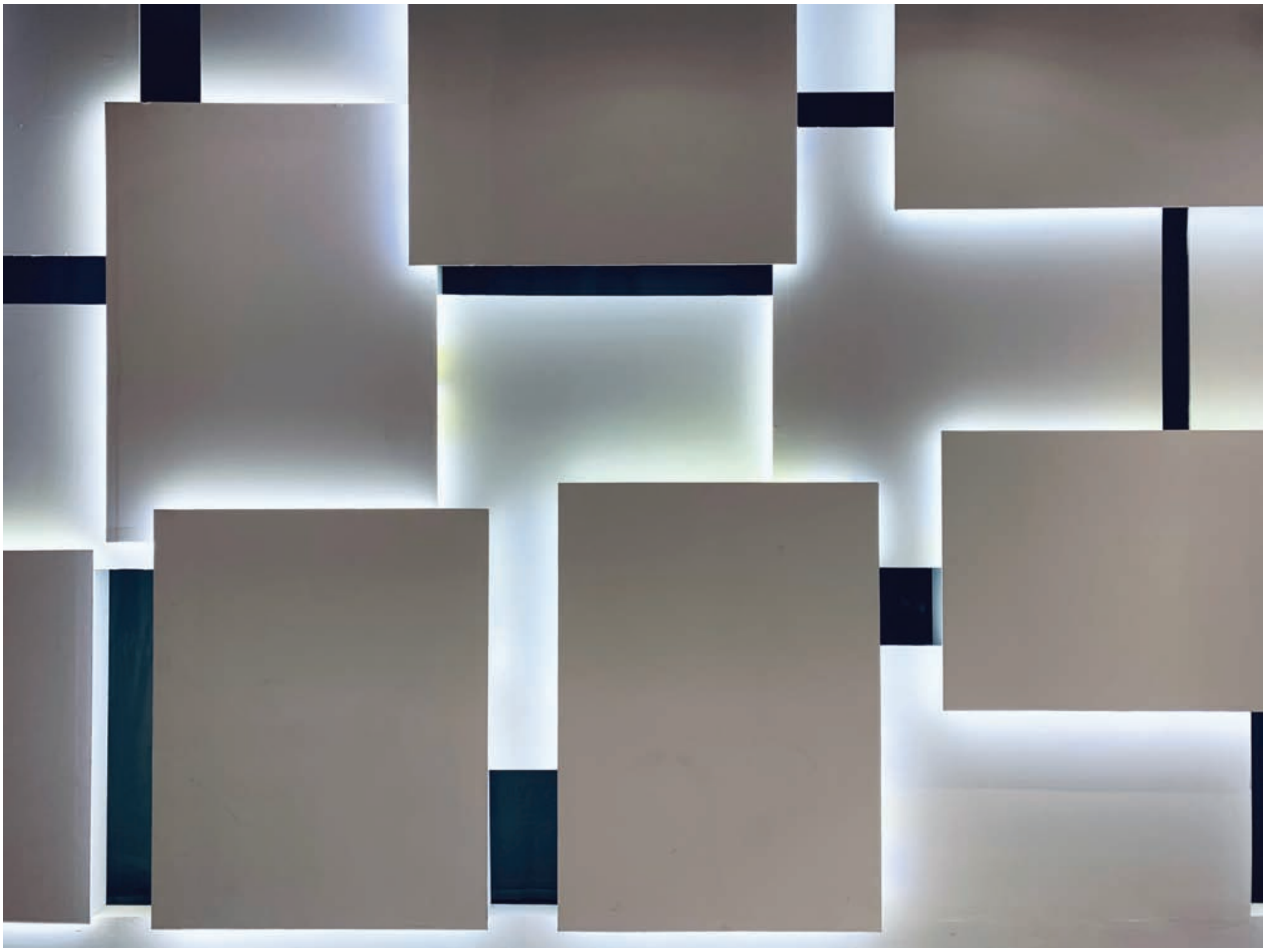
14 [An investigation using technology]
Some of the previous questions involve summing the areas of lower and upper rectangles to approximate a definite integral. Many software programs can do this automatically, using any prescribed number of rectangles. Steadily increasing the number of rectangles will show the sums of the lower and upper rectangles converging to the exact area, which can be checked either using area formulae or using the exact value of the definite integral as calculated later in the course.

Investigate some of the definite integrals from Questions 1–3, 6–8 and 11–13 in this way.



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4B The fundamental theorem of calculus

There is a remarkably simple formula for evaluating definite integrals, based on taking the primitive of the function. The formula is called the *fundamental theorem of calculus* because the whole of calculus depends on it. Its proof is in Section 4D, which is marked as Challenge because it is a little more difficult than the rest of this chapter. The formula alone is presented in this section.

Primitives

Let us first review from Section 3H what primitives are, and the first step in finding them.

3 PRIMITIVES

- A function $F(x)$ is called a *primitive* or *anti-derivative* of a function $f(x)$ if its derivative is $f(x)$:

  $F(x)$ is primitive of $f(x)$ if $F'(x) = f(x)$.

- To find the general primitive of a power x^n, where $n \neq -1$:

  $$\text{If } \frac{dy}{dx} = x^n, \text{ then } y = \frac{x^{n+1}}{n+1} + C, \quad \text{for some constant } C.$$

  'Increase the index by 1 and divide by the new index.'

Statement of the fundamental theorem

The fundamental theorem says that a definite integral can be evaluated by writing down any primitive $F(x)$ of $f(x)$, then substituting the upper and lower limits into it and subtracting.

4 THE FUNDAMENTAL THEOREM OF CALCULUS

Let $f(x)$ be a function that is continuous in a closed interval $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ where } F(x) \text{ is any primitive of } f(x).$$

This result is extraordinary because it says that taking areas and taking tangents are inverse processes, which is not obvious.

Using the fundamental theorem to evaluate an integral

The conventional way to set out these calculations is to enclose the primitive in square brackets, writing the lower and upper limits as subscript and superscript respectively.

Example 4 4B

Evaluate these definite integrals.

a $\displaystyle\int_0^2 2x\,dx$ **b** $\displaystyle\int_2^4 (2x - 3)\,dx$

Then draw diagrams to show the regions that they represent, and check the answers using area formulae.

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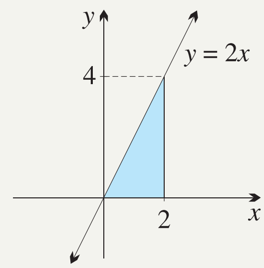
SOLUTION



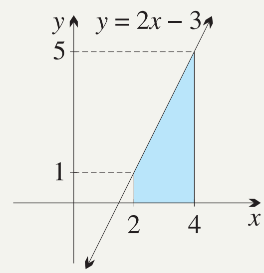
a $\displaystyle\int_0^2 2x\,dx = \Big[x^2\Big]_0^2$ (x^2 is a primitive of $2x$)

$= 2^2 - 0^2$ (substitute 2, then substitute 0 and subract)

$= 4$

This value agrees with the area of the shaded region,

$$\begin{aligned}\text{area of triangle} &= \tfrac{1}{2} \times \text{base} \times \text{height}\\ &= \tfrac{1}{2} \times 2 \times 4\\ &= 4.\end{aligned}$$



b $\displaystyle\int_2^4 (2x - 3)\,dx = \Big[x^2 - 3x\Big]_2^4$ (take the primitive of each term)

$= (16 - 12) - (4 - 6)$ (substitute 4, then substitute 2)

$= 4 - (-2)$

$= 6$

Again, this value agrees with the shaded area,

$$\begin{aligned}\text{area of trapezium} &= \text{average of parallel sides} \times \text{width}\\ &= \frac{1 + 5}{2} \times 2\\ &= 6.\end{aligned}$$

Note: Whenever the primitive has two or more terms, brackets are needed when substituting the upper and lower limits of integration. Set your work out as in line 2 of the solution to part **b**.

Example 5   4B

Evaluate these definite integrals.

a $\displaystyle\int_0^1 x^2\,dx$ **b** $\displaystyle\int_{-2}^2 (x^3 + 8)\,dx$ **c** $\displaystyle\int_0^4 (25 - x^2)\,dx$

SOLUTION

a $\displaystyle\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1$ (increase the index from 2 to 3, then divide by 3)

$= \frac{1}{3} - 0$ (substitute 1, then substitute 0 and subtract)

$= \frac{1}{3}$

This integral was approximated by counting squares in Question 5 of Exercise 1A.

b $\displaystyle\int_{-2}^2 (x^3 + 8)\,dx = \left[\frac{x^4}{4} + 8x\right]_{-2}^2$ (take the primitive of each term)

$= (4 + 16) - (4 - 16)$ (substitute 2, then substitute -2)

$= 20 - (-12)$

$= 32$

c $\displaystyle\int_0^4 (25 - x^2)dx = \Big[25x - \tfrac{1}{3}x^3\Big]_0^4$

$= \left(100 - \frac{1}{3} \times 64\right) - (0 - 0)$ (never omit a substitution of 0)

$= 78\frac{2}{3}$ (this integral was bounded in worked Example 3)

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Expanding brackets in the integrand

As with differentiation, it is often necessary to expand the brackets in the integrand before finding a primitive.

With integration, there is no 'product rule' that could avoid the expansion.

Example 6 4B

Expand the brackets in each integral, then evaluate it.

a $\displaystyle\int_1^6 x(x+1)\,dx$

b $\displaystyle\int_0^3 (x-4)(x-6)\,dx$

SOLUTION

a
$$\begin{aligned}\int_1^6 x(x+1)\,dx &= \int_1^6 (x^2+x)\,dx\\ &= \left[\frac{x^3}{3}+\frac{x^2}{2}\right]_6^1\\ &= (72+18)-\left(\tfrac{1}{3}+\tfrac{1}{2}\right)\\ &= 90-\tfrac{5}{6}\\ &= 89\tfrac{1}{6}\end{aligned}$$

b
$$\begin{aligned}\int_0^3 (x-4)(x-6)\,dx &= \int_0^3 (x^2-10x+24)\,dx\\ &= \left[\frac{x^3}{3}-5x^2+24x\right]_0^3\\ &= (9-45+72)-(0-0+0)\\ &= 36\end{aligned}$$

Note: Parts **a** and **b** show how easily fractions arise in definite integrals because of the fractions in the standard forms for primitives. Care is needed with the resulting common denominators, mixed numerals and cancelling.

Writing the integrand as separate fractions

If the integrand is a fraction with two or more terms in the numerator, it should normally be written as separate fractions, as with differentiation.

With integration, there is no 'quotient rule' that could avoid this.

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Example 7 4B

Write each integrand as two separate fractions, then evaluate the integral.

a $\displaystyle\int_1^2 \frac{3x^4 - 2x^2}{x^2}\,dx$ **b** $\displaystyle\int_{-3}^{-2} \frac{x^3 - 2x^4}{x^3}\,dx$

SOLUTION

a $\displaystyle\int_1^2 \frac{3x^4 - 2x^2}{x^2}\,dx = \int_1^2 (3x^2 - 2)\,dx$ (divide both terms on the top by x^2)

$$= \Big[x^3 - 2x\Big]_1^2$$

$$= (8 - 4) - (1 - 2) = 4 - (-1) = 5$$

b $\displaystyle\int_{-3}^{-2} \frac{x^3 - 2x^4}{x^3}\,dx = \int_{-3}^{-2} (1 - 2x)\,dx$ (divide both terms by x^3)

$$= \Big[x - x^2\Big]_{-3}^{-2}$$

$$= (-2 - 4) - (-3 - 9) = -6 - (-12) = -6 + 12 = 6$$

Negative indices

The fundamental theorem works just as well when the indices are negative. The working, however, requires care when converting between negative powers of x and fractions.

Example 8 4B

Use negative indices to evaluate these definite integrals.

a $\displaystyle\int_1^5 x^{-2}\,dx$ **b** $\displaystyle\int_1^2 \frac{1}{x^4}\,dx$

SOLUTION

a $\displaystyle\int_1^5 x^{-2}\,dx = \left[\frac{x^{-1}}{-1}\right]_1^5$ (increase the index from -2 to -1 and divide by -1)

$$= \left[-\frac{1}{x}\right]_1^5$$ (rewrite x^{-1} as $\frac{1}{x}$ before substitution)

$$= -\tfrac{1}{5} - (-1) = -\tfrac{1}{5} + 1 = \tfrac{4}{5}$$

b $\displaystyle\int_1^2 \frac{1}{x^4}\,dx = \int_1^2 x^{-4}\,dx$ (rewrite $\frac{1}{x^4}$ as x^{-4} before finding the primitive)

$$= \left[\frac{x^{-3}}{-3}\right]_1^2$$ (increase the index to -3 and divide by -3)

$$= \left[-\frac{1}{3x^3}\right]_1^2$$ (rewrite x^{-3} as $\frac{1}{x^3}$ before substitution)

$$= -\tfrac{1}{24} - \left(-\tfrac{1}{3}\right) = -\tfrac{1}{24} + \tfrac{8}{24} = \tfrac{7}{24}$$

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Note: The negative index -1 cannot be handled by this rule, because it would generate division by 0, which is nonsense:

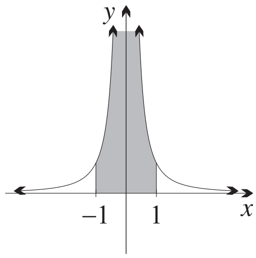
$$\int_1^2 \frac{1}{x}\,dx = \int_1^2 x^{-1}\,dx = \left[\frac{x^0}{0}\right]_2^1 = \text{nonsense.}$$

Chapter 5 on exponential and logarithmic functions will handle this integral.

Warning: Do not integrate across an asymptote

The following calculation seems just as valid as part **b** above:

$$\begin{aligned}\int_{-1}^{1} x^{-4}\,dx &= \left[\frac{x^{-3}}{-3}\right]_{-1}^{1}\\ &= -\tfrac{1}{3} - \tfrac{1}{3}\\ &= -\tfrac{2}{3}.\end{aligned}$$



But in fact the calculation is nonsense — the function has an asymptote at $x = 0$, so it is not even defined there. (And the function $y = x^{-4}$ is always positive, so how could it give a negative answer for an integral?) You cannot integrate across an asymptote, and you always need to be on the lookout for such meaningless integrals.

Exercise 4B

FOUNDATION

Technology: Many programs allow definite integrals to be calculated automatically. This allows not just quick checking of answers, but experimentation with further definite integrals. It would be helpful to generate screen sketches of the graphs and the regions associated with the integrals.

1 Evaluate these definite integrals using the fundamental theorem.

a $\int_0^1 2x\,dx$ **b** $\int_1^4 2x\,dx$ **c** $\int_1^3 4x\,dx$

d $\int_2^5 8x\,dx$ **e** $\int_2^3 3x^2\,dx$ **f** $\int_0^3 5x^4\,dx$

g $\int_1^2 10x^4\,dx$ **h** $\int_0^1 12x^5\,dx$ **i** $\int_0^1 11x^{10}\,dx$

2 **a** Evaluate these definite integrals using the fundamental theorem.

i $\int_0^1 4\,dx$ **ii** $\int_2^7 5\,dx$ **iii** $\int_4^5 dx$

b Check your answers by sketching the graph of the region involved.

3 Evaluate these definite integrals using the fundamental theorem.

a $\int_3^6 (2x + 1)\,dx$ **b** $\int_2^4 (2x - 3)\,dx$ **c** $\int_0^3 (4x + 5)\,dx$

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d $\displaystyle\int_2^3 (3x^2 - 1)\,dx$ **e** $\displaystyle\int_1^4 (6x^2 + 2)\,dx$ **f** $\displaystyle\int_0^1 (3x^2 + 2x)\,dx$

g $\displaystyle\int_1^2 (4x^3 + 3x^2 + 1)\,dx$ **h** $\displaystyle\int_0^2 (2x + 3x^2 + 8x^3)\,dx$ **i** $\displaystyle\int_3^5 (3x^2 - 6x + 5)\,dx$

DEVELOPMENT

4 Evaluate these definite integrals using the fundamental theorem. You will need to take care when finding powers of negative numbers.

a $\displaystyle\int_{-1}^0 (1 - 2x)\,dx$ **b** $\displaystyle\int_{-1}^0 (2x + 3)\,dx$ **c** $\displaystyle\int_{-2}^1 3x^2\,dx$

d $\displaystyle\int_{-1}^2 (4x^3 + 5)\,dx$ **e** $\displaystyle\int_{-2}^2 (5x^4 + 6x^2)\,dx$ **f** $\displaystyle\int_{-2}^{-1} (4x^3 12x^2 - 3)\,dx$

5 Evaluate these definite integrals using the fundamental theorem. You will need to take care when adding and subtracting fractions.

a $\displaystyle\int_1^4 (x + 2)\,dx$ **b** $\displaystyle\int_0^2 (x^2 + x)\,dx$ **c** $\displaystyle\int_0^3 (x^3 + x^2)\,dx$

d $\displaystyle\int_{-1}^1 (x^3 - x + 1)\,dx$ **e** $\displaystyle\int_{-2}^3 (2x^2 - 3x + 1)\,dx$ **f** $\displaystyle\int_{-4}^{-2} (16 - x^3 - x)\,dx$

6 By expanding the brackets first, evaluate these definite integrals.

a $\displaystyle\int_2^3 x(2 + 3x)\,dx$ **b** $\displaystyle\int_0^2 (x + 1)(3x + 1)\,dx$ **c** $\displaystyle\int_{-1}^1 x^2(5x^2 + 1)\,dx$

d $\displaystyle\int_{-1}^2 (x - 3)^2\,dx$ **e** $\displaystyle\int_{-1}^0 x(x - 1)(x + 1)\,dx$ **f** $\displaystyle\int_{-1}^0 (1 - x^2)^2\,dx$

7 Write each integrand as separate fractions, then evaluate the integral.

a $\displaystyle\int_1^3 \frac{3x^3 + 4x^2}{x}\,dx$ **b** $\displaystyle\int_1^2 \frac{4x^4 - x}{x}\,dx$ **c** $\displaystyle\int_2^3 \frac{5x^2 + 9x^4}{x^2}\,dx$

d $\displaystyle\int_1^2 \frac{x^3 + 4x^2}{x}\,dx$ **e** $\displaystyle\int_1^3 \frac{x^3 - x^2 + x}{x}\,dx$ **f** $\displaystyle\int_{-2}^{-1} \frac{x^3 - 2x^5}{x^2}\,dx$

8 Evaluate these definite integrals using the fundamental theorem. You will need to take care when finding powers of fractions.

a $\displaystyle\int_0^{\frac{1}{2}} x^2\,dx$ **b** $\displaystyle\int_0^{\frac{2}{3}} (2x + 3x^2)\,dx$ **c** $\displaystyle\int_{\frac{1}{2}}^{\frac{3}{4}} (6 - 4x)\,dx$

9 **a** Evaluate these definite integrals.

i $\displaystyle\int_5^{10} x^{-2}\,dx$ **ii** $\displaystyle\int_2^3 2x^{-3}\,dx$ **iii** $\displaystyle\int_{\frac{1}{2}}^1 4x^{-5}\,dx$

b By writing them with negative indices, evaluate these definite integrals.

i $\displaystyle\int_1^2 \frac{1}{x^2}\,dx$ **ii** $\displaystyle\int_1^4 \frac{1}{x^3}\,dx$ **iii** $\displaystyle\int_{\frac{1}{2}}^1 \frac{3}{x^4}\,dx$

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10 **a** **i** Show that $\int_2^k 3\,dx = 3k - 6$.

ii Hence find the value of k, given that $\int_2^k 3\,dx = 18$.

b **i** Show that $\int_0^k x\,dx = \frac{1}{2}k^2$.

ii Hence find the positive value of k, given that $\int_0^k x\,dx = 18$.

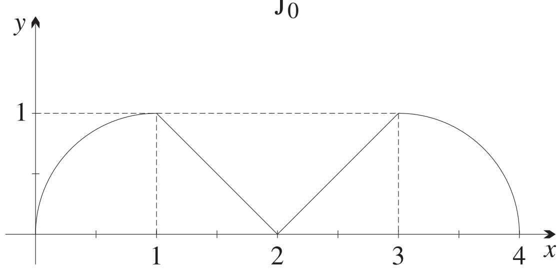
11 Find the value of k if $k > 0$ and:

a $\int_k^3 2\,dx = 4$ **b** $\int_k^8 3\,dx = 12$ **c** $\int_2^3 (k - 3)\,dx = 5$

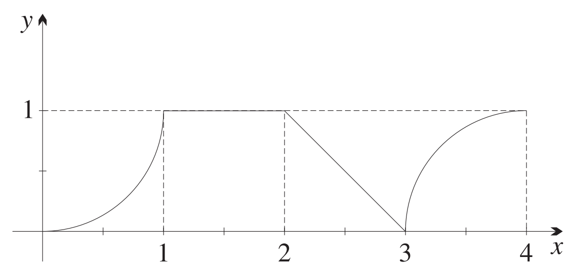
d $\int_3^k (x - 3)\,dx = 0$ **e** $\int_1^k (x + 1)\,dx = 6$ **f** $\int_1^k (k + 3x)\,dx = \frac{13}{2}$

12 Use area formulae to find $\int_0^4 f(x)\,dx$ in each sketch of $f(x)$.

a



b



CHALLENGE

13 Write each integrand as separate fractions, then evaluate the definite integral.

a $\int_1^2 \frac{1 + x^2}{x^2}\,dx$ **b** $\int_{-2}^{-1} \frac{1 + 2x}{x^3}\,dx$ **c** $\int_{-3}^{-1} \frac{1 - x^3 - 4x^5}{2x^2}\,dx$

14 Evaluate these definite integrals.

a $\int_1^3 \left(x + \frac{1}{x}\right)^2 dx$ **b** $\int_1^2 \left(x^2 + \frac{1}{x^2}\right)^2 dx$ **c** $\int_{-2}^{-1} \left(\frac{1}{x^2} + \frac{1}{x}\right)^2 dx$

15 **a** Explain why the function $y = \frac{1}{x^2}$ is never negative.

b Sketch the integrand and explain why the argument below is invalid.

$$\int_{-1}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^{1} = -1 - 1 = -2.$$

c Without evaluating any integrals, say which of these integrals are meaningless.

i $\int_0^2 \frac{1}{(3 - x)^2}\,dx$ **ii** $\int_2^4 \frac{1}{(3 - x)^2}\,dx$ **iii** $\int_4^6 \frac{1}{(3 - x)^2}\,dx$

16 **a** Use the differential form of the fundamental theorem $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ to find:

i $\frac{d}{dx}\int_1^x t^2\,dt$ **ii** $\frac{d}{dx}\int_2^x (t^3 + 3t)\,dt$ **iii** $\frac{d}{dx}\int_a^x \frac{1}{t}\,dt$ **iv** $\frac{d}{dx}\int_a^x (t^3 - 3)^4\,dt$

b Confirm your answers to parts (i) and (ii) above by evaluating the definite integral and then differentiating with respect to x.

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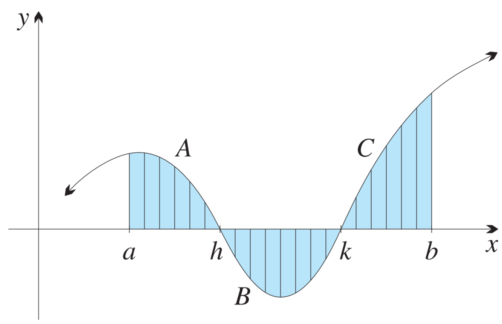
4C The definite integral and its properties

This section will first extend the theory to functions with negative values. Then some simple properties of the definite integral will be established using arguments about the dissection of regions.

Integrating functions with negative values

When a function has negative values, its graph is below the x-axis, so the 'heights' of the little rectangles in the dissection are negative numbers. This means that any areas below the x-axis should contribute negative values to the value of the final integral.



For example, in the diagram above, where $a < h < k < b$, the region B is below the x-axis because the function $f(x)$ is negative in $[h, k]$. So the heights of all the infinitesimal strips making up B are negative, and B therefore contributes a negative number to the definite integral,

$$\int_a^b f(x)\,dx = +\text{ area } A - \text{ area } B + \text{ area } C.$$

Thus we attach the sign $+$ or $-$ to each area, depending on whether the curve, and therefore the region, is above or below the x-axis. For this reason, the three terms in the sum above are often referred to as *signed areas under the curve*, because a sign has been attached to the area of each region.

5 THE DEFINITE INTEGRAL AS THE SUM OF SIGNED AREAS

Let $f(x)$ be a function continuous in the closed interval $[a, b]$, where $a \le b$, and suppose that we are taking the definite integral over $[a, b]$.

- For regions where the curve is above the x-axis, we attach $+$ to the area.
  For regions where the curve is below the x-axis, we attach $-$ to the area.
  These areas, with signs attached, are called *signed areas under the curve*.
- The *definite integral* $\int_a^b f(x)\,dx$ is the the sum of these signed areas under the curve in the interval $[a, b]$.

The whole definite integral $\int_a^b f(x)\,dx$ is often also referred to as *the signed area under the curve*.

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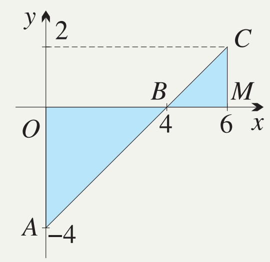
Example 9 4C

Evaluate these definite integrals.

a $\int_0^4 (x - 4)\,dx$ **b** $\int_4^6 (x - 4)\,dx$ **c** $\int_0^6 (x - 4)\,dx$

Sketch the graph of $y = x - 4$ and shade the regions associated with these integrals. Then explain how each result is related to the shaded regions.

SOLUTION



a $\int_0^4 (x - 4)\,dx = \left[\tfrac{1}{2}x^2 - 4x\right]_0^4$

$= (8 - 16) - (0 - 0)$

$= -8$

Triangle OAB has area $8\,\text{u}^2$ and is below the x-axis. so the value of the integral is -8.

b $\int_4^6 (x - 4)\,dx = \left[\tfrac{1}{2}x^2 - 4x\right]_4^6$

$= (18 - 24) - (8 - 16)$

$= -6 - (-8)$

$= 2$

Triangle BMC has area $2\,\text{u}^2$ and is above the x-axis, so the value of the integral is 2.

c $\int_0^6 (x - 4)\,dx = \left[\tfrac{1}{2}x^2 - 4x\right]_0^6$

$= (18 - 24) - (0 - 0)$

$= -6$

This integral represents the area of $\triangle BMC$ minus the area of $\triangle OAB$, so the value of the integral is $2 - 8 = -6$.

Odd and even functions

In the first example below, the function $y = x^3 - 4x$ is an odd function, with point symmetry in the origin. Thus the area of each shaded hump is the same. Hence the whole integral from $x = -2$ to $x = 2$ is zero, because the equal humps above and below the x-axis cancel out.

In the second diagram, the function $y = x^2 + 1$ is even, with line symmetry in the y-axis. Thus the areas to the left and right of the y-axis are equal, so there is a doubling instead of a cancelling.

6 INTEGRATING ODD AND EVEN FUNCTIONS

- If $f(x)$ is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.
- If $f(x)$ is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

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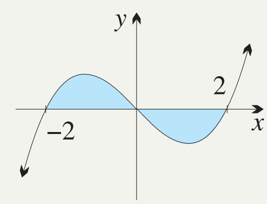
Example 10 4C

Sketch these integrals, then evaluate them using symmetry.

a $\displaystyle\int_{-2}^{2} (x^3 - 4x)\, dx$ **b** $\displaystyle\int_{-2}^{2} (x^2 + 1)\, dx$

SOLUTION

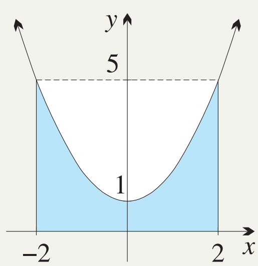
a $\displaystyle\int_{-2}^{2} (x^3 - 4x)\, dx = 0,$ because the integrand is odd.



Without this simplification, the calculation is:

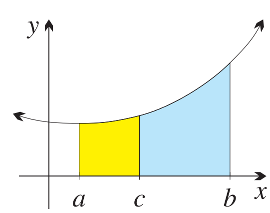
$$\int_{-2}^{2} (x^3 - 4x)\, dx = \Big[\tfrac{1}{4}x^4 - 2x^2\Big]_{-2}^{2}$$
$$= (4 - 8) - (4 - 8)$$
$$= 0, \quad \text{as before.}$$

b Because the integrand is even,



$$\int_{-2}^{2} (x^2 + 1)\, dx = 2\int_{0}^{2} (x^2 + 1)\, dx$$
$$= 2\Big[\tfrac{1}{3}x^3 + x\Big]_{0}^{2}$$
$$= 2\Big(\big(2\tfrac{2}{3} + 2\big) - (0 + 0)\Big)$$
$$= 9\tfrac{1}{3}.$$

Dissection of the interval



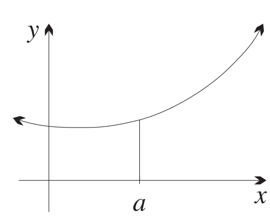
When a region is dissected, its area remains the same. In particular, we can always dissect the region by dissecting the interval $a \le x \le b$ over which we are integrating.

Thus if $f(x)$ is being integrated over the closed interval $[a, b]$ and the number c lies in this interval, then:

7 DISSECTION OF THE INTERVAL

$$\int_{a}^{b} f(x)\, dx = \int_{a}^{c} f(x)\, dx + \int_{c}^{b} f(x)\, dx$$

Intervals of zero width



Suppose that a function is integrated over an interval $a \le x \le a$ of width zero, and that the function is defined at $x = a$. In this situation, the region also has width zero, so the integral is zero.

8 INTERVALS OF ZERO WIDTH

$$\int_{a}^{a} f(x)\, dx = 0$$

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Running an integral backwards from right to left

A further small qualification must be made to the definition of the definite integral. Suppose that the limits of the integral are reversed, so that the integral 'runs backwards' from right to left over the interval. Then its value reverses in sign:

9 REVERSING THE INTERVAL

Let $f(x)$ be continuous in the closed interval $[a, b]$, where $a \le b$. Then we define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

This agrees perfectly with calculations using the fundamental theorem, because

$$F(a) - F(b) = -\big(F(b) - F(a)\big).$$

Example 11 4C

Evaluate and compare these two definite integrals using the fundamental theorem.

a $\displaystyle\int_2^4 (x - 1)\,dx$ **b** $\displaystyle\int_4^2 (x - 1)\,dx$

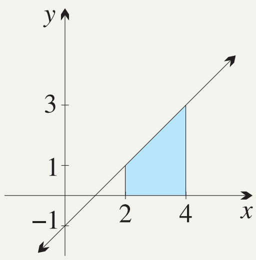
SOLUTION

a $$\int_2^4 (x - 1)\,dx = \left[\frac{x^2}{2} - x\right]_2^4$$
$$= (8 - 4) - (2 - 2)$$
$$= 4,$$

which is positive, because the region is above the x-axis.

b $$\int_4^2 (x - 1)\,dx = \left[\frac{x^2}{2} - x\right]_4^2$$
$$= (2 - 2) - (8 - 4)$$
$$= -4,$$

which is the opposite of part **a**, because the integral runs backwards from right to left, from $x = 4$ to $x = 2$.



Sums of functions

When two functions are added, the two regions are piled on top of each other, so that:

10 INTEGRAL OF A SUM

$$\int_a^b \big(f(x) + g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

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Example 12 4C

Evaluate these two expressions and show that they are equal.

a $\int_0^1 (x^2 + x + 1)\,dx$

b $\int_0^1 x^2\,dx + \int_0^1 x\,dx + \int_0^1 1\,dx$

SOLUTION

a
$$\begin{aligned}\int_0^1 (x^2 + x + 1)\,dx &= \left[\frac{x^3}{3} + \frac{x^2}{2} + x\right]_0^1\\ &= \left(\tfrac{1}{3} + \tfrac{1}{2} + 1\right) - (0 + 0 + 0)\\ &= 1\tfrac{5}{6}.\end{aligned}$$

b
$$\begin{aligned}\int_0^1 x^2\,dx + \int_0^1 x\,dx + \int_0^1 1\,dx &= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^2}{2}\right]_0^1 + \left[x\right]_0^1\\ &= \left(\tfrac{1}{3} - 0\right) + \left(\tfrac{1}{2} - 0\right) + (1 - 0)\\ &= 1\tfrac{5}{6}, \quad \text{the same as in part } \mathbf{a}.\end{aligned}$$

Multiples of functions

Similarly, when a function is multiplied by a constant, the region is stretched vertically by that constant, so that:

11 INTEGRAL OF A MULTIPLE

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

Example 13 4C

Evaluate these two expressions and show that they are equal.

a $\int_1^3 10x^3\,dx$

b $10\int_1^3 x^3\,dx$

SOLUTION

a
$$\begin{aligned}\int_1^3 10x^3\,dx &= \left[\frac{10x^4}{4}\right]_1^3\\ &= \frac{810}{4} - \frac{10}{4}\\ &= \frac{800}{4}\\ &= 200\end{aligned}$$

b
$$\begin{aligned}10\int_1^3 x^3\,dx &= 10 \times \left[\frac{x^4}{4}\right]_1^3\\ &= 10 \times \left(\frac{81}{4} - \frac{1}{4}\right)\\ &= 10 \times \frac{80}{4}\\ &= 200\end{aligned}$$

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Inequalities with definite integrals

Suppose that a curve $y = f(x)$ is always underneath another curve $y = g(x)$ in an interval $a \le x \le b$. Then the area under the curve $y = f(x)$ from $x = a$ to $x = b$ is less than the area under the curve $y = g(x)$.

In the language of definite integrals:

12 INEQUALITY

If $f(x) \le g(x)$ in the closed interval $[a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

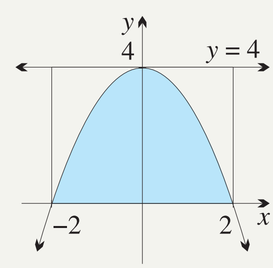
Example 14 — 4C

a Sketch the graph of $f(x) = 4 - x^2$, for $-2 \le x \le 2$.

b Explain why $0 \le \int_{-2}^{2} (4 - x^2)\,dx \le 16$.

SOLUTION

a The parabola and line are sketched opposite.



b Clearly $0 \le 4 - x^2 \le 4$ over the interval $-2 \le x \le 2$.
Hence the region associated with the integral is inside the square of side length 4 in the diagram opposite.

Exercise 4C — FOUNDATION

Technology: All the properties of the definite integral discussed in this section have been justified visually from sketches of the graphs. Computer sketches of the graphs in this exercises would be helpful in reinforcing these explanations. The simplification of integrals of even and odd functions is particularly important and is easily demonstrated visually by graphing programs.

1 Evaluate $\int_4^5 (2x - 3)\,dx$ and $\int_5^4 (2x - 3)\,dx$. What do you notice?

2 Show, by evaluating the definite integrals, that:

a $\int_0^1 6x^2\,dx = 6\int_0^1 x^2\,dx$

b $\int_{-1}^{2} (x^3 + x^2)\,dx = \int_{-1}^{2} x^3\,dx + \int_{-1}^{2} x^2\,dx$

c $\int_0^3 (x^2 - 4x + 3)\,dx = \int_0^2 (x^2 - 4x + 3)\,dx + \int_2^3 (x^2 - 4x + 3)\,dx$

3 Without evaluating the definite integrals, explain why:

a $\int_2^2 (x^2 - 3x)\,dx = 0$

b $\int_{-2}^{2} x\,dx = 0$

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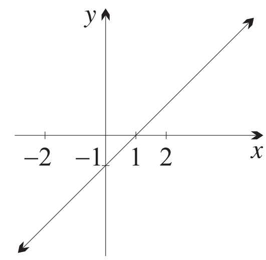
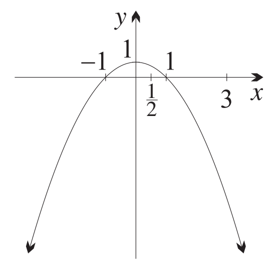


4 The diagram to the right shows the line $y = x - 1$. Without evaluating the definite integrals, explain why:

a $\int_0^1 (x - 1)\, dx$ is negative, **b** $\int_1^2 (x - 1)\, dx$ is positive,

c $\int_0^2 (x - 1)\, dx$ is zero, **d** $\int_{-2}^2 (x - 1)\, dx$ is negative.

5 The diagram to the right shows the parabola $y = 1 - x^2$. Without evaluating the definite integrals, explain why:



a $\int_{-1}^1 (1 - x^2)\, dx$ is positive,

b $\int_1^3 (1 - x^2)\, dx$ is negative,

c $\int_{-1}^0 (1 - x^2)\, dx = \int_0^1 (1 - x^2)\, dx$,

d $\int_0^{\frac{1}{2}} (1 - x^2)\, dx > \int_{\frac{1}{2}}^1 (1 - x^2)\, dx$.

6 **a** If $\int_1^3 f(x)\, dx = 7$, what is the value of $\int_3^1 f(x)\, dx$?

b If $\int_{-1}^{-2} g(x)\, dx = 5$, what is the value of $\int_{-2}^{-1} g(x)\, dx$?

7 **a** Evaluate $\int_1^{-1} (x^2 - 1)\, dx$.

b The graph of $y = x^2 - 1$ is below the x-axis for $-1 \le x \le 1$, and yet the integral is positive. Explain this.

8 Use a diagram to explain why $\int_0^1 2x\, dx > \int_0^1 x\, dx$.

9 Write $\int_{-2}^0 x^3\, dx + \int_0^1 x^3\, dx$ as a single integral, and then use a diagram to explain why this definite integral is negative.

DEVELOPMENT

10 In each part, evaluate the definite integrals. Then use the properties of the definite integral to explain the relationships amongst the integrals within that part.

a **i** $\int_0^2 (3x^2 - 1)\, dx$ **ii** $\int_2^0 (3x^2 - 1)\, dx$

b **i** $\int_0^1 20x^3\, dx$ **ii** $20\int_0^1 x^3\, dx$

c **i** $\int_1^4 (4x + 5)\, dx$ **ii** $\int_1^4 4x\, dx$ **iii** $\int_1^4 5\, dx$

d **i** $\int_0^2 12x^3\, dx$ **ii** $\int_0^1 12x^3\, dx$ **iii** $\int_1^2 12x^3\, dx$

e **i** $\int_3^3 (4 - 3x^2)\, dx$ **ii** $\int_{-2}^{-2} (4 - 3x^2)\, dx$

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11 Use the properties of the definite integral to evaluate each integral without using a primitive function. Give reasons.

a $\int_3^3 \sqrt{9 - x^2}\, dx$ **b** $\int_4^4 (x^3 - 3x^2 + 5x - 7)\, dx$ **c** $\int_{-1}^1 x^3\, dx$

d $\int_{-5}^5 (x^3 - 25x)\, dx$ **e** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\, dx$ **f** $\int_{-2}^2 \frac{x}{1 + x^2}\, dx$

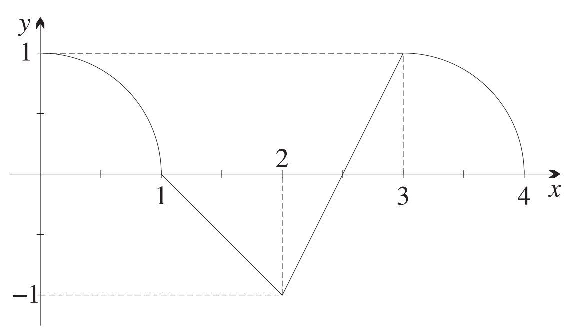
12 **a** On one set of axes sketch $y = x^2$ and $y = x^3$, clearly showing the points of intersection.

b Hence explain why $0 < \int_0^1 x^3\, dx < \int_0^1 x^2\, dx < 1$.

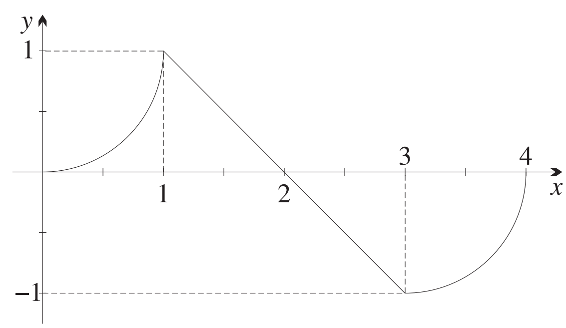
c Check the inequality in part **b** by evaluating each integral.

13 Use area formulae to find $\int_0^4 f(x)\, dx$, given the following sketches of $f(x)$.

a



b



CHALLENGE

14 Using the properties of the definite integral, explain why:

a $\int_{-k}^k (ax^5 + cx^3 + e)\, dx = 0$

b $\int_{-k}^k (bx^4 + dx^2 + f)\, dx = 2\int_0^k (bx^4 + dx^2 + f)\, dx$

15 Sketch a graph of each integral and hence determine whether each statement is true or false.

a $\int_{-1}^1 2^x\, dx = 0$ **b** $\int_0^2 3^x > 0$ **c** $\int_{-2}^{-1} \frac{1}{x}\, dx > 0$ **d** $\int_2^1 \frac{1}{x}\, dx > 0$

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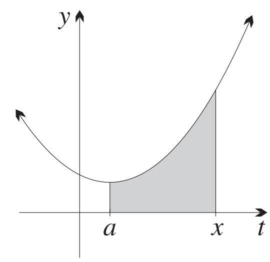
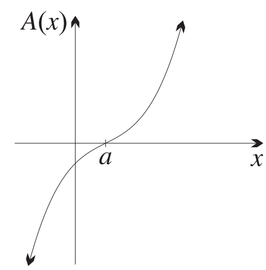
4D Challenge — proving the fundamental theorem

This section develops a proof of the fundamental theorem of calculus, as stated and used in Sections 4B and 4C. The section is challenging, and readers may prefer to leave it to a second reading of the chapter at a later time.

The definite integral as a function of its upper limit



The value of a definite integral $\int_a^b f(x)\,dx$ changes when the value of b changes. This means that it is a function of its upper limit b. When we want to emphasise the functional relationship with the upper limit, we usually replace the letter b by the letter x, which is conventionally the variable of a function.



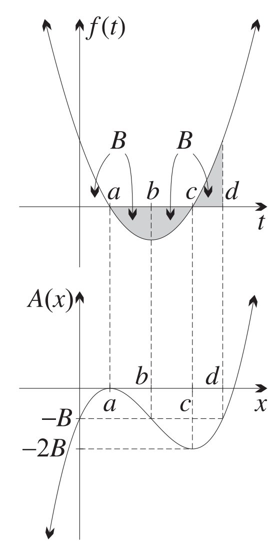
In turn, the original letter x needs to be replaced by some other letter, usually t. Then the definite integral is clearly represented as a function of its upper limit x, as in the first diagram above. This function is called the *signed area function* for $f(x)$ starting at $x = a$, and is defined by

$$A(x) = \int_a^x f(t)\,dt$$

The function $A(x)$ is always zero at $x = a$. For the function sketched above, $A(x)$ increases at an increasing rate — see the second sketch. $A(x)$ is also defined to the left of a, where it is negative because the integrals run backwards.

The signed area function

The function in the sketch above was never negative. But the definite integral is the *signed area* under the curve, meaning that a negative sign is attached to areas of regions below the x-axis (provided that the integral is not running backwards).



The upper graph of $f(t)$ to the right is a parabola with axis of symmetry $x = b$. The four areas marked B are all equal. Here are some properties of the signed area function

$$A(x) = \int_a^x f(t)\,dt.$$

- $A(a) = 0$, as always.
- In the interval $[a, b]$, $f(t)$ is negative and decreasing, so $A(x)$ decreases at an increasing rate.
- In the interval $[b, c]$, $f(t)$ is negative and increasing, so $A(x)$ decreases at a decreasing rate.
- In the interval $[c, \infty)$, $f(t)$ is positive and increasing, so $A(x)$ increases at an increasing rate.
- $A(b) = -B$ and $A(c) = -2B$ and $A(d) = -B$.
- $A(x)$ is also defined in the interval $(-\infty, a]$ to the left of a, where it is negative because the integrals run backwards and the curve is above the x-axis.

The lower diagram above is a sketch of the *signed area function* $A(x)$ of $f(x)$.

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13 THE SIGNED AREA FUNCTION

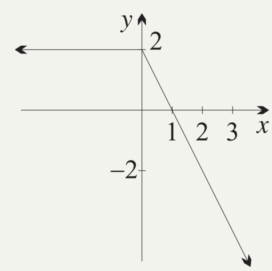
Suppose that $f(x)$ is a function defined in some interval I containing a. The *signed area function* for $f(x)$ starting at a is the function defined by the definite integral

$$A(x) = \int_a^x f(t)\,dt, \qquad \text{for all } x \text{ in the interval } I.$$

In Chapter 10 on continuous probability distributions, the cumulative distribution function of a continuous distribution will be defined in this way.

Example 15 — 4D

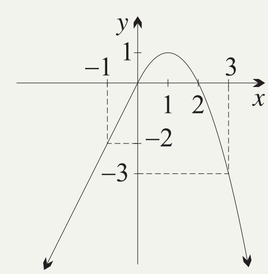
Let $A(x) = \int_0^x f(t)\,dt$ be the signed area function starting at $t = 0$ for the graph sketched to the right. Use area formulae to draw up a table of values for $y = A(x)$ in the interval $[-3, 3]$, then sketch $y = A(x)$.



SOLUTION

Use triangles for $x > 0$ and rectangles for $x < 0$.

x	-3	-2	-1	0	1	2	3
$A(x)$	-6	-4	-2	0	1	0	-3



For $x = -2$ and $x = -1$, $A(x)$ is negative because the integrals run backwards and the curve is above the x-axis. The area function $A(x)$ is increasing for $t < 1$ because $y > 0$, and is decreasing for $x > 1$ because $y < 0$.

The fundamental theorem — differential form

We can now state and prove the *differential form* of the fundamental theorem of calculus, from which we will derive the *integral form* used already in Sections 4B and 4C.

Theorem: If $f(x)$ is continuous, then the signed area function for $f(x)$ is a primitive of $f(x)$. That is,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

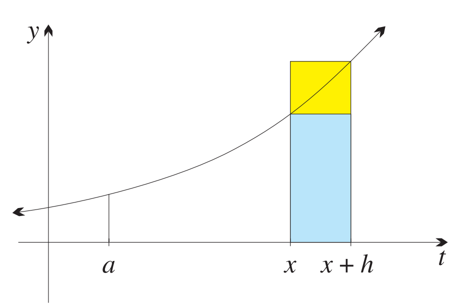
Proof: Because the theorem is so fundamental, its proof must begin with the definition of the derivative as a limit,

$$A'(x) = \lim_{h \to 0} \frac{A(x + h) - A(x)}{h}.$$

Subtracting areas in the diagram to the right,

$$A(x + h) - A(x) = \int_x^{x+h} f(t)\,dt,$$

so
$$A'(x) = \lim_{h \to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$



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This limit is handled by means of a clever sandwiching technique.
Suppose that $f(t)$ is increasing in the interval $[x, x + h]$, as in the diagram above.
Then the lower rectangle on the interval $[x, x + h]$ has height $f(x)$,
and the upper rectangle on the interval $[x, x + h]$ has height $f(x + h)$,

so using areas, $h \times f(x) \le \int_x^{x+h} f(t)\,dt \le h \times f(x + h)$

$\boxed{\div\, h}$ $$f(x) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(x + h). \qquad (1)$$

Thus the middle expression is 'sandwiched' between $f(x)$ and $f(x + h)$.

Because $f(x)$ is continuous, $f(x + h) \to f(x)$ as $h \to 0$,

so by (1), $\lim\limits_{h \to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$, meaning that $A'(x) = f(x)$, as required.

If $f(x)$ is decreasing in the interval $[x, x + h]$, the same argument applies, but with the inequalities reversed.

Note: This theorem shows that the signed area function $A(x) = \int_a^x f(t)\,dt$ is a primitive of $f(x)$. It is therefore often written as $F(x)$ rather than $A(x)$.

Example 16 — 4D

Use the differential form of the fundamental theorem to simplify these expressions. Do not try to evaluate the integral and then differentiate it.

a $\dfrac{d}{dx}\displaystyle\int_0^x (t^2 + 1)^3\,dt$ **b** $\dfrac{d}{dx}\displaystyle\int_4^x (\log_e t)\,dt$ **c** $\dfrac{d}{dx}\displaystyle\int_{-3}^x e^{-t^2}\,dt$

SOLUTION

The differential form says that $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$. Hence:

a $\dfrac{d}{dx}\displaystyle\int_0^x (t^2 + 1)^3\,dt = (x^2 + 1)^3$

b $\dfrac{d}{dx}\displaystyle\int_4^x (\log_e t)\,dt = \log_e x$

c $\dfrac{d}{dx}\displaystyle\int_{-3}^x e^{-t^2}\,dt = e^{-x^2}$

The fundamental theorem — integral form

The integral form of the fundamental theorem is the familiar form that we have been using in Sections 4B and 4C.

Theorem: Suppose that $f(x)$ is continuous in the closed interval $[a, b]$, and that $F(x)$ is a primitive of $f(x)$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

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Proof: We now know that $F(x)$ and $\int_a^x f(t)\,dt$ are both primitives of $f(x)$.

Because any two primitives differ only by a constant,

$$\int_a^x f(t)\,dt = F(x) + C, \quad \text{for some constant } C.$$

Substituting $x = a$, $\int_a^a f(t)\,dt = F(a) + C$,

but $\int_a^a f(t)\,dt = 0$, because the area in this definite integral has zero width,

so
$$0 = F(a) + C$$
$$C = -F(a)$$

Thus
$$\int_a^x f(t)\,dt = F(x) - F(a).$$

and changing letters from x to b and from t to x gives

$$\int_a^b f(x)\,dx = F(b) - F(a), \quad \text{as required.}$$

Example 17 4D

Use the integral form of the fundamental theorem to evaluate each integral. Then differentiate your result, thus confirming the consistency of the discussion above.

a $\dfrac{d}{dx}\displaystyle\int_1^x 6t^2\,dt$ **b** $\dfrac{d}{dx}\displaystyle\int_{-2}^x (t^3 - 9t^2 + 5)\,dt$ **c** $\dfrac{d}{dx}\displaystyle\int_4^x \frac{1}{t^2}\,dt$

SOLUTION

a
$$\int_1^x 6t^2\,dt = \Big[2t^3\Big]_1^x$$
$$= 2x^3 - 2,$$

so
$$\frac{d}{dx}\int_1^x 6t^2\,dt = \frac{d}{dx}(2x^3 - 2)$$
$$= 6x^2, \quad \text{consistent with the differential form.}$$

b
$$\int_{-2}^x (t^3 - 9t^2 + 5)\,dt = \Big[\tfrac{1}{4}t^4 - 3t^3 + 5t\Big]_{-2}^x$$
$$= \left(\tfrac{1}{4}x^4 - 3x^3 + 5x\right) - (4 + 24 - 10)$$
$$= \tfrac{1}{4}x^4 - 3x^3 + 5x - 18,$$

so
$$\frac{d}{dx}\int_{-2}^x (t^3 - 9t^2 + 5)\,dt = \frac{d}{dx}\left(\tfrac{1}{4}x^4 - 3x^3 + 5x - 18\right)$$
$$= x^3 - 9x^2 + 5, \quad \text{consistent with the differential form.}$$

c
$$\int_4^x \frac{1}{t^2}\,dt = \int_4^x t^{-2}\,dt$$
$$= \Big[-t^{-1}\Big]_4^x$$
$$= -x^{-1} + \tfrac{1}{4},$$

so
$$\frac{d}{dx}\int_4^x \frac{1}{t^2}\,dt = \frac{d}{dx}\left(-x^{-1} + \tfrac{1}{4}\right)$$
$$= x^{-2}, \quad \text{consistent with the differential form.}$$

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Super challenge — continuous functions

In Section 8K of the Year 11 volume, we defined continuity at a point — you can draw the curve through the point without lifting the pencil off the paper.

Then throughout this chapter, we have been using the phrase, 'continuous in the closed interval $[a, b]$'. This idea is also straightforward, and informally means that you can place the pencil on the left-hand endpoint $\big(a, f(a)\big)$ and draw the curve to the right-hand endpoint $\big(b, f(b)\big)$ without lifting the pencil off the paper.

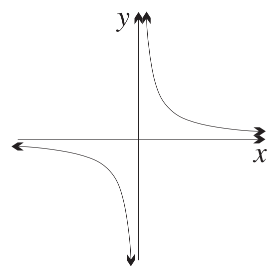
There is, however, a global notion of a *continuous function*:

A continuous function is a function continuous at every value in its domain.

This may look obvious, like so many definitions in mathematics, but it is not.

It means, for example that the reciprocal function $y = \dfrac{1}{x}$ is a continuous function.

This is because $x = 0$ is not in its domain, so the function is continuous at every value of x in its domain.



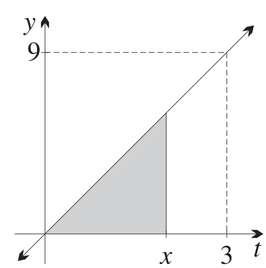
Thus $y = \dfrac{1}{x}$ is a 'continuous function with a discontinuity'.

(Or perhaps the word 'discontinuity' is redefined). We recommend avoiding the concept completely unless some question specifically requires it.

Exercise 4D

FOUNDATION

1 The graph to the right shows $y = 3t$, for $0 \le t \le 3$.



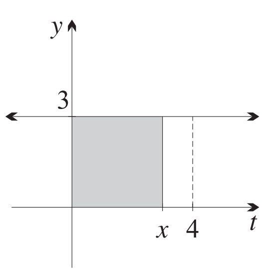
a Use the triangle area formula to find the signed area function

$$A(x) = \int_0^x 3t\,dt, \text{ for } 0 \le x \le 3.$$

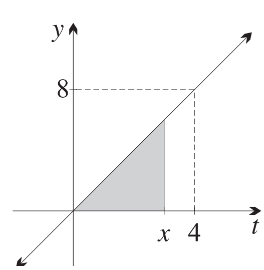
b Differentiate $A(x)$ to show that $A'(x)$ is the original function, apart from a change of letter.

2 Write down the equation of each function, then use area formula, not integration, to calculate the signed area function $A(x) = \displaystyle\int_0^x f(t)\,dt$, for $0 \le t \le 4$. Then differentiate $A(x)$ to confirm that $A'(x)$ is the original function, apart from a change of letter.

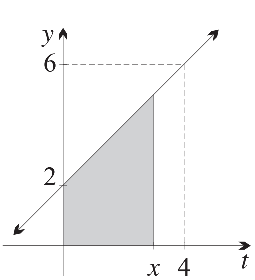
a



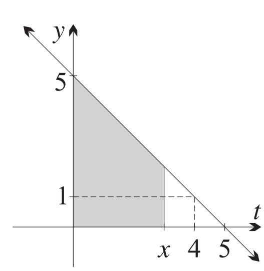
b



c



d



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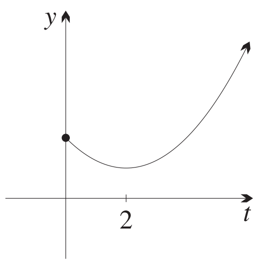
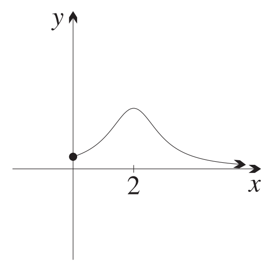
DEVELOPMENT

3 For each function sketched below, describe the behaviour of the signed area function $A(x) = \int_0^x f(t)\,dt$, for all $x \geq 0$, in the interval $[0, 2]$ and in the interval $[2, \infty)$. Then draw a freehand sketch of $y = A(x)$.

a



b



4 The differential form $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt$ of the fundamental theorem tells us that the derivative of the integral is the original function, with a change of letter. Use this to simplify these expressions. Do not attempt to find primitives.

a $\dfrac{d}{dx}\displaystyle\int_1^x \frac{1}{t}\,dt$ **b** $\dfrac{d}{dx}\displaystyle\int_0^x \frac{1}{1+t^3}\,dt$ **c** $\dfrac{d}{dx}\displaystyle\int_0^x e^{-\frac{1}{2}t^2}\,dt$

5 Use the differential form of the fundamental theorem to simplify these expressions. Then confirm the consistency of the discussion in this section by performing the integration and then differentiating.

a $\dfrac{d}{dx}\displaystyle\int_1^x (3t^2 - 12)\,dt$ **b** $\dfrac{d}{dx}\displaystyle\int_2^x (t^3 + 4t)\,dt$ **c** $\dfrac{d}{dx}\displaystyle\int_2^x \frac{1}{t^2}\,dt$

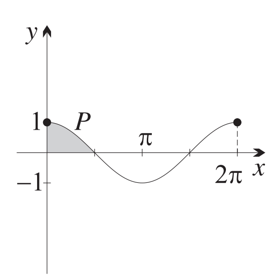
6 **a** Sketch $y = e^t$, then sketch the signed area function $A(x) = \int_0^x e^t\,dt$. How would you describe the behaviour of $y = A(x)$?

b Sketch $y = \log_e t$, then sketch the signed area function $A(x) = \int_1^x \log_e t\,dt$. How would you describe the behaviour of $y = A(x)$?

c Sketch $y = \frac{1}{t}$, then sketch the signed area function $A(x) = \int_1^x \frac{1}{t}\,dt$. How would you describe the behaviour of $y = A(x)$?

CHALLENGE

7 **a** Sketched to the right is $y = \cos x$, for $0 \leq x \leq 2\pi$. Copy and complete the table of values for the signed area function $A(x) = \int_0^t \cos t\,dt$, for $0 \leq t \leq 2\pi$, given that the region marked P has area exactly 1 (this is proven in Chapter 6). Then sketch $y = A(x)$.



x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$A(x)$					

What is your guess for the equation of $A(x)$, and what does this suggest the derivative of $\sin x$ is?

b Sketch $y = \sin t$, for $0 \leq t \leq 2\pi$, and repeat the procedures in part **a**.

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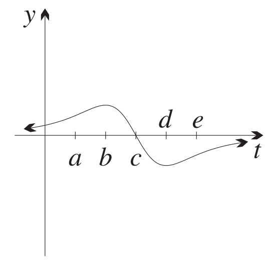
8 The function $y = f(t)$ sketched to the right has point symmetry in $(c, 0)$.



Let $A(x) = \int_a^x f(t)\,dt$.

a Where is $A(x)$ increasing, and when it is decreasing?

b Where does $A(x)$ have a maximum turning point, and where does $A(x)$ have a minimum turning point?

c Where does $A(x)$ have inflections?

d Where are the zeroes of $A(x)$?

e Where is $A(x)$ positive, and where it is negative?

f Sketch $y = A(x)$.

9 This 'super challenge' question may illuminate the definition of a continuous function:

A continuous function is a function that is continuous at every number in its domain.

Classify these functions as continuous or not continuous according to the definition above.

a $y = x - 2$

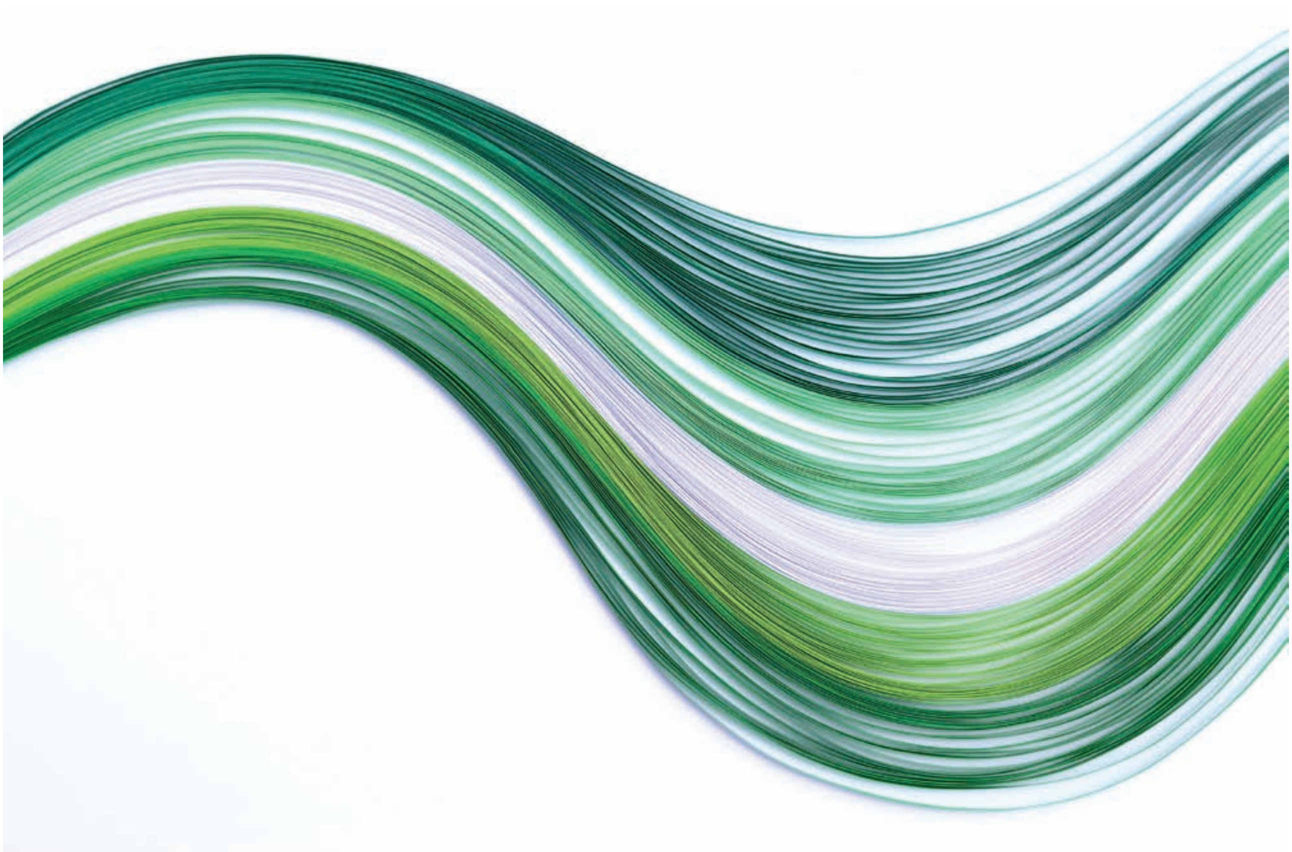
b $y = \dfrac{1}{x - 2}$

c $y = \begin{cases} \dfrac{1}{x-2}, & \text{for } x \neq 2, \\ 0, & \text{for } x = 2. \end{cases}$

d $y = \sqrt{x}$

e $y = \dfrac{1}{\sqrt{x}}$

f $y = \begin{cases} \dfrac{1}{\sqrt{x}}, & \text{for } x > 0, \\ 0, & \text{for } x = 0. \end{cases}$



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4E The indefinite integral

Now that primitives have been established as the key to calculating definite integrals, this section turns again to the task of finding primitives. First, a new and convenient notation for the primitive is introduced.

The indefinite integral

Because of the close connection established by the fundamental theorem between primitives and definite integrals, the term *indefinite integral* is often used for the general primitive. The usual notation for the indefinite integral of a function $f(x)$ is an integral sign without any upper or lower limits. For example, *the primitive* or *the indefinite integral* of $x^2 + 1$ is

$$\int (x^2 + 1)\, dx = \frac{x^3}{3} + x + C, \quad \text{for some constant } C.$$

The word 'indefinite' suggests that the integral cannot be evaluated further because no limits for the integral have yet been specified.

The constant of integration

A definite integral ends up as a pure number. An indefinite integral, on the other hand, is a function of x — the pronumeral x is carried across to the answer.

It also contains an unknown constant C (or c, as it is often written) and the indefinite integral can also be regarded as a function of C (or of c). The constant is called a 'constant of integration' and is an important part of the answer — it must always be included.

The only exception to including the constant of integration is when calculating definite integrals, because in that situation any primitive can be used.

Note: Strictly speaking, the words 'for some constant C' or 'where C is a constant' should follow the first mention of the new pronumeral C, because no pronumeral should be used without having been formally introduced. There is a limit to one's patience, however (in this book there is often no room), and usually it is quite clear that C is the constant of integration. If another pronumeral such as D is used, it would be wise to introduce it formally.

Standard forms for integration

The two rules for finding primitives given in Section 3H can now be restated in this new notation.

14 STANDARD FORMS FOR INTEGRATION

Suppose that $n \neq -1$. Then

- $$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, \quad \text{for some constant } C.$$
- $$\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, \quad \text{for some constant } C.$$

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The word 'integration' is commonly used to refer both to the finding of a primitive, and to the evaluating of a definite integral. Similarly, the unqualified term 'integral' is used to refer both to the indefinite integral and to the definite integral.

Example 18 4E

Use the standard form $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$ to find:

a $\int 9\,dx$

b $\int 12x^3\,dx$

SOLUTION

a $\int 9\,dx = 9x + C$, for some constant C

Note: We know that $9x$ is the primitive of 9, because $\dfrac{d}{dx}(9x) = 9$.

But the formula still gives the correct answer, because $9 = 9x^0$,
so increasing the index to 1 and dividing by this new index 1,

$$\begin{aligned}\int 9x^0\,dx &= \frac{9x^1}{1} + C, \quad \text{for some constant } C\\ &= 9x + C.\end{aligned}$$

b
$$\begin{aligned}\int 12x^3\,dx &= 12 \times \frac{x^4}{4} + C, \quad \text{for some constant } C\\ &= 3x^4 + C\end{aligned}$$

Example 19 4E

Use the standard form $\int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C$ to find:

a $\int (3x+1)^5\,dx$

b $\int (5-2x)^2\,dx$

SOLUTION

a
$$\begin{aligned}\int (3x+1)^5\,dx &= \frac{(3x+1)^6}{3\times 6} + C \qquad (\text{here } n = 5 \text{ and } a = 3 \text{ and } b = 1)\\ &= \tfrac{1}{18}(3x+1)^6 + C\end{aligned}$$

b
$$\begin{aligned}\int (5-2x)^2\,dx &= \frac{(5-2x)^3}{(-2)\times 3} + C \qquad (\text{here } n = 2 \text{ and } a = -2 \text{ and } b = 5)\\ &= -\tfrac{1}{6}(5-2x)^3 + C\end{aligned}$$

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Negative indices

Both standard forms apply with negative indices as well as positive indices, as in the next worked example.

The exception is the index -1, where the rule is nonsense because it results in division by zero. We shall deal with the integration of x^{-1} in Chapter 5.

Example 20 4E

Use negative indices to find these indefinite integrals.

a $\displaystyle\int \frac{12}{x^3}\,dx$

b $\displaystyle\int \frac{dx}{(3x+4)^2}$

SOLUTION

a
$$\begin{aligned}\int \frac{12}{x^3}\,dx &= \int 12x^{-3}\,dx\\ &= 12 \times \frac{x^{-2}}{-2} + C \qquad \text{(increase the index to } -2\text{, then divide by } -2)\\ &= -\frac{6}{x^2} + C\end{aligned}$$

b
$$\begin{aligned}\int \frac{dx}{(3x+4)^2} &= \int (3x+4)^{-2}\,dx\\ &= \frac{(3x+4)^{-1}}{3 \times (-1)} + C \qquad \text{(here } a = 3 \text{ and } b = 4)\\ &= -\frac{1}{3(3x+4)} + C\end{aligned}$$

Special expansions

In many integrals, brackets must be expanded before the indefinite integral can be found. The next worked example uses the special expansions. Part **b** also requires negative indices.

Example 21 4E

Find these indefinite integrals.

a $\displaystyle\int (x^3 - 1)^2\,dx$

b $\displaystyle\int \left(3 - \frac{1}{x^2}\right)\left(3 + \frac{1}{x^2}\right)dx$

SOLUTION

a
$$\begin{aligned}\int (x^3 - 1)^2\,dx &= \int (x^6 - 2x^3 + 1)\,dx \qquad \text{(using } (A+B)^2 = A^2 + 2AB + B^2)\\ &= \frac{x^7}{7} - \frac{x^4}{2} + x + C\end{aligned}$$

b
$$\begin{aligned}\int \left(3 - \frac{1}{x^2}\right)\left(3 + \frac{1}{x^2}\right)dx &= \int \left(9 - \frac{1}{x^4}\right)dx \qquad \text{(using } (A-B)(A+B) = A^2 - B^2)\\ &= \int (9 - x^{-4})\,dx\\ &= 9x - \frac{x^{-3}}{-3} + C = 9x + \frac{1}{3x^3} + C\end{aligned}$$

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Fractional indices

The standard forms for finding primitives of powers also apply to fractional indices. These calculations require quick conversions between fractional indices and surds. The next worked example finds each indefinite integral and then uses it to evaluate a definite integral.

Example 22 4E

Use fractional and negative indices to evaluate:

a $\displaystyle\int_1^4 \sqrt{x}\,dx$

b $\displaystyle\int_1^4 \frac{1}{\sqrt{x}}\,dx$

SOLUTION

a
$$\begin{aligned}\int_1^4 \sqrt{x}\,dx &= \int_1^4 x^{\frac{1}{2}}\,dx && \left(\text{rewrite } \sqrt{x} \text{ as } x^{\frac{1}{2}} \text{ before finding the primitive}\right)\\ &= \tfrac{2}{3}\Big[x^{\frac{3}{2}}\Big]_1^4 && \left(\text{increase the index to } \tfrac{3}{2} \text{ and divide by } \tfrac{3}{2}\right)\\ &= \tfrac{2}{3} \times (8 - 1) && \left(4^{\frac{3}{2}} = 2^3 = 8 \text{ and } 1^{\frac{3}{2}} = 1\right)\\ &= 4\tfrac{2}{3}\end{aligned}$$

b
$$\begin{aligned}\int_1^4 \frac{1}{\sqrt{x}}\,dx &= \int_1^4 x^{-\frac{1}{2}}\,dx && \left(\text{rewrite } \frac{1}{\sqrt{x}} \text{ as } x^{-\frac{1}{2}} \text{ before finding the primitive}\right)\\ &= \tfrac{2}{1}\Big[x^{\frac{1}{2}}\Big]_1^4 && \left(\text{increase the index to } \tfrac{1}{2} \text{ and divide by } \tfrac{1}{2}\right)\\ &= 2 \times (2 - 1) && \left(4^{\frac{1}{2}} = \sqrt{4} = 2 \text{ and } 1^{\frac{1}{2}} = 1\right)\\ &= 2\end{aligned}$$

Example 23 4E

a Use index notation to express $\dfrac{1}{\sqrt{9 - 2x}}$ as a power of $9 - 2x$.

b Hence find the indefinite integral $\displaystyle\int \frac{dx}{\sqrt{9 - 2x}}$.

SOLUTION

a $\dfrac{1}{\sqrt{9 - 2x}} = (9 - 2x)^{-\frac{1}{2}}$.

b Hence
$$\begin{aligned}\int \frac{1}{\sqrt{9 - 2x}} &= \int (9 - 2x)^{-\frac{1}{2}}\,dx\\ &= \frac{(9 - 2x)^{\frac{1}{2}}}{-2 \times \frac{1}{2}} + C, \quad \text{using} \int (ax + b)^n\,dx = \frac{(ax + b)^{n+1}}{a(n + 1)},\\ &= -\sqrt{9 - 2x} + C.\end{aligned}$$

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Exercise 4E

FOUNDATION

Technology: Many programs that can perform algebraic manipulation are also able to deal with indefinite integrals. They can be used to check the questions in this exercise and to investigate the patterns arising in such calculations.

1 Find these indefinite integrals.

a $\int 4\,dx$ **b** $\int 1\,dx$ **c** $\int 0\,dx$ **d** $\int (-2)\,dx$

e $\int x\,dx$ **f** $\int x^2\,dx$ **g** $\int x^3\,dx$ **h** $\int x^7\,dx$

2 Find the indefinite integral of each function. Use the notation of the previous question.

a $2x$ **b** $4x$ **c** $3x^2$ **d** $4x^3$

e $10x^9$ **f** $2x^3$ **g** $4x^5$ **h** $3x^8$

3 Find these indefinite integrals.

a $\int (x + x^2)\,dx$ **b** $\int (x^4 - x^3)\,dx$ **c** $\int (x^7 + x^{10})\,dx$

d $\int (2x + 5x^4)\,dx$ **e** $\int (9x^8 - 11)\,dx$ **f** $\int (7x^{13} + 3x^8)\,dx$

g $\int (4 - 3x)\,dx$ **h** $\int (1 - x^2 + x^4)\,dx$ **i** $\int (3x^2 - 8x^3 + 7x^4)\,dx$

4 Find the indefinite integral of each function. (Leave negative indices in your answers.)

a x^{-2} **b** x^{-3} **c** x^{-8}

d $3x^{-4}$ **e** $9x^{-10}$ **f** $10x^{-6}$

5 Find these indefinite integrals. (Leave fractional indices in your answers.)

a $\int x^{\frac{1}{2}}\,dx$ **b** $\int x^{\frac{1}{3}}\,dx$ **c** $\int x^{\frac{1}{4}}\,dx$

d $\int x^{\frac{2}{3}}\,dx$ **e** $\int x^{-\frac{1}{2}}\,dx$ **f** $\int 4x^{\frac{1}{2}}\,dx$

DEVELOPMENT

6 By first expanding the brackets, find these indefinite integrals.

a $\int x(x + 2)\,dx$ **b** $\int x(4 - x^2)\,dx$ **c** $\int x^2(5 - 3x)\,dx$

d $\int x^3(x - 5)\,dx$ **e** $\int (x - 3)^2\,dx$ **f** $\int (2x + 1)^2\,dx$

g $\int (1 - x^2)^2\,dx$ **h** $\int (2 - 3x)(2 + 3x)\,dx$ **i** $\int (x^2 - 3)(1 - 2x)\,dx$

7 Write each integrand as separate fractions, then perform the integration.

a $\int \frac{x^2 + 2x}{x}\,dx$ **b** $\int \frac{x^7 + x^8}{x^6}\,dx$ **c** $\int \frac{2x^3 - x^4}{4x}\,dx$

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8 Write these functions with negative indices and hence find their indefinite integrals.

a $\frac{1}{x^2}$ **b** $\frac{1}{x^3}$ **c** $\frac{1}{x^5}$ **d** $\frac{1}{x^{10}}$

e $\frac{3}{x^4}$ **f** $\frac{5}{x^6}$ **g** $\frac{7}{x^8}$ **h** $\frac{1}{3x^2}$

i $\frac{1}{7x^5}$ **j** $-\frac{1}{5x^3}$ **k** $\frac{1}{x^2} - \frac{1}{x^5}$ **l** $\frac{1}{x^3} + \frac{1}{x^4}$

9 Write these functions with fractional indices and hence find their indefinite integrals.

a $\sqrt{x}$ **b** $\sqrt[3]{x}$ **c** $\frac{1}{\sqrt{x}}$ **d** $\sqrt[3]{x^2}$

10 Use the indefinite integrals of the previous question to evaluate:

a $\int_0^9 \sqrt{x}\, dx$ **b** $\int_0^8 \sqrt[3]{x}\, dx$ **c** $\int_{25}^{49} \frac{1}{\sqrt{x}}\, dx$ **d** $\int_0^1 \sqrt[3]{x^2}\, dx$

11 By using the rule $\int (ax + b)^n\, dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + C$, find:

a $\int (x + 1)^5\, dx$ **b** $\int (x + 2)^3\, dx$ **c** $\int (4 - x)^4\, dx$

d $\int (3 - x)^2\, dx$ **e** $\int (3x + 1)^4\, dx$ **f** $\int (4x - 3)^7\, dx$

g $\int (5 - 2x)^6\, dx$ **h** $\int (1 - 5x)^7\, dx$ **i** $\int (2x + 9)^{11}\, dx$

j $\int 3(2x - 1)^{10}\, dx$ **k** $\int 4(5x - 4)^6\, dx$ **l** $\int 7(3 - 2x)^3\, dx$

12 By using the rule $\int (ax + b)^n\, dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + C$, find:

a $\int \left(\tfrac{1}{3}x - 7\right)^4 dx$ **b** $\int \left(\tfrac{1}{4}x - 7\right)^6 dx$ **c** $\int \left(1 - \tfrac{1}{5}x\right)^3 dx$

13 By using the rule $\int (ax + b)^n\, dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + C$, find:

a $\int \frac{1}{(x + 1)^3}\, dx$ **b** $\int \frac{1}{(x - 5)^4}\, dx$ **c** $\int \frac{1}{(3x - 4)^2}\, dx$

d $\int \frac{1}{(2 - x)^5}\, dx$ **e** $\int \frac{3}{(x - 7)^6}\, dx$ **f** $\int \frac{8}{(4x + 1)^5}\, dx$

g $\int \frac{2}{(3 - 5x)^4}\, dx$ **h** $\int \frac{4}{5(1 - 4x)^2}\, dx$ **i** $\int \frac{7}{8(3x + 2)^5}\, dx$

14 By expanding the brackets, find:

a $\int \sqrt{x}\left(3\sqrt{x} - x\right) dx$ **b** $\int \left(\sqrt{x} - 2\right)\left(\sqrt{x} + 2\right) dx$ **c** $\int \left(2\sqrt{x} - 1\right)^2 dx$

15 **a** Evaluate these definite integrals.

i $\int_0^1 x^{\frac{1}{2}}\, dx$ **ii** $\int_1^4 x^{-\frac{1}{2}}\, dx$ **iii** $\int_0^8 x^{\frac{1}{3}}\, dx$

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b By writing them with fractional indices, evaluate these definite integrals.

i $\int_0^4 \sqrt{x}\, dx$ **ii** $\int_1^9 x\sqrt{x}\, dx$ **iii** $\int_1^9 \frac{dx}{\sqrt{x}}$

16 Expand the brackets and hence find:

a $\int_2^4 (2 - \sqrt{x})(2 + \sqrt{x})\, dx$ **b** $\int_0^1 \sqrt{x}(\sqrt{x} - 4)\, dx$ **c** $\int_4^9 (\sqrt{x} - 1)^2\, dx$

CHALLENGE

17 Explain why the indefinite integral $\int \frac{1}{x}\, dx$ cannot be found in the usual way using the standard form $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$.

18 Find each indefinite integral.

a $\int \sqrt{2x - 1}\, dx$ **b** $\int \sqrt{7 - 4x}\, dx$

c $\int \sqrt[3]{4x - 1}\, dx$ **d** $\int \frac{1}{\sqrt{3x + 5}}\, dx$

19 Evaluate these definite integrals

a $\int_0^2 (x + 1)^4\, dx$ **b** $\int_2^3 (2x - 5)^3\, dx$ **c** $\int_{-2}^2 (1 - x)^5\, dx$

d $\int_0^5 \left(1 - \frac{x}{5}\right)^4 dx$ **e** $\int_0^1 \sqrt{9 - 8x}\, dx$ **f** $\int_2^7 \frac{1}{\sqrt{x + 2}}\, dx$

g $\int_{-2}^0 \sqrt[3]{x + 1}\, dx$ **h** $\int_1^5 \sqrt{3x + 1}\, dx$ **i** $\int_{-3}^0 \sqrt{1 - 5x}\, dx$

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4F Finding areas by integration

The aim of this section and the next is to use definite integrals to find the areas of regions bounded by curves, lines and the coordinate axes.

Sections 4F–4G ignore integrals that run backwards. Running an integral backwards reverses its sign, which would confuse the discussion of areas in these sections. When finding areas, we decide what integrals to create, and we naturally avoid integrals that run backwards.

Areas and definite integrals

Areas and definite integrals are closely related, but they are not the same thing.

- An area is always positive, whereas a definite integral may be positive or negative, depending on whether the curve is above or below the x-axis.

Problems on areas require care when finding the required integral or combination of integrals. Some particular techniques are listed below, but the general rule is to draw a diagram first to see which pieces need to be added or subtracted.

15 FINDING AN AREA

When using integrals to find the area of a region:

1. Draw a sketch of the curves, showing relevant intercepts and intersections.
2. Create and evaluate the necessary definite integral or integrals.
3. Write a conclusion, giving the required area in square units.

Regions above the x-axis

When a curve lies entirely above the x-axis, the relevant integral will be positive, and the area will be equal to the integral, apart from needing units.

Example 24 **4F**

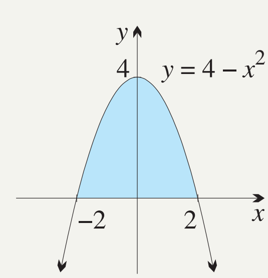
Find the area of the region bounded by the curve $y = 4 - x^2$ and the x-axis. (This was the example sketched in the introduction to this chapter.)

SOLUTION

The curve meets the x-axis at $(2, 0)$ and $(-2, 0)$.

The region lies entirely above the x-axis and the relevant integral is

$$\begin{aligned}\int_{-2}^{2}(4 - x^2)\,dx &= \left[4x - \frac{x^3}{3}\right]_{-2}^{2}\\ &= \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right)\\ &= 5\tfrac{1}{3} - \left(-5\tfrac{1}{3}\right)\\ &= 10\tfrac{2}{3},\end{aligned}$$



which is positive because the region lies entirely above the x-axis.

Hence the required area is $10\frac{2}{3}$ square units.

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Regions below the x-axis

When a curve lies entirely below the x-axis, the relevant integral will be negative, and the area will be the opposite of this.

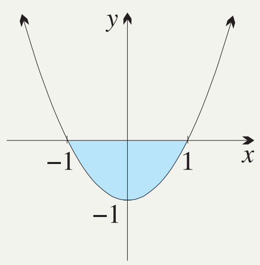
Example 25 4F

Find the area of the region bounded by the curve $y = x^2 - 1$ and the x-axis.

SOLUTION

The curve meets the x-axis at $(1, 0)$ and $(-1, 0)$.

The region lies entirely below the x-axis and the relevant integral is



$$\begin{aligned}\int_{-1}^{1} (x^2 - 1)\,dx &= \left[\frac{x^3}{3} - x\right]_{-1}^{1} \\ &= \left(\tfrac{1}{3} - 1\right) - \left(-\tfrac{1}{3} + 1\right) \\ &= -\tfrac{2}{3} - \tfrac{2}{3} \\ &= -1\tfrac{1}{3},\end{aligned}$$

which is negative, because the region lies entirely below the x-axis.

Hence the required area is $1\frac{1}{3}$ square units.

Curves that cross the x-axis

When a curve crosses the x-axis, the area of the region between the curve and the x-axis cannot usually be found by means of a single integral. This is because integrals representing regions below the x-axis have negative values.

Example 26 4F

a Sketch the cubic curve $y = x(x + 1)(x - 2)$, showing the x-intercepts.

b Shade the region enclosed between the x-axis and the curve, and find its area.

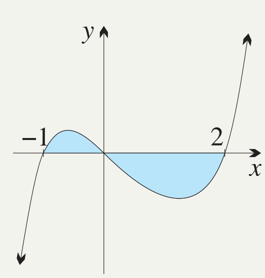
c Find $\displaystyle\int_{-1}^{2} x(x + 1)(x - 2)\,dx$ and explain why this integral does not represent the area of the region described in part **b**.

SOLUTION

a The curve has x-intercepts $x = -1$, $x = 0$ and $x = 2$, and is graphed below.

b Expanding the cubic,
$$\begin{aligned}y &= x(x + 1)(x - 2) \\ &= x(x^2 - x - 2) \\ &= x^3 - x^2 - 2x.\end{aligned}$$

For the region above the x-axis,



$$\begin{aligned}\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} \\ &= (0 - 0 - 0) - \left(\tfrac{1}{4} + \tfrac{1}{3} - 1\right) \\ &= \tfrac{5}{12},\end{aligned}$$

so area above $= \frac{5}{12}$ square units.

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For the region below the x-axis,

$$\begin{aligned}\int_0^2 (x^3 - x^2 - 2x)\,dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_0^2 \\ &= \left(4 - 2\tfrac{2}{3} - 4\right) - (0 - 0 - 0) \\ &= -2\tfrac{2}{3},\end{aligned}$$

so area below $= 2\frac{2}{3}$ square units.

Adding these, total area $= \frac{5}{12} + 2\frac{2}{3}$

$= 3\frac{1}{12}$ square units.

c $$\begin{aligned}\int_{-1}^2 x(x+1)(x-2)\,dx &= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^2 \\ &= \left(4 - 2\tfrac{2}{3} - 4\right) - \left(\tfrac{1}{4} + \tfrac{1}{3} - 1\right) \\ &= -2\tfrac{2}{3} + \tfrac{5}{12} \\ &= -2\tfrac{1}{4}.\end{aligned}$$

This integral represents the difference $2\frac{2}{3} - \frac{5}{12} = 2\frac{1}{4}$ of the two areas, and is negative because the area below is larger than the area above.

Areas associated with odd and even functions

As always in mathematics, these calculations are often much easier if symmetries can be recognised.

Example 27 4F

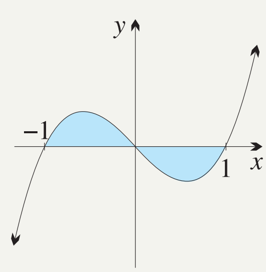
a Show that $y = x^3 - x$ is an odd function.

b Using part **a**, find the area between the curve $y = x^3 - x$ and the x-axis.

SOLUTION

a Let $f(x) = x^3 - x.$

Then $$\begin{aligned} f(-x) &= (-x)^3 - (-x) \\ &= -x^3 + x \\ &= -f(x),\text{ so } f(x) \text{ is odd.}\end{aligned}$$



b Factoring, $$\begin{aligned} y &= x(x^2 - 1) \\ &= x(x-1)(x+1),\end{aligned}$$

so the x-intercepts are $x = -1$, $x = 0$ and $x = 1$.

The two shaded regions have equal areas because the function is odd.

First, $$\begin{aligned}\int_0^1 (x^3 - x)\,dx &= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 \\ &= \left(\tfrac{1}{4} - \tfrac{1}{2}\right) - (0 - 0) \\ &= -\tfrac{1}{4},\end{aligned}$$

so area below the x-axis $= \frac{1}{4}$ square units.

Doubling, total area $= \frac{1}{2}$ square units.

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Area between a graph and the y-axis

Integration with respect to y rather than x can often give a result more quickly without the need for subtraction. When x is a function of y:

- A definite integral with respect to y represents the signed area of the region between the curve and the y-axis.
- This means that the definite integral is the sum of areas of regions to the right of the y-axis, minus the sum of areas of regions to the left of the y-axis.
- The limits of integration are values of y rather than of x.

16 THE DEFINITE INTEGRAL AND INTEGRATION WITH RESPECT TO y

Let x be a continuous function of y in some closed interval $a \le y \le b$.

Then the definite integral $\int_a^b x\,dy$ is the sum of the areas of regions to the right of the y-axis, from $y = a$ to $y = b$, minus the sum of the areas of regions to the left of the y-axis.

Example 28 4F

a Sketch the lines $y = x + 1$ and $y = 5$, and shade the region between these lines to the right of the y-axis.

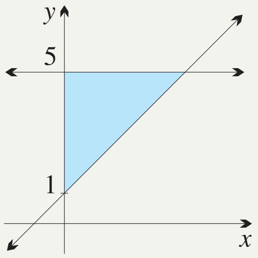
b Write the equation of the line so that x is a function of y.

c Use integration with respect to y to find the area of this region.

d Confirm the result by area formulae.

SOLUTION

a The lines are sketched below. They meet at $(4, 5)$.



b The given equation is $y = x + 1$.
Solving for x, $x = y - 1$.

c The required integral is

$$\begin{aligned}\int_1^5 (y - 1)\,dy &= \left[\frac{y^2}{2} - y\right]_1^5 \\ &= \left(\frac{25}{2} - 5\right) - \left(\frac{1}{2} - 1\right) \\ &= 7\tfrac{1}{2} - \left(-\tfrac{1}{2}\right) \\ &= 8,\end{aligned}$$

which is positive, because the region is to the right of the y-axis.
Hence the required area is $8\,\text{u}^2$.

d $$\begin{aligned}\text{Area of triangle} &= \tfrac{1}{2} \times \text{base} \times \text{height} \\ &= \tfrac{1}{2} \times 4 \times 4 \\ &= 8\,\text{u}^2.\end{aligned}$$

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Example 29 4F

The curve in the diagram below is the cubic $y = x^3$.

a Write the equation of the cubic so that x is a function of y.

b Use integration with respect to y to find the areas of the shaded regions to the right and left of the y-axis.

c Find the total area of the two shaded regions.

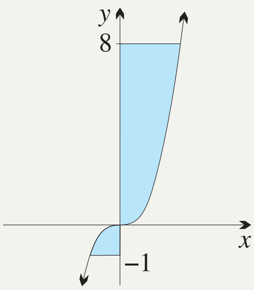
SOLUTION

a The given equation is $y = x^3$.

Solving for x, $x^3 = y$

$$x = y^{\frac{1}{3}}.$$

b For the region to the right of the y-axis,



$$\begin{aligned}\int_0^8 y^{\frac{1}{3}}\,dy &= \tfrac{3}{4}\Big[y^{\frac{4}{3}}\Big]_0^8\\ &= \tfrac{3}{4} \times (16 - 0) \qquad \text{(because } 8^{\frac{4}{3}} = 2^4 = 16)\\ &= 12,\end{aligned}$$

so area $= 12$ square units.

For the region to the left of the y-axis,

$$\begin{aligned}\int_{-1}^0 y^{\frac{1}{3}}\,dy &= \tfrac{3}{4}\Big[y^{\frac{4}{3}}\Big]_{-1}^0\\ &= \tfrac{3}{4} \times (0 - 1) \qquad \text{(because } (-1)^{\frac{4}{3}} = (-1)^4 = 1)\\ &= -\tfrac{3}{4},\end{aligned}$$

so area $= \frac{3}{4}$ square units.

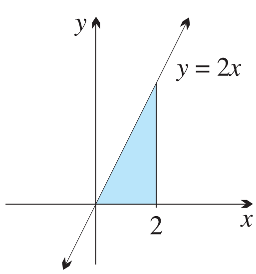
c Adding these, total area $= 12\frac{3}{4}$ square units.

Exercise 4F FOUNDATION

Technology: Graphing software will help in identifying the definite integrals that need to be evaluated to find the area of a given region.

1 Find the area of each shaded region below by evaluating the appropriate integral.

a



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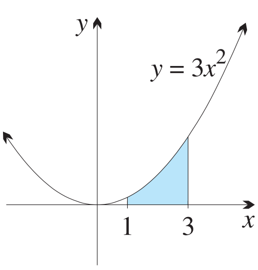

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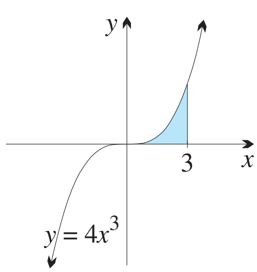

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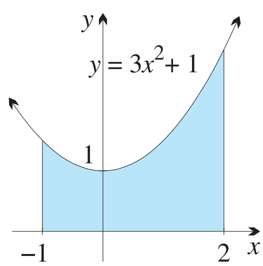

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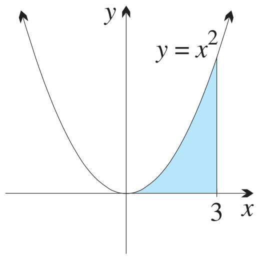

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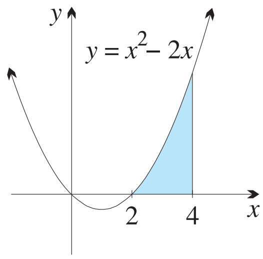

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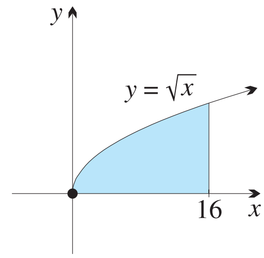

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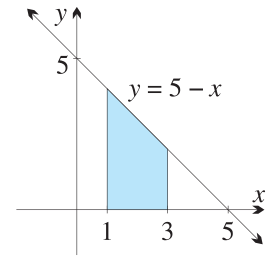

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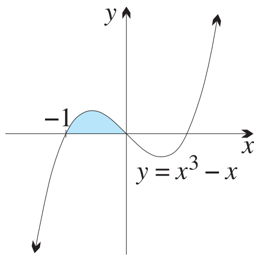

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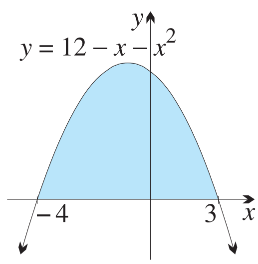

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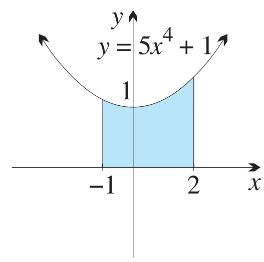

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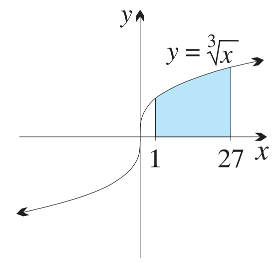
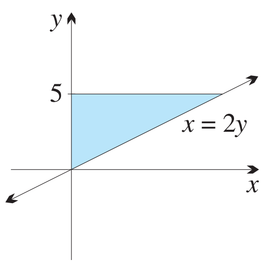
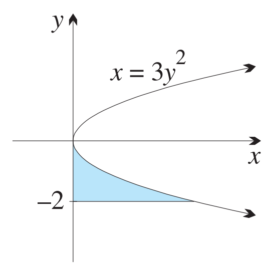


Note: In Questions 2–4, you will be finding areas between a curve and the y-axis. In each such case, the equation of the curve has already been given with x as the subject.

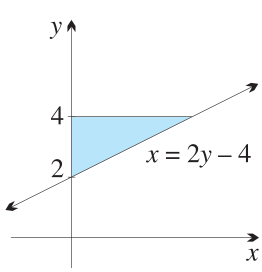
2 **a**


Explain why the shaded area is given by the integral $\int_0^5 2y\, dy$. Then find the area.

b


Explain why the shaded area is given by the integral $\int_{-2}^0 3y^2\, dy$. Then find the area.

3 Find the area of each shaded region below by evaluating the appropriate integral. The equation of the curve has already been given with x as the subject.

a


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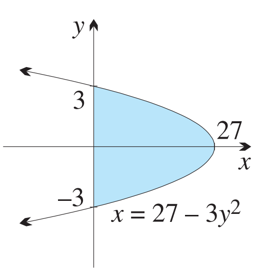

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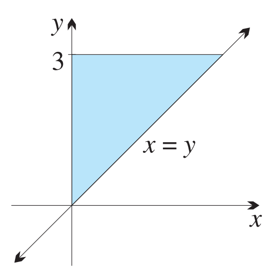

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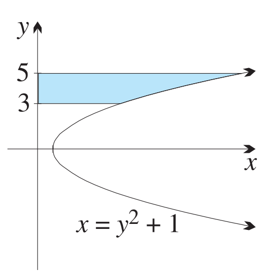
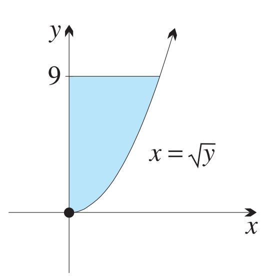


e


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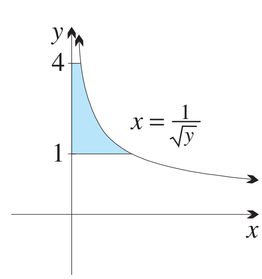

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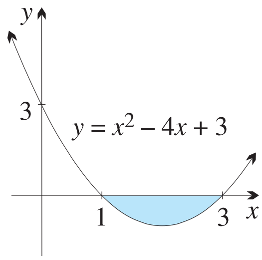
4 Find the area of each shaded region below by evaluating the appropriate integral.

a


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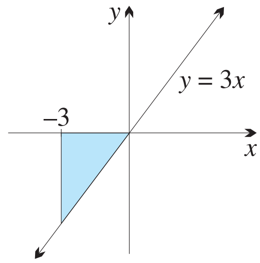

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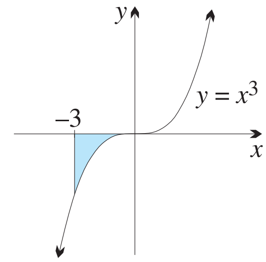

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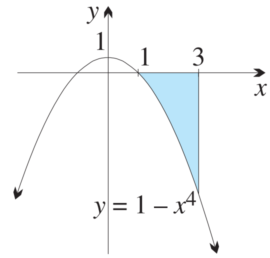
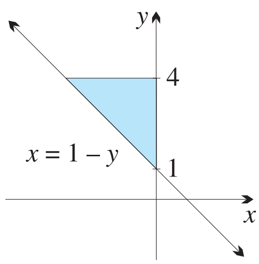


5 Find the area of each shaded region below by evaluating the appropriate integral.

a


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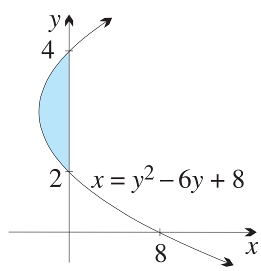

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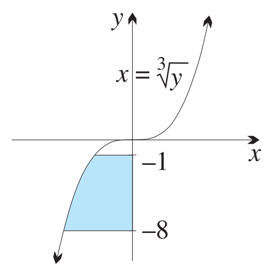

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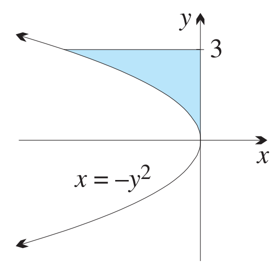
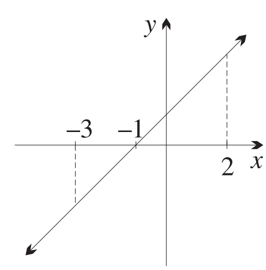


DEVELOPMENT

6 The line $y = x + 1$ is graphed on the right.



a Copy the diagram, and shade the region between the line $y = x + 1$ and the x-axis from $x = -3$ to $x = 2$.

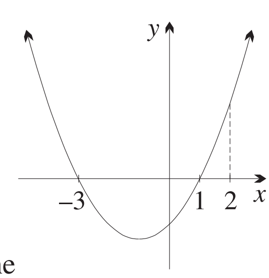
b By evaluating $\int_{-1}^{2} (x + 1)\, dx$, find the area of the shaded region above the x-axis.

c By evaluating $\int_{-3}^{-1} (x + 1)\, dx$, find the area of the shaded region below the x-axis.

d Hence find the area of the entire shaded region.

e Find $\int_{-3}^{2} (x + 1)\, dx$, and explain why this integral does not give the area of the shaded region.

7 The curve $y = (x - 1)(x + 3) = x^2 + 2x - 3$ is graphed.



a Copy the diagram, and shade the region between the curve $y = (x - 1)(x + 3)$ and the x-axis from $x = -3$ to $x = 2$.

b By evaluating $\int_{-3}^{1} (x^2 + 2x - 3)\, dx$, find the area of the shaded region below the x-axis.

c By evaluating $\int_{1}^{2} (x^2 + 2x - 3)\, dx$, find the area of the shaded region above the x-axis.

d Hence find the area of the entire shaded region.

e Find $\int_{-3}^{2} (x^2 + 2x - 3)\, dx$, and explain why this integral does not give the area of the shaded region.

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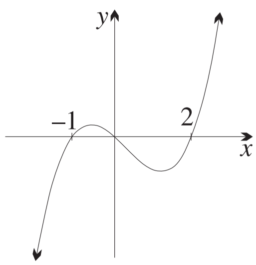
8 The curve $y = x(x + 1)(x - 2) = x^3 - x^2 - 2x$ is graphed.



a Copy the diagram, and shade the region bounded by the curve and the x-axis.

b By evaluating $\int_0^2 (x^3 - x^2 - 2x)\,dx$, find the area of the shaded region below the x-axis.

c By evaluating $\int_{-1}^0 (x^3 - x^2 - 2x)\,dx$, find the area of the shaded region above the x-axis.

d Hence find the area of the entire region you have shaded.

e Find $\int_{-1}^2 (x^3 - x^2 - 2x)\,dx$, and explain why this integral does not give the area of the shaded region.

9 In each part below, find the area of the region bounded by the graph of the given function and the x-axis between the specified values. Remember that areas above and below the x-axis must be calculated separately.

a $y = x^2$, between $x = -3$ and $x = 2$

b $y = 2x^3$, between $x = -4$ and $x = 1$

c $y = 3x(x - 2)$, between $x = 0$ and $x = 2$

d $y = x - 3$, between $x = -1$ and $x = 4$

e $y = (x - 1)(x + 3)(x - 2)$, between $x = -3$ and $x = 2$

f $y = -2x(x + 1)$, between $x = -2$ and $x = 2$

10 In each part below, find the area of the region bounded by the graph of the given function and the y-axis between the specified values. Remember that areas to the right and to the left of the y-axis must be calculated separately.

a $x = y - 5$, between $y = 0$ and $y = 6$

b $x = 3 - y$, between $y = 2$ and $y = 5$

c $x = y^2$, between $y = -1$ and $y = 3$

d $x = (y - 1)(y + 1)$, between $y = 3$ and $y = 0$

11 In each part below you should sketch the curve and look carefully for any symmetries that will simplify the calculation.

a Find the area of the region bounded by the given curve and the x-axis.

i $y = x^7$, for $-2 \le x \le 2$

ii $y = x^3 - 16x = x(x - 4)(x + 4)$, for $-4 \le x \le 4$

iii $y = x^4 - 9x^2 = x^2(x - 3)(x + 3)$, for $-3 \le x \le 3$

b Find the area of the region bounded by the given curve and the y-axis.

i $x = 2y$, for $-5 \le y \le 5$

ii $x = y^2$, for $-3 \le y \le 3$

iii $x = 4 - y^2 = (2 - y)(2 + y)$, for $-2 \le y \le 2$

12 Find the area of the region bounded by $y = |x + 2|$ and the x-axis, for $-6 \le x \le 2$.

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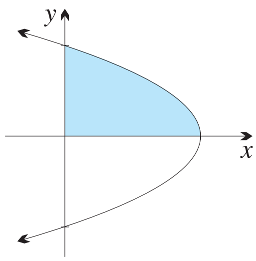
CHALLENGE

13 The diagram shows the parabola $y^2 = 16(2 - x)$.



a Find the x-intercept and the y-intercepts.

b Find the exact area of the shaded region:

i by integrating $y = 4\sqrt{2 - x}$ with respect to x,

ii by integrating with respect to y. (You will need to make x the subject of the equation.)

14 The gradient of a curve is $y' = x^2 - 4x + 3$, and the curve passes through the origin.

a Find the equation of the curve.

b Show that the curve has turning points at $\left(1, 1\frac{1}{3}\right)$ and $(3, 0)$, and sketch its graph.

c Find the area of the region bounded by the curve and the x-axis between the two turning points.

15 Sketch $y = x^2$ and mark the points $A(a, a^2)$, $B(-a, a^2)$, $P(a, 0)$ and $Q(-a, 0)$.

a Show that $\int_0^a x^2\,dx = \frac{2}{3}$ (area $\triangle OAP$).

b Show that $\int_{-a}^a x^2\,dx = \frac{1}{3}$ (area of rectangle $ABQP$).

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4G Areas of compound regions

When a region is bounded by two or more different curves, some dissection process is usually needed before integrals can be used to calculate its area.

Thus a preliminary sketch of the region becomes all the more important.

Areas of regions under a combination of curves

Some regions are bounded by different curves in different parts of the x-axis.

Example 30 4G

a Sketch the curves $y = x^2$ and $y = (x - 2)^2$ on one set of axes.

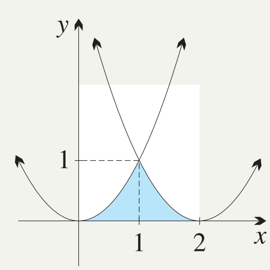
b Shade the region bounded by $y = x^2$, $y = (x - 2)^2$ and the x-axis.

c Find the area of this shaded region.

SOLUTION

a The two curves intersect at (1, 1), because it is easily checked by substitution that this point lies on both curves.

b The whole region is above the x-axis, but it will be necessary to find separately the areas of the regions to the left and right of $x = 1$.



c First,

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \tfrac{1}{3}.$$

Secondly,

$$\int_1^2 (x - 2)^2\,dx = \left[\frac{(x - 2)^3}{3}\right]_1^2 = 0 - \left(-\tfrac{1}{3}\right) = \tfrac{1}{3}.$$

Combining these,

$$\text{area} = \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{2}{3} \text{ square units.}$$

Note: In this worked example, the second parabola is the first shifted right 2, and a parabola is symmetric about its axis of symmetry. This is why the two pieces have the same area.

Areas of regions between curves

Suppose that one curve $y = f(x)$ is always below another curve $y = g(x)$ in an interval $a \le x \le b$. Then the area of the region between the curves from $x = a$ to $x = b$ can be found by subtraction.

17 AREA BETWEEN CURVES

If $f(x) \le g(x)$ in the interval $a \le x \le b$, then

$$\text{area between the curves} = \int_a^b \big(g(x) - f(x)\big)\,dx.$$

That is, take the integral of the top curve minus the bottom curve.

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The assumption that $f(x) \leq g(x)$ is important. If the curves cross each other, then separate integrals will need to be taken or else the areas of regions where different curves are on top will begin to cancel each other out.

Example 31 4G

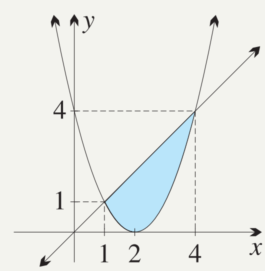
a Find the two points where the curve $y = (x - 2)^2$ meets the line $y = x$.
b Draw a sketch and shade the area of the region between these two graphs.
c Find the shaded area.

SOLUTION

a Substituting $y = x$ into $y = (x - 2)^2$ gives

$$\begin{aligned} (x-2)^2 &= x \\ x^2 - 4x + 4 &= x \\ x^2 - 5x + 4 &= 0 \\ (x-1)(x-4) &= 0, \\ x &= 1 \text{ or } 4, \end{aligned}$$

so the two graphs intersect at $(1, 1)$ and $(4, 4)$.



b The sketch is drawn to the right.

c In the shaded region, the line is above the parabola.

$$\begin{aligned} \text{Hence area} &= \int_1^4 \left(x - (x-2)^2\right) dx \\ &= \int_1^4 \left(x - (x^2 - 4x + 4)\right) dx \\ &= \int_1^4 (-x^2 + 5x - 4)\, dx \\ &= \left[-\frac{x^3}{3} + \frac{5x^2}{2} - 4x\right]_1^4 \\ &= \left(-21\tfrac{1}{3} + 40 - 16\right) - \left(-\tfrac{1}{3} + 2\tfrac{1}{2} - 4\right) \\ &= 2\tfrac{2}{3} + 1\tfrac{5}{6} \\ &= 4\tfrac{1}{2} \text{ square units.} \end{aligned}$$

Note: The formula given in Box 17 on the previous page for the area of the region between two curves holds even if the region crosses the x-axis.

To illustrate this point, the next example is the previous example shifted down 2 units so that the region between the line and the parabola crosses the x-axis. The area of course remains the same — and notice how the formula still gives the correct answer.

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Example 32 — 4G

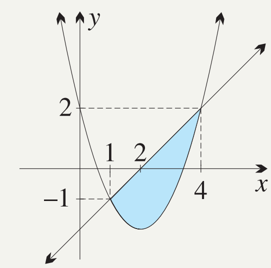
a Find the two points where the curves $y = x^2 - 4x + 2$ and $y = x - 2$ meet.
b Draw a sketch and find the area of the region between these two curves.

SOLUTION

a Substituting $y = x - 2$ into $y = x^2 - 4x + 2$ gives

$$x^2 - 4x + 2 = x - 2$$
$$x^2 - 5x + 4 = 0$$
$$(x - 1)(x - 4) = 0$$
$$x = 1 \text{ or } 4,$$

so the two graphs intersect at $(1, -1)$ and $(4, 2)$.



b Again, the line is above the parabola,

$$\text{so area} = \int_1^4 \Big((x - 2) - (x^2 - 4x + 2)\Big)\,dx$$
$$= \int_1^4 (-x^2 + 5x - 4)\,dx$$
$$= \left[-\frac{x^3}{3} + \frac{5x^2}{2} - 4x\right]_1^4$$
$$= \left(-21\tfrac{1}{3} + 40 - 16\right) - \left(-\tfrac{1}{3} + 2\tfrac{1}{2} - 4\right)$$
$$= 4\tfrac{1}{2} \text{ square units.}$$

Areas of regions between curves that cross

Now suppose that one curve $y = f(x)$ is sometimes above and sometimes below another curve $y = g(x)$ in the relevant interval. In this case, separate integrals will need to be calculated.

Example 33 — 4G

The diagram below shows the curves

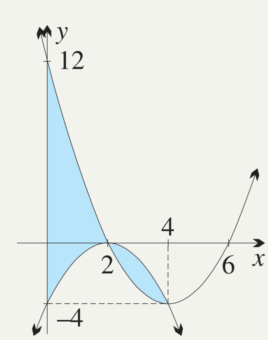
$$y = -x^2 + 4x - 4 \qquad \text{and} \qquad y = x^2 - 8x + 12$$

meeting at the points $(2, 0)$ and $(4, -4)$. Find the area of the shaded region.

SOLUTION

In the left-hand region, the second curve is above the first.

$$\text{Hence} \qquad \text{area} = \int_0^2 \Big((x^2 - 8x + 12) - (-x^2 + 4x - 4)\Big)\,dx$$
$$= \int_0^2 (2x^2 - 12x + 16)\,dx$$
$$= \left[\frac{2x^3}{3} - 6x^2 + 16x\right]_0^2$$
$$= 5\tfrac{1}{3} - 24 + 32$$
$$= 13\tfrac{1}{3}\,\text{u}^2.$$



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In the right-hand region, the first curve is above the second,

$$\text{so} \qquad \text{area} = \int_2^4 \Big((-x^2 + 4x - 4) - (x^2 - 8x + 12) \Big)\, dx$$
$$= \int_2^4 (-2x^2 + 12x - 16)\, dx$$
$$= \Big[-\frac{2x^3}{3} + 6x^2 - 16x \Big]_2^4$$
$$= \Big(-42\tfrac{2}{3} + 96 - 64\Big) - \Big(-5\tfrac{1}{3} + 24 - 32\Big)$$
$$= -10\tfrac{2}{3} + 13\tfrac{1}{3}$$
$$= 2\tfrac{2}{3}\,\text{u}^2.$$

$$\text{Hence total area} = 13\tfrac{1}{3} + 2\tfrac{2}{3}$$
$$= 16\,\text{u}^2.$$

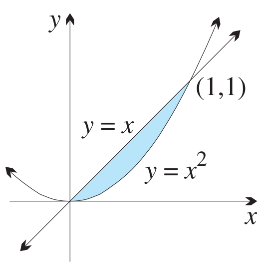
Exercise 4G

FOUNDATION

Technology: Graphing programs are particularly useful with compound regions because they allow the separate parts of the region to be identified clearly.

1 Find the area of the shaded region in each diagram below.

a



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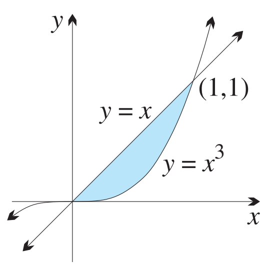
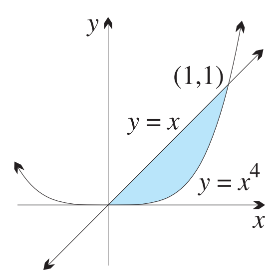


c



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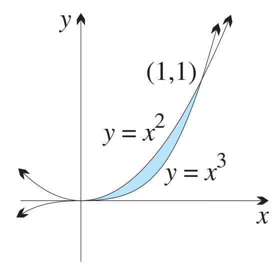
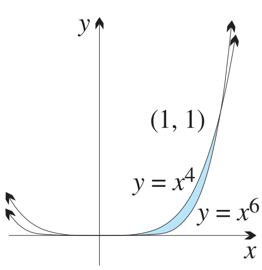


e



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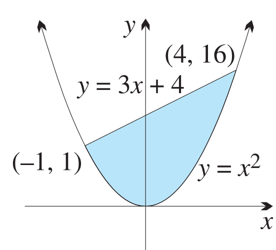
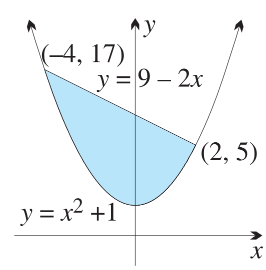
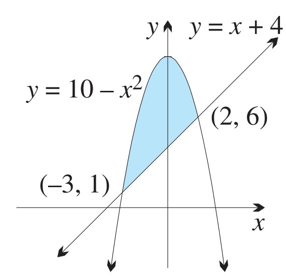


g

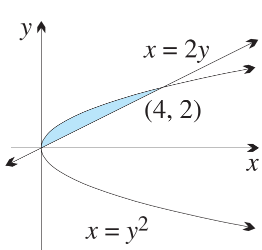


h

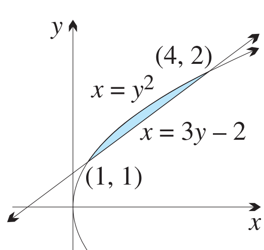


2 By considering regions between the curves and the y-axis, find the area of the shaded region in each diagram below. Notice that the equation of each curve has already been solved for x.

a



b



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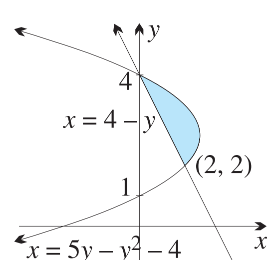

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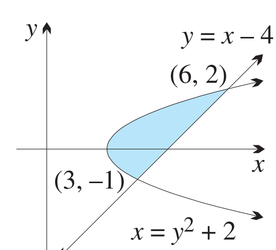

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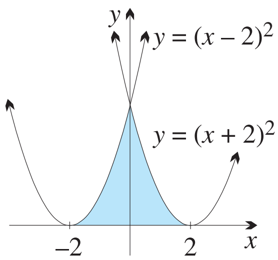
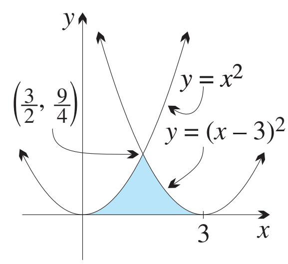
3 Find the areas of the shaded regions in the diagrams below. In each case you will need to find two areas and add them.

a

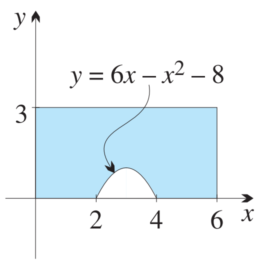


b

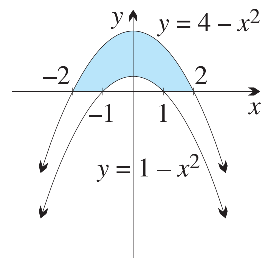


4 Find the areas of the shaded regions in the diagrams below. In each case you will need to find two areas and subtract one from the other.

a



b



DEVELOPMENT

5 **a** By solving the equations simultaneously, show that the parabola $y = x^2 + 4$ and the line $y = x + 6$ intersect at the points $(-1, 5)$ and $(2, 8)$.

b Sketch the parabola and the line on the same diagram, and shade the region enclosed between them.

c Show that this region has area

$$\int_{-1}^{2} \big((x + 6) - (x^2 + 4)\big)\, dx = \int_{-1}^{2} (x - x^2 + 2)\, dx$$

and evaluate the integral.

6 **a** By solving the equations simultaneously, show that the parabola $y = 3x - x^2 = x(3 - x)$ and the line $y = x$ intersect at the points $(0, 0)$ and $(2, 2)$.

b Sketch the parabola and the line on the same diagram, and shade the region enclosed between them.

c Show that this region has area

$$\int_{0}^{2} (3x - x^2 - x)\, dx = \int_{0}^{2} (2x - x^2)\, dx$$

and evaluate the integral.

7 **a** By solving the equations simultaneously, show that the parabola $y = (x - 3)^2$ and the line $y = 14 - 2x$ intersect at the points $(-1, 16)$ and $(5, 4)$.

b Sketch the parabola and the line on the same diagram, and shade the region enclosed between them.

c Show that this region has area

$$\int_{-1}^{5} \big((14 - 2x) - (x - 3)^2\big)\, dx = \int_{-1}^{5} (4x + 5 - x^2)\, dx,$$

and evaluate the integral.

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8 Solve simultaneously to find the points of intersection of each pair of graphs. Then sketch the graphs on the same diagram, and shade the region enclosed between them. By evaluating the appropriate definite integral, find the area of the shaded region in each case.

a $y = x + 3$ and $y = x^2 + 1$

b $y = 9 - x^2$ and $y = 3 - x$

c $y = x^2 - x + 4$ and $y = -x^2 + 3x + 4$

9 **a** By solving the equations simultaneously, show that the parabola $y = x^2 + 2x - 8$ and the line $y = 2x + 1$ intersect at the points $(3, 7)$ and $(-3, -5)$.

b Sketch both graphs on the same diagram, and shade the region enclosed between them.

c Despite the fact that it crosses the x-axis, the region has area given by

$$\int_{-3}^{3} \big((2x + 1) - (x^2 + 2x - 8)\big)\, dx = \int_{-3}^{3} (9 - x^2)\, dx.$$

Evaluate the integral and hence find the area of the region enclosed between the curves.

10 **a** By solving the equations simultaneously, show that the parabola $y = x^2 - x - 2$ and the line $y = x - 2$ intersect at the points $(0, -2)$ and $(2, 0)$.

b Sketch both graphs on the same diagram, and shade the region enclosed between them.

c Despite the fact that it is below the x-axis, the region has area given by

$$\int_{0}^{2} \big((x - 2) - (x^2 - x - 2)\big)\, dx = \int_{0}^{2} (2x - x^2)\, dx.$$

Evaluate this integral and hence find the area of the region between the curves.

11 Solve simultaneously to find the points of intersection of each pair of graphs. Then sketch the graphs on the same diagram, and shade the region enclosed between them. By evaluating the appropriate definite integral, find the area of the shaded region in each case.

a $y = x^2 - 6x + 5$ and $y = x - 5$

b $y = -3x$ and $y = 4 - x^2$

c $y = x^2 - 1$ and $y = 7 - x^2$

12 **a** On the same number plane, sketch the graphs of the parabolas $y = x^2$ and $x = y^2$, clearly indicating their points of intersection. Shade the region enclosed between them.

b Explain why the area of this region is given by $\int_{0}^{1} \left(\sqrt{x} - x^2\right) dx$.

c Find the area of the region bounded by the two curves.

CHALLENGE

13 Tangents are drawn to the parabola $x^2 = 8y$ at the points $A(4, 2)$ and $B(-4, 2)$.

a Draw a diagram of the situation and note the symmetry about the y-axis.

b Find the equation of the tangent at the point A.

c Find the area of the region bounded by the curve and the tangents.

14 **a** Show that the tangent to the curve $y = x^3$ at the point where $x = 2$ has equation $y = 12x - 16$.

b Show by substitution that the tangent and the curve intersect again at the point $(-4, -64)$.

c Find the area of the region enclosed between the curve and the tangent.

15 Consider the curves $y = x^3 - 3$ and $y = -x^2 + 10x - 11$.

a Show by substitution that the curves intersect at three points whose x-values are -4, 1 and 2.

b Sketch the curves showing clearly their intersection points.

c Find the area of the region enclosed by the two curves.

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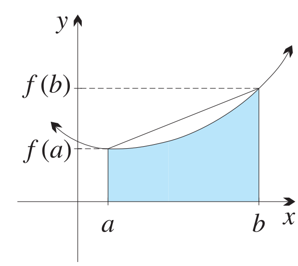
4H The trapezoidal rule

Methods of approximating definite integrals become necessary when exact calculations using primitives are not possible. This can happen for two reasons.

- The primitives of many important functions cannot be written down in a formula suitable for calculation — this is the case for the important normal distribution in Chapter 10.
- Some values of a function may be known only from experiments, and the function formula may be unknown.

The trapezoidal rule

Besides taking upper and lower rectangles, the most obvious way to approximate an integral is to replace the curve by a straight line, that is, by a chord joining $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$. The resulting region is then a trapezium, so this approximation method is called the *trapezoidal rule*.



Consider the trapezium in the diagram to the right.

$$\text{Here} \qquad \text{width} = b - a,$$

$$\text{and average of parallel sides} = \frac{f(a) + f(b)}{2}.$$

Hence area of trapezium = width × average of parallel sides

$$= \frac{b-a}{2}\big(f(a) + f(b)\big).$$

The area of this trapezium is taken as an approximation of the integral.

18 THE TRAPEZOIDAL RULE USING ONE SUBINTERVAL

Let $f(x)$ be a function that is continuous in the closed interval $[a, b]$.

- Approximating the curve from $x = 1$ to $x = b$ by a chord allows the region under the curve to be approximated by a trapezium, giving

$$\int_a^b f(x)\,dx \doteqdot \frac{b-a}{2}\big(f(a) + f(b)\big).$$

- If the function is linear, then the chord coincides with the curve and the formula is exact.
- Always start a trapezoidal-rule calculation by constructing a table of values.

Subdividing the interval

Given an integral over an interval $[a, b]$, we can split that interval $[a, b]$ up into a number of subintervals and apply the trapezoidal rule to each subinterval in turn. This will usually improve the accuracy of the approximation.

Here is the method applied to the reciprocal function $y = \dfrac{1}{x}$, whose primitive we will only establish in Chapter 5.

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Example 34 4H

Find approximations of $\int_1^5 \frac{1}{x}\,dx$ using the trapezoidal rule with:

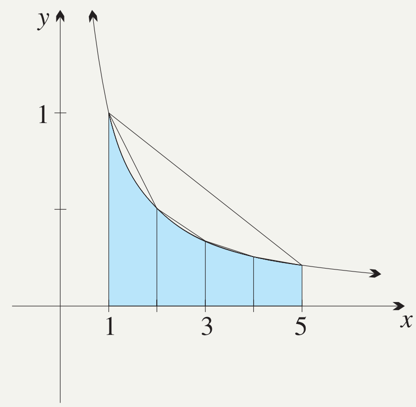
a one subinterval,

b four subintervals.

SOLUTION

Always begin with a table of values of the function.

x	1	2	3	4	5
$\frac{1}{x}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$



a One application of the trapezoidal rule, using the whole interval as the subinterval, requires just two values of the function.

$$\begin{aligned}\int_1^5 \frac{1}{x}\,dx &\doteqdot \frac{5-1}{2} \times \big(f(1) + f(5)\big) \\ &\doteqdot 2 \times \left(1 + \tfrac{1}{5}\right) \\ &\doteqdot 2\tfrac{2}{5}\end{aligned}$$

b Four applications of the trapezoidal rule require five values of the function. Dividing the interval $1 \le x \le 5$ into four equal subintervals,

$$\int_1^5 \frac{1}{x}\,dx = \int_1^2 \frac{1}{x}\,dx + \int_2^3 \frac{1}{x}\,dx + \int_3^4 \frac{1}{x}\,dx + \int_4^5 \frac{1}{x}\,dx$$

Each subinterval has width 1, so applying the trapezoidal rule to each integral,

$$\begin{aligned}\int_1^5 \frac{1}{x}\,dx &\doteqdot \tfrac{1}{2}\big(f(1) + f(2)\big) + \tfrac{1}{2}\big(f(2) + f(3)\big) + \tfrac{1}{2}\big(f(3) + f(4)\big) + \tfrac{1}{2}\big(f(4) + f(5)\big) \\ &\doteqdot \tfrac{1}{2}\left(\tfrac{1}{1} + \tfrac{1}{2}\right) + \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{3}\right) + \tfrac{1}{2}\left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \tfrac{1}{2}\left(\tfrac{1}{4} + \tfrac{1}{5}\right) \\ &\doteqdot 1\tfrac{41}{60}.\end{aligned}$$

Note: Always take subintervals of equal width unless otherwise indicated.

Concavity and the trapezoidal rule

The curve in the example above is concave up, so every chord is above the curve, and every approximation found using the trapezoidal rule is therefore greater than the integral.

Similarly, if a curve is concave down, then every chord is below the curve, and every trapezoidal-rule approximation is less than the integral. The second derivative can be used to test concavity.

19 CONCAVITY AND THE TRAPEZOIDAL RULE

- If the curve is concave up, the trapezoidal rule overestimates the integral.
- If the curve is concave down, the trapezoidal rule underestimates the integral.
- If the curve is linear, the trapezoidal rule gives the exact value of the integral.

The second derivative $\dfrac{d^2y}{dx^2}$ can be used to test the concavity.

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Example 35 4H

a Use the trapezoidal rule with one subinterval (that is, two function values) to approximate $\displaystyle\int_1^5 (200x - x^4)\,dx$.

b Use the second derivative to explain why the approximation underestimates the integral.

SOLUTION

a Construct a table of values for $y = 200x - x^4$.

x	1	5
y	199	375

$$\begin{aligned}\int_1^5 (200x - x^4)\,dx &\doteqdot \frac{5-1}{2} \times \big(f(1) + f(5)\big)\\ &\doteqdot 2 \times (199 + 375)\\ &\doteqdot 1148\end{aligned}$$

b The function is $y = 200x - x^4$.

Differentiating, $y' = 200 - 4x^3$

and $y'' = -12x^2$.

Because $y'' = -12x^2$ is negative throughout the interval $1 \le x \le 5$, the curve is concave down throughout this interval.

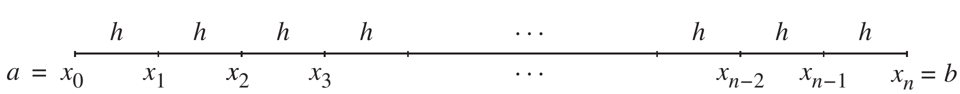
Hence the trapezoidal rule underestimates the integral.

A formula for multiple applications of the trapezoidal rule

When the trapezoidal rule is being applied two or three times, it is easier to perform the two or three calculations required. These separate calculations also reinforce the meaning of the approximation, and help to gain an intuitive understanding of the accuracy of the estimates.

But increasing accuracy with the trapezoidal rule requires larger numbers of applications of the rule, and this can quickly become tedious. Let us then develop a single formula that splits an integral into n subintervals of equal width and applies the trapezoidal rule to each — anyone writing a program or using a spreadsheet to estimate integrals would want to do this.

The first step is to divide the interval $[a, b]$ into n equal subintervals, each of width h, like this:



There are $n + 1$ points altogether, and they divide the interval into n equal subintervals. The endpoints are $a = x_0$ and $b = x_{n+1}$, and the $n - 1$ division points in between are $x_1, x_2, \ldots, x_{n-1}$.

There are n subintervals, so $nh = b - a$, and the width h of each subinterval is

$$h = \frac{b-a}{n}.$$

Thus starting with $a = x_0$, the successive values of the division points are

$$\begin{array}{lcl} x_0 = a & & x_{n-2} = a + (n-2)h \\ x_1 = a + h & \ldots & x_{n-1} = a + (n-1)h \\ x_2 = a + 2h & & x_n = a + nh = a + (b-a) = b \end{array}$$

That is, $x_r = a + rh$, for $r = 0, 1, 2, \ldots, n-1, n$.

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Now we can apply the trapezoidal rule to each subinterval in turn,

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_{n-1}} f(x)\,dx + \int_{x_{n-1}}^{b} f(x)\,dx \\ &\doteqdot \frac{h}{2}\big(f(a) + f(x_1)\big) + \frac{h}{2}\big(f(x_1) + f(x_2)\big) + \cdots \\ &\qquad + \frac{h}{2}\big(f(x_{n-2}) + f(x_{n-1})\big) + \frac{h}{2}\big(f(x_{n-1}) + f(b)\big) \\ &\doteqdot \frac{h}{2}\big(f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(b)\big).\end{aligned}$$

20 TRAPEZOIDAL-RULE FORMULA USING *n* SUBINTERVALS

Let $f(x)$ be a function that is continuous in the closed interval $[a, b]$. Then

$$\int_a^b f(x)\,dx \doteqdot \frac{h}{2}\big(f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(b)\big)$$

where $h = \dfrac{b-a}{n}$ and $x_r = a + rh$, for $r = 1, 2, \ldots, n-1$.

A common rearrangement of this formula, using three sets of nested brackets, is

$$\int_a^b f(x)\,dx \doteqdot \frac{b-a}{2n}\Big(f(a) + f(b) + 2\big(f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1})\big)\Big).$$

Using the formula for the trapezoidal rule

The formula may look complicated at first sight, but it is actually quite straightforward to use, provided that:

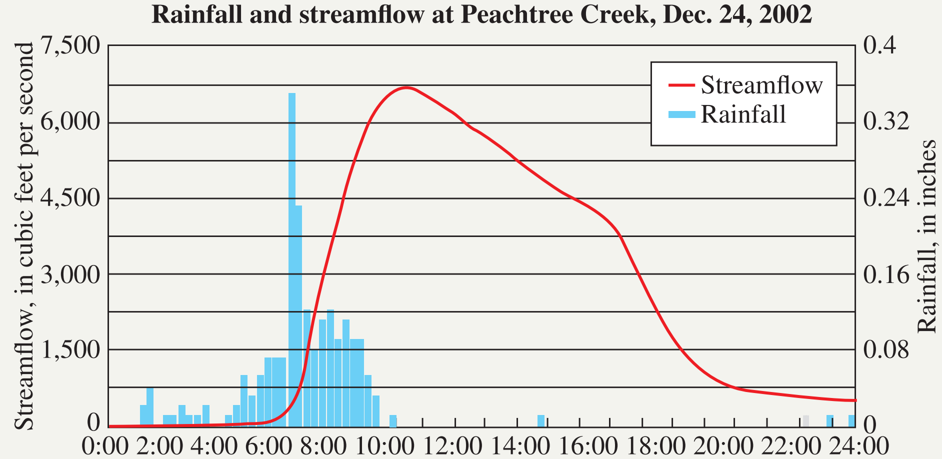
- We begin with a sensible value of the width h of each subinterval.
- We construct a clear table of values to work from.

Here is an example where there is no equation of the function, but simply a set of experimental results gathered by recording equipment.

Example 36 4H

The flow at Peachtree Creek on 4th December 2002 after a storm is shown in the graph below. The flow rate in cubic feet per second is sketched as a function of the time t in hours.

We can estimate the total amount of water that flowed down the creek after the storm that day by integrating from $t = 4$ to $t = 24$. Use the trapezoidal rule with two-hour subintervals to approximate the total amount of water.



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SOLUTION

The graph is very inaccurate, like so much internet data, but here is a rough table of values of the flow rate in cubic feet per second as a function of time t in hours.

t	4	6	8	10	12	14	16	18	20	22	24
Flow rate	100	600	5500	6700	5800	4800	4100	1800	800	600	500

The units need attention. The time is in hours, so the flow rates must be converted to cubic feet per hour by multiplying by $60 \times 60 = 3600$.

To avoid zeroes, let R be the flow rate in millions of cubic feet per hour.

t	4	6	8	10	12	14	16	18	20	22	24
R	0.36	2.16	19.8	24.12	20.88	17.28	14.76	6.48	2.88	2.16	1.8

Here $h = 2$ and $n = 10$. Also $a = x_0 = 4, x_1 = 6, \ldots, x_{n-1} = 22, x_n = b = 24$.

$$\text{Hence} \int_4^{20} R\,dt \doteqdot \frac{2}{2}\Big(f(4) + 2f(6) + 2f(8) + \cdots + 2f(22) + f(24)\Big)$$
$$\doteqdot 0.36 + 4.32 + 39.6 + 48.24 + 34.56 + 29.52 + 12.96 + 5.76 + 4.32 + 1.8$$
$$\doteqdot 223.2.$$

Alternatively, using the second formula,

$$\int_4^{20} R\,dt \doteqdot \frac{24-4}{20}\Big(f(4) + f(24) + 2\big(f(6) + f(8) + \cdots + f(22)\big)\Big)$$
$$\doteqdot 0.36 + 1.8 + 2(2.16 + 19.8 + 24.12 + 17.28 + 14.76 + 62.48 + 1.44 + 1.08)$$
$$= 2.16 + 2 \times 110.52$$
$$= 223.2.$$

Thus about 223 million cubic feet of water flowed down the creek from 4:00 am to midnight.

Using a spreadsheet for calculations

The authors used a spreadsheet for all the calculations above — the trapezoidal-rule formula is well suited for machine computation. The next worked example shows how to use an Excel spreadsheet to carry out such a calculation but any spreadsheet can be used. Note that:

- Excel commands and procedures have been changing over successive versions.
- Mac users will need some adjustments, particularly when implementing the ‘fill down’ and ‘fill right’ commands.

The calculation involves the integration of $\dfrac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$. We will see in Chapter 10 that this function is the probability density function of the normal distribution, and is the most important function in statistics. There is no simple equation for its primitive, so approximations are always necessary.

Example 37 4H

Let $\phi(x) = \dfrac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$. Approximate the integral $\displaystyle\int_0^1 \phi(x)\,dx$ using the trapezoidal rule with 10 subintervals.

(The symbol ϕ is the lower case Greek letter ‘phi’, corresponding to Latin f.)

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SOLUTION

In these instructions, we enter a formula into a cell by typing the = sign as the first character.
The cell in column `D` and row `3` is labelled `D3`, and in formulae, this refers to its contents. Normally leave the top row clear for later titles.

1 On a new sheet in Excel, enter 0 into Cell `A2` and press `Enter`.
— Select Cell `B2` and type `=0.1+A2` and press `Enter`.
— Cell `B2` should now show 0.1.

2 Select Cells `B2:K2` and press `Ctrl+R` to 'fill right'.
— Cells `C2:K2` should now show 0.2, 0.3, . . . , 1.

3 Type `=EXP(-A2*A2/2)/SQRT(2*PI())` into Cell `A3`.
— Cell `A3` should now show $\phi(0) = 0.398942$.

4 Select Cells `A3:K3` and press `Ctrl+R` to 'fill right'.
— Cells `B3:K3` should now show $\phi(0.1) = 0.396953, \ldots, \phi(1) = 0.241971$.

We now have the table of values for the function $\phi(x)$, and we need to add

$$\phi(0) + 2\phi(0.1) + 2\phi(0.2) + \cdots + 2\phi(0.9) + \phi(1)$$

5 Select Cell `A4` and enter `=A3`. This should duplicate the value in `A3`.
— Select Cell `K4` and enter `=K3`. Again, this duplicates the value in `K3`.
— Select Cell `B4` and enter `=2*B3`. This should double the value in `B3`.
— Select Cells `B4:J4` and press `Ctrl+R` to 'fill right'.
— Add the row by selecting Cell `L4` and typing `=SUM(A4:K4)`.

In this case, $h = 0.1$ so we multiply by $\frac{1}{2}h = 0.05$.

6 Select Cell `L5` and type `=L4*0.05` — this shows the final answer.

Hence $\displaystyle\int_0^1 \phi(x)\,dx \doteqdot 0.341$. We shall find in Chapter 10 that this is approximately the probability that a score in a normal distribution lies between the mean 0 and one standard deviations above the mean.
The correct approximation to three decimal places in 0.398 — we will see in Chapter 10 that the curve is concave down in the interval [0, 1], which explains why our estimate is a little smaller than it should be.

Readers may like to repeat the calculations above using 100 subintervals and see how close the approximation is then.

Exercise 4H

FOUNDATION

Technology: It is not difficult to write (or download) a program that will allow the calculations of the trapezoidal rule to be automated. It can then be applied to many examples from this exercise. The number of subintervals used can be steadily increased, and the approximations may then converge to the exact value of the integral. An accompanying screen sketch showing the curve and the chords would be helpful in giving a visual impression of the size and the sign of the error.

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1 Approximate $\int_2^6 f(x)\,dx$ in each part by using the formula $\frac{1}{2}(a+b)h$ for the area of a trapezium.

a

x	2	6
$f(x)$	8	12

b

x	2	6
$f(x)$	6.2	4.8

c

x	2	6
$f(x)$	-4	-9

2 Three function values are given in the table below.

x	2	6	10
$f(x)$	12	20	30

a Approximate $\int_2^{10} f(x)\,dx$ by calculating the areas of two trapezia and then adding.

b Check your answer to the previous part by using the formula for the trapezoidal rule.

3 Three function values are given in the table below.

x	-5	0	5
$f(x)$	2.4	2.6	4.4

a Approximate $\int_{-5}^{5} f(x)\,dx$ by adding the areas of two trapezia.

b Check your answer to the previous part by using the formula for the trapezoidal rule.

4 Show, by means of a diagram, that the trapezoidal rule will:

a overestimate $\int_a^b f(x)\,dx$, if $f''(x) > 0$ for $a \le x \le b$,

b underestimate $\int_a^b f(x)\,dx$, if $f''(x) < 0$ for $a \le x \le b$.

5 **a** Complete this table for the function $y = x(4-x)$:

x	0	1	2	3	4
y					

b Hence use the trapezoidal rule with five function values to approximate $\int_0^4 x(4-x)\,dx$.

c What is the exact value of $\int_0^4 x(4-x)\,dx$, and why does it exceed the approximation? Sketch the curve and the four chords involved.

d Calculate the percentage error in the approximation (that is, divide the error by the exact answer and convert to a percentage).

6 **a** Complete this table for the function $y = \dfrac{6}{x}$.

x	1	2	3	4	5
y					

b Use the trapezoidal rule with the five function values above, that is with four subintervals, to approximate $\int_1^5 \frac{6}{x}\,dx$.

c Show that the second derivative of $y = \dfrac{6}{x}$ is $y'' = 12x^{-3}$, and use this result to explain why the approximation will exceed the exact value of the integral.

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7 **a** Complete this table correct to four decimal places for the function $y = \sqrt{x}$.

x	4	5	6	7	8	9
y						

b Approximate $\int_4^9 \sqrt{x}\,dx$, using the trapezoidal rule with the six function values above, that is with five subintervals. Answer correct to three significant figures.

c What is the exact value of $\int_4^9 \sqrt{x}\,dx$? Show that the second derivative of $y = x^{\frac{1}{2}}$ is $y'' = -\frac{1}{4}x^{-\frac{3}{2}}$, and use this result to explain why the approximation is less than the value of the definite integral.

DEVELOPMENT

8 Use the trapezoidal rule with three function values to approximate each definite integral, writing your answer correct to two significant figures where necessary.

a $\int_0^1 2^{-x}\,dx$ **b** $\int_{-2}^0 2^{-x}\,dx$ **c** $\int_1^3 \sqrt[3]{9-2x}\,dx$ **d** $\int_{-13}^{-1} \sqrt{3-x}\,dx$

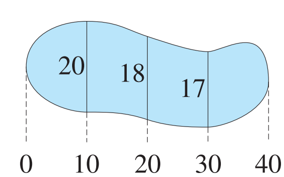
9 Use the trapezoidal rule with four subintervals to approximate each definite integral, writing your answer correct to three significant figures where necessary.

a $\int_2^6 \frac{1}{x}\,dx$ **b** $\int_0^2 \frac{1}{2+\sqrt{x}}\,dx$ **c** $\int_4^8 \sqrt{x^2-3}\,dx$ **d** $\int_1^2 \log_{10} x\,dx$

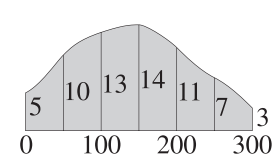
10 An object is moving along the x-axis with values of the velocity v in m/s at various times t given in the table to the right. Given that the distance travelled may be found by calculating the area under the velocity/time graph, use the trapezoidal rule to estimate the distance travelled by the particle in the first 5 seconds.

t	0	1	2	3	4	5
v	1.5	1.3	1.4	2.0	2.4	2.7

11 The diagram to the right shows the width of a lake at 10-metre intervals. Use the trapezoidal rule to estimate the surface area of the water.



12 The diagram to the right shows a vertical rock cutting of length 300 metres alongside a straight horizontal section of highway. The heights of the cutting are measured at 50-metre intervals. Use the trapezoidal rule to estimate the area of the vertical rock cutting.



CHALLENGE

13 **a** Use the trapezoidal rule with five function values to approximate $\int_0^1 \sqrt{1-x^2}\,dx$, giving your answer correct to four decimal places.

b Use part **a** and the fact that $y = \sqrt{1-x^2}$ is a semi-circle to approximate π. Give your answer correct to one decimal place, and explain why your approximation is less than π.

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14 Use the trapezoidal rule with four subintervals, together with appropriate log laws, to show that
$$\int_1^5 \ln x\,dx \doteqdot \ln 54.$$

15 **a** Evaluate $\int_{-1}^{1} (x^3 + 1)\,dx$ using the fundamental theorem, and then using the trapezoidal rule with three function values.

b Explain from the graph why the trapezoidal rule gives the correct answer in this case.

XLS

An investigation using a spreadsheet for trapezoidal rule calculations

16 Work through the spreadsheet example just above this exercise. Then use a spreasheet to estimate these integrals using the trapezoidal rule with 5, 10, 20 and perhaps more subintervals.

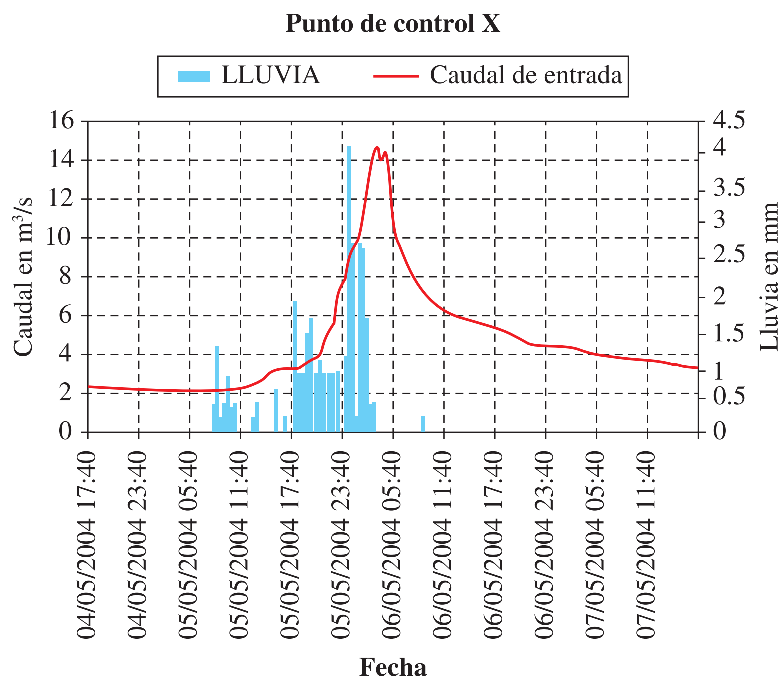
a $\int_1^{11} \frac{1}{x}\,dx$ **b** $\int_1^{11} \log_e x\,dx$ **c** $\int_0^{10} e^{-x^2}\,dx$

You will need to look at the results and perhaps vary the number of decimal places that you are using in the calculations and recording in your answers.

Possible spreadsheet projects

It is possible to program a spreadsheet so that the number of subintervals can be entered as a single variable. The construction of such a program and similar programs could be incorporated into a longer project examining the usefulness and accuracy of the trapezoidal rule, or examining some physical phenomena.

In the next diagram, it is clear that in parts of the graph where there is a lot of activity, the subintervals should be quite narrow, whereas in other calmer parts they can be far wider. Such variability could also be incorporated into the spreadsheet and its formulae.



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An investigation (and possible project) integrating a graph from the web by the trapezoidal rule
There is a great deal of data available on the web for a sustained investigation of river flow. The following question suggests some interesting questions about one such situation, but there are many more situations and questions. Integrating graphs of all kinds from the web using the trapezoidal rule could be the basis of various different projects.

17 The hydrograph above shows the rate of flow through Control Point X on the Turia River in Spain over a three-day period in May 2004. The rate of flow ('Caudal') is given as a function of the date–times ('Fecha') — notice that the successive date–times on the horizontal axis are separated by exactly 6 hours. The rainfall ('Lluvia') is given by the vertical bars.

The units of time are hours, and the units of the flow rate are 'cubic metres per second'. The flow rate R should be converted to units of 'thousands of cubic meters per hour' so that time is in hours and there are fewer zeroes — multiply by $\dfrac{60 \times 60}{1000} = 3.6$.

a From the graph, copy and complete the table of values of the flow rate R at the first four date–times, 04/05/2004 17:40 to 05/05/2004 11:40.

t	17:40	23:40	05:40	11:40
R				

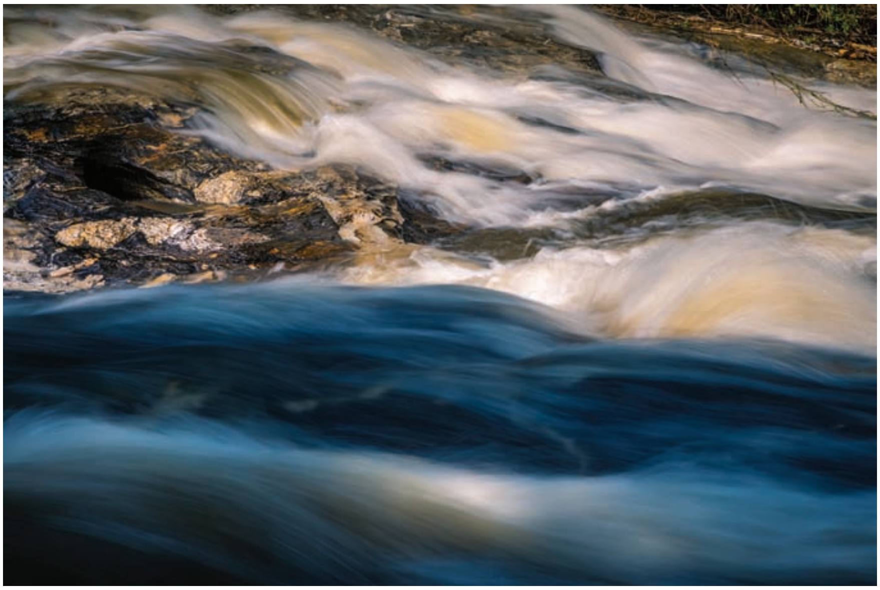
Then use the trapezoidal rule to estimate the total volume of water that flowed through the control point in those 18 hours.

b Draw up a similar table for the 18 hours of heavy flow from 05/05/2004 17:40 to 06/05/2004 11:40, but use 3 hours as the separation between successive times.

t	17:40	20:40	23:40	02:40	05:40	08:40	11:40
R							

Then use the trapezoidal rule to estimate the total volume of water that flowed through the Control Point in those 18 hours. Why are 3 hours suggested here in part **b** for the width of the subintervals, where 6 hours was used in part **a**?

c How many times more water flowed down the river in the second 18-hour period? Look at the rainfall record, and discuss how the river flow responded to the rainfall.



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4I The reverse chain rule

When we use the chain rule to differentiate a composite function, the result is a product of two terms. For example, in the first worked example below,

$$\frac{d}{dx}(x^3 + 5)^4 = 4(x^3 + 5)^3 \times 3x^2.$$

This section deals with the problem of reversing a chain-rule differentiation. The section is demanding, and as with Section 4D, readers may prefer to leave it for a second reading of the chapter at a later time.

Reversing a chain-rule differentiation

Finding primitives is the reverse process of differentiation. Thus once any differentiation has been performed, the process can then be reversed to give a primitive.

Example 38 4I

a Differentiate $(x^3 + 5)^4$ with full setting-out of the chain rule.
b Hence find a primitive of $12x^2(x^3 + 5)^3$.
c Hence find the primitive of $x^2(x^3 + 5)^3$.

SOLUTION

a Let $y = (x^3 + 5)^4$.

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$= 4(x^3 + 5)^3 \times 3x^2$
$= 12x^2(x^3 + 5)^3$.

Let $u = x^3 + 5$.
Then $y = u^4$.
Hence $\frac{du}{dx} = 3x^2$
and $\frac{dy}{du} = 4u^3$.

b By part **a**, $\frac{d}{dx}(x^3 + 5)^4 = 12x^2(x^3 + 5)^3$.

Reversing this, $\int 12x^2(x^3 + 5)^3\,dx = (x^3 + 5)^4$.

c Dividing by 12, $\int x^2(x^3 + 5)^3\,dx = \frac{1}{12}(x^3 + 5)^4 + C$, for some constant C.

Note: It is best not to add the arbitrary constant until the last line, because it would be pointless to divide C by 12 as well.

Example 39 4I

a Differentiate $\frac{1}{1 + x^2}$ with full setting-out of the chain rule.

b Hence find the primitive of $\frac{x}{(1 + x^2)^2}$.

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SOLUTION

a Let $y = \dfrac{1}{1 + x^2}$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= -(1 + x^2)^{-2} \times 2x$

$= \dfrac{-2x}{(1 + x^2)^2}$.

Let $u = 1 + x^2$.
Then $y = u^{-1}$.
Hence $\dfrac{du}{dx} = 2x$
and $\dfrac{dy}{du} = -u^{-2}$.

b By part **a**, $\dfrac{d}{dx}\left(\dfrac{1}{1 + x^2}\right) = \dfrac{-2x}{(1 + x^2)^2}$.

Reversing this, $\displaystyle\int \frac{-2x}{(1 + x^2)^2}\,dx = \frac{1}{1 + x^2}, + C,$ for some constant C,

$\boxed{\div (-2)}$ $\displaystyle\int \frac{x}{(1 + x^2)^2}\,dx = \frac{-1}{2(1 + x^2)} + C.$

21 REVERSING A CHAIN-RULE DIFFERENTIATION

Once a chain-rule differentiation, or any differentiation, has been performed, the result can be written down in reverse as an indefinite integral.

A formula for the reverse chain rule

There is a formula for the reverse chain rule. Start with the formula for differentiating a function using the chain rule — we gave the formula in two forms:

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} \qquad \frac{d}{dx}\big(f(x)\big)^n = n\big(f(x)\big)^{n-1} f'(x)$$

and we can reverse both forms of the formula,

$$\int u^{n-1}\frac{du}{dx}\,dx = \frac{u^n}{n} \qquad \int \big(f(x)\big)^{n-1} f'(x)\,dx = \frac{\big(f(x)\big)^n}{n}.$$

Then replacing $n - 1$ by n and n by $n + 1$,

$$\int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n + 1} \qquad \text{OR} \qquad \int \big(f(x)\big)^n f'(x)\,dx = \frac{\big(f(x)\big)^{n+1}}{n + 1}.$$

Take your pick which formula you prefer to use. The difficult part is recognising what u or $f(x)$ should be in the function that you are integrating.

We shall do worked Example 38 **c** again using the formula.

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Example 40 4I

Use the formula for the reverse chain rule to find $\int x^2(x^3 + 5)^3\,dx$.

SOLUTION

The key to finding this integral is to realise that x^2 is a multiple of the derivative of $x^3 + 5$. At this point, we can put $u = x^3 + 5$ or $f(x) = x^3 + 5$, and everything works smoothly after that. Here are two strong recommendations:

- Show working identifying u or $f(x)$ and its derivative on the right-hand side.
- Write down the standard form above substituting the particular value of n.

Using the first formula,

$$\begin{aligned}\int x^2(x^3 + 5)^3\,dx &= \tfrac{1}{3}\int 3x^2(x^3 + 5)^3\,dx \\ &= \tfrac{1}{3} \times \tfrac{1}{4}(x^3 + 5)^4 + C, \\ &\quad \text{for some constant } C, \\ &= \tfrac{1}{12}(x^3 + 5)^4 + C.\end{aligned}$$

Let $u = x^3 + 5$.
Then $\dfrac{du}{dx} = 3x^2$.
Here $\displaystyle\int u^3 \frac{du}{dx}\,dx = \tfrac{1}{4}u^4$.

Using the second formula,

$$\begin{aligned}\int x^2(x^3 + 5)^3\,dx &= \tfrac{1}{3}\int 3x^2(x^3 + 5)^3\,dx \\ &= \tfrac{1}{3} \times \tfrac{1}{4}(x^3 + 5)^4 + C, \\ &\quad \text{for some constant } C, \\ &= \tfrac{1}{12}(x^3 + 5)^4 + C.\end{aligned}$$

Let $f(x) = x^3 + 5$.
Then $f'(x) = 3x^2$.
Here $\displaystyle\int \big(f(x)\big)^3 f'(x)\,dx = \tfrac{1}{4}\big(f(x)\big)^4$.

Notice that the two notations differ only in the working in the right-hand column.

Example 41 4I

Use the formula for the reverse chain rule to find:

a $\displaystyle\int x\sqrt{1 - x^2}\,dx$

b $\displaystyle\int_0^2 x\sqrt{1 - x^2}\,dx$

SOLUTION

a This integral is based on the recognition that $\dfrac{d}{dx}(1 - x^2) = -2x$.

Using the first formula,

$$\begin{aligned}\int x\sqrt{1 - x^2}\,dx &= -\tfrac{1}{2}\int (-2x) \times (1 - x^2)^{\frac{1}{2}}\,dx \\ &= -\tfrac{1}{2} \times \tfrac{2}{3}(1 - x^2)^{\frac{3}{2}} + C, \\ &\quad \text{for some constant } C, \\ &= -\tfrac{1}{3}(1 - x^2)^{\frac{3}{2}} + C.\end{aligned}$$

Let $u = 1 - x^2$.
Then $\dfrac{du}{dx} = -2x$.
Here $\displaystyle\int u^{\frac{1}{2}} \frac{du}{dx}\,dx = \tfrac{2}{3}u^{\frac{3}{2}}$.

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Using the second formula,

$$\int x\sqrt{1-x^2}\,dx = -\tfrac{1}{2}\int(-2x)\times(1-x^2)^{\frac{1}{2}}$$
$$= -\tfrac{1}{2}\times\tfrac{2}{3}(1-x^2)^{\frac{3}{2}}$$
$$= -\tfrac{1}{3}(1-x^2)^{\frac{3}{2}} + C,$$
for some constant C.

Let $f(x) = 1 - x^2$.
Then $f'(x) = -2x$.
Here $\int\left(f(x)^{\frac{1}{2}}\right)f'(x)\,dx = \tfrac{2}{3}\left(f(x)\right)^{\frac{3}{2}}$.

b The definite integral is meaningless because $\sqrt{1-x^2}$ is undefined for $x > 1$.

22 A FORMULA FOR THE REVERSE CHAIN RULE

- The reversing of the chain rule can be written as a formula in two ways:

$$\int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} + C \qquad \text{OR} \qquad \int\left(f(x)\right)^n f'(x)\,dx = \frac{\left(f(x)\right)^{n+1}}{n+1} + C.$$

- The vital step in using this formula is to identity u or $f(x)$ and its derivative.

To use the formula, we have to write the integrand as a product. One factor is a power of a function $f(x)$ or u. The other factor is the derivative of that function.

Exercise 4I

FOUNDATION

1 **a** Find $\dfrac{d}{dx}(2x+3)^4$.

b Hence find:

i $\int 8(2x+3)^3\,dx$ **ii** $\int 16(2x+3)^3\,dx$

2 **a** Find $\dfrac{d}{dx}(3x-5)^3$.

b Hence find:

i $\int 9(3x-5)^2\,dx$ **ii** $\int 27(3x-5)^2\,dx$

3 **a** Find $\dfrac{d}{dx}(1+4x)^5$.

b Hence find:

i $\int 20(1+4x)^4\,dx$ **ii** $\int 10(1+4x)^4\,dx$

4 **a** Find $\dfrac{d}{dx}(1-2x)^4$.

b Hence find:

i $\int -8(1-2x)^3\,dx$ **ii** $\int -2(1-2x)^3\,dx$

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5 **a** Find $\frac{d}{dx}(4x+3)^{-1}$.

b Hence find:

i $\int -4(4x+3)^{-2}\,dx$

ii $\int (4x+3)^{-2}\,dx$

6 **a** Find $\frac{d}{dx}(2x-5)^{\frac{1}{2}}$.

b Hence find:

i $\int (2x-5)^{-\frac{1}{2}}\,dx$

ii $\int \frac{1}{3}(2x-5)^{-\frac{1}{2}}\,dx$

DEVELOPMENT

7 **a** Find $\frac{d}{dx}(x^2+3)^4$.

b Hence find:

i $\int 8x(x^2+3)^3\,dx$

ii $\int 40x(x^2+3)^3\,dx$

8 **a** Find $\frac{d}{dx}(x^3-1)^5$.

b Hence find:

i $\int 15x^2(x^3-1)^4\,dx$

ii $\int 3x^2(x^3-1)^4\,dx$

9 **a** Find $\frac{d}{dx}\sqrt{2x^2+3}$.

b Hence find:

i $\int \frac{2x}{\sqrt{2x^2+3}}\,dx$

ii $\int \frac{x}{\sqrt{2x^2+3}}\,dx$

10 **a** Find $\frac{d}{dx}\left(\sqrt{x}+1\right)^3$.

b Hence find:

i $\int \frac{3\left(\sqrt{x}+1\right)^2}{2\sqrt{x}}\,dx$

ii $\int \frac{\left(\sqrt{x}+1\right)^2}{\sqrt{x}}\,dx$

11 **a** Find $\frac{d}{dx}(x^3+3x^2+5)^4$.

b Hence find:

i $\int 12(x^2+2x)(x^3+3x^2+5)^3\,dx$

ii $\int (x^2+2x)(x^3+3x^2+5)^3\,dx$

12 **a** Find $\frac{d}{dx}(5-x^2-x)^7$.

b Hence find:

i $\int (-14x-7)(5-x^2-x)^6\,dx$

ii $\int (2x+1)(5-x^2-x)^6\,dx$

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13 Find these indefinite integrals using the reverse chain rule in either form

$$\int f'(x)\left(f(x)\right)^n dx = \frac{\left(f(x)\right)^{n+1}}{n+1} + C \qquad \text{OR} \qquad \int u^n \frac{du}{dx}\, dx = \frac{u^{n+1}}{n+1} + C.$$

a $\int 5(5x+4)^3\, dx$ (Let $f(x) = 5x+4$ or $u = 5x+4$.)

b $\int -3(1-3x)^5\, dx$ (Let $f(x) = 1-3x$ or $u = 1-3x$.)

c $\int 2x(x^2-5)^7\, dx$ (Let $f(x) = x^2-5$ or $u = x^2-5$.)

d $\int 3x^2(x^3+7)^4\, dx$ (Let $f(x) = x^3+7$ or $u = x^3+7$.)

e $\int \frac{6x}{(3x^2+2)^2}\, dx$ (Let $f(x) = 3x^2+2$ or $u = 3x^2+2$.)

f $\int \frac{-6x^2}{\sqrt{9-2x^3}}\, dx$ (Let $f(x) = 9-2x^3$ or $u = 9-2x^3$.)

CHALLENGE

14 Find these indefinite integrals using the reverse chain rule.

a $\int 10x(5x^2+3)^2\, dx$

b $\int 2x(x^2+1)^3\, dx$

c $\int 12x^2(1+4x^3)^5\, dx$

d $\int x(1+3x^2)^4\, dx$

e $\int x^3(1-x^4)^7\, dx$

f $\int 3x^2\sqrt{x^3-1}\, dx$

g $\int x\sqrt{5x^2+1}\, dx$

h $\int \frac{2x}{\sqrt{x^2+3}}\, dx$

i $\int \frac{x+1}{\sqrt{4x^2+8x+1}}\, dx$

j $\int \frac{x}{(x^2+5)^3}\, dx$

15 Evaluate these definite integrals using the reverse chain rule.

a $\int_{-1}^{1} x^2(x^3+1)^4\, dx$

b $\int_0^1 \frac{x}{(5x^2+1)^3}\, dx$

c $\int_0^{\frac{1}{2}} x\sqrt{1-4x^2}\, dx$

d $\int_{-3}^{-1} (x+5)(x^2+10x+3)^2 dx$

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Chapter 4 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 4 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 Evaluate these definite integrals, using the fundamental theorem.

a $\displaystyle\int_0^1 3x^2\,dx$ **b** $\displaystyle\int_1^2 x\,dx$ **c** $\displaystyle\int_2^5 4x^3\,dx$

d $\displaystyle\int_{-1}^1 x^4\,dx$ **e** $\displaystyle\int_{-4}^{-2} 2x\,dx$ **f** $\displaystyle\int_{-3}^{-1} x^2\,dx$

g $\displaystyle\int_0^2 (x+3)\,dx$ **h** $\displaystyle\int_{-1}^4 (2x-5)\,dx$ **i** $\displaystyle\int_{-3}^1 (x^2-2x+1)\,dx$

2 By expanding the brackets where necessary, evaluate these definite integrals.

a $\displaystyle\int_1^3 x(x-1)\,dx$ **b** $\displaystyle\int_{-1}^0 (x+1)(x-3)\,dx$ **c** $\displaystyle\int_0^1 (2x-1)^2\,dx$

3 Write each integrand as separate fractions, then evaluate the integral.

a $\displaystyle\int_1^2 \frac{x^2-3x}{x}\,dx$ **b** $\displaystyle\int_2^3 \frac{3x^4-4x^2}{x^2}\,dx$ **c** $\displaystyle\int_{-2}^{-1} \frac{x^3-2x^4}{x^2}\,dx$

4 **a** **i** Show that $\displaystyle\int_4^k 5\,dx = 5k-20$.

ii Hence find the value of k if $\displaystyle\int_4^k 5\,dx = 10$.

b **i** Show that $\displaystyle\int_0^k (2x-1)\,dx = k^2-k$.

ii Hence find the positive value of k for which $\displaystyle\int_0^k (2x-1)\,dx = 6$.

5 Without finding a primitive, use the properties of the definite integral to evaluate these integrals, stating reasons.

a $\displaystyle\int_3^3 (x^3-5x+4)\,dx$ **b** $\displaystyle\int_{-2}^2 x^3\,dx$ **c** $\displaystyle\int_{-3}^3 (x^3-9x)\,dx$

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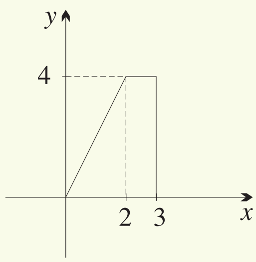
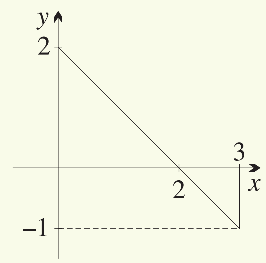
6 Use area formulae to find $\int_0^3 f(x)\,dx$, given the following sketches of $f(x)$.

a



b



7 (From Section 4D, which is marked as Challenge).

a Find each signed area function.

i $A(x) = \int_{-2}^{x} (4 - t)\,dt$ **ii** $A(x) = \int_{2}^{x} t^{-2}\,dt$

b Differentiate the results in part **a** to find:

i $\dfrac{d}{dx}\int_{-2}^{x} (4 - t)\,dt$ **ii** $\dfrac{d}{dx}\int_{2}^{x} t^{-2}\,dt$

c Without first performing the integration, use the fundamental theorem of calculus to find these functions.

i $\dfrac{d}{dx}\int_{7}^{x} (t^5 - 5t^3 + 1)\,dt$ **ii** $\dfrac{d}{dx}\int_{3}^{x} \dfrac{t^2 + 4}{t^2 - 1}\,dt$

8 Find these indefinite integrals.

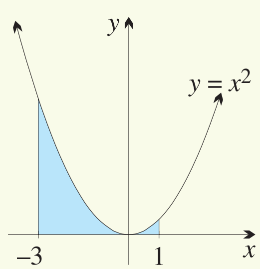
a $\int (x + 2)\,dx$ **b** $\int (x^3 + 3x^2 - 5x + 1)\,dx$ **c** $\int x(x - 1)\,dx$

d $\int (x - 3)(2 - x)\,dx$ **e** $\int x^{-2}\,dx$ **f** $\int \dfrac{1}{x^7}\,dx$

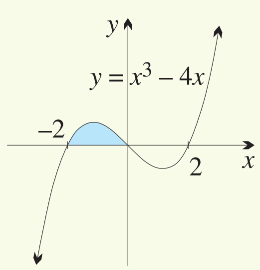
g $\int \sqrt{x}\,dx$ **h** $\int (x + 1)^4\,dx$ **i** $\int (2x - 3)^5\,dx$

9 Find the area of each shaded region below by evaluating a definite integral.

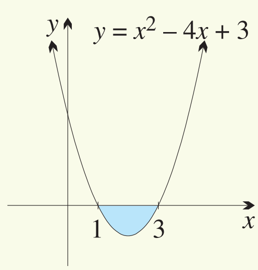
a



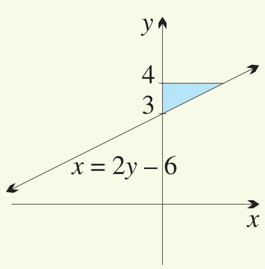
b



c



d



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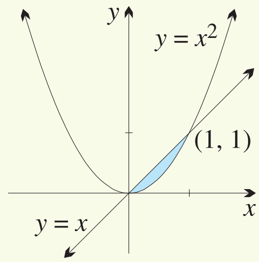
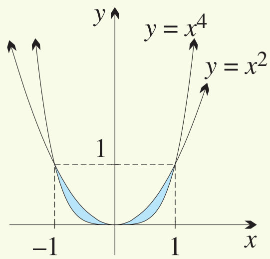
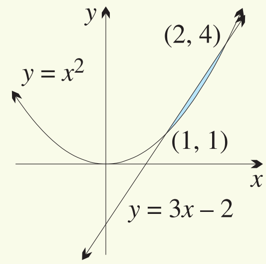
e



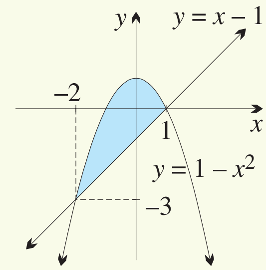
f



g



h



10 a By solving the equations simultaneously, show that the curves $y = x^2 - 3x + 5$ and $y = x + 2$ intersect at the points $(1, 3)$ and $(3, 5)$.

b Sketch both curves on the same diagram and find the area of the region enclosed between them.

11 a Use the trapezoidal rule with two subintervals to approximate $\int_1^3 2^x\,dx$.

b Use the trapezoidal rule with five function values to approximate $\int_1^3 \log_{10} x\,dx$. Give your answer correct to two significant figures.

12 a Find $\dfrac{d}{dx}(3x + 4)^6$.

b Hence find:

i $\int 18(3x + 4)^5\,dx$ **ii** $\int 9(3x + 4)^5\,dx$

13 a Find $\dfrac{d}{dx}(x^2 - 1)^3$.

b Hence find:

i $\int 6x(x^2 - 1)^2\,dx$ **ii** $\int x(x^2 - 1)^2\,dx$

14 Find these indefinite integrals using the reverse chain rule.

a $\int 3x^2(x^3 + 1)^4\,dx$ **b** $\int \dfrac{2x}{(x^2 - 5)^3}\,dx$

15 Use the reverse chain rule to show that $\int_0^1 \dfrac{x}{\sqrt{x^2 + 3}}\,dx = 2 - \sqrt{3}$.

16 Explain why these integrals are meaningless.

a $\int_0^4 \dfrac{1}{(x - 1)^2}\,dx$ **b** $\int_0^4 \sqrt{9 - 3x}\,dx$ **c** $\int_0^4 \log_e(x - 2)\,dx$ **d** $\int_0^4 \dfrac{1}{2 - 2^x}\,dx$

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5 The exponential and logarithmic functions

Chapter 9 of the Year 11 book began to extend calculus beyond algebraic functions to exponential functions and trigonometric functions. This chapter completes what is needed of the calculus of exponential functions, and introduces the calculus of the logarithmic functions. Chapter 6 will then bring the trigonometric functions into calculus as well.

The special number $e \doteqdot 2.7183$ was introduced as the most satisfactory base to use for the powers and logarithms discussed in Chapter 7 last year, and we established the two standard derivatives,

$$\frac{d}{dx}e^x = e^x \qquad \text{and} \qquad \frac{d}{dx}e^{ax+b} = ae^{ax+b}.$$

We sketched the graphs of $y = e^x$ and its inverse function $y = \log_e x$, transformed them in various ways, and developed some ideas about exponential growth and decay. All this is assumed knowledge in the present chapter and is quickly reviewed in Section 5A and 5F, apart from exponential growth and decay, which will be reviewed in Chapter 7 on motion and rates.

Sections 5A–5E deal mostly with exponential functions base e, Sections 5F–5J deal mostly with logarithmic functions base e, and the final Section 5K uses the change-of-base formula to extend the topic to exponential and logarithmic functions with bases other than e.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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5A Review of exponential functions base e

Section 5A and Section 5F will review the ideas in Sections 9A–9F in the Year 11 book. Two small topics, however, are new in these two review sections.

- Dilations of exponential (Section 5A) and logarithmic (Section 5F) functions.
- Exponential and logarithmic equations reducible to quadratics (Section 5F).

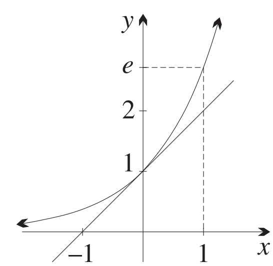
The text will not mention again the index laws and the logarithmic laws that were covered in Chapter 7 of that book and revisited in Chapter 9, but some early exercises will review them.

The exponential function $y = e^x$ is the subject of Sections 5A–5E. Section 5F then brings the logarithmic function $y = \log_e x$ into the discussion.

The number e and the function $y = e^x$

The fundamental result established in Chapter 9 of the Year 11 book is that the function $y = e^x$ is its own derivative,

$$\frac{d}{dx}e^x = e^x, \qquad \text{that is,} \qquad \text{gradient equals height.}$$



The number $e \doteqdot 2.7183$ is defined to be the base so that the exponential graph $y = e^x$ has gradient exactly 1 at the y-intercept. It is an irrational number, and it plays a role in exponential functions similar to the role that π plays in trigonometric functions.

To the right is a sketch of $y = e^x$. Its most significant properties are listed in Box 1.

1 THE FUNCTION $y = e^x$

- There is only one exponential function $y = e^x$ that is its own derivative, and the number $e \doteqdot 2.7183$ is defined to be the base of this function. Thus
  $$\frac{d}{dx}e^x = e^x, \qquad \text{that is,} \qquad \text{at each point, gradient equals height.}$$
- The gradient at the y-intercept is 1.
- The domain is all real numbers, and the range is $y > 0$.
- The line $y = 0$ is a horizontal asymptote.
- The function is one-to-one, that is, its inverse relation is a function.
- Differentiating again, $\dfrac{d^2}{dx^2}e^x = e^x$,
  so the function is always concave up, increasing at an increasing rate.

Sections 5A–5E occasionally require the inverse function $\log_e x$ of e^x, and we need the two inverse function identities:

$$\log_e e^x = x \quad \text{for all real } x \qquad \text{and} \qquad e^{\log_e x} = x \quad \text{for } x > 0.$$

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Using the calculator

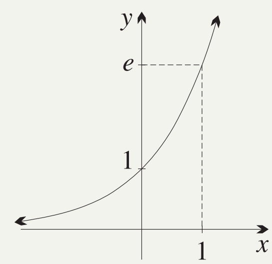
On the calculator, $\boxed{\ln}$ means $\log_e x$ and $\boxed{\log}$ means $\log_{10} x$. The function e^x is usually on the same button as $\log_e x$, and is accessed using $\boxed{\text{shift}}$ followed by $\boxed{\ln}$, or by some similar sequence.

Transformations of $y = e^x$

We applied translations and reflections to the curves, as in the next worked examples. The first part shows the graph of $y = e^{-x}$, which is just as important in science as $y = e^x$ because $y = e^x$ governs exponential growth, and $y = e^{-x}$ governs exponential decay.

Example 1 5A

Sketch each function using a transformation of the graph of $y = e^x$ sketched to the right. Describe the transformation, show and state the y-intercept and the horizontal asymptote, and write down the range.

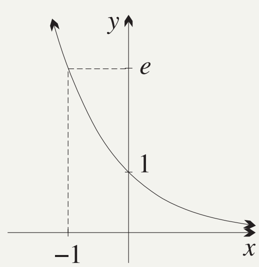


a $y = e^{-x}$ **b** $y = e^x + 3$ **c** $y = e^{x-2}$

Which transformations can also be done using a dilation?

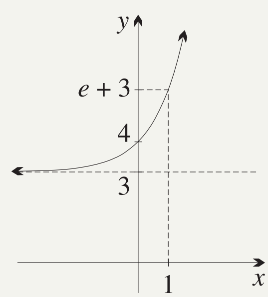
SOLUTION

a



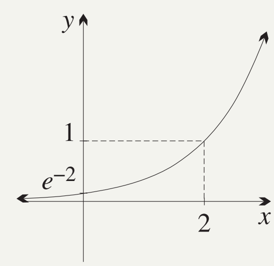
To graph $y = e^{-x}$,
reflect $y = e^x$ in y-axis.
y-intercept: $(0, 1)$
asymptote: $y = 0$
range: $y > 0$

b



To graph $y = e^x + 3$,
shift $y = e^x$ up 3.
y-intercept: $(0, 4)$
asymptote: $y = 3$
range: $y > 3$

c



To graph $y = e^{x-2}$,
shift $y = e^x$ right 2.
y-intercept: $(0, e^{-2})$
asymptote: $y = 0$
range: $y > 0$

- The equation $y = e^{-x}$ in part **a** is a reflection in the y-axis, and any reflection in the y-axis can be regarded as a horizontal dilation with factor -1.
- The equation $y = e^{x-2}$ in part **c** can be written as $y = e^{-2} \times e^x$, so it is also a vertical dilation of $y = e^x$ with factor $e^{-2} \doteqdot 0.135$.

Dilations of $y = e^x$

Dilations were only introduced in Section 2G of this book. In the context of exponential and logarithmic functions, dilations need further attention because some of them have an interesting property — they can be done with a shift in the other direction, as we have already seen in part **c** above.

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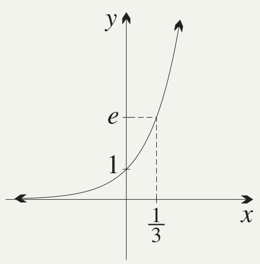
Example 2 5A

Use dilations of $y = e^x$ to generate a sketch of each function. Identify which dilation is also a shift in the other direction.

a $y = e^{3x}$ **b** $y = 3e^x$

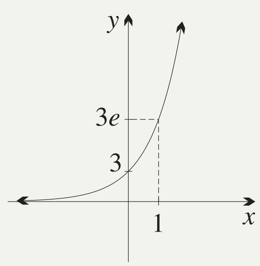
SOLUTION

a



Dilate $y = e^x$ vertically with factor $\frac{1}{3}$.

b



Dilate $y = e^x$ vertically with factor 3.

- $y = 3e^x$ can be written as $y = e^{\log_e 3} \times e^x = e^{x + \log_e 3}$, so it can also be regarded as a shift left by $\log_e 3$.

Tangents and normals to the exponential function

We applied the derivative to sketches of exponential functions. Here is a shortened form of the worked example (Example 7) given in Section 9D of the Year 11 book.

Example 3 5A

Let A be the point on the curve $y = 2e^x$ where $x = 1$.

a Find the equations of the tangent and normal at the point A.

b Show that the tangent at A passes through the origin, and find the point B where the normal meets the x-axis.

c Sketch the situation and find the area of $\triangle AOB$.

SOLUTION

a Substituting into $y = 2e^x$ shows that $A = (1, 2e)$.

Differentiating $y = 2e^x$ gives $y' = 2e^x$,

so at $A(1, 2e)$, where $x = 1$, $y' = 2e$ (which we know because gradient = height).

Hence, using point–gradient form, the tangent at A is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2e &= 2e(x - 1) \\ y &= 2ex. \end{aligned}$$

The normal at A has gradient $-\dfrac{1}{2e}$ (it is perpendicular to the tangent),

so its equation is $\quad y - 2e = -\dfrac{1}{2e}(x - 1)$

$\boxed{\times 2e}$ $\quad 2ey - 4e^2 = -x + 1$

$$x + 2ey = 4e^2 + 1.$$

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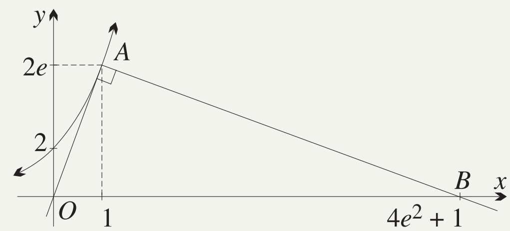
b The tangent passes through the origin O because its y-intercept is zero.
To find the x-intercept B of the normal,
put $y = 0$,
thus $x = 4e^2 + 1$,
so B has coordinates $(4e^2 + 1, 0)$.



c Hence area $\triangle AOB = \frac{1}{2} \times \text{base} \times \text{height}$
$= \frac{1}{2} \times (4e^2 + 1) \times 2e$
$= e(4e^2 + 1)$ square units.

Exercise 5A

FOUNDATION

Note: You will need the $\boxed{e^x}$ function on your calculator. This will require $\boxed{\text{shift}}$ followed by $\boxed{\ln}$, or some similar sequence of keys.

1 Simplify these expressions using the index laws.
a $2^3 \times 2^7$ **b** $e^4 \times e^3$ **c** $2^6 \div 2^2$ **d** $e^8 \div e^5$ **e** $(2^3)^4$ **f** $(e^5)^6$

2 Simplify these expressions using the index laws.
a $e^{2x} \times e^{5x}$ **b** $e^{10x} \div e^{8x}$ **c** $(e^{2x})^5$ **d** $e^{2x} \times e^{-7x}$ **e** $e^x \div e^{-4x}$ **f** $(e^{-3x})^4$

3 Write each expression as a power of e, then use your calculator to approximate it correct to four significant figures.
a e^2 **b** e^{-3} **c** e **d** $\dfrac{1}{e}$ **e** $\sqrt{e}$ **f** $\dfrac{1}{\sqrt{e}}$

4 **a** Write down the first and second derivatives of $y = e^x$.
b Hence copy and complete the sentence, 'The curve $y = e^x$ is always concave . . . , and is always . . . at . . . rate.'

5 **a** Find the gradient of the tangent to $y = e^x$ at $P(1, e)$, then find the equation of the tangent at P and show that it has x-intercept 0.
b Similarly find the equation of the tangent at $Q(0, 1)$, and show that its x-intercept is -1.
c Find the equation of the tangent at $R(-1, \frac{1}{e})$, and show that its x-intercept is -2.

6 **a** What is the y-coordinate of the point P on the curve $y = e^x - 1$ where $x = 1$?
b Find $\dfrac{dy}{dx}$ for this curve, and the value of $\dfrac{dy}{dx}$ when $x = 1$.
c Hence find the equations of the tangent and normal at P (in general form).

7 Sketch each curve using a single transformation of $y = e^x$, and describe the transformation.
a $y = e^x + 1$ **b** $y = e^x - 2$ **c** $y = \frac{1}{3}e^x$ **d** $y = e^{\frac{1}{2}x}$

8 Sketch each curve using a single transformation of $y = e^{-x}$, and describe the transformation.
a $y = e^{-x} - 1$ **b** $y = -e^{-x}$ **c** $y = e^{-2x}$

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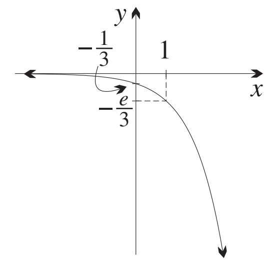
DEVELOPMENT

9 The graph to the right is a dilation of $y = e^x$. Describe the dilation, and write down the equation of the curve.



10 Expand and simplify:

a $(e^x + 1)(e^x - 1)$ **b** $(e^{4x} + 3)(e^{2x} + 3)$

c $(e^{-3x} - 2)e^{3x}$ **d** $(e^{-2x} + e^{2x})^2$

11 Write as a sum of powers of e:

a $\dfrac{e^{4x} + e^{3x}}{e^{2x}}$ **b** $\dfrac{e^{2x} - e^{3x}}{e^{4x}}$ **c** $\dfrac{e^{10x} + 5e^{20x}}{e^{-10x}}$ **d** $\dfrac{6e^{-x} + 9e^{-2x}}{3e^{3x}}$

12 **a** What is the gradient of the tangent to $y = e^x$ at its y-intercept?

b What transformation maps $y = e^x$ to $y = e^{-x}$?

c Use this transformation to find the gradient of $y = e^{-x}$ at its y-intercept.

d Sketch $y = e^x$ and $y = e^{-x}$ on one set of axes.

e How can the transformation be interpreted as a dilation?

13 Write down the first four derivatives of each function. For which curves is it true that at each point on the curve, the gradient equals the height?

a $y = e^x + 5$ **b** $y = e^x + x^3$ **c** $y = 4e^x$ **d** $y = 5e^x + 5x^2$

14 Find the gradient, and the angle of inclination correct to the nearest minute, of the tangent to $y = e^x$ at the points where:

a $x = 0$ **b** $x = 1$ **c** $x = -2$ **d** $x = 5$

Draw a diagram of the curve and the four tangents, showing the angles of inclination.

15 **a** What is the y-coordinate of the point P on the curve $y = e^x - 1$ where $x = 1$?

b Find $\dfrac{dy}{dx}$ for this curve, and the value of $\dfrac{dy}{dx}$ when $x = 1$.

c Hence find the equation of the tangent at the point P found in part **a**.

CHALLENGE

16 **a** Use, and describe, a dilation to sketch $y = e^{2x}$.

b Use, and describe, a subsequent translation to sketch $y = e^{2(x-1)}$.

c Use, and describe, a subsequent dilation to sketch $y = \frac{1}{2}e^{2(x-1)}$.

d Use, and describe, a subsequent translation to sketch $y = \frac{1}{2}e^{2(x-1)} - 2$.

17 **a** Interpret the transformation from $y = e^x$ to $y = e^{x+2}$ as a translation. Then interpret it as a dilation.

b Interpret the transformation from $y = e^x$ to $y = 2e^x$ as a dilation. Then interpret it as a translation by first writing the coefficient 2 as $e^{\log_e 2}$.

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5B Differentiation of exponential functions

We can now develop the calculus of functions involving e^x, picking up the story at differentiation, where two standard forms were established in Chapter 9 (Year 11),

$$\frac{d}{dx}e^x = e^x \qquad \text{and} \qquad \frac{d}{dx}e^{ax+b} = ae^{ax+b}.$$

Using the two standard forms

The second standard form above requires the chain rule with $u = ax + b$. It is proven again in worked Example 5 **c** below.

Example 4 5B

Differentiate:

a $y = e^x + e^{-x}$ **b** $y = 5e^{4x-3}$ **c** $y = e^{2-\frac{1}{2}x}$ **d** $y = \sqrt{e^x} + \dfrac{1}{\sqrt{e^x}}$

SOLUTION

a Given $y = e^x + e^{-x}$.
For e^{-x}, $a = -1$ and $b = 0$,
so $y' = e^x - e^{-x}$.

b Given $y = 5e^{4x-3}$.
Here $a = 4$ and $b = -3$,
so $y' = 20e^{4x-3}$.

c Given $y = e^{2-\frac{1}{2}x}$.
Here $a = -\frac{1}{2}$ and $b = 2$,
so $y' = -\frac{1}{2}e^{2-\frac{1}{2}x}$.

d Here $y = \sqrt{e^x} + \dfrac{1}{\sqrt{e^x}}$
$y = e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}$,
so $y' = \frac{1}{2}e^{\frac{1}{2}x} - \frac{1}{2}e^{-\frac{1}{2}x}$.

Differentiating using the chain rule

The chain rule can be applied in the usual way. As always, the full setting out should continue to be used until readers are very confident with missing some of the steps.

Example 5 5B

Use the chain rule to differentiate:

a $y = e^{1-x^2}$ **b** $y = (e^{2x} - 3)^4$ **c** $y = e^{ax+b}$ (the standard form)

SOLUTION

a Here $y = e^{1-x^2}$.
Applying the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= -2xe^{1-x^2}.$$

Let $u = 1 - x^2$.
Then $y = e^u$.
Hence $\dfrac{du}{dx} = -2x$
and $\dfrac{dy}{du} = e^u$.

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b Here $y = (e^{2x} - 3)^4$.
Applying the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= 4(e^{2x} - 3)^3 \times 2e^{2x}\\ &= 8e^{2x}(e^{2x} - 3)^3.\end{aligned}$$

Let $u = e^{2x} - 3$.
Then $y = u^4$.
Hence $\dfrac{du}{dx} = 2e^{2x}$
and $\dfrac{dy}{du} = 4u^3$.

c Here $y = e^{ax+b}$.
Applying the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= ae^{ax+b}.\end{aligned}$$

Let $u = ax + b$.
Then $y = e^u$.
Hence $\dfrac{du}{dx} = a$
and $\dfrac{dy}{du} = e^u$.

A formula for the chain rule

Some people prefer to learn a formula for chain rule differentiation that can be used for part **a** above. The formula can be written in two ways, using u and using $f(x)$,

$$\frac{d}{dx}e^u = e^u\frac{du}{dx} \qquad \text{OR} \qquad \frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x).$$

This is not recommended, particularly at first. But it is perfectly valid — use one or the other form if you prefer. In the next worked example, part **a** of the previous example is done again using both forms of the formula. Make sure that you are using the right formula, and that you show at least u or $f(x)$ on the right.

Notice that part **b** requires the formula for differentiating powers of functions of x, as reviewed in Section 4I,

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx} \qquad \text{OR} \qquad \frac{d}{dx}\big(f(x)\big)^n = n\big(f(x)\big)^{n-1}f'(x).$$

Example 6 — 5B

Use the chain rule, with a shorter setting out, to differentiate:

a $y = e^{1-x^2}$

b $y = (e^{2x} - 3)^4$

SOLUTION

a
$$\begin{aligned}y &= e^{1-x^2}\\ y' &= -2xe^{1-x^2}\end{aligned}$$

Let $u = 1 - x^2$.
Then $\dfrac{du}{dx} = -2x$
$\dfrac{d}{dx}e^u = e^u\dfrac{du}{dx}$.

OR

Let $f(x) = 1 - x^2$.
Then $f'(x) = -2x$
$\dfrac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$.

b
$$\begin{aligned}y &= (e^{2x} - 3)^4\\ y' &= 8e^{2x} \times (e^{2x} - 3)^3\end{aligned}$$

Let $u = e^{2x} - 3$.
Then $\dfrac{du}{dx} = 2e^{2x}$
$\dfrac{d}{dx}u^4 = 4u^3\dfrac{du}{dx}$.

OR

Let $f(x) = e^{2x} - 3$.
Then $f'(x) = 2e^{2x}$
$\dfrac{d}{dx}\big(f(x)\big)^4 = 4\big(f(x)\big)^3f'(x)$.

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2 THREE STANDARD DERIVATIVES FOR EXPONENTIAL FUNCTIONS

- $\dfrac{d}{dx}e^x = e^x$
- $\dfrac{d}{dx}e^{ax+b} = ae^{ax+b}$
- $\dfrac{d}{dx}e^u = e^u\dfrac{du}{dx}$ OR $\dfrac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$

Using the product rule

A function such as $y = x^3e^x$ is the product of the two functions $u = x^3$ and $v = e^x$. Thus it can be differentiated by the product rule.

Often the result can be factored, allowing any stationary points to be found.

Example 7 5B

Find the derivatives of these functions. Then factor the derivative and write down all the stationary points.

a $y = x^3e^x$

b $y = xe^{5x-2}$

SOLUTION

a Here $y = x^3e^x$.

Applying the product rule,

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= e^x \times 3x^2 + x^3 \times e^x,$$

and taking out the common factor x^2e^x,

$$\frac{dy}{dx} = x^2e^x(3 + x).$$

Hence $\dfrac{dy}{dx}$ has zeroes at $x = 0$ and $x = -3$,

and the stationary points are $(0, 0)$ and $(-3, -27e^{-3})$.

Let $u = x^3$
and $v = e^x$.
Then $\dfrac{du}{dx} = 3x^2$
and $\dfrac{dv}{dx} = e^x$.

b Here $y = xe^{5x-2}$.

Applying the product rule,

$$y' = vu' + uv'$$
$$= e^{5x-2} \times 1 + x \times 5e^{5x-2},$$

and taking out the common factor e^{5x-2},

$$y' = e^{5x-2}(1 + 5x).$$

Hence y' has a zero at $x = -\frac{1}{5}$,

and the stationary point is $\left(-\frac{1}{5}, -\frac{1}{5}e^{-3}\right)$.

Let $u = x$
and $v = e^{5x-2}$.
Then $u' = 1$
and $v' = 5e^{5x-2}$.

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Using the quotient rule

A function such as $y = \dfrac{e^{5x}}{x}$ is the quotient of the two functions $u = e^{5x}$ and $v = x$. Thus it can be differentiated by the quotient rule.

Example 8 5B

Differentiate these functions, then find the x-values all stationary points.

a $\dfrac{e^{5x}}{x}$ **b** $\dfrac{e^x}{1 - x^2}$

SOLUTION

a Let $y = \dfrac{e^{5x}}{x}$. Then applying the quotient rule,

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} = \frac{5xe^{5x} - e^{5x}}{x^2},$$

Let $u = e^{5x}$
and $v = x$.
Then $\dfrac{du}{dx} = 5e^{5x}$
and $\dfrac{dv}{dx} = 1$.

and taking out the common factor e^{5x} in the numerator,

$$\frac{dy}{dx} = \frac{e^{5x}(5x - 1)}{x^2}.$$

Hence there is a stationary point where $x = \frac{1}{5}$.

b Let $y = \dfrac{e^x}{1 - x^2}$. Then applying the quotient rule,

$$y' = \frac{vu' - uv'}{v^2} = \frac{(1 - x^2)e^x + 2xe^x}{(1 - x^2)^2},$$

Let $u = e^x$
and $v = 1 - x^2$.
Then $u' = e^x$
and $v' = -2x$.

and taking out the common factor e^x in the numerator,

$$y' = \frac{e^x(1 + 2x - x^2)}{(1 - x^2)^2}.$$

Hence there is a stationary point where $x^2 - 2x - 1 = 0$,
and calculating $\triangle = 8$ first, $x = 1 + \sqrt{2}$ or $x = 1 - \sqrt{2}$.

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Exercise 5B

FOUNDATION

Technology: Programs that perform algebraic differentiation can be used to confirm the answers to many of these questions.

1 Use the standard form $\frac{d}{dx}e^{ax+b} = ae^{ax+b}$ to differentiate:

a $y = e^{7x}$ **b** $y = 4e^{3x}$ **c** $y = -e^{5x}$
d $y = 6e^{\frac{1}{3}x}$ **e** $y = -e^{-7x}$ **f** $y = -\frac{1}{2}e^{-2x}$

2 Use the same standard form to differentiate:

a $y = e^{x-3}$ **b** $y = e^{3x+4}$ **c** $y = e^{2x-1}$
d $y = e^{4x-3}$ **e** $y = e^{-3x+4}$ **f** $y = e^{-2x-7}$

3 Differentiate:

a $y = e^{x} + e^{-x}$ **b** $y = e^{2x} - e^{-3x}$ **c** $y = \dfrac{e^{x} - e^{-x}}{2}$
d $y = \dfrac{e^{x} + e^{-x}}{3}$ **e** $y = \dfrac{e^{2x}}{2} + \dfrac{e^{3x}}{3}$ **f** $y = \dfrac{e^{4x}}{4} + \dfrac{e^{5x}}{5}$

4 Use the index laws to write each expression as a single power of e, then differentiate it.

a $y = e^{x} \times e^{2x}$ **b** $y = e^{3x} \times e^{-x}$ **c** $y = (e^{x})^{2}$ **d** $y = (e^{2x})^{3}$
e $y = \dfrac{e^{4x}}{e^{x}}$ **f** $y = \dfrac{e^{x}}{e^{2x}}$ **g** $y = \dfrac{1}{e^{3x}}$ **h** $y = \dfrac{1}{e^{5x}}$

5 **a** **i** For the function $f(x) = e^{-x}$, find $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$.
ii What is the pattern in these derivatives?
b **i** For the function $f(x) = e^{2x}$, find $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$.
ii What is the pattern in these derivatives?

DEVELOPMENT

6 Expand the brackets and then differentiate:

a $e^{x}(e^{x} + 1)$ **b** $e^{-x}(2e^{-x} - 1)$ **c** $(e^{x} + 1)^{2}$
d $(e^{x} + 3)^{2}$ **e** $(e^{x} - 1)^{2}$ **f** $(e^{x} - 2)^{2}$
g $(e^{x} + e^{-x})(e^{x} - e^{-x})$ **h** $(e^{5x} + e^{-5x})(e^{5x} - e^{-5x})$

7 Use the chain rule with full setting-out to differentiate:

a $y = e^{2x+1}$ **b** $y = e^{x^{2}}$ **c** $y = e^{-\frac{1}{2}x^{2}}$ **d** $y = e^{x^{2}+1}$
e $y = e^{1-x^{2}}$ **f** $y = e^{x^{2}+2x}$ **g** $y = e^{6+x-x^{2}}$ **h** $y = \frac{1}{2}e^{3x^{2}-2x+1}$

8 Use the product rule to differentiate:

a $y = xe^{x}$ **b** $y = xe^{-x}$ **c** $y = (x - 1)e^{x}$ **d** $y = (x + 1)e^{3x-4}$
e $y = x^{2}e^{-x}$ **f** $y = (2x - 1)e^{2x}$ **g** $y = (x^{2} - 5)e^{x}$ **h** $y = x^{3}e^{2x}$

9 Use the quotient rule to differentiate:

a $y = \dfrac{e^{x}}{x}$ **b** $y = \dfrac{x}{e^{x}}$ **c** $y = \dfrac{e^{x}}{x^{2}}$ **d** $y = \dfrac{x^{2}}{e^{x}}$
e $y = \dfrac{e^{x}}{x + 1}$ **f** $y = \dfrac{x + 1}{e^{x}}$ **g** $y = \dfrac{x - 3}{e^{2x}}$ **h** $y = \dfrac{1 - x^{2}}{e^{x}}$

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10 Expand and simplify each expression, then differentiate it.

a $(e^x + 1)(e^x + 2)$ **b** $(e^{2x} + 3)(e^{2x} - 2)$ **c** $(e^{-x} + 2)(e^{-x} + 4)$

d $(e^{-3x} - 1)(e^{-3x} - 5)$ **e** $(e^{2x} + 1)(e^x + 1)$ **f** $(e^{3x} - 1)(e^{-x} + 4)$

11 Use the chain rule to differentiate:

a $y = (1 - e^x)^5$ **b** $y = (e^{4x} - 9)^4$ **c** $y = \dfrac{1}{e^x - 1}$ **d** $y = \dfrac{1}{(e^{3x} + 4)^2}$

12 **a** Show by substitution that the function $y = e^{5x}$ satisfies the equation $\dfrac{dy}{dx} = 5y$.

b Show by substitution that the function $y = 3e^{2x}$ satisfies the equation $\dfrac{dy}{dx} = 2y$.

c Show by substitution that the function $y = 5e^{-4x}$ satisfies the equation $\dfrac{dy}{dx} = -4y$.

d Show by substitution that the function $y = 2e^{-3x}$ satisfies the equation $\dfrac{dy}{dx} = -3y$.

13 Determine the first and second derivatives of each function below. Then evaluate both derivatives at the value given.

a $f(x) = e^{2x+1}$ at $x = 0$ **b** $f(x) = e^{-3x}$ at $x = 1$

c $f(x) = xe^{-x}$ at $x = 2$ **d** $f(x) = e^{-x^2}$ at $x = 0$

14 Use the standard form $\dfrac{d}{dx}e^{ax+b} = ae^{ax+b}$ to differentiate:

a $y = e^{ax}$ **b** $y = e^{-kx}$ **c** $y = Ae^{kx}$ **d** $y = Be^{-\ell x}$

e $y = e^{px+q}$ **f** $y = Ce^{px+q}$ **g** $y = \dfrac{e^{px} + e^{-qx}}{r}$ **h** $\dfrac{e^{ax}}{a} + \dfrac{e^{-px}}{p}$

15 Use the product, quotient and chain rules as appropriate to differentiate:

a $y = (e^x + 1)^3$ **b** $y = (e^x + e^{-x})^4$ **c** $y = (1 + x^2)e^{1+x}$

d $y = (x^2 - x)e^{2x-1}$ **e** $y = \dfrac{e^x}{e^x + 1}$ **f** $y = \dfrac{e^x + 1}{e^x - 1}$

16 Write each expression as the sum of powers of e, then differentiate it.

a $y = \dfrac{e^x + 1}{e^x}$ **b** $y = \dfrac{e^{2x} + e^x}{e^x}$ **c** $y = \dfrac{2 - e^x}{e^{2x}}$

d $y = \dfrac{3 + e^x}{e^{4x}}$ **e** $y = \dfrac{e^x + e^{2x} - 3e^{4x}}{e^x}$ **f** $y = \dfrac{e^{2x} + 2e^x + 1}{e^{2x}}$

CHALLENGE

17 Differentiate these functions.

a $y = \sqrt{e^x}$ **b** $y = \sqrt[3]{e^x}$ **c** $y = \dfrac{1}{\sqrt{e^x}}$ **d** $y = \dfrac{1}{\sqrt[3]{e^x}}$

e $e^{\sqrt{x}}$ **f** $e^{-\sqrt{x}}$ **g** $e^{\frac{1}{x}}$ **h** $e^{-\frac{1}{x}}$

i $e^{x-\frac{1}{x}}$ **j** e^{e^x}

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18 Define the two functions $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

a Show that $\dfrac{d}{dx}\cosh x = \sin hx$ and $\dfrac{d}{dx}\sinh x = \cosh x$.

b Find the second derivative of each function, and show that they both satisfy $y'' = y$.

c Show that $\cosh^2 x - \sinh^2 x = 1$.

19 **a** Show that $y = 2e^{3x}$ is a solution of each equation by substituting separately into the LHS and RHS:

i $y' = 3y$ **ii** $y'' - 9y = 0$

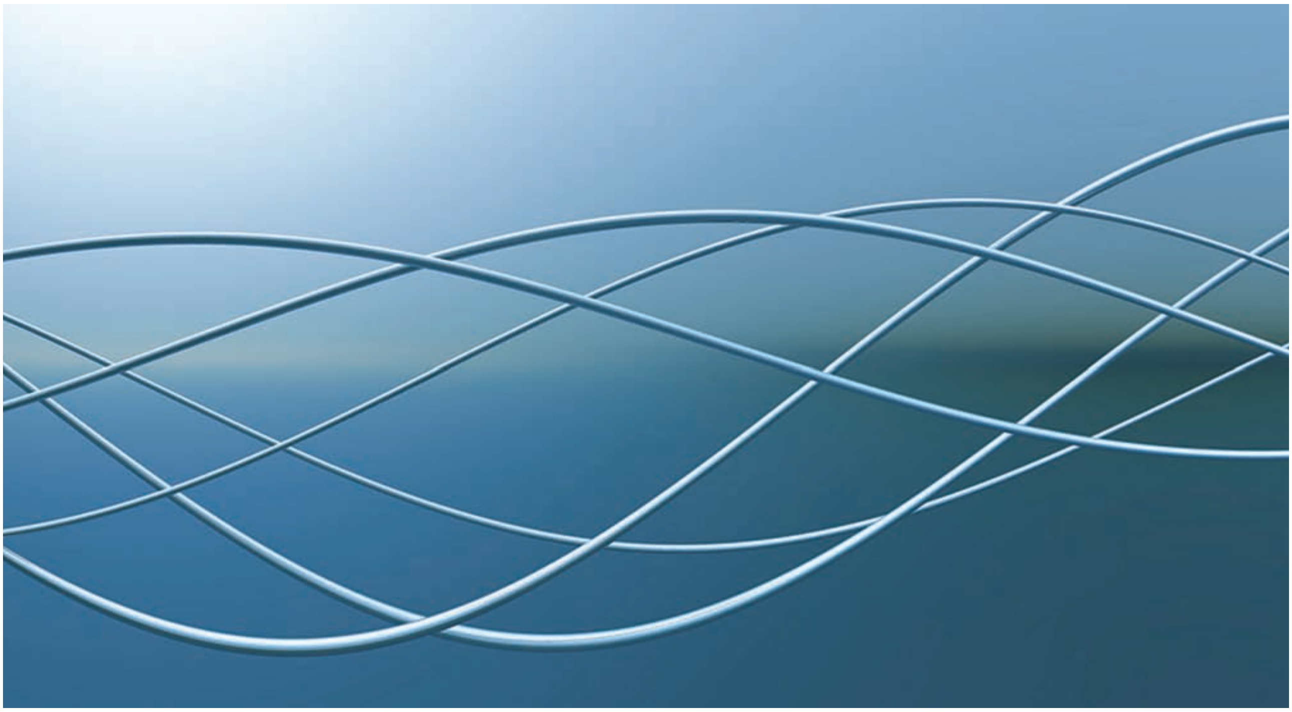
b Show that $y = \frac{1}{2}e^{-3x} + 4$ is a solution of $\dfrac{dy}{dx} = -3(y - 4)$ by substituting y into each side of the equation.

c Show by substitution that each function is a solution of the equation $y'' + 2y' + y = 0$.

i $y = e^{-x}$ **ii** $y = xe^{-x}$

20 Find the values of λ that make $y = e^{\lambda x}$ a solution of:

a $y'' + 3y' - 10y = 0$ **b** $y'' + y' - y = 0$



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5C Applications of differentiation

Differentiation can now be applied in the usual ways to examine functions involving e^x. Sketching of such curves is an important application. Some of these sketches require some subtle limits that would normally be given if a question needed them.

The graphs of e^x and e^{-x}

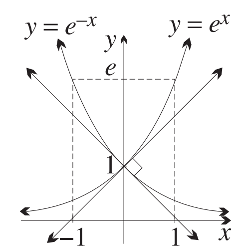
The graphs of $y = e^x$ and $y = e^{-x}$ are the fundamental graphs of this chapter. Because x is replaced by $-x$ in the second equation, the two graphs are reflections of each other in the y-axis.

For $y = e^x$:

x	-2	-1	0	1	2
y	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2

For $y = e^{-x}$:

x	-2	-1	0	1	2
y	e^2	e	1	$\frac{1}{e}$	$\frac{1}{e^2}$



The two curves cross at $(0, 1)$. The gradient of $y = e^x$ at $(0, 1)$ is 1, so by reflection, the gradient of $y = e^{-x}$ at $(0, 1)$ is -1. This means that the curves are perpendicular at their point of intersection.

As remarked earlier, the function $y = e^{-x}$ is just as important as $y = e^x$ in applications. It describes a great many physical situations where a quantity 'dies away exponentially', like the dying away of the sound of a plucked string.

An example of curve sketching

The following curve-sketching example illustrates the use of the six steps of our informal curve-sketching menu in the context of exponential functions. One special limit is given in part **d** so that the sketch may be completed.

Example 9 5C

Sketch the graph of $y = xe^{-x}$ after carrying out these steps.

a Write down the domain.

b Test whether the function is even or odd or neither.

c Find any zeroes of the function, and examine its sign.

d Examine the function's behaviour as $x \to \infty$ and as $x \to -\infty$, noting any asymptotes. (You may assume that as $x \to \infty$, $xe^{-x} \to 0$.)

e Find any stationary points and examine their nature.

f Find any points of inflection, and examine the concavity.

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SOLUTION

a The domain of $y = xe^{-x}$ is the whole real number line.

b $f(-x) = -xe^{x}$, which is neither $f(x)$ nor $-f(x)$, so the function is neither even nor odd.

c The only zero is $x = 0$. From the table of signs, y is positive for $x > 0$ and negative for $x < 0$.

x	-1	0	1
y	$-e$	0	e^{-1}
sign	$-$	0	$+$

d As given in the question, $y \to 0$ as $x \to \infty$, so the x-axis is a horizontal asymptote on the right. Also, $y \to -\infty$ as $x \to -\infty$.

e Differentiating using the product rule,

$$f'(x) = vu' + uv'$$
$$= e^{-x} - xe^{-x}$$
$$= e^{-x}(1 - x).$$

Let $u = x$
and $v = e^{-x}$.
Then $u' = 1$
and $v' = -e^{-x}$.

Hence $f'(x) = 0$ when $x = 1$ (notice that e^{-x} can never be zero), so $(1, \frac{1}{e})$ is the only stationary point.

Differentiating again by the product rule,

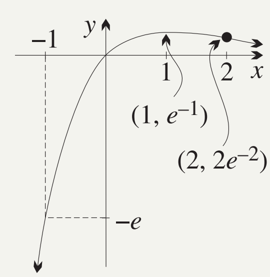
$$f''(x) = vu' + uv'$$
$$= -e^{-x} - (1 - x)e^{-x}$$
$$= e^{-x}(x - 2),$$

Let $u = 1 - x$
and $v = e^{-x}$.
Then $u' = -1$
and $v' = -e^{-x}$.

so $f''(1) = -e^{-1} < 0$, and $(1, e^{-1})$ is thus a maximum turning point.

f $f''(x) = e^{-x}(x - 2)$ has a zero at $x = 2$, and taking test values around $x = 2$,

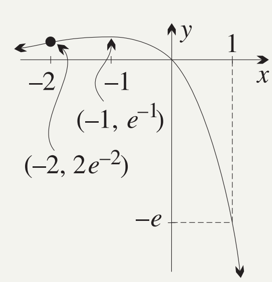
x	0	2	3
$f''(x)$	-2	0	e^{-3}
	$\frown$	$\cdot$	$\smile$



Thus there is an inflection at $(2, 2e^{-2}) \doteqdot (2, 0.27)$.
The curve is concave down for $x < 2$ and concave up for $x > 2$.

Example 10 5C

[Transforming graphs]
Use a suitable transformation of the graph sketched in the previous worked example to sketch $y = -xe^{x}$.



SOLUTION

$y = xe^{-x}$ becomes $y = -xe^{x}$ when x is replaced by $-x$. Graphically, this transformation is a reflection in the y-axis, hence the new graph is as sketched to the right.

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A difficulty with the limits of xe^x and xe^{-x}

Sketching the graph of $y = xe^{-x}$ above required knowing the behaviour of xe^{-x} as $x \to \infty$. This limit is puzzling, because when x is a large number, e^{-x} is a small positive number, and the product of a large number and a small number could be large, small, or anything in between.

In fact, e^{-x} gets small as $x \to \infty$ much more quickly than x gets large, and the product xe^{-x} gets small. The technical term for this is that e^{-x} *dominates* x. A table of values should make it reasonably clear that $\lim_{x\to\infty} xe^{-x} = 0$.

x	0	1	2	3	4	5	6	7	$\cdots$
xe^{-x}	0	$\frac{1}{e}$	$\frac{2}{e^2}$	$\frac{3}{e^3}$	$\frac{4}{e^4}$	$\frac{5}{e^5}$	$\frac{6}{e^6}$	$\frac{7}{e^7}$	$\cdots$
approx	0	0.37	0.27	0.15	0.073	0.034	0.015	0.006	$\cdots$

Limits such as this would normally be given in any question where they were needed.

Similarly, when x is a large negative number, e^x is a very small number, so it is unclear whether xe^x is large or small. Again, e^x dominates x, meaning that $xe^x \to 0$ as $x \to -\infty$. A similar table should make this reasonably obvious.

x	0	-1	-2	-3	-4	-5	-6	-7	$\cdots$
xe^x	0	$-\frac{1}{e}$	$-\frac{2}{e^2}$	$-\frac{3}{e^3}$	$-\frac{4}{e^4}$	$-\frac{5}{e^5}$	$-\frac{6}{e^6}$	$-\frac{7}{e^7}$	$\cdots$
approx	0	-0.37	-0.27	-0.15	-0.073	-0.034	-0.015	-0.006	$\cdots$

Again, this limit would normally be given in any question where it is needed.

Exercise 5C

FOUNDATION

Technology: Graphing programs can be used in this exercise to sketch the curves and then investigate the effects on the curve of making small changes in the equations. It is advisable, however, to puzzle out most of the graphs first using the standard methods of the curve-sketching menu.

1 **a** Find the y-coordinate of the point A on the curve $y = e^{2x-1}$ where $x = \frac{1}{2}$.
b Find the derivative of $y = e^{2x-1}$, and show that the gradient of the tangent at A is 2.
c Hence find the equation of the tangent at A, and prove that it passes through O.

2 **a** Write down the coordinates of the point R on the curve $y = e^{3x+1}$ where $x = -\frac{1}{3}$.
b Find $\frac{dy}{dx}$ and hence show that the gradient of the tangent at R is 3.
c What is the gradient of the normal at R?
d Hence find the equation of the normal at R in general form.

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3 a Find the gradient of the tangent to $y = e^{-x}$ at the point $P(-1, e)$.
b Thus write down the gradient of the normal at this point.
c Hence determine the equation of this normal.
d Find the x- and y-intercepts of the normal.
e Find the area of the triangle whose vertices lie at the intercepts and the origin.

4 a Use the derivative to find the gradient of the tangent to $y = e^x$ at $B(0, 1)$.
b Hence find the equation of this tangent and show that it meets the x-axis at $F(-1, 0)$.
c Use the derivative to find the gradient of the tangent to $y = e^{-x}$ at $B(0, 1)$.
d Hence find the equation of this tangent and show that it meets the x-axis at $G(1, 0)$.
e Sketch $y = e^x$ and $y = e^{-x}$ on the same set of axes, showing the two tangents.
f What sort of triangle is $\triangle BFG$, and what is its area?

5 a Find the gradient of the tangent to $y = x - e^x$ at $x = 1$.
b Write down the equation of the tangent, and show that it passes through the origin.

DEVELOPMENT

6 a Find the first and second derivatives for the curve $y = x - e^x$.
b Deduce that the curve is concave down for all values of x.
c Find any stationary points, then determine their nature using the second derivative.
d Sketch the curve and write down its range.
e Finally, sketch $y = e^x - x$ by recognising the simple transformation.

7 a Use the product rule to differentiate $y = (1 - x)e^x$.
b Find the equation of the tangent to $y = (1 - x)e^x$ at $x = -1$.
c Hence find the x-intercept of the tangent.

8 a Show that the equation of the tangent to $y = (x + 1)e^{-x}$ at $x = -1$ is $y = e(x + 1)$.
b Find the x-intercept and y-intercept of the tangent.
c Hence find the area of the triangle with its vertices at the two intercepts and the origin.

9 a Find the first and second derivatives of $y = e^{3x-6}$.
b Explain why every tangent to the curve has positive gradient, and why the curve is concave up at every point.
c Find the point on the curve where the gradient is 3.
d Find the gradients of the tangent and normal at the y-intercept.

10 a Use the chain rule to differentiate $y = e^{-x^2}$.
b Find the equation of the normal to $y = e^{-x^2}$ at the point where $x = 1$.
c Determine the x-intercept of the normal.

11 a Show that the equation of the tangent to $y = 1 - e^{-x}$ at the origin is $y = x$.
b Deduce the equation of the normal at the origin without further use of calculus.
c What is the equation of the asymptote of this curve?
d Sketch the curve, showing the points T and N where the tangent and normal respectively cut the asymptote.
e Find the area of $\triangle OTN$.

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12 **a** Show that the tangent to $y = e^x$ at $T(t, e^t)$ has gradient e^t.

b Find the equation of the tangent at $x = t$, and show that its x-intercept is $t - 1$.

c Compare this result with Question 4 above, and explain geometrically what is happening.

13 [Technology]

This question is intended to justify and extend the remarks made in the text about *dominance*. Any curve-sketching question would normally give these limits if they are required.

a Use your calculator to complete the table of values for $y = \dfrac{e^x}{x}$ to the right. (This table is intended to confirm that $\dfrac{e^x}{x} \to \infty$ as $x \to \infty$.)

x	2	5	10	20	40
y					

b Use your calculator to complete the table of values for $y = xe^x$ to the right. Then use the table to help you guess the value of $\lim\limits_{x \to -\infty} xe^x$.

x	-2	-5	-10	-20	-40
y					

c Use your calculator to complete the table of values for $y = \dfrac{e^{-x}}{x}$ to the right. (This table is intended to confirm that $\dfrac{e^{-x}}{x} \to -\infty$ as $x \to -\infty$.)

x	-2	-5	-10	-20	-40
y					

d Use your calculator to complete the table of values for $y = xe^{-x}$ to the right. Then use the table to help you guess the value of $\lim\limits_{x \to \infty} xe^{-x}$.

x	2	5	10	20	40
y					

These questions can all be easily extended and plotted using a spreadsheet.

14 Consider the curve $y = xe^x$.

a Where is the function zero, positive and negative? Is it even, odd or neither?

b Show that $y' = (1 + x)e^x$ and $y'' = (2 + x)e^x$.

c Show that there is one stationary point, and determine its nature.

d Find the coordinates of the lone point of inflection.

e What happens to y, y' and y'' as $x \to \infty$?

f Given that $y \to 0$ as $x \to -\infty$, sketch the curve, then write down its range.

g Hence also sketch $y = -xe^{-x}$ by recognising the simple transformation.

15 The function $y = e^{-\frac{1}{2}x^2}$ is the basis of the normal distribution in Chapter 10.

a Show that the function is even. When is it zero, positive and negative?

b Show that $y' = -xe^{-\frac{1}{2}x^2}$ and $y'' = (x^2 - 1)e^{-\frac{1}{2}x^2}$.

c Show that this curve has a maximum turning point at its y-intercept.

d Find the two points of inflection.

e Explain why $y \to 0$ as $x \to \infty$ and as $x \to -\infty$.

f Sketch the graph and write down its range.

g Hence also sketch $y = 1 + e^{-\frac{1}{2}x^2}$ by recognising the simple transformation.

16 Consider the function $y = (1 - x)e^x$.

a Find the zero and draw up a table of signs.

b Show that $y' = -xe^x$ and $y'' = -(x + 1)e^x$.

c Show that this curve has a maximum turning point at its y-intercept, and a point of inflection at $(-1, 2e^{-1})$.

d What happens to y, y' and y'' as $x \to \infty$?

e Given that $y \to 0$ as $x \to -\infty$, sketch the graph and write down its range.

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CHALLENGE

17 We define the new function $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

a Show that $y = \cosh x$ is an even function, and is always positive.

b Find $\dfrac{dy}{dx}$ and show there is a stationary point at the y-intercept.

c Show that the function is always concave up.

d What happens to y as $x \to \infty$?

e Sketch the graph of $y = \cosh x$.

18 **a** Given that $y = x^2 e^{-x}$, show that $y' = x(2 - x)e^{-x}$ and $y'' = (2 - 4x + x^2)e^{-x}$.

b Show that the function has a minimum turning point at the origin and a maximum turning point at $(2, 4e^{-2})$.

c **i** Show that $y'' = 0$ at $x = 2 - \sqrt{2}$ and $x = 2 + \sqrt{2}$.

ii Use a table of values for y'' to show that there are inflection points at these values.

d Given that $y \to 0$ as $x \to \infty$, sketch the graph and write down its range.

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5D Integration of exponential functions

Finding primitives is the reverse of differentiation. Thus the new standard forms for differentiation can now be reversed to provide standard forms for integration.

Standard forms for integration

Reversing the standard forms for differentiating exponential functions gives the standard forms for integrating them.

Reversing $\dfrac{d}{dx}e^x = e^x$ gives $\displaystyle\int e^x\,dx = e^x + C$, for some constant C.

Reversing $\dfrac{d}{dx}e^{ax+b} = a\,e^{ax+b}$ gives $\displaystyle\int a\,e^{ax+b}\,dx = e^{ax+b}$,

and dividing through by a, $\displaystyle\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$, for some constant C.

3 STANDARD FORMS FOR INTEGRATION

- $\displaystyle\int e^x\,dx = e^x + C$, for some constant C
- $\displaystyle\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$, for some constant C

There is also an associated formula for the reverse chain rule, but it is not required in the course. For reference, this formula has the usual two forms:

$$\int e^u \frac{du}{dx}\,dx = e^u + C \qquad \text{OR} \qquad \int e^{f(x)} f'(x)\,dx = e^{f(x)} + C.$$

Example 11 5D

Find these indefinite integrals.

a $\displaystyle\int e^{3x+2}\,dx$ **b** $\displaystyle\int (1 - x + e^x)\,dx$

SOLUTION

a $\displaystyle\int e^{3x+2}\,dx = \tfrac{1}{3}e^{3x+2} + C$ ($a = 3$ and $b = 2$)

b $\displaystyle\int (1 - x + e^x)\,dx = x - \tfrac{1}{2}x^2 + e^x + C$ (integrating each term separately)

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Definite integrals

Definite integrals are evaluated in the usual way by finding the primitive and substituting.

Example 12 5D

Evaluate these definite integrals.

a $\displaystyle\int_0^2 e^x\,dx$

b $\displaystyle\int_2^3 e^{5-2x}\,dx$

SOLUTION

a
$$\begin{aligned}\int_0^2 e^x\,dx &= \Big[e^x\Big]_0^2\\ &= e^2 - e^0\\ &= e^2 - 1\end{aligned}$$

b
$$\begin{aligned}\int_2^3 e^{5-2x}\,dx &= -\tfrac{1}{2}\Big[e^{5-2x}\Big]_2^3 \qquad (a = -2 \text{ and } b = 5)\\ &= -\tfrac{1}{2}(e^{-1} - e)\\ &= -\frac{1}{2}\left(\frac{1}{e} - e\right)\\ &= \frac{e^2 - 1}{2e}\end{aligned}$$

Given the derivative, find the function

As before, if the derivative of a function is known, and the value of the function at one point is also known, then the whole function can be determined.

Example 13 5D

It is known that $f'(x) = e^x$ and that $f(1) = 0$.

a Find the original function $f(x)$.

b Hence find $f(0)$.

SOLUTION

a It is given that $f'(x) = e^x$.

Taking the primitive, $f(x) = e^x + C$, for some constant C.

It is known that $f(1) = 0$, so substituting $x = 1$,

$$0 = e^1 + C$$
$$C = -e.$$

Hence $f(x) = e^x - e$.

b Substituting $x = 0$ into this function,

$$\begin{aligned}f(0) &= e^0 - e\\ &= 1 - e.\end{aligned}$$

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Example 14 5D

a If $f'(x) = 1 + 2e^{-x}$ and $f(0) = 1$, find $f(x)$.
b Hence find $f(1)$.

SOLUTION

a It is given that $f'(x) = 1 + 2e^{-x}$.
Taking the primitive, $f(x) = x - 2e^{-x} + C$, for some constant C.
It is known that $f(0) = 1$, so substituting $x = 0$,

$$1 = 0 - 2e^{0} + C$$
$$1 = 0 - 2 + C$$
$$C = 3.$$

Hence $f(x) = x - 2e^{-x} + 3$.

b Substituting $x = 1$ into this function,

$$\begin{aligned} f(1) &= 1 - 2e^{-1} + 3 \\ &= 4 - 2e^{-1}. \end{aligned}$$

Given a derivative, find an integral

The result of any differentiation can be reversed. This often allows a new primitive to be found.

Example 15 5D

a Use the chain rule to differentiate e^{x^2}.
b Hence find $\int_{-1}^{1} 2xe^{x^2}\,dx$.

SOLUTION

a Let $y = e^{x^2}$.
Applying the chain rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 2xe^{x^2}. \end{aligned}$$

Let $u = x^2$.
Then $y = e^{u}$.
Hence $\dfrac{du}{dx} = 2x$
and $\dfrac{dy}{du} = e^{u}$.

b From part **a**, $\dfrac{d}{dx}e^{x^2} = 2xe^{x^2}$.
Reversing this to give a primitive,

$$\int 2xe^{x^2}\,dx = e^{x^2}.$$

Hence

$$\begin{aligned} \int_{-1}^{1} 2xe^{x^2}\,dx &= \Big[e^{x^2}\Big]_{-1}^{1} \\ &= e^{1} - e^{1} \\ &= 0. \end{aligned}$$

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Note: The fact that the definite integral is zero could have been discovered without ever finding the primitive. The function $f(x) = xe^{x^2}$ is an odd function, because

$$\begin{aligned} f(-x) &= (-x)e^{(-x)^2} \\ &= -xe^{x^2} \\ &= -f(x). \end{aligned}$$

Hence the definite integral over the interval $[-1, 1]$ is zero.

Using a formula for the reverse chain rule

There are some situations where the reverse chain rule formula from Section 4I can be used.

Example 16 — 5D

Use the reverse chain rule formula to find $\displaystyle\int \frac{e^{2x}}{(1 - e^{2x})^3}\,dx$.

SOLUTION

$$\begin{aligned} &\int \frac{e^{2x}}{(1 - e^{2x})^3}\,dx \\ &= -\tfrac{1}{2}\int \frac{-2e^{2x}}{(1 - e^{2x})^3}\,dx \\ &= -\tfrac{1}{2} \times \left(-\tfrac{1}{2}\right) \times (1 - e^{2x})^{-2} \end{aligned}$$

Let $u = 1 - e^{2x}$.
Then $u' = -2e^{2x}$.

$$\int u^{-3}\frac{du}{dx}\,dx = \frac{u^{-2}}{-2}$$

OR

Let $f(x) = 1 - e^{2x}$.
Then $f'(x) = -2e^{2x}$.

$$\int \big(f(x)\big)^{-3} f'(x)\,dx = \frac{\big(f(x)\big)^{-2}}{-2}$$

$$= \frac{1}{4(1 - e^{2x})^2} + C, \quad \text{for some constant } C.$$

Exercise 5D

FOUNDATION

Technology: Some algebraic programs can display the primitive and evaluate the exact value of an integral. These can be used to check the questions in this exercise and also to investigate the effect of making small changes to the function or to the limits of integration.

1 Use the standard form $\displaystyle\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$ to find each indefinite integral.

a $\displaystyle\int e^{2x}\,dx$ b $\displaystyle\int e^{3x}\,dx$ c $\displaystyle\int e^{\frac{1}{3}x}\,dx$

d $\displaystyle\int e^{\frac{1}{2}x}\,dx$ e $\displaystyle\int 10e^{2x}\,dx$ f $\displaystyle\int 12e^{3x}\,dx$

2 Use the standard form $\displaystyle\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$ to find each indefinite integral.

a $\displaystyle\int e^{4x+5}\,dx$ b $\displaystyle\int e^{4x-2}\,dx$ c $\displaystyle\int 6e^{3x+2}\,dx$

d $\displaystyle\int 4e^{4x+3}\,dx$ e $\displaystyle\int e^{7-2x}\,dx$ f $\displaystyle\int \tfrac{1}{2}e^{1-3x}\,dx$

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3 Evaluate these definite integrals.

a $\int_0^1 e^x\,dx$ **b** $\int_1^2 e^x\,dx$ **c** $\int_{-1}^3 e^{-x}\,dx$

d $\int_{-2}^0 e^{-x}\,dx$ **e** $\int_0^2 e^{2x}\,dx$ **f** $\int_{-1}^1 e^{3x}\,dx$

g $\int_{-1}^2 20e^{-5x}\,dx$ **h** $\int_{-3}^1 8e^{-4x}\,dx$ **i** $\int_{-1}^3 9e^{6x}\,dx$

4 Evaluate these definite integrals.

a $\int_0^2 e^{x-1}\,dx$ **b** $\int_{-1}^1 e^{2x+1}\,dx$ **c** $\int_{-2}^0 e^{4x-3}\,dx$

d $\int_{-2}^{-1} e^{3x+2}\,dx$ **e** $\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{3-2x}\,dx$ **f** $\int_{-\frac{1}{3}}^{\frac{1}{3}} e^{2+3x}\,dx$

g $\int_1^2 6e^{3x+1}\,dx$ **h** $\int_2^3 12e^{4x-5}\,dx$ **i** $\int_1^2 12e^{8-3x}\,dx$

5 Express each function using negative indices instead of fractions, and hence find its primitive.

a $\dfrac{1}{e^x}$ **b** $\dfrac{1}{e^{2x}}$ **c** $\dfrac{1}{e^{3x}}$ **d** $-\dfrac{3}{e^{3x}}$ **e** $\dfrac{6}{e^{2x}}$ **f** $\dfrac{8}{e^{-2x}}$

6 **a** A function $f(x)$ has derivative $f'(x) = e^{2x}$. Find the equation of $f(x)$, which will involve an arbitrary constant.

b It is also known that $f(0) = -2$. Find the arbitrary constant and hence write down the equation of $f(x)$.

c Find $f(1)$ and $f(2)$.

DEVELOPMENT

7 Find $f(x)$ and then find $f(1)$, given that:

a $f'(x) = 1 + 2e^x$ and $f(0) = 1$ **b** $f'(x) = 1 - 3e^x$ and $f(0) = -1$

c $f'(x) = 2 + e^{-x}$ and $f(0) = 0$ **d** $f'(x) = 4 - e^{-x}$ and $f(0) = 2$

e $f'(x) = e^{2x-1}$ and $f\left(\frac{1}{2}\right) = 3$ **f** $f'(x) = e^{1-3x}$ and $f\left(\frac{1}{3}\right) = \frac{2}{3}$

g $f'(x) = e^{\frac{1}{2}x+1}$ and $f(-2) = -4$ **h** $f'(x) = e^{\frac{1}{3}x+2}$ and $f(-6) = 2$

8 Expand the brackets and then find primitives of:

a $e^x(e^x + 1)$ **b** $e^x(e^x - 1)$ **c** $e^{-x}(2e^{-x} - 1)$

d $(e^x + 1)^2$ **e** $(e^x + 3)^2$ **f** $(e^x - 1)^2$

g $(e^x - 2)^2$ **h** $(e^x + e^{-x})(e^x - e^{-x})$ **i** $(e^{5x} + e^{-5x})(e^{5x} - e^{-5x})$

9 Use the standard form $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$ to find these indefinite integrals.

a $\int e^{2x+b}\,dx$ **b** $\int e^{7x+q}\,dx$ **c** $\int e^{3x-k}\,dx$

d $\int e^{6x-\lambda}\,dx$ **e** $\int e^{ax+3}\,dx$ **f** $\int e^{sx+1}\,dx$

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g $\int e^{mx-2}\,dx$ **h** $\int e^{kx-1}\,dx$ **i** $\int pe^{px+q}\,dx$

j $\int me^{mx+k}\,dx$ **k** $\int Ae^{sx-t}\,dx$ **l** $\int Be^{kx-\ell}\,dx$

10 Express each function below as a power of e, and hence find its primitive.

a $\dfrac{1}{e^{x-1}}$ **b** $\dfrac{1}{e^{3x-1}}$ **c** $\dfrac{1}{e^{2x+5}}$

d $\dfrac{4}{e^{2x-1}}$ **e** $\dfrac{10}{e^{2-5x}}$ **f** $\dfrac{12}{e^{3x-5}}$

11 By writing each integrand as the sum of powers of e, find:

a $\int \dfrac{e^x+1}{e^x}\,dx$ **b** $\int \dfrac{e^{2x}+1}{e^x}\,dx$ **c** $\int \dfrac{e^x-1}{e^{2x}}\,dx$

d $\int \dfrac{e^x-3}{e^{3x}}\,dx$ **e** $\int \dfrac{2e^{2x}-3e^x}{e^{4x}}\,dx$ **f** $\int \dfrac{2e^x-e^{2x}}{e^{3x}}\,dx$

12 **a** Find y as a function of x if $y' = e^{x-1}$, and $y = 1$ when $x = 1$. What is the y-intercept of this curve?

b The gradient of a curve is given by $y' = e^{2-x}$, and the curve passes through the point $(0, 1)$. What is the equation of this curve? What is its horizontal asymptote?

c It is known that $f'(x) = e^x + \dfrac{1}{e}$ and that $f(-1) = -1$. Find $f(0)$.

d Given that $f'(x) = e^x - e^{-x}$ and that $y = f(x)$ is horizontal as it passes through the origin, find $f(x)$.

13 By first writing each integrand as a sum of powers of e, find:

a $\int_0^1 e^x(2e^x - 1)\,dx$ **b** $\int_{-1}^1 (e^x + 2)^2\,dx$

c $\int_0^1 (e^x - 1)(e^{-x} + 1)\,dx$ **d** $\int_{-1}^1 (e^{2x} + e^{-x})(e^{2x} - e^{-x})\,dx$

e $\int_0^1 \dfrac{e^{3x}+e^x}{e^{2x}}\,dx$ **f** $\int_{-1}^1 \dfrac{e^x-1}{e^{2x}}\,dx$

CHALLENGE

14 **a** **i** Differentiate e^{x^2+3}. **ii** Hence find $\int 2xe^{x^2+3}\,dx$.

b **i** Differentiate e^{x^2-2x+3}. **ii** Hence find $\int (x-1)e^{x^2-2x+3}\,dx$.

c **i** Differentiate e^{3x^2+4x+1}. **ii** Hence find $\int (3x+2)e^{3x^2+4x+1}\,dx$.

d **i** Differentiate $y = e^{x^3}$. **ii** Hence find $\int_{-1}^0 x^2e^{x^3}\,dx$.

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15 Write each integrand as a power of e, and hence find the indefinite integral.

a $\displaystyle\int \frac{1}{(e^x)^2}\,dx$ **b** $\displaystyle\int \frac{1}{(e^x)^3}\,dx$ **c** $\displaystyle\int \sqrt{e^x}\,dx$

d $\displaystyle\int \sqrt[3]{e^x}\,dx$ **e** $\displaystyle\int \frac{1}{\sqrt{e^x}}\,dx$ **f** $\displaystyle\int \frac{1}{\sqrt[3]{e^x}}\,dx$

16 **a** **i** Differentiate $y = xe^x - e^x$. **ii** Hence find $\displaystyle\int_0^2 xe^x\,dx$.

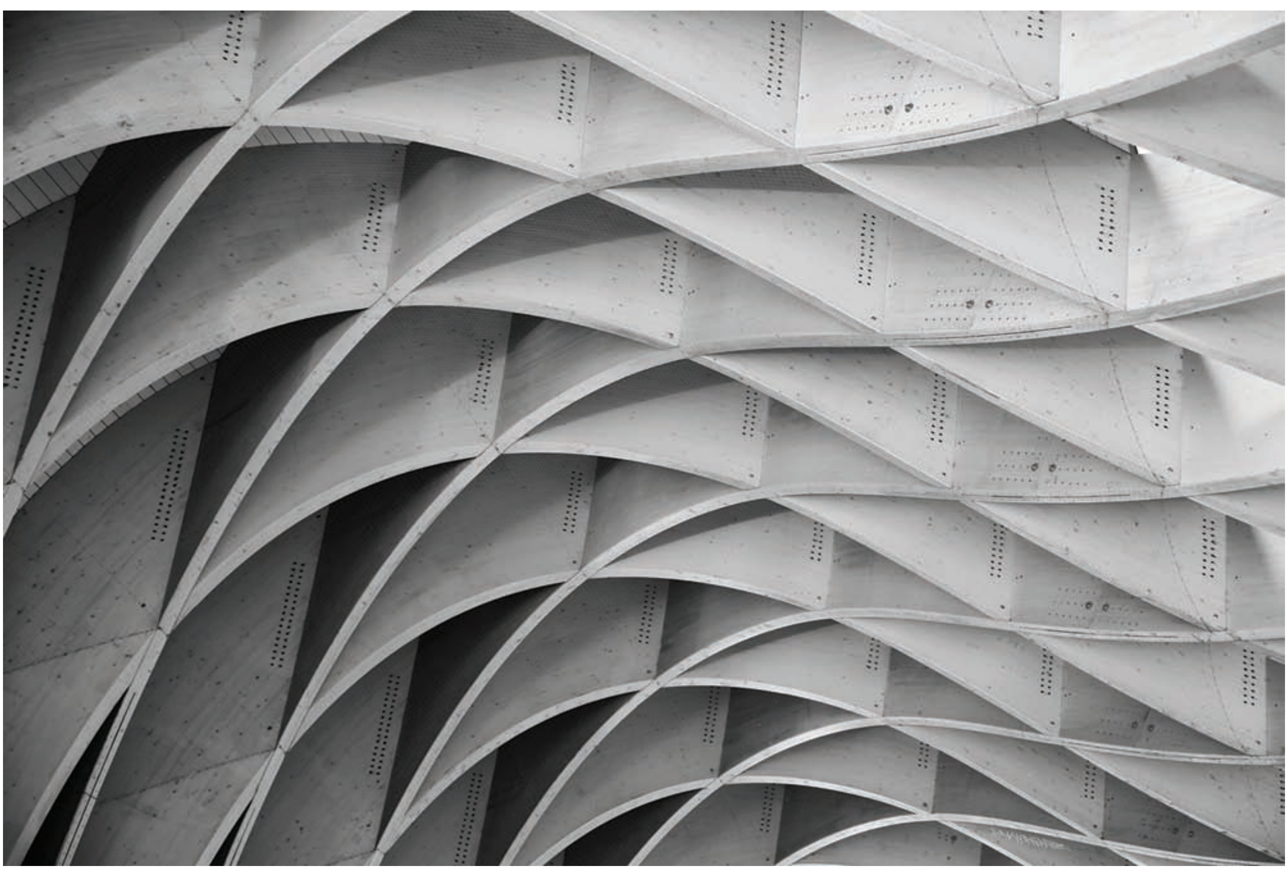
b **i** Differentiate $y = xe^{-x} + e^{-x}$. **ii** Hence find $\displaystyle\int_{-2}^0 xe^{-x}\,dx$.

17 By first simplifying each integrand, determine:

a $\displaystyle\int \frac{e^x - e^{-x}}{\sqrt{e^x}}\,dx$ **b** $\displaystyle\int \frac{e^x + e^{-x}}{\sqrt[3]{e^x}}\,dx$

18 **a** Show that $f(x) = xe^{-x^2}$ is an odd function.

b Hence evaluate $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} xe^{-x^2}\,dx$ without finding a primitive.



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5E Applications of integration

The normal methods of finding areas by integration can now be applied to functions involving e^x.

Finding the area between a curve and the *x*-axis

A sketch is essential here, because the definite integral attaches a negative sign to the area of any region below the x-axis (provided that the integral does not run backwards).

Example 17 5E

a Use shifting to sketch $y = e^x - e$, showing the intercepts and asymptote.

b Find the area of the region between this curve, the x-axis and the y-axis.

SOLUTION

a Move the graph of $y = e^x$ down e units.

To find the y-intercept, put $x = 0$,

then $y = e^0 - e$

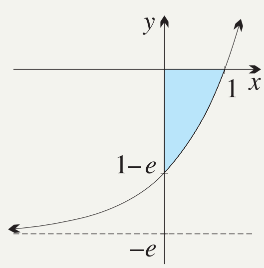
$= 1 - e$.

To find the x-intercept, put $y = 1$,

then $e^x = e$

$x = 1$.

The horizontal asymptote moves down to $y = -e$.



b $\displaystyle\int_0^1 (e^x - e)\,dx = \Big[e^x - ex\Big]_0^1$ (the number e is a constant)

$= (e^1 - e) - (e^0 - 0)$

$= (e - e) - (1 - 0)$

$= -1$.

This integral is negative because the region is below the x-axis.

Hence the required area is 1 square unit.

Finding areas between curves

If a curve $y = f(x)$ is always above $y = g(x)$ in an interval $a \le x \le b$, then the area of the region between the curves is

$$\text{area between the curves} = \int_a^b \big(f(x) - g(x)\big)\,dx.$$

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Example 18 5E

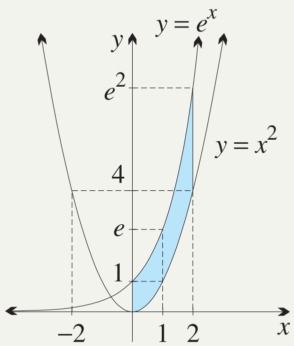
a Sketch the curves $y = e^x$ and $y = x^2$ in the interval $-2 \le x \le 2$.
b Find the area of the region between the curves, from $x = 0$ to $x = 2$.

SOLUTION

a The graphs are drawn to the right.
Note that for $x > 0$, $y = e^x$ is always above $y = x^2$.

b Using the standard formula above,

$$\begin{aligned} \text{area} &= \left[e^x - \tfrac{1}{3}x^3\right]_0^2 \\ &= \left(e^2 - \tfrac{8}{3}\right) - (e^0 - 0) \\ &= e^2 - 3\tfrac{2}{3} \text{ square units.} \end{aligned}$$



Exercise 5E

FOUNDATION

Technology: Graphing programs that can calculate the areas of specified regions may make the problems in this exercise clearer, particularly when no diagram has been given.

1 **a** Use the standard form $\int e^x dx = e^x + C$ to evaluate each definite integral. Then approximate it correct to two decimal places.

i $\int_0^1 e^x\,dx$ **ii** $\int_{-1}^0 e^x\,dx$ **iii** $\int_{-2}^0 e^x\,dx$ **iv** $\int_{-3}^0 e^x\,dx$

b The graph below shows $y = e^x$ from $x = -5$ to $x = 1$, with a scale of 10 divisions to 1 unit, so that 100 little squares equal 1 square unit.
By counting squares under the curve from $x = 0$ to $x = 1$, find an approximation to $\int_0^1 e^x dx$, and compare it with the approximation obtained in part **a**.

c Count squares to the left of the y-axis to obtain approximations to:

i $\int_{-1}^0 e^x dx$, **ii** $\int_{-2}^0 e^x dx$, **iii** $\int_{-3}^0 e^x dx$,

and compare the results with the approximations obtained in part **a**.

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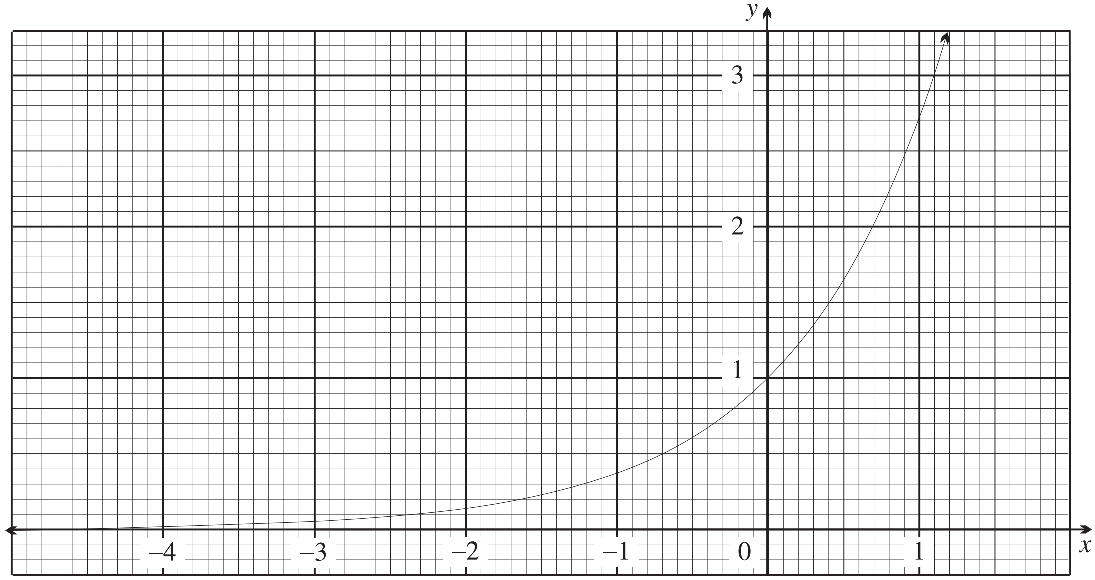
d Continue counting squares to the left of $x = -3$, and estimate the total area under the curve to the left of the y-axis.



2 Find the area between $y = e^x$ and the x-axis for:

a $-1 \le x \le 0$ **b** $1 \le x \le 3$ **c** $-1 \le x \le 1$ **d** $-2 \le x \le 1$

3 Answer these questions first in exact form, then correct to four significant figures. In each case use the standard form $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$.

a Find the area between the curve $y = e^{2x}$ and the x-axis:

i from $x = 0$ to $x = 3$, **ii** from $x = -3$ to $x = 0$.

b Find the area between the curve $y = e^{-x}$ and the x-axis:

i from $x = 0$ to $x = 1$, **ii** from $x = -1$ to $x = 0$.

c Find the area between the curve $y = e^{\frac{1}{3}x}$ and the x-axis:

i from $x = 0$ to $x = 3$, **ii** from $x = -3$ to $x = 0$.

4 In each case find the area between the x-axis and the given curve between the given x-values. Use the standard form $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$.

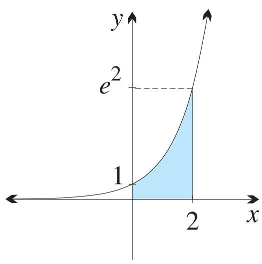
a $y = e^{x+1}$, for $0 \le x \le 2$ **b** $y = e^{x+3}$, for $-2 \le x \le 0$

c $y = e^{2x-1}$, for $0 \le x \le 1$ **d** $y = e^{3x-5}$, for $1 \le x \le 2$

e $y = e^{-x+1}$, for $-1 \le x \le 1$ **f** $y = e^{-2x-1}$, for $-2 \le x \le -1$

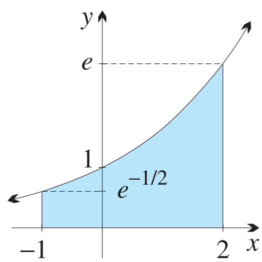
g $y = e^{\frac{1}{3}x+2}$, for $0 \le x \le 3$ **h** $y = e^{\frac{1}{2}x-1}$, for $-2 \le x \le 2$

5 **a**



Find the area of the region bounded by the curve $y = e^x$, the x-axis, the y-axis and the line $x = 2$.

b



Find the area of the region bounded by the curve $y = e^{\frac{1}{2}x}$, the x-axis, and the lines $x = -1$ and $x = 2$.

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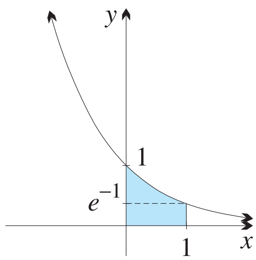
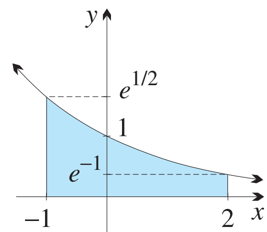
c



Find the area of the region bounded by the curve $y = e^{-x}$, the x-axis, the y-axis and the line $x = 1$.

d



Find the area of the region bounded by the curve $y = e^{-\frac{1}{2}x}$, the x-axis, and the lines $x = -1$ and $x = 2$.

DEVELOPMENT

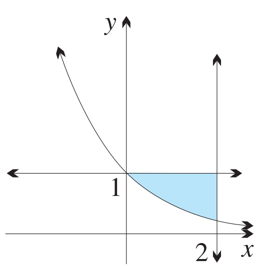
6 **a** Find the area between the curve $y = e^{-x} + 1$ and the x-axis, from $x = 0$ to $x = 2$.

b Find the area between the curve $y = 1 - e^{x}$ and the x-axis, from $x = -1$ to $x = 0$.

c Find the area between the curve $y = e^{x} + e^{-x}$ and the x-axis, from $x = -2$ to $x = 2$.

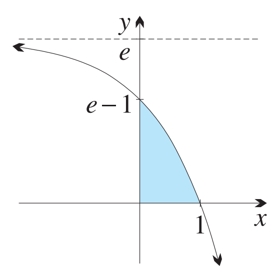
d Find the area between the curve $y = x^2 + e^{x}$ and the x-axis, from $x = -3$ to $x = 3$.

7 **a**



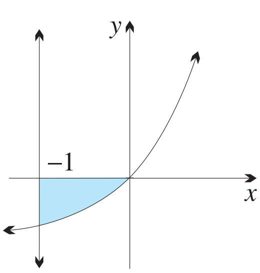
Find the area of the region bounded by the curve $y = e^{-x}$ and the lines $x = 2$ and $y = 1$
.

b



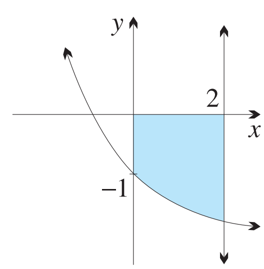
Find the area of the region in the first quadrant bounded by the coordinate axes and the curve $y = e - e^{x}$.

c



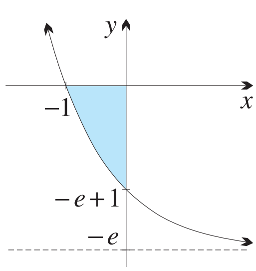
Find the area between the x-axis, the curve $y = e^{x} - 1$ and the line $x = -1$.

d



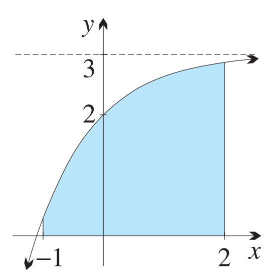
What is the area bounded by $x = 2$, $y = e^{-x} - 2$, the x-axis and the y-axis?

e



Find the area of the region bounded by the curve $y = e^{-x} - e$ and the coordinate axes.

f



Find the area of the region bounded by the curve $y = 3 - e^{-x}$, the x-axis, and the lines $x = -1$ and $x = 2$.

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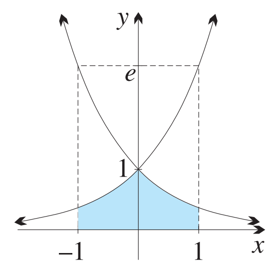
8 **a** Sketch the curves $y = e^x$ and $y = x + 1$, and shade the region between them, from $x = 0$ to $x = 1$. Then write down the area of this region as an integral and evaluate it.

b Sketch the curves $y = e^x$ and $y = 1 - x$, and shade the region between them, from $x = 0$ to $x = 1$. Then write down the area of this region as an integral and evaluate it.

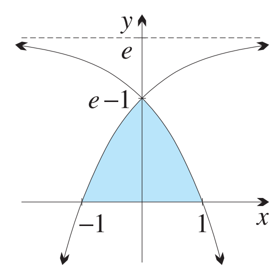
9 The diagram to the right shows the region above the x-axis, below both $y = e^x$ and $y = e^{-x}$, between $x = -1$ and $x = 1$.



a Explain why the area of this region may be written as area $= 2\int_0^1 e^{-x}\,dx$.

b Hence find the area of this region.

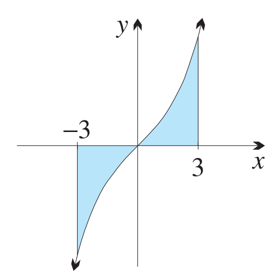
10 The diagram to the right shows the region above the x-axis, below both $y = e - e^{-x}$ and $y = e - e^x$.



a Explain why the area of this region may be written as area $= 2\int_0^1 (e - e^x)\,dx$.

b Hence find the area of this region.

11 The diagram to the right shows the region between the curve $y = e^x - e^{-x}$, the x-axis and the lines $x = -3$ and $x = 3$.



a Show that $y = e^x - e^{-x}$ is an odd function.

b Hence write down the value of $\int_{-3}^{3} (e^x - e^{-x})\,dx$ without finding a primitive.

c Explain why the area of this region may be written as area $= 2\int_0^3 (e^x - e^{-x})\,dx$.

d Hence find the area of this region.

12 **a** Show that the curves $y = x^2$ and $y = e^{x+1}$ intersect at $x = -1$.

b Hence sketch the region in the second quadrant between these two curves and the y-axis.

c Find its area.

13 **a** Show that the curves $y = e^x$ and $y = (e - 1)x + 1$ meet at $A(0, 1)$ and $B(1, e)$.

b Sketch the graphs, and find the area contained between the line and the curve.

14 Sketch the region between the graphs of $y = e^x$ and $y = x$, between the y-axis and $x = 2$, then find its area.

15 In this question, give all approximations correct to four decimal places.

a Find the area between the curve $y = e^x$ and the x-axis, for $0 \le x \le 1$, by evaluating an appropriate integral. Then approximate the result.

b Estimate the area using the trapezoidal rule with two subintervals (that is, with three function values).

c Is the trapezoidal-rule approximation greater than or less than the exact value? Give a geometric explanation.

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16 In this question, give all approximations correct to four decimal places.

a Use the trapezoidal rule with five function values to approximate the area between the curve $y = e^{-x^2}$ and the x-axis, from $x = 0$ to $x = 4$.

b Use the trapezoidal rule with four subintervals to approximate the area between the curve $y = e^{\frac{1}{x}}$ and the x-axis, from $x = 1$ to $x = 3$.

CHALLENGE

17 **a** **i** Evaluate the integral $\int_N^0 e^x\,dx$.

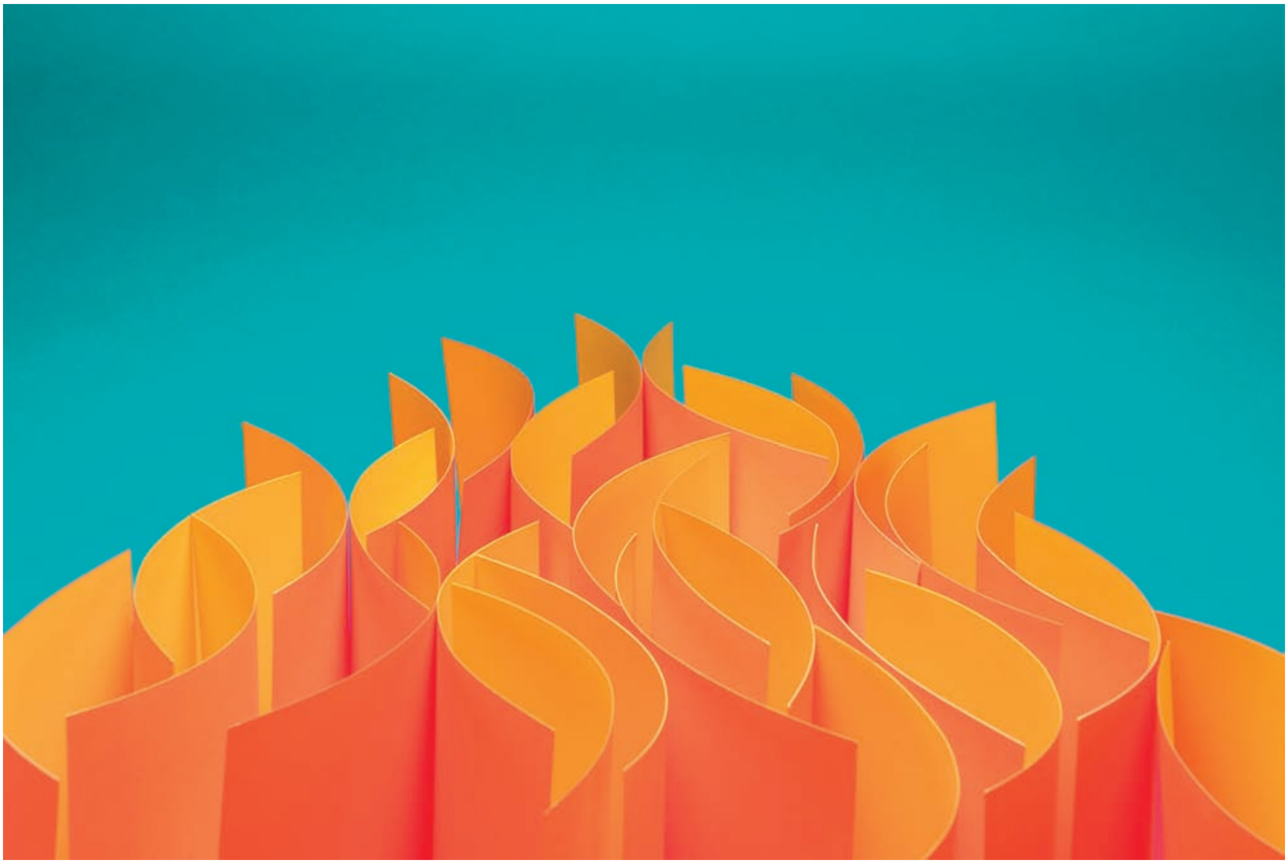
ii What is its limit as $N \to -\infty$?

b **i** Evaluate the integral $\int_0^N e^{-x}\,dx$.

ii What is its limit as $N \to \infty$?

18 **a** Differentiate e^{-x^2} and hence write down a primitive of xe^{-x^2}.

b Hence find the area between the curve $y = xe^{-x^2}$ and the x-axis from $x = 0$ to $x = 2$, and from $x = -2$ to $x = 2$.



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5F Review of logarithmic functions

Section 5A reviewed exponential functions from Sections 9A–9F of the Year 11 book, and this section will complete the review of those sections with a summary of logarithms base e. The two small topics that are new are:

- Dilations of logarithmic functions.
- Exponential and logarithmic equations reducible to quadratics.

The function $y = \log_e x$

As discussed in the Year 11 book, an exponential function is one-to-one, and logarithmic functions are the inverse functions of exponential functions. One should remember that

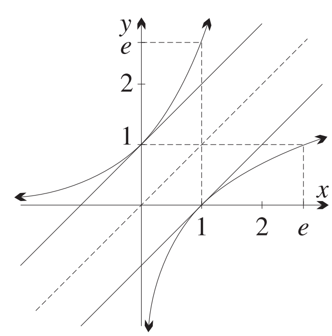
$$3 = \log_2 8 \text{ means } 8 = 2^3 \qquad \text{and} \qquad y = \log_8 x \text{ means } x = e^y.$$

'The log is the index, when the number is written as a power of the base.'

Algebraically, the fact that $y = \log_e x$ is the inverse function of $y = e^x$ means that the composite of the two functions, in either order, is the identity function,

$$\log_e e^x = x, \text{ for all real } x \qquad \text{and} \qquad e^{\log_e x} = x, \text{ for all } x > 0.$$

Geometrically, when the functions are sketched on one graph, they are reflections of each other in the diagonal line $y = x$.



- Both graphs have gradient 1 at their intercepts, $y = e^x$ at its y-intercept, and $y = \log_e x$ at its x-intercept.
- Their domains and ranges are reversed, which is more easily seen with bracket interval notation:
  For $y = e^x$, domain $= (-\infty, \infty)$, range $= (0, \infty)$.
  For $y = \log_e x$, domain $= (0, \infty)$, range $= (-\infty, \infty)$.
- $y = e^x$ has a horizontal asymptote $y = 0$.
  $y = \log_e x$ has a vertical asymptote $x = 0$.
- Both are increasing throughout their domain, $y = e^x$ at an increasing rate, $y = \log_e x$ at a decreasing rate.

4 THE FUNCTION $y = \log_e x$ OR $\ln x$

- The function $y = \log_e x$ is the inverse function of $y = e^x$,
  $y = \log_e x$ means that $x = e^y$.
- The composition of the functions $y = e^x$ and $y = \log_e x$, in any order, is the identity function,
  $\log_e e^x = x$, for all real x and $e^{\log_e x} = x$, for all $x > 0$.
- The graphs of $y = e^x$ and $y = \log_e x$ are reflections of each other in $y = x$.
- This reflection exchanges the domain and range, exchanges the asymptotes, and exchanges the intercepts with the axes.
- The tangents to both curves at their intercepts have gradient 1.
- $y = e^x$ is always concave up, and $y = \log_e x$ is always concave down.
- Both graphs are one-to-one, and both graphs are increasing, $y = e^x$ at an increasing rate, $y = \log_e x$ at a decreasing rate.

The derivative of $y = \log_e x$ will be obtained in Section 5G.

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Notation and the calculator

Write the function as $y = \log_e x$ or as $y = \ln x$ ('logs naperian' or 'logs natural'). We have used the notation $\log_e x$ more often than $\ln x$ in order to emphasise to readers that the base is e, but $\ln x$ is also standard notation.

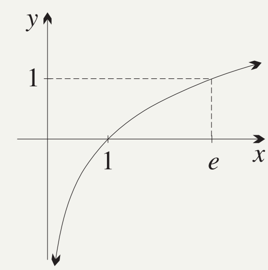
In mathematics, but not elsewhere, interpret $\log x$ as $\log_e x$. Be particularly careful on the calculator, where $\boxed{\ln}$ means $\log_e x$ and $\boxed{\log}$ means $\log_{10} x$.

Example 19 5F

Sketch each function using a transformation of the graph of $y = \log_e x$ sketched to the right. Describe the transformation, write down the domain, and show and state the x-intercept and the vertical asymptote.

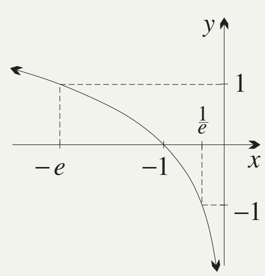
a $y = \log_e(-x)$ **b** $y = \log_e x - 2$ **c** $y = \log_e(x + 3)$

Which transformations can also be done using a dilation?



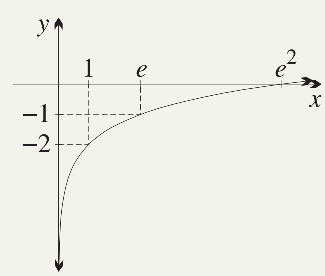
SOLUTION

a



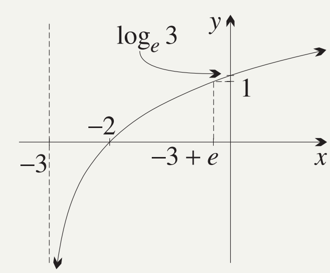
To graph $y = \log_e(-x)$,
reflect $y = \log_e x$ in y-axis.
domain: $x < 0$
x-intercept: $(-1, 0)$
asymptote: $x = 0$

b



To graph $y = \log_e x - 2$,
shift $y = \log_e x$ down 2.
domain: $x > 0$
x-intercept: $(e^2, 0)$
asymptote: $x = 0$

c



To graph $y = \log_e(x + 3)$,
shift $y = \log_e x$ left 3.
domain: $x > -3$
x-intercept: $(-2, 0)$
asymptote: $x = -3$

- The equation $y = \log_e(-x)$ in part **a** is a reflection in the y-axis, and any reflection in the y-axis can be regarded as a horizontal dilation with factor -1.
- In part **b**, the equation $y = \log_e x - 2 = \log_e x - \log_e e^2 = \log_e(e^{-2}x)$ can be regarded as a horizontal dilation with factor $e^2 \doteqdot 7.39$.

Dilations of $y = \log_e x$

Dilations of logarithmic functions share an interesting property with dilations of exponential functions — some of them can be done with a shift in the other direction, as we saw in part **b** above.

Example 20 5F

Use dilations of $y = \log_e x$ to generate a sketch of each function. Identify which dilation is also a shift in the other direction.

a $y = \log_e 3x$ **b** $y = 3 \log_e x$

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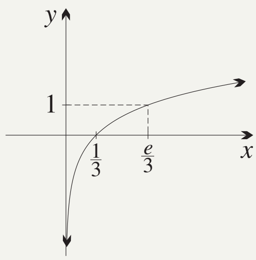
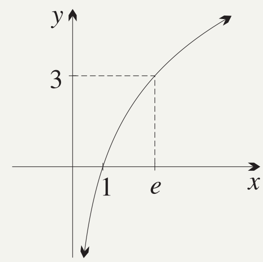
SOLUTION

a $y = \log_e 3x$



Dilate $y = \log_e x$ horizontally factor $\frac{1}{3}$.

b $y = 3\log_e x$



Dilate $y = \log_e x$ vertically factor 3.

- $y = \log_e 3x$ can be written as $y = \log_e x + \log_e 3$, so it is shift up $\log_e 3$.

Using the inverse identities

We conclude with a review of some of the manipulations needed when using logarithms base e. First, some simple examples of using the two inverse identities

$$\log_e e^x = x \text{ for all real } x \qquad \text{and} \qquad e^{\log_e x} = x \text{ for all } x > 0.$$

Example 21 — 5F

Simplify:

a $\log_e e^6$ **b** $\log_e e$ **c** $\log_e \frac{1}{e}$ **d** $\log_e \frac{1}{\sqrt{e}}$ **e** $e^{\log_e 10}$ **f** $e^{\log_e 0.1}$

SOLUTION

a $\log_e e^6 = 6$

b $\log_e e = \log_e e^1$
$= 1$

c $\log_e \frac{1}{e} = \log_e e^{-1}$
$= -1$

d $\log_e \frac{1}{\sqrt{e}} = \log_e e^{-\frac{1}{2}}$
$= -\frac{1}{2}$

e $e^{\log_e 10} = 10$

f $e^{\log_e 0.1} = 0.1$

Conversion between exponential statements and logarithm statements

We recommended that the following sentence be committed to memory:

$$\log_2 8 = 3 \qquad \text{because} \qquad 8 = 2^3.$$

- The base of the power is the base of the log.
- The log is the index, when the number is written as a power of the base.

This pattern applies in exactly the same way when the base is e.

$$\log_e x = y \qquad \text{means} \qquad x = e^y.$$

Example 22 — 5F

Convert each statement to the other form.

a $x = e^3$ **b** $\log_e x = -1$ **c** $x = \log_e 10$ **d** $e^x = \frac{1}{2}$

SOLUTION

a $\log_e x = 3$ **b** $x = \frac{1}{e}$ **c** $e^x = 10$ **d** $x = \log_e \frac{1}{2}$

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The change-of-base formula

We developed the general change of base formula. What is needed here is conversion to base e from a base b, which must be a positive number not equal to 1,

$$\log_b x = \frac{\log_e x}{\log_e b}, \qquad \text{for all } x > 0.$$

Example 23 — 5F

a Locate $\log_2 100$ and $\log_3 100$ between two whole numbers.

b Use logarithms base e to solve $2^x = 100$ and $3^x = 100$ correct to three decimal places.

SOLUTION

a $2^6 < 100 < 2^7$, so $\log_2 100$ lies between 6 and 7.
$3^4 < 100 < 3^5$, so $\log_3 100$ lies between 4 and 5.

b

$$\begin{aligned} 2^x &= 100 \\ x &= \log_2 100 \\ &= \frac{\log_e 100}{\log_e 2} \\ &\doteqdot 6.644 \end{aligned} \qquad\qquad \begin{aligned} 3^x &= 100 \\ x &= \log_3 100 \\ &= \frac{\log_e 100}{\log_e 3} \\ &\doteqdot 4.192 \end{aligned}$$

Alternatively, take logarithms base e of both sides.

5 THE CHANGE-OF-BASE FORMULA

Suppose that the new base b is a positive number not equal to 1. Then

$$\log_b x = \frac{\log_e x}{\log_e b}.$$

'The log of the number over the log of the base.'

Exponential and logarithmic equations reducible to quadratics

Exponential and logarithmic equations can sometimes be reduced to quadratics with a substitution (although the working is sometimes easier without the substitution). This approach can be used whether or not the base is e.

Example 24 — 5F

a Use the substitution $u = 2^x$ to solve the equation $4^x - 7 \times 2^x + 12 = 0$.

b Use the substitution $u = e^x$ to solve the equation $3e^{2x} - 11e^x - 4 = 0$.

c Solve $\log_e x - \dfrac{9}{\log_e x} = 0$ with and without the substitution $u = \log_e x$.

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SOLUTION

a Writing $4^x = (2^x)^2$, the equation becomes

$$(2^x)^2 - 7 \times 2^x + 12 = 0.$$

Substituting $u = 2^x$,

$$\begin{aligned} u^2 - 7u + 12 &= 0 \\ (u - 4)(u - 3) &= 0 \\ u &= 4 \text{ or } 3, \end{aligned}$$

and returning to x,

$$\begin{aligned} 2^x &= 4 \text{ or } 2^x = 3 \\ x &= 2 \text{ or } \log_2 3. \end{aligned}$$

b Writing $e^{2x} = (e^x)^2$, the equation becomes

$$3(e^x)^2 - 11e^x - 4 = 0.$$

Substituting $u = e^x$,

$$\begin{aligned} 3u^2 - 11u - 4 &= 0 \quad (\alpha + \beta = -11,\ \alpha\beta = 3 \times (-4) = -12) \\ 3u^2 - 12u + u - 4 &= 0 \quad (\alpha \text{ and } \beta \text{ are } -12 \text{ and } 1) \\ 3u(u - 4) + (u - 4) &= 0 \\ (3u + 1)(u - 4) &= 0 \\ u &= -\tfrac{1}{3} \text{ or } 4, \end{aligned}$$

and returning to x,

$$e^x = -\tfrac{1}{3} \text{ or } e^x = 4.$$

Because e^x is never negative,

$$\begin{aligned} e^x &= 4 \\ x &= \log_e 4. \end{aligned}$$

c The equation is

$$\log_e x - \frac{9}{\log_e x} = 0.$$

Substituting $u = \log_e x$,

$$u - \frac{9}{u} = 0$$

$\boxed{\times\ u}$

$$\begin{aligned} (u^2 - 9) &= 0 \\ (u - 3)(u + 3) &= 0 \\ u &= 3 \text{ or } -3, \end{aligned}$$

and returning to x,

$$\log_e x = 3 \text{ or } -3.$$

Hence

$$x = e^3 \text{ or } e^{-3}.$$

Alternatively,

$$\log_e x - \frac{9}{\log_e x} = 0$$

$\boxed{\times\ \log_e x}$

$$\begin{aligned} (\log_e x)^2 - 9 &= 0 \\ (\log_e x)^2 &= 9 \\ \log_e x &= 3 \text{ or } -3 \\ x &= e^3 \text{ or } e^{-3}. \end{aligned}$$

Exercise 5F

Remember that on the calculator, $\boxed{\ln}$ means $\log_e x$ and $\boxed{\log}$ means $\log_{10} x$. We have used the notation $\log_e x$ more often than $\ln x$ in order to emphasise the base.

1 Use the calculator's $\boxed{\ln}$ button to approximate, correct to four significant figures:

a $\log_e 10$ **b** $\log_e 0.1$ **c** $\ln 123\,456$ **d** $\ln 0.000\,006$ **e** $\log_e 50$ **f** $\log_e 0.02$

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2 Use the identities $\log_e e^x = x$ for all real x, and $e^{\log_e x} = x$ for $x > 0$, to simplify:

a $\log_e e^3$ **b** $\log_e e^{-1}$ **c** $\log_e \frac{1}{e^2}$ **d** $\log_e \sqrt{e}$

e $e^{\ln 5}$ **f** $e^{\ln 0.05}$ **g** $e^{\ln 1}$ **h** $e^{\ln e}$

3 **a** Use your calculator to confirm that $\log_e 1 = 0$.

b Write 1 as a power of e, then use the identities in question 2 to explain why $\log_e 1 = 0$.

c Use your calculator to confirm that $\log_e e = 1$. (You will need to find $e = e^1$ first.)

d Write e as a power of e, then use the identities in question 2 to explain why $\log_e e = 1$.

4 Convert each exponential statement to logarithm form, and each logarithmic statement to exponential form.

a $x = e^6$ **b** $\log_e x = -2$ **c** $x = \ln 24$ **d** $e^x = \frac{1}{3}$

5 Use the change-of base formula to express each logarithms in terms of logarithms base e. Then approximate it correct to four significant figures.

a $\log_2 7$ **b** $\log_{10} 25$ **c** $\log_3 0.04$

6 **a** What transformation maps $y = e^x$ to $y = \log_e x$, and how can this transformation be used to find the gradient of $y = \log_e x$ at its x-intercept?

b What transformation maps $y = \log_e x$ to $y = \log_e(-x)$, and how can this transformation also be interpreted as a dilation?

c Sketch $y = \log_e x$ and $y = \log_e(-x)$ on one set of axes.

7 Sketch each curve using a single transformation of $y = \log_e x$, and describe the transformation.

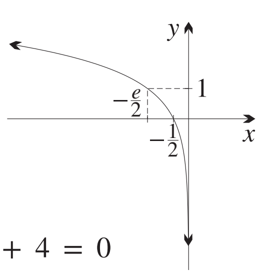
a $y = \log_e x + 1$ **b** $y = \log_e x - 2$ **c** $y = \log_e\left(\frac{1}{2}x\right)$ **d** $y = \frac{1}{3}\log_e x$

8 Sketch each curve using a single transformation of $y = \log_e(-x)$, and describe the transformation.

a $y = \log_e(-x) - 1$ **b** $y = -\log_e(-x)$ **c** $y = 3\log_e(-x)$

DEVELOPMENT

9 The graph drawn to the right is a dilation of $y = \log_e(-x)$. Describe the dilation, and write down the equation of the curve.



10 **a** Use the substitution $u = 2^x$ to solve $4^x - 9 \times 2^x + 14 = 0$.

b Use the substitution $u = 3^x$ to solve $3^{2x} - 8 \times 3^x - 9 = 0$.

c Use similar substitutions, or none, to solve:

i $25^x - 26 \times 5^x + 25 = 0$ **ii** $9^x - 5 \times 3^x + 4 = 0$

iii $3^{2x} - 3^x - 20 = 0$ **iv** $7^{2x} + 7^x + 1 = 0$

v $3^{5x} = 9^{x+3}$ **vi** $4^x - 3 \times 2^{x+1} + 2^3 = 0$

11 Use the substitution $u = e^x$ or $u = e^{2x}$ to reduce these equations to quadratics and solve them. Write your answers as logarithms base e, unless they can be further simplified.

a $e^{2x} - 2e^x + 1 = 0$ **b** $e^{2x} + e^x - 6 = 0$

c $e^{4x} - 10e^{2x} + 9 = 0$ **d** $e^{4x} - e^{2x} = 0$

12 Use a substitution, or none, to solve:

a $(\log_e x)^2 - 5\log_e x + 4 = 0$ **b** $(\log_e x)^2 = 3\log_e x$

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13 **a** Use the laws for logarithms to simplify:

i $\log_e e^e$ **ii** $\log_e(\log_e e^e)$ **iii** $\log_e(\log_e(\log_e e^e))$

b Use the laws for logarithms to express as a single logarithm:

i $\ln 5 + \ln 4$ **ii** $\ln 30 - \log_e 6$ **iii** $\ln 12 - \ln 15 + \ln 10$

CHALLENGE

14 Use a substitution such as $u = 4^x$ to solve each equation. Give each solution as a rational number, or approximate correct to three decimal places.

a $2^{4x} - 7 \times 2^{2x} + 12 = 0$

b $100^x - 10^x - 1 = 0$

c $\left(\frac{1}{5}\right)^{2x} - 7 \times \left(\frac{1}{5}\right)^x + 10 = 0$

15 **a** Use, and describe, a dilation of $y = \log_e x$ to sketch $y = \log_e 2x$.

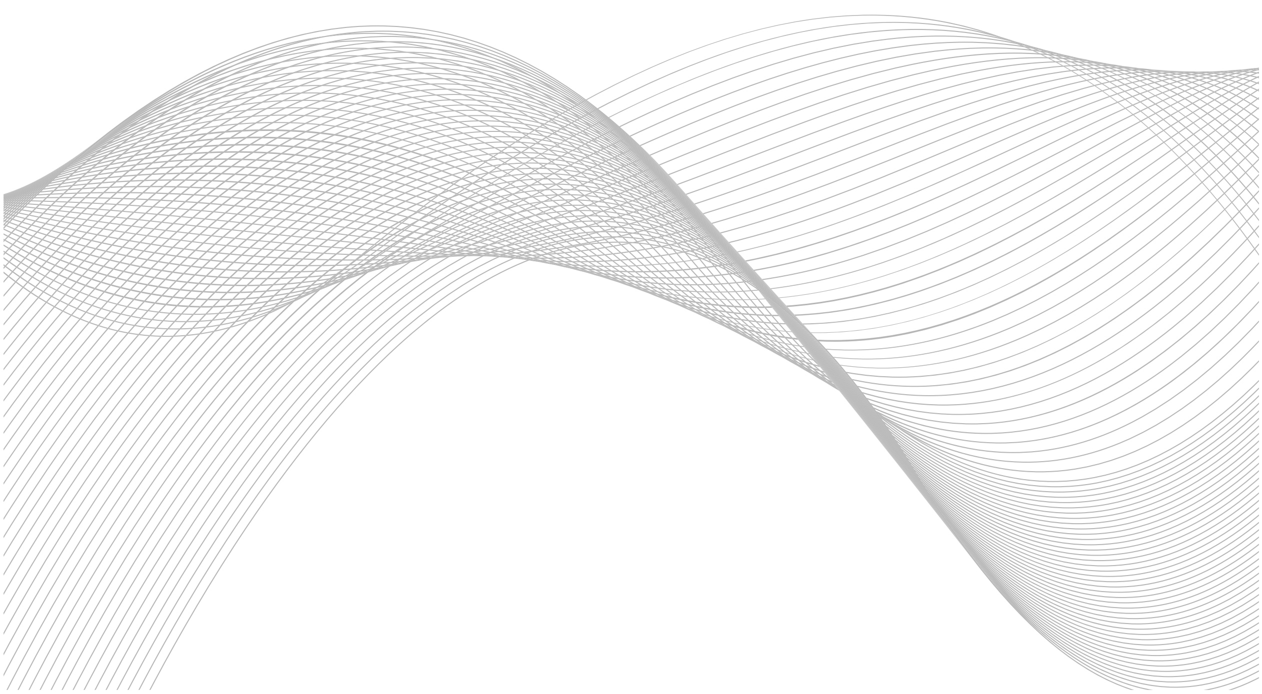
b Use, and describe, a subsequent translation to sketch $y = \log_e 2(x - 1)$.

c Use, and describe, a subsequent dilation to sketch $y = \frac{1}{2}\log_e 2(x - 1)$.

d Use, and describe, a subsequent translation to sketch $y = \frac{1}{2}\log_e 2(x - 1) - 2$.

16 **a** Interpret the transformation from $y = \log_e x$ to $y = \log_e(5x)$ as a dilation. Then interpret it as a translation.

b Interpret the transformation from $y = \log_e x$ to $y = \log_e x + 2$ as a translation. Then interpret it as a dilation by writing 2 as $\log_e e^2$.



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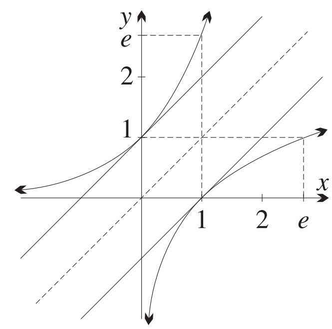
5G Differentiation of logarithmic functions



Calculus with the exponential function $y = e^x$ requires also the calculus of its inverse function $y = \log_e x$.

The diagram to the right shows once again the graphs of both curves drawn on the same set of axes — they are reflections of each other in the diagonal line $y = x$. Using this reflection, the important features of $y = \log_e x$ are:

- The domain is $x > 0$ and the range is all real x.
- The x-intercept is 1, and the gradient there is 1.
- The y-axis is a vertical asymptote.
- As $x \to \infty$, $y \to \infty$ (look at its reflection $y = e^x$ to see this).
- Throughout its whole domain, $\log_e x$ is increasing at a decreasing rate.

Differentiating the logarithmic function

The logarithmic function $= \log_e x$ can be differentiated easily using the known derivative of its inverse function e^x.

Let $\quad y = \log_e x$.

Then $\quad x = e^y$, by the definition of logarithms.

Differentiating, $\quad \dfrac{dx}{dy} = e^y$, because the exponential function is its own derivative,

$\quad = x$, because $e^y = x$,

and taking reciprocals, $\quad \dfrac{dy}{dx} = \dfrac{1}{x}$.

Hence the derivative of the logarithmic function is the reciprocal function.

6 THE DERIVATIVE OF THE LOGARITHMIC FUNCTION IS THE RECIPROCAL FUNCTION

$$\frac{d}{dx}\log_e x = \frac{1}{x}$$

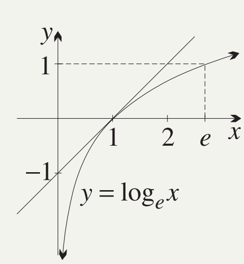
The next worked example uses the derivative to confirm that $y = \log_e x$ has two properties that were already clear from the reflection in the diagram above.

Example 25 5G

a Find the gradient of the tangent to $y = \log_e x$ at its x-intercept.

b Prove that $y = \log_e x$ is always increasing, and always concave down.

SOLUTION



a The function is $\quad y = \log_e x$.

Differentiating, $\quad y' = \dfrac{1}{x}$.

The graph crosses the x-axis at $(1, 0)$, and substituting $x = 1$ into y', gradient at x-intercept $= 1$.

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b The domain is $x > 0$, and $y' = \frac{1}{x}$ is positive for all $x > 0$.

Differentiating again, $\quad y'' = -\frac{1}{x^2}$, which is negative for all $x > 0$.

Hence $y = e^x$ is always increasing, and always concave down.

Example 26 5G

Differentiate these functions using the standard form above.

a $y = x + \log_e x$ **b** $y = 5x^2 - 7\log_e x$

SOLUTION

a
$$y = x + \log_e x$$
$$\frac{dy}{dx} = 1 + \frac{1}{x}$$

b
$$y = 5x^2 - 7\log_e x$$
$$\frac{dy}{dx} = 10x - \frac{7}{x}$$

Further standard forms

The next worked example uses the chain rule to develop two further standard forms for differentiation.

Example 27 5G

Differentiate each function using the chain rule. (Part **b** is a standard form.)

a $\log_e(3x + 4)$ **b** $\log_e(ax + b)$ **c** $\log_e(x^2 + 1)$

SOLUTION

a Let $y = \log_e(3x + 4)$.

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ (chain rule)

$$= \frac{1}{3x + 4} \times 3$$
$$= \frac{3}{3x + 4}.$$

Let $u = 3x + 4$.
Then $y = \log_e u$.
Hence $\frac{du}{dx} = 3$
and $\frac{dy}{du} = \frac{1}{u}$.

b Let $y = \log_e(ax + b)$.

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ (chain rule)

$$= \frac{1}{ax + b} \times a$$
$$= \frac{a}{ax + b}.$$

Let $u = ax + b$.
Then $y = \log_e u$.
Hence $\frac{du}{dx} = \mathrm{a}$
and $\frac{dy}{du} = \frac{1}{u}$.

c Let $y = \log_e(x^2 + 1)$.

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ (chain rule)

$$= \frac{1}{x^2 + 1} \times 2x$$
$$= \frac{2x}{x^2 + 1}.$$

Let $u = x^2 + 1$.
Then $y = \log_e u$.
Hence $\frac{du}{dx} = 2x$
and $\frac{dy}{du} = \frac{1}{u}$.

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Standard forms for differentiation

It is convenient to write down two further standard forms for differentiation based on the chain rule, giving three forms altogether.

7 THREE STANDARD FORMS FOR DIFFERENTIATING LOGARITHMIC FUNCTIONS

$$\frac{d}{dx}\log_e x = \frac{1}{x}$$

$$\frac{d}{dx}\log_e(ax + b) = \frac{a}{ax + b}$$

$$\frac{d}{dx}\log_e u = \frac{u'}{u} \qquad \text{OR} \qquad \frac{d}{dx}\log_e f(x) = \frac{f'(x)}{f(x)}$$

The second of these standard forms was proven in part **b** of the previous worked example. Part **a** was an example of it.

The third standard form is a more general chain-rule extension — part **c** of the previous worked example was a good example of it. This standard form will be needed later for integration. For now, either learn it — in one of its two forms — or apply the chain rule each time.

Example 28 5G

Using the standard forms developed above, differentiate:

a $y = \log_e(4x - 9)$ **b** $y = \log_e(1 - \frac{1}{2}x)$ **c** $y = \log_e(4 + x^2)$

SOLUTION

a For $y = \log_e(4x - 9)$, use the second standard form with $ax + b = 4x - 9$.

Thus $y' = \dfrac{4}{4x - 9}$.

b For $y = \log_e(1 - \frac{1}{2}x)$, use the second standard form with $ax + b = -\frac{1}{2}x + 1$.

Thus $y' = \dfrac{-\frac{1}{2}}{-\frac{1}{2}x + 1}$

$= \dfrac{1}{x - 2}$, after multiplying top and bottom by -2.

c For $y = \log_e(4 + x^2)$,

$y' = \dfrac{2x}{4 + x^2}$.

Let $u = 4 + x^2$.
Then $u' = 2x$.
$\dfrac{d}{dx}\log_e u = \dfrac{u'}{u}$

OR

Let $f(x) = 4 + x^2$.
Then $f'(x) = 2x$.
$\dfrac{d}{dx}\log_e f(x) = \dfrac{f'(x)}{f(x)}$

Alternatively, use the chain rule, as in the previous worked example.

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Using the product and quotient rules

These two rules are used in the usual way.

Example 29 5G

Differentiate:

a $x^3 \ln x$ by the product rule,

b $\dfrac{\ln(1+x)}{x}$ by the quotient rule.

SOLUTION

a Let $y = x^3 \ln x$.

Then $y' = vu' + uv'$

$= 3x^2 \ln x + x^3 \times \dfrac{1}{x}$

$= x^2(1 + 3\ln x)$.

Let $u = x^3$

and $v = \ln x$.

Then $u' = 3x^2$

and $v' = \dfrac{1}{x}$.

b Let $y = \dfrac{\ln(1+x)}{x}$.

Then $y' = \dfrac{vu' - uv'}{v^2}$

$= \dfrac{\dfrac{x}{1+x} - \ln(1+x)}{x^2}$

$= \dfrac{x - (1+x)\ln(1+x)}{x^2(1+x)}$.

Let $u = \ln(1+x)$

and $v = x$.

Then $u' = \dfrac{1}{1+x}$

and $v' = 1$.

Using the log laws to make differentiation easier

The next worked example shows the use of the log laws to avoid a combination of the chain and quotient rules.

Example 30 5G

Use the log laws to simplify each expression, then differentiate it.

a $\log_e 7x^2$

b $\log_e(3x-7)^5$

c $\log_e \dfrac{1+x}{1-x}$.

SOLUTION

a Let $y = \log_e 7x^2$.

Then $y = \log_e 7 + \log_e x^2$ (log of a product is the sum of the logs)

$= \log_e 7 + 2\log_e x$ (log of a power is the multiple of the log),

so $\dfrac{dy}{dx} = \dfrac{2}{x}$ ($\log_e 7$ is a constant, with derivative zero).

b Let $y = \log_e(3x-7)^5$.

Then $y = 5\log_e(3x-7)$ (log of a power is the multiple of the log),

so $\dfrac{dy}{dx} = \dfrac{15}{3x-7}$.

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c Let $y = \log_e \dfrac{1+x}{1-x}$.

Then $y = \log_e(1+x) - \log_e(1-x)$ (log of a quotient is the difference of the logs),

so $\dfrac{dy}{dx} = \dfrac{1}{1+x} + \dfrac{1}{1-x}$.

Exercise 5G

FOUNDATION

Note: Remember that on the calculator, $\boxed{\ln}$ means $\log_e x$ and $\boxed{\log}$ means $\log_{10} x$. We have used the notation $\log_e x$ more often than $\ln x$ in order to emphasise the base.

1 Use the standard form $\dfrac{d}{dx}\log_e(ax+b) = \dfrac{a}{ax+b}$ to differentiate:

a $y = \log_e(x+2)$ **b** $y = \log_e(x-3)$ **c** $y = \log_e(3x+4)$
d $y = \log_e(2x-1)$ **e** $y = \log_e(-4x+1)$ **f** $y = \log_e(-3x+4)$
g $y = \ln(-2x-7)$ **h** $y = 3\ln(2x+4)$ **i** $y = 5\ln(3x-2)$

2 Differentiate these functions.

a $y = \log_e 2x$ **b** $y = \log_e 5x$ **c** $y = \log_e 3x$ **d** $y = \log_e 7x$
e $y = 4\ln 7x$ **f** $y = 3\ln 5x$ **g** $y = 4\ln 6x$ **h** $y = 3\ln 9x$

3 Find $\dfrac{dy}{dx}$ for each function. Then evaluate $\dfrac{dy}{dx}$ at $x = 3$.

a $y = \log_e(x+1)$ **b** $y = \log_e(2x-1)$ **c** $y = \log_e(2x-5)$
d $y = \log_e(4x+3)$ **e** $y = 5\ln(x+1)$ **f** $y = 6\ln(2x+9)$

4 Differentiate these functions.

a $2 + \log_e x$ **b** $5 - \log_e(x+1)$ **c** $x + 4\log_e x$
d $2x^4 + 1 + 3\log_e x$ **e** $\ln(2x-1) + 3x^2$ **f** $x^3 - 3x + 4 + \ln(5x-7)$

DEVELOPMENT

5 Use the log laws to simplify each function, then differentiate it.

a $y = \ln x^3$ **b** $y = \ln x^2$ **c** $y = \ln x^{-3}$
d $y = \ln x^{-2}$ **e** $y = \ln\sqrt{x}$ **f** $y = \ln\sqrt{x+1}$

6 Differentiate these functions.

a $y = \log_e \frac{1}{2}x$ **b** $y = \log_e \frac{1}{3}x$ **c** $y = 3\log_e \frac{1}{5}x$
d $y = -6\log_e \frac{1}{2}x$ **e** $y = x + \log_e \frac{1}{7}x$ **f** $y = 4x^3 - \log_e \frac{1}{5}x$

7 Use the full setting-out of the chain rule to differentiate:

a $\ln(x^2+1)$ **b** $\ln(2-x^2)$ **c** $\ln(1+e^x)$

8 Use the standard form $\dfrac{d}{dx}\log_e u = \dfrac{u'}{u}$ OR $\dfrac{d}{dx}\log_e f(x) = \dfrac{f'(x)}{f(x)}$ to differentiate:

a $\log_e(x^2+3x+2)$ **b** $\log_e(1+2x^3)$ **c** $\ln(e^x-2)$
d $x + 3 - \ln(x^2+x)$ **e** $x^2 + \ln(x^3-x)$ **f** $4x^3 - 5x^2 + \ln(2x^2-3x+1)$

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9 Find the gradient, and the angle of inclination correct to the nearest minute, of the tangent to $y = \ln x$ at the points where:

a $x = 1$ **b** $x = 3$ **c** $x = \frac{1}{2}$ **d** $x = 4$

Draw a diagram of the curve and the four tangents, showing the angles of inclination.

10 Differentiate these functions using the product rule.

a $x \log_e x$ **b** $x \log_e (2x + 1)$ **c** $(2x + 1) \log_e x$ **d** $x^4 \log_e x$
e $(x + 3) \log_e (x + 3)$ **f** $(x - 1) \log_e (2x + 7)$ **g** $e^x \log_e x$ **h** $e^{-x} \log_e x$

11 Differentiate these functions using the quotient rule.

a $y = \dfrac{\log_e x}{x}$ **b** $y = \dfrac{\log_e x}{x^2}$ **c** $y = \dfrac{x}{\log_e x}$
d $y = \dfrac{x^2}{\log_e x}$ **e** $y = \dfrac{\log_e x}{e^x}$ **f** $y = \dfrac{e^x}{\log_e x}$

12 Use the log laws to simplify each function, then differentiate it.

a $y = \log_e 5x^3$ **b** $y = \log_e 3x^4$ **c** $y = \log_e \sqrt[3]{x}$ **d** $y = \log_e \sqrt[4]{x}$
e $y = \log_e \dfrac{3}{x}$ **f** $y = \log_e \dfrac{2}{5x}$ **g** $y = \ln \sqrt{2 - x}$ **h** $y = \ln \sqrt{5x + 2}$

13 Find the first and second derivatives of each function, then evaluate both derivatives at the value given.

a $f(x) = \log (x - 1)$ at $x = 3$ **b** $f(x) = \log (2x + 1)$ at $x = 0$
c $f(x) = \log x^2$ at $x = 2$ **d** $f(x) = x \log x$ at $x = e$

14 Differentiate each function using the chain, product or quotient rules. Then find any values of x for which the derivative is zero.

a $y = x \log_e x - x$ **b** $y = x^2 \log_e x$ **c** $y = \dfrac{\log_e x}{x}$
d $y = (\log_e x)^2$ **e** $y = (\log_e x)^4$ **f** $y = \dfrac{1}{1 + \log_e x}$
g $y = (2 \log_e x - 3)^4$ **h** $y = \dfrac{1}{\log_e x}$ **i** $y = \log_e (\log_e x)$

15 Find the point(s) where the tangent to each curve is horizontal.

a $y = x \ln x$ **b** $y = \dfrac{1}{x} + \ln x$

CHALLENGE

16 **a** Find the derivative of $y = \dfrac{x}{\ln x}$.

b Hence show that $y = \dfrac{x}{\ln x}$ is a solution of the equation $\dfrac{dy}{dx} = \left(\dfrac{y}{x}\right) - \left(\dfrac{y}{x}\right)^2$ by substituting separately into the LHS and the RHS.

17 Use the log laws to simplify these functions, then differentiate them.

a $y = \log_e(x + 2)(x + 1)$ **b** $y = \log_e(x + 5)(3x - 4)$ **c** $y = \ln \dfrac{1 + x}{1 - x}$
d $y = \ln \dfrac{3x - 1}{x + 2}$ **e** $y = \log_e \dfrac{(x - 4)^2}{3x + 1}$ **f** $y = \log_e x\sqrt{x + 1}$

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18 Use the log laws to simplify these functions, then differentiate them.

a $y = \log_e 2^x$ **b** $y = \log_e e^x$ **c** $y = \log_e x^x$

19 This result will be used in Section 5I.

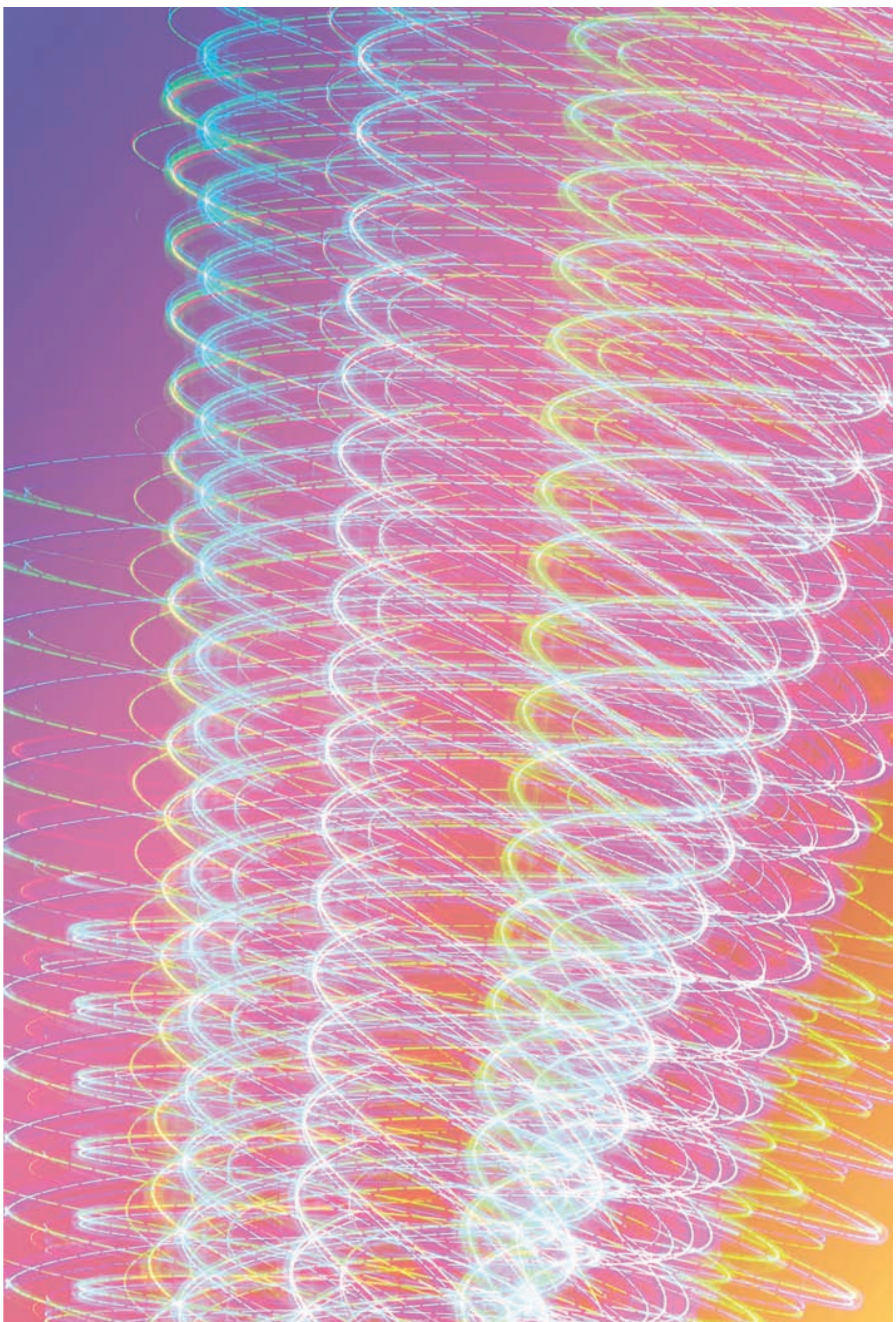
a Copy and complete the statement $\log_e |x| = \begin{cases} \ldots, & \text{for } x > 0, \\ \ldots, & \text{for } x < 0. \end{cases}$

b Use part **a** to sketch the curve $y = \log_e |x|$.

c By differentiating separately the two branches in part **a**, show that

$$\frac{d}{dx} \log_e |x| = \frac{1}{x}, \text{ for all } x \neq 0.$$

d Why was $x = 0$ excluded in this discussion?



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5H Applications of differentiation of $\log_e x$

Differentiation can now be applied in the usual way to study the graphs of functions involving $\log_e x$.

The geometry of tangents and normals

The derivative can be used as usual to investigate the geometry of tangents and normals to a curve.

Example 31 5H

a Show that the tangent to $y = \log_e x$ at $T(e, 1)$ has equation $x = ey$.

b Find the equation of the normal to $y = \log_e x$ at $T(e, 1)$.

c Sketch the curve, the tangent and the normal, and find the area of the triangle formed by the y-axis and the tangent and normal at T.

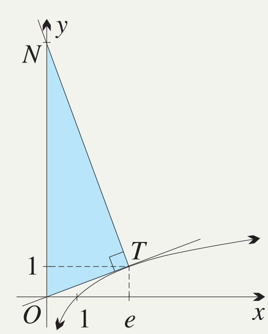
SOLUTION

a Differentiating, $\dfrac{dy}{dx} = \dfrac{1}{x}$,

so the tangent at $T(e, 1)$ has gradient $\dfrac{1}{e}$,

and the tangent is $y - 1 = \dfrac{1}{e}(x - e)$

$$ey - e = x - e$$
$$x = ey$$
$$y = \frac{x}{e}.$$



Notice that this tangent has gradient $\frac{1}{e}$ and passes through the origin.

b The tangent at $T(e, 1)$ has gradient $\frac{1}{e}$, so the normal there has gradient $-e$.

Hence the normal has equation $y - 1 = -e(x - e)$

$$y = -ex + (e^2 + 1).$$

c Substituting $x = 0$, the normal has y-intercept $N(0, e^2 + 1)$.
Hence the base ON of $\triangle ONT$ is $(e^2 + 1)$ and its altitude is e.
Thus the triangle $\triangle ONT$ has area $\frac{1}{2}e(e^2 + 1)$ square units.

An example of curve sketching

Here are the six steps of our informal curve-sketching menu applied to the function $y = x \log_e x$.

Example 32 5H

Sketch the graph of $y = x \log_e x$ after carrying out these steps.

a Write down the domain.

b Test whether the function is even or odd or neither.

c Find any zeroes of the function and examine its sign.

d Examine the function's behaviour as $x \to \infty$ and as $x \to -\infty$, noting any asymptotes. (You may assume that $x \log_e x \to 0$ as $x \to 0^+$.)

e Find any stationary points and examine their nature.

f Find any points of inflection, and examine the concavity.

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SOLUTION

a The domain is $x > 0$, because $\log_e x$ is undefined for $x \le 0$.

b The function is undefined when x is negative, so it is neither even nor odd.

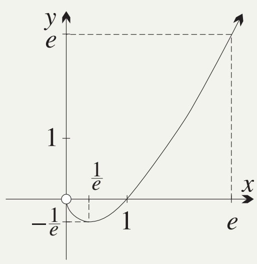
c The only zero is at $x = 1$, and the curve is continuous for $x > 0$.

We take test values at $x = e$ and at $= \dfrac{1}{e}$.

When $x = e$, $y = e\log_e e$
$\quad = e \times 1$
$\quad = e.$

When $x = e^{-1}$, $y = e^{-1}\log_e e^{-1}$
$\quad = e^{-1} \times (-1)$
$\quad = -e^{-1}.$

x	0	e^{-1}	1	e
y	*	$-e^{-1}$	0	e
sign	*	$-$	0	$+$



Hence y is negative for $0 < x < 1$ and positive for $x > 1$.

d As given in the hint, $y \to 0$ as $x \to 0^+$.
Also, $y \to \infty$ as $x \to \infty$.

e Differentiating by the product rule,

$$\begin{aligned} f'(x) &= vu' + uv' \\ &= \log_e x + x \times \frac{1}{x} \\ &= \log_e x + 1, \end{aligned}$$

and $f''(x) = \dfrac{1}{x}$.

Let $u = x$
and $v = \log_e x$.
Then $u' = 1$
and $v' = \dfrac{1}{x}$.

Putting $f'(x) = 0$ gives $\log_e x = -1$
$\qquad x = e^{-1}$.

Substituting, $\quad f''(e^{-1}) = e > 0$
and $\quad f(e^{-1}) = -e^{-1}$, as above,
so $(e^{-1}, -e^{-1})$ is a minimum turning point.
(A point: $f'(x) \to -\infty$ as $x \to 0^+$, so the curve becomes vertical near the origin.)

f Because $f''(x)$ is always positive, there are no inflections, and the curve is always concave up.

A difficulty with the limits of $x\log_e x$ and $\dfrac{\log_e x}{x}$

The curve-sketching example above involved knowing the behaviour of $x\log_e x$ as $x \to 0^+$. When x is a small positive number, $\log_e x$ is a large negative number, so it is not immediately clear whether the product $x\log_e x$ becomes large or small as $x \to 0^+$.

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In fact, $x \log_e x \to 0$ as $x \to 0^+$, and x is said to *dominate* $\log_e x$, in the same way that e^x dominated x in Section 5C. Here is a table of values that should make it reasonably clear that $\lim_{x \to 0^+} x \log_e x = 0$:

x	$\frac{1}{e}$	$\frac{1}{e^2}$	$\frac{1}{e^3}$	$\frac{1}{e^4}$	$\frac{1}{e^5}$	$\frac{1}{e^6}$	$\frac{1}{e^7}$	...
$x \log_e x$	$-\frac{1}{e}$	$-\frac{2}{e^2}$	$-\frac{3}{e^3}$	$-\frac{4}{e^4}$	$-\frac{5}{e^5}$	$-\frac{6}{e^6}$	$-\frac{7}{e^7}$	...
approx.	-0.37	-0.27	-0.15	-0.073	-0.034	-0.015	-0.006	...

Such limits would normally be given in any question where they are needed.

A similar problem arises with the behaviour of $\dfrac{\log_e x}{x}$ as $x \to \infty$, because both top and bottom get large when x is large. Again, x dominates $\log_e x$, meaning that $\dfrac{\log_e x}{x} \to 0$ as $x \to \infty$, as the following table should make reasonably obvious:

x	e	e^2	e^3	e^4	e^5	e^6	e^7	...
$\dfrac{\log_e x}{x}$	$\frac{1}{e}$	$\frac{2}{e^2}$	$\frac{3}{e^3}$	$\frac{4}{e^4}$	$\frac{5}{e^5}$	$\frac{6}{e^6}$	$\frac{7}{e^7}$	...
approx.	0.37	0.27	0.15	0.073	0.034	0.015	0.006	...

Again, this limit would normally be given if it is needed.

Exercise 5H

FOUNDATION

1 **a** Write down the derivative of $y = \log_e x$.
b Use the derivative to find the gradient of the tangent to $y = \log_e x$ at $P(e, 1)$.
c Hence find the equation of the tangent at P, and prove that it passes through O.

2 **a** Similarly find the gradient of the tangent to $y = \log_e x$ at $Q(1, 0)$.
b Hence find the equation of the tangent at Q, and prove that it passes through $A(0, -1)$.

3 **a** Find the gradient of the tangent to $y = \log_e x$ at $R(\frac{1}{e}, -1)$.
b Hence find the equation of the tangent at R, and prove that it passes through $B(0, -2)$.

4 **a** Find the gradient of the tangent to $y = \log_e x$ at the point $A(1, 0)$.
b Show that the gradient of the normal is -1.
c Hence find the equation of the normal at A, and its y-intercept.

5 Find, giving answers in the form $y = mx + b$, the equations of the tangent and normal to:
a $y = 4\log_e x$ at the point $Q(1, 0)$,
b $y = \log_e x + 3$ at the point $R(1, 3)$,
c $y = 2\log_e x - 2$ at the point $S(1, -2)$,
d $y = 1 - 3\log_e x$ at the point $T(1, 1)$.

6 **a** Show that the point $P(1, 0)$ lies on the curve $y = \log_e(3x - 2)$.
b Find the gradients of the tangent and normal at P.
c Find the equations of the tangent and normal at P, and their y-intercepts.
d Find the area of the triangle formed by the tangent, the normal and the y-axis.

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DEVELOPMENT

7 **a** Find the gradient of the tangent to $y = \ln x - \frac{x}{2} + 1$ at $x = 1$.

b Write down the equation of the tangent, and show that it passes through the origin.

8 **a** Find the equation of the tangent to $y = (2 - x)\ln x$ at $x = 2$.

b Hence find the y-intercept of the tangent.

9 **a** Write down the domain of $y = \log_e x$ and the derivative of $y = \log_e x$.

b Hence explain why the gradient of every tangent to $y = \log_e x$ is positive.

c Explain also why the gradient of every normal to $y = \log_e x$ is negative.

d Draw the graph of $y = \log_e x$ to confirm your answers to parts **c** and **d**.

e Find y'' and show that it is always negative. What aspect of the curve does this describe?

10 **a** Find the coordinates of the point on $y = \log_e x$ where the tangent has gradient $\frac{1}{2}$. Then find the equation of the tangent and normal there, in the form $y = mx + b$.

b Find the coordinates of the point on $y = \log_e x$ where the tangent has gradient 2. Then find the equation of the tangent and normal there, in the form $y = mx + b$.

11 **a** In Question 1 you showed that the tangent at $P(e, 1)$ on the curve $y = \log_e x$ passes through the origin. Sketch the graph, showing the tangent, and explain graphically why no other tangent passes through the origin.

b Again arguing geometrically from the graph, classify the points in the plane according to whether 0, 1 or 2 tangents pass through them.

12 **a** Write down the natural domain of $y = x - \log_e x$. What does this answer tell you about whether the function is even, odd or neither?

b Determine its first two derivatives.

c Show that the curve is concave up for all values of x in its domain.

d Find the minimum turning point.

e Sketch the curve and write down its range.

f Finally sketch the curve $y = \log_e x - x$ by recognising the simple transformation.

13 **a** Write down the domain of $y = \frac{1}{x} + \ln x$.

b Show that the first and second derivatives may be expressed as single fractions as $y' = \frac{x - 1}{x^2}$ and $y'' = \frac{2 - x}{x^3}$.

c Show that the curve has a minimum at $(1, 1)$ and an inflection at $(2, \frac{1}{2} + \ln 2)$.

d Sketch the graph and write down its range.

14 This question will confirm the remarks about *dominance* in the text of this section. Such limits would normally be given in a question that needed them.

x	2	5	10	20	40	4000
y						

a Use your calculator to complete the table of values for $y = \frac{\log_e x}{x}$ to the right. Then use the table to help you guess the value of $\lim_{x \to \infty} \frac{\log_e x}{x}$.

b Use your calculator to complete the table of values for $y = x \log_e x$ to the right. Then use the table to help you guess the value of $\lim_{x \to 0^+} x \log_e x$.

x	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{4000}$
y						

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15 Consider the curve $y = x \log_e x$.

a Write down the domain and x-intercept.

b Show that $y' = 1 + \log_e x$ and find y''.

c Hence show there is one stationary point and determine its nature.

d Given that $y \to 0^-$ as $x \to 0^+$ and that the tangent becomes closer and closer to vertical as $x \to 0^+$, sketch the curve and write down its range.

CHALLENGE

16 Consider the curve $y = x \log_e x - x$.

a Write down the domain and x-intercept.

b Draw up a table of signs for the function.

c Show that $y' = \log_e x$ and find y''.

d Hence show that there is one stationary point and determine its nature.

e What does y'' tell you about the curve?

f Given that $y \to 0^-$ as $x \to 0^+$, and that the tangent approaches vertical as $x \to 0^+$, sketch the curve and write down its range.

17 a Write down the domain of $y = \log_e(1 + x^2)$.

b Is the curve, even, odd or neither?

c Find where the function is zero, and explain what its sign is otherwise.

d Show that $y' = \dfrac{2x}{1 + x^2}$ and $y'' = \dfrac{2(1 - x^2)}{(1 + x^2)^2}$.

e Hence show that $y = \log_e(1 + x^2)$ has one stationary point, and determine its nature.

f Find the coordinates of the two points of inflection.

g Hence sketch the curve, and then write down its range.

18 a Find the domain of $y = (\ln x)^2$.

b Find where the function is zero, and explain what its sign is otherwise.

c Find y' and show that $y'' = \dfrac{2(1 - \ln x)}{x^2}$.

d Hence show that the curve has an inflection at $x = e$.

e Classify the stationary point at $x = 1$, sketch the curve, and write down the range.

19 a Write down the domain of $y = \dfrac{\log_e x}{x}$.

b Show that $y' = \dfrac{1 - \log_e x}{x^2}$ and $y'' = \dfrac{2\log_e x - 3}{x^3}$.

c Find any stationary points and determine their nature.

d Find the exact coordinates of the lone point of inflection.

e Sketch the curve, and write down its range. You may assume that $y \to 0$ as $x \to \infty$, and that $y \to -\infty$ as $x \to 0^+$.

20 a Show that the tangent to $y = \log_e x$ at $A(a, \log_e a)$ is $x - ay = a(1 - \log_e a)$.

b Hence show that the only point on $y = \log_e x$ where the tangent passes through the origin is $(e, 1)$.

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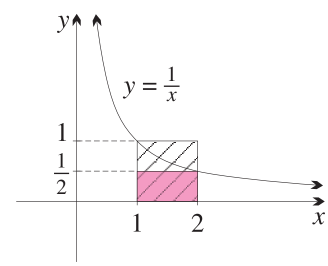
5I Integration of the reciprocal function

The reciprocal function $y = \dfrac{1}{x}$ is an important function — we have seen that it is required whenever two quantities are inversely proportional to each other. So far, however, it has not been possible to integrate the reciprocal function, because the usual rule for integrating powers of x gives nonsense:

$$\text{When } n = -1, \int x^n \, dx = \frac{x^{n+1}}{n+1} \quad \text{gives} \int x^{-1} \, dx = \frac{x^0}{0},$$

which is nonsense because of the division by zero.



Yet the graph of $y = \dfrac{1}{x}$ to the right shows that there should be no problem with definite integrals involving $\dfrac{1}{x}$, provided that the integral does not cross the discontinuity at $x = 0$. For example, the diagram shows the integral $\displaystyle\int_1^2 \frac{1}{x} \, dx$, which the little rectangles show has a value between $\frac{1}{2}$ and 1.

Integration of the reciprocal function

Reversing the standard form for differentiating log x will now give the necessary standard forms for integrating $\dfrac{1}{x}$.

We know that $\quad \dfrac{d}{dx} \log_e x = \dfrac{1}{x}$,

and reversing this, $\quad \displaystyle\int \frac{1}{x} \, dx = \log_e x + C$, for some constant C.

This is a new standard form for integrating the reciprocal function.

The only qualification is that $x > 0$, otherwise $\log_e x$ is undefined, so we have

$$\int \frac{1}{x} \, dx = \log_e x + C, \text{ provided that } x > 0.$$

Example 33 — 5I

a Find the definite integral $\displaystyle\int_1^2 \frac{1}{x} \, dx$ sketched above.

b Approximate the integral correct to three decimal places and verify that

$$\tfrac{1}{2} < \int_1^2 \frac{1}{x} \, dx < 1.$$

SOLUTION

a
$$\begin{aligned} \int_1^2 \frac{1}{x} \, dx &= \Big[\log_e x\Big]_1^2 \\ &= \log_e 2 - \log_e 1 \\ &= \log_e 2, \quad \text{because } \log_e 1 = 0. \end{aligned}$$

b Hence $\displaystyle\int_1^2 \frac{1}{x} \, dx \doteqdot 0.693$,

which is indeed between $\frac{1}{2}$ and 1, as the diagram above indicated.

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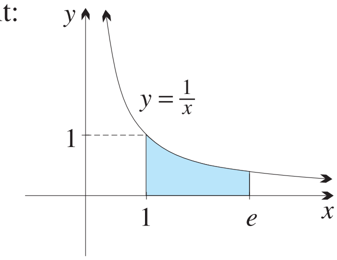
A characterisation of e

Integrating the reciprocal function from 1 to e gives an amazingly simple result:

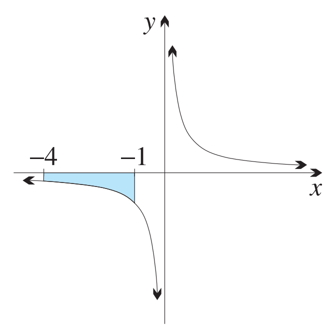


$$\begin{aligned}\int_1^e \frac{1}{x}\,dx &= \Big[\log_e x\Big]_1^e \\ &= \log_e e - \log_e 1 \\ &= 1 - 0 \\ &= 1.\end{aligned}$$

The integral is sketched to the right. The example is very important because it characterises e as the real number satisfying $\int_1^e \frac{1}{x}\,dx = 1$. In other expositions of the theory, this integral is taken as the definition of e.

The primitive of $y = \dfrac{1}{x}$ on both sides of the origin

So far our primitive is restricted by the condition $x > 0$, meaning that we can only deal with definite integrals on the right-hand side of the origin. The full graph of the reciprocal function $y = 1/x$, however, is a hyperbola, with two disconnected branches separated by the discontinuity at $x = 0$.



Clearly there is no reason why we should not integrate over a closed interval such as $-4 \le x \le -1$ on the left-hand side of the origin. We can take any definite integrals of $\frac{1}{x}$ provided only that we do not work across the asymptote at $x = 0$. If x is negative, then $\log(-x)$ is well defined, and using our previous standard forms,

$$\frac{d}{dx}\log(-x) = -\left(\frac{1}{-x}\right) = \frac{1}{x},$$

and reversing this, $\log(-x)$ is a primitive of $\frac{1}{x}$ when x is negative,

$$\int \frac{1}{x}\,dx = \log_e(-x) + C, \quad \text{provided that } x < 0.$$

The absolute value function is designed for just these situations. We can combine the two results into one standard form for the whole reciprocal function,

$$\int \frac{1}{x}\,dx = \log_e |x| + C, \quad \text{provided that } x \ne 0.$$

Question 19 of Exercise 9G gives more detail about this standard form.

Challenge: each branch may have its own constant of integration

Careful readers will realise that because $y = \frac{1}{x}$ has two disconnected branches, there can be different constants of integration in the two branches. So the general primitive of $\frac{1}{x}$ is

$$\int \frac{1}{x}\,dx = \begin{cases}\log_e x + A, & \text{for } x > 0,\\ \log(-x) + B, & \text{for } x < 0,\end{cases} \qquad \text{where } A \text{ and } B \text{ are constants.}$$

If an initial or boundary condition is given for one branch, this has no implication at all for the constant of integration in the other branch.

In any physical interpretation, however, the function would normally have meaning in only one of the two branches, so the complication discussed here is rarely needed, and the over-simplified forms in Box 8 below are standard and generally used — the qualification is understood and taken account of when necessary.

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Three standard forms

As always, reversing the other standard forms for differentiation gives two more standard forms.

8 STANDARD FORMS FOR INTEGRATING RECIPROCAL FUNCTIONS

- $$\int \frac{1}{x}\,dx = \log_e |x| + C$$
- $$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e |ax+b| + C$$
- $$\int \frac{u'}{u}\,dx = \log_e |u| + C \qquad \text{OR} \qquad \int \frac{f'(x)}{f(x)}\,dx = \log_e |f(x)| + C$$

No calculation involving these primitives may cross an asymptote.

The final warning always applies to the primitive of any function, but it is mentioned here because it is such an obvious issue.

Example 34 5I

Evaluate these definite integrals using the first two standard forms above.

a $\displaystyle\int_e^{e^2} \frac{5}{x}\,dx$ **b** $\displaystyle\int_1^4 \frac{1}{1-2x}\,dx$ **c** $\displaystyle\int_1^5 \frac{1}{x-2}\,dx$

SOLUTION

a
$$\begin{aligned}\int_e^{e^2} \frac{5}{x}\,dx &= 5\Big[\log_e |x|\Big]_e^{e^2}\\ &= 5(\log_e e^2 - \log_e e)\\ &= 5(2-1)\\ &= 5\end{aligned}$$

b
$$\begin{aligned}\int_1^4 \frac{1}{1-2x}\,dx &= -\tfrac{1}{2}\Big[\log_e |1-2x|\Big]_1^4 \qquad (\text{here } a = -2 \text{ and } b = 1)\\ &= -\tfrac{1}{2}(\log_e |-7| - \log_e |-1|)\\ &= -\tfrac{1}{2}(\log_e 7 - 0)\\ &= -\tfrac{1}{2}\log_e 7\end{aligned}$$

c This definite integral is meaningless because it crosses the asymptote at $x = 2$.

Using the third standard form

The vital point in using the third standard form,

$$\int \frac{u'}{u}\,dx = \log_e |u| \qquad \text{OR} \qquad \int \frac{f'(x)}{f(x)}\,dx = \log_e |f(x)|,$$

is that the top must be the derivative of the bottom. Choose whichever form of the reverse chain rule you are most comfortable with.

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Example 35 5I

Evaluate these definite integrals using the third standard form above.

a $\displaystyle\int_0^1 \frac{2x}{x^2+2}\,dx$ **b** $\displaystyle\int_4^5 \frac{x}{9-x^2}\,dx$ **c** $\displaystyle\int_0^2 \frac{3x}{1-x^3}\,dx$

SOLUTION

a Let $u = x^2 + 2$ OR $f(x) = x^2 + 2$.
Then $u' = 2x$ $f'(x) = 2x$.

Hence in the fraction $\dfrac{2x}{x^2+1}$, the top is the derivative of the bottom.

Thus, using $\displaystyle\int \frac{u'}{u}\,dx = \log_e|f(x)|$ OR $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log_e|f(x)|$,

$$\begin{aligned}\int_0^1 \frac{2x}{x^2+2}\,dx &= \Big[\log_e(x^2+2)\Big]_0^1\\ &= \log_e 3 - \log_e 2.\end{aligned}$$

Note: The use of absolute value signs here is unnecessary (but is not wrong) because $x^2 + 2$ is never negative.

b Let $u = 9 - x^2$ OR $f(x) = 9 - x^2$.
Then $u' = -2x$ $f'(x) = -2x$.

The first step is to make the top the derivative of the bottom:

$$\begin{aligned}\int_4^5 \frac{x}{9-x^2}\,dx &= -\tfrac{1}{2}\int_4^5 \frac{-2x}{9-x^2}\,dx, \quad \text{which has the form } \int \frac{f'(x)}{f(x)}\,dx,\\ &= -\tfrac{1}{2}\Big[\log_e|9-x^2|\Big]_4^5\\ &= -\tfrac{1}{2}(\log_e|-16| - \log_e|-7|)\\ &= -\tfrac{1}{2}(4\log_e 2 - \log_e 7)\\ &= -2\log_e 2 + \tfrac{1}{2}\log_e 7.\end{aligned}$$

c This definite integral is meaningless because it crosses the asymptote at $x = 1$.

Given the derivative, find the function

Finding the function from the derivative involves a constant that can be found if the value of y is known for some value of x.

Example 36 5I

a Find $f(x)$, if $f'(x) = \dfrac{2}{3-x}$ and the graph passes through the origin.

b Hence find $f(2)$.

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SOLUTION

a Here $f'(x) = \dfrac{2}{3 - x}$.

Taking the primitive, $f(x) = -2 \ln |3 - x| + C$, for some constant C.

Because $f(0) = 0$,

$$\begin{aligned} 0 &= -2 \ln 3 + C \\ C &= 2 \ln 3. \end{aligned}$$

Hence $f(x) = 2 \ln 3 - 2 \ln |3 - x|$.

b Substituting $x = 2$ gives

$$\begin{aligned} f(2) &= 2 \ln 3 - 2 \ln 1 \\ &= 2 \ln 3. \end{aligned}$$

Note: As remarked above, this working is over-simplified, because each branch may have its own constant of integration. But there is no problem in this question because the asymptote is at $x = 3$, and the given point $(0, 0)$ on the curve, and the value $x = 2$ in part **b**, are both on the same side of the asymptote.

A primitive of $\log_e x$

The next worked example is more difficult, but it is important because it produces a primitive of $\log_e x$, which the theory has not yielded so far. There is no need to memorise the result.

Example 37 5I

a Differentiate $x \log_e x$ by the product rule.

b Show by differentiation that $x \log_e x - x$ is a primitive of $\log_e x$.

c Use this result to evaluate $\displaystyle\int_1^e \log_e x \, dx$.

SOLUTION

a Differentiating by the product rule,

$$\begin{aligned} \frac{d}{dx}(x \log_e x) &= vu' + uv' \\ &= \log_e x + x \times \frac{1}{x}, \\ &= 1 + \log_e x. \end{aligned}$$

Let $u = x$ and $v = \log_e x$.
Then $u' = 1$ and $v' = \dfrac{1}{x}$.

b Let $y = x \log_e x - x$.

Then $y' = (1 + \log_e x) - 1$, using the result of part **a**,

$= \log_e x$.

Reversing this result gives the primitive of $\log_e x$,

$$\int \log_e x \, dx = x \log_e x - x + C.$$

c Part **b** can now be used to find the definite integral,

$$\begin{aligned} \int_1^e \log_e x \, dx &= \Big[x \log_e x - x\Big]_1^e \\ &= (e \log_e e - e) - (1 \log_e 1 - 1) \\ &= (e \log_e e - e) - (0 - 1) \\ &= (e - e) + 1 \\ &= 1. \end{aligned}$$

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Exercise 5I

FOUNDATION

1 First rewrite each integral using the result $\int \frac{k}{x}\,dx = k\int \frac{1}{x}\,dx$, where k is a constant. Then use the standard form $\int \frac{1}{x}\,dx = \log_e |x| + C$ to integrate it.

a $\int \frac{2}{x}\,dx$ **b** $\int \frac{5}{x}\,dx$ **c** $\int \frac{1}{2x}\,dx$

d $\int \frac{1}{3x}\,dx$ **e** $\int \frac{4}{5x}\,dx$ **f** $\int \frac{3}{2x}\,dx$

2 Use the standard form $\int \frac{1}{ax + b}\,dx = \frac{1}{a}\log_e |ax + b| + C$ to find these indefinite integrals.

a $\int \frac{1}{4x + 1}\,dx$ **b** $\int \frac{1}{5x - 3}\,dx$ **c** $\int \frac{6}{3x + 2}\,dx$

d $\int \frac{15}{5x + 1}\,dx$ **e** $\int \frac{4}{4x + 3}\,dx$ **f** $\int \frac{dx}{3 - x}$

g $\int \frac{dx}{7 - 2x}$ **h** $\int \frac{4\,dx}{5x - 1}$ **i** $\int \frac{12\,dx}{1 - 3x}$

3 Evaluate these definite integrals. Simplify your answers where possible.

a $\int_1^5 \frac{1}{x}\,dx$ **b** $\int_1^3 \frac{1}{x}\,dx$ **c** $\int_{-8}^{-2} \frac{1}{x}\,dx$

d $\int_{-3}^{9} \frac{1}{x}\,dx$ **e** $\int_1^4 \frac{dx}{2x}$ **f** $\int_{-15}^{-5} \frac{dx}{5x}$

4 Evaluate these definite integrals, then use the function labelled $\boxed{\ln}$ on your calculator to approximate each integral correct to four significant figures.

a $\int_0^1 \frac{dx}{x + 1}$ **b** $\int_{-7}^{-5} \frac{dx}{x + 2}$ **c** $\int_4^{18} \frac{dx}{x - 2}$

d $\int_1^3 \frac{dx}{3x - 1}$ **e** $\int_{-5}^{-2} \frac{dx}{2x + 3}$ **f** $\int_1^2 \frac{3}{5 - 2x}\,dx$

g $\int_{-1}^{1} \frac{3}{7 - 3x}\,dx$ **h** $\int_1^4 \frac{6}{4x - 1}\,dx$ **i** $\int_0^{11} \frac{5}{2x - 11}\,dx$

5 Evaluate these definite integrals. Simplify your answers where possible.

a $\int_1^e \frac{dx}{x}$ **b** $\int_1^{e^2} \frac{dx}{x}$ **c** $\int_1^{e^4} \frac{dx}{x}$ **d** $\int_{\sqrt{e}}^{e} \frac{dx}{x}$

6 Find primitives of these function by first writing them as separate fractions.

a $\frac{x + 1}{x}$ **b** $\frac{x + 3}{5x}$ **c** $\frac{2 - x}{3x}$ **d** $\frac{1 - 8x}{9x}$

e $\frac{3x^2 - 2x}{x^2}$ **f** $\frac{2x^2 + x - 4}{x}$ **g** $\frac{3x^3 + 4x - 1}{x^2}$ **h** $\frac{x^4 - x + 2}{x^2}$

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DEVELOPMENT

7 In each case show that the numerator is the derivative of the denominator. Then use the result $\int \frac{u'}{u}\,dx = \log_e |f(x)|$ OR $\int \frac{f'(x)}{f(x)}\,dx = \log_e |f(x)| + C$ to integrate the expression.

a $\dfrac{2x}{x^2 - 9}$ b $\dfrac{6x + 1}{3x^2 + x}$ c $\dfrac{2x + 1}{x^2 + x - 3}$

d $\dfrac{5 - 6x}{2 + 5x - 3x^2}$ e $\dfrac{x + 3}{x^2 + 6x - 1}$ f $\dfrac{3 - x}{12x - 3 - 2x^2}$

g $\dfrac{e^x}{1 + e^x}$ h $\dfrac{e^{-x}}{1 + e^{-x}}$ i $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Why is it unnecessary (but not wrong) to use absolute value signs in the answers to parts g–i?

8 Find $f(x)$, and then find $f(2)$, given that:

a $f'(x) = 1 + \dfrac{2}{x}$ and $f(1) = 1$ b $f'(x) = 2x + \dfrac{1}{3x}$ and $f(1) = 2$

c $f'(x) = 3 + \dfrac{5}{2x - 1}$ and $f(1) = 0$ d $f'(x) = 6x^2 + \dfrac{15}{3x + 2}$ and $f(1) = 5\ln 5$

9 a Find y as a function of x if $y' = \dfrac{1}{4x}$ and $y = 1$ when $x = e^2$. Where does this curve meet the x-axis on the right-hand side of the origin?

b The gradient of a curve is given by $y' = \dfrac{2}{x + 1}$, and the curve passes through the point $(0, 1)$. What is the equation of this curve?

c Find $y(x)$, given that $y' = \dfrac{2x + 5}{x^2 + 5x + 4}$ and $y = 1$ when $x = 1$. Hence evaluate $y(0)$.

d Write down the equation of the family of curves with the property $y' = \dfrac{2 + x}{x}$. Hence find the curve that passes through $(1, 1)$ and evaluate y at $x = 2$ for this curve.

e Given that $f''(x) = \dfrac{1}{x^2}$, $f'(1) = 0$ and $f(1) = 3$, find $f(x)$ and hence evaluate $f(e)$.

10 Use the standard form $\int \frac{1}{ax + b}\,dx = \frac{1}{a}\log_e |ax + b| + C$ to find these integrals.

a $\displaystyle\int \frac{1}{2x + b}\,dx$ b $\displaystyle\int \frac{1}{3x - k}\,dx$ c $\displaystyle\int \frac{1}{ax + 3}\,dx$

d $\displaystyle\int \frac{1}{mx - 2}\,dx$ e $\displaystyle\int \frac{p}{px + q}\,dx$ f $\displaystyle\int \frac{A}{sx - t}\,dx$

11 Use one of the forms $\int \frac{u'}{u}\,dx = \log_e |u| + C$ or $\int \frac{f'(x)}{f(x)}\,dx = \log_e |f(x)| + C$ to find:

a $\displaystyle\int \frac{3x^2}{x^3 - 5}\,dx$ b $\displaystyle\int \frac{4x^3 + 1}{x^4 + x - 5}\,dx$ c $\displaystyle\int \frac{x^3 - 3x}{x^4 - 6x^2}\,dx$

d $\displaystyle\int \frac{10x^3 - 7x}{5x^4 - 7x^2 + 8}\,dx$ e $\displaystyle\int_2^3 \frac{3x^2 - 1}{x^3 - x}\,dx$ f $\displaystyle\int_e^{2e} \frac{2x + 2}{x^2 + 2x}\,dx$

12 a Given that the derivative of $f(x)$ is $\dfrac{x^2 + x + 1}{x}$ and $f(1) = 1\frac{1}{2}$, find $f(x)$.

b Given that the derivative of $g(x)$ is $\dfrac{2x^3 - 3x - 4}{x^2}$ and $g(2) = -3\ln 2$, find $g(x)$.

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CHALLENGE

13 Find $\displaystyle\int_1^e \left(x + \frac{1}{x^2}\right)^2 dx$.

14 **a** Differentiate $y = x \log_e x - x$.

b Hence find:

i $\displaystyle\int \log_e x\, dx$,

ii $\displaystyle\int_{\sqrt{e}}^{e} \log_e x\, dx$.

15 **a** Show that the derivative of $y = 2x^2 \log_e x - x^2$ is $y' = 4x \log_e x$.

b Hence write down a primitive of $x \log_e x$.

c Use this result to evaluate $\displaystyle\int_e^2 x \log_e x\, dx$.

16 **a** Differentiate $(\log_e x)^2$ using the chain rule.

b Hence determine $\displaystyle\int_{\sqrt{e}}^{e} \frac{\log_e x}{x}\, dx$.

17 Differentiate $\ln(\ln x)$ and hence determine the family of primitives of $\dfrac{1}{x \ln x}$.

18 Stella found the primitive of the function $\dfrac{1}{5x}$ by taking out a factor of $\frac{1}{5}$,

$$\int \frac{1}{5x}\, dx = \frac{1}{5}\int \frac{1}{x}\, dx = \tfrac{1}{5}\log_e |x| + C_1, \text{ for some constant } C_1.$$

Magar used the second standard form in Box 6 with $a = 5$ and $b = 0$,

$$\int \frac{1}{5x}\, dx = \tfrac{1}{5}\log_e |5x| + C_2, \text{ for some constant } C_2.$$

Explain what is going on. Will this affect their result when finding a definite integral?

19 **a** Find the value of a if a is positive and:

i $\displaystyle\int_1^a \frac{1}{x}\, dx = 5$,

ii $\displaystyle\int_a^e \frac{1}{x}\, dx = 5$.

b Find the value of a if a is negative and:

i $\displaystyle\int_a^{-1} \frac{1}{x}\, dx = -2$,

ii $\displaystyle\int_{-e}^{a} \frac{1}{x}\, dx = -2$.

20 Although it is not required in this course, it can be shown that:

$$\int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \log_e \left(x + \sqrt{x^2 + a^2}\right) + C$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \log_e \left|x + \sqrt{x^2 - a^2}\right| + C$$

Use these results to find:

a $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2 + 1}}\, dx$

b $\displaystyle\int_4^8 \frac{1}{\sqrt{x^2 - 16}}\, dx$

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5J Applications of integration of $1/x$

The usual applications of integration can now be applied to the reciprocal function, whose primitive was previously unavailable.

Finding areas by integration

The next worked example involves finding the area between two given curves.

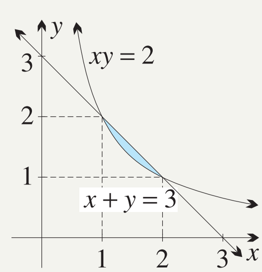
Example 38 5J

a Show that the hyperbola $xy = 2$ and the line $x + y = 3$ meet at the points $A(1, 2)$ and $B(2, 1)$.
b Sketch the situation.
c Find the area of the region between the two curves, in exact form and correct to three decimal places.

SOLUTION

a Substituting $A(1, 2)$ into the hyperbola $xy = 2$,
$$\text{LHS} = 1 \times 2 = 2 = \text{RHS},$$
and substituting $A(1, 2)$ into the line $x + y = 3$,
$$\text{LHS} = 1 + 2 = 3 = \text{RHS},$$
so $A(1, 2)$ lies on both curves.
Similarly, $B(2, 1)$ lies on both curves.

b The hyperbola $xy = 2$ has both axes as aymptotes.
The line $x + y = 3$ has x-intercept $(3, 0)$ and y-intercept $(0, 3)$.



c
$$\begin{aligned}\text{Area} &= \int_1^2 (\text{top curve} - \text{bottom curve})\,dx \\ &= \int_1^2 \left((3 - x) - \frac{2}{x}\right) dx \\ &= \left[3x - \tfrac{1}{2}x^2 - 2\log_e |x|\right]_1^2 \\ &= \left(6 - 2 - 2\log_e 2\right) - \left(3 - \tfrac{1}{2} - 2\log_e 1\right) \\ &= \left(4 - 2\log_e 2\right) - \left(2\tfrac{1}{2} - 0\right) \\ &= \left(1\tfrac{1}{2} - 2\log_e 2\right) \text{ square units} \\ &\doteqdot 0.114 \text{ square units.}\end{aligned}$$

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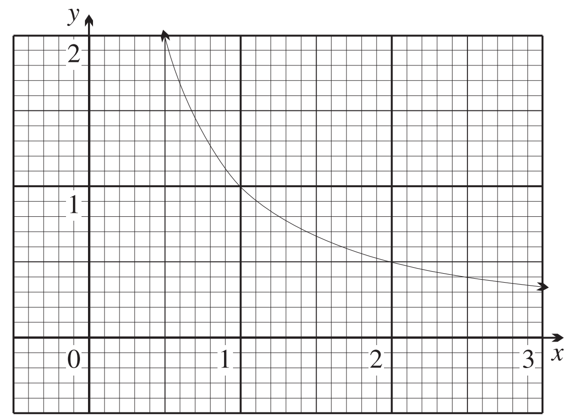
Exercise 5J

FOUNDATION

1 **a** Show that $\int_1^e \frac{1}{x}\,dx = 1$.

b This question uses the result in part **a** to estimate e from a graph of $y = \frac{1}{x}$.

The diagram to the right shows the graph of $y = \frac{1}{x}$ from $x = 0$ to $x = 3$.



The graph been drawn on graph paper with a scale of 10 little divisions to 1 unit, so that 100 of the little squares make 1 square unit.

Count the number of squares in the column from $x = 1.0$ to 1.1, then the squares in the column from $x = 1.1$ to 1.2, and so on.

Continue until the number of squares equals 100 — the x-value at this point will be an estimate of e.

2 Answer each question by first giving your answer in exact form and then finding an approximation correct to four significant figures.

a Find the area between the curve $y = \frac{1}{x}$ and the x-axis for:

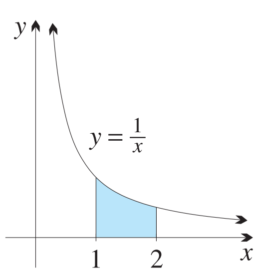
i $1 \le x \le e$ **ii** $1 \le x \le 5$

b Find the area between the curve $y = \frac{1}{x}$ and the x-axis for:

i $e \le x \le e^2$ **ii** $2 \le x \le 8$

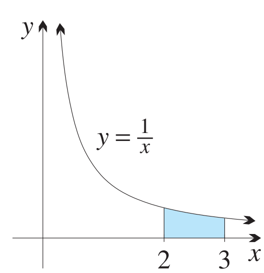
iii $1 \le x \le e^2$ **iv** $1 \le x \le 25$

3 **a**



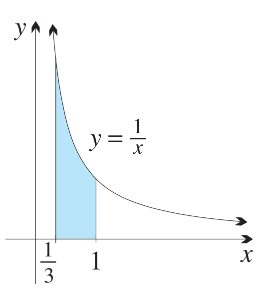
Find the area of the region bounded by the curve $y = \frac{1}{x}$, the x-axis, and the lines $x = 1$ and $x = 2$.

b



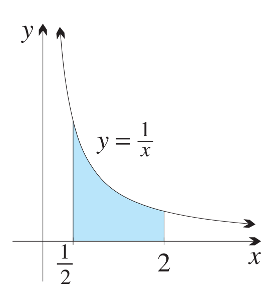
Find the area of the region bounded by the curve $y = \frac{1}{x}$, the x-axis, and the lines $x = 2$ and $x = 3$.

c



Find the area of the region bounded by the curve $y = \frac{1}{x}$, the x-axis, and the lines $x = \frac{1}{3}$ and $x = 1$.

d



Find the area of the region bounded by the curve $y = \frac{1}{x}$, the x-axis, and the lines $x = \frac{1}{2}$ and $x = 2$.

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4 Answer each question by first giving your answer in exact form and then finding an approximation correct to four significant figures. In each case you will need to use the standard form

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e|ax+b| + C.$$

a Find the area between $y = \dfrac{1}{2x+1}$ and the x-axis for:

i $2 \le x \le 5$ **ii** $1 \le x \le 4$

b Find the area between $y = \dfrac{1}{3x+2}$ and the x-axis for:

i $0 \le x \le 1$ **ii** $0 \le x \le 6$

c Find the area between $y = \dfrac{1}{2x-5}$ and the x-axis for:

i $3 \le x \le 4$ **ii** $4 \le x \le 16$

d Find the area between $y = \dfrac{3}{x-1}$ and the x-axis for:

i $2 \le x \le e^3 + 1$ **ii** $3 \le x \le 12$

DEVELOPMENT

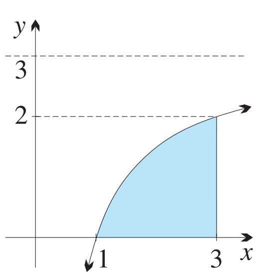
5 **a** Find the area between the graph of $y = \dfrac{1}{x} + 1$ and the x-axis, from $x = 1$ to $x = 2$.

b Find the area between the graph of $y = \dfrac{1}{x} + x$ and the x-axis, from $x = \frac{1}{2}$ to $x = 2$.

c Find the area between the graph of $y = \dfrac{1}{x} + x^2$ and the x-axis, from $x = 1$ to $x = 3$.

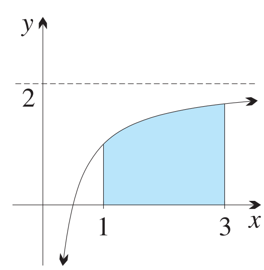
6 Give your answers to each question below in exact form.

a



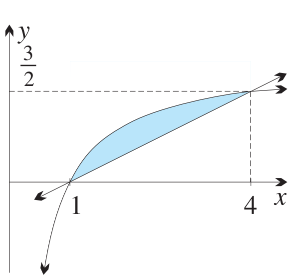
Find the area of the region bounded by $y = 3 - \dfrac{3}{x}$, the x-axis and $x = 3$.

b



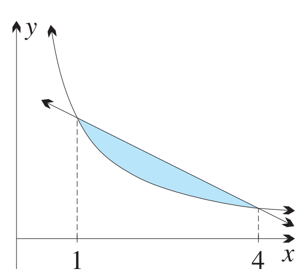
Find the area of the region bounded by $y = 2 - \dfrac{1}{x}$, the x-axis, $x = 1$ and $x = 3$.

7 **a**



Find the area of the region bounded by $y = 2 - \dfrac{2}{x}$ and the line $y = \frac{1}{2}(x-1)$.

b



Find the area of the region between $y = \dfrac{2}{x}$ and the line $x + 2y - 5 = 0$.

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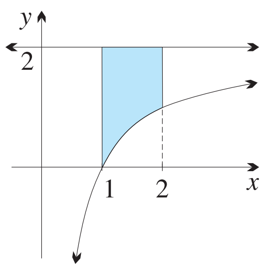
8 **a** Sketch the region bounded by $y = 1$, $x = 8$ and the curve $y = \dfrac{4}{x}$.

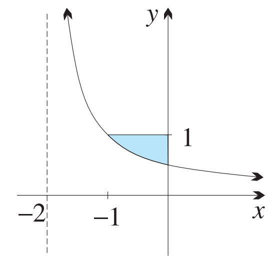
b Determine the area of this region with the aid of an appropriate integral.

9 **a**



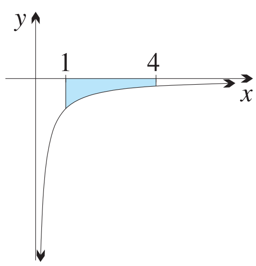
Find the area of the region in the first quadrant bounded by $y = 2 - \dfrac{2}{x}$ and $y = 2$, and lying between $x = 1$ and $x = 2$.

b



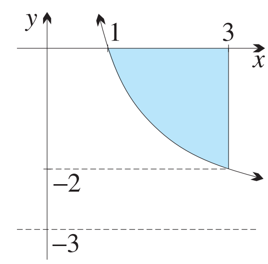
Find the area of the region bounded by the curve $y = \dfrac{1}{x + 2}$, the y-axis and the horizontal line $y = 1$.

10 **a**



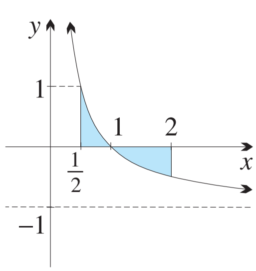
Find the area of the region bounded by $y = -\dfrac{1}{x}$, the x-axis, $x = 1$ and $x = 4$.

b



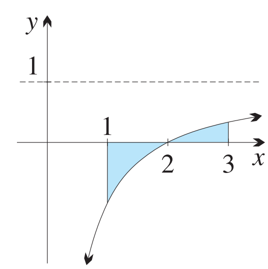
Find the area of the region bounded by $y = \dfrac{3}{x} - 3$, the x-axis and $x = 3$.

11 **a**



Find the area of the region bounded by $y = \dfrac{1}{x} - 1$, the x-axis, $x = \dfrac{1}{2}$ and $x = 2$.

b



Find the area of the region bounded by $y = 1 - \dfrac{2}{x}$, the x-axis, $x = 1$ and $x = 3$.

12 **a** Find the two intersection points of the curve $y = \dfrac{1}{x}$ with the line $y = 4 - 3x$.

b Determine the area between these two curves.

13 **a** What is the derivative of $x^2 + 1$?

b Find the area under the graph $y = \dfrac{x}{x^2 + 1}$, between $x = 0$ and $x = 2$.

14 **a** Find the derivative of $x^2 + 2x + 3$.

b Find the area under the graph $y = \dfrac{x + 1}{x^2 + 2x + 3}$, between $x = 0$ and $x = 1$.

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15 **a** Sketch the region bounded by the x-axis, $y = x$, $y = \frac{1}{x}$ and $x = e$.

b Hence find the area of this region by using two appropriate integrals.

16 **a** Find the exact value of $\int_1^2 \frac{1}{x}\, dx$, then approximate it correct to three decimal places.

b Use the trapezoidal rule with function values at $x = 1, \frac{3}{2}$ and 2 to approximate the area found in part **a**.

17 In this question, give any approximations correct to four decimal places.

a Find the area between the curve $y = \frac{1}{x}$ and the x-axis, for $1 \le x \le 3$, by evaluating an appropriate integral. Then approximate the result.

b Estimate the area using the trapezoidal rule with two subintervals (that is, three function values).

18 Use the trapezoidal rule with four subintervals to approximate the area between the curve $y = \ln x$ and the x-axis, between $x = 1$ and $x = 5$. Answer correct to four decimal places.

CHALLENGE

19 **a** Sketch $y = \log_e x$, for $0 \le x \le e$.

b Evaluate the area between the curve and the y-axis, between $y = 0$ and $y = 1$.

c Hence find the area between the curve and the x-axis, between $x = 1$ and $x = e$.

20 Consider the two curves $y = 6e^{-x}$ and $y = e^x - 1$.

a Let $u = e^x$. Show that the x-coordinate of the point of intersection of these two curves satisfies $u^2 - u - 6 = 0$.

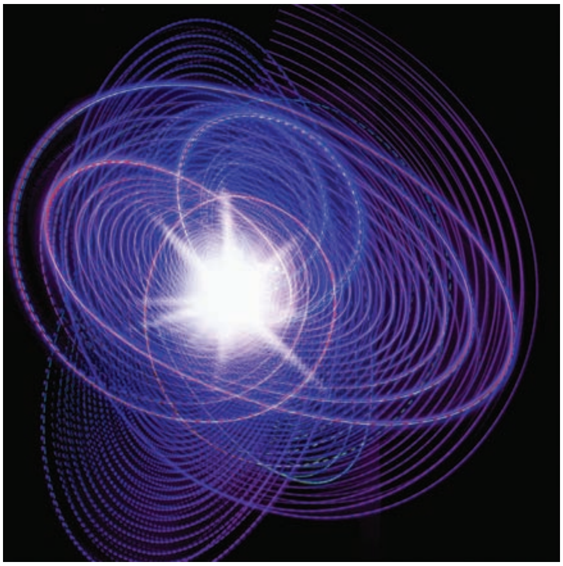
b Hence find the coordinates of the point of intersection.

c Sketch the curves on the same number plane, and shade the region bounded by them and the y-axis.

d Find the area of the shaded region.

21 The hyperbola $y = \frac{1}{x} + 1$ meets the x-axis at $(-1, 0)$. Find the area contained between the x-axis and the curve from:

a $x = -e$ to $x = -1$, **b** $x = -1$ to $x = -e^{-1}$, **c** $x = -e$ to $x = -e^{-1}$.



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5K Calculus with other bases

In applications of exponential functions where calculus is required, the base e can generally be used. For example, the treatment of exponential growth in Chapter 9 of the Year 11 book was done entirely using base e.

The change-of-base formula, however, allows calculus to be applied to exponential and logarithmic functions of any base without conversion to base e. In this section, we develop three further standard forms that allows calculus to be applied straightforwardly to functions such as $y = 2^x$ and $y = 10^x$.

Throughout this section, the other base a must be positive and not equal to 1.

Logarithmic functions to other bases

Any logarithmic functions can be expressed easily in terms of $\log_e x$ by using the change-of-base formula. For example,

$$\log_2 x = \frac{\log_e x}{\log_e 2}.$$

Thus every other logarithmic function is just a constant multiple of $\log_e x$. This allows any other logarithmic function to be differentiated easily.

Example 39 — 5K

a Express the function $y = \log_5 x$ in terms of the function $\log_e x$.

b Hence use the calculator function labelled $\boxed{\ln}$ to approximate, correct to four decimal places:

i $\log_5 30$ **ii** $\log_5 2$ **iii** $\log_5 0.07$

c Check the results of part **b** using the function labelled $\boxed{x^y}$.

SOLUTION

a $\log_5 x = \dfrac{\log_e x}{\log_e 5}$

b i $\log_5 30 = \dfrac{\log_e 30}{\log_e 5} \doteqdot 2.1133$

ii $\log_5 2 = \dfrac{\log_e 2}{\log_e 5} \doteqdot 0.4307$

iii $\log_5 0.07 = \dfrac{\log_e 0.07}{\log_e 5} \doteqdot -1.6523$

c Checking these results using the function labelled $\boxed{x^y}$:

i $5^{2.1133} \doteqdot 30$ **ii** $5^{0.4307} \doteqdot 2$ **iii** $5^{-1.6523} \doteqdot 0.07$

9 LOGARITHMIC FUNCTIONS WITH OTHER BASES

Every logarithmic function can be written as a multiple of a logarithmic function base e:

$$\log_a x = \frac{\log_e x}{\log_e a}, \qquad \text{that is} \qquad \log_a x = \frac{1}{\log_e a} \times \log_e x.$$

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Differentiating logarithmic functions with other bases

Once the function is expressed as a multiple of a logarithmic function base e, it can be differentiated using the previous standard forms.

Example 40 5K

Use the change-of-base formula to differentiate:

a $y = \log_2 x$

b $y = \log_a x$

SOLUTION

a Here $y = \log_2 x$.

Using the change-of-base formula,

$$y = \frac{\log_e x}{\log_e 2}.$$

Because $\log_e 2$ is a constant,

$$\frac{dy}{dx} = \frac{1}{x} \times \frac{1}{\log_e 2}$$

$$= \frac{1}{x \log_e 2}.$$

b Here $y = \log_a x$.

Using the change-of-base formula,

$$y = \frac{\log_e x}{\log_e a}.$$

Because $\log_e a$ is a constant,

$$\frac{dy}{dx} = \frac{1}{x} \times \frac{1}{\log_e a}$$

$$= \frac{1}{x \log_e a}.$$

Part **b** above gives the formula in the general case:

10 DIFFERENTIATING LOGARITHMIC FUNCTIONS WITH OTHER BASES

- Either use the change-of-base formula to convert to logarithms base e.
- Or use the standard form $\dfrac{d}{dx} \log_a x = \dfrac{1}{x \log_e a}$.

Example 41 5K

a Differentiate $\log_{10} x$.

b Differentiate $\log_{1.05} x$.

SOLUTION

a $\dfrac{d}{dx} \log_{10} x = \dfrac{1}{x \log_e 10}$

b $\dfrac{d}{dx} \log_{1.05} x = \dfrac{1}{x \log_e 1.05}$

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A characterisation of the logarithmic function

We have already discussed in Section 5F that the tangent to $y = \log_e x$ at the x-intercept has gradient exactly 1.

The worked example below shows that this property distinguishes the logarithmic function base e from all other logarithmic functions.

Example 42 5K

a Show that the tangent to $y = \log_a x$ at the x-intercept has gradient $\dfrac{1}{\log_e a}$.

b Show that the function $y = \log_e x$ is the only logarithmic function whose gradient at the x-intercept is exactly 1.

SOLUTION

a Here $\qquad y = \log_a x.$

When $y = 0$, $\qquad \log_a x = 0$

$\qquad x = 1,$

so the x-intercept is $(1, 0)$.

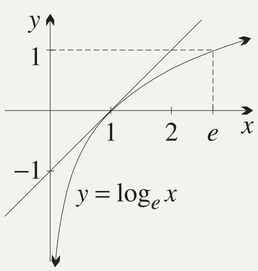
Differentiating, $\quad y' = \dfrac{1}{x \log_e a},$

so when $x = 1$, $\quad y' = \dfrac{1}{\log_e a},$ as required.

b The gradient at the x-intercept is 1 if and only if

$$\begin{aligned} \log_e a &= 1 \\ a &= e^1 \\ &= e, \end{aligned}$$

that is, if and only if the original base a is equal to e.



11 THE GRADIENT AT THE x-INTERCEPT

The function $y = \log_e x$ is the only logarithmic function whose gradient at the x-intercept is exactly 1.

Exponential functions with other bases

Before calculus can be applied to an exponential function $y = a^x$ with base a different from e, it must be written as an exponential function with base e. The important identity used to do this is

$$e^{\log_e a} = a,$$

which simply expresses the fact that the functions e^x and $\log_e x$ are inverse functions. Now a^x can be written as

$$\begin{aligned} a^x &= (e^{\log_e a})^x, && \text{replacing } a \text{ by } e^{\log_e a}, \\ &= e^{x \log_e a}, && \text{using the index law } (e^k)^x = e^{kx}. \end{aligned}$$

Thus a^x has been expressed in the form e^{kx}, where $k = \log_e a$ is a constant.

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12 EXPONENTIAL FUNCTIONS WITH OTHER BASES

- Every positive real number can be written as a power of e:
$$a = e^{\log_e a}$$
- Every exponential function can be written as an exponential function base e:
$$a^x = e^{x \log_e a}$$

Example 43 5K

Express these numbers and functions as powers of e.

a 2 **b** 2^x **c** 5^{-x}

SOLUTION

a $2 = e^{\log_e 2}$

b $$\begin{aligned} 2^x &= \left(e^{\log_e 2}\right)^x \\ &= e^{x \log_e 2} \end{aligned}$$

c $$\begin{aligned} 5^{-x} &= \left(e^{\log_e 5}\right)^{-x} \\ &= e^{-x \log_e 5} \end{aligned}$$

Differentiating and integrating exponential functions with other bases

Write the function as a power of e. It can then be differentiated and integrated.

First,
$$\begin{aligned} a^x &= e^{\log_e a^x} \\ &= e^{x \log_e a}. \end{aligned}$$

Differentiating,
$$\begin{aligned} \frac{d}{dx} a^x &= \frac{d}{dx} e^{x \log_e a} \\ &= e^{x \log_e a} \times \log_e a, \quad \text{because } \frac{d}{dx} e^{kx} = k e^{kx}, \\ &= a^x \log_e a, \quad \text{because } e^{x \log_e a} = a^x. \end{aligned}$$

Integrating,
$$\begin{aligned} \int a^x \, dx &= \int e^{x \log_e a} \, dx \\ &= \frac{e^{x \log_e a}}{\log_e a}, \quad \text{because } \int e^{kx} = \frac{1}{k} e^{kx}, \\ &= \frac{a^x}{\log_e a}, \quad \text{because } e^{x \log_e a} = a^x. \end{aligned}$$

This process can be carried through every time, or the results can be remembered as standard forms.

13 DIFFERENTIATION AND INTEGRATION WITH OTHER BASES

There are two approaches.

- Write all powers with base e before differentiating or integrating.
- Alternatively, use the standard forms:
$$\frac{d}{dx} a^x = a^x \log_e a \qquad \text{and} \qquad \int a^x \, dx = \frac{a^x}{\log_e a} + C$$

Note: The formulae for differentiating and integrating a^x both involve the constant $\log_e a$. This constant $\log_e a$ is 1 when $a = e$, so the formulae are simplest when the base is e. Again, this indicates that e is the appropriate base to use for calculus with exponential functions.

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Example 44

Differentiate $y = 2^x$. Hence find the gradient of $y = 2^x$ at the y-intercept, correct to three significant figures.

SOLUTION

Here $\quad y = 2^x$.

Using the standard form, $\quad y' = 2^x \log_e 2$.

Hence when $x = 0$, $\quad y' = 2^0 \times \log_e 2$

$$= \log_e 2$$

$$\doteqdot 0.693.$$

Note: This result may be compared with the results of physically measuring this gradient in Question 1 of Exercise 9A in the Year 11 book.

Example 45 5K

a Show that the line $y = x + 1$ meets the curve $y = 2^x$ at $A(0, 1)$ and $B(1, 2)$.

b Sketch the two curves and shade the region contained between them.

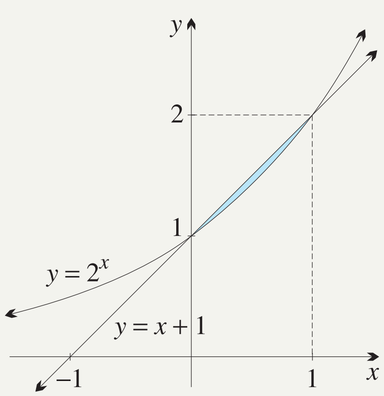
c Find the area of this shaded region, correct to four significant figures.

SOLUTION

a Simple substitution of $x = 0$ and $x = 1$ into both functions verifies the result.

b The graph is drawn to the right.

c
$$\text{Area} = \int_0^1 (\text{upper curve} - \text{lower curve})\, dx$$
$$= \int_0^1 (x + 1 - 2^x)\, dx$$
$$= \left[\tfrac{1}{2}x^2 + x - \frac{2^x}{\log_e 2}\right]_0^1$$
$$= \left(\tfrac{1}{2} + 1 - \frac{2}{\log_e 2}\right) - \left(0 + 0 - \frac{1}{\log_e 2}\right)$$
$$= 1\tfrac{1}{2} - \frac{1}{\log_e 2} \text{ square units}$$
$$\doteqdot 0.05730 \text{ square units.}$$



Exercise 5K

FOUNDATION

1 Use the change-of-base formula $\log_a x = \dfrac{\log_e x}{\log_e a}$ and the function labelled $\boxed{\text{ln}}$ on your calculator to evaluate each expression correct to three significant figures. Then check your answers using the function labelled $\boxed{x^y}$.

a $\log_2 3$ **b** $\log_2 10$ **c** $\log_5 26$ **d** $\log_3 0.0047$

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2 Use the change-of-base formula to express these with base e, then differentiate them.

a $y = \log_2 x$ **b** $y = \log_{10} x$ **c** $y = 3\log_5 x$

3 Use the standard form $\dfrac{d}{dx}\log_a x = \dfrac{1}{x\log_e a}$ to differentiate:

a $y = \log_3 x$ **b** $y = \log_7 x$ **c** $y = 5\log_6 x$

4 Express these functions as powers of e, then differentiate them.

a $y = 3^x$ **b** $y = 4^x$ **c** $y = 2^x$

5 Use the standard form $\dfrac{d}{dx}a^x = a^x\log_e a$ to differentiate:

a $y = 10^x$ **b** $y = 8^x$ **c** $y = 3 \times 5^x$

6 Convert each integrand to a power of e and then integrate.

a $\int 2^x\,dx$ **b** $\int 6^x\,dx$ **c** $\int 7^x\,dx$ **d** $\int 3^x\,dx$

7 Use the result $\int a^x\,dx = \dfrac{a^x}{\log_e a} + C$ to find each primitive, then evaluate the definite integral correct to four significant figures.

a $\int_0^1 2^x\,dx$ **b** $\int_0^1 3^x\,dx$ **c** $\int_{-1}^1 5^x\,dx$ **d** $\int_0^2 4^x\,dx$

8 **a** Complete the table of values to the right, giving your answers correct to two decimal places where necessary.

b Use this table of values to sketch the three curves $y = \log_2 x$, $y = \log_e x$ and $y = \log_4 x$ on the same set of axes.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_2 x$					
$\log_e x$					
$\log_4 x$					

DEVELOPMENT

9 **a** Differentiate $y = \log_2 x$. Hence find the gradient of the tangent to the curve at $x = 1$.

b Hence find the equation of the tangent there.

c Do likewise for:

i $y = \log_3 x$, **ii** $y = \log_5 x$.

10 Give the exact value of each integral, then evaluate it correct to four decimal places.

a $\int_1^3 2^x\,dx$ **b** $\int_{-1}^1 (3^x + 1)\,dx$ **c** $\int_0^2 (10^x - 10x)\,dx$

11 Use the change-of-base formula to express $y = \log_{10} x$ with base e, and hence find y'.

a Find the gradient of the tangent to this curve at the point $(10, 1)$.

b Thus determine the equation of this tangent in general form.

c At what value of x will the tangent have gradient 1?

12 **a** Find the equations of the tangents to each of $y = \log_2 x$, $y = \log_e x$ and $y = \log_4 x$ at the points where $x = 3$.

b Show that the three tangents all meet at the same point on the x-axis.

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13 a Show that the curves $y = 2^x$ and $y = 1 + 2x - x^2$ intersect at $A(0, 1)$ and $B(1, 2)$.
b Sketch the curves and find the area between them.

14 Find the intercepts of the curve $y = 8 - 2^x$, and hence find the area of the region bounded by this curve and the coordinate axes.

15 a Sketch the curve $y = 3 - 3^x$, showing the intercepts and asymptote.
b Find the area contained between the curve and the axes.

16 a Show that the curves $y = x + 1$ and $y = 4^x$ intersect at the y-intercept and at $\left(-\frac{1}{2}, \frac{1}{2}\right)$.
b Write the area of the region enclosed between these two curves as an integral.
c Evaluate the integral found in part **b**.

CHALLENGE

17 a Show that the tangent to $y = \log_3 x$ at $x = e$ passes through the origin.
b Show that the tangent to $y = \log_5 x$ at $x = e$ passes through the origin.
c Show that the same is true for $y = \log_a x$, for any base a.

18 a Differentiate $x \log_e x - x$, and hence find $\int \log_e x \, dx$.
b Use the change-of-base formula and the integral in part **a** to evaluate $\int_1^{10} \log_{10} x \, dx$.

19 As always, the three standard forms in this section have linear extensions. The pronumeral m is used here instead of the usual a because a is being used for the base.

a Use the standard form $\frac{d}{dx} \log_a(mx + b) = \frac{m}{(mx + b)\log_e a}$ to differentiate:

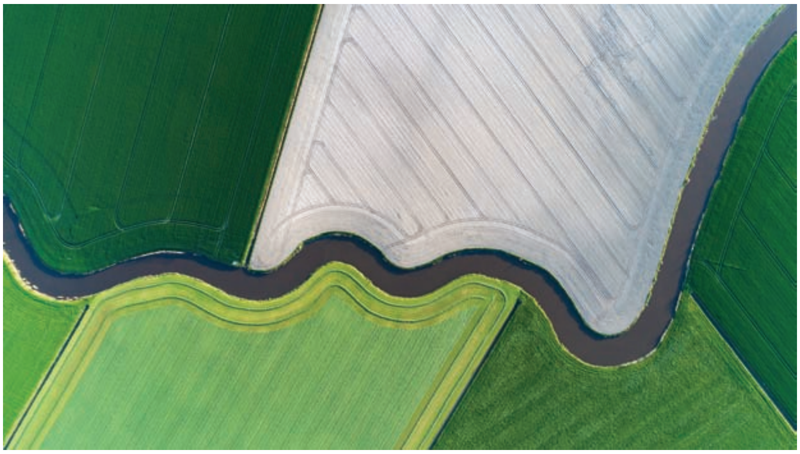
i $y = \log_3 x$ **ii** $y = \log_7(2x + 3)$ **iii** $y = 5 \log_6(4 - 9x)$

b Use the standard form $\frac{d}{dx} a^{mx+b} = ma^{mx+b} \log_e a$ to differentiate:

i $y = 10^x$ **ii** $y = 8^{4x-3}$ **iii** $y = 3 \times 5^{2-7x}$

c Use the standard form $\int a^{mx+b} dx = \frac{a^{mx+b}}{m \log_e a} + C$ to find:

i $\int 3^{5x} dx$ **ii** $\int 6^{2x+7} dx$ **i** $\int 5 \times 7^{4-9x} dx$



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Chapter 5 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 5 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 **a** Sketch the graphs of $y = e^x$ and $y = e^{-x}$ on the same number plane. Add the line that reflects each graph onto the other graph. Then draw the tangents at the y-intercepts, and mark the angle between them.

b Sketch the graphs of $y = e^x$ and $y = \log_e x$ on the same number plane. Add the line that reflects each graph onto the other graph. Then draw the tangents at the intercepts with the axes.

2 Use your calculator, and in some cases the change-of base formula, to approximate each expression correct to four significant figures.

a e^4 **b** e **c** $e^{-\frac{3}{2}}$ **d** $\log_e 2$

e $\log_{10} \frac{1}{2}$ **f** $\log_2 0.03$ **g** $\log_{1.05} 586$ **h** $\log_8 3\frac{3}{7}$

3 Use logarithms to solve these equations correct to four significant figures. You will need to apply the change-of-base formula before using your calculator.

a $3^x = 14$ **b** $2^x = 51$ **c** $4^x = 1345$ **d** $5^x = 132$

4 Simplify:

a $e^{2x} \times e^{3x}$ **b** $e^{7x} \div e^x$ **c** $\dfrac{e^{2x}}{e^{6x}}$ **d** $(e^{3x})^3$

5 Solve each equation using a suitable substitution to reduce it to a quadratic.

a $9^x - 7 \times 3^x - 18 = 0$ **b** $e^{2x} - 11e^x + 28 = 0$

6 Sketch the graph of each function on a separate number plane, and state its range.

a $y = e^x$ **b** $y = e^{-x}$ **c** $y = e^x + 1$ **d** $y = e^{-x} - 1$

7 **a** **i** Explain how $y = e^{x-3}$ can be obtained by translating $y = e^x$, and sketch it.

ii Explain how $y = e^{x-3}$ can be obtained by dilating $y = e^x$.

b **i** Explain how $y = \log_e 3x$ can be obtained by dilating $y = \log_e x$, and sketch it.

ii Explain how $y = \log_e 3x$ can be obtained by translating $y = \log_e x$.

8 Differentiate:

a $y = e^x$ **b** $y = e^{3x}$ **c** $y = e^{2x+3}$ **d** $y = e^{-x}$

e $y = e^{-3x}$ **f** $y = 3e^{2x+5}$ **g** $y = 4e^{\frac{1}{2}x}$ **h** $y = \frac{2}{3}e^{6x-5}$

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Review

9 Write each function as a single power of e, and then differentiate it.

a $y = e^{3x} \times e^{2x}$ **b** $y = \dfrac{e^{7x}}{e^{3x}}$ **c** $y = \dfrac{e^{x}}{e^{4x}}$ **d** $y = (e^{-2x})^3$

10 Differentiate each function using the chain, product and quotient rules as appropriate.

a $y = e^{x^3}$ **b** $y = e^{x^2-3x}$ **c** $y = xe^{2x}$ **d** $y = (e^{2x} + 1)^3$

e $y = \dfrac{e^{3x}}{x}$ **f** $y = x^2e^{x^2}$ **g** $y = (e^x - e^{-x})^5$ **h** $y = \dfrac{e^{2x}}{2x + 1}$

11 Find the first and second derivatives of:

a $y = e^{2x+1}$ **b** $y = e^{x^2+1}$

12 Find the equation of the tangent to the curve $y = e^x$ at the point where $x = 2$, and find the x-intercept and y-intercept of this tangent.

13 Consider the curve $y = e^{-3x}$.

a Find the gradient of the normal to the curve at the point where $x = 0$.

b Find y'' and hence determine the concavity of the curve at the point where $x = 0$.

14 Consider the curve $y = e^x - x$.

a Find y' and y''.

b Show that there is a stationary point at $(0, 1)$, and determine its nature.

c Explain why the curve is concave up for all values of x.

d Sketch the curve and write down its range.

15 Find the stationary point on the curve $y = xe^{-2x}$ and determine its nature.

16 Find:

a $\displaystyle\int e^{5x}\,dx$ **b** $\displaystyle\int 10e^{2-5x}\,dx$ **c** $\displaystyle\int e^{\frac{1}{5}x}\,dx$ **d** $\displaystyle\int 3e^{5x-4}\,dx$

17 Find the exact value of:

a $\displaystyle\int_0^2 e^x\,dx$ **b** $\displaystyle\int_0^1 e^{2x}\,dx$ **c** $\displaystyle\int_{-1}^0 e^{-x}\,dx$

d $\displaystyle\int_{-\frac{2}{3}}^0 e^{3x+2}\,dx$ **e** $\displaystyle\int_0^{\frac{1}{2}} e^{3-2x}\,dx$ **f** $\displaystyle\int_0^2 2e^{\frac{1}{2}x}\,dx$

18 Find the primitive of:

a $\dfrac{1}{e^{5x}}$ **b** $e^{3x} \times e^x$ **c** $\dfrac{6}{e^{3x}}$ **d** $(e^{3x})^2$

e $\dfrac{e^{3x}}{e^{5x}}$ **f** $\dfrac{e^{3x} + 1}{e^{2x}}$ **g** $e^{2x}(e^x + e^{-x})$ **h** $(1 + e^{-x})^2$

19 Find the exact value of:

a $\displaystyle\int_0^1 (1 + e^{-x})\,dx$ **b** $\displaystyle\int_0^2 (e^{2x} + x)\,dx$ **c** $\displaystyle\int_0^1 \frac{2}{e^x}\,dx$

d $\displaystyle\int_0^{\frac{1}{3}} e^{3x}(1 - e^{-3x})\,dx$ **e** $\displaystyle\int_0^1 \frac{e^{2x} + 1}{e^x}\,dx$ **f** $\displaystyle\int_0^1 (e^x + 1)^2\,dx$

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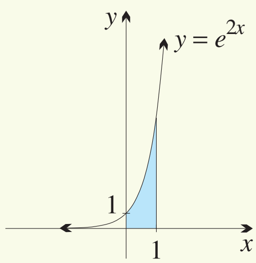
20 If $f'(x) = e^{x} - e^{-x} - 1$ and $f(0) = 3$, find $f(x)$ and then find $f(1)$.

21 **a** Differentiate e^{x^3}.

b Hence find $\int_0^1 x^2 e^{x^3}\,dx$.

22 Find the area of each region correct to three significant figures.

a



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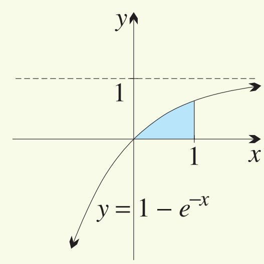
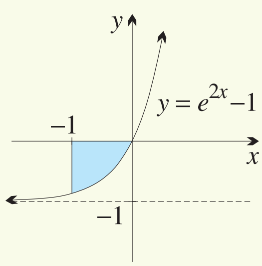


23 Find the exact area of the shaded region.

a



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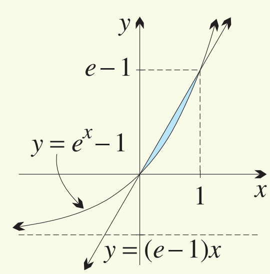

24 Sketch graphs of these functions, clearly indicating the vertical asymptote in each case.

a $y = \log_2 x$ **b** $y = -\log_2 x$ **c** $y = \log_2(x - 1)$ **d** $y = \log_2(x + 3)$

25 Sketch graphs of these functions, clearly indicating the vertical asymptote in each case.

a $y = \log_e x$ **b** $y = \log_e(-x)$ **c** $y = \log_e(x - 2)$ **d** $y = \log_e x + 1$

26 Use the log laws to simplify:

a $e \log_e e$ **b** $\log_e e^3$ **c** $\ln \dfrac{1}{e}$ **d** $2e \ln \sqrt{e}$

27 Differentiate these functions.

a $\log_e x$ **b** $\log_e 2x$ **c** $\log_e(x + 4)$

d $\log_e(2x - 5)$ **e** $2\log_e(5x - 1)$ **f** $x + \log_e x$

g $\ln(x^2 - 5x + 2)$ **h** $\ln(1 + 3x^5)$ **i** $4x^2 - 8x^3 + \ln(x^2 - 2)$

28 Use the log laws to simplify each function and then find its derivative.

a $\log_e x^3$ **b** $\log_e \sqrt{x}$ **c** $\ln x(x + 2)$ **d** $\ln \dfrac{x}{x - 1}$

29 Differentiate these functions using the product or quotient rule.

a $x \log x$ **b** $e^{x} \log x$ **c** $\dfrac{x}{\ln x}$ **d** $\dfrac{\ln x}{x^2}$

30 Find the equation of the tangent to the curve $y = 3\log_e x + 4$ at the point $(1, 4)$.

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31 Consider the function $y = x - \log_e x$.

a Show that $y' = \dfrac{x-1}{x}$.

b Hence show that the graph of $y = x - \log_e x$ has a minimum turning point at $(1, 1)$.

32 Find these indefinite integrals.

a $\displaystyle\int \frac{1}{x}\,dx$ **b** $\displaystyle\int \frac{3}{x}\,dx$ **c** $\displaystyle\int \frac{1}{5x}\,dx$ **d** $\displaystyle\int \frac{1}{x+7}\,dx$

e $\displaystyle\int \frac{1}{2x-1}\,dx$ **f** $\displaystyle\int \frac{1}{2-3x}\,dx$ **g** $\displaystyle\int \frac{2}{2x+9}\,dx$ **h** $\displaystyle\int \frac{8}{1-4x}\,dx$

33 Evaluate these definite integrals.

a $\displaystyle\int_0^1 \frac{1}{x+2}\,dx$ **b** $\displaystyle\int_1^4 \frac{1}{4x-3}\,dx$ **c** $\displaystyle\int_1^e \frac{1}{x}\,dx$ **d** $\displaystyle\int_{e^2}^{e^3} \frac{1}{x}\,dx$

34 Use the standard form $\displaystyle\int \frac{u'}{u}\,dx = \log|u| + C$ or $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log_e|f(x)| + C$ to find:

a $\displaystyle\int \frac{2x}{x^2+4}\,dx$ **b** $\displaystyle\int \frac{3x^2-5}{x^3-5x+7}\,dx$ **c** $\displaystyle\int \frac{x}{x^2-3}\,dx$ **d** $\displaystyle\int \frac{x^3-1}{x^4-4x}\,dx$

35 Find the area of the region bounded by the curve $y = \dfrac{1}{x}$, the x-axis and the lines $x = 2$ and $x = 4$.

36 a By solving the equations simultaneously, show that the curve $y = \dfrac{5}{x}$ and the line $y = 6 - x$ intersect at the points $(1, 5)$ and $(5, 1)$.

b By sketching both graphs on the same number plane, find the area of the region enclosed between them.

37 Find the derivatives of:

a e^x **b** 2^x **c** 3^x **d** 5^x

38 Find these indefinite integrals.

a $\displaystyle\int e^x\,dx$ **b** $\displaystyle\int 2^x\,dx$ **c** $\displaystyle\int 3^x\,dx$ **d** $\displaystyle\int 5^x\,dx$

39 a Differentiate $x\log_e x$, and hence find $\displaystyle\int \log_e x\,dx$.

b Differentiate $x\,e^x$, and hence find $\displaystyle\int xe^x\,dx$.

c Hence prove that $\displaystyle\int_1^e \frac{1}{x}\,dx = \int_1^e \log_e x\,dx = \int_1^e xe^x\,dx = 1$.

40 a Find the gradient of $y = 2^x$ at $A(3, 8)$.

b Find the gradient of $y = \log_2 x$ at $B(8, 3)$.

c Explain geometrically why the two gradients are reciprocals of each other.

41 a Find $\displaystyle\int_0^3 2^x\,dx$ and $\displaystyle\int_{-3}^0 2^x\,dx$.

b Explain geometrically why the first is 8 times the second.

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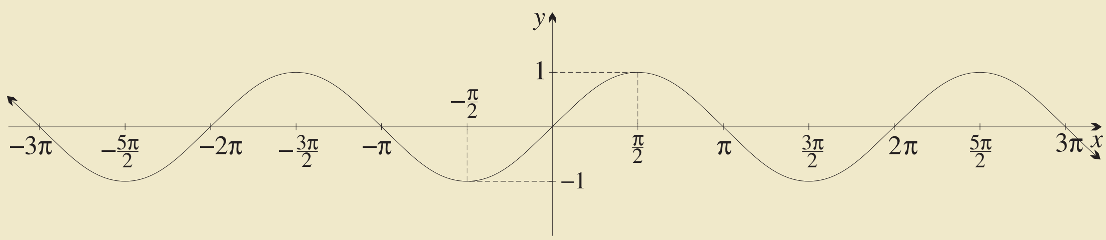
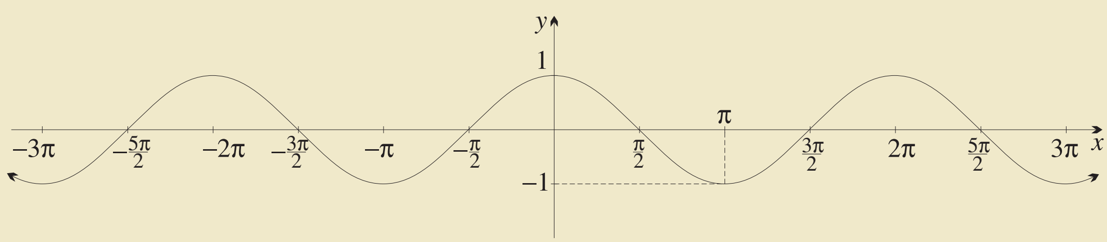
6 The trigonometric functions

This chapter extends calculus to the trigonometric functions. The sine and cosine functions are extremely important because their graphs are waves. They are therefore essential in the modelling of all the many wave-like phenomena such as sound waves, light and radio waves, vibrating strings, tides, and economic cycles. The alternating current that we use in our homes fluctuates in a sine wave. Most of the attention in this chapter is given to these two functions.

$y = \sin x$



$y = \cos x$



In the second half of Chapter 9 of the Year 11 book, we introduced radian measure, promising that it was the correct way to measure angles when doing calculus. We drew the six trigonometric graphs in radians and discussed their symmetries in some detail, and also developed area formula for calculating arc length and the areas of sectors and segments. Then in the last section of Chapter 2, we applied translations, reflections and dilations to the trigonometric graphs and developed the four ideas of amplitude, period, phase and mean value. All this previous work is required in the present chapter.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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6A The behaviour of sin x near the origin

This section proves an important limit that is the crucial step in finding the derivative of $\sin x$ in the next section. This limit establishes that the curve $y = \sin x$ has gradient 1 when it passes through the origin. Geometrically, this means that the line $y = x$ is the tangent to $y = \sin x$ at the origin.

Note: The limit established in this section provides the geometric basis for differentiating the trigonometric functions, but the material is not easy, and the section could well be left to a second reading of the chapter at a later time.

A fundamental inequality

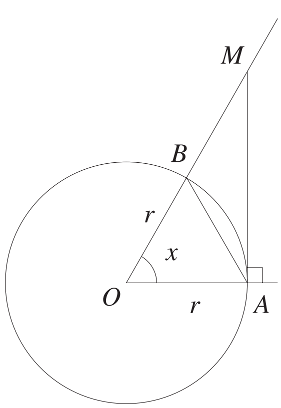
First, an appeal to geometry is needed to establish an inequality concerning x, $\sin x$ and $\tan x$.

1 AN INEQUALITY FOR sin x AND tan x NEAR THE ORIGIN

- For all acute angles x, $\quad \sin x < x < \tan x$.
- For $-\frac{\pi}{2} < x < 0$, $\quad \sin x > x > \tan x$.

Proof

A Let x be an acute angle.
Construct a circle of centre O and any radius r,
and a sector AOB subtending the angle x at the centre O.
Let the tangent at A meet the radius OB at M
(the radius OB will need to be produced) and join the chord AB.



In $\triangle OAM$, $\quad \dfrac{AM}{r} = \tan x$,

so $\quad AM = r\tan x$.

It is clear from the diagram that

$$\text{area}\ \triangle OAB < \text{area sector}\ OAB < \text{area}\ \triangle OAM,$$

and using area formulae for triangles and sectors,

$$\tfrac{1}{2}r^2\sin x < \tfrac{1}{2}r^2 x < \tfrac{1}{2}r^2\tan x$$

$$\boxed{\div\ \tfrac{1}{2}r^2} \quad \sin x < x < \tan x.$$

B Because x, $\sin x$ and $\tan x$ are all odd functions,

$$\sin x > x > \tan x, \quad \text{for } -\tfrac{\pi}{2} < x < 0.$$

The main theorem

This inequality now allows two fundamental limits to be proven:

2 TWO FUNDAMENTAL LIMITS

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x\to 0}\frac{\tan x}{x} = 1$$

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Proof: When x is acute, $\sin x < x < \tan x$.

Dividing through by $\sin x$, $1 < \dfrac{x}{\sin x} < \dfrac{1}{\cos x}$.

As $x \to 0^+$, $\cos x \to 1$, so $\dfrac{x}{\sin x} \to 1$ as $x \to 0^+$.

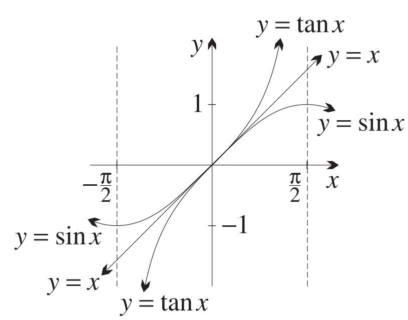
But $\dfrac{x}{\sin x}$ is even, so $\dfrac{x}{\sin x} \to 1$ as $x \to 0^-$.

Combining these two limits, $\dfrac{x}{\sin x} \to 1$ as $x \to 0$.

Finally,
$$\frac{\tan x}{x} = \frac{\sin x}{x} \times \frac{1}{\cos x}$$
$$\to 1 \times 1, \text{ as } x \to 0.$$

The diagram to the right shows what has been proven about the graphs of $y = x$, $y = \sin x$ and $y = \tan x$ near the origin.

- The line $y = x$ is a common tangent at the origin to both $y = \sin x$ and $y = \tan x$.
- On both sides of the origin, $y = \sin x$ curls away from the tangent towards the x-axis.
- On both sides of the origin, $y = \tan x$ curls away from the tangent in the opposite direction.



3 THE BEHAVIOUR OF sin x AND tan x NEAR THE ORIGIN

- The line $y = x$ is a tangent to both $y = \sin x$ and $y = \tan x$ at the origin.
- When $x = 0$, the derivatives of both $\sin x$ and $\tan x$ are exactly 1.

Approximations to the trigonometric functions for small angles

For 'small' angles, positive or negative, the limits above yield good approximations for the three trigonometric functions (the angle must, of course, be expressed in radians).

4 SMALL-ANGLE APPROXIMATIONS

- Suppose that x is a 'small' angle (written in radians). Then

$$\sin x \doteqdot x \qquad \text{and} \qquad \cos x \doteqdot 1 \qquad \text{and} \qquad \tan x \doteqdot x.$$

In order to use these approximations, one needs to get some idea about how good the approximations are. Two questions in Exercise 6A below ask for tables of values for $\sin x$, $\cos x$ and $\tan x$ for progressively smaller angles.

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Example 1 6A

Use the small-angle approximations in Box 4 to give approximate values of:

a $\sin 1°$ **b** $\cos 1°$ **c** $\tan 1°$

SOLUTION

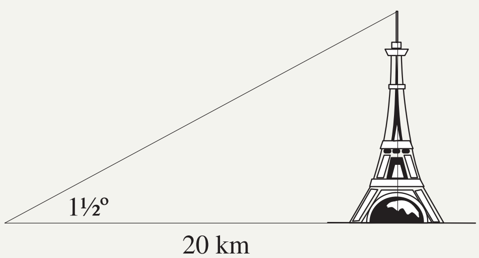
The 'small angle' of $1°$ is $\frac{\pi}{180}$ radians. Hence, using the approximations above:

a $\sin 1° \doteqdot \frac{\pi}{180}$ **b** $\cos 1° \doteqdot 1$ **c** $\tan 1° \doteqdot \frac{\pi}{180}$

Example 2 6A

Approximately how high is a tower that subtends an angle of $1\frac{1}{2}°$ when it is 20 km away?

SOLUTION



Convert 20 km to 20 000 metres.

Then from the diagram, using simple trigonometry,

$$\frac{\text{height}}{20000} = \tan 1\tfrac{1}{2}°$$
$$\text{height} = 20000 \times \tan 1\tfrac{1}{2}°.$$

But the 'small' angle $1\frac{1}{2}°$ expressed in radians is $\frac{\pi}{120}$,

so $\tan 1\frac{1}{2}° \doteqdot \frac{\pi}{120}$.

Hence, approximately, $\text{height} \doteqdot 20000 \times \dfrac{\pi}{120}$

$$\doteqdot \frac{500\pi}{3} \text{ metres}$$
$$\doteqdot 524 \text{ metres.}$$

Example 3 6A

The sun subtends an angle of $0°31'$ at the Earth, which is 150 000 000 km away. What is the sun's approximate diameter?

Note: This problem can be done similarly to the previous problem, but like many small-angle problems, it can also be done by approximating the diameter to an arc of the circle.

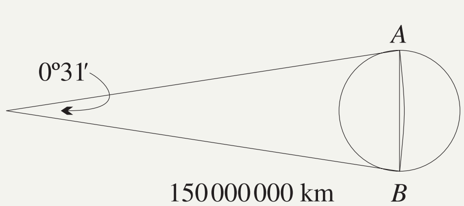
SOLUTION

First, $0°31' = \dfrac{31°}{60}$

$$= \frac{31}{60} \times \frac{\pi}{180} \text{ radians.}$$

Because the diameter AB is approximately equal to the arc length AB,

$$\text{diameter} \doteqdot r\theta$$
$$\doteqdot 150000000 \times \frac{31}{60} \times \frac{\pi}{180}$$
$$\doteqdot 1353000 \text{ km.}$$



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Exercise 6A

FOUNDATION

1 **a** Copy and complete the following table of values, giving entries correct to six decimal places. (Your calculator must be in radian mode.)

angle size in radians	1	0.5	0.2	0.1	0.08	0.05	0.02	0.01	0.005	0.002
$\sin x$										
$\dfrac{\sin x}{x}$										
$\tan x$										
$\dfrac{\tan x}{x}$										
$\cos x$										

b What are the limits of $\dfrac{\sin x}{x}$ and $\dfrac{\tan x}{x}$ approach as $x \to 0$?

2 [Technology] The previous question is perfect for a spreadsheet approach. The spreadsheet columns can be identical to the rows above. Various graphs can then be drawn using the data from the spreadsheet.

3 **a** Express $2°$ in radians.

b Explain why $\sin 2° \doteqdot \dfrac{\pi}{90}$.

c Taking π as 3.142, find $\sin 2°$, correct to four decimal places, *without* using a calculator.

DEVELOPMENT

4 **a** Copy and complete the following table of values, giving entries correct to four significant figures. For each column, hold x in the calculator's memory until the column is complete.

angle size in degrees	$60°$	$30°$	$10°$	$5°$	$2°$	$1°$	$20'$	$5'$	$1'$	$30''$	$10''$
angle size x in radians											
$\sin x$											
$\dfrac{\sin x}{x}$											
$\tan x$											
$\dfrac{\tan x}{x}$											
$\cos x$											

b Write x, $\sin x$ and $\tan x$ in ascending order, for acute angles x.

c Although $\sin x \to 0$ and $\tan x \to 0$ as $x \to 0$, what are the limits, as $x \to 0$, of:

i $\dfrac{\sin x}{x}$,

ii $\dfrac{\tan x}{x}$?

d Experiment with your calculator, or a spreadsheet, to find how small x must be in order for $\dfrac{\sin x}{x} > 0.999$ to be true.

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5 [Technology] A properly prepared spreadsheet makes it easy to ask a sequence of questions like part **d** of the previous question. One can ask how small x must be for each of the following three functions to be closer to 1 than 0.1, 0.001, 0.0001, 0.00001, . . .

$$\frac{\sin x}{x} \quad \text{and} \quad \frac{\tan x}{x} \quad \text{and} \quad \cos x.$$

6 A car travels 1 km up a road that is inclined at 5° to the horizontal. Through what vertical distance has the car climbed? (Use the fact that $\sin x \doteqdot x$ for small angles, and give your answer correct to the nearest metre.)

7 A tower is 30 metres high. What angle, correct to the nearest minute, does it subtend at a point 4 km away? (Use the fact that when x is small, $\tan x \doteqdot x$.)

CHALLENGE

8 [Technology] Draw on one screen the graphs $y = \sin x$, $y = \tan x$ and $y = x$, noting how the two trigonometric graphs curl away from $y = x$ in opposite directions. Zoom in on the origin until the three graphs are indistinguishable.

9 [Technology] Draw the graph of $y = \dfrac{\sin x}{x}$. It is undefined at the y-intercept, but the curve around this point is flat, and clearly has limit 1 as $x \to 0$. Other features of the graph can be explained, and the exercise can be repeated with the function $y = \dfrac{\tan x}{x}$.

10 The moon subtends an angle of $31'$ at an observation point on Earth, 400 000 km away. Use the fact that the diameter of the moon is approximately equal to an arc of a circle whose centre is the point of observation to show that the diameter of the moon is approximately 3600 km. (Hint: Use a diagram like that in Example 3 in the notes above.)

11 A regular polygon of 300 sides is inscribed in a circle of radius 60 cm. Show that each side is approximately 1.26 cm.

12 [A better approximation for cos x when x is small] The chord AB of a circle of radius r subtends an angle x at the centre O.

a Find AB^2 by the cosine rule, and find the length of the arc AB.

b By equating arc and chord, show that for small angles, $\cos x \doteqdot 1 - \dfrac{x^2}{2}$.

Explain whether the approximation is bigger or smaller than $\cos x$.

c Check the accuracy of the approximation for angles of 1°, 10°, 20° and 30°.

13 [Technology] Sketch on one screen the graphs of $y = \cos x$ and $y = 1 - \frac{1}{2}x^2$ as discussed in the previous question. Which one is larger, and why? A spreadsheet may help you to identify the size of the error for different values of x.

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6B Differentiating the trigonometric functions

Using the limit from Section 6A, we can now establish the derivatives of the three trigonometric functions $\sin x$, $\cos x$ and $\tan x$. Because the proofs of these standard forms are difficult, they have been placed in an Appendix at the end of the chapter. Using them to differentiate further trigonometric functions, however, is straightforward, and is the subject of this section.

Standard forms

Here are the formulae, proven in the Appendix, for the derivatives of the first three trigonometric functions.

5 STANDARD DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

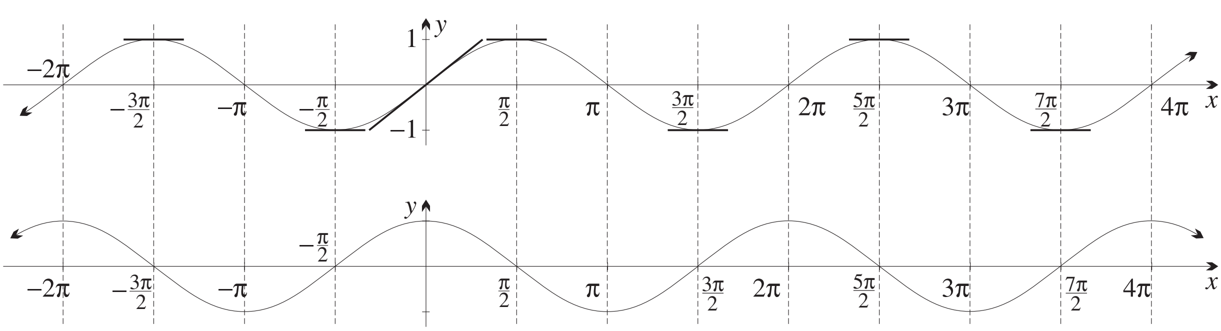
- $\dfrac{d}{dx}\sin x = \cos x$
- $\dfrac{d}{dx}\cos x = -\sin x$
- $\dfrac{d}{dx}\tan x = \sec^2 x$

The exercises ask for derivatives of the secant, cosecant and cotangent functions.

A graphical demonstration that the derivative of sin *x* is cos *x*

The upper graph in the sketch below is $y = \sin x$. The lower graph is a rough sketch of the derivative of $y = \sin x$. This second graph is straightforward to construct simply by paying attention to where the gradients of tangents to $y = \sin x$ are zero, maximum and minimum. The lower graph is periodic, with period 2π, and has a shape unmistakably like a cosine graph.

Moreover, it was proven in the previous section that the gradient of $y = \sin x$ at the origin is exactly 1. This means that the lower graph has a maximum of 1 when $x = 0$. By symmetry, all its maxima are 1 and all its minima are -1. Thus the lower graph not only has the distinctive shape of the cosine curve, but has the correct amplitude as well.



This doesn’t prove conclusively that the derivative of $\sin x$ is $\cos x$, but it is very convincing.

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Differentiating using the three standard forms

These worked examples use the standard forms to differentiate functions involving $\sin x$, $\cos x$ and $\tan x$.

Example 4 6B

Differentiate these functions.

a $y = \sin x + \cos x$

b $y = x - \tan x$

Hence find the gradient of each curve when $x = \frac{\pi}{4}$.

SOLUTION

a The function is $y = \sin x + \cos x$.

Differentiating, $y' = \cos x - \sin x$.

When $x = \frac{\pi}{4}$,
$$\begin{aligned} y' &= \cos\tfrac{\pi}{4} - \sin\tfrac{\pi}{4} \\ &= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \\ &= 0. \end{aligned}$$

b The function is $y = x - \tan x$.

Differentiating, $y' = 1 - \sec^2 x$.

When $x = \frac{\pi}{4}$,
$$\begin{aligned} y' &= 1 - \sec^2\tfrac{\pi}{4} \\ &= 1 - \left(\sqrt{2}\right)^2 \\ &= -1. \end{aligned}$$

Example 5 6B

If $f(x) = \sin x$, find $f'(0)$. Hence find the equation of the tangent to $y = \sin x$ at the origin, then sketch the curve and the tangent.

SOLUTION

Here $f(x) = \sin x$,

and substituting $x = 0$, $f(0) = 0$,

so the curve passes through the origin.

Differentiating, $f'(x) = \cos x$,

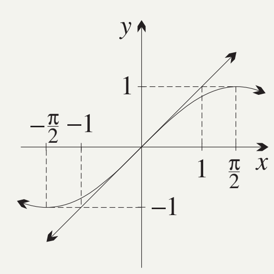
and substituting $x = 0$, $f'(0) = \cos 0$

$= 1$,

so the tangent to $y = \sin x$ at the origin has gradient 1.

Hence its equation is $y - 0 = 1(x - 0)$

$y = x$.



Note: This result was already clear from the limit $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ proven in the previous section. The simplicity of the result confirms that radian measure is the correct measure to use for angles when doing calculus.

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Using the chain rule to generate more standard forms

A simple pattern emerges when the chain rule is used to differentiate functions such as $\cos(3x + 4)$, where the angle $3x + 4$ is a linear function.

Example 6 6B

Use the chain rule to differentiate:

a $y = \cos(3x + 4)$ **b** $y = \tan(5x - 1)$ **c** $y = \sin(ax + b)$

SOLUTION

a Here $y = \cos(3x + 4)$.
Applying the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= -\sin(3x + 4) \times 3\\ &= -3\sin(3x + 4).\end{aligned}$$

Let $u = 3x + 4$.
Then $y = \cos u$.
Hence $\dfrac{du}{dx} = 3$
and $\dfrac{dy}{du} = -\sin u$.

b Here $y = \tan(5x - 1)$.
Applying the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= \sec^2(5x - 1) \times 5\\ &= 5\sec^2(5x - 1).\end{aligned}$$

Let $u = 5x - 1$.
Then $y = \tan u$.
Hence $\dfrac{du}{dx} = 5$
and $\dfrac{dy}{du} = \sec^2 u$.

c Here $y = \sin(ax + b)$.
Applying the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= \cos(ax + b) \times a\\ &= a\cos(ax + b).\end{aligned}$$

Let $u = ax + b$.
Then $y = \sin u$.
Hence $\dfrac{du}{dx} = a$
and $\dfrac{dy}{du} = \cos u$.

The last result in the previous worked example can be extended to the other trigonometric functions, giving the following standard forms:

6 STANDARD DERIVATIVES OF FUNCTIONS OF $ax + b$

- $\dfrac{d}{dx}\sin(ax + b) = a\cos(ax + b)$
- $\dfrac{d}{dx}\cos(ax + b) = -a\sin(ax + b)$
- $\dfrac{d}{dx}\tan(ax + b) = a\sec^2(ax + b)$

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Example 7 6B

Use the extended standard forms given in Box 6 above to differentiate:

a $y = \cos 7x$ **b** $y = 4\sin\left(3x - \frac{\pi}{3}\right)$ **c** $y = \tan \frac{3}{2}x$

SOLUTION

a The function is $y = \cos 7x$, so $a = 7$ and $b = 0$,

and $\dfrac{dy}{dx} = -7\sin 7x$.

b The function is $y = 4\sin\left(3x - \frac{\pi}{3}\right)$, so $a = 3$ and $b = -\frac{\pi}{3}$,

and $\dfrac{dy}{dx} = 12\cos\left(3x - \frac{\pi}{3}\right)$.

c The function is $y = \tan \frac{3}{2}x$, so $a = \frac{3}{2}$ and $b = 0$,

and $\dfrac{dy}{dx} = \frac{3}{2}\sec^2 \frac{3}{2}x$

Using the chain rule with trigonometric functions

The chain rule can also be applied in the usual way to differentiate compound functions.

Example 8 6B

Use the chain rule to differentiate:

a $y = \tan^2 x$ **b** $y = \sin\left(x^2 - \frac{\pi}{4}\right)$

SOLUTION

a Here $y = \tan^2 x$.

Applying the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 2\tan x \sec^2 x.$$

Let $u = \tan x$.
Then $y = u^2$.
Hence $\dfrac{du}{dx} = \sec^2 x$
and $\dfrac{dy}{du} = 2u$.

b Here $y = \sin\left(x^2 - \frac{\pi}{4}\right)$.

Applying the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 2x\cos\left(x^2 - \frac{\pi}{4}\right).$$

Let $u = x^2 - \frac{\pi}{4}$.
Then $y = \sin u$.
Hence $\dfrac{du}{dx} = 2x$
and $\dfrac{dy}{du} = \cos u$.

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Using the product rule with trigonometric functions

A function such as $y = e^x \cos x$ is the product of the two functions $u = e^x$ and $v = \cos x$. It can therefore be differentiated using the product rule.

Example 9 6B

Use the product rule to differentiate:

a $y = e^x \cos x$

b $y = 5\cos 2x \cos \frac{1}{2}x$

SOLUTION

a Here $y = e^x \cos x$.

Applying the product rule,

$$\begin{aligned}\frac{dy}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx}\\ &= e^x\cos x - e^x \sin x\\ &= e^x(\cos x - \sin x).\end{aligned}$$

Let $u = e^x$

and $v = \cos x$.

Then $\dfrac{du}{dx} = e^x$

and $\dfrac{dv}{dx} = -\sin x$.

b Here $y = 5\cos 2x \cos \frac{1}{2}x$.

Applying the product rule,

$$\begin{aligned}y' &= vu' + uv'\\ &= -10\sin 2x \cos \tfrac{1}{2}x - \tfrac{5}{2}\cos 2x \sin \tfrac{1}{2}x.\end{aligned}$$

Let $u = 5\cos 2x$

and $v = \cos \frac{1}{2}x$.

Then $u' = -10\sin 2x$

and $v' = -\frac{1}{2}\sin\frac{1}{2}x$.

Using the quotient rule with trigonometric functions

A function such as $y = \dfrac{\sin x}{x}$ is the quotient of the two functions $u = \sin x$ and $v = x$. Thus it can be differentiated using the quotient rule.

Example 10 6B

Use the quotient rule to differentiate:

a $y = \dfrac{\sin x}{x}$

b $y = \dfrac{\cos 2x}{\cos 5x}$

SOLUTION

a Here $y = \dfrac{\sin x}{x}$.

Applying the quotient rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\\ &= \frac{x\cos x - \sin x}{x^2}.\end{aligned}$$

Let $u = \sin x$

and $v = x$.

Then $\dfrac{du}{dx} = \cos x$

and $\dfrac{dv}{dx} = 1$.

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b Here $y = \dfrac{\cos 2x}{\cos 5x}$.

Applying the quotient rule,

$$y' = \frac{vu' - uv'}{v^2}$$

$$= \frac{-2\sin 2x \cos 5x + 5\cos 2x \sin 5x}{\cos^2 5x}.$$

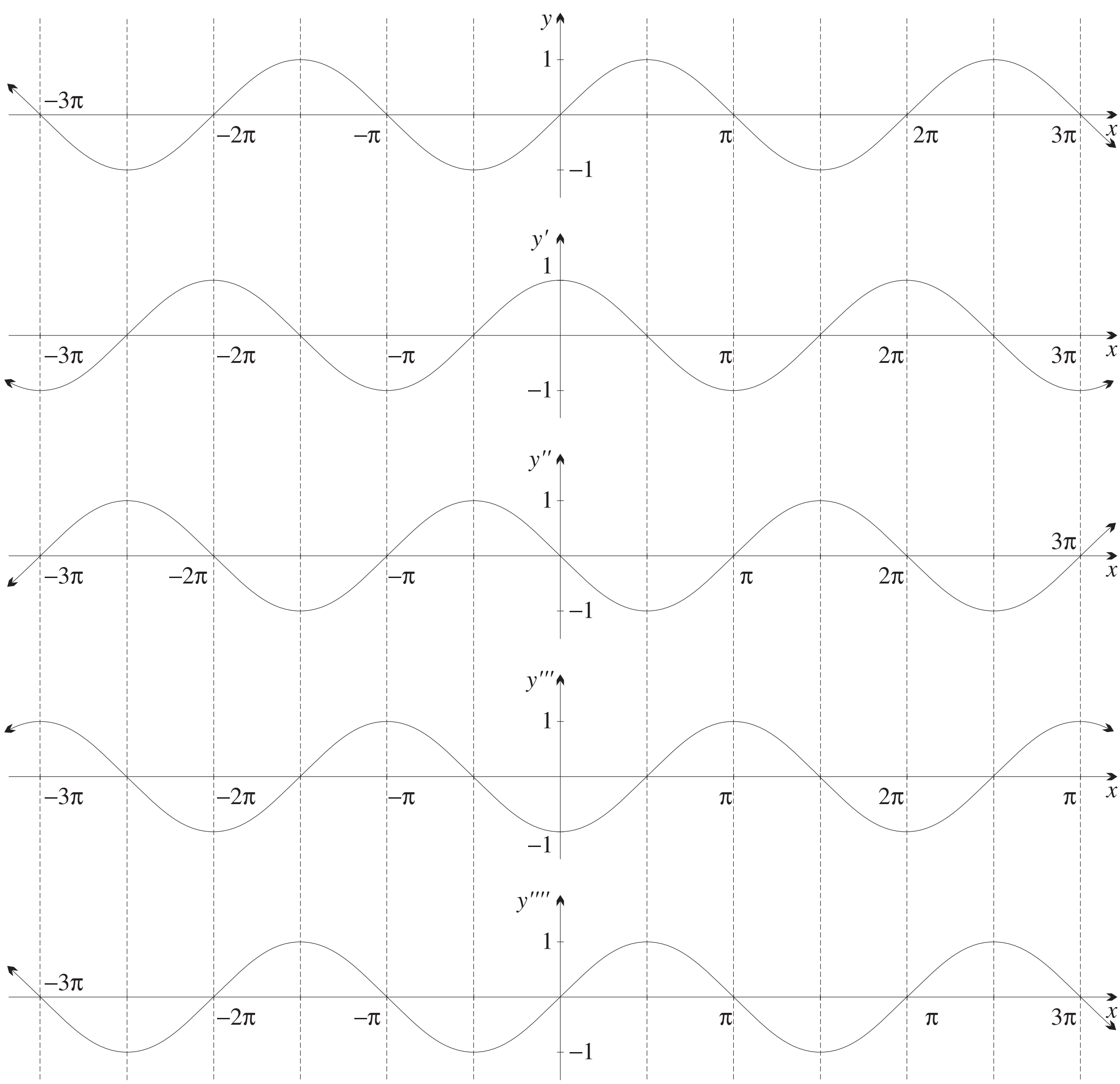
Let $u = \cos 2x$
and $v = \cos 5x$.
Then $u' = -2\sin 2x$
and $v' = -5\sin 5x$.

Successive differentiation of sine and cosine

Differentiating $y = \sin x$ repeatedly,

$$\frac{dy}{dx} = \cos x, \qquad \frac{d^2y}{dx^2} = -\sin x, \qquad \frac{d^3y}{dx^3} = -\cos x, \qquad \frac{d^4y}{dx^4} = \sin x.$$

Thus differentiation is an *order 4 operation* on the sine function, meaning that when differentiation is applied four times, the original function returns. Sketched below are the graphs of $y = \sin x$ and its first four derivatives.



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Each application of differentiation shifts the wave left $\frac{\pi}{2}$, which is a quarter of the period 2π. Thus differentiation advances the phase by $\frac{\pi}{2}$, meaning that

$$\frac{d}{dx}\sin x = \cos x = \sin\left(x + \tfrac{\pi}{2}\right) \quad \text{and} \quad \frac{d}{dx}\cos x = -\sin x = \cos\left(x + \tfrac{\pi}{2}\right)$$

Double differentiation shifts the wave left π, which is a half the period 2π, and thus advances the phase by π. Double differentiation also exchanges $y = \sin x$ with its opposite function $y = -\sin x$, with each graph being the reflection of the other in the x-axis. It has similar effects on the cosine function. Thus both $y = \sin x$ and $y = \cos x$ satisfy the equation $y'' = -y$.

Four differentiations shift the wave left 2π, which is one full period, where it coincides with itself again. Thus the differentiation transformation acting on the sine and cosine functions has *order 4*, and both $y = \sin x$ and $y = \cos x$ satisfy the equation $y'''' = y$.

7 DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS AS PHASE SHIFT

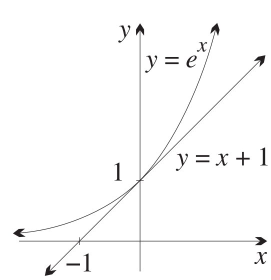
- Differentiation of $y = \sin x$ and $y = \cos x$ shifts each curve left $\frac{\pi}{2}$, advancing the phase $\frac{\pi}{2}$,
$$\frac{d}{dx}\sin x = \cos x = \sin\left(x + \tfrac{\pi}{2}\right) \quad \text{and} \quad \frac{d}{dx}\cos x = -\sin x = \cos\left(x + \tfrac{\pi}{2}\right)$$
- The second derivatives of $\sin x$ and $\cos x$ reflect each curve in the x-axis,
$$\frac{d^2}{dx^2}\sin x = -\sin x \quad \text{and} \quad \frac{d^2}{dx^2}\cos x = -\cos x$$
- Differentiation of $\sin x$ and $\cos x$ has order 4,
$$\frac{d^4}{dx^4}\sin x = \sin x \quad \text{and} \quad \frac{d^4}{dx^4}\cos x = \cos x$$

The properties of the exponential function $y = e^x$ are quite similar. The first derivative of $y = e^x$ is $y' = e^x$ and the second derivative of $y = e^{-x}$ is $y'' = e^{-x}$. This means there are now four functions whose fourth derivatives are equal to themselves:

$$y = \sin x, \qquad y = \cos x, \qquad y = e^x, \qquad y = e^{-x}.$$

This is one clue amongst many others in the course that the trigonometric functions and the exponential functions are closely related. See also Question 14(d) in Exercise 6B.

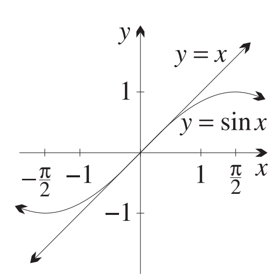
Some analogies between π and e



In the previous chapter, and in Chapter 9 of the Year 11 book, we discussed how choosing the special number e as the base of the exponential function makes the derivative of $y = e^x$ is exactly $y' = e^x$.

In particular, the tangent to $y = e^x$ at the y-intercept has gradient exactly 1.

The choice of radian measure, based on the special number π, was motivated in exactly the same way. As has just been explained, the derivative of $y = \sin x$ using radian measure is exactly $y' = \cos x$.



In particular, the tangent to $y = \sin x$ at the origin has gradient exactly 1.

Both numbers $\pi = 3.141\,592\ldots$ and $e = 2.718\,281\ldots$ are irrational. The number π is associated with the area of a circle and e is associated with areas under the rectangular hyperbola. These things are further hints of connections between trigonometric and exponential functions.

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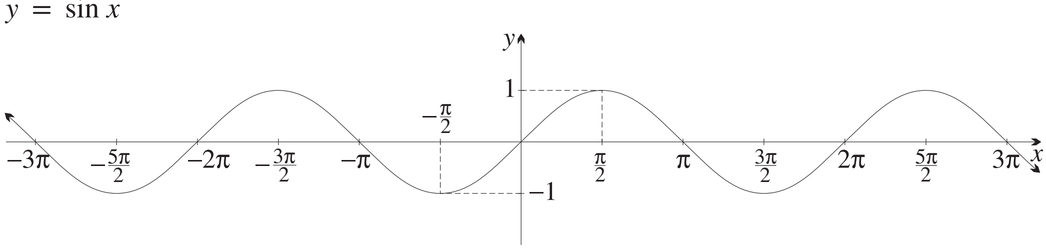
Sketches of the six trigonometric functions in radians

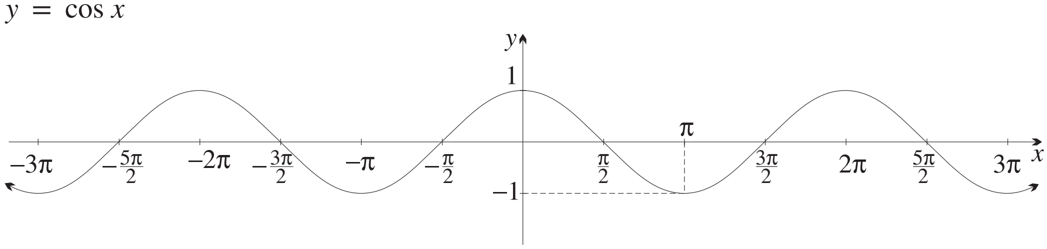
These graphs are repeated from Year 11, Section 9J and were investigated in the exercise. Their key properties:

- $\sin x$ and $\cos x$ each have amplitude 1. The others do not have an amplitude.
- $\sin x$ and $\cos x$ (and their reciprocals $\sec x$ and $\operatorname{cosec} x$ each have period 2π, $\tan x$ (and its reciprocal $\cot x$) have period π.

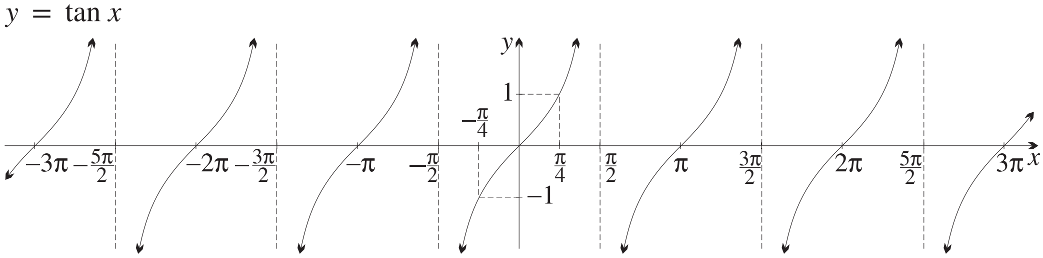
$y = \sin x$



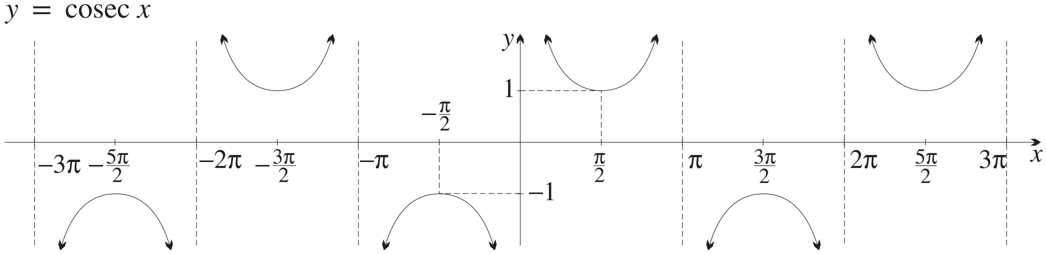
$y = \cos x$



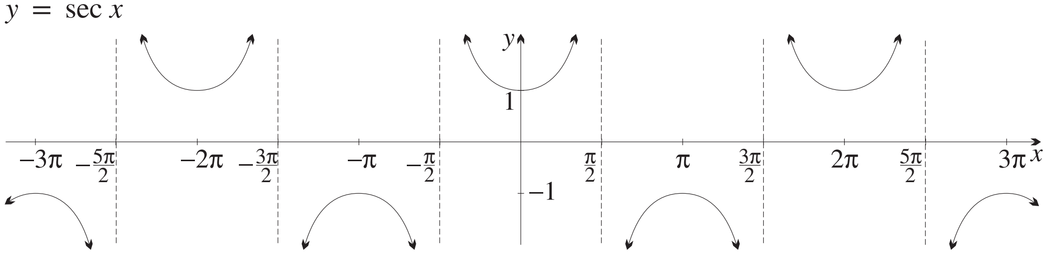
$y = \tan x$



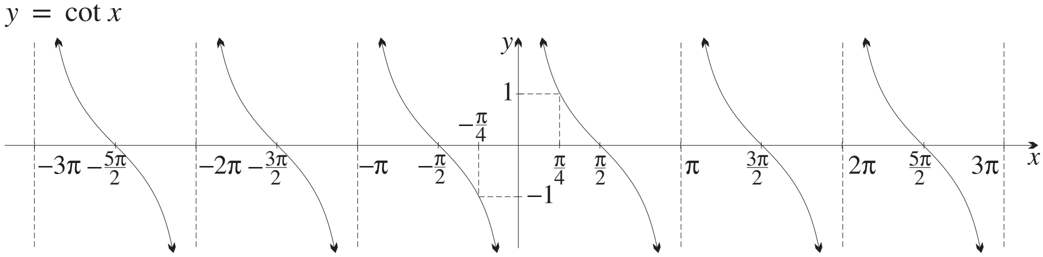
$y = \operatorname{cosec} x$



$y = \sec x$



$y = \cot x$



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Exercise 6B

FOUNDATION

1 Use the standard forms to differentiate with respect to x:

a $y = \sin x$ **b** $y = \cos x$ **c** $y = \tan x$
d $y = 2\sin x$ **e** $y = \sin 2x$ **f** $y = 3\cos x$
g $y = \cos 3x$ **h** $y = \tan 4x$ **i** $y = 4\tan x$
j $y = 2\sin 3x$ **k** $y = 2\tan 2x$ **l** $y = 4\cos 2x$
m $y = -\sin 2x$ **n** $y = -\cos 2x$ **o** $y = -\tan 2x$
p $y = \tan \frac{1}{2}x$ **q** $y = \cos \frac{1}{2}x$ **r** $y = \sin \frac{x}{2}$
s $y = 5\tan \frac{1}{5}x$ **t** $y = 6\cos \frac{x}{3}$ **u** $y = 12\sin \frac{x}{4}$

2 Differentiate with respect to x:

a $\sin 2\pi x$ **b** $\tan \frac{\pi}{2}x$ **c** $3\sin x + \cos 5x$
d $4\sin \pi x + 3\cos \pi x$ **e** $\sin(2x - 1)$ **f** $\tan(1 + 3x)$
g $2\cos(1 - x)$ **h** $\cos(5x + 4)$ **i** $7\sin(2 - 3x)$
j $10\tan(10 - x)$ **k** $6\sin\left(\frac{x+1}{2}\right)$ **l** $15\cos\left(\frac{2x+1}{5}\right)$

3 Find the first, second, third and fourth derivatives of:

a $y = \sin 2x$ **b** $y = \cos 10x$ **c** $y = \sin \frac{1}{2}x$ **d** $y = \cos \frac{1}{3}x$

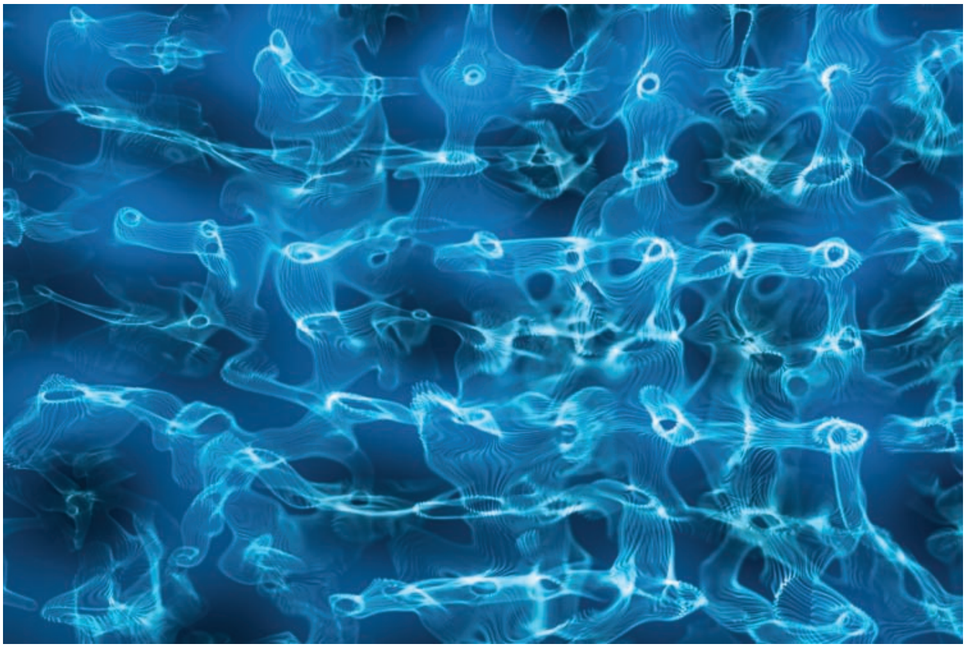
In parts **a** and **d**, write down the amplitudes of the four resulting functions.

4 If $f(x) = \cos 2x$, find $f'(x)$ and then find:

a $f'(0)$ **b** $f'\left(\frac{\pi}{12}\right)$ **c** $f'\left(\frac{\pi}{6}\right)$ **d** $f'\left(\frac{\pi}{4}\right)$

5 If $f(x) = \sin\left(\frac{1}{4}x + \frac{\pi}{2}\right)$, find $f'(x)$ and then find:

a $f'(0)$ **b** $f'(2\pi)$ **c** $f'(-\pi)$ **d** $f'(\pi)$



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DEVELOPMENT

6 Find $\dfrac{dy}{dx}$ using the product rule.

a $y = x \sin x$ **b** $y = 2x \tan 2x$ **c** $y = x^2 \cos 2x$ **d** $y = x^3 \sin 3x$

7 Find $\dfrac{dy}{dx}$ using the quotient rule.

a $y = \dfrac{\sin x}{x}$ **b** $y = \dfrac{\cos x}{x}$ **c** $y = \dfrac{x^2}{\cos x}$ **d** $y = \dfrac{x}{1 + \sin x}$

8 Find $\dfrac{dy}{dx}$ using the chain rule. Remember that $\cos^2 x$ means $(\cos x)^2$.

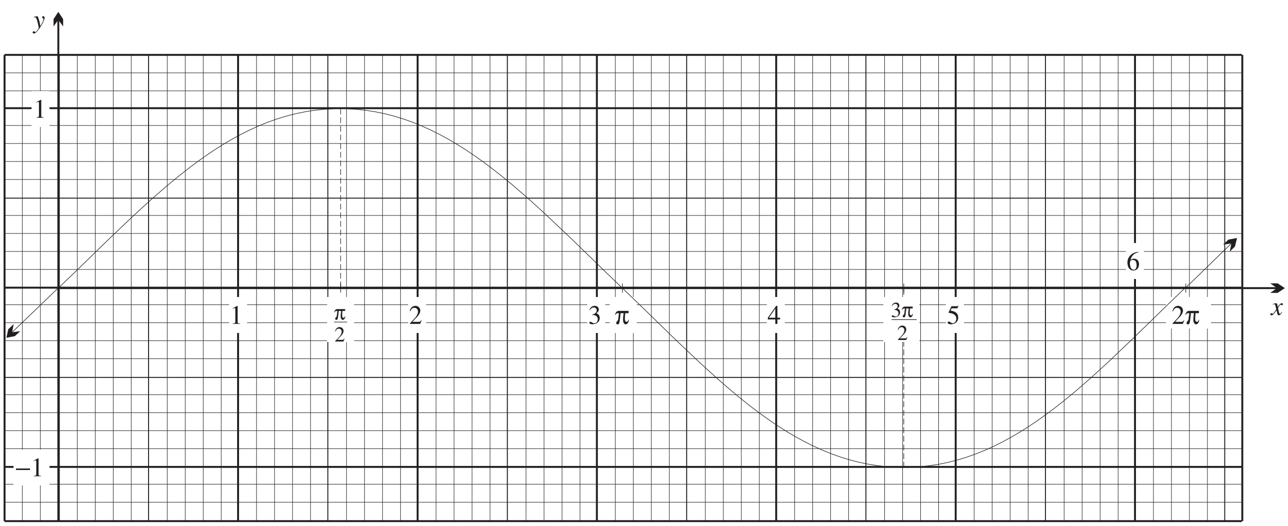
a $y = \sin(x^2)$ **b** $y = \sin(1 - x^2)$

c $y = \cos(x^3 + 1)$ **d** $y = \sin \dfrac{1}{x}$

e $y = \cos^2 x$ **f** $y = \sin^3 x$

g $y = \tan^2 x$ **h** $y = \tan\sqrt{x}$

9



a Photocopy the sketch above of $f(x) = \sin x$. Carefully draw tangents at the points where $x = 0, 0.5, 1, 1.5, \ldots, 3$, and also at $x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

b Measure the gradient of each tangent correct to two decimal places, and copy and complete the following table.

x	0	0.5	1	1.5	$\frac{\pi}{2}$	2	2.5	3	π	3.5	4	4.5	$\frac{3\pi}{2}$	5	5.5	6	2π
$f'(x)$																	

c Use these values to plot the graph of $y = f'(x)$.

d What is the equation of this graph?

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10 [Technology] Most graphing programs can graph the derivative of a function. Start with $y = \sin x$, as in the previous question, then graph y', y'', y''' and y'''', and compare your results with the graphs printed in the theory introducing this exercise.

11 Differentiate:

a $f(x) = e^{\tan x}$ **b** $f(x) = e^{\sin 2x}$
c $f(x) = \sin(e^{2x})$ **d** $f(x) = \log_e(\cos x)$
e $f(x) = \log_e(\sin x)$ **f** $f(x) = \log_e(\cos 4x)$

12 Differentiate these functions.

a $y = \sin x \cos x$ **b** $y = \sin^2 7x$
c $y = \cos^5 3x$ **d** $y = (1 - \cos 3x)^3$
e $y = \sin 2x \sin 4x$ **f** $y = \tan^3(5x - 4)$

13 Find $f'(x)$, given that:

a $f(x) = \dfrac{1}{1 + \sin x}$ **b** $f(x) = \dfrac{\sin x}{1 + \cos x}$
c $f(x) = \dfrac{1 - \sin x}{\cos x}$ **d** $f(x) = \dfrac{\cos x}{\cos x + \sin x}$

14 **a** Sketch $y = \cos x$, for $-3\pi \le x \le 3\pi$.
b Find y', y'', y''' and y'''', and sketch them underneath the first graph.
c What geometric relationship between the two graphs is indicated by the fact that:
i $y'' = -y$? **ii** $y'''' = y$?
d Find which of the function $y = e^x$, $y = e^{-x}$, $y = \sin x$ and $y = x^n$ satisfy:
i $y' = y$ **ii** $y'' = y$ **iii** $y''' = y$ **iv** $y'''' = y$

15 [Technology] The previous question is well suited to a graphing program, and the results should be compared with those of successive differentiation of $\sin x$.

16 **a** If $y = e^x \sin x$, find y' and y'', and show that $y'' - 2y' + 2y = 0$.
b If $y = e^{-x}\cos x$, find y' and y'', and show that $y'' + 2y' + 2y = 0$.

17 Consider the function $y = \frac{1}{3}\tan^3 x - \tan x + x$.

a Show that $\dfrac{dy}{dx} = \tan^2 x \sec^2 x - \sec^2 x + 1$.

b Hence use the identity $\sec^2 x = 1 + \tan^2 x$ to show that $\dfrac{dy}{dx} = \tan^4 x$.

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CHALLENGE

18 **a** Copy and complete: $\log_b\left(\frac{P}{Q}\right) = \ldots$

b If $f(x) = \log_e\left(\dfrac{1 + \sin x}{\cos x}\right)$, show that $f'(x) = \sec x$.

19 **a** The third standard form is $\dfrac{d}{dx}\tan x = \sec^2 x$. Look at the graph of $y = \tan x$ at the end of the text of this section, and hence draw $y = \dfrac{d}{dx}\tan x$ to confirm the standard form. In your sketches, use the fact that $y = \tan x$ has gradient 1 at the origin.

b **i** Use the quotient rule to prove that $\dfrac{d}{dx}\cot x = -\operatorname{cosec}^2 x$.

ii Repeat the steps of part **a** to confirm this derivative of $\cot x$.

20 **a** By writing $\sec x$ as $(\cos x)^{-1}$, show that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.

b Similarly, show that $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$.

21 Show that $\dfrac{d}{dx}\left(\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x\right) = \sin^4 x \cos^3 x$.

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6C Applications of differentiation

Differentiation of the trigonometric functions can be applied in the usual way to the analysis of a number of functions that are very significant in the practical application of calculus. It can also be used to solve optimisation problems (meaning problems about maximise and minimise).

Tangents and normals

As always, the derivative is used to find the gradients of the relevant tangents, then point–gradient form is used to find their equations.

Example 11 6C

Find the equation of the tangent to $y = 2\sin x$ at the point P where $x = \frac{\pi}{6}$.

SOLUTION

When $x = \frac{\pi}{6}$, $y = 2\sin\frac{\pi}{6}$

$= 1$ $\left(\text{because } \sin\frac{\pi}{6} = \frac{1}{2}\right)$,

so the point P has coordinates $\left(\frac{\pi}{6}, 1\right)$.

Differentiating, $\dfrac{dy}{dx} = 2\cos x$.

When $x = \frac{\pi}{6}$, $\dfrac{dy}{dx} = 2\cos\frac{\pi}{6}$

$= \sqrt{3}$ $\left(\text{because } \cos\frac{\pi}{6} = \frac{1}{2}\sqrt{3}\right)$,

so the tangent at $P\left(\frac{\pi}{6}, 1\right)$ has gradient $\sqrt{3}$.

Hence its equation is $y - y_1 = m(x - x_1)$ (point–gradient form)

$$y - 1 = \sqrt{3}\left(x - \tfrac{\pi}{6}\right)$$

$$y = x\sqrt{3} + 1 - \tfrac{\pi}{6}\sqrt{3}.$$

Example 12 6C

a Find the equations of the tangents and normals to the curve $y = \cos x$ at $A\left(-\frac{\pi}{2}, 0\right)$ and $B\left(\frac{\pi}{2}, 0\right)$.

b Show that the four lines form a square, sketch, and find the other two vertices.

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SOLUTION

a The function is $y = \cos x$,
and the derivative is $y' = -\sin x$.

Hence gradient of tangent at $A\left(-\frac{\pi}{2}, 0\right) = -\sin\left(-\frac{\pi}{2}\right) = 1$,

and gradient of normal at $A\left(-\frac{\pi}{2}, 0\right) = -1$.

Similarly, gradient of tangent at $B\left(\frac{\pi}{2}, 0\right) = -\sin\frac{\pi}{2} = -1$,

and gradient of normal at $B\left(\frac{\pi}{2}, 0\right) = 1$.

Hence the tangent at A is $y - 0 = 1 \times \left(x + \frac{\pi}{2}\right)$

$y = x + \frac{\pi}{2}$,

and the normal at A is $y - 0 = -1 \times \left(x + \frac{\pi}{2}\right)$

$y = -x - \frac{\pi}{2}$.

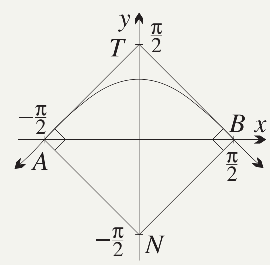
Similarly, the tangent at B is $y - 0 = -1 \times \left(x - \frac{\pi}{2}\right)$

$y = -x + \frac{\pi}{2}$,

and the normal at B is $y - 0 = 1 \times \left(x - \frac{\pi}{2}\right)$

$y = x - \frac{\pi}{2}$.

b Hence the two tangents meet on the y-axis at $T\left(0, \frac{\pi}{2}\right)$, and the two normals meet on the y-axis at $N\left(0, -\frac{\pi}{2}\right)$. Because adjacent sides are perpendicular, $ANBT$ is a rectangle, and because the diagonals are perpendicular, it is also a rhombus, so the quadrilateral $ANBT$ is a square.



Example 13 6C

a Find the equation of the tangent to $y = \tan 2x$ at the point on the curve where $x = \frac{\pi}{8}$.
b Find the x-intercept and y-intercept of this tangent.
c Sketch the situation.
d Find the area of the triangle formed by this tangent and the coordinate axes.

SOLUTION

a The function is $y = \tan 2x$,
and differentiating, $y' = 2\sec^2 2x$.
When $x = \frac{\pi}{8}$, $y = \tan\frac{\pi}{4}$
$= 1$
and $y' = 2\sec^2\frac{\pi}{4}$
$= 2 \times \left(\sqrt{2}\right)^2$
$= 4$,
so the tangent is $y - 1 = 4\left(x - \frac{\pi}{8}\right)$
$y = 4x - \frac{\pi}{2} + 1$.

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b When $x = 0$, $y = 1 - \frac{\pi}{2}$

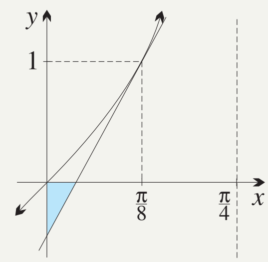
$$= \frac{2 - \pi}{2},$$

and when $y = 0$, $0 = 4x - \frac{\pi}{2} + 1$

$$4x = \frac{\pi}{2} - 1$$

$$4x = \frac{\pi - 2}{2}$$

$\boxed{\div 4}$ $$x = \frac{\pi - 2}{8}.$$



c The sketch is drawn opposite.

d Area of triangle $= \frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times \frac{\pi - 2}{2} \times \frac{\pi - 2}{8}$$

$$= \frac{(\pi - 2)^2}{32} \text{ square units.}$$

Curve sketching

Curve-sketching problems involving trigonometric functions can be long, with difficult details. Nevertheless, the usual steps of the 'curve-sketching menu' still apply and the working of each step is done exactly the same as usual.

Sketching these curves using either a computer package or a graphics calculator would greatly aid understanding of the relationships between the equations of the curves and their graphs.

Note: With trigonometric functions, it is often easier to determine the nature of stationary points from an examination of the second derivative than from a table of values of the first derivative.

Example 14 6C

Consider the curve $y = \sin x + \cos x$ in the interval $0 \le x \le 2\pi$.

a Find the values of the function at the endpoints of the domain.

b Find the x-intercepts of the graph.

c Find any stationary points and determine their nature.

d Find any points of inflection and sketch the curve.

SOLUTION

a When $x = 0$, $y = \sin 0 + \cos 0 = 1$,

and when $x = 2\pi$, $y = \sin 2\pi + \cos 2\pi = 1$.

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b To find the x-intercepts, put $y = 0$.

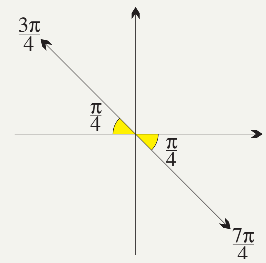
Then $\sin x + \cos x = 0$

$$\sin x = -\cos x$$

$$\tan x = -1 \quad \text{(dividing through by } \cos x\text{)}.$$

Hence x is in quadrant 2 or 4, with related angle $\frac{\pi}{4}$,

so $\quad x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$.



c Differentiating, $\quad y' = \cos x - \sin x$,

so y' has zeroes when $\sin x = \cos x$,

that is, $\quad \tan x = 1$ (dividing through by $\cos x$).

Hence x is in quadrant 1 or 3, with related angle $\frac{\pi}{4}$,

so $\quad x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.

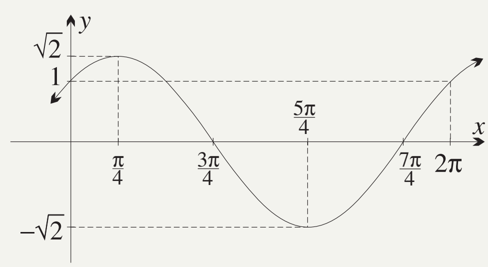
When $x = \frac{\pi}{4}$, $\quad y = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$

$$= \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}$$

$$= \sqrt{2},$$

and when $x = \frac{5\pi}{4}$, $\quad y = -\tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}\sqrt{2}$

$$= -\sqrt{2}.$$



Differentiating again, $\quad y'' = -\sin x - \cos x$,

so when $x = \frac{\pi}{4}$, $\quad y'' = -\sqrt{2}$,

and when $x = \frac{5\pi}{4}$, $\quad y'' = \sqrt{2}$.

Hence $\left(\frac{\pi}{4}, \sqrt{2}\right)$ is a maximum turning point,

and $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$ is a minimum turning point.

d The second derivative y'' has zeroes when $-\sin x - \cos x = 0$,

that is, at the zeroes of y, which are $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{6}$.

x	0	$\frac{3\pi}{4}$	π	$\frac{7\pi}{4}$	2π
y''	-1	0	1	0	-1
	$\frown$	$\cdot$	$\smile$	$\cdot$	$\frown$

Hence the x-intercepts $\left(\frac{3\pi}{4}, 0\right)$ and $\left(\frac{7\pi}{4}, 0\right)$ are also inflections.

Note: The final graph is simply a wave with the same period 2π as $\sin x$ and $\cos x$, but with amplitude $\sqrt{2}$. It is actually $y = \sqrt{2}\cos x$ shifted right by $\frac{\pi}{4}$. Any function of the form $y = a\sin x + b\cos x$ has a similar graph.

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Example 15 6C

[A harder example]

Sketch the graph of $f(x) = x - \sin x$ after carrying out these steps.

a Write down the domain.

b Test whether the function is even or odd or neither.

c Find any zeroes of the function and examine its sign.

d Examine the function's behaviour as $x \to \infty$ and as $x \to -\infty$.

e Find any stationary points and examine their nature.

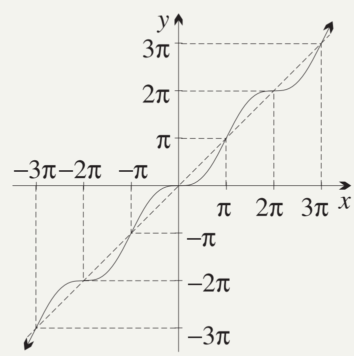
f Find any points of inflection.

Note: This function is essentially the function describing the area of a segment, if the radius in the formula $A = \frac{1}{2}r^2(x - \sin x)$ is held constant while the angle x at the centre varies.

SOLUTION

a The domain of $f(x) = x - \sin x$ is the set of all real numbers.

b $f(x)$ is odd, because both $\sin x$ and x are odd.



c The function is zero at $x = 0$ and nowhere else,
because $\sin x < x$, for $x > 0$,
and $\sin x > x$, for $x < 0$.

d The value of $\sin x$ always remains between -1 and 1,
so for $f(x) = x - \sin x$, $f(x) \to \infty$ as $x \to \infty$,
and $f(x) \to -\infty$ as $x \to -\infty$.

e Differentiating, $f'(x) = 1 - \cos x$,
so $f'(x)$ has zeroes whenever $\cos x = 1$,
that is, for $x = \ldots, -2\pi, 0, 2\pi, 4\pi, \ldots$.
But $f'(x) = 1 - \cos x$ is never negative, because $\cos x$ is never greater than 1,
thus the curve $f(x)$ is always increasing except at its stationary points.
Hence each stationary point is a stationary inflection,
and these points are $\ldots, (-2\pi, -2\pi), (0, 0), (2\pi, 2\pi), (4\pi, 4\pi), \ldots$.

f Differentiating again, $f''(x) = \sin x$,
which is zero for $x = \ldots, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$.
We know that $\sin x$ changes sign around each of these points,
so $\ldots, (-\pi, -\pi), (\pi, \pi), (3\pi, 3\pi), \ldots$ are also inflections.
Because $f'(\pi) = 1 - (-1) = 2$, the gradient at these other inflections is 2.

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Exercise 6C

FOUNDATION

Technology: The large number of sketches in this exercise should allow many of the graphs to be drawn first on a computer. Such sketching should be followed by an algebraic explanation of the features.

Many graphing packages allow tangents and normals to be drawn at specific points so that diagrams can be drawn of the earlier questions in the exercise.

1 Find the gradient of the tangent to each curve at the point indicated.

a $y = \sin x$ at $x = 0$ **b** $y = \cos x$ at $x = \frac{\pi}{2}$ **c** $y = \sin x$ at $x = \frac{\pi}{3}$

d $y = \cos x$ at $x = \frac{\pi}{6}$ **e** $y = \sin x$ at $x = \frac{\pi}{4}$ **f** $y = \tan x$ at $x = 0$

g $y = \tan x$ at $x = \frac{\pi}{4}$ **h** $y = \cos 2x$ at $x = \frac{\pi}{4}$ **i** $y = -\cos \frac{1}{2}x$ at $x = \frac{2\pi}{3}$

j $y = \sin \frac{x}{2}$ at $x = \frac{2\pi}{3}$ **k** $y = \tan 2x$ at $x = \frac{\pi}{6}$ **l** $y = \sin 2x$ at $x = \frac{\pi}{12}$

2 **a** Show that the line $y = x$ is the tangent to the curve $y = \sin x$ at $(0, 0)$.

b Show that the line $y = x$ is the tangent to the curve $y = \tan x$ at $(0, 0)$.

c Show that the line $y = \frac{\pi}{2} - x$ is the tangent to the curve $y = \cos x$ at $\left(\frac{\pi}{2}, 0\right)$.

DEVELOPMENT

3 Find the equation of the tangent at the given point on each curve.

a $y = \sin x$ at $(\pi, 0)$ **b** $y = \tan x$ at $\left(\frac{\pi}{4}, 1\right)$

c $y = \cos x$ at $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$ **d** $y = \cos 2x$ at $\left(\frac{\pi}{4}, 0\right)$

e $y = \sin 2x$ at $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$ **f** $y = x \sin x$ at $(\pi, 0)$

4 Find, in the domain $0 \le x \le 2\pi$, the x-coordinates of the points on each curve where the gradient of the tangent is zero.

a $y = 2 \sin x$ **b** $y = 2 \sin x - x$

c $y = 2 \cos x + x$ **d** $y = 2 \sin x + \sqrt{3}x$

5 The point $P\left(\frac{\pi}{6}, \frac{1}{2}\right)$ lies on the curve $y = 2 \sin x - \cos 2x$.

a Show that the tangent at P has equation $2\sqrt{3}x - y = \frac{1}{3}\pi\sqrt{3} - \frac{1}{2}$.

b Show that the normal at P has equation $x + 2\sqrt{3}y = \frac{\pi}{6} + \sqrt{3}$.

6 **a** Show that $y = \sin^2 x$ has derivative $y' = 2 \sin x \cos x$.

b Find the gradients of the tangent and normal to $y = \sin^2 x$ at the point where $x = \frac{\pi}{4}$.

c Find the equations of the tangent and normal to $y = \sin^2 x$ at the point where $x = \frac{\pi}{4}$.

d Suppose that the tangent meets the x-axis at P, the normal meets the y-axis at Q and O is the origin. Show that $\triangle OPQ$ has area $\frac{1}{32}(\pi^2 - 4)$ units ^{2}.

7 **a** Differentiate $y = e^{\sin x}$.

b Hence find, in the domain $[0, 2\pi]$, the x-coordinates of the points on the curve $y = e^{\sin x}$ where the tangent is horizontal.

8 **a** Differentiate $y = e^{\cos x}$.

b Hence find, in the domain $[0, 2\pi]$, the x-coordinates of the points on the curve $y = e^{\cos x}$ where the tangent is horizontal.

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9 **a** Find the first and second derivatives of $y = \cos x + \sqrt{3}\sin x$.
b Find the stationary points in the domain $0 \le x \le 2\pi$, and use the second derivative to determine their nature.
c Find the points of inflection.
d Hence sketch the curve, for $0 \le x \le 2\pi$.

10 **a** Repeat the previous question for $y = \cos x - \sin x$.
b Verify your results by sketching $y = \cos x$ and $y = -\sin x$ on the same diagram, and then sketching $y = \cos x - \sin x$ by addition of heights.

11 **a** Find the derivative of $y = x + \sin x$, and show that $y'' = -\sin x$.
b Find the stationary points in the domain $-2\pi < x < 2\pi$, and determine their nature.
c Find the points of inflection.
d Hence sketch the curve, for $-2\pi \le x \le 2\pi$.

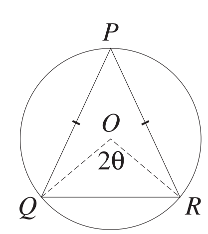
12 Repeat the steps of the previous question for $y = x - \cos x$.

13 A conical tent with top T is being designed to have a slant height of 3 metres. Let $\theta = \angle TPO$, where O is the centre of the base and P is any point at the ground on the edge of the tent.
a Draw a diagram, and show that the vertical height of the tent is $h = 3\sin\theta$, and that the base radius is $r = 3\cos\theta$.
b Use the formula $V = \frac{1}{3}\pi r^2 h$ for the volume V of a cone to show that
$$V = 3\pi(\sin\theta - \sin^3\theta).$$
c Find $\dfrac{dV}{d\theta}$, and hence find in degrees, correct to two decimal places, the angle θ so that the cone has maximum volume.
d What is the exact value of the maximum volume of the tent?

CHALLENGE

14 Find any stationary points and inflections of the curve $y = 2\sin x + x$ in the interval $0 \le x \le 2\pi$, then sketch the curve.

15 An isosceles triangle PQR is inscribed in a circle with centre O of radius 1 unit, as shown in the diagram to the right. Let $\angle QOR = 2\theta$, where θ is acute.



a Join PO and extend it to meet QR at M. Then prove that $QM = \sin\theta$ and $OM = \cos\theta$.
b Show that the area A of $\triangle PQR$ is $A = \sin\theta(\cos\theta + 1)$.
c Hence show that, as θ varies, $\triangle PQR$ has its maximum possible area when it is equilateral.

16 **a** Show that $\dfrac{d}{d\theta}\left(\dfrac{2 - \sin\theta}{\cos\theta}\right) = \dfrac{2\sin\theta - 1}{\cos^2\theta}$.
b Hence find the maximum and minimum values of the expression $\dfrac{2 - \sin\theta}{\cos\theta}$ in the interval $0 \le \theta \le \frac{\pi}{4}$, and state the values of θ for which they occur.

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6D Integrating the trigonometric functions

As always, the standard forms for differentiation can be reversed to give standard forms for integration.

The standard forms for integrating the trigonometric functions

When the standard forms for differentiating $\sin x$, $\cos x$ and $\tan x$ are reversed, they give three new standard integrals.

First, $\dfrac{d}{dx}\sin x = \cos x$, and reversing this, $\displaystyle\int \cos x\, dx = \sin x$.

Secondly, $\dfrac{d}{dx}\cos x = -\sin x$, and reversing this, $\displaystyle\int (-\sin x)\, dx = \cos x$

$\boxed{\times\ (-1)}$ $\displaystyle\int \sin x\, dx = -\cos x$.

Thirdly, $\dfrac{d}{dx}\tan x = \sec^2 x$, and reversing this, $\displaystyle\int \sec^2 x\, dx = \tan x$.

This gives three new standard integrals. These three standard forms should be carefully memorised — pay attention to the signs in the first two standard forms.

8 STANDARD TRIGONOMETRIC INTEGRALS

- $\displaystyle\int \cos x\, dx = \sin x + C$, for some constant C
- $\displaystyle\int \sin x\, dx = -\cos x + C$, for some constant C
- $\displaystyle\int \sec^2 x\, dx = \tan x + C$, for some constant C

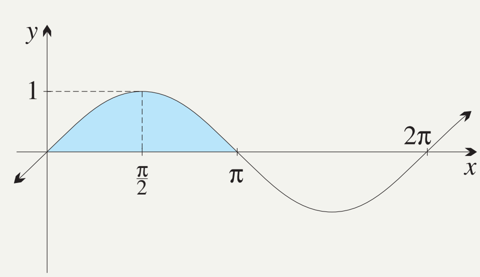
No calculation involving a primitive may cross an asymptote.

Example 16 6D

The curve $y = \sin x$ is sketched below. Show that the first arch of the curve, as shaded in the diagram, has area 2 square units.

SOLUTION

Because the region is entirely above the x-axis,



$$\begin{aligned}\text{area} &= \int_0^{\pi} \sin x\, dx \\ &= \Big[-\cos x\Big]_0^{\pi} \\ &= -\cos \pi + \cos 0 \\ &= -(-1) + 1\end{aligned}$$

(the graph of $y = \cos x$ shows that $\cos \pi = -1$)

$= 2$ square units.

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Note: This simple answer confirms again that radians are the right units to use for calculus with trigonometric functions. Similar simple results were obtained earlier when e was used as the base for powers. For example, Question **33c** of the Chapter 5 Review gathered together three remarkably simple results:

$$\int_1^e \frac{1}{x}\,dx = \int_1^e \log_e x\,dx = \int_0^1 xe^x\,dx = 1$$

Example 17 6D

Evaluate these definite integrals.

a $\displaystyle\int_0^{\pi} \cos x\,dx$ **b** $\displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x\,dx$ **c** $\displaystyle\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sec^2 x\,dx$

SOLUTION

a $\displaystyle\int_0^{\pi} \cos x\,dx = \Big[\sin x\Big]_0^{\pi}$
$= \sin\pi - \sin 0$
$= 0$

(Use the graph of $y = \sin x$ to see that $\sin\pi = 0$ and $\sin 0 = 0$.)

b $\displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x\,dx = \Big[\tan x\Big]_0^{\frac{\pi}{3}}$
$= \tan\frac{\pi}{3} - \tan 0$
$= \sqrt{3}$

(Here $\tan\frac{\pi}{3} = \sqrt{3}$ and $\tan 0 = 0$.)

c This integral is meaningless because it crosses the asymptote at $x = \frac{\pi}{2}$.

Replacing x by $ax + b$

Reversing the standard forms for derivatives in Section 6B gives a further set of standard forms. Again, the constants of integration have been ignored until the boxed statement of the standard forms.

First, $\quad\dfrac{d}{dx}\sin(ax + b) = a\cos(ax + b)$,

so $\quad\displaystyle\int a\cos(ax + b)\,dx = \sin(ax + b)$

and dividing by a, $\quad\displaystyle\int \cos(ax + b)\,dx = \frac{1}{a}\sin(ax + b)$.

Secondly, $\quad\dfrac{d}{dx}\cos(ax + b) = -a\sin(ax + b)$,

so $\quad\displaystyle\int -a\sin(ax + b)\,dx = \cos(ax + b)$

and dividing by $-a$, $\quad\displaystyle\int \sin(ax + b)\,dx = -\frac{1}{a}\cos(ax + b)$.

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Thirdly, $\dfrac{d}{dx}\tan(ax + b) = a\sec^2(ax + b)$,

so $\displaystyle\int a\sec^2(ax + b)\,dx = \tan(ax + b)$.

and dividing by a, $\displaystyle\int \sec^2(ax + b)\,dx = \frac{1}{a}\tan(ax + b)$.

The result is extended forms of the three standard integrals. These extended standard forms should also be carefully memorised.

9 STANDARD INTEGRALS FOR FUNCTIONS OF *ax* + *b*

- $\displaystyle\int \cos(ax + b)\,dx = \frac{1}{a}\sin(ax + b) + C$, for some constant C
- $\displaystyle\int \sin(ax + b)\,dx = -\frac{1}{a}\cos(ax + b) + C$, for some constant C
- $\displaystyle\int \sec^2(ax + b)\,dx = \frac{1}{a}\tan(ax + b) + C$, for some constant C

Example 18 6D

Evaluate these definite integrals.

a $\displaystyle\int_0^{\frac{\pi}{6}} \cos 3x\,dx$ **b** $\displaystyle\int_\pi^{2\pi} \sin \tfrac{1}{4}x\,dx$ **c** $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2(2x + \pi)\,dx$

SOLUTION

a
$$\begin{aligned}\int_0^{\frac{\pi}{6}} \cos 3x\,dx &= \tfrac{1}{3}\Big[\sin 3x\Big]_0^{\frac{\pi}{6}}\\ &= \tfrac{1}{3}\Big(\sin \tfrac{\pi}{2} - 3\sin 0\Big)\\ &= \tfrac{1}{3}\end{aligned}$$

b
$$\begin{aligned}\int_\pi^{2\pi} \sin \tfrac{1}{4}x\,dx &= -4\Big[\cos \tfrac{1}{4}x\Big]_\pi^{2\pi} \qquad \text{(because the reciprocal of } \tfrac{1}{4} \text{ is 4)}\\ &= -4\cos\tfrac{\pi}{2} + 4\cos\tfrac{\pi}{4}\\ &= 0 + 4 \times \frac{\sqrt{2}}{2}\\ &= 2\sqrt{2}\end{aligned}$$

c
$$\begin{aligned}\int_0^{\frac{\pi}{8}} \sec^2(2x + \pi)\,dx &= \tfrac{1}{2}\Big[\tan(2x + \pi)\Big]_0^{\frac{\pi}{8}}\\ &= \tfrac{1}{2}\Big(\tan\tfrac{5\pi}{4} - \tan\pi\Big)\\ &= \tfrac{1}{2}(1 - 0) \qquad \Big(\tfrac{5\pi}{4} \text{ is in quadrant 3 with related angle } \tfrac{\pi}{4}\Big)\\ &= \tfrac{1}{2}\end{aligned}$$

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The primitives of tan *x* and cot *x*

The primitives of tan x and cot x can be found by using the ratio formulae $\tan x = \dfrac{\sin x}{\cos x}$ and $\cot x = \dfrac{\cos x}{\sin x}$ and then applying the standard form from the previous chapter, in either one of its two versions,

$$\int \frac{u'}{u}\,dx = \log_e |u| + C \qquad \text{or} \qquad \int \frac{f'(x)}{f(x)}\,dx = \log_e |f(x)| + C.$$

Example 19 6D

Find primitives of these functions.

a $\cot x$ **b** $\tan x$

SOLUTION

a $\int \cot x\,dx$

$= \int \dfrac{\cos x}{\sin x}\,dx$

$= \log_e |\sin x| + C.$

Let $u = \sin x$.
Then $u' = \cos x$.
$\int \dfrac{u'}{u}\,dx = \log_e |u|$

OR

Let $f(x) = \sin x$.
Then $f'(x) = \cos x$.
$\int \dfrac{f'(x)}{f(x)}\,dx = \log_e |f(x)|$

b $\int \tan x\,dx$

$= \int \dfrac{\sin x}{\cos x}\,dx$

$= -\int \dfrac{-\sin x}{\cos x}\,dx$

$= -\log_e |\cos x| + C.$ (This can also be written as $\log_e |\sec x|$.)

Let $u = \cos x$.
Then $u' = -\sin x$.
$\int \dfrac{u'}{u}\,dx = \log_e |u|$

OR

Let $f(x) = \cos x$.
Then $f'(x) = -\sin x$.
$\int \dfrac{f'(x)}{f(x)}\,dx = \log_e |f(x)|$

Note: Do not run across a zero of $\sin x$ when using part **a**, or a zero of $\cos x$ when using part **b**.

Finding a function whose derivative is known

If the derivative of a function is known, and the value of the function at one point is also known, then the whole function can be found.

Example 20 6D

The derivative of a certain function is $y' = \cos x$, and the graph of the function has y-intercept $(0, 3)$. Find the original function $f(x)$ and then find $f\left(\frac{\pi}{2}\right)$.

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SOLUTION

Here $y' = \cos x$,

and taking the primitive, $y = \sin x + C$, for some constant C.

When $x = 0$, $y = 3$, so substituting $x = 0$,

$$3 = \sin 0 + C$$
$$C = 3.$$

Hence $y = \sin x + 3$.

When $x = \frac{\pi}{2}$, $y = \sin \frac{\pi}{2} + 3$

$= 4$, because $\sin \frac{\pi}{2} = 1$.

Example 21 6D

Given that $f'(x) = \sin 2x$ and $f(\pi) = 1$:

a find the function $f(x)$,

b find $f\left(\frac{\pi}{4}\right)$.

SOLUTION

a Here $f'(x) = \sin 2x$,

and taking the primitive, $f(x) = -\frac{1}{2}\cos 2x + C$, for some constant C.

It is known that $f(\pi) = 1$, so substituting $x = \pi$,

$$1 = -\tfrac{1}{2}\cos 2\pi + C$$
$$1 = -\tfrac{1}{2} \times 1 + C$$
$$C = 1\tfrac{1}{2}.$$

Hence $f(x) = -\frac{1}{2}\cos 2x + 1\frac{1}{2}$.

b Substituting $x = \frac{\pi}{4}$, $f\left(\frac{\pi}{4}\right) = -\frac{1}{2} \times \cos \frac{\pi}{2} + 1\frac{1}{2}$

$= 1\frac{1}{2}$, because $\cos \frac{\pi}{2} = 0$.

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Given a chain-rule derivative, find an integral

As always, the results of a chain-rule differentiation can be reversed to give a primitive.

Example 22 6D

a Use the chain rule to differentiate $\cos^5 x$.

b Hence find $\int_0^{\pi} \sin x \cos^4 x \, dx$.

SOLUTION

a Let $y = \cos^5 x$.

By the chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= -5 \sin x \cos^4 x$.

Let $u = \cos x$.
Then $y = u^5$.
Hence $\dfrac{du}{dx} = -\sin x$
and $\dfrac{dy}{du} = 5u^4$.

b From part **a**, $\dfrac{d}{dx}(\cos^5 x) = -5 \sin x \cos^4 x$.

Reversing this, $\int (-5 \sin x \cos^4 x) \, dx = \cos^5 x$.

$\boxed{\div (-5)}$ $\int \sin x \cos^4 x \, dx = -\tfrac{1}{5} \cos^5 x$.

Hence $\int_0^{\pi} \sin x \cos^4 x \, dx = -\tfrac{1}{5}\Big[\cos^5 x\Big]_0^{\pi}$

$= -\tfrac{1}{5}(-1 - 1)$

$= \tfrac{2}{5}$.

Using a formula for the reverse chain rule

The integral in the example above could have been done using the reverse chain rule for powers of u or $f(x)$.

Example 23 6D

Use the reverse chain rule to find $\int \sin x \cos^4 x \, dx$.

SOLUTION

$\int_0^{\pi} \sin x \cos^4 x \, dx$

$= -\int (-\sin x) \cos^4 x \, dx$

$= -\tfrac{1}{5} \cos^5 x + C$.

Let $u = \cos x$.
Then $u' = -\sin x$.

$\int u^n \dfrac{du}{dx} \, dx = \dfrac{u^{n+1}}{n+1}$

OR

Let $f(x) = \cos x$.
Then $f'(x) = -\sin x$.

$\int \big(f(x)\big)^n \dfrac{du}{dx} \, dx = \dfrac{\big(f(x)\big)^{n+1}}{n+1}$

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Exercise 6D

FOUNDATION

1 Find these indefinite integrals.

a $\int \sec^2 x\,dx$ **b** $\int \cos x\,dx$ **c** $\int \sin x\,dx$

d $\int -\sin x\,dx$ **e** $\int 2\cos x\,dx$ **f** $\int \cos 2x\,dx$

g $\int \frac{1}{2}\cos x\,dx$ **h** $\int \cos \frac{1}{2}x\,dx$ **i** $\int \sin 2x\,dx$

j $\int \sec^2 5x\,dx$ **k** $\int \cos 3x\,dx$ **l** $\int \sec^2 \frac{1}{3}x\,dx$

m $\int \sin \frac{x}{2}\,dx$ **n** $\int -\cos \frac{1}{5}x\,dx$ **o** $\int -4\sin 2x\,dx$

p $\int \frac{1}{4}\sin \frac{1}{4}x\,dx$ **q** $\int 12\sec^2 \frac{1}{3}x\,dx$ **r** $\int 2\cos \frac{x}{3}\,dx$

2 Find the value of:

a $\int_0^{\frac{\pi}{2}} \cos x\,dx$ **b** $\int_0^{\frac{\pi}{6}} \cos x\,dx$ **c** $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\,dx$

d $\int_0^{\frac{\pi}{3}} \sec^2 x\,dx$ **e** $\int_0^{\frac{\pi}{4}} 2\cos 2x\,dx.$ **f** $\int_0^{\frac{\pi}{3}} \sin 2x\,dx$

g $\int_0^{\frac{\pi}{2}} \sec^2\left(\frac{1}{2}x\right)dx$ **h** $\int_{\frac{\pi}{3}}^{\pi} \cos\left(\frac{1}{2}x\right)dx$ **i** $\int_0^{\pi} (2\sin x - \sin 2x)\,dx$

3 **a** The gradient function of a certain curve is given by $\dfrac{dy}{dx} = \sin x$.

i If the curve passes through the origin, find its equation.

ii If the curve passes through $\left(\frac{\pi}{2}, 3\right)$ find its equation.

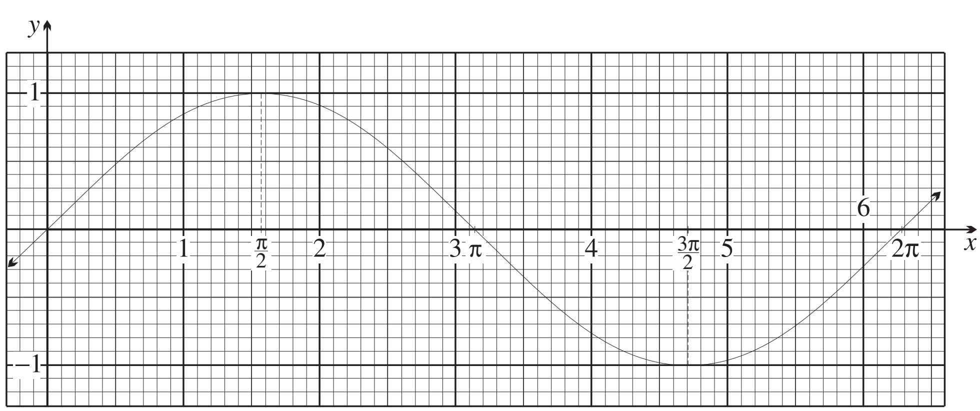
iii What translation moves the curve in part **ii** to the curve in part **iii**?

b Another curve passing through the origin has gradient function $y' = \cos x - 2\sin 2x$. Find its equation.

c If $\dfrac{dy}{dx} = \sin x + \cos x$, and $y = -2$ when $x = \pi$, find y as a function of x.

DEVELOPMENT

4



The graph of $y = \sin x$ is sketched above.

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a Worked Example 18 in the notes for this section proved that $\int_0^{\pi} \sin x \, dx = 2$. Count squares on the graph of $y = \sin x$ above to confirm this result.

b On the same graph of $y = \sin x$, count squares and use symmetry to find:

i $\int_0^{\frac{\pi}{4}} \sin x \, dx$ **ii** $\int_0^{\frac{\pi}{2}} \sin x \, dx$ **iii** $\int_0^{\frac{3\pi}{4}} \sin x \, dx$

iv $\int_0^{\frac{5\pi}{4}} \sin x \, dx$ **v** $\int_0^{\frac{3\pi}{2}} \sin x \, dx$ **vi** $\int_0^{\frac{7\pi}{4}} \sin x \, dx$

c Evaluate these integrals using the fact that $-\cos x$ is a primitive of $\sin x$, and confirm the results of part **b**.

5 [Technology] Programs that sketch the graph and then approximate definite integrals would help reinforce the previous very important investigation. The investigation could then be continued past $x = \pi$, after which the definite integral decreases again.

Similar investigation with the graphs of $\cos x$ and $\sec^2 x$ would also be helpful, comparing the results of computer integration with the exact results obtained by integration using the standard primitives.

6 Find these indefinite integrals.

a $\int \cos(x + 2) \, dx$ **b** $\int \cos(2x + 1) \, dx$ **c** $\int \sin(x + 2) \, dx$

d $\int \sin(2x + 1) \, dx$ **e** $\int \cos(3x - 2) \, dx$ **f** $\int \sin(7 - 5x) \, dx$

g $\int \sec^2(4 - x) \, dx$ **h** $\int \sec^2\left(\frac{1 - x}{3}\right) dx$ **i** $\int \sin\left(\frac{1 - x}{3}\right) dx$

7 **a** Find $\int \left(6 \cos 3x - 4 \sin \tfrac{1}{2}x\right) dx$.

b Find $\int \left(8 \sec^2 2x - 10 \cos \tfrac{1}{4}x + 12 \sin \tfrac{1}{3}x\right) dx$.

8 **a** If $f'(x) = \pi \cos \pi x$ and $f(0) = 0$, find $f(x)$ and $f\left(\tfrac{1}{3}\right)$.

b If $f'(x) = \cos \pi x$ and $f(0) = \frac{1}{2\pi}$, find $f(x)$ and $f\left(\tfrac{1}{6}\right)$.

c If $f''(x) = 18 \cos 3x$ and $f'(0) = f\left(\tfrac{\pi}{2}\right) = 1$, find $f(x)$.

9 Find these indefinite integrals, where a, b, u and v are constants.

a $\int a \sin(ax + b) \, dx$ **b** $\int \pi^2 \cos \pi x \, dx$

c $\int \frac{1}{u} \sec^2(v + ux) \, dx$ **d** $\int \frac{a}{\cos^2 ax} \, dx$

10 **a** Copy and complete $1 + \tan^2 x = \ldots$, and hence find $\int \tan^2 x \, dx$.

b Simplify $1 - \sin^2 x$, and hence find the value of $\int_0^{\frac{\pi}{3}} \frac{2}{1 - \sin^2 x} \, dx$.

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11 **a** Copy and complete $\int \frac{f'(x)}{f(x)}\,dx = \ldots.$

b Hence show that $\int_0^{\frac{\pi}{6}} \frac{\cos x}{1 + \sin x}\,dx \doteqdot 0.4.$

12 **a** Use the fact that $\tan x = \frac{\sin x}{\cos x}$ to show that $\int_0^{\frac{\pi}{4}} \tan x\,dx = \frac{1}{2}\ln 2.$

b Use the fact that $\cot x = \frac{\cos x}{\sin x}$ to find $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x\,dx.$

13 **a** **i** Find $\frac{d}{dx}(\sin x^2).$ **ii** Hence find $\int 2x\cos x^2\,dx.$

b **i** Find $\frac{d}{dx}(\cos x^3).$ **ii** Hence find $\int x^2\sin x^3\,dx.$

c **i** Find $\frac{d}{dx}\left(\tan\sqrt{x}\right).$ **ii** Hence find $\int \frac{1}{\sqrt{x}}\sec^2\sqrt{x}\,dx.$

CHALLENGE

14 **a** Find $\frac{d}{dx}(\sin^5 x)$, and hence find $\int \sin^4 x\cos x\,dx.$

b Find $\frac{d}{dx}(\tan^3 x)$, and hence find $\int \tan^2 x\sec^2 x\,dx.$

15 **a** Differentiate $e^{\sin x}$, and hence find the value of $\int_0^{\frac{\pi}{2}} \cos x e^{\sin x}\,dx.$

b Differentiate $e^{\tan x}$, and hence find the value of $\int_0^{\frac{\pi}{4}} \sec^2 x e^{\tan x}\,dx.$

16 **a** Show that $\frac{d}{dx}(\sin x - x\cos x) = x\sin x$, and hence find $\int_0^{\frac{\pi}{2}} x\sin x\,dx.$

b Show that $\frac{d}{dx}\left(\frac{1}{3}\cos^3 x - \cos x\right) = \sin^3 x$, and hence find $\int_0^{\frac{\pi}{3}} \sin^3 x\,dx.$

17 Find $\frac{d}{dx}\left(\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right)$, and hence find $\int_0^{\frac{\pi}{4}} x\cos 2x\,dx.$

18 Find these definite integrals, if they exist, giving reasons. (The primitives of $\tan x$ and $\cot x$ are mentioned in the text, and occurred also in Question 12.)

a $\int_0^{\pi} \sec^2 x\,dx$ **b** $\int_0^{\pi} \tan x\,dx$ **c** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cot x\,dx$

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6E Applications of integration

The trigonometric integrals can now be used to find areas in the usual way.

Finding areas by integration

As always, *a sketch is essential*, because areas below the x-axis are represented as a negative number by the definite integral.

It is best to evaluate the separate integrals first and then make a conclusion about areas.

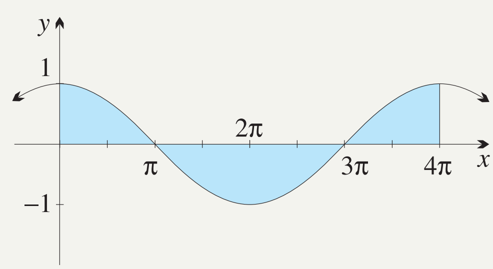
Example 24 6E

a Sketch $y = \cos \frac{1}{2}x$ in the interval $0 \le x \le 4\pi$, marking both x-intercepts.

b Hence find the area between the curve and the x-axis, for $0 \le x \le 4\pi$.

SOLUTION

a The curve $y = \cos \frac{1}{2}x$ has amplitude 1, and the period is $2\pi \div \frac{1}{2} = 4\pi$.
The two x-intercepts in the interval are $x = \pi$ and $x = 3\pi$.



b We must integrate separately over the three intervals $[0, \pi]$ and $[\pi, 3\pi]$ and $[3\pi, 4\pi]$.

First,
$$\begin{aligned}\int_0^{\pi} \cos \tfrac{1}{2}x\, dx &= \Big[2\sin \tfrac{1}{2}x\Big]_0^{\pi} \\ &= 2\sin \tfrac{\pi}{2} - 2\sin 0 \\ &= 2 - 0 \\ &= 2,\end{aligned}$$

which is positive, because the curve is above the x-axis for $0 \le x \le \pi$.

Secondly,
$$\begin{aligned}\int_{\pi}^{3\pi} \cos \tfrac{1}{2}x\, dx &= \Big[2\sin \tfrac{1}{2}x\Big]_{\pi}^{3\pi} \\ &= 2\sin \tfrac{3\pi}{2} - 2\sin \tfrac{\pi}{2} \\ &= -2 - 2 \\ &= -4,\end{aligned}$$

which is negative, because the curve is below the x-axis for $\pi \le x \le 3\pi$.

Thirdly,
$$\begin{aligned}\int_{3\pi}^{4\pi} \cos \tfrac{1}{2}x\, dx &= \Big[2\sin \tfrac{1}{2}x\Big]_{3\pi}^{4\pi} \\ &= 2\sin 2\pi - 2\sin \tfrac{3\pi}{2} \\ &= 0 - (-2) \\ &= 2,\end{aligned}$$

which is positive, because the curve is above the x-axis for $3\pi \le x \le 4\pi$.

Hence
$$\begin{aligned}\text{total area} &= 2 + 4 + 2 \\ &= 8 \text{ square units.}\end{aligned}$$

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Finding areas between curves

The next worked example uses the principle that if $y = f(x)$ is above $y = g(x)$ throughout some interval $a \le x \le b$, then the area between the curves is given by the formula

$$\text{area between the curves} = \int_a^b (f(x) - g(x))\,dx.$$

Example 25 — 6E

a Show that the curves $y = \sin x$ and $y = \sin 2x$ intersect when $x = \frac{\pi}{3}$.

b Sketch these curves in the interval $[0, \pi]$.

c Find the area contained between the curves in the interval $\left[0, \frac{\pi}{3}\right]$.

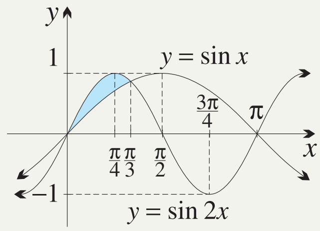
SOLUTION

a The curves intersect at $x = \frac{\pi}{3}$ because $\sin\frac{\pi}{3} = \sin\frac{2\pi}{3} = \frac{1}{2}\sqrt{3}$.

b The curves are sketched to the right below.

c In the interval $0 \le x \le \frac{\pi}{3}$, the curve $y = \sin 2x$ is always above $y = \sin x$,

$$\begin{aligned}\text{so area between} &= \int_0^{\frac{\pi}{3}} (\sin 2x - \sin x)\,dx \\ &= \Big[-\tfrac{1}{2}\cos 2x + \cos x\Big]_0^{\frac{\pi}{3}} \\ &= \left(-\tfrac{1}{2}\cos\tfrac{2\pi}{3} + \cos\tfrac{\pi}{3}\right) - \left(-\tfrac{1}{2}\cos 0 + \cos 0\right).\end{aligned}$$



Because $\cos 0 = 1$ and $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\cos\frac{2\pi}{3} = -\frac{1}{2}$,

$$\begin{aligned}\text{area} &= \left(\tfrac{1}{4} + \tfrac{1}{2}\right) - \left(-\tfrac{1}{2} + 1\right) \\ &= \tfrac{1}{4} \text{ square units.}\end{aligned}$$

Example 26 — 6E

a Show that in the interval $0 \le x \le 2\pi$, the curves $y = \sin x$ and $y = \cos x$ intersect when $x = \frac{\pi}{4}$ and when $x = \frac{5\pi}{4}$.

b Sketch the curves in this interval and find the area contained between them.

SOLUTION

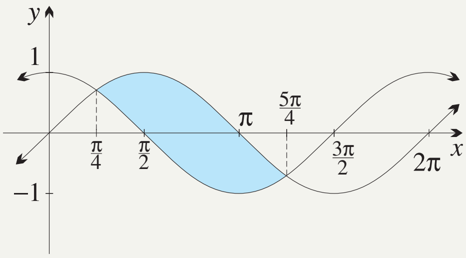
a Put $\sin x = \cos x$.

Then $\tan x = 1$

$x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$,

so the curves intersect at the points

$\left(\frac{\pi}{4}, \frac{1}{2}\sqrt{2}\right)$ and $\left(\frac{5\pi}{4}, -\frac{1}{2}\sqrt{2}\right)$.



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b Area between $= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx$

$$= \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= -\cos\tfrac{5\pi}{4} - \sin\tfrac{5\pi}{4} + \cos\tfrac{\pi}{4} + \sin\tfrac{\pi}{4}$$

$$= -\left(-\tfrac{1}{2}\sqrt{2}\right) - \left(-\tfrac{1}{2}\sqrt{2}\right) + \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}$$

$$= 2\sqrt{2} \text{ square units.}$$

Exercise 6E

FOUNDATION

Technology: Some graphing programs can perform numerical integration on specified regions. Such programs would help to confirm the integrals in this exercise and to investigate quickly further integrals associated with these curves.

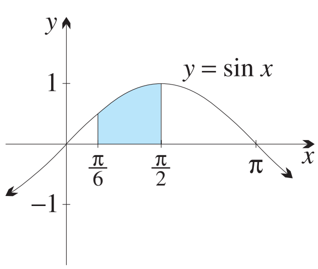
1 Find the exact area between the curve $y = \cos x$ and the x-axis:
a from $x = 0$ to $x = \frac{\pi}{2}$,
b from $x = 0$ to $x = \frac{\pi}{6}$.

2 Find the exact area between the curve $y = \sec^2 x$ and the x-axis:
a from $x = 0$ to $x = \frac{\pi}{4}$,
b from $x = 0$ to $x = \frac{\pi}{3}$.

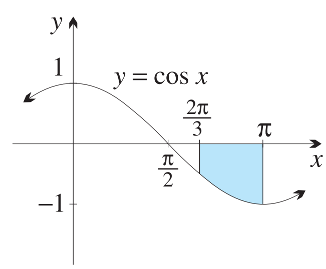
3 Find the exact area between the curve $y = \sin x$ and the x-axis:
a from $x = 0$ to $x = \frac{\pi}{4}$,
b from $x = 0$ to $x = \frac{\pi}{6}$.

4 Find the area of each shaded region (then observe that the two regions have equal area).

a



b



DEVELOPMENT

5 Find the area enclosed between each curve and the x-axis over the specified domain.
a $y = \sin x$, from $x = \frac{\pi}{3}$ to $x = \frac{\pi}{2}$
b $y = \sin 2x$, from $x = \frac{\pi}{4}$ to $x = \frac{\pi}{2}$
c $y = \cos x$, from $x = \frac{\pi}{3}$ to $x = \frac{\pi}{2}$
d $y = \cos 3x$, from $x = \frac{\pi}{12}$ to $x = \frac{\pi}{6}$
e $y = \sec^2 x$, from $x = \frac{\pi}{6}$ to $x = \frac{\pi}{3}$
f $y = \sec^2 \frac{1}{2}x$, from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$

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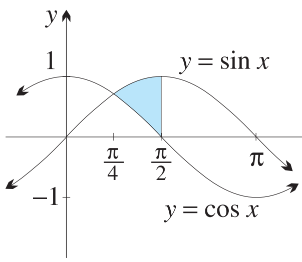
6 Calculate the area of the shaded region in each diagram.

a



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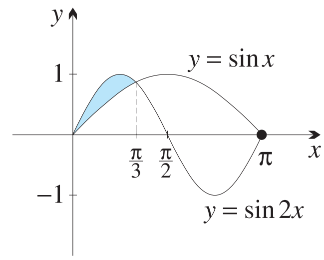
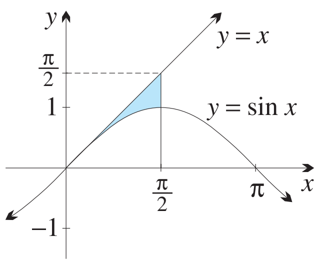


c



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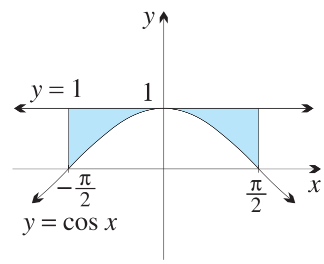
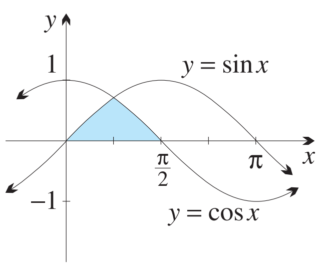


7 Calculate the area of the shaded region in each diagram.

a



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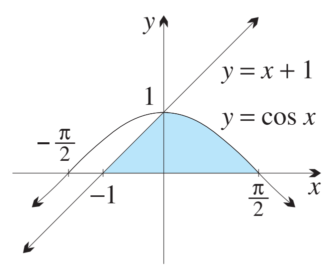


8 Find, using a diagram, the area bounded by one arch of each curve and the x-axis.

a $y = \sin x$

b $y = \cos 2x$

9 Sketch the area enclosed between each curve and the x-axis over the specified domain, and then find the exact value of the area. (Make use of symmetry wherever possible.)

a $y = \cos x$, from $x = 0$ to $x = \pi$

b $y = \sin x$, from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$

c $y = \cos 2x$, from $x = 0$ to $x = \pi$

d $y = \sin 2x$, from $x = \frac{\pi}{3}$ to $x = \frac{2\pi}{3}$

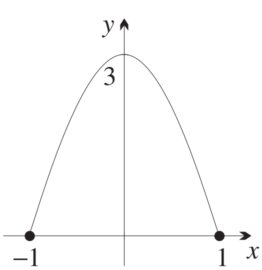
e $y = \sin x$, from $x = -\frac{5\pi}{6}$ to $x = \frac{7\pi}{6}$

f $y = \cos 3x$, from $x = \frac{\pi}{6}$ to $x = \frac{2\pi}{3}$

10 a Sketch the curve $y = 2\cos \pi x$ in the interval $[-1, 1]$, clearly marking the two x-intercepts.

b Find the exact area bounded by the curve $y = 2\cos \pi x$ and the x-axis, between the two x-intercepts.

11 An arch window 3 metres high and 2 metres wide is made in the shape of the curve $y = 3\cos\left(\frac{\pi}{2}x\right)$, as shown to the right. Find the area of the window in square metres, correct to one decimal place.



12 The graphs of $y = x - \sin x$ and $y = x$ are sketched together in worked Example 15 in Section 6C. Find the total area enclosed between these graphs, from $x = 0$ to $x = 2\pi$.

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13 The region R is bounded by the curve $y = \tan x$, the x-axis and the vertical line $x = \frac{\pi}{3}$. Show that R has area $\ln 2$ square units.

14 a Sketch the region bounded by the graphs of $y = \sin x$ and $y = \cos x$, and by the vertical lines $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{6}$.

b Find the area of the region in part (a).

15 a Show by substitution that $y = \sin x$ and $y = \cos 2x$ meet at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{6}$.

b On the same number plane, sketch $y = \sin x$ and $y = \cos 2x$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{6}\right]$.

c Hence find the area of the region bounded by the two curves.

CHALLENGE

16 a Show that for all positive integers n:

i $\displaystyle\int_0^{2\pi} \sin nx\, dx = 0$ **ii** $\displaystyle\int_0^{2\pi} \cos nx\, dx = 0$

b Sketch each graph. Then find the area between the curve and the x-axis, from $x = 0$ to $x = 2\pi$:

i $y = \sin x$ **ii** $y = \sin 2x$ **iii** $y = \sin 3x$

iv $y = \sin nx$ **v** $y = \cos nx$

17 a Show that $\displaystyle\int_0^{n} (1 + \sin 2\pi x)\, dx = n$, for all positive integers n.

b Sketch $y = 1 + \sin 2\pi x$, and interpret the result geometrically.

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Chapter 6 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 6 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 Differentiate with respect to x:

a $y = 5\sin x$ **b** $y = \sin 5x$ **c** $y = 5\cos 5x$ **d** $y = \tan(5x - 4)$

e $y = x\sin 5x$ **f** $y = \dfrac{\cos 5x}{x}$ **g** $y = \sin^5 x$ **h** $y = \tan(x^5)$

i $y = e^{\cos 5x}$ **j** $y = \log_e(\sin 5x)$

2 Find the gradient of the tangent to $y = \cos 2x$ at the point on the curve where $x = \frac{\pi}{3}$.

3 a Find the equation of the tangent to $y = \tan x$ at the point where $x = \frac{\pi}{3}$.

b Find the equation of the tangent to $y = x\cos x$ at the point where $x = \frac{\pi}{2}$.

4 Find the x-coordinates of the stationary points on each curve, for $0 \le x \le 2\pi$.

a $y = x + \cos x$ **b** $y = \sin x - \cos x$

5 Find:

a $\displaystyle\int 4\cos x\, dx$ **b** $\displaystyle\int \sin 4x\, dx$ **c** $\displaystyle\int \sec^2 \tfrac{1}{4}x\, dx$

6 Find the value of:

a $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 x\, dx$ **b** $\displaystyle\int_0^{\frac{\pi}{4}} \cos 2x\, dx$ **c** $\displaystyle\int_0^{\frac{1}{3}} \pi \sin \pi x\, dx$

7 Find the value of $\displaystyle\int_0^{\frac{1}{4}} \sin 3x\, dx$, correct to three decimal places.

8 A curve has gradient function $y' = \cos\frac{1}{2}x$ and passes through the point $(\pi, 1)$. Find its equation.

9 a Sketch the curve $y = 2\sin 2x$ in the interval $[0, \pi]$, and then shade the area between the curve and the x-axis from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$.

b Calculate the shaded area in part (a).

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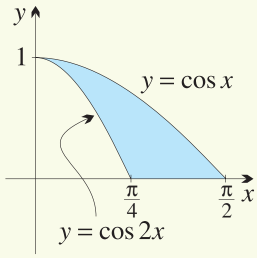
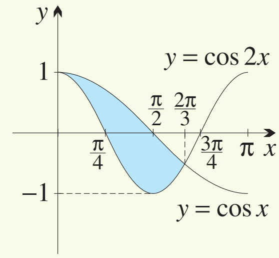
Review

10 Find the area of the shaded region in each diagram below.

a

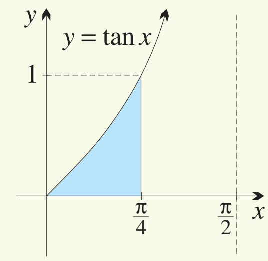


b



11 a Write $\tan x$ in terms of $\sin x$ and $\cos x$.

b Hence find the exact area of the shaded region in the diagram to the right.



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Appendix: Differentiating trigonometric functions

It is reasonably clear from the graph that the derivative of $\sin x$ is $\cos x$, but even after proving in Section 6A that $\sin x$ has gradient 1 at the origin, completing the proof requires several steps. The first-principles differentiation formula is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

and applying this formula to the function $f(x) = \sin x$ gives

$$\frac{d}{dx}(\sin x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}.$$

We can't go any further, however, until we can expand $\sin(x + h)$.

The expansion of sin $(x + h)$

This formula is not otherwise required in the course and has therefore not been boxed as a formula to remember. The formula is

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta, \text{ for all angles } \alpha \text{ and } \beta.$$

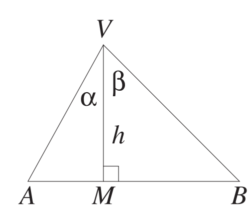
Proof: We shall only prove this formula here where α and β are both acute.

Construct a triangle ABV with altitude VM so that

$$\angle AVM = \alpha \quad \text{and} \quad \angle BVM = \beta.$$

Then $\dfrac{AV}{h} = \dfrac{1}{\cos\alpha}$ and $\dfrac{BV}{h} = \dfrac{1}{\cos\beta}$

so $AV = \dfrac{h}{\cos\alpha}$ and $BV = \dfrac{h}{\cos\beta}$.



Now area ΔAVB = area ΔAVM + area ΔBVM,

and applying the area formula in each of the three triangles,

$$\tfrac{1}{2} AV \times BV \times \sin(\alpha+\beta) = \tfrac{1}{2} AV \times MV \sin\alpha + \tfrac{1}{2} BV \times MV \sin\beta$$

$$\tfrac{1}{2}\left(\frac{h}{\cos\alpha}\right)\left(\frac{h}{\cos\beta}\right)\sin(\alpha+\beta) = \tfrac{1}{2}h\left(\frac{h}{\cos\alpha}\right)\sin\alpha + \tfrac{1}{2}h\left(\frac{h}{\cos\beta}\right)\sin\beta$$

$\boxed{\div\ \tfrac{1}{2}h^2}$ $$\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}.$$

$\boxed{\times\ \cos\alpha\cos\beta}$ $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, as required.

The derivative of sin x is cos x

The proof that the derivative of $\sin x$ is $\cos x$ depends on the fundamental limit proven in Section 6A,

$$\lim_{h \to 0} \frac{\sin h}{h} = 1.$$

Theorem: The derivative of $\sin x$ is $\cos x$,

$$\frac{d}{dx}\sin x = \cos x$$

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Proof: We saw above that

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h}.$$

Now
$$\begin{aligned}\frac{\sin(x+h) - \sin x}{h} &= \frac{\sin x\cos h + \cos x\sin h - \sin x}{h}\\
&= \frac{\cos x\sin h}{h} + \frac{\sin x(\cos h - 1)}{h}\\
&= \cos x \times \frac{\sin h}{h} + \sin x \times \frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)}\\
&= \cos x \times \frac{\sin h}{h} + \sin x \times \frac{\cos^2 h - 1}{h(\cos h + 1)}\\
&= \cos x \times \frac{\sin h}{h} - \sin x \times \frac{\sin^2 h}{h(\cos h + 1)}\\
&= \cos x \times \frac{\sin h}{h} - \sin x \times \frac{\sin h}{h} \times \frac{\sin h}{\cos h + 1}.\end{aligned}$$

As $h \to 0$, the first term has limit $\cos x$, because $\lim\limits_{h\to 0}\frac{\sin h}{h} = 1$,

and the second term has limit 0, because $\lim\limits_{h\to 0}\frac{\sin h}{h} = 1$, and $\lim\limits_{h\to 0}\frac{\sin h}{\cos h + 1} = \frac{0}{2}$.

Hence $\frac{d}{dx}(\sin x) = \cos x - 0$, as required.

The derivatives of cos *x* and tan *x*

These calculations are now straightforward:

$$\frac{d}{dx}\cos x = -\sin x \qquad \text{and} \qquad \frac{d}{dx}\tan x = \sec^2 x$$

Proof:

A. Let $y = \cos x$.

Then $y = \sin\left(\frac{\pi}{2} - x\right)$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \quad \text{(chain rule)}\\
&= -\cos\left(\tfrac{\pi}{2} - x\right)\\
&= -\sin x.\end{aligned}$$

Let $u = \frac{\pi}{2} - x$.

Then $y = \sin u$.

Hence $\frac{du}{dx} = -1$

and $\frac{dy}{du} = \cos u$.

B. Let $y = \tan x$.

Then $y = \frac{\sin x}{\cos x}$.

$$\begin{aligned}y' &= \frac{vu' - uv'}{v^2} \quad \text{(quotient rule)}\\
&= \frac{\cos x\cos x + \sin x\sin x}{\cos^2 x}\\
&= \frac{1}{\cos^2 x}, \qquad \text{because } \cos^2 x + \sin^2 x = 1,\\
&= \sec^2 x.\end{aligned}$$

Let $u = \sin x$

and $v = \cos x$.

Then $u' = \cos x$

and $v' = -\sin x$.

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7 Motion and rates

Anyone watching objects in motion can see that they often make patterns with a striking simplicity and predictability. These patterns are related to the simplest objects in geometry and arithmetic. A thrown ball traces out a parabolic path. A cork bobbing in flowing water traces out a sine wave. A rolling billiard ball moves in a straight line, rebounding symmetrically off the table edge. The stars and planets move in more complicated, but highly predictable, paths across the sky. The relationship between physics and mathematics, logically and historically, begins with these and many similar observations.

The first three sections of this chapter, however, only begin to introduce the relationship between calculus and motion. Because this is a mathematics course, not a physics course, our attention will not be on the nature of space and time, but on the striking alternative interpretations that the physical world brings to the first and second derivatives. The first derivative of displacement is velocity, which we can see. The second derivative is acceleration, which we can feel.

Motion is just one example of a rate. We have met rates briefly several times in the Year 11 book — Section 7G (exponential rates), Section 8J (using the derivative) and Section 9F (exponential growth and decay). The last three sections of this chapter unify and extend examples of rates in general, using now a much larger array of functions. Rates also provide the context to clarify the ideas of increasing and concave up in an interval, rather than at a point as in Chapter 3.

The examples of motion and rates in this chapter also provide models of the linear, quadratic, exponential and trigonometric functions, because all these functions can be brought into play at once.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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7A Average velocity and speed

This first section sets up the mathematical description of motion in one dimension, using a function to describe the relationship between time and the position of an object in motion. Average velocity is the gradient of the chord on this displacement–time graph. This will lead, in the next section, to the description of instantaneous velocity as the gradient of a tangent.

Motion in one dimension

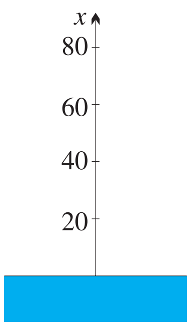
When a particle is moving in one dimension (meaning along a line) its position is varying over time. That position can be specified at any time t by a single number x, called the *displacement*, and the whole motion can be described by giving x as a function of the *time* t.

Suppose, for example, that a ball is hit vertically upwards from ground level and lands 8 seconds later in the same place. Its motion can be described approximately by the following quadratic equation and table of values,

$$x = 5t(8 - t)$$

t	0	2	4	6	8
x	0	60	80	60	0

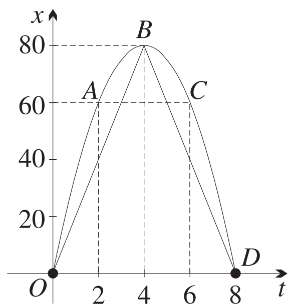
Here x is the height in metres of the ball above the ground t seconds after it is thrown. The diagram to the right shows the path of the ball up and down along the same vertical line.



This vertical line has been made into a number line, with the ground as the origin, upwards as the positive direction, and metres as the units of displacement.

Time has also become a number line. The origin of time is when the ball is thrown, and the units of time are seconds.

The graph to the right is the resulting graph of the equation of motion $x = 5t(8 - t)$. The horizontal axis is time and the vertical axis is displacement — the graph must not be mistaken as a picture of the ball's path.



The graph is a section of a parabola with vertex at (4, 80), which means that the ball achieves a maximum height of 80 metres after 4 seconds. When $t = 8$, the height is zero, and the ball strikes the ground again. The equation of motion therefore has quite restricted domain and range,

$$0 \le t \le 8 \qquad \text{and} \qquad 0 \le x \le 80.$$

Most equations of motion have this sort of restriction on the domain of t. In particular, *it is a convention of this course that negative values of time are excluded unless the question specifically allows it*.

1 MOTION IN ONE DIMENSION

- Motion in one dimension is specified by giving the displacement x on the number line as a function of time t after time zero.
- Negative values of time are excluded unless otherwise stated.

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Example 1 7A

Consider the example above, where $x = 5t(8 - t)$.

a Find the height of the ball after 1 second.

b At what other time is the ball at this same height above the ground?

SOLUTION

a When $t = 1$, $x = 5 \times 1 \times 7$
$= 35.$

Hence the ball is 35 metres above the ground after 1 second.

b To find when the height is 35 metres, solve the equation $x = 35$.
Substituting into $x = 5t(8 - t)$ gives

$$\begin{aligned} 5t(8 - t) &= 35 \\ \boxed{\div 5} \qquad t(8 - t) &= 7 \\ 8t - t^2 - 7 &= 0 \\ \boxed{\times (-1)} \qquad t^2 - 8t + 7 &= 0 \\ (t - 1)(t - 7) &= 0 \\ t &= 1 \text{ or } 7. \end{aligned}$$

Hence the ball is 35 metres high after 1 second and again after 7 seconds.

Average velocity

During its ascent, the ball in the worked example above moved 80 metres upwards. This is a change in displacement of +80 metres in 4 seconds, giving an average velocity of 20 metres per second.

Average velocity thus equals the gradient *of the chord OB* on the displacement–time graph (be careful, because there are different scales on the two axes). Hence the formula for average velocity is the familiar gradient formula.

2 AVERAGE VELOCITY

Suppose that a particle has displacement $x = x_1$ at time $t = t_1$, and displacement $x = x_2$ at time $t = t_2$. Then

$$\text{average velocity} = \frac{\text{change in displacement}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}.$$

That is, on the displacement−time graph,

average velocity = gradient of the chord.

During its descent, the ball moved 80 metres downwards in 4 seconds, which is a change in displacement of $0 - 80 = -80$ metres. The average velocity is therefore -20 metres per second, which is equal to the gradient of the chord BD.

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Example 2 7A

Consider again the example $x = 5t(8 - t)$. Find the average velocities of the ball:

a during the first second,

b during the fifth second.

SOLUTION

The first second stretches from $t = 0$ to $t = 1$ and the fifth second stretches from $t = 4$ to $t = 5$. The displacements at these times are given in the table to the right.

t	0	1	4	5
x	0	35	80	75

a Average velocity during 1st second

$$= \frac{x_2 - x_1}{t_2 - t_1}$$

$$= \frac{35 - 0}{1 - 0}$$

$$= 35 \text{ m/s.}$$

b Average velocity during 5th second

$$= \frac{x_2 - x_1}{t_2 - t_1}$$

$$= \frac{75 - 80}{5 - 4}$$

$$= -5 \text{ m/s.}$$

Distance travelled

The change in displacement can be positive, negative or zero. *Distance*, however, is always positive or zero. In the previous example, the change in displacement during the 4 seconds from $t = 4$ to $t = 8$ is -80 metres, but the distance travelled is 80 metres.

The *distance travelled* by a particle also takes into account any journey and return. Thus the total distance travelled by the ball is $80 + 80 = 160$ metres, even though the ball's change in displacement over the first 8 seconds is zero because the ball is back at its original position on the ground.

3 DISTANCE TRAVELLED

- The *distance travelled* takes into account any journey and return.
- Distance travelled can never be negative.

Average speed

The *average speed* is the distance travelled divided by the time taken. Average speed, unlike average velocity, can never be negative.

4 AVERAGE SPEED

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

Average speed can never be negative.

During the 8 seconds of its flight, the change in displacement of the ball is zero, but the distance travelled is 160 metres, so

$$\text{average velocity} = \frac{0 - 0}{8 - 0} = 0 \text{ m/s,}$$

$$\text{average speed} = \frac{160}{8} = 20 \text{ m/s.}$$

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Example 3 7A

Find the average velocity and the average speed of the ball:

a during the eighth second,

b during the last six seconds.

SOLUTION

The eighth second stretches from $t = 7$ to $t = 8$ and the last six seconds stretch from $t = 2$ to $t = 8$. The displacements at these times are given in the table to the right.

t	0	2	7	8
x	0	60	35	0

a During the eighth second, the ball moves 35 metres down from $x = 35$ to $x = 0$.

$$\begin{aligned} \text{Hence}\quad \text{average velocity} &= \frac{0 - 35}{8 - 7} \\ &= -35 \text{ m/s.} \\ \text{Also}\quad \text{distance travelled} &= 35 \text{ metres,} \\ \text{so}\quad \text{average speed} &= 35 \text{ m/s.} \end{aligned}$$

b During the last six seconds, the ball rises 20 metres from $x = 60$ to $x = 80$, and then falls 80 metres from $x = 80$ to $x = 0$.

$$\begin{aligned} \text{Hence}\quad \text{average velocity} &= \frac{0 - 60}{8 - 2} \\ &= -10 \text{ m/s.} \\ \text{Also}\quad \text{distance travelled} &= 20 + 80 \\ &= 100 \text{ metres,} \\ \text{so}\quad \text{average speed} &= \frac{100}{6} \\ &= 16\tfrac{2}{3} \text{ m/s.} \end{aligned}$$

Exercise 7A

FOUNDATION

1 A particle is moving with displacement function $x = t^2 + 2$, where time t is in seconds and displacement x is in metres.

a Find the position when $t = 0$.

b Find the position when $t = 4$.

c Find the average velocity during the first 4 seconds, using the definition

$$\text{average velocity} = \frac{\text{change in displacement}}{\text{change in time}}.$$

2 For each displacement function below, copy and complete the table of values to the right. Hence find the average velocity during the first 2 seconds. The units in each part are seconds and centimetres.

t	0	2
x		

a $x = 12t - t^2$

b $x = (t - 2)^2$

c $x = t^3 - 4t + 3$

d $x = 2^t$

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3 A particle moves according to the equation $x = t^3 - 4t$, where x is the displacement in centimetres from the origin O at time t seconds after time zero.

a Copy and complete the table of values to the right.

t	0	1	2	3	4
x					

b Hence find the average velocity:

i during the first second (that is, from $t = 0$ to $t = 1$),

ii during the second second (that is, from $t = 1$ to $t = 2$),

iii during the third second (that is, from $t = 2$ to $t = 3$),

iv during the fourth second (that is, from $t = 3$ to $t = 4$).

4 A particle moves according to the equation $x = t^2 - 4$, where x is the displacement in metres from the origin O at time t seconds after time zero.

a Copy and complete the table of values to the right.

t	0	1	2	3
x				

b Hence find the average velocity:

i during the first second,

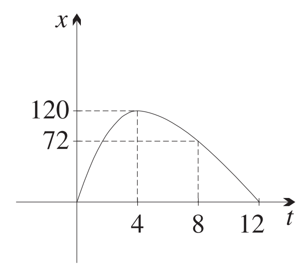
ii during the first two seconds,

iii during the first three seconds,

iv during the third second.

c Use the table of values above to sketch the displacement–time graph. Then add the chords corresponding to the average velocities calculated in part **b**.

5 A piece of cardboard is shot 120 metres vertically into the air by an explosion and floats back to the ground, landing at the same place. The graph to the right gives its height x metres above the ground t seconds after the explosion.



a Copy and complete the following table of values.

t	0	4	8	12
x				

b What is the total distance travelled by the cardboard?

c Find the average speed of the cardboard during its travels, using the formula

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}.$$

d Find the average velocity during:

i the ascent, **ii** the descent, **iii** the full 12 seconds.

6 A particle moves according to the equation $x = 4t - t^2$, where distance is in metres and time is in seconds.

a Copy and complete the table of values to the right.

t	0	1	2	3	4
x					

b Hence sketch the displacement–time graph.

c Find the total distance travelled during the first 4 seconds. Then find the average speed during this time.

d Find the average velocity during the time:

i from $t = 0$ to $t = 2$, **ii** from $t = 2$ to $t = 4$, **iii** from $t = 0$ to $t = 4$.

e Add to your graph the chords corresponding to the average velocities in part **d**.

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DEVELOPMENT

7 Michael the mailman rides his bicycle 1 km up a hill at a constant speed of 10 km/hr. He then turns around and rides back down the hill at a constant speed of 30 km/hr.

a How many minutes does he take to travel:

i the first kilometre, when he is riding up the hill,

ii the second kilometre, when he is riding back down again?

b Use these values to draw a displacement–time graph, with the time axis in minutes.

c What is his average speed over the total 2 km journey?

d What is the average of his speeds up and down the hill?

8 Sadie the snail is crawling up a 6-metre-high wall. She takes an hour to crawl up 3 metres, then falls asleep for an hour and slides down 2 metres, repeating the cycle until she reaches the top of the wall. Let x be Sadie's height in metres after t hours.

a Copy and complete the table of values of Sadie's height up the wall.

t	0	1	2	3	4	5	6	7
x								

b Hence sketch the displacement–time graph.

c How long does Sadie take to reach the top?

d What total distance does she travel, and what is her average speed?

e What is her average velocity over this whole time?

f Which places on the wall does she visit exactly three times?

9 A particle moves according to the equation $x = 2\sqrt{t}$, for $t \geq 0$, where distance x is in centimetres and time t is in seconds.

a Copy and complete the table of values to the right, and sketch the curve.

t					
x	0	2	4	6	8

b Hence find the average velocity as the particle moves:

i from $x = 0$ to $x = 2$,

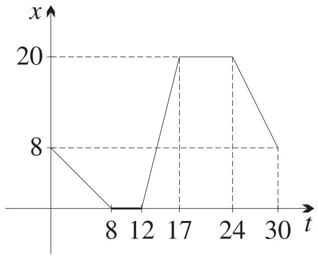
ii from $x = 2$ to $x = 4$,

iii from $x = 4$ to $x = 6$,

iv from $x = 0$ to $x = 6$.

c What does the equality of the answers to parts **ii** and **iv** of part **b** tell you about the corresponding chords in part **c**?

10 Eleni is practising reversing in her driveway. Starting 8 metres from the gate, she reverses to the gate, and pauses. Then she drives forward 20 metres, and pauses. Then she reverses to her starting point. The graph to the right shows her distance x in metres from the front gate after t seconds.



a What is her average velocity:

i during the first 8 seconds,

ii while she is driving forwards,

iii while she is reversing the second time?

b Find the total distance she travelled, and her average speed, over the 30 seconds.

c Find her change in displacement, and her average velocity, over the 30 seconds.

d What would her average speed have been if she had not paused at the gate and at the garage?

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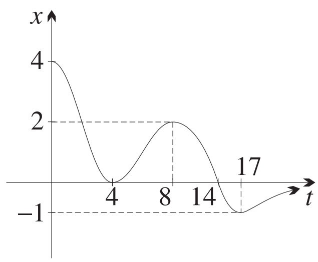
11 A girl is leaning over a bridge 4 metres above the water, playing with a weight on the end of a spring. The graph shows the height x in metres of the weight above the water as a function of time t seconds after she first drops it.



a How many times is the weight:

i at $x = 3$, **ii** at $x = 1$, **iii** at $x = -\frac{1}{2}$?

b At what times is the weight:

i at the water surface, **ii** above the water surface?

c How far above the water does it rise again after it first touches the water, and when does it reach this greatest height?

d What is the weight's greatest depth under the water and when does it occur?

e What happens to the weight eventually?

f What is its average velocity:

i during the first 4 seconds, **ii** from $t = 4$ to $t = 8$, **iii** from $t = 8$ to $t = 17$?

g What distance does it travel:

i over the first 4 seconds, **ii** over the first 8 seconds,

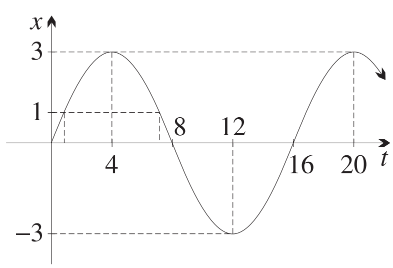
iii over the first 17 seconds, **iv** eventually?

h What is its average speed over the first:

i 4, **ii** 8, **iii** 17 seconds?

CHALLENGE

12 A particle is moving according to $x = 3 \sin \frac{\pi}{8}t$, in units of centimetres and seconds. Its displacement–time graph is sketched to the right.



a Use $T = \dfrac{2\pi}{n}$ to confirm that the period is 16 seconds.

b Find the maximum and minimum values of the displacement.

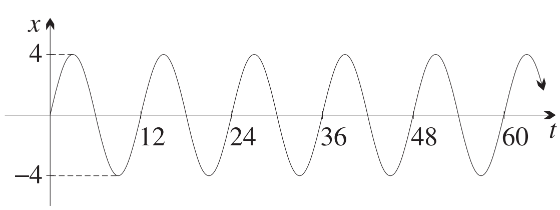
c Find the first two times when the displacement is maximum.

d Find the first two times when the particle returns to its initial position.

e When, during the first 20 seconds, is the particle on the negative side of the origin?

f Find the total distance travelled during the first 16 seconds, and the average speed.

13 A particle is moving according to the equation $x = 4 \sin \frac{\pi}{6}t$, in units of metres and seconds. The graph of its displacement for the first minute is sketched to the right.



a Find the amplitude and period.

b How many times does the particle return to the origin by the end of the first minute?

c Find at what times it visits $x = 4$ during the first minute.

d Find how far it travels during the first 12 seconds, and its average speed in that time.

e Find the values of x when $t = 0$, $t = 1$ and $t = 3$. Hence show that the average speed during the first second is twice the average speed during the next 2 seconds.

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14 A balloon rises so that its height h in metres after t minutes is $h = 8000\left(1 - e^{-0.06t}\right)$.

a What height does it start from, and what happens to the height as $t \to \infty$?

b Copy and complete the table to the right, correct to the nearest metre.

t	0	10	20	30
h				

c Sketch the displacement–time graph of the motion.

d Find the balloon's average velocity during the first 10 minutes, the second 10 minutes and the third 10 minutes, correct to the nearest metre per minute.

e Use your calculator to show that the balloon has reached 99% of its final height after 77 minutes, but not after 76 minutes.

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7B Velocity and acceleration as derivatives

If I drive the 160 km from Sydney to Newcastle in 2 hours, my average velocity is 80 km per hour. But my *instantaneous velocity* during the journey, as displayed on the speedometer, may range from zero at traffic lights to 110 km per hour on expressways. Just as an average velocity corresponds to the gradient of a chord on the displacement–time graph, so an instantaneous velocity corresponds to the gradient of a tangent.

Instantaneous velocity and speed

From now on, the words *velocity* and *speed* alone will mean instantaneous velocity and instantaneous speed.

5 INSTANTANEOUS VELOCITY AND INSTANTANEOUS SPEED

- The *instantaneous velocity* v of a particle in motion is the gradient of the tangent on the displacement–time graph,

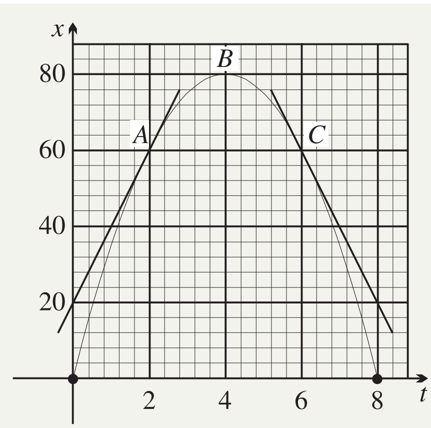
$$v = \frac{dx}{dt} \qquad \text{which can also be written as} \qquad v = \dot{x}.$$

- The dot over any symbol means differentiation with respect to time.
- The *instantaneous speed* is the absolute value $|v|$ of the velocity.

The notation $\dot{x}$, originally introduced by Newton, is yet another way of writing the derivative. The dot over the x, or over any symbol, stands for differentiation with respect to time t. Thus the symbols v, $\dfrac{dx}{dt}$ and $\dot{x}$ all mean velocity.

Example 4 7B

Here again is the displacement–time graph of the ball moving with equation $x = 5t(8 - t)$.



a Differentiate to find the equation for the velocity v, draw up a table of values at 2-second intervals and sketch the velocity–time graph.

b Measure the gradients of the tangents that have been drawn at A, B and C on the displacement–time graph and compare your answers with the table of values in part **a**.

c With what velocity was the ball originally hit?

d What is its impact speed when it hits the ground?

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SOLUTION

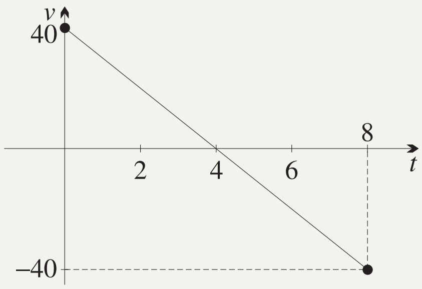
a The equation of motion is $x = 5t(8 - t)$

$$x = 40t - 5t^2,$$

and differentiating, $v = 40 - 10t.$

The graph of velocity is a straight line, with v-intercept 40 and gradient -10.

t	0	2	4	6	8
v	40	20	0	-20	-40



b These values agree with the measurements of the gradients of the tangents at A where $t = 2$, at B where $t = 4$, and at C where $t = 6$.
(Be careful to take account of the different scales on the two axes.)

c When $t = 0$, $v = 40$, so the ball was originally hit upwards at 40 m/s.

d When $t = 8$, $v = -40$, so the ball hits the ground again at 40 m/s.

Vector and scalar quantities

Displacement and velocity are *vector quantities*, meaning that they have a direction built into them. In the example above, a negative velocity means the ball is going downwards and a negative displacement would mean it was below ground level. Distance and speed, however, are called *scalar quantities* — distance is the magnitude of the displacement, and speed is the magnitude of the velocity — and neither can be negative.

Example 5 7B

A particle moves with displacement $x = t^3 - 6t - 2$, in units of metres and seconds.

a Differentiate to find the velocity function.

b Find the displacement, distance from the origin, velocity and speed when:

i $t = 0$, **ii** $t = 3$.

SOLUTION

a The displacement equation is $x = t^3 - 6t - 2$,
and differentiating, $v = 3t^2 - 6$.

b **i** When $t = 0$, $x = -2$
and $v = -6$.
Thus when $t = 0$, the displacement is -2 metres, the particle is 2 metres from the origin, the velocity is -6 m/s, and the speed is 6 m/s.

ii When $t = 3$, $x = 27 - 18 - 2$
$= 7$,
and $v = 27 - 6$
$= 21$.
Thus when $t = 3$, the displacement is 7 metres, the particle is 7 metres from the origin, the velocity is 21 m/s, and the speed is 21 m/s.

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Finding when a particle is stationary

A particle is said to be *stationary* when its velocity v is zero, that is, when $\dfrac{dx}{dt} = 0$. This is the origin of the word 'stationary point' that we have been using to describe a point on a graph where the derivative is zero. For example, the ball in the first example was stationary for an instant at the top of its flight when $t = 4$, because the velocity was zero at the instant when its motion changed from upwards to downwards.

6 FINDING WHEN A PARTICLE IS STATIONARY

- A particle is *stationary* when its velocity is zero.
- To find when a particle is stationary, put $v = 0$ and solve for t.

Example 6 7B

A particle moves so that its distance in metres from the origin at time t seconds is given by $x = \frac{1}{3}t^3 - 6t^2 + 27t - 18$.

a Find the times when the particle is stationary.

b Find its distance from the origin at these times.

SOLUTION

a The displacement function is $x = \frac{1}{3}t^3 - 6t^2 + 27t - 18$

and differentiating, $v = t^2 - 12t + 27$

$= (t - 3)(t - 9)$,

so the particle is stationary after 3 seconds and after 9 seconds.

b When $t = 3$, $x = 9 - 54 + 81 - 18$

$= 18$,

and when $t = 9$, $x = 243 - 486 + 243 - 18$

$= -18$,

Thus the particle is 18 metres from the origin on both occasions.

Acceleration as the second derivative

A particle is said to be *accelerating* if its velocity is changing. The *acceleration* of an object is defined to be the rate at which the velocity is changing. Thus the acceleration a is the derivative $\dfrac{dv}{dt} = \dot{v}$ of the velocity with respect to time.

Because velocity is the derivative of displacement, the acceleration is the second derivative $\dfrac{d^2x}{dt^2} = \ddot{x}$ of displacement.

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7 ACCELERATION AS A DERIVATIVE

- Acceleration is the first derivative of velocity with respect to time:

$$a = \frac{dv}{dt} = \dot{v}.$$

- Acceleration is the second derivative of displacement with respect to time:

$$a = \frac{d^2x}{dt^2} = \ddot{x}.$$

Again, the dot stands for differentiation with respect to time t. Thus

$$\dot{x} \text{ means } \frac{dx}{dt}, \qquad \ddot{x} \text{ means } \frac{d^2x}{dt^2}, \qquad \dot{v} \text{ means } \frac{dv}{dt}.$$

and the symbols a, $\ddot{x}$, $\dot{v}$, $\dfrac{d^2x}{dt^2}$ and $\dfrac{dv}{dt}$ all mean the acceleration.

Note: Be very careful with the symbol a, because in this context a is the acceleration function, whereas elsewhere the letter a is usually used for a constant.

Example 7 7B

Consider again the ball moving with displacement function $x = 5t(8 - t)$.

a Find the velocity function v and the acceleration function a.

b Sketch the graph of the acceleration function.

c Find and describe the displacement, velocity and acceleration when $t = 2$.

d State when the ball is speeding up and when it is slowing down, explaining why this can happen when the acceleration is constant.

SOLUTION

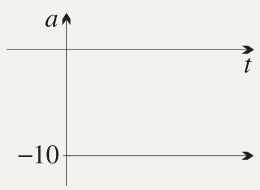
a The function is $\quad x = 40t - 5t^2.$

Differentiating, $\quad v = 40 - 10t.$

Differentiating again, $\quad a = -10,$ which is a constant.

b Hence the acceleration is always 10 m/s^2 downwards.

The graph is drawn to the right.



c Substitute $t = 2$ into the functions x, v and a.

When $t = 2$, $x = 60$,

$v = 20$,

$a = -10$.

Thus when $t = 2$, the displacement is 60 metres above the ground, the velocity is 20 m/s upwards, and the acceleration is 10 m/s^2 downwards.

d During the first 4 seconds, the ball has positive velocity, meaning that it is rising, and the ball is slowing down by 10 m/s every second.

During the last 4 seconds, however, the ball has negative velocity, meaning that it is falling, and the ball is speeding up by 10 m/s every second.

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Units of acceleration

In the previous example, the particle's velocity was decreasing by 10 m/s every second. The particle is said to be 'accelerating at -10 metres per second, per second', written in symbols as $-10\ \text{m/s}^2$ or as $-10\ \text{ms}^{-2}$. The units of acceleration correspond with the indices of the second derivative $\dfrac{d^2x}{dt^2}$.

Acceleration should normally be regarded as a vector quantity, that is, with a direction built into it. This is why the particle's acceleration is written with a minus sign as $-10\ \text{m/s}^2$. Alternatively, one can omit the minus sign and specify the direction instead, writing '10 m/s^2 in the downwards direction'.

Example 8 — 7B

In worked Example 6, we examined the function $x = \frac{1}{3}t^3 - 6t^2 + 27t - 18$.

a Find the acceleration function and find when the acceleration is zero.

b Where is the particle at this time and what is its velocity?

SOLUTION

a The displacement function is $x = \frac{1}{3}t^3 - 6t^2 + 27t - 18$.

Differentiating, $v = t^2 - 12t + 27$,

and differentiating again, $a = 2t - 12$

$= 2(t - 6)$.

Thus the acceleration is zero when $t = 6$.

b When $t = 6$, $v = 36 - 72 + 27$

$= -9$,

and $x = 72 - 216 + 162 - 18$

$= 0$.

Thus when $t = 6$, the particle is at the origin, moving with velocity -9 m/s.

Trigonometric equations of motion

When a particle's motion is described by a sine or cosine function, it moves backwards and forwards, and is therefore stationary over and over again.

The resulting wavy graphs of x, v and a are very helpful in interpreting the particle's motion. In fact, in the next worked example, it is possible to solve all the trigonometric equations simply by looking at these three graphs.

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Example 9 7B

A particle's displacement function is $x = 2 \sin \pi t$.

a Find its velocity and acceleration functions.

b Graph all three functions in the time interval $0 \le t \le 2$.

c Find the times within the time interval $0 \le t \le 2$ when the particle is at the origin, and find its speed and acceleration at those times.

d Find the times within the time interval $0 \le t \le 2$ when the particle is stationary, and find its displacement and acceleration at those times.

e Briefly describe the motion.

SOLUTION

a The displacement function is $x = 2 \sin \pi t$,

which has amplitude 2 and period $\dfrac{2\pi}{\pi} = 2$.

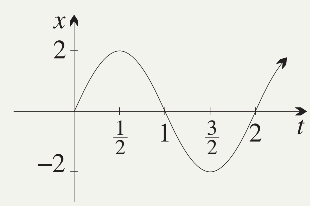
Differentiating, $\qquad v = 2\pi \cos \pi t$,

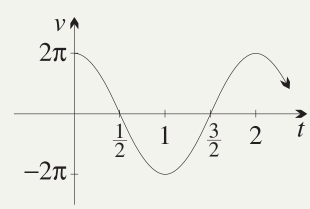
which has amplitude 2π and period 2.

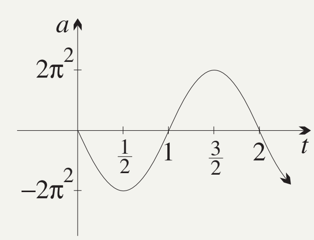
Differentiating again, $\qquad a = -2\pi^2 \sin \pi t$,

which has amplitude $2\pi^2$ and period 2.

b The three graphs are drawn to the right.







c • The *instantaneous velocity* v of a particle in motion is the gradient of the tangent on the displacement–time graph,

$$v = \frac{dx}{dt} \qquad \text{which can also be written as} \qquad v = \dot{x}.$$

• The dot over any symbol means differentiation with respect to time.

• The *instantaneous speed* is the absolute value $|v|$ of the velocity.

d The particle is stationary when $v = 0$,

and reading from the velocity graph above,

this occurs when $\qquad t = \frac{1}{2}$ or $1\frac{1}{2}$.

Reading from the displacement and acceleration graphs,

when $t = \frac{1}{2}$, $\qquad x = 2$, $\quad$ and $\quad a = -2\pi^2$,

and when $t = 1\frac{1}{2}$, $\qquad x = -2$ $\quad$ and $\quad a = 2\pi^2$.

e The particle oscillates forever between $x = -2$ and $x = 2$, with period 2, beginning at the origin and moving first to $x = 2$.

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Motion with exponential functions — limiting values of displacement and velocity

Sometimes a question will ask what happens to the particle 'eventually', or 'as time goes on'. This just means taking the limit of the displacement and the velocity as $t \to \infty$. Particles whose motion is described by an exponential function are the most usual examples of this. Remember that $e^{-x} \to 0$ as $x \to \infty$.

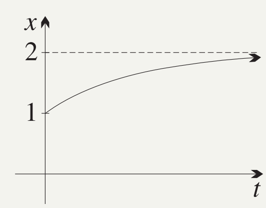
Example 10 7B

A particle is moving so that its height x metres above the ground at time t seconds after time zero is $x = 2 - e^{-3t}$.

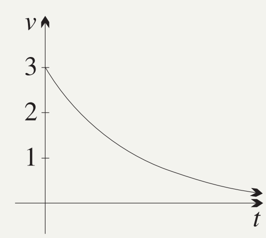
a Find the velocity and acceleration functions.
b Sketch the three graphs of displacement, velocity and acceleration.
c Find the initial values of displacement, velocity and acceleration.
d What happens to the displacement, velocity and acceleration eventually?
e Briefly describe the motion.

SOLUTION

a The displacement function is $\quad x = 2 - e^{-3t}$.
Differentiating, $\quad v = 3e^{-3t}$,
and differentiating again, $\quad a = -9e^{-3t}$.

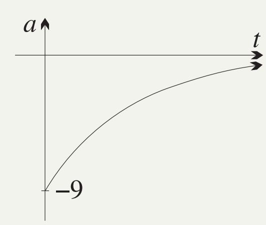


b The three graphs are drawn to the right.



c Substitute $t = 0$ and use the fact that $e^0 = 1$.
Thus initially, $x = 1$,
$v = 3$,
$a = -9$.

d As t increases, that is, as $t \to \infty$, $\quad e^{-3t} \to 0$.
Hence *eventually* (meaning as $t \to \infty$), $\quad x \to 2$,
$v \to 0$,
$a \to 0$.



e The particle starts 1 metre above the ground,
with initial velocity of 3 m/s upwards.
It is constantly slowing down,
and it moves towards a limiting position at height 2 metres.

Extension — Newton's second law of motion

Newton's second law of motion — a law of physics, not of mathematics — says that when a force is applied to a body that is free to move, the body accelerates with an acceleration proportional to the force and inversely proportional to the mass of the body. Written symbolically,

$$F = ma,$$

where m is the mass of the body, F is the sum of all the force applied, and a is the resulting acceleration. (The units of force are chosen to make the constant of proportionality 1 — in units of kilograms, metres and seconds, the units of force are, appropriately, called *newtons*.)

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This means that acceleration is felt in our bodies as a force, as we all know when a car we are in accelerates away from the lights or comes to a stop quickly. In this way, the second derivative becomes directly observable to our senses as a force, just as the first derivative, velocity, is observable to our sight.

Although these things are not part of the Advanced course, it is helpful to have an intuitive idea that force and acceleration are closely related.

Exercise 7B

FOUNDATION

Note: Most questions in this exercise are long in order to illustrate how the physical situation of the particle's motion is related to the mathematics and the graph. The mathematics should be well known, but the physical interpretations can be confusing.

1 A particle is moving with displacement function $x = 20 - t^2$, in units of metres and seconds.

- **a** Differentiate to find the velocity v as a function of time t.
- **b** Differentiate again to find the acceleration a.
- **c** Find the displacement, velocity and acceleration when $t = 3$.
- **d** What are the distance from the origin and the speed when $t = 3$?

2 For each displacement function below, differentiate to find the velocity function v, and differentiate again to find the acceleration function a. Then find the displacement, velocity and acceleration when $t = 1$. The units are metres and seconds.

a $x = 5t^2 - 10t$ **b** $x = 3t - 2t^3$ **c** $x = t^4 - t^2 + 4$

3 A particle's displacement function is $x = t^2 - 10t$, in units of centimetres and seconds.

- **a** Differentiate to find v as a function of t.
- **b** What are the displacement, the distance from the origin, the velocity, and the speed after 3 seconds?
- **c** When is the particle stationary and where is it then?

4 A particle moves on a horizontal line so that its displacement x cm to the right of the origin at time t seconds is $x = t^3 - 6t^2$.

- **a** Differentiate to find v as a function of t, and differentiate again to find a.
- **b** Where is the particle initially and what are its speed and acceleration then?
- **c** At time $t = 3$, is the particle to the left or to the right of the origin?
- **d** At time $t = 3$, is the particle travelling to the left or to the right?
- **e** At time $t = 3$, is the particle accelerating to the left or to the right?
- **f** Show that the particle is stationary when $t = 4$ and find where it is at this time.
- **g** Show that the particle is at the origin when $t = 6$ and find its velocity and speed at this time.

5 For each displacement function below, differentiate to find the velocity function v, and differentiate again to find the acceleration function a. Then find the displacement, velocity and acceleration when $t = \frac{\pi}{2}$. The units are centimetres and seconds.

a $x = \sin t$ **b** $x = \cos t$

6 For each displacement function below, find by differentiation the velocity function v and the acceleration function a. Then find the displacement, velocity and acceleration when $t = 1$. The units are metres and seconds.

a $x = e^t$ **b** $x = e^{-t}$

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7 A cricket ball is thrown vertically upwards. Its height x in metres at time t seconds after it is thrown is given by $x = 20t - 5t^2$.

- **a** Find v and a as functions of t, and show that the ball is always accelerating downwards. Then sketch graphs of x, v and a against t.
- **b** Find the speed at which the ball was thrown.
- **c** Find when it returns to the ground (that is, when $x = 0$) and show that its speed then is equal to the initial speed.
- **d** Find its maximum height above the ground and the time taken to reach this height.
- **e** Find the acceleration at the top of the flight, and explain why the acceleration can be non-zero when the ball is stationary.

DEVELOPMENT

8 If $x = e^{-4t}$, find the velocity function $\dot{x}$ and the acceleration function $\ddot{x}$.

- **a** Explain why none of the functions x, $\dot{x}$ and $\ddot{x}$ can ever change sign, and state their signs.
- **b** Using the displacement function, find where the particle is:
  - **i** initially (substitute $t = 0$),
  - **ii** eventually (take the limit as $t \to \infty$).
- **c** What are the particle's velocity and acceleration:
  - **i** initially,
  - **ii** eventually?

9 Find the velocity function v and the acceleration function a for a particle P moving according to $x = 2 \sin \pi t$.

- **a** Show that P is at the origin when $t = 1$ and find its velocity and acceleration then.
- **b** In what direction is the particle:
  - **i** moving,
  - **ii** accelerating, when $t = \frac{1}{3}$?

10 A particle moves according to $x = t^2 - 8t + 7$, in units of metres and seconds.

- **a** Find the velocity $\dot{x}$ and the acceleration $\ddot{x}$ as functions of time t.
- **b** Sketch the graphs of the displacement x, velocity $\dot{x}$ and acceleration $\ddot{x}$.
- **c** When is the particle:
  - **i** at the origin,
  - **ii** stationary?
- **d** What is the maximum distance from the origin, and when does it occur:
  - **i** during the first 2 seconds,
  - **ii** during the first 6 seconds,
  - **iii** during the first 10 seconds?
- **e** What is the particle's average velocity during the first 7 seconds? When and where is its instantaneous velocity equal to this average?
- **f** How far does it travel during the first 7 seconds, and what is its average speed?

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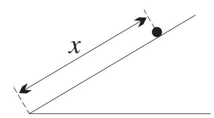
11 A smooth piece of ice is projected up a smooth inclined surface, as shown to the right. Its distance x in metres up the surface at time t seconds is $x = 6t - t^2$.



a Find the functions for velocity v and acceleration a.

b Sketch the graphs of displacement x and velocity v.

c In which direction is the ice moving, and in which direction is it accelerating:

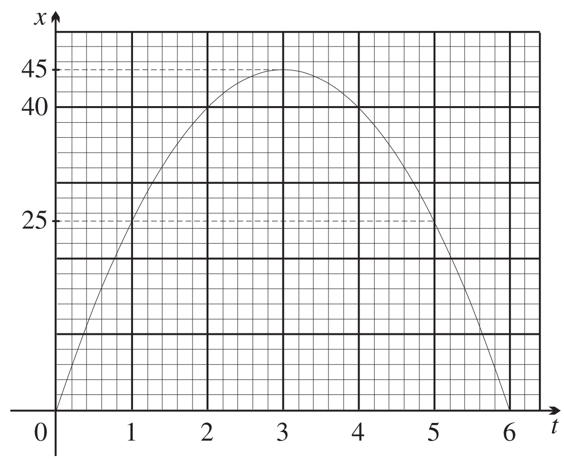
i when $t = 2$? **ii** when $t = 4$?

d When is the ice stationary, for how long is it stationary, where is it then, and is it accelerating then?

e Show that the average velocity over the first 2 seconds is 4 m/s. Then find the time and place at which the instantaneous velocity equals this average velocity.

f Show that the average speed during the first 3 seconds, the next 3 seconds, and the first 6 seconds are all the same.

12 A stone was thrown vertically upwards. The graph to the right shows its height x metres at time t seconds after it was thrown.



a What was the stone's maximum height, how long did it take to reach it, and what was its average speed during this time?

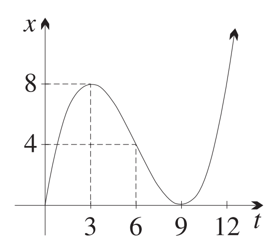
b Draw tangents and measure their gradients to find the velocity of the stone at times $t = 0,\ 1,\ 2,\ 3,\ 4,\ 5$ and 6.

c For what length of time was the stone stationary at the top of its flight?

d The graph is concave down everywhere. How is this relevant to the motion?

e Draw a graph of the instantaneous velocity of the stone from $t = 0$ to $t = 6$. What does this velocity–time graph tell you about what happened to the velocity during these 6 seconds?

13 A particle is moving horizontally so that its displacement x metres to the right of the origin at time t seconds is given by the graph to the right.



a In the first 10 seconds, what is its maximum distance from the origin and when does it occur?

b By examining the gradient, find when the particle is:

i stationary,

ii moving to the right,

iii moving to the left.

c When does it return to the origin, what is its velocity then, and in which direction is it accelerating?

d When is its acceleration zero, where is it then, and in what direction is it moving?

e By examining the concavity, find the time interval during which the particle's acceleration is negative.

f At about what times are:

i the displacement,

ii the velocity, about the same as those at $t = 2$?

g Sketch (roughly) the graphs of velocity v and acceleration a.

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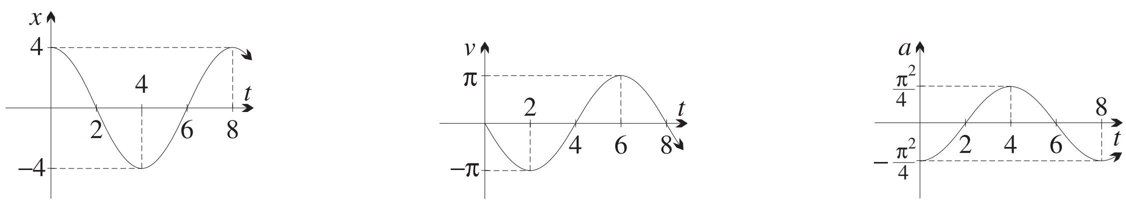
CHALLENGE

14 A particle is moving according to $x = 4\cos\frac{\pi}{4}t$, where the units are metres and seconds. The displacement, velocity and acceleration graphs are drawn below, for $0 \le t \le 8$.



a Differentiate to find the functions for the velocity v and the acceleration a.

b What are the particle's maximum displacement, velocity and acceleration, and when, during the first 8 seconds, do they occur?

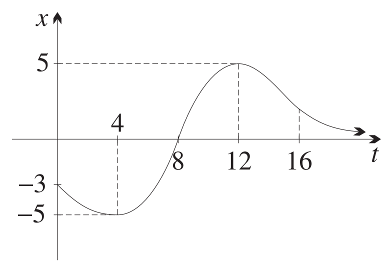
c How far does it travel during the first 20 seconds, and what is its average speed?

d Show by substitution that $x = 2$ when $t = 1\frac{1}{3}$ and when $t = 6\frac{2}{3}$. Hence use the graph to find when $x < 2$ during the first 8 seconds.

e When, during the first 8 seconds, is:

i $v = 0$, **ii** $v > 0$?

15 A particle is moving vertically according to the graph shown to the right, where upwards has been taken as positive.



a At what times is this particle:

i below the origin,

ii moving downwards,

iii accelerating downwards?

b At about what time is its speed greatest?

c At about what times are:

i the distance from the origin, **ii** the velocity, about the same as those at $t = 3$?

d How many times between $t = 4$ and $t = 12$ is the instantaneous velocity equal to the average velocity during this time?

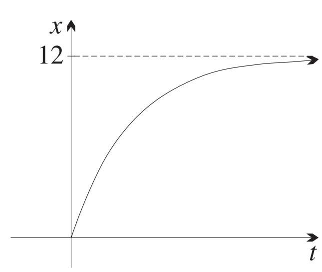
e How far will the particle eventually travel?

f Draw an approximate sketch of the graph of v as a function of time.

16 A large stone is falling through a layer of mud. Its depth x metres below ground level at time t minutes is

$$x = 12 - 12e^{-0.5t}.$$

Its displacement–time graph is drawn to the right.



a Show that the velocity and acceleration functions are

$$\dot{x} = 6e^{-0.5t} \quad \text{and} \quad \ddot{x} = -3e^{-0.5t}.$$

b In which direction is the stone always:

i travelling, **ii** accelerating?

c What happens to the position, velocity and acceleration of the particle as $t \to \infty$?

d Show that the stone is halfway between the origin and its final position at the time when $e^{-0.5t} = \frac{1}{2}$, and solve this equation for t. Show that its speed is then half its initial speed, and its acceleration is half its initial acceleration.

e Use your calculator to show that the stone is within 1 mm of its final position after 19 minutes, but not after 18 minutes.

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7C Integrating with respect to time

The inverse process of differentiation is integration. Thus if the acceleration function is known, integration will generate the velocity function. In the same way, if the velocity function is known, integration will generate the displacement function.

Using initial conditions

Taking the primitive of a function always involves a constant of integration. Determining such a constant requires an *initial condition* (or *boundary condition*) to be known. For example, the problem may tell us the velocity when $t = 0$, or give us the displacement when $t = 3$.

In this chapter, the constants of integration cannot ever be omitted.

8 INTEGRATING WITH RESPECT TO TIME

- Given the acceleration function a, integrate to find the velocity function v.
- Given the velocity function v, integrate to find the displacement function x.
- Never omit the constants of integration.
- Use an initial or boundary condition to evaluate each constant of integration.

In the next worked example, the velocity function is given. Integration, using the initial condition, gives the displacement function. Then differentiation gives the acceleration function.

Example 11 7C

A particle is moving so that its velocity t seconds after time zero is $v = 2t - 2$ m/s. Initially it is at $x = 1$.

a Integrate, substituting the initial condition, to show that $x = (t - 1)^2$.
b Find when the particle is at the origin and its velocity then.
c Explain why the particle is never on the negative side of the origin.
d Differentiate to find the acceleration, and show that it is constant.

SOLUTION

a The given velocity function is $v = 2t - 2$. (1)
Integrating, $x = t^2 - 2t + C$, for some constant C.
When $t = 0, x = 1$, so $1 = 0 - 0 + C$,
so $C = 1$ and $x = t^2 - 2t + 1$
$x = (t - 1)^2$. (2)

b Put $x = 0$.
Then from (2), $(t - 1)^2 = 0$
$t = 1$.
Hence the particle is at the origin when $t = 1$,
and substituting $t = 1$ into (1), $v = 2 - 2 = 0$ m/s.

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c Because $x = (t - 1)^2$ is a square, the value of x can never be negative, so the particle is never on the negative side of the origin.

d Differentiating the velocity function $v = 2t - 2$
gives $a = 2, \quad (3)$
so the acceleration is a constant $2\ \text{m/s}^2$.

Example 12 7C

A particle's acceleration function is $a = 24t$. Initially it is at the origin, moving with velocity -12 cm/s.

a Integrate, substituting the initial condition, to find the velocity function.
b Integrate again to find the displacement function.
c Find when the particle is stationary and find the displacement then.
d Find when the particle returns to the origin and the acceleration then.

SOLUTION

a The given acceleration function is $a = 24t$. $\quad (1)$
Integrating, $v = 12t^2 + C$, for some constant C.
When $t = 0$, $v = -12$, so $-12 = 0 + C$,
so $C = -12$ and $v = 12t^2 - 12$. $\quad (2)$

b Integrating again, $x = 4t^3 - 12t + D$, for some constant D.
When $t = 0$, $x = 0$, so $0 = 0 - 0 + D$,
so $D = 0$ and $x = 4t^3 - 12t$. $\quad (3)$

c Put $v = 0$. Then from (2),

$$12t^2 - 12 = 0$$
$$t^2 = 1$$
$$t = 1 \quad (\text{because } t \geq 0).$$

Hence the particle is stationary after 1 second.
When $t = 1$, $x = -8$, so at this time its displacement is $x = -8$ cm.

d Put $x = 0$. Then using (3),

$$4t^3 - 12t = 0$$
$$4t(t^2 - 3) = 0,$$

so

$$t = 0 \text{ or } t = \sqrt{3} \quad (\text{because } t \geq 0).$$

Hence the particle returns to the origin after $\sqrt{3}$ seconds,
and at this time, $a = 24\sqrt{3}\ \text{cm/s}^2$.

The acceleration due to gravity

Since the time of Galileo, it has been known that near the surface of the Earth, a body that is free to fall accelerates downwards at a constant rate, whatever its mass and whatever its velocity, provided that air resistance is ignored. This acceleration is called the *acceleration due to gravity* and is conventionally given the symbol g. The value of this acceleration is about $9.8\ \text{m/s}^2$, or in rounder figures, $10\ \text{m/s}^2$.

The acceleration is downwards. Thus if upwards is taken as positive, the acceleration is $-g$, but if downwards is taken as positive, the acceleration is g.

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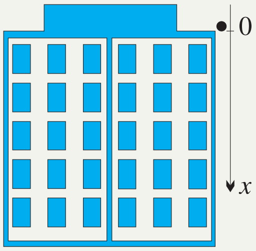
9 THE ACCELERATION DUE TO GRAVITY

- A body that is falling accelerates downwards at a constant rate $g \doteqdot 9.8\ \text{m/s}^2$, provided that air resistance is ignored.
- If upwards is taken as positive, start with the function $a = -g$ and integrate.
- If downwards is taken as positive, start with the function $a = g$ and integrate.

Example 13 — 7C

A stone is dropped from the top of a high building. How far has it travelled, and how fast is it going, after 5 seconds? Take $g = 9.8\ \text{m/s}^2$.



SOLUTION

Let x metres be the distance travelled t seconds after the stone is dropped. This simple sentence puts the origin of space at the top of the building, it puts the origin of time at the instant when the stone is dropped, and it makes downwards the positive direction. It also defines the units of space and time.

Then $\quad a = 9.8$ (given). $\quad$ (1)

Integrating, $\quad v = 9.8t + C$, for some constant C.
Because the stone was dropped, its initial speed was zero,
and substituting, $\quad 0 = 0 + C$,
so $C = 0$, and $\quad v = 9.8t$. $\quad$ (2)

Integrating again, $x = 4.9t^2 + D$, for some constant D.
Because the initial displacement of the stone was zero,
$\quad 0 = 0 + D$,
so $D = 0$, and $\quad x = 4.9t^2$. $\quad$ (3)

When $t = 5$, $\quad v = 49$ $\quad$ (substituting into (2) above)
and $\quad x = 122.5$ $\quad$ (substituting into (3) above).
Hence the stone has fallen 122.5 metres and is moving downwards at 49 m/s.

Making a convenient choice of the origin and the positive direction

Physical problems do not come with origins and directions attached. Thus it is up to us to choose the origins of displacement and time, and the positive direction, so that the arithmetic is as simple as possible.

The previous worked example made reasonable choices, but the next worked example makes quite different choices. In all such problems, the physical interpretation of negatives and displacements is the mathematician's responsibility, and the final answer should be given in ordinary language.

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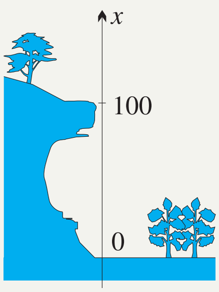
Example 14 7C

A cricketer is standing on a lookout that projects out over the valley floor 100 metres below him. He throws a cricket ball vertically upwards at a speed of 40 m/s and it falls back past the lookout onto the valley floor below.

How long does it take to fall, and with what speed does it strike the ground? (Take $g = 10$ m/s^2.)



SOLUTION

Let x metres be the distance above the valley floor t seconds after the stone is thrown. Again, this simple sentence puts the origin of space at the valley floor, and puts the origin of time at the instant when the stone is thrown. It also makes upwards positive, so that $a = -10$ because the acceleration is downwards.

As discussed, $\quad a = -10. \qquad (1)$

Integrating, $\quad v = -10t + C$, for some constant C.
Because $v = 40$ when $t = 0$, $\quad 40 = 0 + C$,
so $C = 40$, and $\quad v = -10t + 40. \qquad (2)$

Integrating again, $\quad x = -5t^2 + 40t + D$, for some constant D.
Because $x = 100$ when $t = 0$, $100 = 0 + 0 + D$,
so $D = 100$, and $\quad x = -5t^2 + 40t + 100. \qquad (3)$

The stone hits the ground when $x = 0$, so using (3) above,

$$\begin{aligned} -5t^2 + 40t + 100 &= 0 \\ t^2 - 8t - 20 &= 0 \\ (t - 10)(t + 2) &= 0 \\ t &= 10 \text{ or } -2. \end{aligned}$$

The ball was not in flight at $t = -2$, so the ball hits the ground after 10 seconds.
Substituting $t = 10$ into equation (2), $v = -100 + 40 = -60$,
so the ball hits the ground at 60 m/s.

Formulae from physics cannot be used

This course requires that even problems where the acceleration is constant, such as the two above, must be solved by integrating the acceleration function. Many readers will know of three very useful equations for motion with constant acceleration a:

$$v = u + at \qquad \text{and} \qquad s = ut + \tfrac{1}{2}at^2 \qquad \text{and} \qquad v^2 = u^2 + 2as.$$

These equations automate the integration process, and so cannot be used in this course. Question 18 in Exercise 7C develops a proper proof of these results.

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Integrating trigonometric functions

The next worked example applies the same methods of integration to motion involving trigonometric functions.

Example 15 7C

The velocity of a particle initially at the origin is $v = \sin\frac{1}{4}t$, in units of metres and seconds.

a Find the displacement function.
b Find the acceleration function.
c Find the values of displacement, velocity and acceleration when $t = 4\pi$.
d Briefly describe the motion, and sketch the displacement–time graph.

SOLUTION

a The velocity is $v = \sin\frac{1}{4}t$. (1)

Integrating (1), $x = -4\cos\frac{1}{4}t + C$, for some constant C.

Substituting $x = 0$ when $t = 0$,

$$0 = -4 \times 1 + C,$$
$$C = 4.$$

Thus $x = 4 - 4\cos\frac{1}{4}t$. (2)

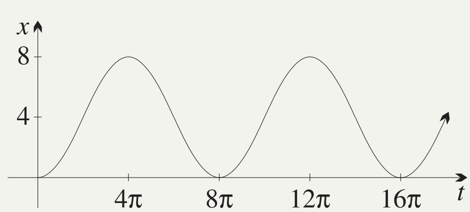
b Differentiating (1), $a = \frac{1}{4}\cos\frac{1}{4}t$. (3)

c When $t = 4\pi$, $x = 4 - 4 \times \cos\pi$, using (2),
$= 8$ metres.

Also $v = \sin\pi$, using (1),
$= 0$ m/s,

and $a = \frac{1}{4}\cos\pi$, using (3),
$= -\frac{1}{4}$ m/s^2.

d The particle oscillates between $x = 0$ and $x = 8$ with period 8π seconds.



Integrating exponential functions

The next worked example involves exponential functions. The velocity function approaches a limit 'as time goes on'.

Example 16 7C

The acceleration of a particle is given by $a = e^{-2t}$ (in units of metres and seconds), and the particle is initially stationary at the origin.

a Find the velocity and displacement functions.
b Find the displacement when $t = 10$.
c Sketch the velocity–time graph and describe briefly what happens to the velocity of the particle as time goes on.

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SOLUTION

a The acceleration is $a = e^{-2t}$. (1)

Integrating, $v = -\frac{1}{2}e^{-2t} + C$, for some constant C.

It is given that when $t = 0$, $v = 0$,

so $0 = -\frac{1}{2} + C$

$C = \frac{1}{2}$,

and $v = -\frac{1}{2}e^{-2t} + \frac{1}{2}$. (2)

Integrating again, $x = \frac{1}{4}e^{-2t} + \frac{1}{2}t + D$, for some constant D.

It is given that when $t = 0$, $x = 0$,

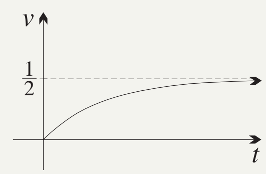
so $0 = \frac{1}{4} + D$

$D = -\frac{1}{4}$,

and $x = \frac{1}{4}e^{-2t} + \frac{1}{2}t - \frac{1}{4}$. (3)

b When $t = 10$, $x = \frac{1}{4}e^{-20} + 5 - \frac{1}{4}$

$= 4\frac{3}{4} + \frac{1}{4}e^{-20}$ metres.



c Using equation (2), the velocity is initially zero,
and increases so that the limiting velocity as time goes on is $\frac{1}{2}$ m/s.

Note: A car moving off from the kerb on level ground obeys this sort of equation if the accelerator is pressed down not too far and kept in that position.

Exercise 7C

FOUNDATION

1 A particle is moving with velocity function $v = 3t^2 - 6t$, in units of metres and seconds. At time $t = 0$, its displacement is $x = 4$.

a Integrate, substituting the initial condition, to find the displacement function.

b Show that the particle is at the origin when $t = 2$ and find its velocity then.

c Differentiate the given velocity function to find the acceleration function.

d Show that the acceleration is zero when $t = 1$, and find the displacement then.

2 A particle is moving with acceleration $a = -6t$, in units of centimetres and seconds. Initially it is at rest at $x = 8$.

a Integrate, substituting the initial condition, to find the velocity function.

b Find the particle's velocity and speed when $t = 5$.

c Integrate again to find the displacement function.

d Show that the particle is at the origin when $t = 2$, and find its acceleration then.

3 A particle is moving with acceleration function $a = 8$. Two seconds after time zero, it is stationary at the origin.

a Integrate to find the velocity function.

b Integrate again to find the displacement function.

c Where was the particle initially, and what were its velocity and speed?

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4 The velocity of a particle is the constant function $v = 6$ m/s. At time $t = 0$, the particle is at $x = -30$.

a Integrate to find the displacement function.

b By solving $x = 0$, find how long it takes the particle to reach the origin.

c What is the acceleration function of the particle?

5 A particle is moving with acceleration function $a = 2$, in units of metres and seconds. Initially, it is at the origin, moving with velocity -20 m/s.

a Find the velocity function.

b Find the displacement function.

c By solving $v = 0$, find when the particle is stationary and find where it is then.

d By solving $x = 0$, find when it returns to the origin, and show that its speed then is equal to its initial speed.

6 A stone is dropped from a lookout 80 metres high. Take $g = 10$ m/s^2 and downwards as positive, so that the acceleration function is $a = 10$.

a Using the lookout as the origin, find the velocity and displacement as functions of t. (Hint: When $t = 0$, $v = 0$ and $x = 0$.)

b Show that the stone takes 4 seconds to fall, and find its impact speed.

c Where is it, and what is its speed, halfway through its flight time?

d Show that it takes $2\sqrt{2}$ seconds to go halfway down, and find its speed then.

7 A stone is thrown downwards from the top of a 120-metre building, with an initial speed of 25 m/s. Take $g = 10$ m/s^2 and take upwards as positive, so that $a = -10$.

a Using the ground as the origin, find the acceleration, velocity and height x of the stone t seconds after it is thrown. (Hint: When $t = 0$, $v = -25$ and $x = 120$.)

b By solving $x = 0$, find the time it takes to reach the ground.

c Find the impact speed.

d What is the average speed of the stone during its descent?

DEVELOPMENT

8 Find the velocity function $\dot{x}$ and the displacement function x of a particle whose initial velocity and displacement are zero if:

a $\ddot{x} = -4$ b $\ddot{x} = 6t$ c $\ddot{x} = e^{\frac{1}{2}t}$ d $\ddot{x} = e^{-3t}$

e $\ddot{x} = 8\sin 2t$ f $\ddot{x} = \cos \pi t$ g $\ddot{x} = \sqrt{t}$ h $\ddot{x} = 12(t + 1)^{-2}$

9 Find the acceleration function a and the displacement function x of a particle whose initial displacement is -2 if:

a $v = -4$ b $v = 6t$ c $v = e^{\frac{1}{2}t}$ d $v = e^{-3t}$

e $v = 8\sin 2t$ f $v = \cos \pi t$ g $v = \sqrt{t}$ h $v = 12(t + 1)^{-2}$

10 A particle is moving with acceleration $\ddot{x} = 12t$. Initially, it has velocity -24 m/s and is 20 metres on the positive side of the origin.

a Find the velocity function $\dot{x}$ and the displacement function $\ddot{x}$.

b When does the particle return to its initial position, and what is its speed then?

c What is the minimum displacement, and when does it occur?

d Find x when $t = 0$, 1, 2, 3 and 4, and sketch the displacement–time graph.

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11 A body is moving with its acceleration proportional to the time elapsed. That is,

$$a = kt,$$

where k is a constant of proportionality. When $t = 1$, $v = -6$, and when $t = 2$, $v = 3$.

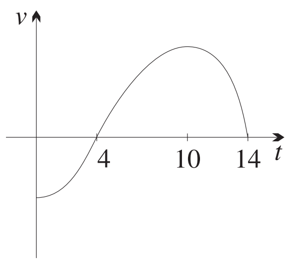
a Integrate the given acceleration function, adding the constant C of integration. Then substitute the two given conditions to find the values of k and C.

b Suppose now that the particle is initially at $x = 2$.

i Integrate again to find the displacement function.

ii When does the body return to its original position?

12 The graph to the right shows a particle's velocity–time graph.



a When is the particle moving forwards?

b When is the acceleration positive?

c When is it furthest from its starting point?

d When is it furthest in the negative direction?

e About when does it return to its starting point?

f Sketch the graphs of acceleration and displacement, assuming that the particle is initially at the origin.

13 A car moves along a straight road from its front gate, where it is initially stationary. During the first 10 seconds, it has a constant acceleration of 2 m/s^2, it has zero acceleration during the next 30 seconds, and it decelerates at 1 m/s^2 for the final 20 seconds until it stops.

a What is the car's speed after 20 seconds?

b Show that the car travels:

i 100 metres during the first 10 seconds,

ii 600 metres during the next 30 seconds,

iii 200 metres during the last 20 seconds.

c Sketch the graphs of acceleration, velocity and distance from the gate.

14 A particle is moving with velocity $\dot{x} = 16 - 4t$ cm/s on a horizontal number line.

a Find $\ddot{x}$ and x. (The function x will have a constant of integration.)

b When does it return to its original position, and what is its speed then?

c When is the particle stationary? Find the maximum distances right and left of the initial position during the first 10 seconds, and the corresponding times and accelerations.

d How far does it travel in the first 10 seconds, and what is its average speed?

CHALLENGE

15 A particle moves from $x = -1$ with velocity $v = \dfrac{1}{t + 1}$.

a Find its displacement and acceleration functions.

b Find how long it takes to reach the origin, and its speed and acceleration then.

c Describe its subsequent motion.

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16 A body moving vertically through air experiences an acceleration $\ddot{x} = -40e^{-2t}$ m/s^2 (we are taking upwards as positive). Initially, it is thrown upwards with speed 15 m/s.

a Taking the origin at the point where it is thrown, find the velocity function $\dot{x}$ and the displacement function x, and find when the body is stationary.

b Find its maximum height and its acceleration then.

c Describe the velocity of the body as $t \to \infty$.

17 A moving particle is subject to an acceleration of $a = -2\cos t$ m/s^2. Initially it is at $x = 2$, moving with velocity 1 m/s, and it travels for 2π seconds.

a Find the velocity function v and the displacement function x.

b When is the acceleration positive?

c When and where is the particle stationary?

d What are the maximum and minimum velocities, and when do they occur?

18 [A proof of three constant-acceleration formulae from physics — not to be used elsewhere.] A particle moves with constant acceleration a. Its initial velocity is u, and at time t it is moving with velocity v and its distance from its initial position is s. Show that:

a $v = u + at$ **b** $s = ut + \frac{1}{2}at^2$ **c** $v^2 = u^2 + 2as$

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7D Rates and differentiation

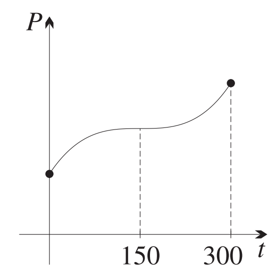
When a quantity varies over time, we saw in Section 8J of the Year 11 book that we can differentiate to find the rate at which it is increasing or decreasing at each time t, and differentiate again to find the rate at which that rate is increasing or decreasing at time t. Then in Chapter 3, we carefully defined a function to be

- *increasing* at $x = a$ if $f'(a) > 0$, and *decreasing* at $x = a$ if $f'(a) < 0$,
- *concave up* at $x = a$ if $f''(a) > 0$, and *concave down* at $x = a$ if $f''(a) < 0$.

These were all precise *pointwise* definitions. But at the same time, from the beginning of Year 11, we have used only vague language about a function being *increasing or decreasing over an interval*. These ideas will now be made precise as well.

Increasing and decreasing in an interval

Suppose that the price P of a share in newly-launched company t days after launching is given by the curve $P = P(t)$ drawn to the right.



The share price P is said to be *increasing in the interval* $0 \le t \le 300$ because every chord slopes upwards, that is

$$P(a) < P(b), \quad \text{for all } a < b \text{ in the interval } [0, 300].$$

Notice that except for the isolated point $t = 150$ where the tangent is horizontal, the function P is increasing at every point in the interval. Such a function is clearly increasing in the interval, because if every other tangent slopes upwards, then every chord without exception slopes upwards.

10 INCREASING AND DECREASING IN AN INTERVAL

Suppose that a function $f(x)$ is defined in an interval I. The interval may be bounded or unbounded, and may be open or closed or neither.

- The function $f(x)$ is called *increasing in the interval I* if every chord within the interval slopes upwards, that is,

$$f(a) < f(b), \quad \text{for all } a < b \text{ in the interval.}$$

- The function $f(x)$ is called *decreasing in the interval I* if every chord within the interval slopes downwards, that is,

$$f(a) > f(b), \quad \text{for all } a < b \text{ in the interval.}$$

If $f(x)$ is differentiable in the interval, and is increasing at every point in the interval (except perhaps for some isolated points where the tangent is horizontal), then clearly it is increasing in the interval.

Concave up and down in an interval

We have also been sloppy when talking about concavity over an interval rather than at a point. The graph above is said to be *concave down in the interval* $[0, 150]$ because every chord lies *under the curve*. The significance of this is that the share price, which is increasing in this interval, is *increasing at a decreasing rate*.

The graph is *concave up in the interval* $[150, 300]$ because every chord in this interval lies *above the curve*. The significance of this is that the share price, which is increasing in this interval as well, is *increasing at an increasing rate*.

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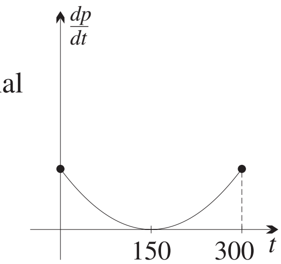
Sketched to the right is the gradient function $\frac{dP}{dt}$ of the share price P. This gradient function is decreasing in the interval $[0, 150]$, corresponding to the fact that the original curve is concave down in the interval $[0, 150]$. The gradient function $\frac{dP}{dt}$ is increasing in the remaining part $[150, 300]$ of the interval, corresponding to the fact that the original function P is concave up in the interval $[150, 300]$.



11 CONCAVE UP AND CONCAVE DOWN IN AN INTERVAL

Suppose that a function $f(x)$ is defined and continuous in an interval I, which may be bounded or unbounded, and may be open or closed or neither.

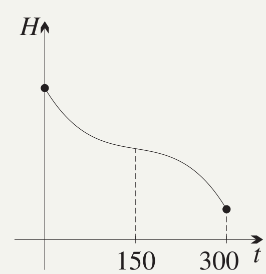
- The function $f(x)$ is called *concave up in the interval I* if every chord within the interval lies above the curve.
- The function $f(x)$ is called *concave down in the interval I* if every chord within the interval lies below the curve.

Concavity is usually a matter of common sense. These may or may not help.

- If $f(x)$ is differentiable in the interval, and $f'(x)$ is increasing in the interval, then $f(x)$ is concave up in the interval.
- If $f(x)$ is doubly differentiable in the interval and concave up at every point in the interval (except perhaps from some isolated points where $f''(x)$ is zero), then it is concave up in the interval.

Example 17 — 7D

Drought hit Kookaburra Valley last year, and the height of the Everyflow River dropped alarmingly, as shown in the graph to the right of the river height H at Emu Bridge t days after 1st January. Use the features of the graph to describe the behaviour of the river height. What happened in the period just before the 150th day?



SOLUTION

The river height is decreasing for the whole 300 days.
The graph is concave up in the interval $[0, 150]$,
and the height is decreasing at a decreasing rate.
The graph is concave down in the interval $[150, 300]$,
and the height is decreasing at an increasing rate.
It probably rained a little in the period just before $t = 150$.

Note: When we say that the height is 'decreasing at a decreasing rate', we are saying that $\frac{dH}{dt}$ is negative and that $\left|\frac{dH}{dt}\right|$ is decreasing. Similarly, when we say that the height is 'decreasing at an increasing rate', we are saying that $\frac{dH}{dt}$ is negative and that $\left|\frac{dH}{dt}\right|$ is increasing. People naturally use the right language here, but thinking about the situation may lead to confusion.

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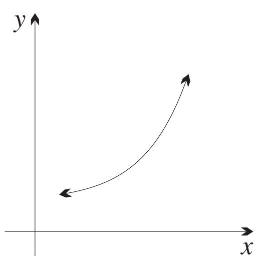
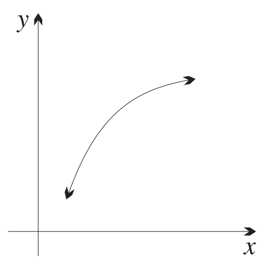
12 INCREASING OR DECREASING AT AN INCREASING OR DECREASING RATE

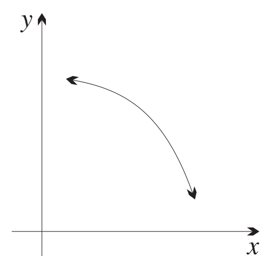
Suppose that a function $f(x)$ is defined and continuous in an interval I.
Increasing at an increasing rate: $f(x)$ is increasing and concave up in I.
Increasing at a decreasing rate: $f(x)$ is increasing and concave down in I.
Decreasing at an increasing rate: $f(x)$ is decreasing and concave down in I.
Decreasing at an decreasing rate: $f(x)$ is decreasing and concave up in I.



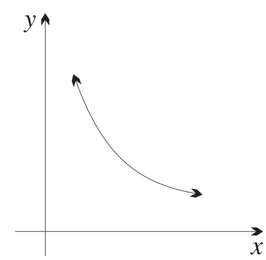
Increasing at an increasing rate



Increasing at a decreasing rate



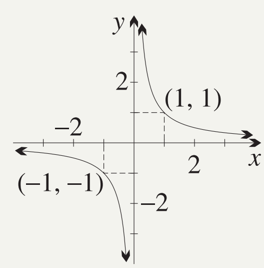
Decreasing at an increasing rate



Decreasing at a decreasing rate

Example 18 7D

Describe the branches of the function $y = \dfrac{1}{x}$ in the terms of this section.



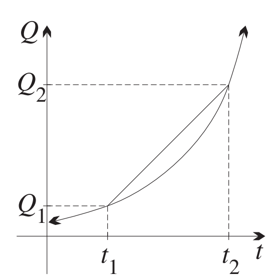
SOLUTION

In the interval $(-\infty, 0)$, the curve is decreasing at an increasing rate, and concave down. In the interval $(0, \infty)$, the curve is decreasing at a decreasing rate, and concave up.

Average rates and instantaneous rates

Worked Example 19 below is an example of a rates question that uses the new language of this section. It also uses differentiation to find a rate, and the second derivative to classify turning points and find inflections. First, however, here is a quick summary of average and instantaneous rates from Section 8J of the Year 11 book.

Suppose that a quantity Q is given as a function of time t, as in the diagram to the right. There are two types of rates.



- An *average rate of change* corresponds to a chord. From the usual gradient formula,

$$\text{average rate} = \frac{Q_2 - Q_1}{t_2 - t_1}.$$

- An *instantaneous rate of change* corresponds to a tangent. The instantaneous rate of change at time t_1 is the value of the derivative $\dfrac{dQ}{dt}$ at time $t = t_1$,

$$\text{instantaneous rate} = \frac{dQ}{dt}, \text{ evaluated at } t = t_1.$$

A rate of change always means the instantaneous rate of change unless otherwise stated, and is the gradient of the corresponding tangent.

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Example 19 7D

For the first 8 months after its first listing, the share price P in cents of the new company Avocado Marketing followed the cubic function $P = (t - 4)^3 - 12x + 100$, where the time t is in months after listing.

a What were its initial share price and its final share price?

b What were the rate of change of the price (as a function), and the rate of change of the rate of change?

c When was the share price at a local maximum or minimum, and what were those values?

d Find any points of inflection, and sketch the curve.

e Describe the behaviour of the price in different intervals of time using the terms in Box 12.

f What was the average rate of increase of the share price over the whole 8 months?

SOLUTION

a When $t = 0$,
$$\begin{aligned} P &= (-4)^3 + 0 + 100 \\ &= 36 \text{ cents.} \end{aligned}$$
When $t = 8$,
$$\begin{aligned} P &= 4^3 - 96 + 100 \\ &= 68 \text{ cents.} \end{aligned}$$

b Differentiating,
$$\begin{aligned} \frac{dP}{dt} &= 3(t - 4)^2 - 12 \\ \frac{d^2P}{dt^2} &= 6(t - 4). \end{aligned}$$

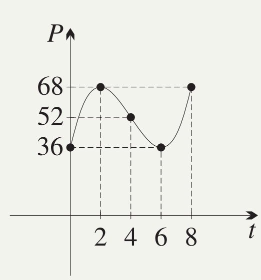
c Put $\dfrac{dP}{dt} = 0$ to find the stationary points,
$$\begin{aligned} 3(t - 4)^2 &= 12 \\ t - 4 &= 2 \text{ or } -2 \\ t &= 2 \text{ or } 6. \end{aligned}$$
When $t = 2$, $\dfrac{d^2P}{dt^2} = -12 < 0$ and $P = -8 - 24 + 100 = 68$,

and when $t = 6$, $\dfrac{d^2P}{dt^2} = 12 > 0$ and $P = 8 - 72 + 100 = 36$,

so the share price had a local maximum of 68 cents after 2 months,
and a local minimum of 36 cents after 6 months.

d There is a zero of $\dfrac{d^2P}{dt^2}$ when $t = 4$, and the value there is $P = 0 - 48 + 100 = 52$.

x	2	4	6
$\dfrac{d^2P}{dt^2}$	-12	0	12
	$\frown$	.	$\smile$



The table shows that $\dfrac{d^2P}{dt^2}$ changes sign around the point, so $(4, 52)$ is an inflection.

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e In the interval $[0, 2]$, the price is increasing at a decreasing rate.
In the interval $[2, 4]$, the price is decreasing at an increasing rate.
In the interval $[4, 6]$, the price is decreasing at a decreasing rate.
In the interval $[6, 8]$, the price is increasing at an increasing rate.

f The price at time $t = 0$ was 36 cents, and at time $t = 8$ it was 68 cents.

$$\begin{aligned}\text{Hence average rate of increase} &= \frac{68 - 36}{8} \\ &= 4 \text{ cents per month.}\end{aligned}$$

Exercise 7D

FOUNDATION

1 Grain is pouring into a storage silo. After t minutes there are V tonnes of grain in the silo, where $V = 20t$.

- **a** How much grain is in the silo after 4 minutes?
- **b** Show that the silo was empty to begin with.
- **c** If the silo takes 18 minutes to fill, what is its capacity?
- **d** Differentiate to find the rate at which the silo is being filled.

2 The amount F litres of fuel in a tank t minutes after it starts to empty is given by $F = 200(20 - t)^2$. Initially the tank is full.

- **a** Find the initial amount of fuel in the tank.
- **b** Find the quantity of fuel in the tank after 15 minutes.
- **c** Find the time taken for the tank to empty, and hence write down the domain of F.
- **d** Show that $\dfrac{dF}{dt} = -400(20 - t)$, and hence find the rate at which the tank is emptying after 5 minutes.
- **e** The value of $\dfrac{dF}{dt}$ is negative for all values of t in the domain. Explain why this is expected in this situation.
- **f** What is the average rate at which the tank is emptying?

3 Grape juice is being pumped into a vat at the rate of $\dfrac{dV}{dt} = 300$ litres per minute, where V litres is the volume of grape juice in the tank after t minutes. The tank already has 1500 litres in it when the pump starts. Rachael correctly guesses that $V = kt + C$, but she does not know the values of k and C.

- **a** Use the initial value to determine C.
- **b** Substitute the formula for V into the equation for $\dfrac{dV}{dt}$ to find the value of k.
- **c** The tank can hold 6000 litres. How long does the pump need to run to fill the tank?
- **d** What is the average rate of flow to fill the tank?

4 Using the graph of $y = 2^x$ and its reflections in the axes, draw graphs to represent functions that are:

- **a** increasing at an increasing rate,
- **b** decreasing at an increasing rate,
- **c** decreasing at a decreasing rate,
- **d** increasing at a decreasing rate.

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5 Consider the function $y = \sin x$ with domain $0 \le x \le 2\pi$. Sketch the function and then answer the following questions.

a For what values of x in the domain is the function:

i increasing at a decreasing rate, **ii** decreasing at an increasing rate,

iii decreasing at a decreasing rate, **iv** increasing at an increasing rate.

b Use the graph or your answers to part **a** to state the intervals in which the function is:

i concave up, **ii** concave down.

6 An object is projected vertically, and its height h metres at time t seconds is given by

$h = 180\left(1 - e^{-\frac{1}{3}t}\right) - 30t$.

a Find the rate at which the height is changing.

b What is the initial speed?

c The object reaches its maximum height at time T. Find T and the maximum height, both correct to four significant figures.

d Find the height, correct to the nearest centimetre, and the speed at time $2T$.

e What is the eventual speed of the object?

DEVELOPMENT

7 When a jet engine starts operating, the rate of fuel burn, R kg per minute, t minutes after startup is given by $R = 10 + \dfrac{10}{1 + 2t}$.

a What is the rate of fuel burn after:

i 2 minutes, **ii** 7 minutes?

b What limiting value does R approach as t increases?

c Show that $\dfrac{dR}{dt} < 0$ and that $\dfrac{d^2R}{dt^2} > 0$ for $t \ge 0$.

d Describe the graph of R against t as either increasing or decreasing at either an increasing or decreasing rate.

e Draw a sketch of R as a function of t to confirm your answer to the previous part.

8 For a certain brand of medicine, the amount M present in the blood after t hours is given by $M = 9te^{-t}\,\mu\text{g}$, for $0 \le t \le 9$. (The symbol μg means 'micrograms'.)

a What are the values of M at $t = 0$ and $t = 9$? Evaluate the latter correct to one decimal place.

b Determine $\dfrac{dM}{dt}$ and hence find the turning point.

c Determine $\dfrac{d^2M}{dt^2}$ and hence find the point of inflection.

d Sketch a graph of M against t, showing these features.

e When is the amount of medicine in the blood a maximum?

f When is the amount of medicine increasing most rapidly?

g When is the amount of medicine decreasing most rapidly?

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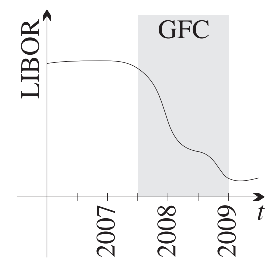
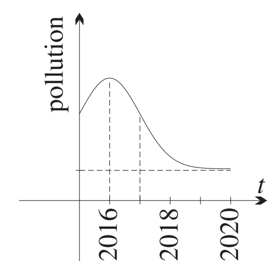
9 To the right is a simplified graph showing the effects of the Global Financial Crisis (GFC), from July in 2007 to the end of 2008, on the London Interbank Offered Rate (LIBOR). The LIBOR is sometimes used as a measure of the strength of the world economy.



a According to this graph, when was the crisis at its most frightening?
b What feature of the graph indicates the end of the crisis in January 2009.
c Why might an economist at the time have been optimistic in July of 2008?
d Sketch a possible graph of $\dfrac{dL}{dt}$, the derivative of the LIBOR, as a function of time t.

10 In 2015, a local council received a report indicating that pollution in a river had been increasing over the previous five years. The council immediately implemented a scheme to reduce the level of pollution in the river. The graph shows the level of pollution in the river between 2015 and 2020, after the scheme was implemented.



Comment briefly on whether this scheme worked and how the level of pollution changed. Include mention of the rate of change.

11 The number U of unemployed people at time t was studied over a period of time. At the start of this period, the number of unemployed was 600 000.

a Throughout the study, $\dfrac{dU}{dt} > 0$. What can be deduced about U over this period?
b The study also found that $\dfrac{d^2U}{dt^2} < 0$. What does this indicate about the changing unemployment level?
c Sketch a graph of U against t, showing this information.

12 A scientist studying an insect colony estimates the number $N(t)$ of insects after t months to be

$$N(t) = \frac{A}{2 + e^{-t}}.$$

a When the scientist begins measuring, the number of insects in the colony is estimated to be 3×10^5. Find A.
b What is the population of the colony one month later?
c How many insects would you expect to find in the nest after a long time?
d Find an expression for the rate at which the population increases with time.
e Hence show that $\dfrac{dN}{dt} = \dfrac{N(A - 2N)}{A}$.

13 The inflation rate I as a percentage can be modelled using the Consumer Price Index C according to the equation

$$I = \frac{100}{C} \times \frac{dC}{dt}\%.$$

The treasury department in the nation of Mercatura has predicted that

$$C(t) = -\tfrac{1}{5}t^3 + 3t^2 + 200$$

for the next eight years, where t is the number of years from now.

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a Find an expression for $I(t)$.

b Hence evaluate $I(4)$ correct to two decimal places.

c According to this model, there are two years in which the inflation rate is 0. What are those years, and why must the later value be rejected?

CHALLENGE

14 The standard normal distribution which will be studied later in this course has probability density function $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.

a Show that $\phi(x)$ is an even function.

b Explain why $\phi(x) > 0$ for all values of x.

c Evaluate $\phi(0)$ and determine $\lim\limits_{x \to \infty} \phi(x)$.

d Show that $\phi'(x) = -x\phi(x)$, and hence find where the function is decreasing.

e Show that $\phi''(x) = (x^2 - 1)\phi(x)$, and hence locate the two points of inflection.

f Sketch the graph of $y = \phi(x)$ showing all these details.

g Using the graph or the signs of $\phi'(x)$ and $\phi''(x)$, determine where in the domain $\phi(x)$ is decreasing at an increasing rate, and where it is decreasing at a decreasing rate.

h How is the latter evident in the graph?

15 [A difficult question] In an ideal situation, a sound wave decays away with distance according to the function $y = 2e^{-ax}\cos x$ for $x \geq 0$, where a is a positive constant.

a Find the y-intercept and the x-intercepts.

b Use the product rule to show that $y' = -2e^{-ax}(a\cos x + \sin x)$.

c Use the product rule again to show that $y'' = 2e^{-ax}\big((a^2 - 1)\cos x + 2a\sin x\big)$.

d Show that when $a = \tan\frac{\pi}{12}$, the stationary points are at $x = \frac{11\pi}{12}, \frac{23\pi}{12}, \frac{35\pi}{12}, \ldots$

e It is known that $\tan\frac{\pi}{12} = 2 - \sqrt{3}$. Show that $y'' = 0$ at $x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \ldots$ You may assume that these are the points of inflection of the curve.

f Sketch the curve showing this information, approximating the y-coordinates of the stationary and points of inflection correct to one decimal place.

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7E Rates and integration

In many situations, what is given is the rate $\frac{dQ}{dt}$ at which a quantity Q is changing. The original function Q can then be found by integration — and as with motion, never omit the constant of integration. To evaluate the constant of integration, an *initial* or *boundary condition*, giving the value of Q at some particular time t.

13 FINDING THE QUANTITY FROM THE RATE

Suppose that the rate of change of a quantity Q is known as a function of time t.

- Integrate to find Q as a function of time.
- Never omit the constant of integration.
- Use an *initial* or *boundary condition* to evaluate the constant of integration.

Example 20 7E

A tank contains 40 000 litres of water. When the draining valve is opened, the volume V in litres of water in the tank decreases at a variable rate given by $\frac{dV}{dt} = -1500 + 30t$, where t is the time in seconds after opening the valve. Once the water stops flowing, the valve shuts off.

a When does the water stop flowing?

b Give a common-sense reason why the rate $\frac{dV}{dt}$ is negative up to this time.

c Integrate to find the volume of water in the tank at time t, and sketch the graph of volume V as a function of time t.

d How much water has flowed out of the tank and how much remains?

SOLUTION

a Put $\quad \frac{dV}{dt} = 0.$

Then $\quad -1500 + 30t = 0$

$\quad t = 50,$

so it takes 50 seconds for the flow to stop.

b During this 50 seconds, the water is flowing out of the tank.

Hence the volume V is decreasing, so the derivative $\frac{dV}{dt}$ is negative.

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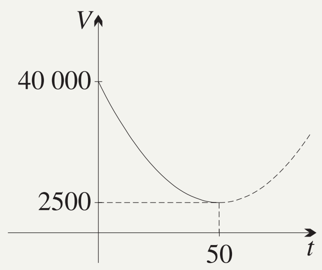
c Integrating, $V = -1500t + 15t^2 + C$, for some constant C.
It is given that when $t = 0$, $V = 40000$,
and substituting, $40000 = 0 + 0 + C$
$C = 40000$.
Hence $V = 40000 - 1500t + 15t^2$.



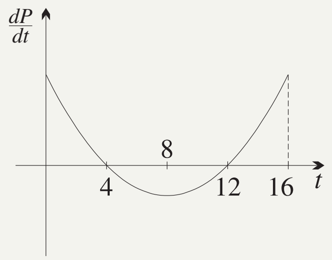
d When $t = 50$, $V = 40000 - (1500 \times 50) + (15 \times 2500)$
$= 2500$.
Hence the tank still holds 2500 litres when the valve closes,
so $40000 - 2500 = 37500$ litres has flowed out during the 50 seconds.

Questions with a diagram or a graph instead of an equation

In some problems about rates, a graph of some function is known, but its equation is unknown. Such problems require careful attention to zeroes and turning points and points of inflection. An approximate sketch of another graph often needs to be drawn.

Example 21 7E

Frog numbers were increasing in the Ranavilla district, but during a long drought, the rate of increase fell and actually became negative for a few years. The rate $\frac{dP}{dt}$ of population growth of the frogs has been graphed to the right as a function of the time t years after careful observations began.



a When was the frog population neither increasing nor decreasing?
b When was the frog population decreasing and when was it increasing?
c When was the frog population decreasing most rapidly?
d When, during the first 12 years, was the frog population at a maximum?
e When, during the years $4 \le t \le 16$, was the frog population at a minimum?
f Draw a possible graph of the frog population P against time t.

SOLUTION

a The graph shows that $\frac{dP}{dt}$ is zero when $t = 4$ and again when $t = 12$.
These are the times when the frog population was neither increasing nor decreasing.

b The graph shows that $\frac{dP}{dt}$ is negative when $4 < t < 12$,
so the population was decreasing during the years $4 < t < 12$.
The graph shows that $\frac{dP}{dt}$ is positive when $0 < t < 4$ and when $12 < t < 16$,
so the population was increasing during the years $0 < t < 4$ and during $12 < t < 16$.

c The frog population was decreasing most rapidly when $t = 8$.

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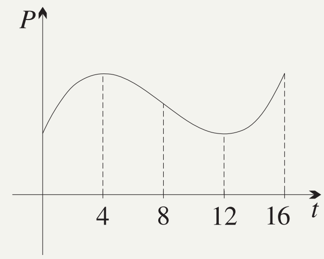
d The population was at a maximum when $x = 4$, because from parts **a** and **b**, the population was rising before this and falling afterwards.

e Similarly, the population was minimum when $x = 12$.

f All that matters is to draw the possible graph of P so that its gradients are consistent with the graph of $\frac{dP}{dT}$. These things were discussed above in parts **a**–**d**. Also, the frog population must never fall below zero.



Rates involving the exponential function

Many natural events involve a quantity that dies away gradually, with an equation that involves the exponential function. The next worked example uses the standard form $\int e^{ax+b} = \frac{1}{a}e^{ax+b} + C$ to evaluate the primitive of $3e^{-0.02t}$. The full working is

$$\begin{aligned}\int 3e^{-0.02t}\,dt &= 3 \times \frac{1}{-0.02} \times e^{-0.02t} + C \\ &= -3 \times \frac{100}{2} \times e^{-0.02t} + C \\ &= -150e^{-0.02t} + C.\end{aligned}$$

Example 22 — 7E

During a drought, the flow rate $\frac{dV}{dt}$ of water from Welcome Well gradually diminishes according to the formula $\frac{dV}{dt} = 3e^{-0.02t}$, where V is the volume in megalitres of water that has flowed out during the first t days after time zero.

a Show that $\frac{dV}{dt}$ is always positive, and explain the physical significance of this.

b Find the volume V as a function of time t.

c How much water will flow from the well during the first 100 days?

d Describe the behaviour of V as $t \to \infty$, and find what percentage of the total flow comes in the first 100 days. Then sketch the function.

SOLUTION

a Because $e^x > 0$ for all x, $\frac{dV}{dt} = 3e^{-0.02t}$ is always positive.

The volume V is always increasing, because V is the volume that has flowed out of the well, and the water doesn't flow backwards into the well.

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b The given rate is $\dfrac{dV}{dt} = 3e^{-0.02t}$.

Integrating, $V = -150e^{-0.02t} + C$ (using the calculation above).

When $t = 0$, no water has flowed out, so $V = 0$,

and substituting, $0 = -150 \times e^{0} + C$

$C = 150.$

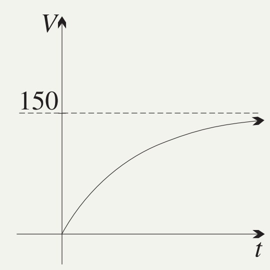
Hence $V = -150e^{-0.02t} + 150$

$= 150\left(1 - e^{-0.02t}\right).$

c When $t = 100$, $V = 150(1 - e^{-2})$

$\doteqdot 129.7$ megalitres.

d As $t \to \infty$, $e^{-0.02t} \to 0$, so $V \to 150$.



Hence $\dfrac{\text{flow in first 100 days}}{\text{total flow}} = \dfrac{150(1 - e^{-2})}{150}$

$= 1 - e^{-2}$

$= 0.86466\ldots$

$\doteqdot 86.5\%.$

Exercise 7E

FOUNDATION

1 Find y as a function of t if:

a $\dfrac{dy}{dt} = 3$, and $y = -1$ when $t = 0$,

b $\dfrac{dy}{dt} = 1 - 2t$, and $y = 2$ when $t = 0$,

c $\dfrac{dy}{dt} = \cos t$, and $y = 1$ when $t = 0$,

d $\dfrac{dy}{dt} = e^{t}$, and $y = 0$ when $t = 0$.

2 Water is being pumped into a tank at the rate of $\dfrac{dV}{dt} = 300$ litres per minute, where V litres is the volume of water in the tank after t minutes of pumping. The tank had 1500 litres of water in it at time $t = 0$.

a Show that $V = 300t + 1500$.

b How long will the pump take to fill the tank if the tank holds 6000 litres?

3 Water is flowing out of a tank at the rate of $\dfrac{dV}{dt} = 10t - 250$, where V is the volume in litres remaining in the tank at time t minutes after time zero.

a When does the water stop flowing?

b Given that the tank still has 20 litres left in it when the water flow stops, show that the volume V at any time is given by $V = 5t^2 - 250t + 3145$.

c How much water was initially in the tank?

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4 A colony of ants is building a nest. The rate at which the ants are moving the earth is given by $\dfrac{dE}{dt} = t + 3$ cubic centimetres per minute.

a At what rate are the ants moving the earth:

i initially, **ii** after 10 minutes?

b Integrate to find E as a function of t. (Hint: Find the constant of integration by assuming that when $t = 0, E = 0$.)

c How much earth is moved by the ants in:

i the first 10 minutes, **ii** the next 10 minutes?

DEVELOPMENT

5 Twenty-five wallabies are released on Wombat Island and the population is observed over the next six years. It is found that the rate of increase in the wallaby population is given by $\dfrac{dP}{dt} = 12t - 3t^2$, where time t is measured in years.

a Show that $P = 25 + 6t^2 - t^3$.

b After how many years does the population reach a maximum? (Hint: Let $\dfrac{dP}{dt} = 0$.)

c What is the maximum population?

d When does the population increase most rapidly? (Hint: Let $\dfrac{d^2P}{dt^2} = 0$.)

6 The rate at which a perfume ball loses its scent over time is $\dfrac{dP}{dt} = -\dfrac{2}{t + 1}$, where t is measured in days.

a Find P as a function of t if the initial perfume content is 6.8.

b How long will it be before the perfume in the ball has run out and it needs to be replaced? Answer correct to the nearest day.

7 A certain brand of medicine tablet is in the shape of a sphere with diameter 5 mm. The rate at which the pill dissolves is $\dfrac{dr}{dt} = -k$, where r is the radius of the sphere at time t hours, and k is a positive constant.

a Show that $r = \frac{5}{2} - kt$.

b The pill dissolves completely in 12 hours. Find k.

8 The velocity of a particle is given by $\dfrac{dx}{dt} = e^{-0.4t}$.

a Does the particle ever stop moving?

b If the particle starts at the origin, show that its displacement x as a function of t is given by $x = \frac{5}{2}(1 - e^{-0.4t})$.

c When does the particle reach $x = 1$? (Answer correct to two decimal places.)

d Where does the particle eventually move to? (That is, find its limiting position.)

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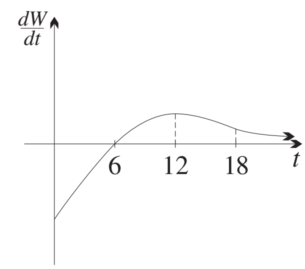
9 A ball is falling through the air and experiences air resistance. Its velocity, in metres per second at time t, is given by $\dfrac{dx}{dt} = 250\left(e^{-0.2t} - 1\right)$, where x is the height above the ground.

a What is its initial speed?

b What is its eventual speed?

c Find x as a function of t, if it is initially 200 metres above the ground.

10 The graph to the right shows the rate $\dfrac{dW}{dt}$ at which the average weight W of bullocks at St Vidgeon station was changing t months after a drought was officially proclaimed.



a When was the average weight decreasing and when was it increasing?

b When was the average weight at a minimum?

c When was the average weight increasing most rapidly?

d What appears to have happened to the average weight as time went on?

e Sketch a possible graph of the average weight W.

11 A tap on a large tank is gradually turned off so as not to create any hydraulic shock. As a consequence, the flow rate while the tap is being turned off is given by $\dfrac{dV}{dt} = -2 + \dfrac{1}{10}t$ m^3/s.

a What is the initial flow rate, when the tap is fully on?

b How long does it take to turn the tap off?

c Given that when the tap has been turned off there are still 500 m^3 of water left in the tank, find V as a function of t.

d Hence find how much water is released during the time it takes to turn the tap off.

e Suppose that it is necessary to let out a total of 300 m^3 from the tank. How long should the tap be left fully on before gradually turning it off?

CHALLENGE

12 James had a full drink bottle containing 500 ml of Gatorade™. He drank from it so that the volume V ml of Gatorade™ in the bottle changed at a rate given by $\dfrac{dV}{dt} = \left(\frac{2}{5}t - 20\right)$ ml/s.

a Find a formula for V.

b Show that it took James 50 seconds to drink the contents of the bottle.

c How long, correct to the nearest second, did it take James to drink half the contents of the bottle?

13 Over spring and summer, the snow and ice on White Mountain is melting with the time of day according to $\dfrac{dI}{dt} = -5 + 4\cos\frac{\pi}{12}t$, where I is the tonnage of ice on the mountain at time t in hours since 2:00 am on 20th October.

a It was estimated at that time that there was still 18 000 tonnes of snow and ice on the mountain. Find I as a function of t.

b Explain, from the given rate, why the ice is always melting.

c The beginning of the next snow season is expected to be four months away (120 days). Show that there will still be snow left on the mountain then.

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14 The flow of water into a small dam over the course of a year varies with time and is approximated by $\dfrac{dW}{dt} = 1.2 - \cos^2 \frac{\pi}{12} t$, where W is the volume of water in the dam, measured in thousands of cubic metres, and t is the time measured in months from the beginning of January.

a What is the maximum flow rate into the dam and when does this happen?

b Given that the dam is initially empty, find W.

c The capacity of the dam is 25 200 m^3. Show that it will be full in three years.

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7F Exponential growth and decay

This topic was briefly introduced in Sections 7G and 9F of the Year 11 book. This section reviews those methods and extends them to more examples.

The most important result about exponential functions is that the derivative of e^x is the same function e^x. When dealing with rates, the result looks more familiar if the pronumerals are replaced by Q and t:

$$\text{If } Q = e^t, \quad \text{then } \frac{dQ}{dt} = e^t. \quad \text{That is, } \frac{dQ}{dt} = Q.$$

Geometrically, this means that the gradient $\frac{dQ}{dt}$ at any point on the graph is equal to the height Q of the graph at that point.

Now replace t by kt. The derivative of the function $Q = e^{kt}$ is k times its derivative:

$$\text{If } Q = e^{kt}, \quad \text{then } \frac{dQ}{dt} = ke^{kt}. \quad \text{That is, } \frac{dQ}{dt} = kQ.$$

Geometrically, this means that the gradient $\frac{dQ}{dt}$ at any point on the graph is equal to k times the height Q of the graph at that point.

These functions $Q = e^{kt}$ are the functions used in this section to model situations.

Exponential growth

Consider a growing population P, of people in some country, or rabbits on an island, or bacteria in a laboratory culture. The more individuals there are in the population, the more new individuals are born in each unit of time. Thus the rate at which the population is growing at any time t should be roughly proportional to the number of individuals in the population at that time. Writing this in symbols,

$$\frac{dP}{dt} = kP, \quad \text{where } k \text{ is a constant of proportionality.}$$

Such a situation is called *exponential growth*, and a population growing in this way is said to obey the *law of exponential growth*.

Geometrically, this means that the gradient of the population graph at any point is proportional to the height of the graph at that point.

The exponential growth theorem

As explained above, exponential functions are exactly what is needed to model such a situation.

14 EXPONENTIAL GROWTH

Suppose that the rate of change of Q is proportional to Q,

$$\frac{dQ}{dt} = kQ, \quad \text{where } k \text{ is a constant of proportionality.}$$

Then $Q = Q_0 e^{kt}$, where Q_0 is the value of Q at time $t = 0$.

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Proof

A Substituting the function $Q = Q_0 e^{kt}$ into the equation $\dfrac{dQ}{dt} = kQ$,

$$\begin{aligned} \text{LHS} &= \frac{dQ}{dt} \\ &= \frac{d}{dt}\left(Q_0 e^{kt}\right) \\ &= kQ_0 e^{kt} \\ &= kQ \\ &= \text{RHS}, \end{aligned}$$

so the function satisfies the differential equation, as required.

B Secondly, substituting $t = 0$, into the function $Q = Q_0 e^{kt}$,

$$\begin{aligned} Q &= Q_0 e^{k \times 0} \\ &= Q_0 \times 1, \quad \text{because } e^0 = 1, \\ &= Q_0, \end{aligned}$$

so the initial value of Q is Q_0, as required.

C It is assumed without further proof that there are no other such functions.

Note: Questions often require a proof that a given function is a solution of the given differential equation. This should be done by substitution of the function into the differential equation, set out as in the proof above and in worked Examples 23 and 24 below.

Problems involving exponential growth

The constant k can usually be calculated from the data in the problem. The approximate value of k should then be held in the memory of the calculator for later use.

Example 23 7F

The rabbit population P on Goat Island was estimated to be 1000 at the start of the year 1995 and 3000 at the start of the year 2000. The population is growing according to the law of exponential growth. That is, $\dfrac{dP}{dt} = kP$, for some constant k, where P is the rabbit population t years after the start of 1995.

a Show that $P = P_0 e^{kt}$ satisfies the differential equation $\dfrac{dP}{dt} = kP$.

b Find the values of P_0 and k, then sketch the graph of P as a function of t.

c How many rabbits are there at the start of 2003? Answer correct to the nearest ten rabbits.

d When will the population be 10 000, correct to the nearest month?

e Find, correct to the nearest 10 rabbits per year, the rate at which the population is increasing:

i when there are 8000 rabbits,

ii at the start of 1997.

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SOLUTION

a Substituting the function $P = P_0 e^{kt}$ into the equation $\dfrac{dP}{dt} = kP$,

$$\begin{aligned}\text{LHS} &= \frac{dP}{dt}\\ &= \frac{d}{dt}\left(P_0 e^{kt}\right)\\ &= kP_0 e^{kt},\\ &= kP\\ &= \text{RHS},\end{aligned}$$

so the function satisfies the differential equation, as required.

b When $t = 0$, $P = 1000$, so $1000 = P_0 \times e^0$

$$P_0 = 1000.$$

When $t = 5$, $P = 3000$, so $3000 = 1000\, e^{5k}$

$$\begin{aligned}e^{5k} &= 3\\ 5k &= \log_e 3\\ k &= \tfrac{1}{5}\log_e 3\\ k &= 0.219722\ldots \quad \text{(store this in the memory)}.\end{aligned}$$

c At the start of 2003, when $t = 8$, $P = 1000e^{8k}$

$$\doteqdot 5800 \text{ rabbits.}$$

d Substituting $P = 10\,000$, $10\,000 = 1000\, e^{kt}$

$$\begin{aligned}e^{kt} &= 10\\ kt &= \log_e 10\\ t &= \frac{\log_e 10}{k}\\ &= 10.479516\ldots\\ &\doteqdot 10 \text{ years and 6 months,}\end{aligned}$$

so the population will reach 10 000 about 6 months into 2005.

e **i** Substituting $P = 8000$ into $\dfrac{dP}{dt} = kP$,

$$\begin{aligned}\frac{dP}{dt} &= 8000k\\ &\doteqdot 1760 \text{ rabbits per year.}\end{aligned}$$

ii Differentiating, $\dfrac{dP}{dt} = 1000k\, e^{kt}$,

so at the start of 1997, when $t = 2$, $\dfrac{dP}{dt} = 1000k\, e^{2k}$

$$\doteqdot 340 \text{ rabbits per year.}$$

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Example 24 7F

The price P of a pair of shoes rises with inflation so that

$$\frac{dP}{dt} = kP, \text{ for some constant } k,$$

where t is the time in years since records were kept.

a Show that $P = P_0 e^{kt}$, where P_0 is the price at time zero, satisfies the given differential equation.

b If the price doubles every 10 years, find k, sketch the curve, and find how long it takes for the price to rise to 10 times its original price.

SOLUTION

a Substituting $P = P_0 e^{kt}$ into the differential equation $\frac{dP}{dt} = kP$,

$$\begin{aligned} \text{LHS} &= \frac{dP}{dt} \\ &= \frac{d}{dt}\left(P_0 e^{kt}\right) \\ &= kP_0 e^{kt}, \\ &= kP \\ &= \text{RHS}, \quad \text{so the function satisfies the differential equation.} \end{aligned}$$

Also, when $t = 0$, $P = P_0 e^0 = P_0 \times 1$, so P_0 is the price at time zero.

b When $t = 10$, we know that $P = 2P_0$,

$$\begin{aligned} \text{so} \quad 2P_0 &= P_0 e^{10k} \\ e^{10k} &= 2 \\ 10k &= \log_e 2 \\ k &= \tfrac{1}{10} \log_e 2 \\ k &= 0.069314\ldots \quad \text{(store this in the memory)}. \end{aligned}$$

Now substituting $P = 10\, P_0$,

$$\begin{aligned} 10\, P_0 &= P_0 e^{kt} \\ e^{kt} &= 10 \\ kt &= \log_e 10 \\ t &= \frac{\log_e 10}{k} \\ &\doteqdot 33.219, \end{aligned}$$

so it takes about 33.2 years for the price of the shoes to rise tenfold.

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Exponential decay

The same method can deal with situations in which some quantity is decreasing at a rate proportional to the quantity. Radioactive substances, for example, decay in this manner. Let M be the mass of the substance, regarded as a function of time t. Because M is decreasing, the derivative $\dfrac{dM}{dt}$ is negative, so

$$\frac{dM}{dt} = -kM, \quad \text{where } k \text{ is a positive constant.}$$

Then applying the theorem, $M = M_0 e^{-kt}$, where M_0 is the mass at time $t = 0$.

15 EXPONENTIAL DECAY

In situations of exponential decay, let the constant of proportionality be $-k$, where k is a positive constant. Then, if M is the quantity that is changing,

$$\frac{dM}{dt} = -kM \qquad \text{and} \qquad M = M_0 e^{-kt},$$

where M_0 is the value of M at time $t = 0$.

Note: It is perfectly acceptable to omit the minus sign and use a negative constant k. Then $\dfrac{dM}{dt} = kM$, where k is a negative constant, and $M = M_0 e^{kt}$. The arithmetic of logarithms, however, is easier if the minus sign is built in.

Example 25 7F

A paddock has been contaminated with strontium-90, which has a half-life of 28 years. (This means that exactly half of any quantity of the isotope will decay in 28 years.) Let M_0 be the original mass present.

a Find the mass of strontium-90 as a function of time.

b Find what proportion of the radioactivity will remain after 100 years (answer correct to the nearest 0.1%).

c How long will it take for the radioactivity to drop to 0.001% of its original value? Answer correct to the nearest year.

SOLUTION

a Let M be the quantity of the isotope at time t years.

Then $\dfrac{dM}{dt} = -kM$, for some positive constant k of proportionality,

so $\quad M = M_0 e^{-kt}$.

After 28 years, only half of the original mass remains,

that is, $M = \frac{1}{2}M_0$ when $t = 28$,

and substituting, $\quad \frac{1}{2}M_0 = M_0 e^{-28k}$

$$e^{-28k} = \tfrac{1}{2}.$$

Taking reciprocals, $\quad e^{28k} = 2 \quad \left(\text{the reciprocal of } e^{-28k} \text{ is } e^{28k}\right)$

$$\begin{aligned} 28k &= \log_e 2 \\ k &= \tfrac{1}{28}\log_e 2 \\ k &= 0.024755\ldots \quad \text{(store this in the memory).} \end{aligned}$$

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b When $t = 100$, $M = M_0\, e^{-100k}$
$\doteqdot 0.084\, M_0$,
so the radioactivity has dropped to about 8.4% of its original value.

c The radioactivity has dropped to 0.001% when $M = 10^{-5} M_0$.

Notice that $0.001\% = \dfrac{0.001}{100}$
$= 0.00001$
$= 10^{-5}$.

Substituting $M = 10^{-5} M_0$ into the equation from part **a**,

$$\begin{aligned} 10^{-5} M_0 &= M_0\, e^{-kt} \\ e^{-kt} &= 10^{-5} \\ -kt &= -5 \log_e 10 \\ t &= \frac{5 \log_e 10}{k} \\ &\doteqdot 465 \text{ years.} \end{aligned}$$

Exercise 7F

FOUNDATION

1 Consider the equation $C = 10\, e^{3t}$.

- **a** Find C correct to the nearest whole number when $t = 2$.
- **b** Find t correct to one decimal place when $C = 10\,000$.
- **c** Show that $\dfrac{dC}{dt} = 3C$.
- **d** Find $\dfrac{dC}{dt}$ when $C = 37.8$.
- **e** Find $\dfrac{dC}{dt}$ correct to the nearest whole number when $t = 1$. (Hint: Find C first.)

2 Consider the equation $M = 40 e^{-\frac{1}{2}t}$.

- **a** Find M correct to four decimal places when $t = 10$.
- **b** Find t correct to two decimal places when $M = 10$.
- **c** Show that $\dfrac{dM}{dt} = -\frac{1}{2}M$.
- **d** Find $\dfrac{dM}{dt}$ when $M = 25$.
- **e** Find $\dfrac{dM}{dt}$ correct to one decimal place when $t = 4$.

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3 Some rabbits were released on Paradise Island. The number R of rabbits after t months can be calculated from the formula $R = 20\,e^{0.1t}$.

a How many rabbits were released onto the island?

b How many rabbits were on the island after 12 months? (Answer correct to the nearest rabbit.)

c In which month did the rabbit population reach 200?

d Show that $\frac{dR}{dt} = 0.1R$, and hence find the rate at which the number of rabbits was increasing when there were 50 rabbits.

4 The mass M kg of a certain radioactive substance is decreasing exponentially according to the formula $M = 100\,e^{-0.04t}$, where t is measured in years.

a What was the initial mass?

b What was the mass after 10 years, correct to the nearest kilogram?

c What was the mass after a further 10 years, correct to the nearest kilogram?

d After how many years was the mass 5 kg?

e Show that $\frac{dM}{dt} = -0.04M$, and hence find the rate at which the mass was decreasing when the mass was 20 kg.

f Find the rate of decrease of the mass after 18 years, correct to the nearest kg/year. (Hint: First find the mass after 18 years.)

5 The population P of a town rose from 1000 at the beginning of 1995 to 2500 at the beginning of 2005. Assume natural growth, that is, $P = 1000 \times e^{kt}$, where t is the time in years since the beginning of 1995.

a Find the value of the positive constant k by using the fact that when $t = 10$, $P = 2500$.

b Sketch the graph of $P = 1000 \times e^{kt}$.

c What was the population of the town at the beginning of 2018, correct to the nearest 10 people?

d In what year does the population reach 10 000?

e Find the rate $\frac{dP}{dt}$ at which the population is increasing at the beginning of that year. Give your answer correct to the nearest whole number.

6 It is found that under certain conditions, the number of bacteria in a sample grows exponentially with time according to the equation $B = B_0\,e^{\frac{1}{10}t}$, where t is measured in hours.

a Show that B satisfies the differential equation $\frac{dB}{dt} = \frac{1}{10}B$.

b Initially, the number of bacteria is estimated to be 1000. Find how many bacteria there are after three hours. Answer correct to the nearest bacterium.

c Use your answers to parts **a** and **b** to find how fast the number of bacteria is growing after three hours.

d By solving $1000\,e^{\frac{1}{10}t} = 10\,000$, find, correct to the nearest hour, when there will be 10 000 bacteria.

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DEVELOPMENT

7 Twenty grams of salt is gradually dissolved in hot water. Assume that the amount S left undissolved after t minutes satisfies the law of natural decay, that is, $\frac{dS}{dt} = -kS$, for some positive constant k.

a Show that $S = 20e^{-kt}$ satisfies the differential equation.

b Given that only half the salt is left after three minutes, show that

$$k = -\tfrac{1}{3}\log_e \tfrac{1}{2} = -\tfrac{1}{3}\log_e 2^{-1} = \tfrac{1}{3}\log_e 2.$$

c Find how much salt is left after five minutes, and how fast the salt is dissolving then. Answer correct to two decimal places.

d After how long, correct to the nearest second, will there be four grams of salt left undissolved?

e Find the amounts of undissolved salt when $t = 0, 1, 2$ and 3, correct to the nearest 0.01 g, show that these values form a GP, and find the common ratio.

8 The population P of a rural town has been declining over the last few years. Five years ago the population was estimated at 30000, and today it is estimated at 21000.

a Assume that the population obeys the law of natural decay $\frac{dP}{dt} = -kP$, for some positive constant k, where t is time in years from the first estimate, and show that $P = 30000\,e^{-kt}$ satisfies this differential equation.

b Find the value of the positive constant k.

c Estimate the population 10 years from now.

d The local bank has estimated that it will not be profitable to stay open once the population falls below 16000. When will the bank close?

9 When a liquid is placed in a refrigerator kept at 0° C, the rate at which it cools is proportional to its temperature h at time t, thus $\frac{dh}{dt} = -kh$, where k is a positive constant.

a Show that $h = h_0\,e^{-kt}$ is a solution of the differential equation.

b Find h_0, given that the liquid is initially at 100° C.

c After 5 minutes the temperature has dropped to 40° C. Find the value of k.

d Find the temperature of the liquid after 15 minutes.

10 The amount A in grams of carbon-14 isotope in a dead tree trunk after t years is given by $A = A_0\,e^{-kt}$, where A_0 and k are positive constants.

a Show that A satisfies the equation $\frac{dA}{dt} = -kA$.

b The amount of isotope is halved every 5750 years. Find the value of k.

c For a certain dead tree trunk, the amount of isotope is only 15% of the original amount in the living tree. How long ago, correct to the nearest 1000 years, did the tree die?

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11 A chamber is divided into two identical parts by a porous membrane. The left part of the chamber is initially more full of a liquid than the right. The liquid is let through at a rate proportional to the difference in the levels x, measured in centimetres. Thus $\frac{dx}{dt} = -kx$.

- **a** Show that $x = Ae^{-kt}$ is a solution of this equation.
- **b** Given that the initial difference in heights is 30 cm, find the value of A.
- **c** The level in the right compartment has risen 2 cm in five minutes, and the level in the left has fallen correspondingly by 2 cm.
  - **i** What is the value of x at this time?
  - **ii** Hence find the value of k.

12 A radioactive substance decays with a half-life of 1 hour. The initial mass is 80 g.

- **a** Write down the mass when $t = 0$, 1, 2 and 3 hours (no need for calculus here).
- **b** Write down the average loss of mass during the 1st, 2nd and 3rd hour, then show that the percentage loss of mass per hour during each of these hours is the same.
- **c** The mass M at any time satisfies the usual equation of natural decay $M = M_0 e^{-kt}$, where k is a constant. Find the values of M_0 and k.
- **d** Show that $\frac{dM}{dt} = -kM$, and find the instantaneous rate of mass loss when $t = 0$, $t = 1$, $t = 2$ and $t = 3$.
- **e** Sketch the M–t graph, for $0 \le t \le 1$, and add the relevant chords and tangents.

13 The height H of a wave decays such that $H = H_0 e^{-\frac{1}{3}t}$, where H_0 is the initial height of the wave. Giving your answer correct to the nearest whole percent, what percentage of the initial height is the height of the wave when:

- **a** $t = 1$?
- **b** $t = 3$?
- **c** $t = 8$?

14 A quantity Q of radium at time t years is given by $Q = Q_0 e^{-kt}$, where k is a positive constant and Q_0 is the amount of radium at time $t = 0$.

- **a** Given that $Q = \frac{1}{2}Q_0$ when $t = 1690$ years, calculate k.
- **b** After how many years does only 20% of the initial amount of radium remain? Give your answer correct to the nearest year.

15 Air pressure P in millibars is a function of the altitude a in metres, with $\frac{dP}{da} = -\mu P$. The pressure at sea level is 1013.25 millibars.

- **a** Show that $P = 1013.25\, e^{-\mu a}$ is a solution of this problem.
- **b** One reference book quotes the pressure at 1500 metres to be 845.6 millibars. Find the value of μ for the data in that book, correct to three significant figures.
- **c** Another reference book quotes the pressure at 6000 metres to be half that at sea level. Find the value of μ in this case, correct to three significant figures.
- **d** Are the data in the two books consistent?
- **e** Assuming the first book to be correct:
  - **i** What is the pressure at 4000 metres?
  - **ii** What is the pressure 1 km down a mine shaft?
  - **iii** At what altitude is the pressure 100 millibars?

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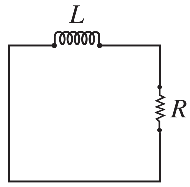
16 A current i_0 is established in the circuit shown to the right. When the source of the current is removed, the current in the circuit decays according to the equation $L\dfrac{di}{dt} = -iR$.



a Show that $i = i_0\, e^{-\frac{R}{L}t}$ is a solution of this equation.

b Given that the resistance is $R = 2$ and that the current in the circuit decays to 37% of the initial current in a quarter of a second, find L. $\left(\text{Notice that } 37\% \doteqdot \frac{1}{e}.\right)$

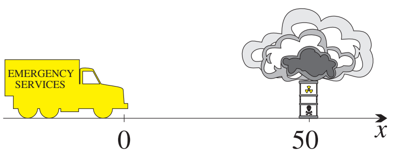
CHALLENGE

17 A certain radioactive isotope decays at such a rate that after 68 minutes only a quarter of the initial amount remains.

a Find the half-life of this isotope.

b What proportion of the initial amount will remain after 3 hours? Give your answer as a percentage, correct to one decimal place.

18 The emergency services are dealing with a toxic gas cloud around a leaking gas cylinder 50 metres away. The prevailing conditions mean that the concentration C in parts per million (ppm) of the gas increases proportionally to the concentration as one moves towards the cylinder. That is, $\dfrac{dC}{dx} = kC$, where x is the distance in metres towards the cylinder from their current position.



a Show that $C = C_0\, e^{kx}$ is a solution of the above equation.

b At the truck, where $x = 0$, the concentration is $C = 20\,000$ ppm. Five metres closer, the concentration is $C = 22\,500$ ppm. Use this information to find the values of the constants C_0 and k. Give k exactly, then correct to three decimal places.

c Find the gas concentration at the cylinder, correct to the nearest part per million.

d The accepted safe level for this gas is 30 parts per million. The emergency services calculate how far back from the cylinder they should keep the public, rounding their answer up to the nearest 10 metres.

i How far back do they keep the public?

ii Why do they round their answer up and not round it in the normal way?

19 In 2000, the population of Bedsworth was $B = 25\,000$ and the population of Yarra was $Y = 12\,500$. That year the mine in Bedsworth was closed, and the population began falling, while the population of Yarra continued to grow, so that

$$B = 25\,000\, e^{-pt} \qquad \text{and} \qquad Y = 12\,500\, e^{qt}.$$

a Ten years later it was found the populations of the two towns were $B = 20\,000$ and $Y = 15\,000$. Find the values of p and q.

b In what year were the populations of the two towns equal?

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Chapter 7 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 7 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 For each displacement function below, copy and complete the table of values to the right. Hence find the average velocity from $t = 2$ to $t = 4$. The units in each part are centimetres and seconds.

t	2	4
x		

a $x = 20 + t^2$
b $x = (t + 2)^2$
c $x = t^2 - 6t$
d $x = 3^t$

2 For each displacement function below, find the velocity function and the acceleration function. Then find the displacement, the velocity and the acceleration of the particle when $t = 5$. All units are metres and seconds.

a $x = 40t - t^2$
b $x = t^3 - 25t$
c $x = 4(t - 3)^2$
d $x = 50 - t^4$
e $x = 4 \sin \pi t$
f $x = 7e^{3t-15}$

3 A ball rolls up an inclined plane and back down again. Its distance x metres up the plane after t seconds is given by $x = 16t - t^2$.

a Find the velocity function v and the acceleration function a.
b What are the ball's position, velocity, speed and acceleration after 10 seconds?
c When does the ball return to its starting point, and what is its velocity then?
d When is the ball farthest up the plane, and where is it then?
e Sketch the displacement–time graph, the velocity–time graph and the acceleration–time graph.

4 Differentiate each velocity function below to find the acceleration function a. Then integrate v to find the displacement function x, given that the particle is initially at $x = 4$.

a $v = 7$
b $v = 4 - 9t^2$
c $v = (t - 1)^2$
d $v = 0$
e $v = 12 \cos 2t$
f $v = 12e^{-3t}$

5 For each acceleration function below, find the velocity function v and the displacement function x, given that the particle is initially stationary at $x = 2$.

a $a = 6t + 2$
b $a = -8$
c $a = 36t^2 - 4$
d $a = 0$
e $a = 5 \cos t$
f $a = 7e^t$

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Review

6 A particle is moving with acceleration function $\ddot{x} = 6t$, in units of centimetres and seconds. Initially it is at the origin and has velocity 12 cm/s in the negative direction.

a Find the velocity function $\dot{x}$ and the displacement function x.
b Show that the particle is stationary when $t = 2$.
c Hence find its maximum distance on the negative side of the origin.
d When does it return to the origin, and what are its velocity and acceleration then?
e What happens to the particle's position and velocity as time goes on?

7 A stone is thrown vertically upwards with velocity 40 m/s from a fixed point B situated 45 metres above the ground. Take $g = 10$ m/s^2.

a Taking upwards as positive, explain why the acceleration function is $a = -10$.
b Using the ground as the origin, find the velocity function v and the displacement function x.
c Hence find how long the stone takes to reach its maximum height, and find that maximum height.
d Show that the time of flight of the stone until it strikes the ground is 9 seconds.
e With what speed does the stone strike the ground?
f Find the height of the stone after 1 second and after 2 seconds.
g Hence find the average velocity of the stone during the 2nd second.

8 The acceleration of a body moving along a line is given by $\ddot{x} = \sin t$, where x is the distance from the origin O at time t seconds.

a Sketch the acceleration–time graph.
b From your graph, state the first two times after $t = 0$ when the acceleration is zero.
c Integrate to find the velocity function, given that the initial velocity is -1 m/s.
d What is the first time when the body stops moving?
e The body is initially at $x = 5$.
  i Find the displacement function x.
  ii Find where the body is when $t = \frac{\pi}{2}$.

9 The velocity of a particle is given by $v = 20e^{-t}$, in units of metres and seconds.

a What is the velocity when $t = 0$?
b Why is the particle always moving in a positive direction?
c Find the acceleration function a.
d What is the acceleration at time $t = 0$?
e The particle is initially at the origin. Find the displacement function x.
f What happens to the acceleration, the velocity and the displacement as t increases?

10 The stud farm at Benromach sold a prize bull to a grazier at Dalmore, 300 kilometres west. The truck delivering the bull left Benromach at 9:00 am, driving over the dirt roads at a constant speed of 50 km/hr. At 10:00 am, the driver realised that he had left the sale documents behind, so he drove back to Benromach at the same speed. He then drove the bull and the documents straight to Dalmore at 60 km/hr.

a Draw the displacement–time graph of his displacement x kilometres west of Benromach at time t hours after 9:00 am.
b What total distance did he travel?
c What was his average road speed for the whole journey?

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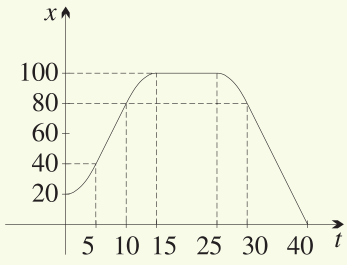
Review

11 Crispin was trying out his bicycle in Abigail Street. The graph below shows his displacement in metres north of the oak tree after t seconds.



a Where did he start from, and what was his initial speed?

b What was his velocity:

i from $t = 5$ to $t = 10$,

ii from $t = 15$ to $t = 25$,

iii from $t = 30$ to $t = 40$?

c In what direction was he accelerating:

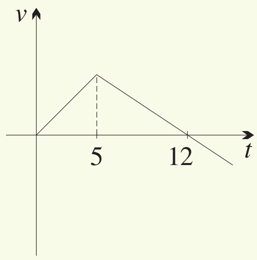
i from $t = 0$ to $t = 5$,

ii from $t = 10$ to $t = 15$,

iii from $t = 25$ to $t = 30$?

d Draw a possible sketch of the velocity–time graph.

12 A small rocket was launched vertically from the ground. The graph to the right shows its velocity–time graph. After a few seconds the motor cut out. A few seconds later the rocket reached its maximum height and then began to fall back towards the ground.



a When did the motor cut out?

b When was the rocket stationary, when was it moving upwards and when was it moving downwards?

c When was the rocket accelerating upwards, and when was it accelerating downwards?

d When was the rocket at its maximum height?

e Sketch the acceleration–time graph.

f Sketch the displacement–time graph.

13 a Draw rough sketches of the following functions for $0 \le x \le \frac{\pi}{2}$.

i $y = \sin x$ **ii** $y = \cos x$ **iii** $y = \tan x$ **iv** $y = \cot x$

b For the graphs you drew, and over the given domain, which functions are:

i increasing at a decreasing rate **ii** decreasing at an increasing rate

iii decreasing at a decreasing rate **iv** increasing at an increasing rate

14 The kakapo is a critically endangered species of parrot in New Zealand. In 2017 the New Zealand Department of Conservation released a number of kakapo on Little Barrier Island. Biologists hoped that the population would grow slowly at first and later more quickly. It was also expected that after several more years the population would then grow at a slower and slower rate. Answer the following questions, assuming the biologists' predictions were correct.

a Explain why a graph of the kakapo population K over time t has an inflection point.

b Sketch a possible graph of K as a function of t, showing the inflection point.

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Review

15 The volume V litres of water in a tank at time t minutes is given by $V = 3(50 - 2t)^2$, for $0 \le t \le 25$.

a What is the initial volume of liquid in the tank?

b Determine $\dfrac{dV}{dt}$.

c Use $\dfrac{dV}{dt}$ to explain why the tank must be emptying.

d Does the outflow increase or decrease with time?

16 The velocity of a particle is given by $\dfrac{dx}{dt} = 3 - 2e^{-\frac{1}{5}t}$, with x measured in metres and t in seconds.

a Draw a graph of the velocity versus time.

b Is the particle accelerating or decelerating? Confirm your answer by finding $\dfrac{d^2x}{dt^2}$.

c What is the eventual velocity of the particle?

d The particle starts at the origin. Find x as a function of t.

17 The population of a town is growing according to the model $P = Ae^{kt}$, where t is time in years since 2015.

a The population at the beginning of 2015 was 13 000, correct to the nearest 100. What is the value of A?

b The population at the start of 2019 was 18 500, correct to the nearest 100. Determine the value of k, correct to two significant figures.

c Assuming the population continues to grow according to this model, and using the value of k found in part **b**, what is the population at the beginning of 2025, correct to the nearest 100.

18 A quantity of a certain radioactive isotope reduces from 300 g to 208 g in 30 minutes.

a Find the half life of this isotope. Give your answer correct to the nearest second.

b How much of this isotope is left after $2\frac{1}{2}$ hours? Give your answer correct to the nearest gram.

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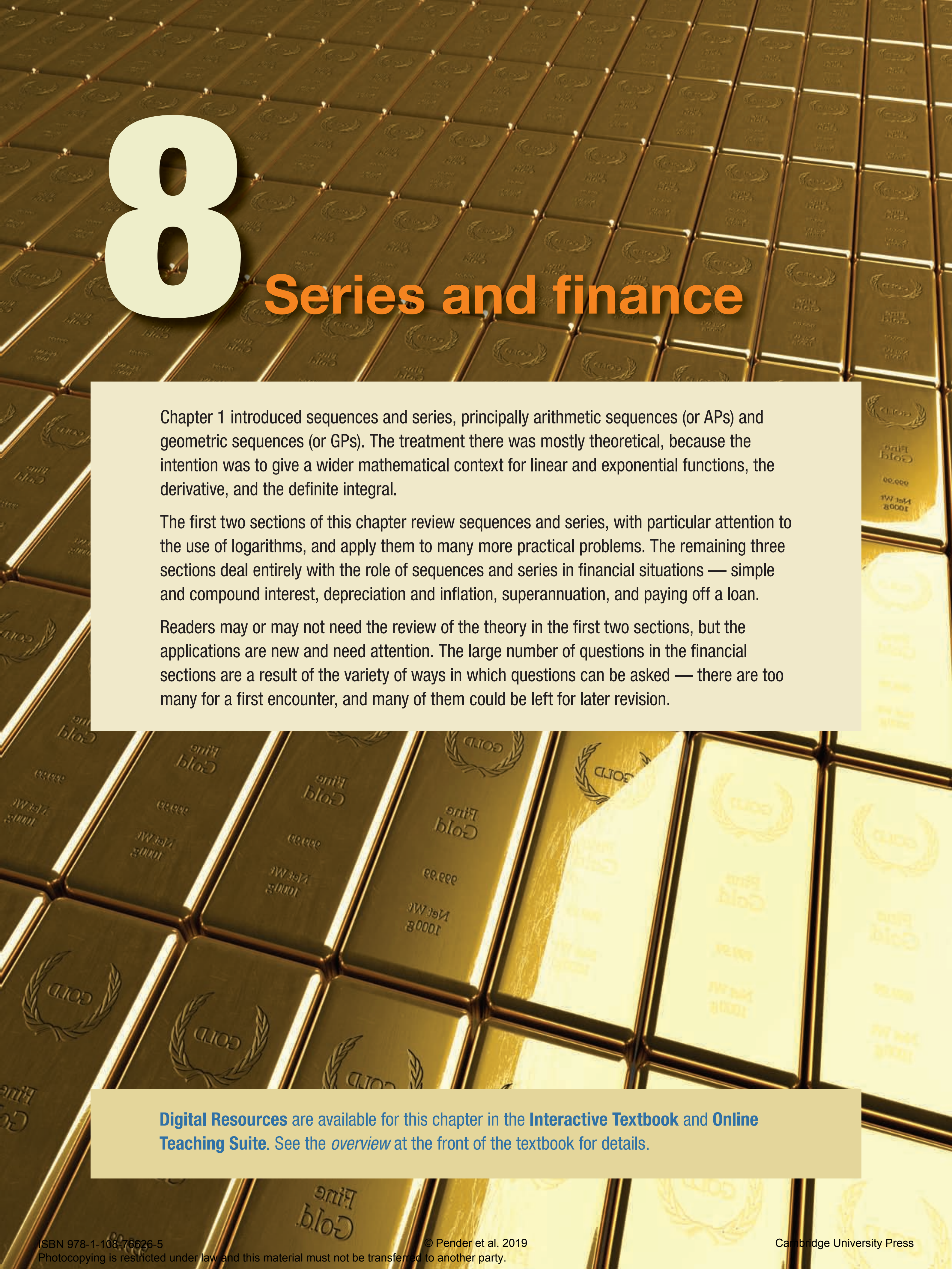

8 Series and finance

Chapter 1 introduced sequences and series, principally arithmetic sequences (or APs) and geometric sequences (or GPs). The treatment there was mostly theoretical, because the intention was to give a wider mathematical context for linear and exponential functions, the derivative, and the definite integral.

The first two sections of this chapter review sequences and series, with particular attention to the use of logarithms, and apply them to many more practical problems. The remaining three sections deal entirely with the role of sequences and series in financial situations — simple and compound interest, depreciation and inflation, superannuation, and paying off a loan.

Readers may or may not need the review of the theory in the first two sections, but the applications are new and need attention. The large number of questions in the financial sections are a result of the variety of ways in which questions can be asked — there are too many for a first encounter, and many of them could be left for later revision.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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8A Applications of APs and GPs

This section will review the main results about APs and GPs from Chapter 1 and apply them to a variety of problems, in preparation for the later sections on finance. Section 8B is particularly concerned with the use of logarithms in solving the exponential equations that arise when working with GPs.

Formulae for arithmetic sequences

Here are the essential definitions and formulae that are needed for problems involving APs.

1 ARITHMETIC SEQUENCES

- A sequence T_n is called an *arithmetic sequence* or *AP* if the difference between successive terms is constant. That is,

  $$T_n - T_{n-1} = d, \text{ for all } n \geq 2,$$

  where d is a constant, called the *common difference*.
- The nth term of an AP with first term a and common difference d is

  $$T_n = a + (n-1)d.$$

- Three numbers a, x and b are in AP if

  $$b - x = x - a, \qquad \text{that is,} \qquad x = \frac{a+b}{2}.$$

- The sum S_n of the first n terms of an AP is

  $$S_n = \tfrac{1}{2}n(a + \ell) \qquad \text{(use when the last term } \ell = T_n \text{ is known),}$$

  $$\text{or } S_n = \tfrac{1}{2}n\big(2a + (n-1)d\big) \qquad \text{(use when the difference } d \text{ is known).}$$

The word *series* is usually used when we are adding up the terms of a sequence. Typically, we will refer to

the sequence $1, 3, 5, \ldots$ and the series $1 + 3 + 5 + \cdots$.

Example 1 [Salaries and APs] 8A

Georgia earned \$50 000 in her first year at Information Holdings, and her salary then increased every year by \$6000. She worked at the company for 12 years.

a What was her annual salary in her final year?

b What were her total earnings over the 12 years?

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SOLUTION

Her annual salaries form a series, $50000 + 56000 + 62000 + \cdots$ with 12 terms. This is an AP with $a = 50000$, $d = 6000$ and $n = 12$.

a Her final salary is the twelfth term T_{12} of the series.

$$\begin{aligned}\text{Final salary} &= a + 11d \qquad \text{(using the formula for } T_{12}\text{)}\\ &= 50000 + 66000\\ &= \$116000.\end{aligned}$$

b Her total earnings are the sum S_{12} of the first twelve terms of the series.

Using the first formula for S_n,

$$\begin{aligned}\text{Total earnings} &= \tfrac{1}{2}n(a + \ell)\\ &= \tfrac{1}{2} \times 12 \times (a + \ell)\\ &= 6 \times (50000 + 116000)\\ &= \$996000.\end{aligned}$$

Using the second formula for S_n,

$$\begin{aligned}\text{Total earnings} &= \tfrac{1}{2}n\big(2a + (n - 1)d\big)\\ &= \tfrac{1}{2} \times 12 \times (2a + 11d)\\ &= 6 \times (100000 + 66000)\\ &= \$996000.\end{aligned}$$

Example 2 **8A**

The Roxanne Cinema has a concession for groups. It charges \$24 for the first ticket and then \$16 for each additional ticket.

a How much would 20 tickets cost?

b Find a formula for the cost of n tickets.

c How many people are in a group whose tickets cost \$600?

SOLUTION

The costs of $1, 2, 3, \ldots$ tickets form the sequence \$24, \$40, \$56, ...

This is an AP with $a = 24$ and $d = 16$.

a The cost of 20 tickets is the 20th term T_{20} of the sequence.

$$\begin{aligned}\text{Cost of 20 tickets} &= a + 19d \qquad \text{(using the formula for } T_{20}\text{)}\\ &= 24 + 19 \times 16\\ &= \$328.\end{aligned}$$

b
$$\begin{aligned}\text{Cost of } n \text{ tickets} &= a + (n - 1)d \qquad \text{(using the formula for } T_n\text{)}\\ &= 24 + (n - 1)16\\ &= 24 + 16n - 16\\ &= 16n + 8 \text{ dollars.}\end{aligned}$$

c Put cost of n tickets $= 600$ dollars.

$$\begin{aligned}\text{Then} \quad 16n + 8 &= 600 \qquad \text{(using the formula found in part } \mathbf{b} \text{ for } n \text{ tickets)}\\ 16n &= 592\\ n &= 37 \text{ tickets.}\end{aligned}$$

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Counting when the years are named

Problems in which events happen in particular named years are notoriously tricky. The following problem becomes clearer when the years are stated in terms of 'years after 2005'.

Example 3 8A

Gulgarindi Council is very happy. It had 2870 complaints in 2006, but only 2170 in 2016. The number of complaints decreased by the same amount each year.

a What was the total number of complaints during these years?
b By how much did the number of complaints decrease each year?
c How many complaints were there in 2008?
d Find a formula for the number of complaints in the nth year.
e If the trend continued, in what year would there be no complaints at all?

SOLUTION

The first year is 2006, the second year is 2007, and the 11th year is 2016.
In general, the nth year of the problem is the nth year after 2005.
The successive numbers of complaints form an AP with $a = 2870$, $\ell = 2170$ and $n = 11$.

a The total number of complaints is the sum S_{11} of the first 11 terms of the series.

$$\begin{aligned}\text{Total number of complaints} &= \tfrac{1}{2} \times 11 \times (a + \ell) \quad \text{(using the first formula for } S_n\text{)}\\ &= \tfrac{1}{2} \times 11 \times (2870 + 2170)\\ &= \tfrac{1}{2} \times 11 \times 5040\\ &= 27\,720.\end{aligned}$$

b This question is asking for the common difference d, which is negative here.

$$\begin{aligned}\text{Put} \qquad T_{11} &= 2170\\ a + 10d &= 2170 \qquad \text{(using the formula for } T_{11}\text{)}\\ 2870 + 10d &= 2170\\ 10d &= -700\\ d &= -70.\end{aligned}$$

Hence the number of complaints decreased by 70 each year.

c The year 2008 is the third year, so we find the third term T_3 of the series.

$$\begin{aligned}\text{Number of complaints in 2008} &= a + 2d \qquad \text{(using the formula for } T_3\text{)}\\ &= 2870 + 2 \times (-70)\\ &= 2730.\end{aligned}$$

d The number of complaints in the nth year is the nth term T_n of the series.

$$\begin{aligned}\text{Number of complaints} &= a + (n-1)d \qquad \text{(using the formula for } T_n\text{)}\\ &= 2870 - 70(n-1)\\ &= 2870 - 70n + 70\\ &= 2940 - 70n.\end{aligned}$$

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e To find the year in which there are no complaints at all,

put $\quad T_n = 0.$

Then $2940 - 70n = 0$ (using the formula found in part **d** for T_n)

$$70n = 2940$$

$$n = 42.$$

Thus there would be no complaints at all in the year $2005 + 42 = 2047$.

Formulae for geometric sequences

The formulae for GPs correspond roughly to the formulae for APs, except that the formula for the limiting sum of a GP has no analogy for arithmetic sequences.

2 GEOMETRIC SEQUENCES

- A sequence T_n is called a *geometric sequence* if the ratio of successive terms is constant. That is,
$$\frac{T_n}{T_{n-1}} = r, \text{ for all } n \geq 2,$$
where r is a constant, called the *common ratio*.
- The nth term of a GP with first term a and common ratio r is
$$T_n = ar^{n-1}.$$
- Neither the ratio nor any term of a GP can be zero.
- Three numbers a, x and b are in GP if
$$\frac{b}{x} = \frac{x}{a}, \qquad \text{that is,} \qquad x^2 = ab.$$
- The sum S_n of the first n terms of a GP is
$$S_n = \frac{a(r^n - 1)}{r - 1} \qquad \text{(easier when } r > 1\text{)},$$
$$S_n = \frac{a(1 - r^n)}{1 - r} \qquad \text{(easier when } r < 1\text{)}.$$
- The limiting sum S_∞ exists if and only if $-1 < r < 1$, that is, $|r| < 1$, and in this case,
$$S_\infty = \frac{a}{1 - r}, \qquad \text{that is,} \qquad \lim_{n \to \infty} S_n = \frac{a}{1 - r}.$$

Example 4 [An example with $r > 1$] 8A

The town of Elgin grew quite fast in the eight years after the new distillery was opened. In the first year afterwards, there were just 15 car accidents, but over these eight years, the number of accidents doubled every year.

a Find the number of accidents in the eighth year after the distillery opened.

b Find the total number of accidents over these eight years.

c What percentage of the total accidents occurred in the final year?

SOLUTION

The number of accidents per year form a sequence 15, 30, 60, ...
This is a GP with $a = 15$ and $r = 2$.

a The number of accidents during the eighth year is the eighth term T_8.

$$\begin{aligned}\text{Hence number of accidents} &= ar^7 \qquad \text{(using the formula for } T_8\text{)}\\ &= 15 \times 2^7\\ &= 15 \times 128\\ &= 1920.\end{aligned}$$

b The total accidents during these eight years is the sum S_8 of the first eight terms.

$$\begin{aligned}\text{Hence number of accidents} &= \frac{a(r^8 - 1)}{r - 1} \qquad \text{(using the first formula for } S_8 \text{ because } r > 1\text{)}\\ &= \frac{15 \times (2^8 - 1)}{2 - 1}\\ &= 15 \times 255\\ &= 3825.\end{aligned}$$

c To find the percentage of the total accidents that occurred in the final year:

$$\begin{aligned}\frac{\text{Accidents in the final year}}{\text{Total accidents}} &= \frac{1920}{3825}\\ &\doteqdot 50.20\% \qquad \text{(correct to the nearest 0.01\%).}\end{aligned}$$

Example 5 [An example with $r < 1$] 8A

Sales from the Gumnut Softdrinks Factory in the mountain town of Wadelbri are declining by 6% every year. In 2016, 50000 bottles were sold.

a How many bottles will be sold in 2025?

b How many bottles will be sold in total in the years 2016–2025?

SOLUTION

Here 2016 is the first year, 2017 is the second year, . . . , and 2025 is the 10th year.
The annual sales form a GP with $a = 50000$ and $r = 0.94$.

a The sales in 2025 are the 10th term T_{10}, because 2016–2025 consists of 10 years.

$$\begin{aligned}\text{Sales in 2025} &= ar^9\\ &= 50000 \times 0.94^9 \qquad \text{(using the formula for } T_{10}\text{)}\\ &\doteqdot 28650 \qquad \text{(correct to the nearest bottle).}\end{aligned}$$

b The total sales in 2016–2025 are the sum S_{10} of the first 10 terms of the series.

$$\begin{aligned}\text{Total sales} &= \frac{a(1 - r^{10})}{1 - r} \qquad \text{(using the second formula for } S_{10} \text{ because } r < 1\text{)}\\ &= \frac{50000 \times (1 - 0.94^{10})}{0.06}\\ &\doteqdot 384487 \qquad \text{(correct to the nearest bottle).}\end{aligned}$$

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Limiting sums

If the ratio of a GP is between -1 and 1, that is, $|r| < 1$, the sum S_n of the first n terms of the GP converges to the limit $S_\infty = \dfrac{a}{1-r}$ as $n \to \infty$. In applications, this allows us to speak about the sum of the terms 'eventually', or 'as time goes on'.

Example 6 8A

Consider again the Gumnut Softdrinks Factory in Wadelbri, where sales are declining by 6% every year and 50000 bottles were sold in 2016. Suppose now that the company continues in business indefinitely.

a What would the total sales from 2016 onwards be eventually?

b What proportion of those sales would occur by the end of 2025?

SOLUTION

The sales form a GP with $a = 50000$ and $r = 0.94$.
Because $-1 < r < 1$, the limiting sum exists.

a
$$\begin{aligned}\text{Eventual sales} &= S_\infty \\ &= \frac{a}{1-r} \\ &= \frac{50000}{0.06} \\ &\doteqdot 833333 \quad \text{(correct to the nearest bottle)}.\end{aligned}$$

b Using the results from part **a**, and from the previous worked example,

$$\begin{aligned}\frac{\text{sales in 2016–2025}}{\text{eventual sales}} &= \frac{50000 \times (1 - 0.94^{10})}{0.06} \times \frac{0.06}{50000} \quad \text{(using the exact values)} \\ &= 1 - 0.94^{10} \\ &\doteqdot 46.14\% \quad \text{(correct to the nearest 0.01\%)}.\end{aligned}$$

Example 7 [A trigonometric application] 8A

Consider the series $1 - \tan^2 x + \tan^2 x - \cdots$, where x is an acute angle.

a For what values of x does the series have a limiting sum?

b What is this limiting sum when it exists?

SOLUTION

a The series is a GP with $a = 1$ and $r = -\tan^2 x$,
so the limiting sum exists when $-1 < r < 1$,
that is, $-1 < \tan^2 x < 1$,
which means that $-1 < \tan x < 1$.
But $\tan 45° = 1$ and $\tan 0° = 0$, and the angle x is acute,
so from the graph of $\tan x$, $0° < x < 45°$.

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b To find the limiting sum:
When the series converges, $S_\infty = \dfrac{a}{1 - r}$
$= \dfrac{1}{1 + \tan^2 x}$
$= \dfrac{1}{\sec^2 x}$
$= \cos^2 x.$

Exercise 8A

FOUNDATION

Note: The theory for this exercise was discussed in Chapter 1 and reviewed in Section 8A. The exercise is a medley of problems on APs and GPs, with six introductory questions to revise the formulae for APs and GPs.

1 **a** Show that 8, 15, 22, . . . is an arithmetic sequence.
b State the first term a and the common difference d.
c Use the formula $T_n = a + (n - 1)d$ to find the 51st term T_{51}.
d Use the formula $S_n = \frac{1}{2}n\big(2a + (n - 1)d\big)$ to find the sum S_{20} of the first 20 terms.

2 Consider the sum $2 + 4 + 6 + \cdots + 1000$ of the first 500 even numbers.
a Show that this is an AP and write down the first term a and common difference d.
b Use the formula $S_n = \frac{1}{2}n(a + \ell)$ to find the sum.

3 **a** Show that 5, 10, 20, 40, . . . is a geometric sequence.
b State the first term a and the common ratio r.
c Use the formula $T_n = ar^{n-1}$ to find the seventh term T_7.
d Use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ to find the sum S_7 of the first seven terms.

4 **a** Show that the sequence 96, 48, 24, . . . is a GP.
b Write down the first term a and the common ratio r.
c Use the formula $T_n = ar^{n-1}$ to find the eighth term T_8.
d Use the formula $S_n = \dfrac{a(1 - r^n)}{1 - r}$ to find the sum S_8 of the first eight terms.
e Give a reason why this series has a limiting sum, and use the formula $S_\infty = \dfrac{a}{1 - r}$ to find it.

5 **a** Consider the series $52 + 58 + 64 + \cdots + 130$.
i Show that it is an AP, and write down the first term and common difference.
ii How many terms are there in this series?
iii Find the sum.
b In a particular arithmetic series, the first term is 15 and the fiftieth term is -10.
i What is the sum of all the terms?
ii What is the common difference?
c Consider the series $100 + 97 + 94 + \cdots$.
i Show that the series is an AP, and find the common difference.
ii Show that the nth term is $T_n = 103 - 3n$, and find the first negative term.
iii Find an expression for the sum S_n of the first n terms, and show that 68 is the minimum number of terms for which S_n is negative.

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6 **a** Consider the sequence 100, 101, 102.01, ...

i Show that it is a geometric sequence, and find the common ratio.

ii Write down the twentieth term and evaluate it, correct to two decimal places.

iii Find the sum of the first 20 terms, correct to two decimal places.

b The first few terms of a particular series are $2000 + 3000 + 4500 + \cdots$.

i Show that it is a geometric series, and find the common ratio.

ii What is the sum of the first five terms?

iii Explain why the series does not have a limiting sum.

c Consider the series $18 + 6 + 2 + \cdots$.

i Show that it is a geometric series, and find the common ratio.

ii Explain why this geometric series has a limiting sum, and find its value.

iii Find the sum of the first ten terms, and show that it and the limiting are approximately equal, correct to the first three decimal places.

7 A secretary starts on an annual salary of \$60000, with annual increments of \$4000.

a Use the AP formulae to find his annual salary, and his total earnings, at the end of 10 years.

b In which year will his salary be \$84000?

8 An accountant receives an annual salary of \$80000, with 5% increments each year.

a Show that her annual salary forms a GP and find the common ratio.

b Find her annual salary, and her total earnings, at the end of ten years, each correct to the nearest dollar.

DEVELOPMENT

9 **a** What can be said about the terms of an AP in which:

i the common difference is zero, **ii** the common difference is negative?

b Why can't the common ratio of a GP be zero?

c What can be said about the terms of a GP with common ratio r in which:

i $r < 0$, **ii** $r = 1$, **iii** $r = -1$, **iv** $0 < |r| < 1$?

10 Lawrence and Julian start their first jobs on low wages. Lawrence starts at \$50000 per annum, with annual increases of \$5000. Julian starts at the lower wage of \$40000 per annum, with annual increases of 15%.

a Find Lawrence's annual wages in each of the first three years and explain why they form an arithmetic sequence.

b Find Julian's annual wages in each of the first three years and explain why they form a geometric sequence.

c Show that the first year in which Julian's annual wage is the greater of the two will be the sixth year, and find the difference, correct to the nearest dollar.

11 **a** An initial salary of \$50000 increases by \$3000 each year.

i Find a formula for T_n, the salary in the nth year.

ii In which year will the salary first be at least twice the original salary?

b An initial salary of \$50000 increases by 4% each year. What will the salary be in the tenth year, correct to the nearest dollar?

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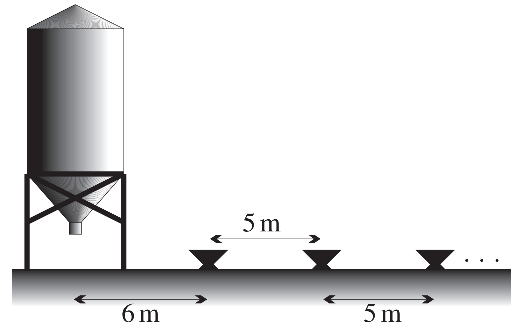
12 A farmhand is filling a row of feed troughs with grain. The distance between adjacent troughs is 5 metres, and the silo that stores the grain is 6 metres from the closest trough. He decides that he will fill the closest trough first and work his way to the far end. (He can only carry enough grain to fill one trough with each trip.)



a How far will the farmhand walk to fill the 1st trough and return to the silo? How far for the 2nd trough? How far for the 3rd trough?

b How far will the farmhand walk to fill the nth trough and return to the silo?

c If he walks a total of 62 metres to fill the furthest trough:

i how many feed troughs are there,

ii what is the total distance he will walk to fill all the troughs?

13 One Sunday, 120 days before Christmas, Aldsworth store publishes an advertisement saying '120 shopping days until Christmas'. Aldsworth subsequently publishes similar advertisements every Sunday until Christmas.

a How many times does Aldsworth advertise?

b Find the sum of the numbers of days published in all the advertisements.

c On which day of the week is Christmas?

14 The number of infections in an epidemic rose from 10000 on 1st July to 160000 on 1st of September.

a If the number of infections increased by a constant difference each month, what was the number of infections on 1st August?

b If the number of infections increased by a constant ratio each month, what was the number of infections on 1st August?

15 Theodor earns \$60000 in his first year of work, and his salary increases each year by a fixed amount \$$D$.

a Find D if his salary in his tenth year is \$117600.

b Find D if his total earnings in the first ten years are \$942000.

c If $D = 4400$, in which year will his salary first exceed \$120000?

d If $D = 4000$, show that his total earnings first exceed \$1200000 during his 14th year.

16 A line of four cones is used in a fitness test. John starts at the first cone. He runs 20 metres to the last cone and runs back again. Then he runs 10 metres to the third cone and runs back again. Finally he runs 5 metres to the 2nd cone and runs back.

a Write down the distances that John travels on each run. Show that they form a GP and write down the first term and the common ratio.

b Suppose that more and more cones are added to continue this pattern of runs. What distance will John eventually travel?

c The coach asks Stewart to run the original course in reverse, which he does. Explain why Stewart does not want more and more cones to be added to continue with his pattern.

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CHALLENGE

17 Margaret opens a hardware store. Sales in successive years form a GP, and sales in the fifth year are half the sales in the first year. Let sales in the first year be $\$F$.

a Find, in exact form, the ratio of the GP.

b Find the total sales of the company as time goes on, as a multiple of the first year's sales, correct to two decimal places.

18 [Limiting sums of trigonometric series]

a Consider the series $1 + \cos^2 x + \cos^4 x + \cdots$.

i Show that the series is a GP, and find its common ratio.

ii For which angles in the domain $0 \le x \le 2\pi$ does this series not converge?

iii Use the formula for the limiting sum of a GP to show that for other angles, the series converges to $S_\infty = \operatorname{cosec}^2 x$.

iv We omitted a qualification. What happens when $\cos x = 0$?

b Consider the series $1 + \sin^2 x + \sin^4 x + \cdots$.

i Show that the series is a GP, and find its common ratio.

ii For which angles in the domain $0 \le x \le 2\pi$ does this series not converge?

iii Use the formula for the limiting sum of a GP to show that for other angles, the series converges to $S_\infty = \sec^2 x$.

iv We omitted a qualification. What happens when $\sin x = 0$?

19

Two bulldozers are sitting in a construction site facing each other. Bulldozer A is at $x = 0$ and bulldozer B is 36 metres away at $x = 36$. There is a bee sitting on the scoop at the very front of bulldozer A.

At 7:00 am the workers start up both bulldozers and start them moving towards each other at the same speed V m/s. The bee is disturbed by the commotion and flies at twice the speed of the bulldozers to land on the scoop of bulldozer B.

a Show that the bee reaches bulldozer B when it is at $x = 24$.

b Immediately the bee lands, it takes off again and flies back to bulldozer A. Where is bulldozer A when the two meet?

c Assume that the bulldozers keep moving towards each other and the bee keeps flying between the two, so that the bee will eventually have three feet on each bulldozer.

i Where will this happen?

ii How far will the bee have flown?

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8B The use of logarithms with GPs

None of the exercises in the previous section asked about the number of terms in a given GP. Such questions require either trial-and-error or logarithms.

Trial-and-error may be easier to understand, but it is a clumsy method when the numbers are larger. Logarithms provide a better approach, but require understanding of the relationship between logarithms and indices.

Solving exponential inequations using trial-and-error

Questions about the number of terms in a GP involve solving an equation with n in the index. The next worked example shows how to solve such an equation using trial-and-error.

Example 8 8B

Use trial-and-error on your calculator to find the smallest integer n such that:

a $3^n > 400000$

b $0.95^n < 0.01$

SOLUTION

a Using the function labelled $\boxed{x^y}$

$$3^{11} = 177147$$
and $3^{12} = 531441$,
so the smallest such integer is 12.

b Using the function labelled $\boxed{x^y}$

$$0.95^{89} = 0.010408\ldots$$
and $0.95^{90} = 0.009888\ldots$,
so the smallest such integer is 90.

Note: In practice, quite a few more trial calculations are usually needed in order to trap the given number between two integer powers.

Notice how the powers of 3 get bigger because the base 3 is greater than 1. The powers of 0.95, however, get smaller because the base 0.95 is less than 1.

Example 9 [Inflation and GPs] 8B

The General Widget Company has sold 2000 widgets per year since its foundation in 2011 when the company charged \$300 per widget. Each year, the company lifts its prices by 5% because of cost increases.

a Find the value of the sales in the nth year after 2010.

b Using trial-and-error, find the first year in which sales exceeded \$900000.

c Find the total sales from the foundation of the company to the end of the nth year.

d Using trial-and-error, find the year during which the total sales of the company since its foundation will first exceed \$20000000.

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SOLUTION

The value of the annual sales in 2011 were $\$300 \times 2000 = \$600000$.
Hence the annual sales form a GP with $a = 600000$ and $r = 1.05$.

a The sales in the nth year after 2010 constitute the nth term T_n of the series,

$$\begin{aligned} \text{and } T_n &= ar^{n-1} \\ &= 600000 \times 1.05^{n-1}. \end{aligned}$$

b

$$\begin{aligned} \text{Sales in 2019} &= T_9 \\ &= 600000 \times 1.05^8 \\ &\doteqdot \$886473. \\ \text{Sales in 2020} &= T_{10} \\ &= 600000 \times 1.05^9 \\ &\doteqdot \$930797. \end{aligned}$$

Hence the sales first exceeded $900000 in the year 2020.

c The total sales since the foundation of the company constitute the sum S_n of the first n terms of the series,

$$\begin{aligned} \text{and } S_n &= \frac{a(r^n - 1)}{r - 1} \quad \text{(using this formula because } r > 1) \\ &= \frac{600000 \times (1.05^n - 1)}{0.05} \\ &= 12000000 \times (1.05^n - 1). \end{aligned}$$

d

$$\begin{aligned} \text{Total sales to 2030} &= S_{20} \\ &= 12000000 \times (1.05^{20} - 1) \quad \text{(using the formula from part } \mathbf{c}) \\ &\doteqdot \$19839572. \\ \text{Total sales to 2031} &= S_{21} \\ &= 12000000 \times (1.05^{21} - 1) \\ &\doteqdot \$21431551. \end{aligned}$$

Hence cumulative sales will first exceed $20000000 during 2031.

The use of logarithms with GPs

To solve an exponential in equation using logarithms, the corresponding equation must be solved. This requires two steps.

- First, convert the exponential equation to a logarithmic equation.
- Secondly, calculators only have logarithms base 10 and e, so logarithms must be converted to logarithms base 10 or base e using the change-of-base formula

$$\log_b x = \frac{\log_{10} x}{\log_{10} b} \qquad \text{OR} \qquad \log_b x = \frac{\log_e x}{\log_e b}.$$

'The log of the number over log of the base.'

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Example 10 8B

Use logarithms to find the smallest integer n such that:

a $3^n > 400000$

b $1.04^n > 2$

SOLUTION

a Put $3^n = 400000$ (beginning with the corresponding equation)

Then $n = \log_3 400000$ (converting to a logarithmic equation).

Using the change-of-base formula,

$$n = \frac{\log_{10} 400000}{\log_{10} 3} \quad \text{OR} \quad \frac{\log_e 400000}{\log_e 3}$$

$$= 11.741\ldots.$$

Thus the smallest such integer is 12, because $3^{11} < 400000$ and $3^{12} > 400000$.

b Put $1.04^n = 2$.

Then $n = \log_{1.04} 2$

$$= \frac{\log_{10} 2}{\log_{10} 1.04} \quad \text{OR} \quad \frac{\log_e 2}{\log_e 1.04}$$

$$= 17.672\ldots.$$

Thus the smallest such integer is 18, because $1.02^{17} < 2$ and $1.02^{18} > 2$.

An alternative approach — taking logarithms of both sides

There is an alternative and equally effective approach — take logarithms base 10 or base e of both sides and then use the logarithmic law, 'the log of the power is the multiple of the base',

$$\log_{10} a^n = n \log_{10} a \qquad \text{OR} \qquad \log_e a^n = n \log_e a.$$

Below is worked Example 10 done again using this alternative approach. The working takes one more line and involves, in effect, a proof of the change-of-base formula. Although the method has not been illustrated again, readers may prefer to adopt it. Practise it also taking logarithms base e.

Example 11 8B

Use logarithms to find the smallest integer n such that:

a $3^n > 400000$

b $1.04^n > 2$

SOLUTION

a Put $3^n = 400000$ (beginning with the corresponding equation).

Then $\log_{10} 3^n = \log_{10} 400000$ (taking logarithms base 10 of both sides)

$n \log_{10} 3 = \log_{10} 400000$ (the log of a power is the multiple of the log)

$$n = \frac{\log_{10} 400000}{\log_{10} 3} \quad \text{(dividing both sides by } \log_{10} 3\text{)}$$

$$= 11.741\ldots.$$

Thus the smallest such integer is 12, because $3^{11} < 400000$ and $3^{12} > 400000$.

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b Put $1.04^n = 2.$

Then
$$\begin{aligned}\log_{10} 1.04^n &= \log_{10} 2\\ n\log_{10} 1.04 &= \log_{10} 2\\ n &= \frac{\log_{10} 2}{\log_{10} 1.04}\\ &= 17.672\ldots.\end{aligned}$$

Thus the smallest such integer is 18, because $1.02^{17} < 2$ and $1.02^{18} > 2$.

Applying logarithms to problems

The next worked example is a typical example where logarithms are used to solve a problem involving a GP.

Example 12 — 8B

The profits of the Extreme Sports Adventure Company have been increasing by 15% every year since its formation, when its profit was \$60000 in the first year.

a Find a formula for its profit in the nth year.

b During which year did its profit first exceed \$1200000?

c Find a formula for its total profit during the first n years.

d During which year did its total profit since foundation first exceed \$4000000?

SOLUTION

The successive profits form a GP with $a = 60000$ and $r = 1.15$.

a
$$\begin{aligned}\text{Profit in the } n\text{th year} &= T_n\\ &= ar^{n-1}\\ &= 60000 \times 1.15^{n-1}.\end{aligned}$$

b
$$\begin{aligned}&\text{Put} & T_n &= 1200000. \quad \text{(the corresponding equation).}\\ &\text{Then} & 60000 \times 1.15^{n-1} &= 1200000\\ &\boxed{\div\ 60000} & 1.15^{n-1} &= 20\\ && n - 1 &= \log_{1.15} 20\\ && n - 1 &= \frac{\log_{10} 20}{\log_{10} 1.15} \quad \text{or} \quad \frac{\log_e 20}{\log_e 1.15}\\ && n - 1 &\doteqdot 21.43\\ &\boxed{+\ 1} & n &\doteqdot 22.43,\end{aligned}$$

so the profit first exceeds \$1200000 during the 23rd year.

c
$$\begin{aligned}\text{Total profit in the first } n \text{ years} &= S_n\\ &= \frac{a(r^n - 1)}{r - 1} \qquad \text{(using this form because } r > 1\text{)}\\ &= \frac{60000 \times (1.15^n - 1)}{0.15}\\ &= 400000 \times (1.15^n - 1).\end{aligned}$$

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d To find the year in which total profit first exceeded \$4 000 000:

$$\begin{aligned}
\text{Put} \quad S_n &= 4\,000\,000 \quad \text{(the corresponding equation).}\\
\text{Then} \quad 400\,000 \times (1.15^n - 1) &= 4\,000\,000\\
\boxed{\div\, 400\,000} \quad 1.15^n - 1 &= 10\\
1.15^n &= 11\\
n &= \log_{1.15} 11\\
&= \frac{\log_{10} 11}{\log_{10} 1.15} \quad \text{OR} \quad \frac{\log_e 11}{\log_e 1.15}\\
&\doteqdot 17.16,
\end{aligned}$$

so the total profit since foundation first exceeds \$4 000 000 during the 18th year.

Using logarithms when the base is less than 1

The successive powers of a base greater than 1 form an increasing sequence. E.g. the powers of 2 are

$$2^1 = 2, \qquad 2^2 = 4, \qquad 2^3 = 8, \qquad 2^4 = 16, \qquad \ldots$$

But when the base is less than 1, the successive powers form a decreasing sequence. E.g. the powers of $\frac{1}{2}$ are

$$\left(\tfrac{1}{2}\right)^1 = \tfrac{1}{2}, \qquad \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4}, \qquad \left(\tfrac{1}{2}\right)^3 = \tfrac{1}{8}, \qquad \left(\tfrac{1}{2}\right)^4 = \tfrac{1}{16}, \qquad \ldots$$

Thus when the base is less than one, questions will be asking for the smallest value of the index that makes the power *less* than some small number.

Example 13 8B

Use logarithms to find the smallest value of n such that:

a $\left(\frac{1}{3}\right)^n < 0.000\,001$ **b** $0.95^n < 0.01$

SOLUTION

a
$$\begin{aligned}
\text{Put} \quad \left(\frac{1}{3}\right)^n &= 0.000\,001 && \text{(the corresponding equation).}\\
\text{Then} \quad n &= \log_{\frac{1}{3}} 0.000\,001 && \text{(converting to a logarithmic equation)}\\
&= \frac{\log_{10} 0.000\,001}{\log_{10} \frac{1}{3}} && \text{(using the change-of-base formula)}\\
&= 12.575\ldots.
\end{aligned}$$

Thus the smallest such integer is 13, because $\left(\frac{1}{3}\right)^{12} > 0.000\,001$ and $\left(\frac{1}{3}\right)^{13} < 0.000\,001$.

b
$$\begin{aligned}
\text{Put} \quad 0.95^n &= 0.01.\\
\text{Then} \quad n &= \log_{0.95} 0.01\\
&= \frac{\log_{10} 0.01}{\log_{10} 0.95} \quad \text{OR} \quad \frac{\log_e 0.01}{\log_e 0.95} \quad \text{(change-of-base formula)}\\
&= 89.781\ldots.
\end{aligned}$$

Thus the smallest such integer is 90, because $0.95^{89} > 0.01$ and $0.95^{90} < 0.01$.

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3 SOLVING EXPONENTIAL INEQUATIONS

To solve an exponential inequation such as $3^n > 400000$ or $0.95^n < 0.01$:

- The first approach is to use trial-and-error with the calculator.
- The second approach is to use logarithms base e or base 10.
  - Write down the corresponding equation $3^n = 400000$ or $0.95^n = 0.01$.
  - Solve for n, giving $n = \log_3 400000$ or $n = \log_{0.95} 0.01$.
  - Convert this to logarithms base e or base 10, and approximate.
  - Then write down the solution of corresponding inequation.

Be aware that powers of 3 get bigger as the index increases because 3 is greater than 1, and powers of 0.95 get smaller because 0.95 is smaller than 1.

A third approach is to take logarithms base e or base 10 of both sides.

Example 14 — 8B

Consider again the slowly failing Gumnut Softdrinks Factory in Wadelbri, where sales are declining by 6% every year, with 50000 bottles sold in 2016.

During which year will sales first fall below 20000?

SOLUTION

The sales form a GP with $a = 50000$ and $r = 0.94$.

Put $T_n = 20000$. (this is the corresponding equation).

Then
$$
\begin{aligned}
ar^{n-1} &= 20000 \\
50000 \times 0.94^{n-1} &= 20000 \\
\boxed{\div 50000} \quad 0.94^{n-1} &= 0.4 \\
n - 1 &= \log_{0.94} 0.4 \quad \text{(converting to a logarithmic equation)} \\
n - 1 &= \frac{\log_{10} 0.4}{\log_{10} 0.94} \quad \text{OR} \quad \frac{\log_e 0.4}{\log_e 0.94} \quad \text{(change-of-base)} \\
n - 1 &\doteqdot 14.81 \\
\boxed{+\ 1} \quad n &\doteqdot 15.81.
\end{aligned}
$$

Hence sales will first fall below 20000 when $n = 16$, that is, in 2031.

Exercise 8B

FOUNDATION

1 Use trial-and-error (and your calculator, where necessary) to find the smallest integer n such that:

a $2^n > 30$ **b** $2^n > 15000$ **c** $2^n > 7000000$

d $3^n > 10$ **e** $3^n > 16000$ **f** $3^n > 5000000$

g $\left(\frac{1}{2}\right)^n < 0.1$ **h** $\left(\frac{1}{2}\right)^n < 0.005$ **i** $\left(\frac{1}{2}\right)^n < 0.0001$

j $\left(\frac{1}{3}\right)^n < 0.2$ **k** $\left(\frac{1}{3}\right)^n < 0.01$ **l** $\left(\frac{1}{3}\right)^n < 0.00001$

2 **a** Show that 10, 11, 12.1, . . . is a geometric sequence.

b State the first term and the common ratio.

c Use the formula $T_n = ar^{n-1}$ to write down the fifteenth term.

d Find the number of terms less than 60 using trial-and-error on your calculator.

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3 An accountant receives an annual salary of \$40 000, with 5% increments each year.

a Show that her annual salary forms a GP, and find the common ratio.

b Find her annual salary, and her total earnings, at the end of ten years, each correct to the nearest dollar.

c In which year will her salary first exceed \$70 000?

4 An initial salary of \$50 000 increases by 4% each year. In which year will the salary first be at least twice the original salary?

DEVELOPMENT

5 Confirm your answers to question 1 by solving each equation using logarithms.

6 Confirm your answers to the last part of each of questions 2, 3 and 4 by using logarithms.

7 A certain company manufactures three types of shade cloth. The product with code SC50 cuts out 50% of harmful UV rays, SC75 cuts out 75% and SC90 cuts out 90% of UV rays. In the following questions, you will need to consider the amount of UV light let through.

a What percentage of UV light does each cloth let through?

b Show that two layers of SC50 would be equivalent to one layer of SC75 shade cloth.

c Use logarithms to find the minimum number of layers of SC50 that would be required to cut out at least as much UV light as one layer of SC90.

d Similarly find how many layers of SC50 would be required to cut out 99% of UV rays.

8 Yesterday, a tennis ball used in a game of cricket in the playground was hit onto the science block roof. Luckily it rolled off the roof. After bouncing on the playground it reached a height of 3 metres. After the next bounce it reached 2 metres, then $1\frac{1}{3}$ metres, and so on.

a What was T_n, the height in metres reached after the nth bounce?

b What was the height of the roof that the ball fell from?

c The last time the ball bounced, its height was below 1 cm for the first time. After that it rolled away across the playground.

i Show that if $T_n < 0.01$, then $\left(\frac{3}{2}\right)^{n-1} > 300$.

ii Use logarithms, and then use trial-and-error on the calculator, to find how many times it bounced.

9 Madeleine opens a business selling computer stationery. In its first year, the business has sales of \$200 000, and each year sales are 20% more than the previous year's sales.

a In which year do annual sales first exceed \$1 000 000?

b In which year do total sales since foundation first exceed \$2 000 000?

10 **a** Explain why 'increasing a quantity by 300%' means 'multiplying the quantity by 4'.

b A population is increasing by 25% every year. How many full years will it take the population to increase by over 300%?

CHALLENGE

11 Consider the geometric series $3, 2, \frac{4}{3}, \ldots$.

a Write down a formula for the sum S_n of the first n terms of the series.

b Explain why the geometric series has a limiting sum, and determine its value S.

c Find the smallest value of n for which $S - S_n < 0.01$.

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8C Simple and compound interest

This section reviews the formulae for simple and compound interest. Simple interest is both an arithmetic sequence and a linear function. Compound interest is both a geometric sequence and an exponential function.

Simple interest, arithmetic sequences and linear functions

The formula for simple interest I should be well known from earlier years,

$$I = PRn,$$

where P is the principal invested, n is the number of units of time (such as days, weeks, months or years), and R is the interest rate per unit time.

If we regard P and R as constants, this formula represents the interest I as a linear function of n. When, however, we substitute the values $1, 2, 3, \ldots,$ the total interest payments over $1, 2, 3, \ldots$ units of time form a sequence

$$T_1 = PR, \quad T_2 = 2PR, \quad T_3 = 3PR, \quad \ldots$$

which is an AP with first term PR and common difference PR. Substituting $n = 0$ gives $T_0 = 0$, which can be regarded as the 0th term of the sequence, when the interest due is still zero.

The simple interest formula gives the interest alone. To find the total amount at the end of n units of time, add the original principal P to the interest.

4 SIMPLE INTEREST

Suppose that a principal $\$P$ earns simple interest at a rate R per unit time.

- The simple interest $\$I$ earned over n units of time is
  $$I = PRn.$$
- Thus the interest is a linear function of n.
- Substituting $n = 1, 2, 3 \ldots$ gives an AP with first term PR and difference PR.
- To find the total amount at the end of n units of time, add the principal P.

Note: The interest rate here is a number. If the rate is given as a percentage, such as 7% pa, then substitute $R = 0.07$. (The abbreviation 'pa' stands for *per annum*, which is Latin for 'per year' — always be careful of the units of time.)

Example 15 **8C**

A principal $\$P$ is invested at 6% pa simple interest.

a If the principal $\$P$ is \$3000, how much money will there be after seven years?

b Find the principal $\$P$, if the total at the end of five years is \$6500.

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SOLUTION

a Using the formula,

$$\begin{aligned} \text{interest} &= PRn \\ &= 3000 \times 0.06 \times 7 \\ &= \$1260. \end{aligned}$$

Hence

$$\begin{aligned} \text{final amount after 7 years} &= 3000 + 1260 \quad \text{(principal + interest)} \\ &= \$4260. \end{aligned}$$

b

$$\begin{aligned} \text{Final amount after 5 years} &= P + PRn \quad \text{(principal + interest)} \\ &= P + P \times 0.06 \times 5 \\ &= P + P \times 0.3 \\ &= P \times 1.3. \end{aligned}$$

Hence

$$\begin{aligned} P \times 1.3 &= 6500 \\ \boxed{\div\ 1.3} \qquad P &= \$5000. \end{aligned}$$

Compound interest, geometric sequences and exponential functions

The formula for compound interest should also be well known from earlier years,

$$A_n = P(1 + R)^n,$$

where A_n is the final amount after n units of time (such as days, weeks, months or years), P is the principal, and R is the interest rate per unit time.

If we regard P and R as constants, this formula represents the final amount A_n as an exponential function of n with base $1 + R$. When, however, we substitute the values $n = 1, 2, 3, \ldots,$ the final amounts after $1, 2, 3, \ldots$ units of time form a sequence

$$A_1 = P(1 + R)^1, \quad A_2 = P(1 + R)^2, \quad A_3 = P(1 + R)^3, \quad \ldots$$

which is a GP with first term $P(1 + R)$ and common ratio $1 + R$. Substituting $n = 0$ gives $A_0 = P$, which can be regarded as the 0th term of the sequence, when the amount due is still equal to the principal.

Thus GPs and exponential functions are closely related, as discussed in Chapter 1.

Sometimes a question will ask what interest was earned on the principal. To find the interest, subtract the principal from the final amount.

5 COMPOUND INTEREST

Suppose that a principal $\$P$ earns compound interest at a rate R per unit time for n units of time, compounded every unit of time.

- The total amount A_n after n units of time is
  $$A_n = P(1 + R)^n.$$
- Thus the final amount is an exponential function with base $1 + R$.
- Substituting $n = 1, 2, 3, \ldots$ gives a GP with first term $P(1 + R)$ and ratio $1 + R$.
- To find the interest, subtract the principal from the final amount.

Note: The formula only works when compounding occurs after every unit of time. For example, if the interest rate is given as 24% per year with interest compounded monthly, then the units of time must be months, and the interest rate must be the rate per month, which is $R = 0.24 \div 12 = 0.02$.

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Proof Although the formula was developed in earlier years, it is important to understand how it arises and how the process of compounding generates a GP.

The initial principal is P, and the interest rate is R per unit time.
Hence the amount A_1 at the end of one unit of time is

$$A_1 = \text{principal} + \text{interest} = P + PR = P(1 + R).$$

This means that adding the interest is effected by multiplying by $1 + R$.
Thus the amount A_2 is obtained by multiplying A_1 by $1 + R$:

$$A_2 = A_1(1 + R) = P(1 + R)^2.$$

Continuing the process for the amounts $A_3, A_4, \ldots,$

$$A_3 = A_2(1 + R) = P(1 + R)^3,$$
$$A_4 = A_3(1 + R) = P(1 + R)^4,$$

so that when the money has been invested for n units of time,

$$A_n = A_{n-1}(1 + R) = P(1 + R)^n.$$

Example 16 8C

Amelda takes out a loan of \$5000 at a rate of 12% pa, compounded monthly. She makes no repayments.

a Find the total amount owing after five years and after ten years.

b Hence find the interest alone after five years and after ten years.

c Use logarithms to find when the amount owing doubles, giving your answer correct to the nearest month.

SOLUTION

Because the interest is compounded every month, the units of time must be months. The interest rate is therefore 1% per month, so $R = 0.01$.

a
$$\begin{aligned} A_{60} &= P(1 + R)^{60} && \text{(converting 5 years to 60 months)} \\ &= 5000 \times 1.01^{60} \\ &\doteqdot \$9083. \\ A_{120} &= P(1 + R)^{120} && \text{(converting 10 years to 120 months)} \\ &= 5000 \times 1.01^{120} \\ &\doteqdot \$16\,502. \end{aligned}$$

b After five years, interest $= \$9083 - \$5000$ (subtracting the principal)
$= \$4083.$

After ten years, interest $= \$16\,502 - \$5000$ (subtracting the principal)
$= \$11\,502.$

c The formula is $A_n = P(1 + R)^n$.
Substitute $P = 5000$, $1 + R = 1.01$ and $A_n = 10\,000$.

$$\begin{aligned} \text{Then} \quad 10\,000 &= 5000 \times 1.01^n \\ \boxed{\div 5000} \quad 1.01^n &= 2 \\ n &= \log_{1.01} 2 && \text{(converting to a logarithmic equation)} \\ &= \frac{\log_{10} 2}{\log_{10} 1.01} && \text{(using the change-of-base formula)} \\ &\doteqdot 70 \text{ months} && \text{(or use trial-and-error).} \end{aligned}$$

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Depreciation

Depreciation is important when a business buys equipment because the equipment becomes worn or obsolete over time and loses its value. The company is required to record this loss of value as an expense in its accounts. Depreciation reduces the company's profit, which in turn reduces also the income tax payable.

Depreciation is usually expressed as the loss per unit time of a percentage of the value of an item. The formula for depreciation is therefore the same as the formula for compound interest, except that the rate is negative.

6 DEPRECIATION

Suppose that goods originally costing $\$P$ depreciate at a rate R per unit time.

- The value A_n of the goods after n units of time is
  $$A_n = P(1 - R)^n.$$
- Thus the final value is an exponential function of n with base $1 - R$.
- Substituting $n = 1, 2, 3, \ldots$ gives a GP with first term $P(1 - R)$ and ratio $1 - R$.
- To find the loss of value, subtract the final value from the original value.

Substituting $n = 0$ gives $A_0 = P$, the original value (0th term of the sequence).

Example 17 8C

An espresso machine bought for \$15 000 on 1st January 2016 depreciates at a rate of $12\frac{1}{2}\%$ pa.

a What will the depreciated value be on 1st January 2025?

b What is the loss of value over those nine years?

c During which year will the value drop below 10% of the original cost?

SOLUTION

This is depreciation with $R = 0.125$, so $1 - R = 0.875$.

a
$$\begin{aligned} \text{Depreciated value} &= A_9 && \text{(from 01/01/2016 to 01/01/2025 is 9 years)} \\ &= P(1 - R)^n \\ &= 15000 \times 0.875^9 \\ &\doteqdot \$4510. \end{aligned}$$

b
$$\begin{aligned} \text{Loss of value} &\doteqdot 15000 - 4510 && \text{(subtracting the depreciated value)} \\ &\doteqdot \$10490. \end{aligned}$$

c The formula is $A_n = P(1 - R)^n$.

Substituting $P = 15000$, $1 - R = 0.875$ and $A_n = 1500$,

$$\begin{aligned} 1500 &= 15000 \times 0.875^n \\ \boxed{\div 15000} \quad 0.875^n &= 0.1 \\ n &= \log_{8.75} 0.1 && \text{(converting to a logarithmic equation)} \\ &= \frac{\log_{10} 0.1}{\log_{10} 0.875} && \text{(using the change-of-base formula)} \\ &\doteqdot 17.24. \end{aligned}$$

Hence the depreciated value will drop below 10% during 2033.

(There are 17 years from 01/01/2016 to 01/01/2033, so the drop occurs during 2033.)

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Exercise 8C

FOUNDATION

1 Use the formula $I = PRn$ to find: **i** the simple interest, **ii** the total amount, when:
a \$5000 is invested at 6% per annum for three years,
b \$300 is invested at 5% per annum for eight years,
c \$10 000 is invested at $7\frac{1}{2}\%$ per annum for five years,
d \$12 000 is invested at 6.15% per annum for seven years.

2 Use the formula $A = P(1 + R)^n$ to find, correct to the nearest cent: **i** the total value,
ii the interest alone, correct to the nearest cent, of:
a \$5000 invested at 6% per annum, compounded annually, for three years,
b \$300 invested at 5% per annum, compounded annually, for eight years,
c \$10 000 invested at $7\frac{1}{2}\%$ per annum, compounded annually, for five years,
d \$12 000 invested at 6.15% per annum, compounded annually, for seven years.

3 Use the formula $A = P(1 - R)^n$ to find, correct to the nearest cent: **i** the final value,
ii the loss of value, correct to the nearest cent, of:
a \$5000 depreciating at 6% per annum for three years,
b \$300 depreciating at 5% per annum for eight years,
c \$10 000 depreciating at $7\frac{1}{2}\%$ per annum for five years,
d \$12 000 depreciating at 6.15% per annum for seven years.

4 First convert the interest rate to the appropriate unit of time, then find the final value, correct to the nearest cent, when:
a \$400 is invested at 12% per annum, compounded monthly, for two years,
b \$1000 is invested at 8% per annum, compounded quarterly, for five years,
c \$750 is invested at 10% per annum, compounded six-monthly, for three years,
d \$10 000 is invested at 7.28% per annum, compounded weekly, for one year.

5 **a** Find the total value of an investment of \$5000 earning 7% per annum simple interest for three years.
b A woman invested an amount for nine years at a rate of 6% per annum. She earned a total of \$13 824 in simple interest. What was the initial amount she invested?
c A man invested \$23 000 at 3.25% per annum simple interest. At the end of the investment period he withdrew all the funds from the bank, a total of \$31 222.50. How many years did the investment last?
d The total value of an investment earning simple interest after six years is \$22 610. If the original investment was \$17 000, what was the interest rate?

6 A man invested \$10 000 at 6.5% per annum simple interest.
a Write down a formula for A_n, the total value of the investment at the end of the nth year.
b Show that the investment exceeds \$20 000 at the end of 16 years, but not at the end of 15 years.

7 A company has just bought several cars for a total of \$229 000. The depreciation rate on these cars is 15% per annum.
a What will be the net worth of the fleet of cars five years from now?
b What will be the loss in value then?

8 Howard is arguing with Juno over who has the better investment. Each invested \$20 000 for one year. Howard has his invested at 6.75% per annum simple interest, while Juno has hers invested at 6.6% per annum compound interest.

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a If Juno's investment is compounded annually, who has the better investment, and what are the final values of the two investments?

b Juno then points out that her interest is compounded monthly, not yearly. Now who has the better investment, and by how much?

DEVELOPMENT

9 To what value does \$1000 grow if invested for a year at 12% per annum compound interest, compounded:

a annually, **b** six-monthly, **c** quarterly, **d** monthly?

10 **a** The final value of an investment, after ten years earning 15% per annum, compounded yearly, was \$32 364. Find the amount invested, correct to the nearest dollar.

b The final value of an investment that earned 7% compound interest per annum for 18 years was \$40 559.20. What was the original amount, correct to the nearest dollar?

c A sum of money is invested at 4.5% interest per annum, compounded monthly. At the end of three years the value is \$22 884.96. Find the amount of the original investment, correct to the nearest dollar.

11 An insurance company recently valued my car at \$14 235. The car is three years old and the depreciation rate used by the insurance company was 10.7% per annum. What was the cost of the car, correct to the nearest dollar, when I bought it?

12 **a** What does \$6000 grow to at 8.25% per annum for three years, compounded monthly?

b How much interest is earned over the three years?

c What rate of simple interest would yield the same amount? Give your answer correct to three significant figures.

13 An amount of \$10 000 is invested for five years at 4% pa interest, compounded monthly.

a Find the final value of the investment.

b What rate of simple interest, correct to two significant figures, would be needed to yield the same final balance?

c How many full months will it take for the money to exceed \$15 000?

14 The present value of a company asset is \$350 000. If it has been depreciating at $17\frac{1}{2}\%$ per annum for the last six years, what was the original value of the asset, correct to the nearest \$1000?

15 Find the smallest integer n for which:

a $8000 \times (1.07)^n > 40000$ **b** $10000 \times (1.1)^n > 35000$

c $20000 \times (0.8)^n < 5000$ **d** $100000 \times (0.75)^n < 10000$

16 Write down the formula for the total value A_n when a principal of \$6000 is invested at 12% pa compound interest for n years. Hence find the smallest number of years required for the investment to:

a double, **b** treble,

c quadruple, **d** increase by a factor of 10.

17 Xiao and Mai win a prize in the lottery and decide to put \$100 000 into a retirement fund offering 8.25% per annum interest, compounded monthly. How long will it be before their money has doubled? Give your answer correct to the nearest month.

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18 [The formulae for compound interest and for natural growth are essentially the same.]
The cost C of an article is rising with inflation in such a way that at the start of every month, the cost is 1% more than it was a month before. Let C_0 be the cost at time zero.

- **a** Use the compound interest formula in Box 4 to construct a formula for the cost C after t months. Hence find, in exact form and then correct to four significant figures:
  - **i** the percentage increase in the cost over twelve months,
  - **ii** the time required for the cost to double.
- **b** The exponential growth formula $C = C_0 e^{kt}$ also models the cost after t months. Use the fact that when $t = 1$, $C = 1.01\,C_0$ to find the value of k. Hence find, in exact form and then correct to four significant figures:
  - **i** the percentage increase in the cost over twelve months,
  - **ii** the time required for the cost to double.

CHALLENGE

---

19 After six years of compound interest, the final value of a \$30 000 investment was \$45 108.91. What was the rate of interest, correct to two significant figures, if it was compounded annually?

20
- **a** Find the interest on \$15 000 invested at 7% per annum simple interest for five years.
- **b** Hence write down the total value of the investment.
- **c** What rate of compound interest would yield the same amount if compounded annually? Give your answer correct to three significant figures.

21 A student was asked to find the original value, correct to the nearest dollar, of an investment earning 9% per annum, compounded annually for three years, given its current value of \$54 391.22.

- **a** She incorrectly thought that because she was working in reverse, she should use the depreciation formula. What value did she get?
- **b** What is the correct answer?

22 A bank customer earned \$7824.73 in interest on a \$40 000 investment at 6% per annum, compounded quarterly.

- **a** Show that $1.015^n \doteqdot 1.1956$, where n is the number of quarters.
- **b** Hence find the period of the investment, correct to the nearest quarter.

A possible project

Interest rates change, sometimes only every couple of years, sometimes every month. The Reserve Bank of Australia sets a benchmark interest rate called the *cash rate*. The historical cash rates are available, and a spreadsheet can be set up that will calculate the value of an amount that has earned this benchmark rate of interest over a number of years. Alternatively, the historical term deposit interest rates from a major bank could be used.

Inflation keeps varying also. A good question to ask is whether the amount has kept up with inflation, which also changes from month to month and needs a spreadsheet to calculate. There is also the problem that an investor has to pay tax on the interest, and the rate of tax varies over time and varies with the investor's income.

Thus even though the dollar value of a monetary asset may have increased over the years, its purchasing power may have gone backwards and taxation will have been lost. All this can be set up in a spreadsheet over the last say 30–40 years using data gathered from the web. A significant comparison is the price of housing over the same period, but there are many other interesting comparisons to be made.

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8D Investing money by regular instalments

Investment schemes such as superannuation — often called annuities — require money to be invested at regular intervals, for example every month or every year. This complicates things because each individual instalment earns compound interest for a different length of time. Calculating the value of these investments at some future time requires adding the terms of a GP.

This section and the next are applications of GPs. Learning new formulae is not recommended, because they will all need to be derived within each question.

Developing the GP and summing it

The most straightforward way to solve these problems is to find what each instalment grows to as it accrues compound interest. These final amounts form a GP, which can then be summed.

7 FINDING THE FUTURE VALUE OF AN INVESTMENT SCHEME

- Find what each instalment will amount to as it earns compound interest.
- Add up all these amounts using the formula for the sum of a GP.

Example 18 8D

Rawen's parents invested \$1000 in his name on the day that he was born. They continued to invest \$1000 for him on each birthday until his 20th birthday. On his 21st birthday they gave him the value of the investment.

If all the money earned interest of 7% compounded annually, what was the final value of the scheme, correct to the nearest dollar?

SOLUTION

The 1st instalment is invested for 21 years, and so amounts to 1000×1.07^{21}.
The 2nd instalment is invested for 20 years, and so amounts to 1000×1.07^{20}.

. . . .

The 20th instalment is invested for 2 years, and so amounts to 1000×1.07^{2}.
The 21st and last instalment is invested for 1 year, and so amounts to 1000×1.07^{1}.
Thus the total amount A_{21} at the end of 21 years is the sum

$$\begin{aligned} A_{21} &= \text{instalments plus interest} \\ &= (1000 \times 1.07^1) + (1000 \times 1.07^2) + \cdots + (1000 \times 1.07^{21}). \end{aligned}$$

This is a GP with first term $a = 1000 \times 1.07$, ratio $r = 1.07$, and 21 terms.

$$\begin{aligned} \text{Hence } A_{21} &= \frac{a(r^{21} - 1)}{r - 1} \qquad \text{(using the formula for } S_n \text{ for a GP with } r > 1) \\ &= \frac{1000 \times 1.07 \times (1.07^{21} - 1)}{0.07} \\ &\doteqdot \$48\,006. \end{aligned}$$

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Example 19 8D

Robin and Robyn are investing \$10000 in a superannuation scheme on 1st July each year, beginning in the year 2010. The money earns compound interest at 8% pa, compounded annually.

a How much will the fund amount to by 30th June 2030?

b How much will the fund amount to by the end of n years?

c Show that 2031 is the year when the fund first exceeds \$500000 on 30th June.

d What annual instalment would have produced \$1000000 by 2030?

SOLUTION

a The 1st instalment is invested for 20 years, and so amounts to 10000×1.08^{20}.
The 2nd instalment is invested for 19 years, and so amounts to 10000×1.08^{19}.
The 19th instalment is invested for 2 years, and so amounts to 10000×1.08^{2}.
The 20th and last is invested for 1 year, and so amounts to 10000×1.08^{1}.
Thus the total amount A_{20} at the end of 20 years is the sum

$$\begin{aligned} A_{20} &= \text{instalments plus interest} \\ &= (10000 \times 1.08^1) + (10000 \times 1.08^2) + \cdots + (10000 \times 1.08^{20}). \end{aligned}$$

This is a GP with first term $a = 10000 \times 1.08$, ratio $r = 1.08$ and 20 terms.

$$\begin{aligned} \text{Hence } A_{20} &= \frac{a(r^{20} - 1)}{r - 1} \qquad \text{(using the GP formula for } S_n \text{ when } r > 1) \\ &= \frac{10000 \times 1.08 \times (1.08^{20} - 1)}{0.08} \\ &\doteqdot \$494229 \qquad \text{(correct to the nearest dollar)}. \end{aligned}$$

b The 1st instalment is invested for n years, and so amounts to 10000×1.08^{n}.
The 2nd instalment is invested for $n - 1$ years, and so amounts to $10000 \times 1.08^{n-1}$.
The n th and last is invested for 1 year, and so amounts to 10000×1.08^{1}.
Thus the total amount A_n at the end of n years is the sum

$$\begin{aligned} A_{n} &= \text{instalments plus interest} \\ &= (10000 \times 1.08^1) + (10000 \times 1.08^2) + \cdots + (10000 \times 1.08^{n}). \end{aligned}$$

This is a GP with first term $a = 10000 \times 1.08$, ratio $r = 1.08$ and n terms.

$$\begin{aligned} \text{Hence } A_{n} &= \frac{a(r^{n} - 1)}{r - 1} \qquad \text{(using the GP formula for } S_n \text{ when } r > 1) \\ &= \frac{10000 \times 1.08 \times (1.08^{n} - 1)}{0.08} \\ &= 135000 \times (1.08^n - 1). \end{aligned}$$

c From part **a**, the total after 20 years is just under \$500000.
Substituting $n = 21$ into the formula in part **b**,

$$\begin{aligned} A_{21} &= 135000 \times (1.08^{21} - 1) \\ &\doteqdot \$544568. \end{aligned}$$

Hence 2031 is the year when the fund first exceeds \$500000 on 30th June.

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d To find the annual instalment that would have produced \$100000 by 2030:
Reworking part **b** with an instalment \$$M$ instead of \$10000 gives the formula

$$A_n = 13.5 \times M \times (1.08^n - 1).$$

Substituting $n = 20$ and $A_{20} = 1000000$ into this formula,

$$1000000 = 13.5 \times M \times (1.08^{20} - 1)$$

$$M = \frac{1000000}{13.5 \times (1.08^{20} - 1)} \quad \text{(making } M \text{ the subject)}$$

$$\doteqdot \$20234 \quad \text{(correct to the nearest dollar).}$$

Example 20 [Using logarithms to find n] 8D

Continuing with the previous example, use logarithms to find the year in which the fund first exceeds \$700000 on 30th June.

SOLUTION

Substituting $M = 10000$ and $A_n = 700000$ into the formula found in part **b**,

$$700000 = 135000 \times (1.08^n - 1)$$

$\boxed{\div\ 135000}$ $$1.08^n - 1 = \tfrac{700}{135}$$

$\boxed{+\ 10}$ $$1.08^n = \tfrac{835}{135}$$

$$n = \log_{1.08}\tfrac{835}{135} \quad \text{(converting to algorithmic equation)}$$

$$= \frac{\log_{10}\frac{835}{135}}{\log_{10}1.08} \quad \text{(using the change-of-base formula)}$$

$$\doteqdot 23.68.$$

Hence the fund first exceeds \$700000 on 30th June when $n = 24$, that is, in 2034.

Example 21 [Monthly and weekly compounding] 8D

Charmaine has a superannuation scheme with monthly instalments of \$600 for 10 years and an interest rate of 7.8% pa, compounded monthly. What will the final value of her investment be?

a Charmaine was offered an alternative scheme with interest of 7.8% pa, compounded weekly, and weekly instalments. What weekly instalments would have yielded the same final value as the scheme in part **a**?

b Which scheme would have cost her more per year?

SOLUTION

a The monthly interest rate is $0.078 \div 12 = 0.0065$.
There are 120 months in 10 years.
The 1st instalment is invested for 120 months and so amounts to 600×1.0065^{120}.
The 2nd instalment is invested for 119 months and so amounts to 600×1.0065^{119}.
The 120th and last is invested for 1 month and so amounts to 600×1.0065^{1}.
Thus the total amount A_{120} at the end of 120 months is the sum

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$$\begin{aligned} A_{120} &= \text{instalments plus interest} \\ &= (600 \times 1.0065^1) + (600 \times 1.0065^2) + \cdots + (600 \times 1.0065^{120}). \end{aligned}$$

This is a GP with first term $a = 600 \times 1.0065$, ratio $r = 1.0065$ and 120 terms.

$$\begin{aligned} \text{Hence } A_{120} &= \frac{a(r^{120} - 1)}{r - 1} \qquad \text{(using the GP formula for } S_n \text{ when } r > 1) \\ &= \frac{600 \times 1.0065 \times (1.0065^{10} - 1)}{0.0065} \\ &\doteqdot \$109\,257 \qquad \text{(retained in the memory for part } \mathbf{b}). \end{aligned}$$

b The weekly interest rate is $0.078 \div 52 = 0.0015$.
Let $\$M$ be the weekly instalment. There are 520 weeks in 10 years.
The 1st instalment is invested for 520 weeks and so amounts to $M \times 1.0015^{520}$.
The 2nd instalment is invested for 519 weeks and so amounts to $M \times 1.0015^{519}$.
The 520th and last is invested for 1 week and so amounts to $M \times 1.0015^1$.
Thus the total amount A_{520} at the end of 520 weeks is the sum

$$\begin{aligned} A_{520} &= \text{instalments plus interest} \\ &= M \times 1.0015 + M \times 1.0015^2 + \cdots + M \times 1.0015^{520}. \end{aligned}$$

This is a GP with first term $a = M \times 1.0015$, ratio $r = 1.0015$ and 520 terms.

$$\text{Hence } A_{520} = \frac{a(r^{520} - 1)}{r - 1} \qquad \text{(Using the GP formula for } S_n \text{ when } r > 1)$$

$$A_{520} = \frac{M \times 1.0015 \times (1.0015^{520} - 1)}{0.0015}.$$

$$A_{520} = \frac{M \times 10015 \times (1.0015^{520} - 1)}{15}.$$

Writing this formula with M as the subject,

$$M = \frac{15 \times A_{520}}{10015 \times (1.0015^{520} - 1)}$$

But the final value A_{520} is to be the same as the final value in part **a**,
so substituting the answer to part **a** for A_{520} gives
$M \doteqdot \$138.65$ (retained in the memory for part **c**).

The weekly scheme in part **b** therefore costs about \$7210.04 per year, compared with \$7200 per year for the monthly scheme in part **a**.

An alternative approach using recursion

There is an alternative approach, using recursion, to developing the GPs involved in these calculations. Because the working is slightly longer, we have chosen not to display this method in the notes. It has, however, the great advantage that its steps follow the progress of a banking statement.

For those interested in the recursive method, it is developed in two structured questions, Questions 17 and 18, at the end of the Challenge section of Exercise 8D. Most of the other questions can also be done recursion if that is preferred (provided, of course, that some structuring within the question is ignored).

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Exercise 8D

FOUNDATION

Note: Questions 1–5 of this exercise have been heavily structured to follow the approach given in the worked examples of this section. There are several other satisfactory approaches, including the recursive method outlined in Questions 17 and 18. If a different approach is chosen, the structuring in the first five questions below can be ignored.

1 Suppose that an instalment of \$500 is invested in a superannuation scheme on 1st January each year for four years, beginning in 2020. The money earns interest at 10% pa, compounded annually.

a **i** What is the value of the first instalment on 31st December 2023?

ii What is the value of the second instalment on 31st December 2023?

iii What is the value of the third instalment on this date?

iv What is the value of the fourth instalment?

v What is the total value of the superannuation on 31st December 2023?

b **i** Write down the four answers to parts **i** to **iv** above in increasing order, and notice that they form a GP.

ii Write down the first term, common ratio and number of terms.

iii Use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ to find the sum of the GP, and hence check your answer to part **a v**.

2 Suppose that an instalment of \$1200 is invested in a superannuation scheme on 1st April each year for five years, beginning in 2015. The money earns interest at 5% pa, compounded annually.

a In each part round your answer correct to the nearest cent.

i What is the value of the first instalment on 31st March 2020?

ii What is the value of the second instalment on this date?

iii Do the same for the third, fourth and fifth instalments.

iv What is the total value of the superannuation on 31st March 2020?

b **i** Write down the answers to parts **i** to **iii** above in increasing order, and notice that they form a GP.

ii Write down the first term, common ratio and number of terms.

iii Use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ to find the sum of the GP, rounding your answer correct to the nearest cent, and hence check your answer to part **a iv**.

3 Joshua makes 15 contributions of \$1500 to his superannuation scheme on 1st April each year. The money earns compound interest at 7% per annum. He calculates what the scheme will be worth at a target date 15 years later.

a Let A_{15} be the total value of the fund at the target date.

i How much does the first instalment amount to at the target date?

ii How much does the second instalment amount to at the target date?

iii How much does the last contribution amount to invested for just one year?

iv Hence write down a series for A_{15}.

b Hence show that the final value of the fund is $A_{15} = \dfrac{1500 \times 1.07 \times (1.07^{15} - 1)}{0.07}$, and evaluate this correct to the nearest dollar.

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4 Laura makes 24 contributions of \$250 to her superannuation scheme on the first day of each month. The money earns interest at 6% per annum, compounded monthly (that is, at 0.5% per month). She calculates the scheme's value at a target date 24 months later.

a Let A_{24} be the total value of the fund at the target date.

i How much does the first instalment amount to at the target date?

ii How much does the second instalment amount to at the target date?

iii What is the value of the last contribution, invested for just one month?

iv Hence write down a series for A_{24}.

b Hence show that the total value of the fund after contributions have been made for two years is $A_{24} = \dfrac{250 \times 1.005 \times (1.005^{24} - 1)}{0.005}$, and evaluate this correct to the nearest dollar.

5 A company makes contributions of \$3000 to the superannuation fund of one of its employees on 1st July each year. The money earns compound interest at 6.5% per annum. In this question, round all currency amounts correct to the nearest dollar.

a Let A_{25} be the value of the fund at the end of 25 years.

i How much does the first instalment amount to at the end of 25 years?

ii How much does the second instalment amount to at the end of 24 years?

iii How much does the last instalment amount to at the end of just one year?

iv Hence write down a series for A_{25}.

b Hence show that $A_{25} = \dfrac{3000 \times 1.065 \times (1.065^{25} - 1)}{0.065}$.

c What will be the value of the fund after 25 years, and what will be the total amount of the contributions?

DEVELOPMENT

6 Finster and Finster Superannuation offer a superannuation scheme with annual contributions of \$12 000 invested at an interest rate of 9% pa, compounded annually. Contributions are paid on 1st of January each year.

a Zoya decides to invest in the fund for the next 20 years. Show that the final value of her investment is given by $A_{20} = \dfrac{12000 \times 1.09 \times (1.09^{20} - 1)}{0.09}$.

b Evaluate A_{20}.

c By how much does this exceed the total contributions Zoya made?

d The company agrees to let Zoya make a higher contribution to the scheme. Let this instalment be M. Show that in this case $A_{20} = \dfrac{M \times 1.09 \times (1.09^{20} - 1)}{0.09}$.

e What would Zoya's annual contribution have to be in order for her superannuation to have a total value of \$1 000 000 at the end of the 20 years?

7 The company that Itsushi works for makes contributions to his superannuation scheme on 1st January each year. Any amount invested in this scheme earns interest at the rate of 7.5% pa.

a Let M be the annual contribution. Show that the value of the investment at the end of the n th year is $A_n = \dfrac{M \times 1.075 \times (1.075^n - 1)}{0.075}$.

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b Itsushi plans to have \$1 500 000 in superannuation when he retires in 25 years' time. Show that the company must contribute \$20 526.52 each year, correct to the nearest cent.

c The first year that Itsushi's superannuation is worth more than \$750 000, he decides to change jobs. Let this year be n.

i Show that n is the smallest integer solution of $(1.075)^n > \dfrac{750000 \times 0.075}{20526.52 \times 1.075} + 1$.

ii Evaluate the right-hand side and hence show that $(1.075)^n > 3.5492$.

iii Use logarithms or trial-and-error to find the value of n.

8 A person invests \$10 000 each year in a superannuation fund. Compound interest is paid at 10% per annum on the investment. The first payment is made on 1st January 2021 and the last payment is made on 1st January 2040.

a How much did the person invest over the life of the fund?

b Calculate, correct to the nearest dollar, the amount to which the 2021 payment has grown by the beginning of 2041.

c Find the total value of the fund when it is paid out on 1st January 2041.

d The person wants to reach a total value of \$1 000 000 in superannuation.

i Find a formula for A_n, the value of the investment after n years.

ii Show that the target is reached when $1.1^n > \dfrac{10}{1.1} + 1$.

iii At the end of which year will the superannuation be worth \$1 000 000?

e Suppose instead that the person wanted to achieve the same total investment of \$1 000 000 after only 20 years. What annual contribution would produce this amount?
(Hint: Let M be the amount of each contribution.)

9 Each year on her birthday, Jane's parents put \$20 into an investment account earning $9\frac{1}{2}\%$ per annum compound interest. The first deposit took place on the day of her birth. On her 18th birthday, Jane's parents gave her the account and \$20 cash in hand.

a How much money had Jane's parents deposited in the account?

b How much money did she receive from her parents on her 18th birthday?

10 A man about to turn 25 is getting married. He has decided to pay \$5000 each year on his birthday into a combination life insurance and superannuation scheme that pays 8% compound interest per annum. If he dies before age 65, his wife will inherit the value of the insurance to that point. If he lives to age 65, the insurance company will pay out the value of the policy in full. Answer these questions correct to the nearest dollar.

a The man is in a dangerous job. What will be the payout if he dies just before he turns 30?

b The man's father died of a heart attack just before age 50. Suppose that the man also dies of a heart attack just before age 50. How much will his wife inherit?

c What will the insurance company pay the man if he survives to his 65th birthday?

11 In 2021, the school fees at a private girls' school are \$20 000 per year. Each year the fees rise by $4\frac{1}{2}\%$ due to inflation.

a Susan is sent to the school, starting in Year 7 in 2021. If she continues through to her HSC year, how much will her parents have paid the school over the six years?

b Susan's younger sister is starting in Year 1 in 2021. How much will they spend on her school fees over the next 12 years if she goes through to her HSC?

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CHALLENGE

12 A woman has just retired with a payment of \$500 000, having contributed for 25 years to a superannuation fund that pays compound interest at the rate of $12\frac{1}{2}\%$ per annum. What was the size of her annual premium, correct to the nearest dollar?

13 At age 20, a woman takes out a life insurance policy under which she agrees to pay premiums of \$500 per year until she turns 65, when she is to be paid a lump sum. The insurance company invests the money and gives a return of 9% per annum, compounded annually. If she dies before age 65, the company pays out the current value of the fund plus 25% of the difference between the current value and what the value would have been had she lived until 65.

a What is the value of the payout, correct to the nearest dollar, at age 65?

b Unfortunately she dies at age 53, just before her 35th premium is due.

i What is the current value of the life insurance?

ii How much does the life insurance company pay her family?

14 A person pays \$2000 into an investment fund every year, and it earns compound interest at a rate of 6% pa.

a How much is the fund worth at the end of 10 years?

b In which year will the fund reach \$70 000?

15 [Technology]

In the first column of a spreadsheet, enter the numbers from 1 to 30 on separate rows. In the first 30 rows of the second column, enter the formula

$$\frac{20256.52 \times 1.075 \times (1.075^n - 1)}{0.075}$$

for the value of a superannuation investment, where n is the value given in the first column.

a Which value of n is the first to give a superannuation amount greater than \$750 000?

b Compare this answer with your answer to question 7(c).

c Try to do question 8(d) in the same way.

16 [Technology]

Try checking your answers to questions 3 to 11 using a spreadsheet and its built-in financial functions. In particular, the built-in Excel™ function `FV(rate, nper, pmt, pv, type)`, which calculates the future value of an investment, seems to produce an answer different from what might be expected. Investigate this and explain the difference.

Note: The last two questions illustrate an alternative approach to superannuation questions, using a recursive method to generate the appropriate GP. The advantage of the method is that its steps follow the progress of a banking statement.

17 [The recursive method]

At the start of each month, Cecilia deposits \$$M$ into a savings scheme paying 1% per month, compounded monthly. Let A_n be the amount in her account at the end of the nth month.

a Explain why $A_1 = 1.01M$.

b Explain why $A_2 = 1.01(M + A_1)$, and why $A_{n+1} = 1.01(M + A_n)$, for $n \geq 2$.

c Use the recursive formulae in part **b**, together with the value of A_1 in part **a**, to obtain expressions for $A_2, A_3, \ldots, A_n$.

d Using the formula for the sum of n terms of a GP, show that $A_n = 101M(1.01^n - 1)$.

e If each deposit is \$100, how much will be in the fund after three years?

f Hence find, correct to the nearest cent, how much each deposit M must be if Cecilia wants the fund to amount to \$30 000 at the end of five years.

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18 [The recursive method]

A couple saves \$100 at the start of each week in an account paying 10.4% pa interest, compounded weekly. Let A_n be the amount in the account at the end of the nth week.

a Explain why $A_1 = 1.002 \times 100$, and why $A_{n+1} = 1.002 \times (100 + A_n)$, for $n \geq 2$.

b Use these recursive formulae to obtain expressions for $A_2, A_3, \ldots, A_n$.

c Using GP formulae, show that $A_n = 50\,100 \times (1.002^n - 1)$.

d Hence find how many weeks it will be before the couple has \$100 000.

A possible project

As discussed at the end of Exercise 8C, interest rates vary over time, and inflation means that the purchasing power of a matured superannuation fund is less than its dollar-value may have suggested some years ago. Taking all this into account requires a spreadsheet. But with superannuation there are many other considerations, and building these things into a spreadsheet as well would involve an extended project.

- Most superannuation funds have an insurance component, insuring against early death or disability. The cost of this insurance is built into the policy, but the details are not straightforward.
- All superannuation funds charge fees, which are calculated in various ways, perhaps depending on the balance, perhaps depending on the future value, perhaps depending on the number of transactions. This could also be investigated and built into the spreadsheet.
- The contributions to the fund are almost certainly a proportion of the salary. Thus some estimates must be made of future salary.

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8E Paying off a loan

Long-term loans such as housing loans are usually paid off by regular instalments, with compound interest charged on the balance owing at any time. The calculations associated with paying off a loan are therefore similar to the investment calculations of the previous section.

Developing the GP and summing it

As with superannuation, the most straightforward method is to calculate the final value of each instalment as it earns compound interest, and then add these final values up as before, using the theory of GPs. But there is an extra complication — these instalments must be balanced against the initial loan, which is growing with compound interest. The loan is finally paid off when the amount owing is zero.

8 CALCULATIONS ASSOCIATED WITH PAYING OFF A LOAN

To find the amount A_n still owing after n units of time:

- Find what the principal, earning compound interest, would amount to if no instalments were paid.
- Find what each instalment will amount to as it earns compound interest, then add up all these amounts, using the formula for the sum of a GP.
- The amount A_n still owing at the end of n units of time is
  $$A_n = (\text{principal plus interest}) - (\text{instalments plus interest}).$$

The loan is paid off when the amount A_n still owing is zero.

Note: When paying off a loan, the first payment is usually made one unit of time after the loan is taken out. But read the question carefully!

Example 22 8E

Yianni and Eleni borrow \$20 000 from the Town and Country Bank to go on a trip to Constantinople. Interest is charged at 12% per annum, compounded monthly. They start repaying the loan one month after taking it out, and their monthly instalments are \$300.

a How much will they still owe the bank at the end of six years?

b How much interest will they have paid in these six years?

SOLUTION

a The monthly interest rate is 1%, so $1 + R = 1.01$.

The initial loan of \$20 000, after 72 months, amounts to 20000×1.01^{72}.

The 1st instalment is invested for 71 months, and so amounts to 300×1.01^{71}.

The 2nd instalment is invested for 70 months, and so amounts to 300×1.01^{70}.

The 71st instalment is invested for 1 month, and so amounts to 300×1.01^{1}.

The 72nd and last instalment is invested for no time at all, and so amounts to 300.

Hence the amount A_{72} still owing at the end of 72 months is

$$\begin{aligned} A_{72} &= (\text{principal plus interest}) - (\text{instalments plus interest}) \\ &= 20000 \times 1.01^{72} - (300 + 300 \times 1.01 + \cdots + 300 \times 1.01^{71}). \end{aligned}$$

The bit in brackets is a GP with first term $a = 300$, ratio $r = 1.01$, and 72 terms.

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$$\text{Hence } A_{72} = 20000 \times 1.01^{72} - \frac{a(r^{72} - 1)}{r - 1} \qquad \text{(finding the sum of the GP)}$$
$$= 20000 \times 1.01^{72} - \frac{300 \times (1.01^{72} - 1)}{0.01}$$
$$= 20000 \times 1.01^{72} - 30000 \times (1.01^{72} - 1)$$
$$\doteqdot \$9529 \text{ (correct to the nearest dollar).}$$

b To find how much interest they will have paid in these six years:

Total instalments over six years $= 300 \times 72$
$= \$21600.$

Reduction in loan over six years $= 20000 - 9529$
$= \$10471.$

Hence interest charged $= 21600 - 10471$
$= \$11129$ (more than half the original loan).

Example 23 [Finding what instalments should be paid] 6E

Ali takes out a loan of \$10000 to buy a car. He will repay the loan in five years, paying 60 equal monthly instalments, beginning one month after he takes out the loan. Interest is charged at 6% pa, compounded monthly.

Find how much the monthly instalment should be, correct to the nearest cent.

SOLUTION

The monthly interest rate is 0.5%, so $1 + R = 1.005$. Let each instalment be \$$M$.
First calculate the amount A_{60} still owing at the end of 60 months.
Then find M by setting A_{60} equal to zero.

The initial loan of \$10000, after 60 months, amounts to 10000×1.005^{60}.
The 1st instalment is invested for 59 months, and so amounts to $M \times 1.005^{59}$.
The 2nd instalment is invested for 58 months, and so amounts to $M \times 1.005^{58}$.
The 59th instalment is invested for 1 month, and so amounts to $M \times 1.005^{1}$.
The 60th and last instalment is invested for no time at all, and so amounts to M.
Hence the amount A_{60} still owing at the end of 60 months is

$$A_{60} = \text{(principal plus interest)} - \text{(instalments plus interest)}$$
$$= 10000 \times 1.005^{60} - (M + M \times 1.005 + \cdots + M \times 1.005^{59}).$$

The bit in brackets is a GP with first term $a = M$, ratio $r = 1.005$, and 60 terms.

$$\text{Hence } A_{60} = 10000 \times 1.005^{60} - \frac{a(r^{60} - 1)}{r - 1}$$
$$= 10000 \times 1.005^{60} - \frac{M(1.005^{60} - 1)}{0.005}$$
$$= 10000 \times 1.005^{60} - 200M(1.005^{60} - 1) \quad \left(\text{because } 0.005 = \tfrac{1}{200}\right).$$

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But the loan is exactly paid off in these 5 years, so $A_{60} = 0$.

Hence $\quad 10000 \times 1.005^{60} - 200M(1.005^{60} - 1) = 0$

$$200M(1.005^{60} - 1) = 10000 \times 1.005^{60}$$

$\boxed{\div 200}$ $\quad M(1.005^{60} - 1) = 50 \times 1.005^{60}$

$$M = \frac{50 \times 1.005^{60}}{1.005^{60} - 1}$$

$$\doteqdot \$193.33.$$

Finding the length of the loan

A loan is fully repaid when the amount A_n still owing is zero. Thus finding the length of a loan means solving an equation for the index n, a process that requires logarithms.

Example 24 — 6E

Natasha and Richard take out a loan of \$200 000 on 1st January 2002 to buy a house. They will repay the loan in monthly instalments of \$2200. Interest is charged at 12% pa, compounded monthly.

a Find a formula for the amount owing at the end of n months.

b How much is owing after five years?

c How long does it takes to repay:

i the full loan? **ii** half the loan?

d Why would instalments of \$1900 per month never repay the loan?

SOLUTION

a The monthly interest rate is 1%, so $1 + R = 1.01$.

The initial loan, after n months, amounts to 200000×1.01^n.

The 1st instalment is invested for $n - 1$ months and so amounts to $2200 \times 1.01^{n-1}$.

The 2nd instalment is invested for $n - 2$ months and so amounts to $2200 \times 1.01^{n-2}$.

The nth and last instalment is invested for no time at all and so amounts to 2200.

Hence the amount A_n still owing at the end of n months is

$$A_n = (\text{principal plus interest}) - (\text{instalments plus interest})$$
$$= 200000 \times 1.01^n - (2200 + 2200 \times 1.01 + \cdots + 2200 \times 1.01^{n-1}).$$

The bit in brackets is a GP with first term $a = 2200$, ratio $r = 1.01$, and n terms.

Hence $A_n = 200000 \times 1.01^n - \dfrac{a(r^n - 1)}{r - 1}$

$$= 200000 \times 1.01^n - \frac{2200 \times (1.01^n - 1)}{0.01}$$
$$= 200000 \times 1.01^n - 220000 \times (1.01^n - 1)$$
$$= 220000 - 20000 \times 1.01^n.$$

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b To find the amount owing after 5 years, substitute $n = 60$,

$$A_{60} = 220000 - 20000 \times 1.01^{60}$$
$$\doteqdot \$183666 \quad \text{(This is still almost as much as the original loan!)}$$

c **i** To find when the loan is repaid, put $A_n = 0$,

$$\begin{aligned}
220000 = 20000 \times 1.01^n &= 0 \\
20000 \times 1.01^n &= 220000 \\
\boxed{\div 20000} \qquad 1.01^n &= 11 \\
n &= \log_{1.01} 11 && \text{(converting to alogarithmic equation)} \\
n &= \frac{\log_{10} 11}{\log_{10} 1.01} && \text{(using the change-of-base formula)} \\
&\doteqdot 20 \text{ years and 1 month.}
\end{aligned}$$

ii To find when the loan is half repaid, put $A_n = 100000$,

$$\begin{aligned}
220000 - 20000 \times 1.01^n &= 100000 \\
20000 \times 1.01^n &= 220000 \\
\boxed{\div\ 20000} \qquad 1.01^n &= 6 \\
n &= \log_{1.01} 6 && \text{(converting to alogarithmic equation)} \\
n &= \frac{\log_{10} 6}{\log_{10} 1.01} && \text{(using the change-of-base formula)} \\
&\doteqdot 15 \text{ years.}
\end{aligned}$$

Notice that this is about three-quarters, not half, the total time of the loan.

d To explain why instalments of \$1900 per month would never repay the loan:
With a loan of \$200000 at an interest rate of 1% per month,

$$\begin{aligned}
\text{initial interest per month} &= 200000 \times 0.01 \\
&= \$2000.
\end{aligned}$$

This means that at the start of the loan, \$2000 of the instalment is required just to pay the interest.
Hence with repayments of only \$1900, the debt would increase rather than decrease.

The alternative approach using recursion

As with superannuation, the GP involved in a loan-repayment calculation can also be developed using a recursive method, whose steps follow the progress of a banking statement.

Again, this method is developed in two structured questions, Questions 18 and 19 at the end of the Challenge section of Exercise 8E, and recursion can easily be applied to the other questions in the exercise provided that some internal structuring is ignored.

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Exercise 8E

FOUNDATION

Note: As in the previous exercise, Questions 1 and 2 have been heavily structured to follow the approach given in the worked examples of this section. There are several other satisfactory approaches, including the recursive method outlined in Questions 18 and 19. If a different approach is chosen, the structuring in the first three questions below can be ignored.

1 On 1st January 2020, Lizbet borrows \$501 from a bank for four years at an interest rate of 10% pa. She repays the loan with four equal instalments of \$158.05 at the end of each year.

a Use the compound interest formula to show that the initial loan amounts to \$733.51 at the end of four years.

b i What is the value of the first instalment on 31st December 2023, having been invested for three years?

ii What is the value of the second instalment on this date?

iii What is the value of the third instalment?

iv What is the value of the fourth (and last) instalment?

v Find the total value of all the instalments on 31st December 2023, and hence show that Lizbet has now repaid the loan.

c i Write down the four answers to parts i–iv above in increasing order, and notice that they form a GP.

ii Write down the first term, common ratio and number of terms.

iii Use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ to find the sum of the GP, rounding your answer correct to the nearest cent, and hence check your answer to part b v.

2 Suppose that on 1st April 2020 a loan of \$5600 is made, which is repaid with equal instalments of \$1293.46 made on 31st March each year for five years, beginning in 2021. The loan attracts interest at 5% pa, compounded annually.

a Use the compound interest formula to show that the initial loan amounts to \$7147.18 by 31st March 2025.

b In each part, round your answer correct to the nearest cent.

i What is the value of the first instalment on 31st March 2025?

ii What is the value of the second instalment on this date?

iii Do the same for the third, fourth and fifth instalments.

iv Find the total value of the instalments on 31st March 2025, and hence show that the loan has been repaid.

c i Write down your answers to parts i–iii above in increasing order, and notice that they form a GP.

ii Write down the first term, common ratio and number of terms.

iii Use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$ to find the sum of the GP, rounding your answer correct to the nearest cent, and hence check your answer to part b iv.

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3 Lome took out a loan with Tornado Credit Union for \$15 000, to be repaid in 15 equal annual instalments of \$1646.92 on 1st April each year. Compound interest is charged at 7% per annum.

a Let A_{15} be the amount owed at the end of 15 years.

i Use the compound interest formula to show that $15\,000 \times (1.07)^{15}$ is owed on the initial loan after 15 years.

ii How much does the first instalment amount to at the end of the loan, having been invested for 14 years?

iii How much does the second instalment amount to at the end of 13 years?

iv What is the value of the second-last instalment?

v What is the worth of the last contribution, invested for no time at all?

vi Hence write down an expression involving a series for A_{15}.

b Show that the final amount owed is

$$A_{15} = 15\,000 \times (1.07)^{15} - \frac{1646.92 \times (1.07^{15} - 1)}{0.07}.$$

c Evaluate A_{15} and hence show that the loan has been repaid.

4 Matts signed a mortgage agreement for \$100 000 with a bank for 20 years at an interest rate of 6% per annum, compounded monthly (that is, at 0.5% per month).

a Let M be the size of each repayment to the bank, and let A_{240} be the amount owing on the loan after 20 years.

i What does the initial loan amount to after 20 years?

ii Write down the amount that the first repayment grows to by the end of the 240th month.

iii Do the same for the second repayment and for the last repayment.

iv Hence write down a series expression for A_{240}.

b Hence show that $A_{240} = 100\,000 \times 1.005^{240} - 200 \times M(1.005^{240} - 1)$.

c Explain why the bank puts $A_{240} = 0$.

d Hence find M, correct to the nearest cent.

e How much will Matts have paid the bank over the period of the loan?

5 I took out a personal loan of \$10 000 with a bank for five years at an interest rate of 18% per annum, compounded monthly (that is, at 1.5% per month).

a Let M be the size of each instalment to the bank, and let A_{60} be the amount owing on the loan after 60 months.

i What does the initial loan amount to after 60 months?

ii Write down the amount that the first instalment grows to by the end of the 60th month.

iii Do the same for the second instalment and for the last instalment.

iv Hence write down a series expression for A_{60}.

b Hence show that $0 = 10000 \times 1.015^{60} - \dfrac{M(1.015^{60} - 1)}{0.015}$.

c Hence find M, correct to the nearest dollar.

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DEVELOPMENT

6 A couple take out a \$165 000 mortgage on a house, and they agree to pay the bank \$1700 per month. The interest rate on the loan is 9% per annum, compounded monthly, and the contract requires that the loan be paid off within 15 years.

a Let A_{180} be the balance on the loan after 15 years. Find a series expression for A_{180}.

b Show that $A_{180} = 165000 \times 1.0075^{180} - \dfrac{1700(1.0075^{180} - 1)}{0.0075}$.

c Evaluate A_{180}, and hence show that the loan is actually paid out in less than 15 years.

7 A couple take out a \$250 000 mortgage on a house, and they agree to pay the bank \$2000 per month. The interest rate on the loan is 7.2% per annum, compounded monthly, and the contract requires that the loan be paid off within 20 years.

a Let A_n be the balance on the loan after n months. Find a series expression for A_n.

b Hence show that $A_n = 250000 \times 1.006^n - \dfrac{2000(1.006^n - 1)}{0.006}$.

c Find the amount owing on the loan at the end of the tenth year, and state whether this is more or less than half the amount borrowed.

d Find A_{240}, and hence show that the loan is actually paid out in less than twenty years.

e If it is paid out after n months, show that $1.006^n = 4$, and hence that $n = \dfrac{\log 4}{\log 1.006}$.

f Find how many months early the loan is paid off.

8 A company borrows \$500 000 from the bank at an interest rate of 5.25% per annum, compounded monthly, to be repaid in monthly instalments. The company repays the loan at the rate of \$10 000 per month.

a Let A_n be the amount owing at the end of the nth month. Show that

$$A_n = 500000 \times 1.004375^n - \frac{10000(1.004375^n - 1)}{0.004375}.$$

b Given that the loan is paid off, use the result in part **a** to show that $1.004375^n = 1.28$.

c Use logarithms or trial-and-error to find how long it will take to pay off the loan. Give your answer in whole months.

9 As can be seen from these questions, the calculations involved with reducible loans are reasonably complex. For that reason, it is sometimes convenient to convert the reducible interest rate into a simple interest rate. Suppose that a mortgage is taken out on a \$180 000 house at 6.6% reducible interest per annum for a period of 25 years, with payments of amount M made monthly.

a Using the usual pronumerals, explain why $A_{300} = 0$.

b Show that $A_{300} = 180000 \times 1.0055^{300} - \dfrac{M(1.0055^{300} - 1)}{0.0055}$.

c Find the size of each repayment to the bank.

d Hence find the total paid to the bank, correct to the nearest dollar, over the life of the loan.

e What amount is therefore paid in interest? Use this amount and the simple interest formula to calculate the simple interest rate per annum over the life of the loan, correct to two significant figures.

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10 A personal loan of \$15 000 is borrowed from the Min Hua Finance Company at a rate of $13\frac{1}{2}\%$ per annum over five years, compounded monthly. Let M be the amount of each monthly instalment.

a Show that $15000\,(1.01125)^{60} - \dfrac{M(1.01125^{60} - 1)}{0.01125} = 0$.

b What is the monthly instalment necessary to pay back the loan? Give your answer correct to the nearest dollar.

11 [Problems with rounding] Most questions so far have asked you to round monetary amounts correct to the nearest dollar. This is not always wise, as this question demonstrates. A personal loan for \$30 000 is approved with the following conditions. The reducible interest rate is 13.3% per annum, with payments to be made at six-monthly intervals over five years.

a Find the size of each instalment, correct to the nearest dollar.

b Using this amount, show that $A_{10} \neq 0$, that is, the loan is not paid off in five years.

c Explain why this has happened.

12 A couple have worked out that they can afford to pay \$19 200 each year in mortgage payments. The current home loan rate is 7.5% per annum, with equal payments made monthly over a period of 25 years.

a Let P be the principal borrowed and A_{300} the amount owing after 25 years. Show that

$$A_{300} = P \times 1.00625^{300} - \frac{1600\,(1.00625^{300} - 1)}{0.00625}.$$

b Hence determine the maximum amount that the couple can borrow and still pay off the loan. Round your answer down to the nearest dollar.

13 The current credit card rate of interest on Bankerscard is 23% per annum, compounded monthly.

a If a cardholder can afford to repay \$1500 per month on the card, what is the maximum value of purchases that can be made in one day if the debt is to be paid off in two months?

b How much would be saved in interest payments if the cardholder instead saved up the money for two months before making the purchase?

CHALLENGE

14 Some banks offer a 'honeymoon' period on their loans. This usually takes the form of a lower interest rate for the first year. Suppose that a couple borrowed \$170 000 for their first house, to be paid back monthly over 15 years. They work out that they can afford to pay \$1650 per month to the bank. The standard rate of interest is $8\frac{1}{2}\%$ pa, but the bank also offers a special rate of 6% pa for one year to people buying their first home. (All interest rates are compounded monthly.)

a Calculate the amount the couple would owe at the end of the first year, using the special rate of interest.

b Use this value as the principal of the loan at the standard rate for the next 14 years. Calculate the value of the monthly payment that is needed to pay the loan off. Can the couple afford to agree to the loan contract?

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15 Over the course of years, a couple have saved \$300 000 in a superannuation fund. Now that they have retired, they are going to draw on that fund in equal monthly pension payments for the next 20 years. The first payment is at the beginning of the first month. At the same time, any balance will be earning interest at $5\frac{1}{2}\%$ per annum, compounded monthly. Let B_n be the balance left immediately after the nth payment, and let M be the amount of the pension instalment. Also, let $P = 300\,000$ and R be the monthly interest rate.

a Show that $B_n = P \times (1 + R)^{n-1} - \dfrac{M((1 + R)^n - 1)}{R}$.

b Why is $B_{240} = 0$?

c What is the value of M?

16 A company buys machinery for \$500 000 and pays it off by 20 equal six-monthly instalments, the first payment being made six months after the loan is taken out. If the interest rate is 12% pa, compounded monthly, how much will each instalment be?

17 [Technology]

In the first column of a spreadsheet, enter the numbers from 1 to 60 on separate rows. In the first 60 rows of the second column, enter the formula

$$500\,000 \times 1.004375^n - \frac{10\,000 \times (1.004375^n - 1)}{0.004375}$$

for the balance of a loan repayment, where n is the value given in the first column.

a Observe the pattern of figures in the second column. Notice that the balance decreases more slowly at first and more quickly towards the end of the loan.

b Which value of n is the first to give a balance less than or equal to zero?

c Compare this answer with your answer to question 8.

d Try to do question 7 **f** in the same way.

Note: The next two questions illustrate the alternative approach to loan repayment questions, using a recursive method to generate the appropriate GP.

18 [The recursive method]

A couple buying a house borrow $\$P = \$150\,000$ at an interest rate of 6% pa, compounded monthly. They borrow the money at the beginning of January, and at the end of every month, they pay an instalment of $\$M$. Let A_n be the amount owing at the end of n months.

a Explain why $A_1 = 1.005P - M$.

b Explain why $A_2 = 1.005A_1 - M$, and why $A_{n+1} = 1.005A_n - M$, for $n \geq 2$.

c Use the recursive formulae in part **b**, together with the value of A_1 in part **a**, to obtain expressions for $A_2, A_3, \ldots, A_n$.

d Using GP formulae, show that $A_n = 1.005^n P - 200M(1.005^n - 1)$.

e Hence find, correct to the nearest cent, what each instalment should be if the loan is to be paid off in 20 years?

f If each instalment is \$1000, how much is still owing after 20 years?

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19 [The recursive method]
Eric and Enid borrow $\$P$ to buy a house at an interest rate of 9.6% pa, compounded monthly. They borrow the money on 15th September, and on the 14th day of every subsequent month, they pay an instalment of $\$M$. Let A_n be the amount owing after n months have passed.

a Explain why $A_1 = 1.008P - M$, and why $A_{n+1} = 1.008A_n - M$, for $n \geq 2$.

b Use these recursive formulae to obtain expressions for $A_2, A_3, \ldots, A_n$.

c Using GP formulae, show that $A_n = 1.008^n P - 125M(1.008^n - 1)$.

d If the maximum instalment they can afford is \$1200, what is the maximum they can borrow, if the loan is to be paid off in 25 years? (Answer correct to the nearest dollar.)

e Put $A_n = 0$ in part **c**, and solve for n. Hence find how long will it take to pay off the loan of \$100 000 if each instalment is \$1000. (Round up to the next month.)

A possible project

The remarks at the end of Exercises 8C and 8D about varying interest rates and inflation hold also for housing loans. Many loans also contain insurance against death or disability or loss of employment, and again there are fees, which are not easily found.

All this historical data can be found on the web and built into a spreadsheet. The spreadsheet could also take into account the increasing value of housing over past years. An interesting comparison could be made between the relative wealth of a couple who rented for a long period and invested their savings elsewhere, and a couple on the same (increasing) salary who purchased a home with a large mortgage.

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Chapter 8 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 8 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 Consider the series $31 + 44 + 57 + \cdots + 226$.

a Show that it is an AP and write down the first term and the common difference.

b How many terms are there in this series?

c Find the sum.

2 Consider the series $24 + 12 + 6 + \cdots$.

a Show that it is a geometric series and find the common ratio.

b Explain why this geometric series has a limiting sum.

c Find the limiting sum and the sum of the first 10 terms, and show that they are approximately equal, correct to the first three significant figures.

3 Use trial-and-error, and probably a calculator, to find the smallest integer such that:

a $2^n > 2000$ **b** $(1.08)^n > 2000$ **c** $(0.98)^n < 0.01$ **d** $\left(\frac{1}{2}\right)^n < 0.0001$

Then repeat parts **a**–**d** using logarithms.

4 On a certain day at the start of a drought, 900 litres of water flowed from the Neverfail Well. The next day, only 870 litres flowed from the well, and each day, the volume of water flowing from the well was $\frac{29}{30}$ of the previous day's volume. Find the total volume of water that would have flowed from the well if the drought had continued indefinitely.

5 The profits of a company are growing at 14% per year. If this trend continues, how many full years will it be before the profit has increased by over 2000%?

6 A chef receives an annual salary of \$35 000, with 4% increments each year.

a Show that her annual salaries form a GP and find the common ratio.

b Find her annual salary, and her total earnings, at the end of 10 years, each correct to the nearest dollar.

7 Darko's salary is \$47 000 at the beginning of 2004, and it will increase by \$4000 each year.

a Find a formula for T_n, his salary in the nth year.

b In which year will Darko's salary first be at least twice what it was in 2004?

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Review

8 Miss Yamada begins her new job in 2005 on a salary of \$53 000, and it is increased by 3% each year. In which year will her salary be at least twice her original salary?

9 a Find the value of a \$12 000 investment that has earned 5.25% per annum, compounded monthly, for five years.
b How much interest was earned over the five years?
c What annual rate of simple interest would yield the same amount? Give your answer correct to three significant figures.

10 A Wolfsrudel car depreciates at 12% per annum. Jake has just bought one that is four years old at its depreciated value of \$25 000.
a What will the car's depreciated value be in another four years?
b Find the average loss in value over the next four years.
c What was the new price of the car?
d Find the average loss in value over the four years from when it was new.

11 Katarina has entered a superannuation scheme into which she makes annual contributions of \$8000. The investment earns interest of 7.5% per annum, compounded annually, with contributions made on 1st October each year.
a Show that after 15 years of contributions, the value of Katarina's investment is given by

$$A_{15} = \frac{8000 \times 1.075 \times (1.075^{15} - 1)}{0.075}.$$

b Evaluate A_{15}.
c By how much does A_{15} exceed the total contributions Katarina made over these years?
d Show that after 17 years of contributions, the value A_{17} of the superannuation is more than double Katarina's contributions over the 17 years.

12 Ahmed wishes to retire with superannuation worth half a million dollars in 25 years' time. On 1st August each year he pays a contribution to a scheme that gives interest of 6.6% per annum, compounded annually.
a Let M be the annual contribution. Show that the value of the investment at the end of the nth year is

$$A_n = \frac{M \times 1.066 \times (1.066^n - 1)}{0.066}.$$

b Hence show that the amount of each contribution is \$7852.46.

13 Alonso takes out a mortgage on a flat for \$159 000, at an interest rate of 6.75% per annum, compounded monthly. He agrees to pay the bank \$1415 each month for 15 years.
a Let A_{180} be the balance of the loan after 15 years. Find a series expression for A_{180}.
b Show that $A_{180} = 159000 \times 1.005625^{180} - \dfrac{1415\,(1.005625^{180} - 1)}{0.005625}$.
c Evaluate A_{180}, and hence show that the loan is actually paid out in less than 15 years.
d What monthly payment, correct to the nearest cent, is needed in order to pay off the loan in 15 years?

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14 May-Eliane borrowed \$1.7 million from the bank to buy some machinery for her farm. She agreed to pay the bank \$18 000 per month. The interest rate is 4.5% per annum, compounded monthly, and the loan is to be repaid in 10 years.

a Let A_n be the balance of the loan after n months. Find a series expression for n.

b Hence show that $A_n = 1700000 \times 1.00375^n - \dfrac{18000(1.00375^n - 1)}{0.00375}$.

c Find the amount owing on the loan at the end of the fifth year, and state whether this is more or less than half the amount borrowed.

d Find A_{120}, and hence show that the loan is actually paid out in less than 10 years.

e If it is paid out after n months (that is, put $A_n = 0$), show that $1.00375^n = 1.5484$, and hence that

$$n = \frac{\log_{10} 1.5484}{\log_{10} 1.00375}.$$

f Find how many months early the loan is paid off.

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9 Displaying and interpreting data

Data bombard us constantly from every direction — prices of cars, daily temperatures and rainfall, birth weights of babies, marks in different subjects, how much milk people have for breakfast — and we struggle to make sense of it all. The subject of statistics begins as *descriptive statistics*, which develops various systematic approaches, using tables, graphs and summary statistics, so that we can see the big picture. Very quickly, however, statistics is combined with *probability theory*, inviting the language of *prediction* to be used, and leading perhaps to a discussion of *causation*, which is so fundamental in science.

In the Year 11 book, the foundations of probability theory and discrete probability distributions were developed, ending with some limited discussion of sampling and the way in which probability theory is related to statistical observations. This chapter and the next work the other way around, beginning in Chapter 9 with the raw data and ways of organising raw data. Then Chapter 10 moves to the relationship of that data to probability theory and continuous distributions.

Sections 9A–9C deal with *univariate* data, meaning that there is just one variable involved, such as the prices of cars or the birth weights of babies. Data may also be *bivariate*, such as when we measure the heights and weights of people to investigate how height and weight are related. Sections 9D–9F introduce the possible *correlation* between two statistical variables and the associated *line of best fit*.

Many opportunities are provided for investigations, for serious use of technology, and for possible projects, particularly in Exercise 9F.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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9A Displaying data

Raw data come in small, large and huge unsorted lists, mostly of numbers, but also of categories. It is usually unrewarding to make much sense of such a list just by scanning through it. The first task of *statistics* is to provide tools for the analysis of data — Chapters 10 and 11 of the Year 11 book began this task.

There are three successive stages to analysing raw data.

- Display the raw data in various *tables* and *graphs* (or *charts*) to gain some overview of it, and perhaps some initial insight into what is happening.
- Carry our calculations of *summary statistics*. For univariate data, the most important are the *measures of location*, such as the mean and median, and the *measures of spread*, such as the variance, standard deviation and interquartile range. For bivariate data, we also use the *correlation* and the *line of best fit*.
- Speculate about patterns, predictions, and possible causal factors of the data, using *probability theory* to calculate theoretical probability distributions, followed by tests as to how well the data fit any suggested distribution.

All this may then be followed by suggestions for further experiments, in which the statistician may well be heavily involved in *designing the experiments* that will yield the next sets of raw data.

This chapter is about data — the table, the graphs, and the summary statistics. The next chapter deals with continuous probability distributions, and in particular with the normal distribution. Discrete probability distributions were discussed in Chapter 11 of the Year 11 book.

A review of random variables

Here is a quick review of random variables from Section 11A of the Year 11 book.

1 EXPERIMENTS AND RANDOM VARIABLES

Random variables

- A *deterministic experiment* is an experiment with one possible outcome.
- A *random experiment* is an experiment with more than one possible outcome.
- A *random variable*, usually denoted by an upper-case letter such as X, is the outcome when a random experiment is run, and the various possible outcomes of the experiment are called the *values* of the random variable.

Scores and frequency

- When an experiment is run many times, the outcomes are called *scores*, and the (finite) list of all the scores is called a *sample*.
- The *frequency* of an outcome or value is the number of times it occurs.

Types of random variables

- A random variable may be *numeric* (if its values are numbers) or *categorical*.
- A random variable is called *discrete* if it is numeric and its values can be *listed*, meaning that it is possible to write them down in a sequence $x_1, x_2, x_3, \ldots$

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Recording the country of birth of a person chosen at random in Australia is a categorical variable. Recording the number of overseas countries visited by that person is a numeric variable, which is discrete because its possible values 0, 1, 2, . . . can be listed.

Recording the height of a person is a numeric variable, but if we regard peoples' heights as real numbers rather than rounded measurements, then the variable is not discrete because we cannot list the set of possible values. This is a *continuous variable*, whose precise definition will be given in Chapter 10.

Frequency tables and cumulative frequency tables

The most basic object for organising and inspecting raw data is a *frequency table*, as introduced in Chapter 11 (Year 11). This table of frequencies can be produced digitally using a spreadsheet or database, or by hand using tallies.

When the data are numeric, we can also produce a *cumulative frequency table*, which tells us the number of scores less than or equal to a given score. For example, at the start of Year 7, Cedar Heights High School gave 40 students a spelling test marked out of 10. Here are the raw results, presented as univariate data:

4, 7, 2, 8, 7, 6, 3, 2, 8, 2, 9, 5, 8, 5, 8, 3, 6, 7, 5, 2,
10, 6, 7, 5, 6, 6, 9, 1, 5, 7, 8, 1, 6, 5, 7, 10, 6, 7, 8, 6,

and here are the tallies, the frequencies and the cumulative frequencies.

Mark	0	1	2	3	4	5	6	7	8	9	10	Sum
Tally		ǀǀ	ǀǀǀǀ	ǀǀ	ǀ	卌 ǀ	卌 ǀǀǀ	卌 ǀǀ	卌 ǀ	ǀǀ	ǀǀ	—
Frequency	0	2	4	2	1	6	8	7	6	2	2	40
Cumulative frequency	0	2	6	8	9	15	23	30	36	38	40	—

A quick glance at the cumulative frequencies suggests that 8–9 students have poor spelling, or perhaps they had little experience in earlier years doing tests.

2 CUMULATIVE FREQUENCY

- For numeric data, the *cumulative frequency* is the number of scores that are less than or equal to a given score.
- A frequency distribution table can be extended to a cumulative frequency distribution table by taking the accumulating sums of the frequencies.

If the values of a categorical dataset have been sorted into some sort of meaningful order, then a cumulative frequency table can also be produced — see the Pareto charts later in this section.

Finding the median from the cumulative frequencies

Two questions dominate the discussion of univariate data, as we saw in Chapter 11 (Year 11).

- Measures of location: Where is the centre of the distribution?
- Measures of spread: How spread out are the data?

We begin with the median, which is a measure of location.

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3 THE MEDIAN — A MEASURE OF LOCATION

The *median* of numeric data — with symbol Q_2 meaning second quartile — is the middle score, when the scores are written out in ascending order. More specifically:

- For an odd number of scores, the median is the middle score.
- For an even number of scores, the median is the average of the two middle scores.

A cumulative frequency table makes it easy to pick out the median.

It may be helpful to review how to find the median by writing the scores out in order. Suppose first that there are an odd number of scores:

$$\begin{array}{ccccccccc} 4 & 7 & 8 & 10 & 10 & 11 & 13 & 15 & 21 \\ & & & & \uparrow & & & & \end{array} \qquad \text{(9 scores)}$$

The median of the 9 scores is the middle score. This is the 5th score in the row, which is 10. Now suppose that there are an even number of scores:

$$\begin{array}{cccccccccc} 4 & 7 & 8 & 10 & 10 & 11 & 11 & 13 & 13 & 21 \end{array} \qquad \text{(10 scores)}$$
$$\uparrow$$

There are now two middle scores — the 5th, which is 10, and the 6th, which is 11. The median is their average, which is $10\frac{1}{2}$.

Example 1 **9A**

Explain how to use the cumulative frequency table above of the 40 spelling test marks to calculate the median of the 40 scores.

SOLUTION

There are 40 scores, so the median is the average of the 20th and 21st scores.

From the table of cumulative frequencies, the 15th score is 5, and all scores from the 16th up to the 23rd score are 6.

Hence the 20th and 21st scores are both 6, so the median is 6.

Thus someone with a score of 6 is in the middle of the class.

Displaying categorical data

A glance at some newspapers shows the variety of tables and graphs that are used to display data. Such tables and graphs should be easy to read because one of their purposes is to display data to people who are interested in the subject matter of the table or graph, but may or may not know any statistics.

Tables and graphs may be intended as the first step of an open-minded analysis of an unknown situation. They may be a quick visual illustration of ideas expounded in accompanying text. They may be intended to argue a contentious point. Whatever their purpose, the first job of the reader is to look out for intended or unintended distortions created by the display. Statistics wrongly used routinely lead people astray.

Two examples displaying categorical data are given here — Pareto charts and two-way graphs. The exercises have examples of other displays, such as pie charts and bar charts. In every example, interpretation is vital.

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Pareto charts

Any set of categorical data, or even discrete data, can be represented on a *Pareto chart*. Its main purpose, however, is to identify which problems in a business are most urgent, and it is classified as one of ‘seven basic tools of quality’.

For example, Secure Roofs often arranges a repair, but for various reasons that repair does not take place on the scheduled day, causing loss of income for the company while salary and other expenses still have to be paid. The manager organised the last 200 such failures into six categories, as in the table to the right.

Problem	Frequency
Blackout	4
Employee not arriving	6
Illness of employee	16
Owner not home	64
Rain	88
Truck breakdown	22
Total	200

To construct a *Pareto chart*, first arrange the categories into descending order of frequency — this places the most serious issues first, because they are the first problems that need to be addressed. Then add a cumulative frequency column.

The Pareto chart consists of two graphs drawn together on the same chart:
- a frequency histogram with columns arranged in this descending order,
- a cumulative frequency polygon.

The chart usually has two vertical axes, one on the left and one on the right. On the left are the frequencies, on the right are the percentage frequencies.

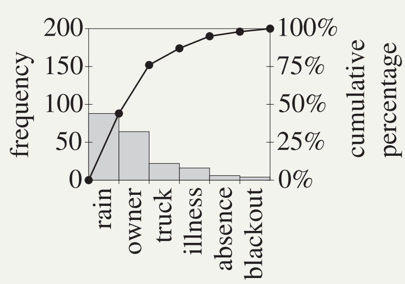
Example 2 9A

a Draw a Pareto chart of the data gathered by Secure Roofs.
b Describe what actions the manager may decide to take using the chart.

SOLUTION

a Here is the cumulative frequency table, with the categories arranged in descending order of frequency, and the cumulative frequencies calculated corresponding to that order. The Pareto chart is on the right.

Problem	Frequency	Cumulative
Rain	88	88
Owner	64	152
Truck	22	174
Illness	16	190
Absence	6	196
Blackout	4	200
Total	200	



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b With this chart, the manager can go through the issues from left to right and attempt to deal with what is causing problems for the business, from the most serious to the least serious.

- First, perhaps, he will first decide only to schedule external roof repairs three days ahead, when forecasts are more reliable.
- Then perhaps he will personally ring each owner two days ahead with a friendly reminder, following up with an SMS the evening before.
- Perhaps he has budgeted for a new truck next year.
- Perhaps he knows that he has little control over the other three issues.

There are no firmly established conventions for drawing the details of a Pareto chart, and the conventions we have chosen are certainly not universal. Here are some details about the chart as we have drawn it.

- The rectangles of the histogram join up with each other.
- The cumulative frequency polygon starts at the left-hand bottom corner of the left-hand rectangle because the initial sum is zero.
  — The next plot is at the right-hand top corner of the first rectangle.
  — Each subsequent plot is above the right-hand side of each rectangle.

We will have more to say about histograms and polygons in Section 9B.

The cumulative frequency polygon in a Pareto chart is always concave down because the issues have been arranged in descending order of frequency. (The term ‘concave down’ is being used more loosely here than in Chapter 8 — here every chord lies ‘below or on the curve’ rather than ‘below the curve’.)

Two-way tables (contingency tables)

A *two-way table* or *contingency table* consists of two or more related frequency tables put together. In its simplest form, it has only four numbers in the table, yet it is surprisingly complicated to interpret. The topic anticipates the discussion of bivariate data in Section 9D, and it involves also conditional probability from Section 10G of Year 11.

A survey asked 200 adults what colour phone cover they preferred. Responses were recorded as black–brown (Dark) or as coloured (Colour), and the gender of the person was also recorded. The resulting *joint frequencies* are tabulated to the right.

	Dark	Colour
Men	38	12
Women	56	94

Let us ask the question, ‘Do men prefer dark colours more than women do?’ If we glanced just at the joint frequencies 38 and 56 under ‘Dark’, we might conclude that women prefer dark colours more than men. The data are deceptive, however, because far more women were questioned than men — for reasons that we have not been told. We need to take this into account.

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Example 3 9A

Explain how to analyse the two-way table above to answer the question, ‘Do men prefer dark colours more than women do?’

SOLUTION

We can find the sums 50 and 150 of the two rows, and the sums 94 and 106 of the two columns. The grand total is 200, which checks the additions. These five numbers are called *marginal frequencies*. The bias of the sample towards women is now clear.

	Dark	Colour	Sum
Men	38	12	50
Women	56	94	150
Sum	94	106	200

Each of the three rows, and each of the three columns, is a frequency table. The last row and the last column are called *marginal distributions*, and the inner two rows and columns are called *conditional distributions* (for reasons explained below).

Now we can answer the question. The proportion of men preferring dark covers is $\frac{38}{50} = 76\%$, and the proportion of women preferring dark covers is $\frac{56}{150} \doteqdot 37\%$, so the survey definitely says that men prefer dark covers more than women do. (Because the survey was biased towards women, the proportion of people surveyed preferring dark covers is $\frac{94}{200} = 47\%$, which is not the mean of 76% and 37%).

Similarly, the proportion of men preferring coloured covers is $\frac{12}{50} = 24\%$, and the proportion of women preferring coloured colours is $\frac{94}{150} \doteqdot 63\%$ (and the proportion of people surveyed preferring coloured covers is $\frac{106}{200} = 53\%$).

Conditional probability in two-way tables

The percentages in the worked example above are probabilities. If we choose a person from the survey at random,

$$P\,(\text{person prefers dark covers}) = \tfrac{94}{200} = 0.47.$$

Section 10G in the Year 11 book introduced *conditional probability*. To find the probability that the person prefers dark covers given that the person is a man (or a woman), we use the *reduced sample space* of 50 men (or 150 women),

$$\begin{aligned} P\,(\text{prefers dark} \mid \text{man}) &= \tfrac{38}{50} = 0.76 \\ P\,(\text{prefers dark} \mid \text{woman}) &= \tfrac{56}{150} \doteqdot 0.37. \end{aligned}$$

We can also find conditional probabilities the other way around. For a person chosen at random from the survey,

$$\begin{aligned} P\,(\text{person is a man}) &= \tfrac{50}{200} = 0.25 \\ P\,(\text{man} \mid \text{prefers dark}) &= \tfrac{38}{94} \doteqdot 0.40 \\ P\,(\text{man} \mid \text{prefers coloured}) &= \tfrac{12}{106} \doteqdot 0.11. \end{aligned}$$

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4 TWO PARTICULAR DATA DISPLAYS

Histograms and polygons, and cumulative histograms and polygons, will be discussed further in the next section. In this section we have particularly looked at:

- *Pareto charts*, which consist of a frequency histogram and a cumulative frequency polygon drawn together, after the categories have been arranged in decreasing order of frequency. They are normally used with categorical data, for the purpose of displaying issues in descending order of importance.
- *Two-way tables* (or *contingency tables*), by which we can investigate whether two variables are related, and make estimates of conditional probability.

The mode and the range

The *mode* is the most popular score, meaning the score with the greatest frequency ('mode' means 'fashion'). It is the simplest measure of location to identify because it is immediately obvious from the frequency table. It is even more obvious from the resulting histogram.

For example, in the frequency table of problems experienced by Secure Roofs, the mode is the problem 'Rain', with a frequency of 88. In the earlier table of spelling test scores, the mode is 6, which happens to coincide with the median, but this is not always the case.

Some frequency tables have two or more scores with the same maximum frequency, and are called *bimodal* or *trimodal* or *multimodal*.

The *range* is only defined for numeric data. It is the difference between the minimum and maximum scores. For example, with the 40 spelling test scores,

$$\text{minimum} = 1, \qquad \text{maximum} = 10, \qquad \text{range} = 10 - 1 = 9.$$

The range is the simplest measure of spread of a dataset.

This meaning of the word 'range' in statistics is quite different from its meaning in the language of functions, where it means the set of output values of a function.

5 MODE (A MEASURE OF LOCATION) AND RANGE (A MEASURE OF SPREAD)

- The *mode* of a dataset is the most popular score, that is, the score with the greatest frequency. A dataset may be *bimodal*, *trimodal* or multimodal.
- The *range* of a dataset is the difference between the minimum and the maximum scores.
- The mode is a measure of location, and the range is a measure of spread.

Exercise 9A

FOUNDATION

1 State whether each random variable is numeric or categorical. If it is numeric, state whether it is discrete or continuous. Comments may be appropriate.

a The favourite day of the week for a person chosen at random in Australia

b Height of Australian professional basketball players

c Age

d Political affiliation

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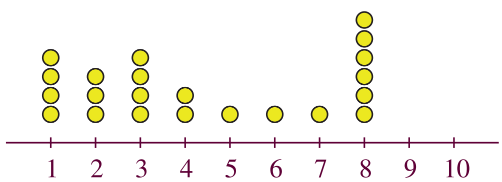
e Colour of a counter drawn from a cup containing 5 red and 6 blue counters.
f Sex (male or female) of a child attending a particular primary school
g Sum of the numbers when two dice are thrown
h Shoe size
i Examination scores

2 Find the median, mode and range of each dataset.
a 10 13 14 14 15 17 18
b 5 7 8 9 10 12 13 15 17
c 3 3 4 5 7 9 10 12 13 15
d 4 4 4 6 6 6 7 7 8 8 9 10
e 4 2 6 4 7 3 4 6 3 4 3 5 2 1 5 7 8
f 2 9 7 6 4 3 2 7 8 9 10 5 4 2 3 6 9 3

3 A shop sells individual cupcakes and keeps a record of how many cupcakes each customer purchases. The results are shown in the dot plot to the right.



a Construct a cumulative frequency table from the data.
b Find the median sales of cupcakes.
c Find the mode of the data.
d The shop intends to pre-package cupcakes to streamline its sales for many customers. Discuss the advantage of selling the cupcakes in packages of **i** 3, **ii** 4, **iii** 8.

4 A basketball coach begins each training session by challenging his star player to shoot as many hoops as he can in two minutes. Over twenty-one sessions he records these results:

4 3 5 4 5 4 6 7 5 6 8 6 6 8 9 10 7 8 9 7 9

a Construct a dot plot of the data. (This can be used as an alternative to a tally.)
b Copy and fill in the following frequency table.

score x	3	4	5	6	7	8	9	10
frequency f								
cumulative								

c What is the median number of hoops shot by the star player?
d In his twenty-second session he shoots 11 hoops in the 2 minutes allowed. What is his new median score?
e Is this frequency table a helpful way of displaying the scores?

5 An operator tracks the number of customers who pay the $4 fee to take his amusement ride on each day of the week. His data is displayed in the following table.

Mon	Tue	Wed	Thu	Fri	Sat	Sun
13	32	35	38	57	75	65

a Draw a bar chart showing the data, with days of the week on the horizontal axis. Use a scale of 1 cm per 10 rides.
b Construct a table of cumulative frequencies, and draw a cumulative bar chart showing the number of rides sold up to and including that day of the week.

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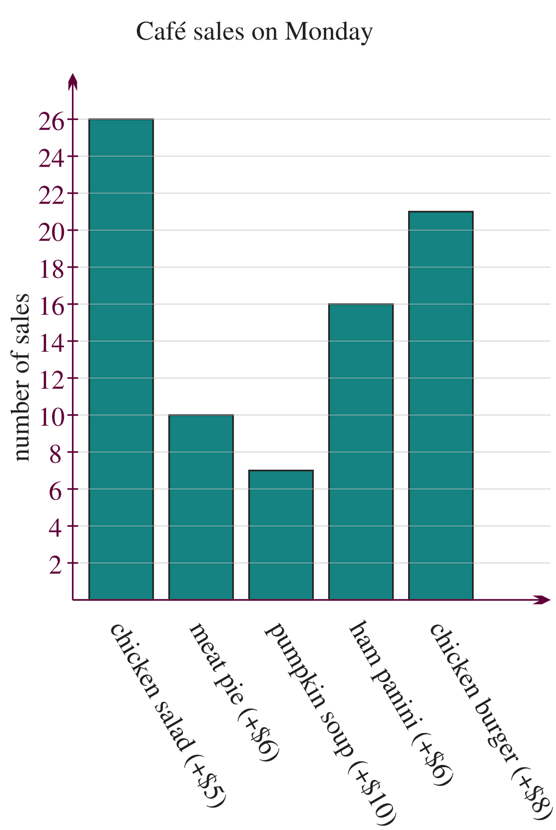
6 A survey of 1000 people in the Netherlands generated the following data shown in a contingency table relating hair and eye colour.

Colour	Blond hair	Red hair	Brown hair	Black hair	Total
Brown eyes	78	4	65	25	172
Blue eyes	324	10	46	9	389
Grey eyes	252	8	47	10	317
Green eyes	74	3	35	10	122
Total	728	25	193	54	1000

a What is the most common hair and eye colour combination in the study?
b What was the least common combination?
c What is the probability that a blond-haired person also has blue eyes?
d What is the probability that a black-haired person also has blue eyes?
e What percentage of people with black hair had brown eyes?
f What percentage of people with dark hair (brown or black) had brown eyes?
g What percentage of people with light hair (blond or red) had lighter coloured eyes (blue, grey or green)?
h Does there appear to be a link between hair colour and eye colour?
i This study was carried out from a particular genetic population. Is it likely that similar results hold everywhere?

7 A café tracks its sales on a certain Monday to find what menu items are selling. It has a limited menu: chicken salad, meat pie, pumpkin soup, ham panini, and chicken burger. The café's results are shown on the graph to the right. The graph also records the markup (profit) on each choice, shown as (+$ markup).

a What is the total number of menu orders for the café on Monday?
b Determine what percentage the sale of each menu option is of the total.
c What is the profit, in dollars, obtained from each of the choices on the menu on the Monday?
d What is the total profit, in dollars, for the café on the day?
e The café has a policy to drop from the menu any choice with sales below 10%. Give two reasons why they should not drop the pumpkin soup from the menu.



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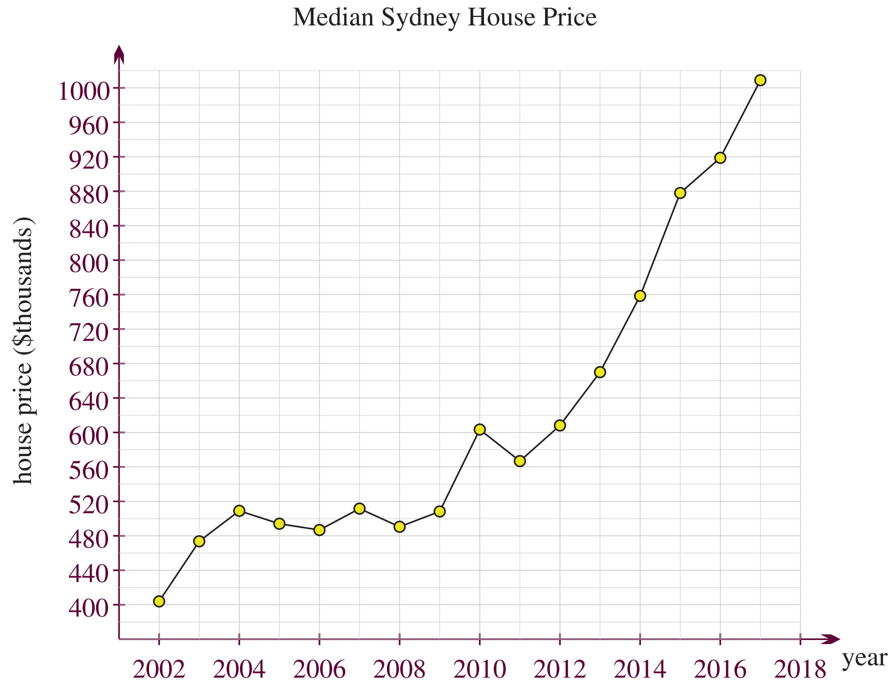
8 The median house price in Sydney from 2002 to 2017 is recorded in the line graph.



a Write down the median price of a Sydney house in 2002 and 2017, correct to the nearest ten thousand dollars.

b What is the percentage increase in house prices in Sydney from 2002 to 2017?

c What was the average increase in house price per year over this time?

d If this trend continues, what do you predict the median house price will be in 2030?

e What year saw the greatest increase in house prices?

f What year saw the greatest decrease in house prices? How much did prices change?

9 Owners of *The Happy Pet* boarding house for pets whose owners are out of town are looking to expand their business. A business analyst has asked them to keep track of the last 400 pets staying at their boarding house to determine what kind of pets they will need to accommodate in their planned expansion. This information is displayed in the Pareto chart to the right.

a What percentage of the last 400 pets at the boarding house were dogs? How many dogs was this?

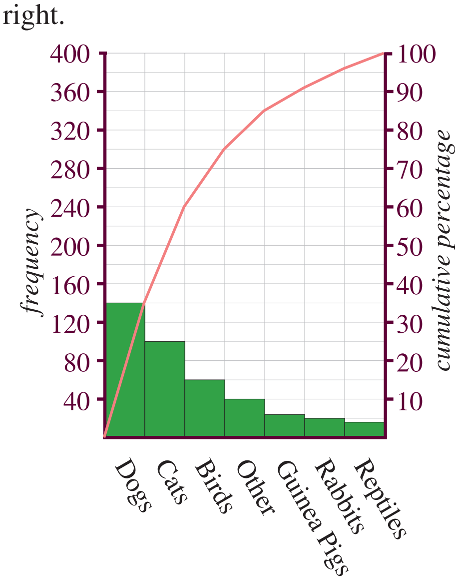
b Rabbits and guinea pigs can stay in the same type of cage. What percentage of the last 400 pets staying were rabbits or guinea pigs?

c To maximise business profits, the owners decide to concentrate on the three most common pets. What percentage of the last 400 pets were one of these three?

d What percentage of pets fell into one of the three least common categories?

e Comment on the size of the 'others' category.

f What other matters should the owners take into account, besides the numbers of pets looking for boarding?



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DEVELOPMENT

10 A school is investigating reasons why students arrive late to class. Students who are late are asked to state a reason. The reasons given by the last 100 students are recorded in the table to the right.

a Construct a table with the categories ordered by decreasing frequency. Add a cumulative frequency column, which will also be the cumulative frequency percentage because there were 100 students in the survey.

b Construct a Pareto chart of the data. Use a scale of 1 cm per 10 units on each vertical scale.

c Explain why the cumulative frequency polygon of a Pareto chart will always be concave down.

d What percentage of the reasons are included in the first three categories?

e Comment on how the school could work to reduce the first three causes of tardiness.

Reasons late to class	Frequency
Didn't hear the bell	20
Held back in last class	27
Cancelled music lesson	10
Lost bag	5
Late back from lunch	3
Summons by senior teacher	2
Medical	3
Distance from last class	20
Other	10

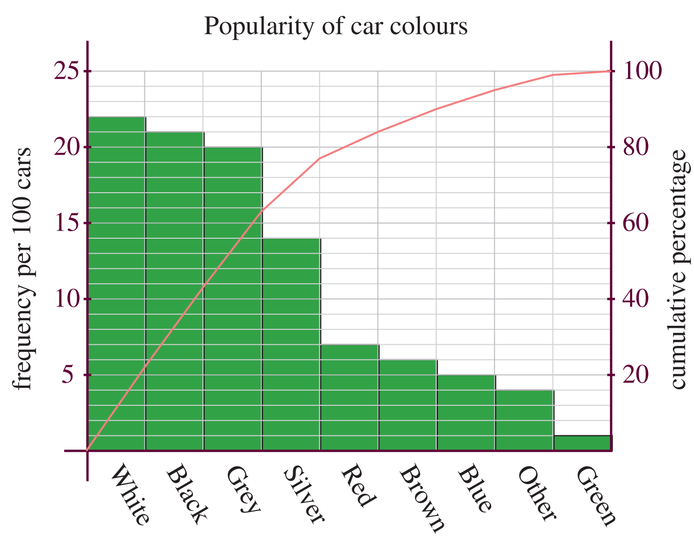
11 The colours of cars on the road are recorded in the following Pareto chart.

a What percentage of cars are brown?

b What percentage of cars are of one of the three most common colours, white, black or grey?

c How many cars are not one of the seven most popular colours?

d This Pareto chart uses different scales on the two axes — is this confusing?



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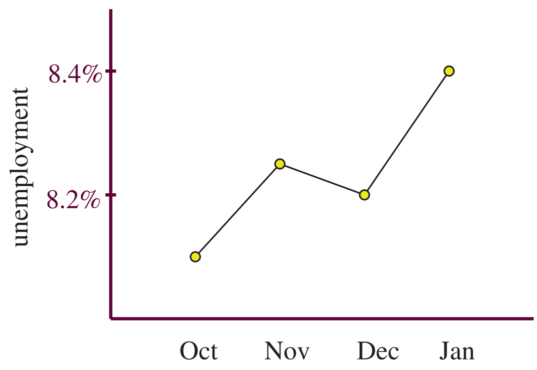
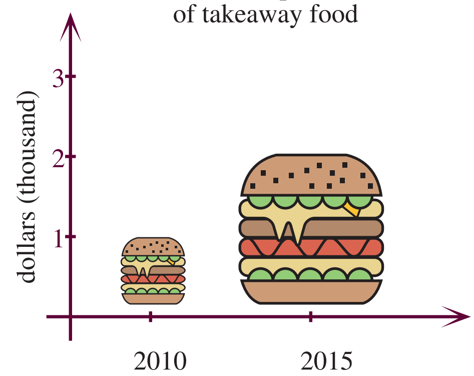
12 Statistics can easily be misinterpreted or deliberately used to mislead.

a Unemployment under new government



The graph copied above was published in a newspaper. Can you suggest why it might be a misleading use of statistics and graphing?

b Increase in purchase of takeaway food



A study has been commissioned into the consumption of fast food. The graph to the right shows one of the results of this study. Discuss why this graph may be misleading if used in a local newspaper or television advertisement.

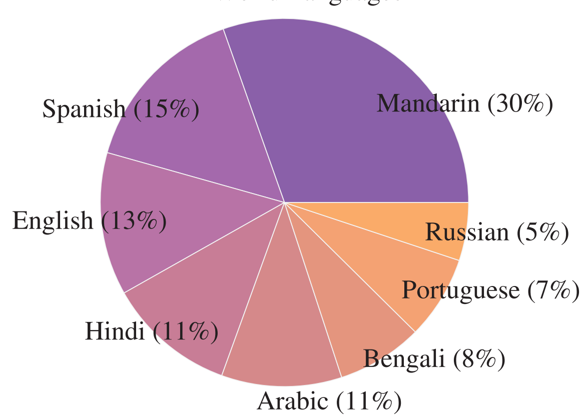
c A survey was designed to collect data to investigate the following scenarios. Comment on problems with the design of the experiment.

i A study is carried out to determine people's musical tastes. People are asked to fill in an online survey, and the results are then collated.

ii To investigate the growth in medical costs, a community group accesses data from their local hospital. The growth in total expenses at the hospital over time is displayed in a line graph.

13 In 2019, around 40% of the world's 7.7 billion population were first-language speakers of one of eight different languages. The sector chart to the right shows the breakdown of first-language speakers as a percentage of the top eight.

Breakdown of the Eight Major World Languages



a What percentage of the 40% speak one of the three most common languages as a first language?

b How many people speak one of these eight as their first language?

c How many people in the world speak Mandarin as their first language?

d Around what percentage of the world's population speak English as their first language?

e Is this chart useful to a school deciding what languages to offer or a student deciding what language to learn?

14 The mean temperature for each month in Sydney (Observatory Hill) and Dubbo (airport) at 3pm is recorded in degrees Celsius on a radial chart.

For example, to read the mean temperatures for February, look at the radius marked 'Feb' — the mean temperature in Dubbo is about 30°C, and in Sydney it is about 25°C. The mean temperatures in January are on the adjacent radius, and each pair of values are joined by an unbroken interval for Sydney, and a broken interval for Dubbo.

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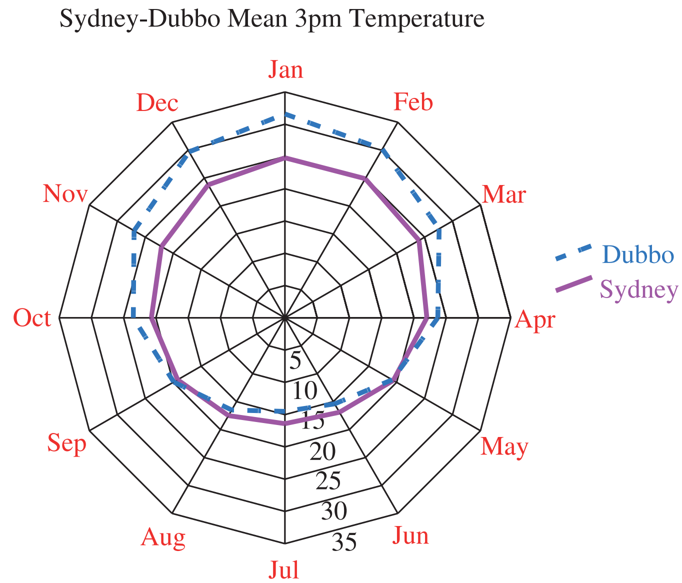
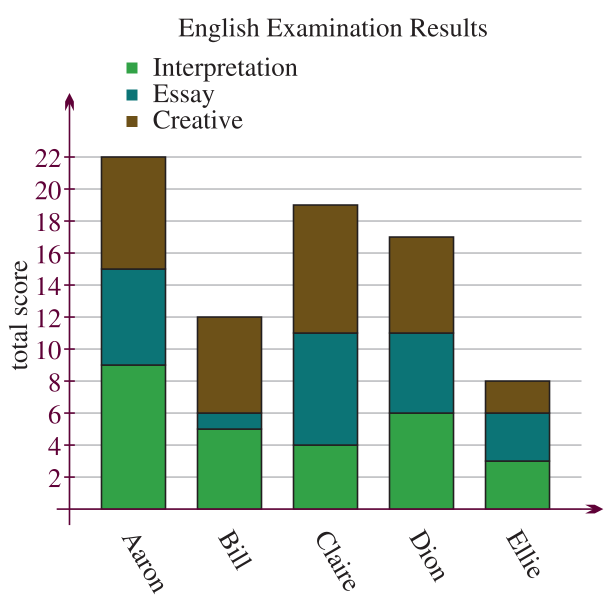
Using the 3pm temperature as a measure of the temperature in Sydney and Dubbo:



- **a** What is the 3pm temperature in Dubbo in
  - **i** July,
  - **ii** December?
- **b** What is the 3pm temperature in Sydney in
  - **i** August,
  - **ii** March?
- **c** What is the maximum 3pm mean temperature difference between the two locations, and in what month does it occur?
- **d** In which months is the mean 3pm temperature in Sydney and Dubbo the same?
- **e** In which months is Dubbo at least 5 degrees hotter than Sydney?
- **f** Are there any months where Dubbo is colder than Sydney?
- **g** Is this a good style of chart to display the data? Would there be a better type of chart to use?
- **h** Why do you think that the designer of this chart chose to use dotted and solid lines, rather than just the cyan and magenta colours, to distinguish the two temperature lines?

15 A small class is working with pupils who have difficulty with English. After students had been in the class for some time, examination results were recorded in the stacked bar chart below. The examination consisted of three sections: an interpretative exercise, an essay on a novel studied in class, and a creative writing exercise. Each section was awarded a mark out of 10.



- **a** What was the examination out of in total?
- **b** What were the highest and lowest scores, as a percentage?
- **c** Identify any of the three sections for which a pupil may need additional help.
- **d** What percentage did Claire receive in Interpretation?
- **e** What percentage did Dion receive in Essay?
- **f** Students who receive 55% overall and at least 50% in each section leave this class. Who will be leaving the class following this examination?

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CHALLENGE

16 The following two-way table shows the highest non-school qualification received by Australians aged between 15–64, with a break-down by age. The entries are percentages.

	15–24	25–34	35–44	45–54	55–64	Total 15–64
Postgraduate	0.4	6.4	6.2	5.3	4.7	4.6
Graduate diploma/cert	0.1	1.9	2.5	3.2	3.1	2.1
Bachelor degree	7.4	26.8	21.2	15.1	13.3	17.0
Adv diploma/diploma	4.2	9.6	11.1	11.7	9.1	9.1
Certificate I-IV	14.0	21.7	24.1	23.3	22.9	21.1
Other	2.1	3.0	2.7	2.5	2.8	2.6
None	71.8	30.6	32.2	38.9	44.1	43.5
Total (percent)	100%	100%	100%	100%	100%	100%
Total (thousands)	3122.5	3215.9	3118.1	2969.3	2422.3	14848.1

a *There is a greater percentage of people in the 25–34 age group with postgraduate degrees, compared with the 55–64 aged group. This suggests that more people are gaining postgraduate degrees in more recent times.* Comment on whether this is a reasonable interpretation of the data.

b What it the probability that an Australian chosen at random from the 45–54 age group has a post-school qualification?

c What it the probability that an Australian aged 15–64 chosen at random from the group with a post-school qualification lies in the age group 45–54?

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9B Grouped data and histograms

The main purpose of organising data into tables and graphs is to see the data as a whole. When a table has too many rows or columns, or a graph has too much detail, such an overview is much more difficult. The usual approach in such situations is to *group* the data. This reduces the number of rows or columns on the tables, and reduces the amount of clutter on the graphs.

Grouping data

Here are the heights of 100 people in centimetres, from a file detailing individuals of all ages from the !Kung people of the Kalahari desert. The underlying random variable here is continuous (assuming that heights are real numbers), but height cannot be measured correct to more than a few significant figures.

151.765	139.7	136.525	156.845	145.415	163.83	149.225	168.91	147.955	165.1
154.305	151.13	144.78	149.9	150.495	163.195	157.48	121.92	105.41	86.36
161.29	156.21	129.54	109.22	146.4	148.59	147.32	137.16	125.73	114.3
147.955	161.925	146.05	146.05	142.875	142.875	147.955	160.655	151.765	171.45
147.32	147.955	144.78	121.92	128.905	97.79	154.305	143.51	146.7	157.48
127	110.49	97.79	165.735	152.4	141.605	158.8	155.575	164.465	151.765
161.29	154.305	145.415	145.415	152.4	163.83	144.145	129.54	129.54	153.67
142.875	146.05	167.005	91.44	165.735	149.86	147.955	137.795	154.94	161.925
147.955	113.665	159.385	148.59	136.525	158.115	144.78	156.845	179.07	118.745
170.18	146.05	147.32	113.03	162.56	133.985	152.4	160.02	149.86	142.875

The data seem to be given correct to 0.005 cm, which seems less than one can reliably measure, and the trailing zeroes that we normally insert are missing — always question the credibility of raw data. Perhaps heights were recorded in inches, then converted to centimetres. We have grouped the data in 10 cm intervals because that results in 10 classes, which is a good number for seeing the big picture. Here is the table of frequencies and cumulative frequencies.

interval	class centre	frequency	cumulative frequency
80–90	85	1	1
90–100	95	3	4
100–110	105	2	6
110–120	115	5	11
120–130	125	8	19
130–140	135	6	25
140–150	145	34	59
150–160	155	22	81
160–170	165	16	97
170–180	175	3	100

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The *class centre* on each row is the midpoint of the interval used in the grouping. The table could just as well have been written with rows instead of columns, and rows have been used later in the calculation of the mean and variance.

This makes the distribution of heights reasonably clear. A frequency distribution table based on the raw data, however, would be practically useless, because the frequency of almost every score is just 1 (and the histograms drawn on the next page would also be useless).

Grouping data is a form of rounding. It is useful because it allows us to see the big picture, but it always involves ignoring information, and the summary statistics for the grouped data will only be an approximation of the summary statistics of the raw data. Never discard the original data.

For example, the median height is the average of the 50th and 51st heights. For the grouped data, both these heights are 145 cm (taking the class centre as the measurement), whereas if we work with the original data, the median is 147.955. Similarly, the range of the grouped data is $175 - 85 = 90$, but the range of the raw data is $179.07 - 86.36 = 92.71$.

When the underlying random variable is continuous, any data are already grouped by the rounding that all measurement involves. When those measurements involve several significant figures, as they do here, further grouping is usually required when displaying the data.

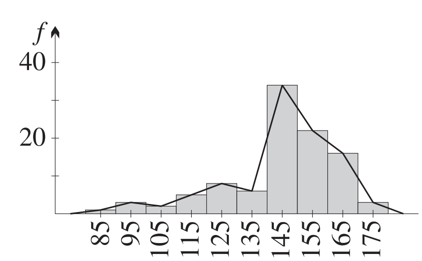
6 GROUPING DATA

- Numeric data, whether discrete or continuous, may be *grouped* so that the resulting tables and graphs give a clearer overview of the data.
- The grouping involves *intervals of equal width* and *class centres*.
- Grouping involves ignoring information. This may or may not be an issue.
- Data on a boundary should be treated consistently, and the treatment noted.

With a continuous variable, there may be data on the boundary, because data are always rounded. You can place these scores in the lower interval — this is consistent with the cumulative frequency convention in Section 9A. Or you can place them in the upper interval — this is also standard practice. But be consistent, and make a note about it if any boundary data actually occurred. This didn't happen with the data above.

Frequency histograms and frequency polygons

Whether or not the data have been grouped, a *frequency histogram* is the most basic way of displaying data in a graph. The diagram to the right shows the histogram of the grouped heights in the previous frequency table.



The *frequency polygon* has been added to the display. The two graphs can be drawn separately or together, and only one may be needed

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Some guidelines when drawing a frequency histogram

- For ungrouped data, each rectangle is centred on the value. For grouped data, each rectangle is centred on the class centre.
- The rectangles join up with no gaps.
- As a practical concern, too many columns in a histogram can make it difficult to interpret. Coarser grouping is the best solution here.
- The subintervals on the horizontal axis are often called *bins*.

Some guidelines when drawing a frequency polygon

- The plotted points are at the centre of the top of each rectangle.
- Join the plotted points with intervals.
- On the left, start the polygon on the horizontal axis, at the previous value or class centre.
- On the right, end the polygon on the horizontal axis, at the next value or class centre.

A question from the graph: Always ask questions about the data display.

- Why is the data so *skewed to the left*, with such low frequencies? Were children's heights included?

Histograms with discrete data

Histograms are designed for data from a continuous variable, which is the main reason why the rectangles should join up. When they are used for discrete data, be aware that the rectangles still have width, that they still join up (according to most conventions), and that they are centred on the values (or on the class centres for grouped data). These conventions will routinely involve numbers such as half-integers that are not possible values of the random variable.

Example 4 — 9B

Section 9A prepared a frequency table for 40 spelling test marks in Year 7.

Mark x	1	2	3	4	5	6	7	8	9	10
Frequency f	2	4	2	1	6	8	7	6	2	2

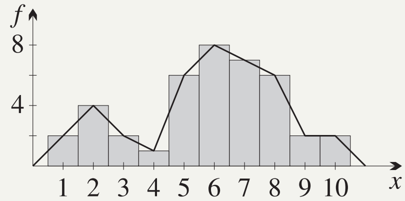
a Group the data by pairing the marks 1–2, 3–4, ...
b Draw a histogram and frequency polygon for the original data.
c Draw a histogram and frequency polygon for the grouped data.
d Comment on what the various displays have shown.

SOLUTION

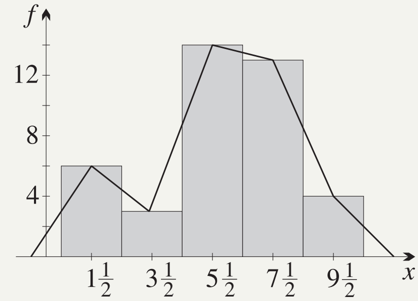
a

Interval	1–2	3–4	5–6	7–8	9–10
Class centre x	$1\frac{1}{2}$	$3\frac{1}{2}$	$5\frac{1}{2}$	$7\frac{1}{2}$	$9\frac{1}{2}$
Frequency f	6	3	14	13	4

b



c

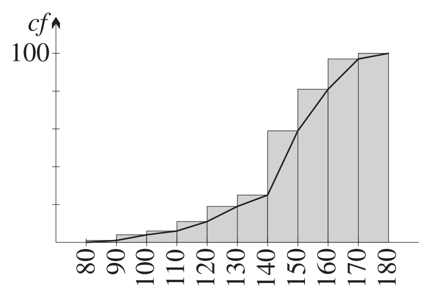


d The histogram of the grouped data perhaps makes it a little clearer that a significant group have difficulties either with spelling or with tests.

Note: The frequency polygon starts and finishes on the horizontal axis at the previous or next value or class centre. For the original data in part **b**, it starts at 0 and finishes at 11. For the grouped data, it starts at $-\frac{1}{2}$ and finishes at $11\frac{1}{2}$.

Cumulative frequency histograms and polygons (ogives)

A *cumulative frequency histogram* is drawn using the same procedures as for the earlier frequency histogram. The *cumulative frequency polygon*, also called an *ogive*, is drawn slightly differently corresponding to its cumulative nature.



We have drawn the two graphs together for the grouped table of heights at the start of this section, but again, each can be drawn separately.

Some guidelines when drawing a cumulative frequency histogram

- The rectangles of the frequency histogram are piled on top of each other to form the cumulative frequency histogram.
- The height of the last rectangle is the total size of the sample.

Some guidelines when drawing a cumulative frequency polygon

- The polygon starts at zero at the bottom left-hand corner of the first rectangle, when no scores have yet been accumulated.
- It passes through the top right-hand corner of each rectangle because it plots the scores less than or equal to the upper bound of the class interval.
- It finishes at the top right-hand corner of the last rectangle, and its height there equals the total size of the sample.

Example 5 9B

Section 9A also prepared a cumulative frequency table for the spelling test marks.

Mark x	1	2	3	4	5	6	7	8	9	10
Frequency f	2	4	2	1	6	8	7	6	2	2
Cumulative	2	6	8	9	15	23	30	36	38	40

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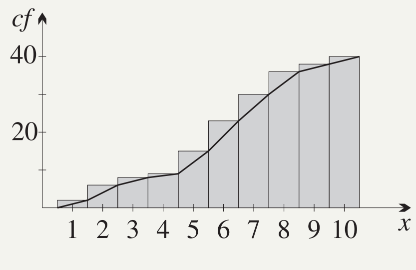
a Group the data by pairing the marks, adding the cumulative frequency.
b Draw a cumulative frequency histogram and ogive for the data.
c Draw a cumulative frequency histogram and ogive for the grouped data.
d Use the cumulative frequency tables to calculate the median using the original data and the grouped data, and compare them.

SOLUTION

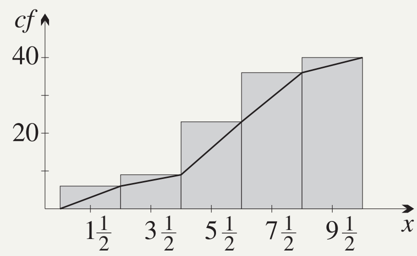
a

Interval	1–2	3–4	5–6	7–8	9–10
Class centre x	$1\frac{1}{2}$	$3\frac{1}{2}$	$5\frac{1}{2}$	$7\frac{1}{2}$	$9\frac{1}{2}$
Frequency f	6	3	14	13	4
Cumulative	6	9	23	36	40

b



c



d We calculated before that the median was the average of the 20th and 21st scores, which the cumulative frequencies tell us are both 6, so the median is 6.

From the grouped data, the class centres of the 20th and 21st scores are both $5\frac{1}{2}$, so the median is $5\frac{1}{2}$. Such discrepancies are normal after grouping.

7 HISTOGRAMS AND POLYGONS

- The rectangles of the *frequency histogram* and the *cumulative frequency histogram* join up. For ungrouped data, they are centred on the value, and for grouped data, they are centred on the class centre.
- The *frequency polygon* passes through the centres of the rectangles.
- On the left and right, the frequency polygon starts and finishes on the horizontal axis, centred on the previous or next value or class interval.
- The *cumulative frequency polygon* or *ogive* starts at the bottom left corner of the first rectangle.
- The ogive passes through the right-hand top corner of each rectangle.

The mean

In Year 11, we calculated the mean and variance of a sample in Section 11D, and we used relative frequencies in those calculations because our attention was on estimating probabilities from relative frequencies. This chapter, however, is about data, so in reviewing mean and variance, we will instead use formulae based only on the frequencies. The review also takes grouped data into account.

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Recall that we use the symbols $\bar{x}$ and s for the mean and standard deviation when we are dealing with a sample, that is, with data, as we are in this chapter. When dealing with a population or a theoretical distribution, we use the symbol μ or $\mathrm{E}(X)$ for the mean, and σ for the standard deviation.

Here again is the grouped frequency table of heights, arranged this time in rows, but it could as well be in columns.

class centre x	85	95	105	115	125	135	145	155	165	175	Sum
frequency f	1	3	2	5	8	6	34	22	16	3	100
$x \times f$	85	285	210	575	1000	810	4930	3410	2640	525	14470

The sum of the scores is the sum of the products $x \times f =$ score $\times$ frequency, except that with grouped data we use the class centres. This sum is 14470. The number of scores is the sum of the frequencies, which is 100. Hence

$$\text{mean} = \frac{\sum xf}{n} = \frac{14470}{100} = 144.70 \text{ cm.} \qquad \left(\sum xf \text{ means add all the products } xf.\right)$$

The authors also calculated the mean of the heights without grouping and obtained 145.352 cm. Thus the grouping reduced the mean by about $6\frac{1}{2}$ mm.

In Section 11D (Year 11) we wrote the mean as a weighted mean of the scores, weighting each score by its relative frequency using the formula $\bar{x} = \sum xf_r$, where f_r is the relative frequency. The relative frequencies are obtained by dividing each frequency by the number n of scores, that is, $f_r = \frac{f}{n}$, so the formula used above and the earlier formula are the same. To prove this, start with the earlier formula,

$$\bar{x} = \sum xf_r = \sum \left(x \times \frac{f}{n}\right) = \frac{\sum xf}{n}.$$

The variance and standard deviation

As explained in the Year 11 book, the variance and standard deviation are measures of spread, meaning that they measure how spread out the data are away from the centre. The standard deviation is the square root of the variance. Thus the variance has symbol s^2 for a sample and σ^2 or $\mathrm{Var}(X)$ for a population or a theoretical distribution. In Section 11D of the Year 11 book, we developed two formulae for the variance of data,

$$s^2 = \sum (x - \bar{x})^2 f_r \qquad \text{and} \qquad s^2 = \sum x^2 f_r - \bar{x}^2.$$

Substituting $f_r = \frac{f}{n}$ for the relative frequency, these two formulae become

$$\begin{aligned} s^2 &= \sum (x - \bar{x})^2 f_r \\ &= \frac{\sum (x - \bar{x})^2 f}{n} \end{aligned} \qquad \text{and} \qquad \begin{aligned} s^2 &= \sum x^2 f_r - \bar{x}^2 \\ &= \frac{\sum x^2 f}{n} - \bar{x}^2. \end{aligned}$$

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It is rare with data that the mean $\bar{x}$ is a round number, so the second form is the recommended form for calculation. The first form, however, makes it clear that the variance is a measure of spread — we are looking at deviations from the mean, squaring them so that they are all positive, then taking their weighted mean, weighted according to the frequencies.

8 MEAN AND STANDARD DEVIATION OF A SAMPLE

Suppose that data have been organised into a frequency table with scores x and frequencies f.

- The mean is $\bar{x} = \dfrac{\sum xf}{n}$ (where $\sum$ says take the sum over the distribution).
- The variance is $s^2 = \dfrac{\sum x^2 f}{n} - \bar{x}^2$.
- The standard deviation is the square root of the variance, and has the same units as the scores.
- With grouped data, use the class centres rather than the scores.

The actual definition of the variance is $s^2 = \dfrac{\sum (x - \bar{x})^2 f}{n}$. This formula is usually less suitable for calculation, but it makes it clear that we are taking the average of the squares of the deviations from the mean.

Here are the calculations for the variance and standard deviations of the heights.

x	85	95	105	115	125	135	145	155	165	175	Sum
f	1	3	2	5	8	6	34	22	16	3	100
xf	85	285	210	575	1000	810	4930	3410	2640	525	14470
x^2f	7225	27075	22050	66125	125000	109350	714850	528550	435600	91875	2127700

Thus $\bar{x} = \dfrac{\sum xf}{n}$

$= \dfrac{14470}{100}$

$= 144.7$ cm,

and $s^2 = \dfrac{\sum x^2 f}{n} - \bar{x}^2$

$= \dfrac{2127700}{100} - 144.7^2$

$= 338.91$,

$s \doteqdot 18.41$ cm.

The authors used technology with the raw scores, and obtained $s^2 = 325.5407$ and $s \doteqdot 18.04$ cm. Grouping has produced results that are slightly different.

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Example 6 9B

a Find the mean, variance and standard deviation of the Year 7 spelling test marks in worked Example 4.
b Calculate them again using the grouped data.

SOLUTION

a

x	1	2	3	4	5	6	7	8	9	10	Sum
f	2	4	2	1	6	8	7	6	2	2	40
$x \times f$	2	8	6	4	30	48	49	48	18	20	233
$x^2 \times f$	2	16	18	16	150	288	343	384	162	200	1579

Hence $\bar{x} = \dfrac{\sum xf}{n} = \dfrac{233}{40} = 5.825,$

and $s^2 = \dfrac{\sum x^2 f}{n} - \bar{x}^2 = \dfrac{1579}{40} - 5.825^2 = 5.544375,$

$s \doteqdot 2.355.$

b

Interval	1–2	3–4	5–6	7–8	9–10	Sum
Class centre x	1.5	3.5	5.5	7.5	9.5	—
Frequency f	6	3	14	13	4	40
$x \times f$	9	10.5	77	97.5	38	232
$x^2 \times f$	13.5	36.75	423.5	731.25	361	1566

Hence $\bar{x} = \dfrac{\sum xf}{n} = \dfrac{232}{40} = 5.8,$

and $s^2 = \dfrac{\sum x^2 f}{n} - \bar{x}^2 = \dfrac{1566}{40} - 5.8^2 = 5.51,$

$s \doteqdot 2.347.$

Again, the differences between the results of parts **a** and **b** arise from grouping.

Challenge — a correction factor for the sample variance

These qualifications may not be required.

We mentioned at the end of Section 11D in the Year 11 book that when we know the theoretical or the population mean μ, and we are sampling to find the variance, there is no problem with the formulae for the sample variance. When, however, we are sampling both to find the mean and to find the variance, then the sample mean will drift very slightly towards the sample results, with the effect that the sample variance will tend to be slightly smaller than it should be.

The standard solution is to multiply the sample variance by a correction factor $\dfrac{n}{n-1}$, where n is the size of the sample. Thus in the example using 100 heights, we were using a sample mean rather than a theoretical or population mean, so the correction factor is $\frac{100}{99}$.

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Using the correction factor would yield

$$\bar{x} = 144.7 \text{ cm} \quad \text{(as before)} \qquad s^2 = 338.91 \times \tfrac{100}{99} \doteqdot 342.3333 \qquad s = \sqrt{s^2} \doteqdot 18.50 \text{ cm.}$$

The larger the size n of the sample, the less difference the correction makes.

On the calculator, a button $\boxed{\sigma_n}$ or something equivalent does not apply this correction, and seems to be all that is required in this course. A button labelled $\boxed{\sigma_{n-1}}$ or equivalent applies the correction factor.

Currently in Excel 365, the function `STDEV.S` (S for 'sample') applies the correction, and the function `STDEV.P` (P for 'population') does not apply the correction, but earlier versions did things differently.

Like so many things in statistics, the distinction between a sample and a population is not straightforward. The Year 7 spelling test marks may be regarded as the record of spelling on that day from every member of the cohort — then the marks are a population. They may also be regarded as one set of estimates of an underlying random variable, 'spelling ability of each child', to be augmented by next week's spelling test and more in later weeks — then the marks are a sample. Scaling software, which massages results into aggregates and positions and ranks, tends to regard marks as a population. A classroom teacher, who is watching students learn and develop and is very aware that the choice and design of test questions are arbitrary, tends to regard marks as a sample.

A possible project

Systematic testing of the validity of this correction factor by taking a large number of samples from a known population could be developed into a project.

Perhaps theoretical probability distributions could also be considered, perhaps both discrete and continuous (as developed in Chapter 10).

Exercise 9B

FOUNDATION

1 a Copy and complete the following table to determine the mean and standard deviation for the data. (The mean is a whole number, so calculations using this formula will be straightforward.)

x	3	5	6	7	8	9	10	Sum
f	1	1	1	3	2	1	1	
$x \times f$								
$(x - \bar{x})^2$								
$(x - \bar{x})^2 f$								

$$\text{Mean} = \bar{x} = \frac{\sum xf}{n} = \ldots$$

$$\text{Variance} = s^2 = \frac{\sum (x - \bar{x})^2 f}{n} = \ldots$$

b Repeat the calculation using the alternative formula for the variance.

x	3	5	6	7	8	9	10	Sum
f	1	1	1	3	2	1	1	
$x \times f$								
$x^2 \times f$								

$$\text{Mean} = \bar{x} = \frac{\sum xf}{n} = \ldots$$

$$\text{Variance} = s^2 = \frac{\sum x^2 f}{n} - \bar{x}^2 = \ldots$$

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2 Use a table as in Question 1 to calculate manually the mean and standard deviation of each dataset. Use the two forms for the variance in different parts — the means here are all whole numbers. Give your answers correct to two decimal places.

a 12, 14, 16, 17, 19, 21, 22, 23
b 2, 3, 3, 3, 6, 6, 7, 8, 8, 8, 9, 9, 10, 10, 13
c 40, 49, 50, 50, 51, 54, 57, 57, 57, 60, 65, 70
d 7, 8, 9, 9, 10, 10, 10, 10, 11, 11, 11, 11, 11, 12, 12, 12, 12, 13, 13, 14, 15

3 Use your calculator to find the mean and standard deviation of the each dataset.

a 3, 7, 9, 10, 3, 4, 6, 8, 13, 6, 5, 12
b 4, 4, 4, 4, 5, 5, 7, 8, 8, 8
c 3.2, 3.6, 1.3, 2.4, 1.9, 4.1, 3.5, 4.1, 3.9, 2.3
d 34, 45, 23, 56, 34, 53, 23, 43, 37, 55, 52, 41, 43, 51, 57, 39

DEVELOPMENT

4 A census was carried out on the houses in Short Street to determine the number x of people in each household. The results are recorded in the frequency table below.

The population in this question is all the houses in the street. Because the data here are determined by a census of the whole population, statisticians use the symbol μ for the mean of the population (called the *population mean*) and the symbol σ for the standard deviation of the population (called the *population standard deviation*).

x	0	1	2	3	4	5	6	7	8
f	1	5	6	7	8	3	3	0	1

a How many houses are there in Short Street?
b Calculate the mean μ and standard deviation σ of the data.
c Group the data into the classes 0–2, 3–5 and 6–8 and construct a grouped frequency table.
d Calculate the mean and standard deviation of this grouped data.
e Why do your results from part **b** and **d** differ?

5 Xiomi recorded her time to get to work each day. Her results in minutes were:

22 30 23.5 27 25 21.5 39 30 32.5 33 35.5 37 42 22 23.5 27 29.5 23 34

a Write the data out in order, and determine the median.
b Group the data into classes by completing the following table.

class	20–24	24–28	28–32	32–36	36–40	40–44
class centre						
frequency						
cumulative						

c Find the median of the grouped data. Does it agree with your answer to part **a**?
d Draw a frequency histogram and polygon on the same chart.
e Draw a cumulative frequency histogram and polygon on the same chart.

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6 In a class experiment, students measured their heights. The results in centimetres were:

155 152 165 162 170 168 165 162 166 154 158 159
163 166 164 164 159 157 163 154 166 158 159 163

a Display the data in a frequency table.
b Calculate the median of the dataset.
c Why would it not be helpful to graph the data without first grouping it into classes?
d Group the data into the intervals 150–154, 154–158, 158–162, 162–166, 166–170 and display your results in a grouped frequency table. Include any scores on a boundaries in the lower group, thus x is in the group 150–154 if $150 < x \le 154$.
e Calculate the median of the dataset from this grouped frequency table.
f Display your grouped data on a histogram with a frequency polygon joining the centres. Construct a cumulative frequency histogram and ogive — remember that the ogive starts at the bottom left-corner of the first rectangle and passes through the right-hand top corner of each rectangle.
g Trace the line at frequency 12 (50%) until it meets the ogive, and check whether this agrees with your answer for the median of the grouped data in part **e**.
h Construct a cumulative frequency histogram and ogive of the *ungrouped* data.
i Compare your grouped and ungrouped cumulative histograms in parts **f** and **h**. How similar are the graphs? Contrast the differences between the histogram of the grouped data in part **f** and what you would expect the histogram of the ungrouped data to look like (not drawn).
j Confirm that the line at frequency 12 meets your ungrouped ogive to give the same median as in part **g**.

CHALLENGE

7 The Australian Bureau of Statistics (ABS) surveys important medical and physical information for the Australian population. According to their 1995 survey, the mean weight of a male over 18 was 82 kg, with a standard deviation of 13.6 kg.

The data were gathered from a sample of the whole population, but the quoted standard deviation has been calculated using the population standard deviation formula. Thus the *population variance* should be multiplied by the *correction factor* $\dfrac{n}{n-1}$, as discussed in the notes at the end of this section, to give the sample variance. This corrects for the drift of the calculated variance towards the sample results and away from the true population standard deviation.

a What would be the corrected sample standard deviation, assuming that the sample only surveyed:
i 10 people,   ii 100 people,   iii 1000 people?
b Actually, the ABS survey involved 10 000 people. What percentage change would the correction factor make to the standard deviation? Give your answer correct to three decimal places.

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9C Quartiles and interquartile range

For numeric data, the spread of the data around the median can also be identified by the quartiles and the interquartile range.

Upper and lower quartiles

Write the scores in increasing order. The lower quartile, the median and the upper quartile attempt to divide this list into four equal parts.

An odd number of scores: Reliable Appliances sell toasters. Here are the numbers of toasters that they sold in each of 15 successive weeks.

19 16 18 15 16 19 17 21 16 16 20 18 30 19 21

First we write them out in a list in increasing order,

$$15\quad 16\quad 16\quad 16\quad 16\quad 17\quad 18\quad \underset{\uparrow}{18}\quad 19\quad 19\quad 19\quad 20\quad 21\quad 21\quad 30$$

The number of scores in the list is 15, which is odd. The median Q_2 is the 8th score 18. Now divide the list into two sublists of 7, with the median in the middle,

$$15\ 16\ 16\ \underset{\uparrow}{16}\ 16\ 17\ 18\qquad 18\qquad 19\ 19\ 19\ \underset{\uparrow}{20}\ 21\ 21\ 30$$

The lower quartile Q_1 is the median of the left-hand list, which is 16, and the upper quartile Q_3 is the median of the right-hand list, which is 20. In summary:

$$Q_1 = 16 \qquad \text{and} \qquad Q_2 = 18 \qquad \text{and} \qquad Q_3 = 20.$$

An even number of scores: On the 16th week they sold 21 toasters, making 16 scores, which is even. The list can now be written out in two equal sublists,

$$15\ 16\ 16\ 16\ 16\ 17\ 18\ 18\qquad 19\ 19\ 19\ 20\ 21\ 21\ 21\ 30$$

The median Q_2 is the average of the 8th and 9th scores, which is $18\frac{1}{2}$. The lower quartile Q_1 is the median of the left-hand list, which is 16, and the upper quartile Q_3 is the median of the right-hand list, which is $20\frac{1}{2}$. In summary:

$$Q_1 = 16 \qquad \text{and} \qquad Q_2 = 18\tfrac{1}{2} \qquad \text{and} \qquad Q_3 = 20\tfrac{1}{2}.$$

Interquartile range: The difference $Q_3 - Q_1$ between the upper and lower quartiles is called the *interquartile range* or IQR. It is the range of the middle 50% of the marks. Thus in the two examples above, the interquartile ranges are

$$\text{IQR} = Q_3 - Q_1 = 20 - 16 = 4 \qquad \text{and} \qquad \text{IQR} = Q_3 - Q_1 = 20\tfrac{1}{2} - 16 = 4\tfrac{1}{2}.$$

9 QUARTILES AND INTERQUARTILE RANGE

Suppose that a set of scores is arranged in increasing order.

An odd number of scores:
- Omit the middle score, thus separating the list into two sublists of equal size.
- The *lower* or *first quartile* Q_1 is the median of the left-hand list,
- The *upper* or *third quartile* Q_3 is the median of the right-hand list.

An even number of scores:
- Separate the list into two sublists of equal size.
- The *lower* or *first quartile* Q_1 is the median of the left-hand list,
- The *upper* or *third quartile* Q_3 is the median of the right-hand list.

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The interquartile range — a measure of spread:
- The *interquartile range* or IQR is the difference $Q_3 - Q_1$.
- The interquartile range is a measure of spread.

Quartiles, like medians, are easily calculated from the cumulative frequency table. They can also be calculated for theoretical distributions, as in worked Example 7.

The five-number summary

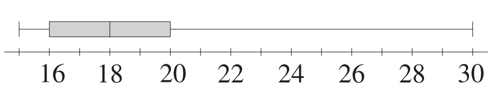
A data distribution can be usefully summarised in a five-number summary, which can then be displayed in a box-and-whisker plot. This summary presents the median, the quartiles, and the two extreme scores, that is,
- the minimum score (sometimes written as Q_0)
- the lower quartile Q_1
- the median Q_2
- the upper quartile Q_3
- the maximum score (sometimes written as Q_4).

Thus the five-number summaries of the two sets of weekly toaster-sale scores are:
15 weekly toaster-sale scores: 15, 16, 18, 20, 30
16 weekly toaster-sale scores: 15, 16, $18\frac{1}{2}$, $20\frac{1}{2}$, 30

Notice that the range is the difference between the first and last numbers, and the interquartile range is the difference between the second and second-last numbers. The symbols Q_0 and Q_4 are convenient, but are not standard notation.

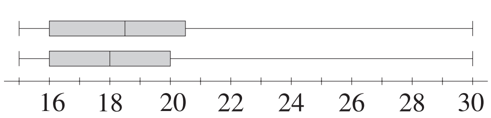
Box-and-whisker plots (box plots)

To the right is the *box-and-whisker plot* (also called *box plot*) of the 15 weekly toaster-sales scores. It displays the five-number summary in a clear diagram.



- The *box* extends from the lower quartile $Q_1 = 16$ to the upper quartile $Q_3 = 20$. Its length is the IQR.
- The vertical line within the box is the median $Q_2 = 18$.
- The *whiskers* extend left to the least score 15, and right to the greatest score 30, showing the range.

The second diagram shows the box plot of the 15 toaster-sale scores underneath, and above it the box plot of the subsequent 16 toaster-sale scores. It is a *parallel box plot*.



The addition of the one extra score 17 has increased the median and the upper quartile. The point here is to see immediately any significant differences in the overall picture.

Outliers

When scientists do experiments, they often end up with data that they secretly wish that they had not collected — perhaps they were not expecting these results, or the data do not fit their theories, or the data are ‘clearly’ the result of an experimental error, or they were ‘wrongly recorded’ by a research assistant, or they just look strange and the scientist doesn’t know what to do about them. These pieces of data are scores that lie a long way from most of the data collected and they consequently muck up the patterns that the other data create.

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Such pieces of data are called *outliers*, and the inclusion or exclusion of these outliers from datasets causes serious arguments wherever statistics is used. And time and time again, outliers have been an indication of an inadequate theory that needed to be reformulated.

There are no generally accepted criteria for outliers, just a few contradictory rules that people argue about. In this course, we shall usually take a criterion based on quartiles and the interquartile range IQR. We usually take an outlier to be a score that lies

more than 1.5 $\times$ IQR below Q_1 or more than 1.5 $\times$ IQR above Q_3.

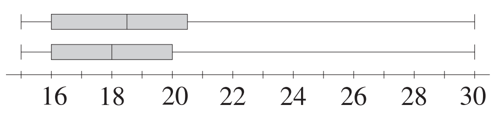
This criterion is very simple and can usually be calculated mentally.

- One problem with this test is that the gap between the suspect outlier and the next score or scores is also an important criterion, and *analysis of such gaps is missing from this test*.
- A second problem is that when there are very large datasets, we expect from the normal laws of probability to have scores many IQRs from the quartiles, but *the size of the sample is missing from this test*.

In the end, nothing can replace careful attention to the scores themselves. In most circumstances, outliers should be left in the dataset, but probably should be displayed differently and commented on.

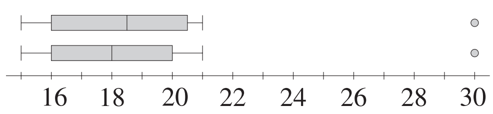
Outliers in box-and-whisker plots

Box-and-whisker plots are easily adapted to show outliers. You will have seen in both the sets of toaster-sales data that one week 30 toasters were sold, whereas the next highest score is 21.



The score of 30 is an outlier by two criteria — it is well separated from the other scores, and (in the case of the 15 scores) it is 10 above the upper quartile, which is 2.5 times the interquartile range.

We have therefore redrawn the parallel box plots to the right, with the right-hand whisker stopping at 20, and a circle placed at 30 as a code for the outlier. Outliers are often indicated on a box plot in this or a similar way.



Explaining outliers is most important — there was a toaster sale that week.

Example 7 9C

a Throw a die until a six occurs, and record the number n of throws needed. Do this 50 times, or use the results of class members, or simulate it using random numbers or a spreadsheet.

b Construct a frequency table and cumulative frequency table.

c Find the median, the quartiles, and the interquartile range.

d Draw a box plot and discuss any outliers.

e Let X be the number of tosses required to get a six. Explain why

$$P(X = 1) = \tfrac{1}{6}, \quad P(X = 2) = \tfrac{5}{6} \times \tfrac{1}{6}, \quad P(X = 3) = \left(\tfrac{5}{6}\right)^2 \times \tfrac{1}{6}, \quad \ldots,$$

and use GP theory to prove that the limiting sum of these probabilities is 1.

f Copy and complete the following cumulative discrete probability table, giving each value correct to three decimal places.

n	1	2	3	4	5	6	7	8	9	10	...
$P(X = n)$	0.167										
$P(X \le n)$	0.167										

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g Hence find the theoretical mean, quartiles and interquartile ranges. What values of n are classified as outliers according to the IQR criterion? Then sketch a box plot of the theoretical results.

h Explain why both the box plot of the data and the box plot of the theoretical distribution are unsymmetric.

SOLUTION

a These results were obtained by simulation using random numbers based on part **e**. Note the gaps in the last two results.

b

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	22	26
f	4	5	9	4	5	2	3	1	2	3	1	2	1	1	1	2	1	1	1	1
Cml	4	9	18	22	27	29	32	33	35	38	39	41	42	43	44	46	47	48	49	50

c The median Q_2 is the average of the 25th and 26th scores, which is 5. The lower quartile Q_1 is the 13th score, which is 3. The upper quartile Q_3 is the 38th score, which is 10.

d The IQR is $10 - 3 = 7$, so

$$Q_3 + 1.5 \times \text{IQR} = 10 + 10\tfrac{1}{2} = 20\tfrac{1}{2}.$$

The IQR criterion for outliers classifies the last two scores 22 and 26 as outliers. This is also a common-sense classification, because these last two scores are well separated from the other scores.

e Using standard probability techniques from Chapter 10 of the Year 11 book,

$$P(X = 1) = \tfrac{1}{6} = \left(\tfrac{5}{6}\right)^0 \times \tfrac{1}{6},$$

$$P(X = 2) = P(\text{TH}) = \tfrac{5}{6} \times \tfrac{1}{6} = \left(\tfrac{5}{6}\right)^1 \times \tfrac{1}{6},$$

$$P(X = 3) = P(\text{TTH}) = \tfrac{5}{6} \times \tfrac{5}{6} \times \tfrac{1}{6} = \left(\tfrac{5}{6}\right)^2 \times \tfrac{1}{6},$$

$$\ldots$$

This is a GP with first term $a = \frac{1}{6}$ and ratio $r = \frac{5}{6}$,

so $$S_\infty = \frac{a}{1 - r} = \tfrac{1}{6} \div \left(1 - \tfrac{5}{6}\right) = 1.$$

f

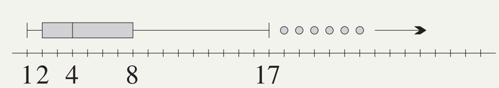
n	1	2	3	4	5	6	7	8	9	10
$P(X = n)$	0.167	0.139	0.116	0.096	0.080	0.067	0.056	0.047	0.039	0.032
$P(X \le n)$	0.167	0.306	0.421	0.518	0.598	0.665	0.721	0.767	0.806	0.838

g The median is $Q_2 = 4$, because it is the first score whose cumulative probability is at least $\frac{1}{2}$.

The lower quartile is $Q_1 = 2$ because it is the first score whose cumulative probability is $\frac{1}{4}$.

The upper quartile is $Q_3 = 8$ because it is the first score whose cumulative probability is $\frac{3}{4}$.

Hence the IQR is $8 - 2 = 6$. Thus the IQR criterion classifies as an outlier every score greater than $Q_3 + 1.5 \times \text{IQR} = 8 + 9 = 17$.



h The frequencies or probabilities are bunched up on the left and spread out or *skewed* on the right.

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Median and quartiles vs mean and standard deviation

We now have two families of summary statistics:

- Mean, variance and standard deviation.
- Median, quartiles and interquartile range.

Earlier in this section, we used 15 weeks of toaster sales and found that

$$Q_1 = 16, \qquad Q_2 = 18, \qquad Q_3 = 20, \qquad \text{IQR} = 4.$$

After further calculation, the mean is 18.73 and the standard deviation is 3.53.

Now suppose that we replace the outlier 30 by 21, the highest of the other scores. The mean, quartiles and interquartile range do not change, but the mean changes to 18.13, and the standard deviation changes dramatically to 2.00. That is, the standard deviation may be very sensitive to outliers, the mean less so.

If Reliable Appliances is tallying up its cash flow and profits, they would use mean and standard deviation. If they were looking at their marketing and want to study toaster purchases outside exceptional situations such as sales, they would use the median and the quartiles, which are *robust* to outliers.

House prices are another much-discussed example. The prices of homes are stretched out or *skewed* on the right by very expensive homes, so that the median is a more useful measure of prices of ordinary homes than the mean. The upper quartile is also not affected by those very expensive homes, so that interquartile range may be a better measure of the spread than the standard deviation.

Summary statistics review

The summary statistics discussed in Sections 9A–9C were:

10 SUMMARY STATISTICS

Measures of location

Mode, median, mean

Measures of spread

Range, interquartile range, variance, standard deviation,

The five-number summary

minimum, first quartile Q_1, median Q_2, third quartile Q_3, maximum

A *box-and-whisker plot* is constructed from the five-number summary.

We have described data several times as being skewed. Recall from earlier years that data are skewed in the direction of the tail, not the peak.

- Skewed to the right, or positively skewed, means that there is bigger tail on the right-hand side.
- Skewed to the left, or negatively skewed, means that there is bigger tail on the left-hand side.

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Exercise 9C

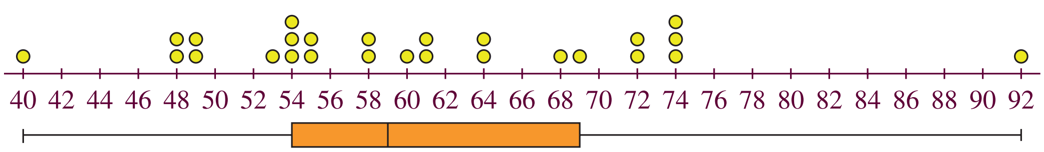
FOUNDATION

1 For each dataset, calculate the three measures of central tendency (mean, median and mode), and calculate the range.
  - a 4, 8, 5, 2, 9, 12, 8
  - b 12, 23, 18, 30, 24, 29, 19, 22, 25, 12
  - c 7, 6, 2, 5, 7, 3, 4, 5, 7, 6
  - d 54, 62, 73, 57, 61, 61, 54, 66, 73

2 Find the middle quartiles Q_1, Q_2, Q_3 and the interquartile range IQR $= Q_3 - Q_1$ for each dataset.
  - a 3, 7, 9, 13, 14, 17, 20
  - b 8, 12, 13, 17, 20, 24, 27, 31
  - c 4, 7, 8, 9, 11, 14, 17, 19, 20
  - d 2, 5, 7, 10, 13, 17
  - e 2, 4, 5, 7, 12, 13, 14
  - f 8, 10, 12, 15, 17, 21, 22
  - g 3, 4, 6, 7, 9, 11, 13, 14, 18
  - h 9, 12, 13, 15, 18, 21

3 Find the IQR of each unordered dataset.
  - a 15, 10, 12, 19, 1, 17, 13, 6, 2
  - b 1, 11, 14, 9, 0, 4
  - c 12, 7, 9, 11, 13, 2, 9
  - d 6, 3, 2, 12, 0, 6, 8, 4
  - e 7, 11, 7, 5, 10, 7
  - f 2, 9, 5, 4, 5, 9, 12
  - g 8, 3, 4, 1, 12, 2, 4, 11
  - h 10, 4, 5, 18, 11, 13, 2, 9, 7

DEVELOPMENT

4 

The combined box plot and dot diagram above shows the exam scores for a small cohort of 26 students.
  - a Use your intuition to identify any *outliers*, thinking here of outliers as scores that are a long way from the rest of the data.
  - b Write down the five-number summary statistics: the minimum value, the lower quartiles Q_1, the median Q_2, the upper quartile Q_3, and the maximum value.
  - c The IQR criterion identifies *outliers* as those values less than $Q_1 - 1.5 \times$ IQR or more than $Q_3 + 1.5 \times$ IQR. Use this criterion to identify any outliers.
  - d Do your answers to part **a** and **c** agree?
  - e Recalculate the three quartiles Q_1, Q_2, Q_3 if we:
    - i omit 40 only,
    - ii omit 92 only,
    - iii omit both 40 and 92.
  - f Do the quartiles and the IQR change much when outliers are removed?
  - g Calculate the mean and standard deviation of the dataset correct to one decimal place:
    - i with all values,
    - ii without 40,
    - iii without 92,
    - iv without 40 and 92.
  - h What is the change in the standard deviation in part **g iv** as a result of removing the outlying values, as a percentage of the standard deviation in part **g i**?

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5 **a** For each dataset, calculate the interquartile range and identify any outliers using the IQR criterion. Then draw a combined dot plot and box-and-whisker plot.

i 3, 5, 6, 7, 7, 9, 18
ii 9, 10, 10, 12, 12, 13, 18
iii 9, 10, 10, 12, 12, 13, 18, 19
iv 1, 3, 7, 9, 9, 9, 10, 10, 11, 12
v 1, 3, 8, 9, 9, 9, 10, 10, 11, 12
vi 5, 7, 7, 8, 8, 8, 8, 9
vii 5, 7, 7, 8, 8, 8, 8, 9, 9
viii 1, 1, 2, 2, 3, 10, 12, 12, 12, 12, 13, 13, 13, 13, 13, 14, 14, 15, 16, 22, 22, 23, 24, 25

b By reference to your answers in part **a**, comment on how well this definition of outliers seems to agree with your intuitive notion of an outlier as an extreme value.

c The dataset in part **a i** contained an outlier. Redraw the box-and-whisker plot for part **a i** with the whiskers excluding the outlier, and with the outlier shown as a single small circle on your diagram.

6 **a** Draw up a frequency table and hence determine the mean and standard deviation for the dataset
16, 12, 13, 14, 14, 14, 12, 16, 15, 24

b Construct a dot plot. Do any of the values seem to be *outliers*, that is isolated points, a long way from the rest of the data?

c Confirm that the IQR criterion of an outlier agrees with your instinct in part **b**.

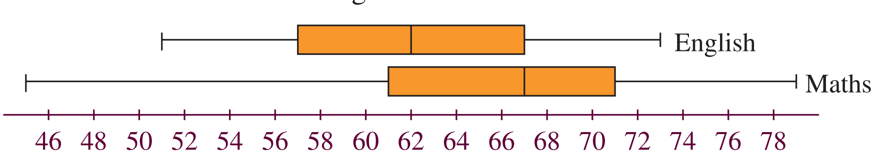
d Remove the outlier you should have found in parts **b**–**c**, and recalculate the mean and standard deviation.

e Has the outlier had a big effect on the mean and standard deviation? Use your tabular calculations to explain why this might be so.

f Has the outlier had a big influence on the median or IQR?

g The interquartile range and the standard deviation are both measures of spread. When might the interquartile range be a better measure than the standard deviation?

7 The results of a class in an English task and a mathematics task are shown in the parallel box plot below.



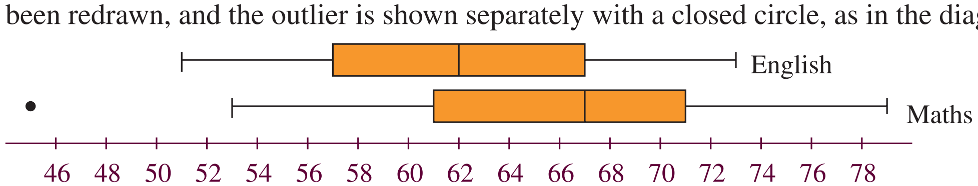
a Emily's result was in the bottom half of the English cohort. What can be said about her mark?

b What percentage of the students fall below 67 in mathematics?

c Comment on the spread of the two distributions by comparing the medians of the two sets of scores, and then by comparing the entire distributions.

d Xavier gets 66 in both English and mathematics. Is this a more impressive result in English or in mathematics, and why?

e Closer analysis shows that there is actually an outlier in the mathematics results. The box plots have been redrawn, and the outlier is shown separately with a closed circle, as in the diagram below:



i What was the outlier score in mathematics?

ii Compare again the spread of the bottom 25% of the students in mathematics and English, this time with the outlier excluded.

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8 An English class completes a writing and a speaking task. The results are displayed in the back-to-back stem-and-leaf plot below.

Writing task		Speaking task
5	3	7
	4	
6 6 5 2	5	1 3 4 7 8 8
9 9 8 8 7 7 4 4	6	3 5 5 6 6 7 8
5 5 4 4 2 1	7	1 1 3 5 7
	8	
1	9	3

- **a** Genjo got 35 in the writing task. What was his score in the speaking task?
- **b** For the writing task:
  - **i** calculate the mean, median and range,
  - **ii** calculate the interquartile range and determine any outliers.
- **c** For the speaking task:
  - **i** calculate the mean, median and range,
  - **ii** calculate the interquartile range and determine any outliers.
- **d** Which set of results was more impressive?

CHALLENGE

9 [An investigation that could become a project] In this section we have given two tests for outliers — the IQR criterion, and graphing the data and applying common sense with particular attention to any gaps. The discussion of outliers is an important subject and books have been written on the subject. Interested students may like to investigate this further. Questions may include the following:

- **a** Some statisticians label scores that are below $Q1 - 3 \times \text{IQR}$ or above $Q3 + 3 \times \text{IQR}$ as *extreme outliers*. Scores below $Q1 - 1.5 \times \text{IQR}$ or above $Q3 + 1.5 \times \text{IQR}$ that are not extreme outliers are called *mild outliers*. What questions in this exercise have included extreme outliers? Generate some data with both and note the difference on a dot plot and a box-and-whisker plot.
- **b** What is the effect of having two or more outliers in a distribution — can they mask each other's existence from an IQR criterion test? Generate datasets to explore this. Include the possibility of multiple outliers at both ends of the dataset, or only on one end.
- **c** Outliers can also be defined by measuring the number of standard deviations from the mean. Calculate the number of standard deviations that the outliers in this exercise are from the mean, and decide if this could be developed into a reasonable criterion. Start with the datasets in Question **5**.

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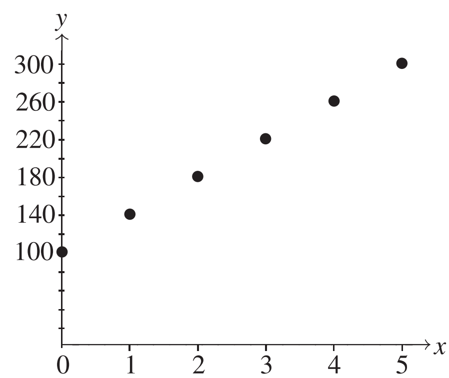
9D Bivariate data

For bivariate data, two further summary statistics are used — correlation, and the line of best fit. This section takes an intuitive approach to correlation and line of best fit, using displays and drawing the line of best fit by eye.

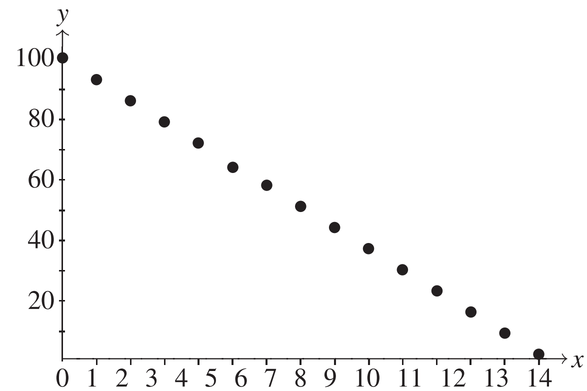
Section 9E (a Challenge section) will introduce formulae for correlation and the least squares version of the line of best fit. The last section, Section 9F, will use technology to analyse and display bivariate data.

Variables can be correlated without being related by a function

This course mostly concerns functions, where a variable y is completely determined by a variable x. Here are two linear functions, one with positive gradient, and one with negative gradient.

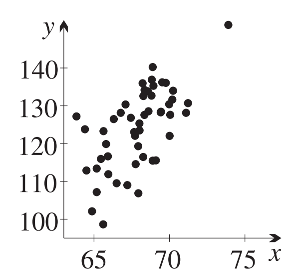


An electrician charges \$100 to visit a home, then \$40 for each power point. His total fee y for installing x power points in a home is $y = 100 + 40x$.



One hundred old cars were dumped in a park, and the council is removing 7 per day. The number y of cars remaining after x days is $y = 100 - 7x$.

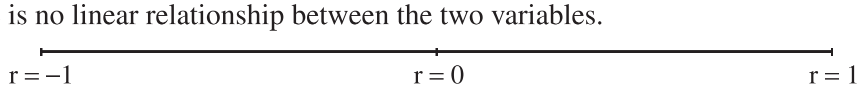
Now think about the relationship between the heights and weights of people. The *scatterplot* to the right plots the height x inches and weights y pounds of a group of people, with x as the *independent variable* and y as the *dependent variable*.



The cluster of dots is spread out because people of the same height don't all have the same weight, that is, y is not completely determined by x. And yet there seems to be more than a random relationship between the heights and weight of the people in the group, because it looks as if taller people tend to be heavier.

Correlation is the word for such a statistical relationship. In this section we will judge correlation by eye after the points are plotted on a scatterplot, then in Sections 9E–9F we will introduce a summary statistic called *Pearson's correlation coefficient* r. When y is determined by x, as in the two graphs above,

- $r = 1$, if they lie on a straight line and increase together,
- $r = -1$, if they lie on a straight line with one increasing, the other decreasing.
- $r = 0$, if there is no linear relationship between the two variables.



Correlations of 1 and -1 are called *perfect correlation*. We will perform some calculations of Pearson's correlation coefficient by hand in Section 9E, and by technology in Section 9F.

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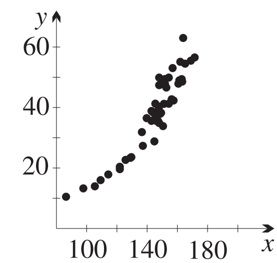
The choice of which variable is independent and which is dependent is sometimes obvious, but is sometimes rather arbitrary, and in either case the choice may well be a matter for a scientist rather than a mathematician. A choice must be made, however, for the methods of these sections to work.

Heights and weights — a positive correlation

The raw data of the heights of people in Section 9B came also with the weights of those people.

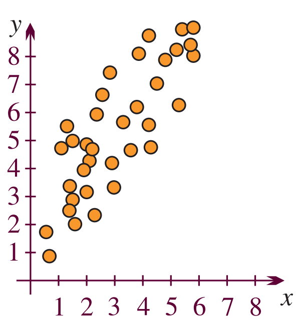


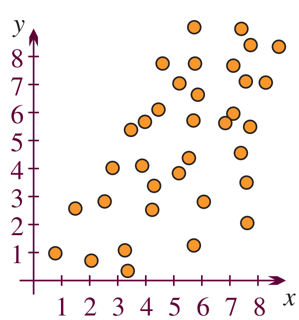
Here is the *scatterplot* of the first 50 pairs of measurements, with the height x cm taken as the independent variable on the horizontal axis, and the weight y kg taken as the dependent variable on the vertical axis.

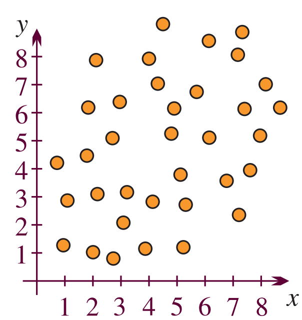
Each dot in the chart represents the height and weight of one person. Notice that the horizontal scale starts at 80 kg, not at zero.

This gives a visual demonstration that the weight in that group is very closely related to the height. We can classify this visually as a *strong correlation* — that the actual value of the correlation here is about 0.928, which is regarded as very strong (the correlation of the previous scatterplot is 0.64 — still strong, according to many published guides). The correlation is also *positive*, meaning that both variables increase together and that the slope of the cluster is positive.

Here is a rough guide to positive correlations — it can only be a rough guide:


strong


moderate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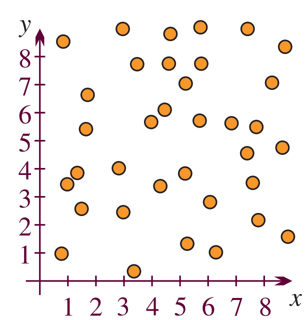
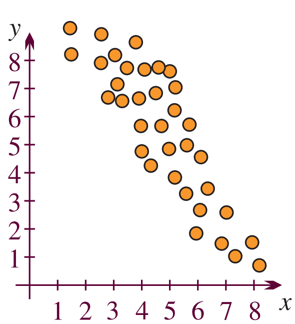

none

And here is a rough guide to negative correlations: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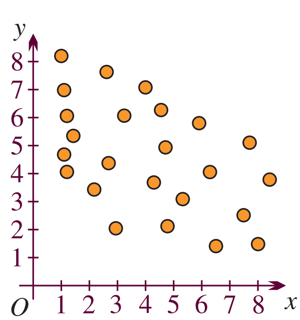
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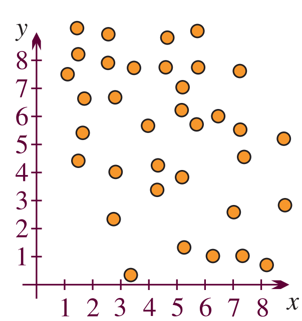
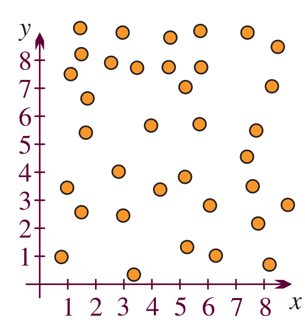

week


none

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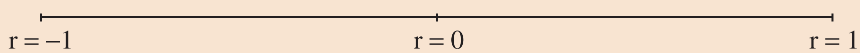
Non-linear correlation

Data are always more complicated than whatever is said about them. The cluster of dots in the heights–weights scatterplot above has a definite curve in it. Perhaps it should be tested not against a straight line, which is all that we will do in this chapter, but against a curve. Should that curve be quadratic, or exponential? Perhaps it should be cubic, because the volume of similar figures is proportional to the cube of the height. Such reasoning shows how prediction and causation are always involved in any discussion about data.

11 CORRELATION

Bivariate data means data in the form of ordered pairs. Suppose that we have identified an *independent variable* x and a *dependent variable* y in bivariate data.

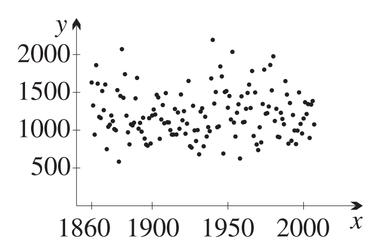
- A *scatterplot* graphs all these pairs on the one coordinate plane.
- *Linear correlation* occurs when the dots in the scatterplot tend to cluster in a shape vaguely like a line.
- Linear correlation is usually measured by *Pearson's correlation coefficient*, or simply the *correlation*, which is a real number r in the interval $-1 \le r \le 1$.



- *Non-linear correlation* occurs when the dots in the scatterplot tend to cluster in a shape vaguely like some other curve.

Rainfall over the years — no correlation

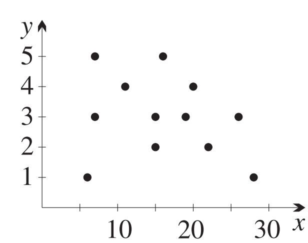
The Bureau of Meteorology has data for rainfall in Sydney for every day of every year since about 1858. We can download and massage that data to find the annual rainfall y in millimetres each year x from 1860–2007 — the scatterplot is drawn to the right.



There is no (linear) correlation. Pearson's correlation coefficient is -0.014, which is virtually zero. But look at the dots! You can see the drought years and the flood years.

Callers put on hold — a negative correlation

Customers of a large technology company often ring about problems, and are put on hold for short or long periods. At the end of their call they are asked to rank the company 1 (very dissatisfied) to 5 (very satisfied). Some preliminary test data on the waiting time x minutes and the rank y were taken, with the following results.



x	7	15	22	11	20	15	7	28	6	16	26	19
y	5	2	2	4	4	3	3	1	1	5	3	3

The correlation here is negative because the cluster slopes backwards, and it can be roughly characterised as weak (the calculated correlation is about -0.26).

The point $(6, 1)$ could be identified as an *outlier*. It has a big effect. Place your finger over this single dot. The correlation now looks moderate (the calculated value is about -0.57). Correlation is very sensitive to outliers when there are few values — both the visual impression and the calculated value.

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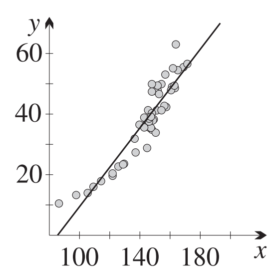
12 POSITIVE, NEGATIVE AND ZERO CORRELATION

- A positive slope in the cluster corresponds to $0 < r \leq 1$, and is called *positive correlation*. A correlation of $r = 1$ means *perfect positive correlation*, that is, y is a linear function of x with some positive gradient.
- A negative slope in the cluster corresponds to $-1 \leq r < 0$, and is called *negative correlation*. A correlation of $r = -1$ means *perfect negative correlation*, that is, y is a linear function of x with some negative gradient.
- $r = 0$ means that there is no linear correlation between the variables.

One qualification about zero correlation is needed. A horizontal line of points, or a cluster in the vague shape of a horizontal line, both have zero correlation.

The line of best fit

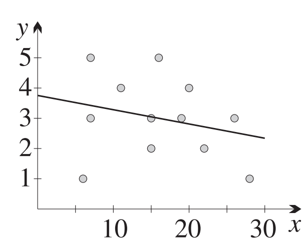
The purpose of correlation is to test whether the cluster of results are suggesting a line, sloping forwards or backwards. The line that they best cluster around is called the *line of best fit* or the *regression line*. The most common way of calculating it is to find the line that minimises the squares of the vertical distances from the points to the line, and the resulting line is called the *least squares regression line*. In Section 9E (a Challenge section) we will use formulae to calculate the gradient and y-intercept of this line of best fit, and in Section 9F we will use technology, but for now, we will estimate it by eye.



Here again is the scatterplot of the heights and weights of 50 individuals. We have drawn on it by eye a line of best fit. It has gradient about 0.65, and x-intercept about 85. Its equation is therefore about

$$y - 0 = 0.65(x - 85)$$
$$y = 0.65x - 55.25$$

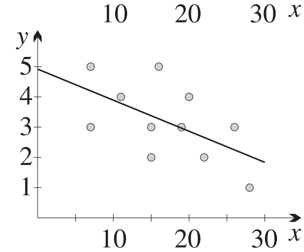
Using the formulae in the next section, the gradient and y-intercept of the least-squares regression line are about 0.649 and -55.52. Again, be careful because the horizontal scale starts at 80 kg, not zero.



The scatterplot of the callers on hold shows only weak correlation, so it is difficult to draw a line of best fit by eye. By the formulae, the gradient is about -0.047, and the y-intercept is about 3.75, giving the line

$$y = -0.037x + 3.75$$

that we have drawn on the graph.



In the second scatterplot, the outlier has been omitted (this is not the recommended procedure). The correlation is now moderate, and this makes it easier to draw the line by eye. The formulae in the next section tell us that the gradient is about -0.10, and the y-intercept is about 4.92. The resulting line

$$y = -0.1x + 4.92$$

has been drawn on the graph.

Why was the outlier there? Perhaps someone rang simply to cancel the service because of all sorts of other problems, and the short waiting time had no effect on his rank of 1. Should it be ignored? That is up to management and what questions they are asking.

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Double, triple and multiple points

None of our diagrams or datasets so far have repeated points, but in practice repeated points often occur. There are different conventions — use ever larger circles, use a code of circles, squares and crosses, place numbers inside the circle, ... Use whatever is convenient.

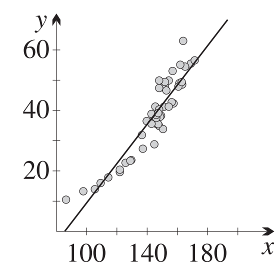
It is important to be aware of repeated points when judging correlation and line of best fit by eye, otherwise one's judgement will be right out. See Question 4 in Exercise 9D.

13 REGRESSION — THE LINE OF BEST FIT

- Given bivariate data, we can calculate the *line of best fit* or *regression line*.
- When correlation can be seen clearly, we can draw the line of best fit by eye. It is important then to identify any multiple points in the scatter graph.
- Outliers may have a significant effect on correlation and the line of best fit.

Interpolation, extrapolation, prediction and causation

Interpolation mean *predicting further results within the range of the variables in the data*. That is reasonably straightforward once the line of best fit has been drawn. Interpolation is justified, provided that we are convinced that the sample we are working from is not biased.



Extrapolation mean *predicting further results outside the range of the variables in the data*. That can present a real problem, because the situation is often quite different outside the range of sample values. For example, given the scatterplot and line of best fit of 50 heights and weights, a person of height 85 cm would be predicted to have zero weight, and a baby of height 40 cm would float away!

Even when data have extremely high correlation — high enough for the relationship to be regarded as a function — extrapolation is dangerous. Newton's laws of motion cannot be extrapolated to speeds approaching the speed of light because of relativity theory, and cannot be extrapolated to very tiny particles because of quantum mechanics.

Causation is best left to scientists. If events A and B are correlated, there are four possibilities — A causes B, B causes A, A and B have a common cause C, and the correlation is a fluke. Events can have multiple causes, particularly in medicine and weather. Many phenomena are chaotic, such as the weather and eddies in flowing water, making prediction impossible and rendering the idea of causation extremely complicated.

14 INTERPOLATION, EXTRAPOLATION, PREDICTION AND CAUSATION

- Provided that the data are reliable, the line of best fit can reasonably be used for *interpolation*, meaning prediction of results within the range of the variables.
- It can also be used for *extrapolation*, meaning prediction of results outside the range of the variables in the data, but this requires caution and common sense because the results of extrapolation are often very misleading.
- Questions of causation are probably best left to scientists.

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Exercise 9D

FOUNDATION

1 In the following relationships, identify the most reasonable choice for: **i** the independent variable, **ii** the dependent variable. Is there any uncertainty in your answer?

a height and weight,

b the area of a circle and its radius,

c the weight of meat purchased and the price,

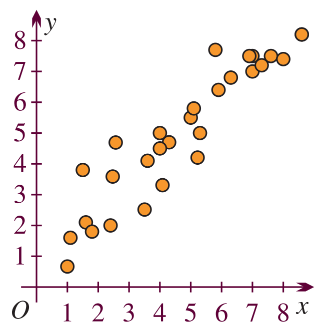
d the outcome for a player at Wimbledon Tennis championships and the player's previous world rank,

e outside temperature and household power consumption,

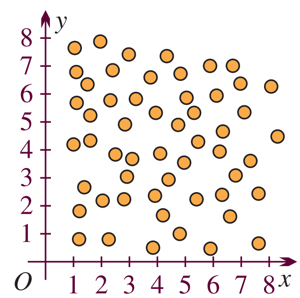
f pronumerals x and y for the many-to-one function $y = f(x)$.

2 By eye, decide if each of these graphs is an example of strong, moderate, weak or no linear correlation. Where there is a correlation, note whether it is a positive or negative correlation.

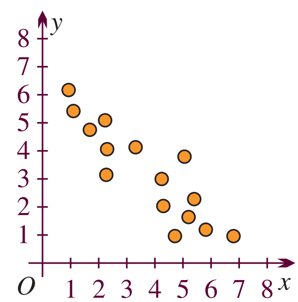
a



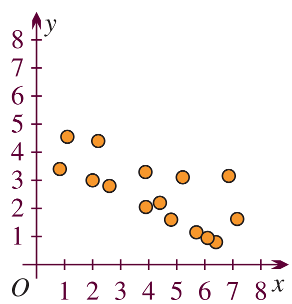
b



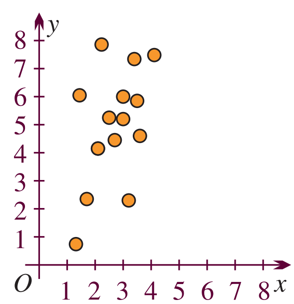
c



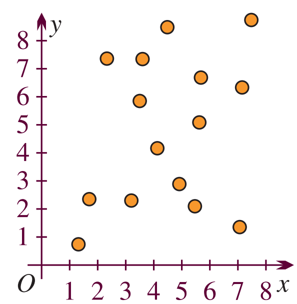
d



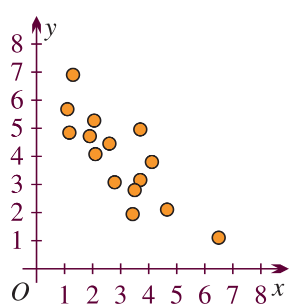
e



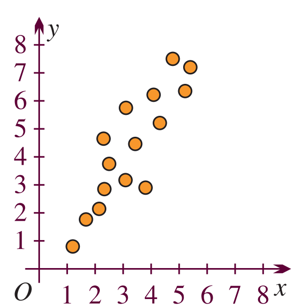
f



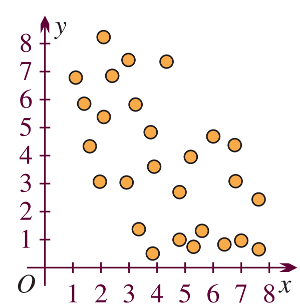
g



h



i



3 Plot each dataset on a separate diagram and draw a line of best fit by eye. Write down the equation of the line of best fit that you have drawn.

a $(0, 3)$ $(1, 2)$ $(2, 4)$ $(3, 3)$ $(4, 5)$ $(5, 5)$ $(6, 7)$ $(9, 7)$

b $(0, 30)$ $(1, 35)$ $(2, 45)$ $(3, 40)$ $(4, 55)$ $(5, 55)$ $(6, 70)$ $(7, 65)$

c $(0, 16)$ $(2, 14)$ $(3, 8)$ $(4, 10)$ $(6, 7)$ $(7, 4)$ $(9, 4)$ $(12, 0)$

d $(0, 6)$ $(1, 8)$ $(2, 7)$ $(3, 5)$ $(4, 4)$ $(5, 5)$ $(6, 4)$ $(7, 1)$

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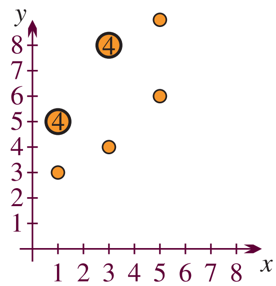
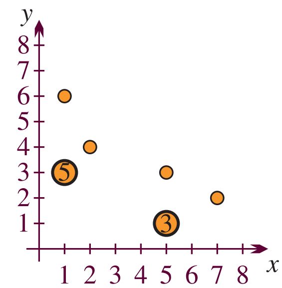
4 The following datasets include repeated points, with the frequency indicated by a number on the plot. For each question, allowing for the extra weighting of the repeated points:

- **i** Estimate the strength of the correlation
- **ii** Copy the diagram and draw in a line of best fit by eye,.

a

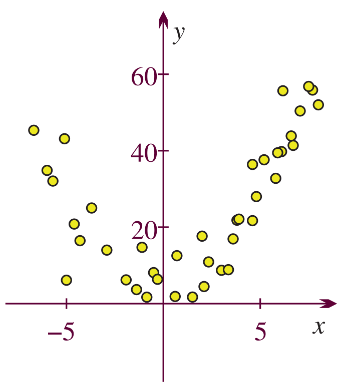


b

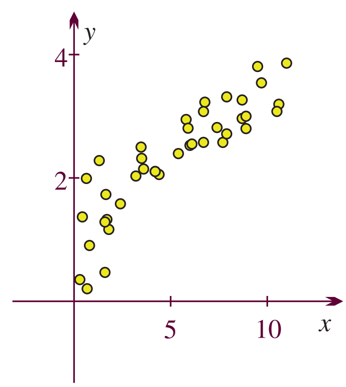


5 Not every dataset shows a linear relationship — some data are best modelled by a quadratic curve (a parabola), a circle (or semi-circle), a hyperbola, a square root, an exponential, or some other such curve. By eye, suggest what type of curve might model each dataset.

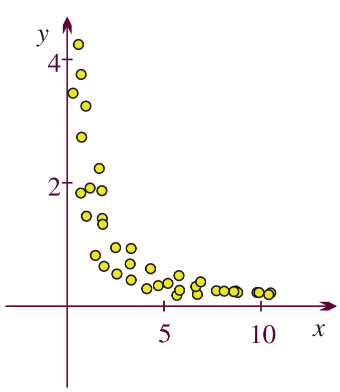
a



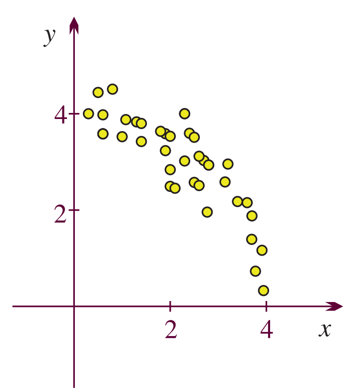
b



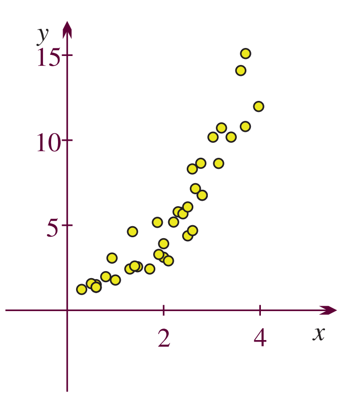
c



d



e



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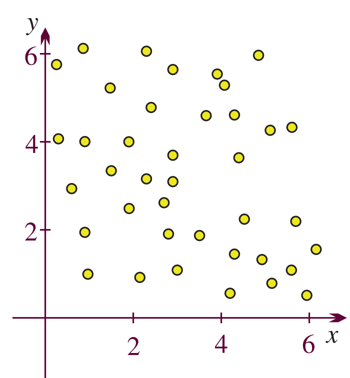

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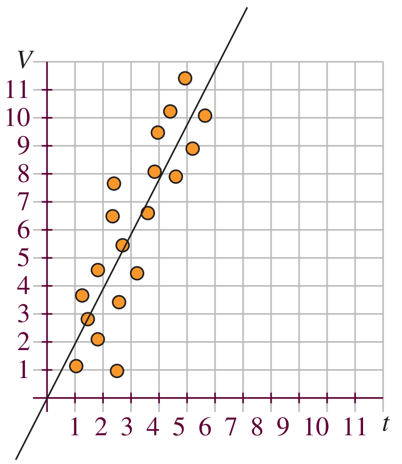
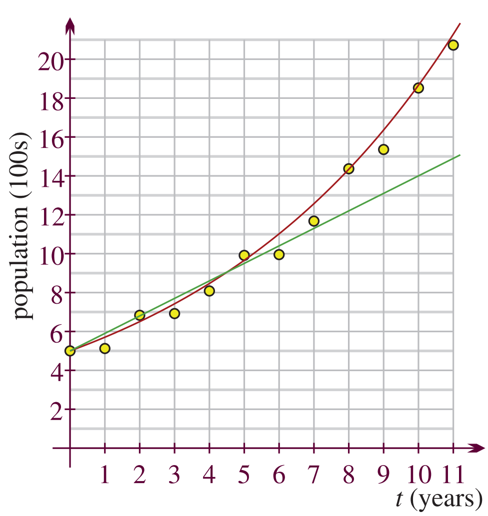
DEVELOPMENT

6 Yasuf has conducted an experiment. He recorded the volume in litres of water that flowed through a pipe in a given time of between 1 and 6 minutes. Then he repeated this procedure many times over a period of several hours in the middle of the day. His results are shown in the scatterplot to the right. Yasuf has also drawn a line of best fit through the data.



- **a** Use interpolation, by reading off the graph, to estimate the volume of water flowing through the pipe in: **i** 3 minutes, **ii** 5 minutes.
- **b** Estimate the equation of the line of best fit.
- **c** What is the value of the V-intercept, and why would you expect that value?
- **d** What is the physical meaning of the gradient?
- **e** Should Yasuf have drawn the line down through the origin into the third quadrant?
- **f** Why do you think the correlation in this experiment is not perfect, that is, why don't all the points lie on a straight line?
- **g** Use the equation of the line of best fit determined by Yasuf to extrapolate the amount of water flowing through the pipe in half an hour. Is this extrapolation reasonable?
- **h** How long would be required for the pipe to disgorge 45 litres of water?
- **i** Yasuf wishes to estimate how much water will flow through the pipe in one 24-hour day. Explain why extrapolating from these results may not be valid.

7 Population in the town of Hammonsville has been growing strongly over the last few years. When town planners first took a census in 2010, the population of people living in the town was 500. The population over the next 11 years is recorded in the scatterplot to the right. Planners have also attempted to fit the data with various curves to model future population growth.



- **a** What was the population five years later in 2015?
- **b** After seven years the planners began to draw a scatterplot, and they added a line of best fit in order to estimate population trends.

  Place your fingers over the last four points on the scatterplot so that you see only the 8-point scatter graph that the planners had after 7 years.

  - **i** Find the equation of the line (population P in 100s as a function of time t years since 2010). Use the fact that the line passes through $(0, 5)$ and $(10, 14)$.
  - **ii** Does this line look a good fit for the seven-year period?
  - **iii** In 2017 predictors used this model to extrapolate the population after a further two years (that is, when $t = 9$). What was the error in their prediction, compared with the plotted population in that year?

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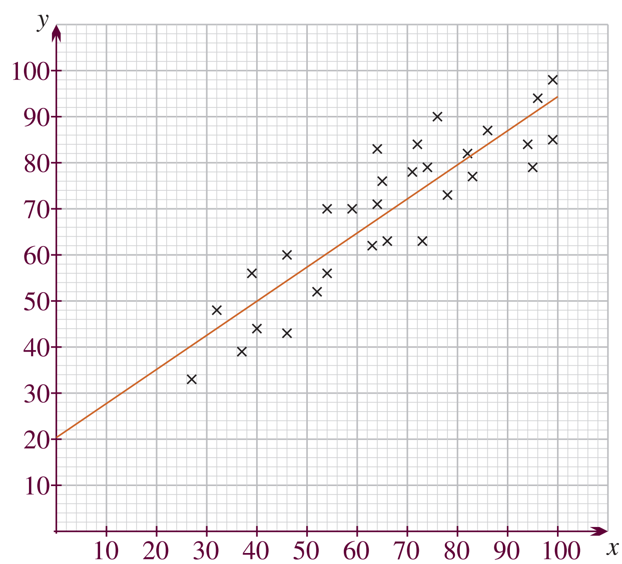
c Experts noticed that their model became an increasingly poor predictor as time went on, and instead attempted to fit the data with an exponential curve $P = 5 \times 2^{0.19t}$, where P is the population t years after 2010.

i Check the accuracy of this model by calculating its population prediction in 2019.

ii Would you expect this model to be accurate for the next 10 years?

d What does this question suggest about the general viability of using a line to fit data (that is, a line of best fit)?

8 Student percentage marks for assessment 1 (x) and assessment 2 (y) have been compared on a scatterplot, and a line of best fit has been drawn. There are no repeated points.



a What was the top mark in each assessment? Were they obtained by the same student?

b What was the bottom mark in each assessment? Were they obtained by the same student?

c Which assessment was more difficult? Give reasons for your claim.

d Some students were absent for one or the other of the assessment tasks, and their missing scores must be estimated. All scores are to be recorded as a whole number. Use interpolation on the line of best fit to estimate the score in the other assessment of a student who received:

i 40 in assessment 1, **ii** 60 in assessment 1, **iii** 80 in assessment 2,

iv 40 in assessment 2, **v** 15 in assessment 2.

e Read off the coordinates of the points on the line at $x = 0$ and $x = 100$. Hence find the equation of the line of best fit.

f Is this an accurate method of estimating a student's missing assessment score?

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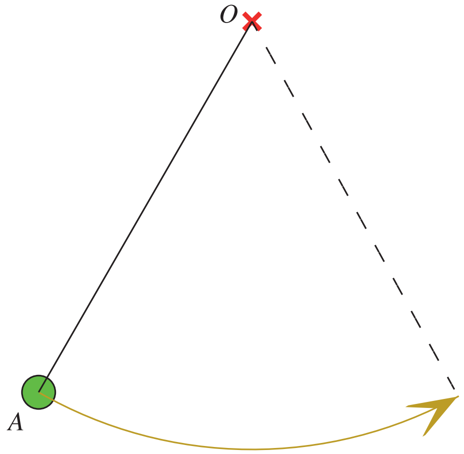
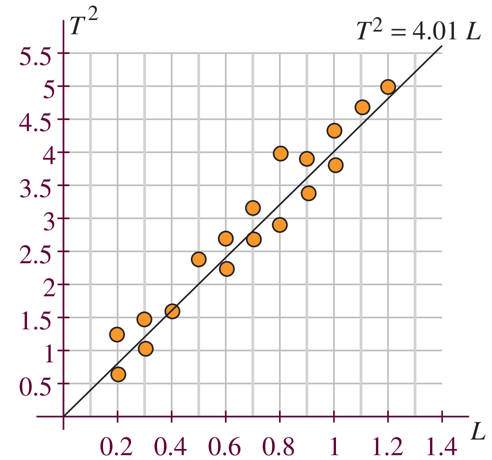
CHALLENGE

9 A class is carrying out an experiment. A weight is attached by a string to a fixed point O. It is drawn aside and released, allowing it to swing and then return to its original position. The time taken to return is called the *period* of the pendulum. The experiment requires students to measure the periods T minutes for different string lengths L metres. Theory suggests that the square of the period is related to the length of the string. The class's results are shown in the scatterplot below, graphing T^2 on the vertical axis and L on the horizontal axis.





a For a given length of string, what is the maximum difference between the square of the period predicted by the linear model and the square of the measured period for a given length of string?

b This particular maximum difference is significantly larger than for other points, and it appears to be an outlier in the data. Comment on possible causes.

c The class has claimed great accuracy in measuring the times for a period, as can be seen from the strong correlation. What methods may they have used to achieve this accuracy?

d Scientist have developed a theoretical model relating T and L. The model predicts that $T = 2\pi\sqrt{\frac{L}{g}}$, where $g \doteqdot 9.8\ \text{m/s}^2$. Does this agree with these experimental results?

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9E Formulae for correlation and regression

You must be able use technology to calculate Pearson's correlation coefficient and the line of best fit from given data. The next section goes into some detail about those procedures.

This section presents the actual formulae for the line of best fit. They are rather elaborate, and calculations using them take more time and paper than the earlier calculations for mean and standard deviation. Nevertheless, calculating at least a few examples by hand can prevent statistics becoming a 'black-box' where the user of the results has no real idea what is happening. In a mathematics course, understanding is key.

This section could all be regarded as Challenge. The very short exercise has just one purpose — familiarity with the formulae.

The formula for Pearson's correlation coefficient

The standard measure of correlation is *Pearson's correlation coefficient* r. It tests only for linear correlation, that is, it gives a measure of how close the data are to being on a line of non-zero gradient,

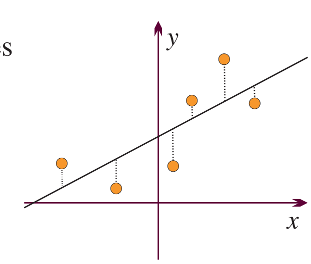
$$r = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sqrt{\sum (x - \bar{x})^2 \sum (y - \bar{y})^2}}.$$

We will not develop this formula, but these remarks should help to understand its significance and how to go about calculating it.

- We must first calculate the means $\bar{x}$ and $\bar{y}$ of the x-values and of the y-values. The point $(\bar{x}, \bar{y})$ will lie in the middle of the cluster on the scatterplot.
- We need all the *deviations from the mean*. These are the deviations $x - \bar{x}$ of the x-values from their mean $\bar{x}$, and the deviations $y - \bar{y}$ of the y-values from their mean $\bar{y}$.
- For the numerator, we take each product $(x - \bar{x})(y - \bar{y})$. This is the key object, because if x and y both lie on the same side of their means, the product is positive, and if they lie on opposite sides, the product is negative. Adding them up gives some sense of whether the variables are working together, or working contrary to each other and cancelling out.
- The denominator is necessary to normalise the quantity and make it a ratio. In particular, notice that the units of x cancel out and the units of y cancel out, so that the resulting quotient r is a pure number.
- The denominator is closely related to the formulae for the standard deviation of x and of y. In fact, the formula for r can be rewritten using the standard deviation, and standard deviation calculations can be re-used here.
- Pearson's correlation coefficient is unaffected by units and gradient (apart from the sign of the gradient). If we change metres to centimetres, or multiply all the y-values by +7, there is no change in r. That is, only the clustering and the sign of the gradient are relevant.

Formulae for the regression line

The standard method of finding the regression line is to find the line that minimises the sum of the squares of the vertical distances from each plot to the line — this line is called the *least squares regression line*. Again, we omit any derivation, and simply state that the line is



$$y = mx + b, \text{ where } m = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (x - \bar{x})^2} \text{ and } b = \bar{y} - m\bar{x}.$$

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- The numerator of the gradient m is the same as the numerator of r.
- The denominator of m has already been calculated when calculating r.
- The value of the y-intercept b simply ensures that the line passes through the point $(\bar{x}, \bar{y})$.

Thus once the calculations required for r have been done, the calculation of the line of best fit is very quick.

Calculations using the callers on hold

The example in Section 9D of the waiting times of callers on hold was deliberately engineered so that the means were whole numbers. Otherwise an alternative form of the formula would need to be developed, which is not appropriate in this course, or machine calculation would be necessary.

The sums of the first and second lines of the table allow the means $\bar{x}$ and $\bar{y}$ of the x-values and y-values to be calculated. These means are needed in the third and fourth lines to calculate the deviations.

													Sum
x	7	15	22	11	20	15	7	28	6	16	26	19	192
y	5	2	2	4	4	3	3	1	1	5	3	3	36
$x - \bar{x}$	−9	−1	6	−5	−4	−1	−9	12	−10	0	10	3	0
$y - \bar{y}$	2	−1	−1	1	1	0	0	−2	−2	2	0	0	0
$(x - \bar{x})^2$	81	1	36	25	16	1	81	144	100	0	100	9	594
$(y - \bar{y})^2$	4	1	1	1	1	0	0	4	4	4	0	0	20
$(x - \bar{x})(y - \bar{y})$	−18	1	−6	−5	4	0	0	−24	20	0	0	0	−28

First, $\bar{x} = \dfrac{\sum x}{n} = \dfrac{192}{12} = 16,$ and $\bar{y} = \dfrac{\sum y}{n} = \dfrac{36}{12} = 3.$

Hence
$$r = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sqrt{\sum (x - \bar{x})^2 \sum (y - \bar{y})^2}} = \frac{-28}{\sqrt{594 \times 20}} \doteqdot -0.25689.$$

For regression,
$$m = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (x - \bar{x})^2} = \frac{-28}{594} \doteqdot -0.04714,$$

and
$$b = \bar{y} - m\bar{x} = 3 + \frac{28 \times 16}{594} \doteqdot 3.75421.$$

Thus the correlation is -0.26, and the line of best fit is $y = -0.047x + 3.754$.

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15 FORMULAE FOR CORRELATION AND REGRESSION

Let $\bar{x}$ and $\bar{y}$ be the means of the x-values and y-values of a set of bivariate data.

- Pearson's correlation coefficient r is given by

$$r = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sqrt{\sum (x - \bar{x})^2 (y - \bar{y})^2}}.$$

- The least-squares regression line is $y = mx + b$, where

$$m = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (x - \bar{x})^2} \qquad \text{and} \qquad b = \bar{y} - m\bar{x}.$$

Other forms of these formulae combine them with the formula for variance.

Classifying correlations

The verbal descriptions used in Section 9D for the strength of correlation are visual impressions of the scatterplot. There is no agreed relationship between those verbal descriptions and the values of the correlation coefficient calculated in this section and the next. The same criteria may not be appropriate for different disciplines or for different experiments within a discipline.

The authors regard the following as reasonably helpful suggestions. For positive correlations (and similarly for negative correlations),

Correlation	0.6–1.0	0.4–0.6	0.1–0.4	0.0–0.1
Classification	strong	moderate	weak	virtually none

There are no rules — have the scatterplot at hand, think about the experiment, and be aware of any outliers.

According to this classification, the caller waiting times and ranks, with a correlation of about -0.26, show weak negative correlation. With the outlier (6, 1) removed, the correlation is about -0.57, which is moderate negative correlation.

The correlations of the set of eight scatterplots in Section 9D are:

Top row: 0.82, 0.59, 0.35, 0.07 Bottom row: -0.87, -0.5, -0.37, -0.09

Exercise 9E

Note: If this exercise is attempted, the best approach is perhaps to do the calculations once or twice to gain some familiarity, and then to combine calculations by hand with calculations by technology. Further opportunities to use technology for such calculations are given in Questions 1–2 of Exercise 9F.

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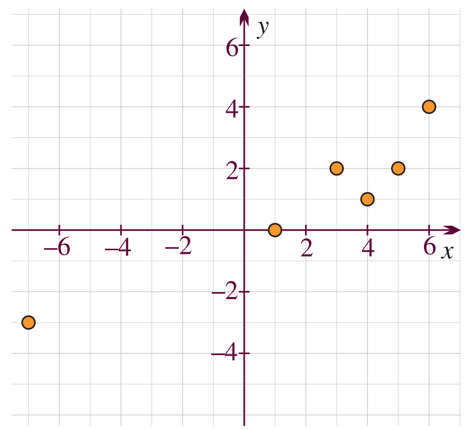
1 A student is given the task of calculating the line of best fit for the small dataset:

$(-7, -3)$ $(1, 0)$ $(3, 2)$ $(4, 1)$ $(5, 2)$ $(6, 4)$

The data are shown on the scatterplot to the right.



a Does the correlation of the data appear linear, and if so, does the correlation appear strong, weak or moderate?

b Copy the scatterplot into your book. By eye, estimate and draw the line of best fit for the data.

c Copy the following table into your book. Complete the sum for the first two rows, and hence calculate the means $\bar{x}$ and $\bar{y}$.

							Sum
x	-7	1	3	4	5	6	
y	-3	0	2	1	2	4	
$x - \bar{x}$							
$y - \bar{y}$							
$(x - \bar{x})^2$							
$(y - \bar{y})^2$							
$(x - \bar{x})(y - \bar{y})$							

d Mark the point $(\bar{x}, \bar{y})$. Does it fall on your estimated line of best fit?

e Complete the last five rows of the table.

f Use the formula

$$r = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sqrt{\sum (x - \bar{x})^2 (y - \bar{y})^2}}$$

to find Pearson's correlation coefficient for the data, correct to two decimal places.

g Is r close to 1 or to -1? This would indicate that the line is a good fit for the data.

h Use the formula to calculate the gradient m of the least squares line of best fit,

$$m = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (x - \bar{x})^2}.$$

i Use the formula $b = \bar{y} - m\bar{x}$ to calculate the y-intercept of the line of best fit,

j Write down the line of best fit calculated from these formula, rounding m and b each correct to one decimal place. If this line differs from your estimate of the line of best fit in part **b**, add this line to your diagram.

2 Repeat the previous question for the following datasets.

a $(-2, 0)$ $(0, 0)$ $(1, 1)$ $(3, 1)$ $(4, 2)$ $(6, 2)$

b $(-3, -4)$ $(-2, -3)$ $(0, 1)$ $(2, 3)$ $(3, 4)$ $(6, 8)$

c $(-4, 7)$ $(-2, 6)$ $(-1, 1)$ $(0, -1)$ $(1, -3)$ $(2, 1)$ $(4, -4)$

d $(-2, 6)$ $(-1, 3)$ $(0, 4)$ $(1, 2)$ $(2, 0)$ $(6, -3)$

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9F Using technology with bivariate data

It is clear from Section 9E that calculations by hand are laborious. The right tools are statistics calculators, spreadsheets and statistical packages, all of which may be online. This section has several purposes, loosely grouped around technology.

- Small datasets suited to a statistics calculator to find r and line of best fit.
- Larger datasets for investigations using spreadsheets and other software.
- Investigation activities such as surveys to generate data for analysis.
- Investigations allowing the reader to search out raw data from the internet.

Waiting times of callers on hold — correlation and regression

The previous section calculated Pearson's correlation coefficient for the waiting times of callers and their ranking of the company (data in Section 9D). If you have a *calculator* that is capable of performing the calculations, then work out now how to use it to get the correlation and the line of best fit as calculated in Section 9E. The various calculators differ from one another, and we have chosen not to give detailed instructions for any particular model. Find the instructions and read them.

For any extended work, however, a spreadsheet is the best tool to use (in the absence of specialised statistical software). Excel is widely used, and we have worked the example here using Excel. This example, and the questions in Exercise 9F, should be easily adapted to other technology, including online versions.

Excel has many different versions, and its functions are complicated. Use the help file or search online for guidance. In particular:

- The functions in Excel for Windows keep changing over time. All the functions used here were different some years ago.
- The Mac versions of Excel have their own peculiarities. For example, 'Fill down' and 'Fill right' are well-known difficulties on a Mac.

In particular, if you cannot find the search box in the fourth dot point below, search for 'Formula Builder' in the help file.

Here are the steps in finding the correlation and regression line using the most recent version of Excel 365. The dataset is drawn from the 'Waiting time of callers' just above Box 12 in Section 9D.

The data: Type the data into Excel from the table in Section 9D.

- Type 'Time x' into cell `A1` and 'Rank y' into cell `B1` — these are the headers.
- Then type the 12 data pairs into the cells `A2:A13` and `B2:B13`.

The means: Type 'Mean x' and 'Mean y' into cells `D1:E1`.

- Place the cursor into cell `D2` and type = into it. This initial character = is the code for Excel to interpret what follows in a cell as a function.
- You will notice that in the top left below the word `File` there is now a text box with a down arrow. Click on the arrow, then click on 'More functions'.
- In the resulting search box, enter 'Average'. Select `AVERAGE` and read its description. Perhaps also click the link to the help file, and perhaps compare this function with the function `AVERAGEA`.
- Double-clicking will bring up a dialogue. You only need Number1, and you can either enter the cells with the x-values as `A2:A13`, or select them in the spreadsheet so that Excel enters the cell labels. The result should be the mean 16.

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- With cell `D2` selected, look at the text box above the row-and-column array. It should give the formula in the cell, which is `=AVERAGE(A2:A13)`.
- Do the same for the mean of the *y*-values in cell `E2` — the mean is 3. But Excel is cleverer than this! Instead, select cells `D2:E2`, and press `Ctrl+R` to `Fill right`. You can check that the formula in cell `D2` has been copied to cell `E2`, except that column `A` in the formula has been changed automatically to column `B`. You can also see immediately what cells have been referenced by selecting cell `E2` and pressing the `F2` key.

The standard deviations: Type 'SD *x*' and 'SD *y*' into cells `D4:E4`.

- Then repeat the steps to insert the means into cells `D5:E5`, except search for 'standard deviation', and select the function `STDEV.P`.
- (As we remarked in an extension note at the end of Section 9B, it may be more correct, because this is a sample, to use the sample standard deviation `STDEV.S` rather than the population standard deviation `STDEV.P`.)

The correlation: Type 'Correlation' into cell `D7`.

- Insert the correlation into cell `D8` as before — type `=` and click on the down arrow, select 'More functions', but search for 'correlation' and select `PEARSON`.
- Enter the *x*-values into the top box and the *y*-values into the second box.

The line of best fit: Type 'Regression' into cell `D10`, 'Gradient' into cell `D11` and 'Intercept' into cell `E11`.

- To insert the gradient of the regression line into cell `D12`, search for 'Regression' and select `SLOPE`. Be careful here because things are reversed! Enter the *y*-values into the top box, and the *x*-values into the second box.
- To insert the *y*-intercept of the regression line into the cell `E12`, search for 'Regression' and select `INTERCEPT`. As before, enter the *y*-values into the top box, and the *x*-values into the second box.

	A	B	C	D	E
1	Time *x*	Rank *y*		Mean *x*	Mean *y*
2	7	5		16	3
3	15	2			
4	22	2		SD *x*	Sd *y*
5	11	4		7.035624	1.290994
6	20	4			
7	15	3		Correlation	
8	7	3		-0.25689	
9	28	1			
10	6	1		Regression	
11	16	5		Gradient	Intercept
12	26	3		-0.04714	3.754209
13	19	3			

Waiting times of callers on hold — the scatterplot and regression line

Excel can draw a scatterplot of the data with the regression line inserted.

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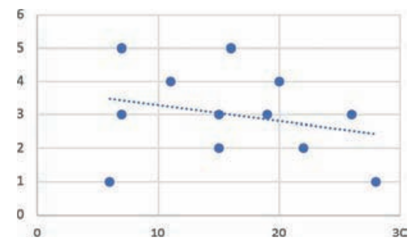
The scatterplot: First select the 24 cells `A2:B2` down to `A13:B13` that contain all the data.

- Click on the 'Insert' tab at the top of Excel. Find the 'Charts' group, and click 'Recommended charts'. Then click on the 'All charts' tab.
- You will see all sorts of charts there (including box-and-whisker that we discussed in Section 9C), but the one we want is 'X Y Scatter'. Click on it, place the cursor on the second of the two charts that just shows the 12 dots all in one colour, and double-click.
- You now have a scatterplot placed onto your spreadsheet. You can move it and resize it in the usual ways. When you double-click on the chart, a menu appears on the far right allowing other changes to be made.

The line of best fit: Click on the chart again, and a new tab 'Design' appears on the very top of the Excel window.



- Click on the 'Design' tab, find the group 'Chart layouts' (on the far left), and select 'Add chart element'.
- Go to 'Trendline' and select 'Linear'. Now the least squares regression line appears, calculated as in Section 9E.

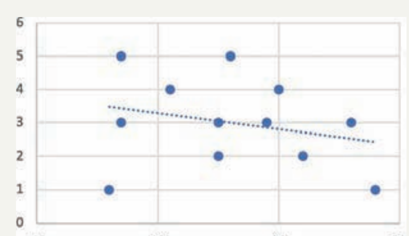
Saving the chart: You can export the chart to other software such as Word or image-processing software.

- Click on the chart, then right-click on the top border, then click 'Copy'.
- Paste the file where you want it.

Example 8 9F

Repeat all the steps above for the call waiting times with the outlier removed. Alternatively, your spreadsheet should allow you to remove one data point.

SOLUTION



- The means are about 16.91 for the x-values and 3.18 for the y values.
- The standard deviations are about 6.64 for the x-values and 1.19 for the y values.
- The correlation is about -0.57.
- The regression line is about $y = -0.1x + 4.92$.
- Excel's scatterplot with the regression line is copied to the right.

Internet data — investigations and possible projects

Exercise 9F is mostly concerned with processing data from the internet using technology. The methods are the same, whether there are 12 or 12 000 pairs of data.

Internet data, however, can be very rough. Always look carefully through the data for obvious anomalies. For example, in the heights and weights data introduced in Section 9D, many entries had the height field or the weight field missing, or both, and there were a few entries that seemed to have become corrupted — these are two of the few good reasons for omitting outliers.

The first few questions use technology for correlation and regression. Then the exercise is intended to provide investigations of various types. Most questions can easily be extended to projects by asking further questions and using and comparing several sources of data.

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Exercise 9F

INVESTIGATION

Calculation of the gradient and the y-intercept of the line of best fit, and the calculation of Pearson's correlation coefficient, are best done using technology. These calculations may be done using a calculator with a regression analysis mode (also sometimes labelled '2 Var Stats' or '$A + Bx$'), with a spreadsheet program (such as Excel), with statistical software (such as the software package R), or with an online program designed to analyse such data. A number of such free online programs are available if you type *online line of regression* into a search engine.

The datasets in the first four questions are small, to enable practice with the technology before attempting to analyse some more realistic data. Further practice with technology is provided by the datasets in Exercise 5E.

1 Use technology to calculate Pearson's correlation coefficient and the equation of the line of best fit for each dataset.

a $(1, 1.7)$ $(2, 1.9)$ $(3, 3.7)$ $(4, 4)$ $(5, 4.5)$ $(6, 6.7)$

b $(-2, 3)$ $(0, 2)$ $(2, 2.3)$ $(3, 3.1)$ $(4, 4)$ $(5, 5.7)$ $(6, 6.1)$

c $(3.6, 3.2)$ $(5.9, 3.9)$ $(7, 1)$ $(4.4, 5.2)$ $(3.4, 6.2)$ $(2.2, 5.7)$ $(5, 2.3)$ $(1.2, 8.2)$

d $(6.3, 0.7)$ $(3.6, 4.9)$ $(4, 2.6)$ $(5.4, 1.2)$ $(9, 2.3)$ $(1.9, 1.8)$ $(1.4, 7.4)$ $(0.4, 3.6)$

e $(2.1, 3)$ $(4.7, 6.9)$ $(2.8, 4)$ $(3.3, 5.5)$ $(1.3, 3.3)$ $(2.7, 4.8)$ $(4.9, 7.8)$ $(1.7, 2)$ $(3.8, 6.2)$ $(0.5, 1.8)$ $(3.3, 6.1)$ $(4.4, 6.6)$

2 The following datasets each include an *outlier point*. Recall that for our purposes, outliers are points that are a large vertical distance from the line of best fit in relation to the other points. For each dataset:

i Repeat the previous question and also look carefully at a scatterplot of the data.

ii Note any points that appear to be outliers.

iii Remove the outliers and calculate the correlation and line of best fit again.

a $(0.9, 5.2)$ $(6.7, 8.8)$ $(3.9, 1.1)$ $(1.8, 6.7)$ $(4.6, 8.7)$ $(0.8, 3.9)$

b $(2.9, 3.7)$ $(1.4, 5.6)$ $(4.4, 2.6)$ $(4.3, 5.6)$ $(2.5, 5.1)$ $(6.4, 1.3)$

c $(4.7, 8.3)$ $(2.4, 2)$ $(5.3, 7.1)$ $(1.3, 2.9)$ $(3.6, 6.4)$ $(5.5, 9.5)$ $(1.8, 0.5)$ $(1.3, 2.9)$ $(6.5, 10.5)$ $(3.6, 2.2)$ $(6.1, 8.9)$ $(2.6, 4.9)$

3 Explain why in Question 2 Pearson's correlation coefficient and the line of best fit seemed less affected by the outlier in part **c** than in the other datasets.

4 This question presents two datasets, each with a repeated point, which can have a significant and possibly overlooked effect on the line of best fit.

Dataset 1: $(1, 3)$ $(3, 3)$ $(3, 5)$ $(5, 5)$ and repeated point $(5, 7)$ with frequency 5

Dataset 2: $(5, 7)$ $(6, 8)$ $(4, 7)$ $(1, 4)$ $(3, 4)$ $(4, 4)$ $(1, 2)$ $(8, 7)$ $(0, 2)$ $(5, 5)$ and repeated point $(3, 6)$ with frequency 5

a For each dataset, calculate the equation of the line of best fit, and Pearson's correlation coefficient:

i with the dataset as given,

ii if the repeated point is only included once in the dataset.

b Comment on the strength of correlation in these examples.

c Comment on the effect of the repeated point on the equations of the line of best fit.

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5 [Pareto charts]
Each of these suggested investigations involves a survey or collection of data, which is then collated into a Pareto chart.

- A preliminary class discussion or sample survey with a small number of respondents may be necessary to decide on categories for the data and chart.
- You will need to consider how to design your survey or data collection so that it is random — that you are, for example, collecting data in multiple locations, or conducting your school survey with respondents from a range of year groups.
- An online survey (search for *online questionnaire*) may be a good way to get a range of data, if the target group all have access to the online survey.
- How many respondents do you think you need to get accurate results?

a In Exercise 9A, a question explored the most common car colours and showed this information on a Pareto chart.
  i Design your own experiment where you record the colour of the first 100 cars that pass the school. Let several groups do the experiment simultaneously and check agreement. You may need to discuss the colour categories carefully — it can be a matter of opinion if a car is silver or grey.
  ii Do your results agree with the results shown on the chart in Exercise 9A?

b Investigate causes of customer dissatisfaction with the school canteen.
  i First discuss in class suitable categories, such as prices, variety and opening hours. A limited number of categories is best.
  ii Survey people to find their major cause of dissatisfaction — only one category can be chosen. In a good survey, it is important to survey a range of interest groups, such as all year groups and the teachers.
  iii Alternatively, do a survey on causes of customer *satisfaction*.

c Investigate the reasons students are late to school.
  i Decide on a number of categories in a class discussion, such as, ‘slept in’, ‘traffic’, ‘unwell’.
  ii Design your questionnaire and show your results on a Pareto chart.

d Do the people at your school use recycling? Investigate the reasons why they don’t. This could be a question about recycling at school, at home, or across the spectrum of their lives.
  i Decide on categories. Some examples are: unaware of environmental value of recycling, too inconvenient, no recycling bins at school, not aware how to recycle (because of issues such as bin labelling or lack of information on how to recycle old electronic items).
  ii Complete your results and draw up a Pareto chart.

e Pareto charts are regarded as one of the *seven basic tools of quality control*, because they are frequently used to investigate questions about quality control, customer satisfaction, and so forth. Investigate what the other six tools are, and see if you can apply a number of them to a quality issue in your school or environment.

6 [Contingency tables]
Contingency tables investigate the interrelationship between different variables in a complex dataset. They work best where there are a limited number of categories (such as male/female and blond/red/brown/black hair).

a Survey a number of students to find their favourite style of music, recording their gender.
  i Display your results on a contingency table.
  ii Investigate whether the favourite style of music appears to differ between genders in your sample.

b Consider other surveys that might be explored using contingency tables. Some examples: favourite music and year group, gender and time spent playing video games (group these in categories such 1 hour per week and 2–3 hours per week), gender and time spent on homework, favourite winter sport and favourite summer sport, favourite social media and age.

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7 [Scatterplots]
Many of these investigations generate large amounts of data. Excel or another spreadsheet program could be used to generate a scatterplot and to draw the line of best fit.
- a Investigate the ages of students at your school and their heights.
  - i Plot their age in months on the horizontal axis and their height in centimetres on the vertical axis. Do they appear to be correlated? Can you draw a reasonable line of best fit through the data?
  - ii If your program does not allow you to shift the intersection of the axes, it may be clearer to plot their age in months above 11 years and their height in centimetres above 130 cm.
  - iii Find the equation of the line of best fit and Pearson's correlation coefficient using technology.
- b Measure the lengths of students' forearms and their heights.
  - i Draw a scatterplot of height x and forearm length y.
  - ii Construct the line of best fit, and use Pearson's coefficient to decide if there is a good linear correlation between the quantities.
  - iii Vary this experiment to: leg length and height, stride length and height, foot length and height, arm span and height, hand and forearm length, length of thumb and middle finger, height of student and their father (or mother), circumference of head and height.

Large datasets: The following questions involve the analysis of large sets of data. There are many such datasets available on the internet. For your convenience, some of datasets for these questions may be downloaded from the interactive textbook.

No solutions are provided, and it is recommended that the solutions to this exercise should be discussed in class.

8 There is frequent mention in the media of rising sea levels over recent years, because of melting ice sheets and glaciers, and because of the expansion of seawater as it warms. This question uses satellite data provided by NASA to investigate the rise in sea levels since 1995. Further records are also available on the website if you wish to pursue historical records from coastal tide gauge records.

Source: https://climate.nasa.gov/vital-signs/sea-level/

XLS

- a Download the data files provided in the interactive textbook. Alternatively, press the button marked 'DOWNLOAD DATA' on the NASA webpage given above, and massage the data into a spreadsheet.

  Explanation is provided in the spreadsheet about what the various columns mean and how the data are collected.
- b Open the spreadsheet. Copy column C and column F to a new sheet at columns A and B. Column C is the date since 1993, in year and fraction of a year, and column F is the sea level measure known as Global Mean Sea Level (GMSL) with a 20-year reference mean taken as 0. Because our concern is the change in sea level, the zero reference point is not important.
- c Construct a scatterplot of the data, with date on the horizontal axis and sea level on the vertical level.
- d Find the line of best fit.
- e Get Excel to display the R^2 value (which is the square of Pearson's correlation). How good is the fit?
- f Get Excel to display the equation of the line of best fit. What is the meaning of the gradient here? Is the y-intercept significant?

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g It is probably more meaningful at a glance to adjust the vertical axis to be *change in sea level since the mean height in 1993* (when our data starts).

  i Calculate the mean of the sea level values in 1993, storing this in cell `E1`.

  ii Add a new column `C`, defined by `=B1-$E$1`, and fill this value down to the rest of the cells in column `C`.

  iii Construct a new scatterplot from columns `A` and `C`.

h For further investigation and calculation:

  i Eighty percent of the Maldives is less than one metre above sea level. How long will it take if this trend continues for that eighty percent to be under water?

  ii Find the height above sea level of your current location, and estimate when it will be under water, if the trend continues.

9 An interesting investigation would be to repeat this question using instead the data from Fort Denison in Sydney, http://www.bom.gov.au/oceanography/projects/ntc/monthly/. The data at this URL go back to 1914, and is regarded worldwide as providing one of the most reliable set of measurements of past sea level. The data are presented quite differently, so you will need to adapt your methods and your questions.

XLS

10 Economists make use of linear correlation and regression to forecast a number of economic indicators. This question examines data from the Australian Bureau of Statistics (http://stat.data.abs.gov.au/) on the gross operating profits of the Australian mining industry, collected from 1994–2018.

a Download the spreadsheet from the interactive textbook. Open the tab labelled Data1 and copy Columns A and B to a new spreadsheet. Delete rows 1–12, leaving the data from 1995–2018. (The data from 1994 are incomplete, so we shall begin in 1995).

b In cell C1 enter the formula `=(A1-DATE(1995,1,1))/365+1995`. This converts the date in cell `A1` to a year-and-decimal-fraction-of-a-year format, so that 1/Mar/1995 should convert to 1995.162 because it is $1/6 \doteqdot 0.162$ of the way through 1995.

c Fill the formula in C1 down to the rest of the column, to convert all the dates to this more useful format.

d Create a scatterplot with column `C` on the horizontal axis, and column `B` on the vertical axis. It may be easier in your version of Excel if you first copy column `B` to column `D`, so that the data are in the expected order — first x-values, then y-values. Create the scatterplot of columns `C` and `D`.

e Determine the R^2 value and the formula for the line of best fit. How well are the data correlated to the straight line model?

f What are the correct units for the company profits on the vertical axis?

g The data do not fit perfectly on a straight line, but is it still a useful model for economic prediction? `3*LOG`

11 Astronomers have discovered that for a certain large class of stars, brightness is well correlated with colour. These stars are on the so-called *main sequence*, which omits the very massive red giants and the relatively light white dwarves. Astronomers measure brightness by both apparent magnitude and absolute magnitude, the second of which is adjusted so that the measure of brightness does not depend on a star's distance from Earth. Colour is measured on the BV colour scale, which gives each colour a number. (The initials BV comes from the way the colour is determined by the use of Blue and Visible light filters.)

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–0.33	–0.30	–0.02	0.30	0.58	0.81	0.40

BV Colour Indices

a Download the spreadsheet from the interactive textbook.

b Copy column `C` to column `J`. In cell `K2` enter the formula `=B2+5*LOG(D2*10)`. This formula converts the star's apparent magnitude to its absolute magnitude. Fill cell `K2` down to the rest of the cells in column `K`, down to `K6221`. Add the heading *Absolute Magnitude* in `K1`. (You can find out more about this conversion formula if you search for *Convert Apparent Magnitude absolute magnitude* in a web browser).

c Create a scatterplot of the data in columns `J` and `K`.

d Because of the large number of points, the default size that Excel uses to display a point is too large. Click on a point and change the pointer option to a smaller size.

e Adjust the scale on your scatterplot so that the bulk of the data is visible on the plot, say horizontally from -0.5 to 2 and vertically -5 to 20. Excel will allow you to double-click on each axis and set the range of the axis.

f Traditionally, the vertical axis should show the axis flipped, with the negative numbers above the positive. Excel includes an option on the axis to display *Values in reverse order*.

g This dataset will NOT give a correlation coefficient close to -1, because the data include stars off the main sequence.

- **i** Copy columns `J—K` to `O—P`, Replace `P2` by the formula

  `=IF(ABS(B2$+5*LOG($D2*10)-7.5*O2)>3,NA(),B2$+5*LOG($D2*10)).`

  This code is designed to remove all points (stars) vertically well separated from the apparent line of best fit $y = 7.5x$, which seems to model the main sequence. The values that have been eliminated are marked #NA, meaning the value is ignored.
- **ii** Draw a scatterplot of these new data and check if the result gives a good correlation. Remember to adjust the point size and the default axes range, and to flip the direction of the vertical axis.
- **iii** Is it valid to eliminate much of our data in this way?

h Compare your resulting plots with those found online under a search for Hertzsprung–Russell Diagram.

i [Extension] Investigate further the formula relating apparent and absolute magnitude, the BV colour scale, the parsec and arcsec scale for measuring distance, and the 'correct' way to choose stars on the main sequence, by mass.

j [Extension] The magnitude data may be separated into different columns on the basis of the colour index. When Excel constructs the scatterplot, data from separate columns may be coloured independently, illustrating the differing star colours on the main sequence.

12 Weather is notoriously difficult to model, but it is such an important phenomenon that much effort has been applied to modelling its behaviour. The Bureau of Meterology keeps historical and recent data on Australian weather in the data section of its website

http://www.bom.gov.au/climate/data/

Investigate correlation between data on temperature and rainfall for May (or some other month) over a number of years. Choose other variables on the BOM website for similar investigations of correlation.

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Chapter 9 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 9 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 Write down the mean, median, mode and range of each dataset.

a 4 7 9 2 4 5 8 6 1 4

b 16 17 14 13 18 15 16 15 11

2 A market gardener is testing a new fertiliser on his coriander. Plant beds are set aside to test the new fertiliser. One set of beds have the new fertiliser, while others continue to use the old fertiliser. The total average yield in kilograms in each set of beds is recorded.

Old fertiliser: 1.8 1.4 1.6 2.1 1.9 2.3 1.8 2.1 1.9 1.8

New fertiliser: 2.1 1.6 1.7 2.2 1.9 2.2 1.9 2.2 2.0 1.8

a Find the median and the two quartiles Q_1 and Q_3 for the two datasets.

b Construct a parallel box-and-whisker diagram for the data.

c Do you have any recommendations for the market gardener?

3 **a** Draw up a dot plot of the data in the following list:

4 9 2 6 7 8 2 2 13 5 7 8 6 4 3 6 9 7 4 15 4 6 7 8 9 7 6

b Examine your plot and decide if would you call any of the data points outliers.

c Calculate the interquartile range and determine if the IQR criterion agrees with your assessment in part **b**.

d Draw up a box-and-whisker plot, showing any outliers by a separate dot outside the whiskers.

4 An athlete records his times in seconds running 100 m over a number of events and trials.

11.22 11.43 11.17 11.58 12.10 12.53 11.45 12.04

11.29 13.04 11.31 11.67 12.45 12.14 12.24 11.46

a Calculate the mean and standard deviation for the data using your calculator.

b Construct a frequency table for the data grouped into classes

10.8–11.2, 11.2–11.6, 11.6–12.0,12.0–12.4, 12.4–12.8, 12.8–13.2.

c Calculate the mean and standard deviation for the grouped data. How does your answer in part **a** compare?

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d Draw a frequency histogram for the data.
e Explain why you probably wouldn't want to use a wider class width, such as 0.5.
f Construct a cumulative frequency histogram and ogive for the grouped data.
g Use your cumulative frequency polygon (ogive) to estimate the median for the dataset.

5 A restaurant wishes to streamline the rate at which it gets food out to its customers at their tables. It divides the evening service into two sittings — the first sitting (arrive 6:00 pm–leave 8:00 pm) and the second sitting (after 8:00 pm). Data are gathered over the course of one night on whether a customer orders an entrée or not.

	first sitting	second sitting	Total
order entrée	45	42	
no entrée	38	28	
Total			

a Copy and complete the table.
b How many people attended the restaurant that night?
c What is the probability that a customer ordered an entrée?
d What percentage of the customers attended the first sitting?
e The manager claims that someone attending the first session is more likely to order an entrée than someone attending the second sitting. Is this correct? Explain.
f It is discovered at the end of the evening that one customer absconded without paying her bill. Given that she ordered an entrée, what is the probability that she was in the first session?
g The next day the restaurant is expecting an exceptionally busy evening for the second sitting, with 90 customers placing a booking. Estimate how many entrées will be ordered.
h Can you suggest why there might be a difference in the ordering habits of those attending the restaurant?
i Why might this survey contain useful information for a restaurant?

6 These small datasets are suitable for calculation by hand or by technology.
  i Plot the points and estimate a line of best fit;
  ii Calculate Pearson's correlation coefficient and the gradient and y-intercept of the line of best fit, correct to two decimal places.

a (1, 5) (2, 4) (6, 2) (5, 3) (3, 4) (7, 1) (4, 2)
b (7, 5) (0, 1) (2, 2) (6, 4) (3, 3) (5, 3) (1, 2) (4, 4)

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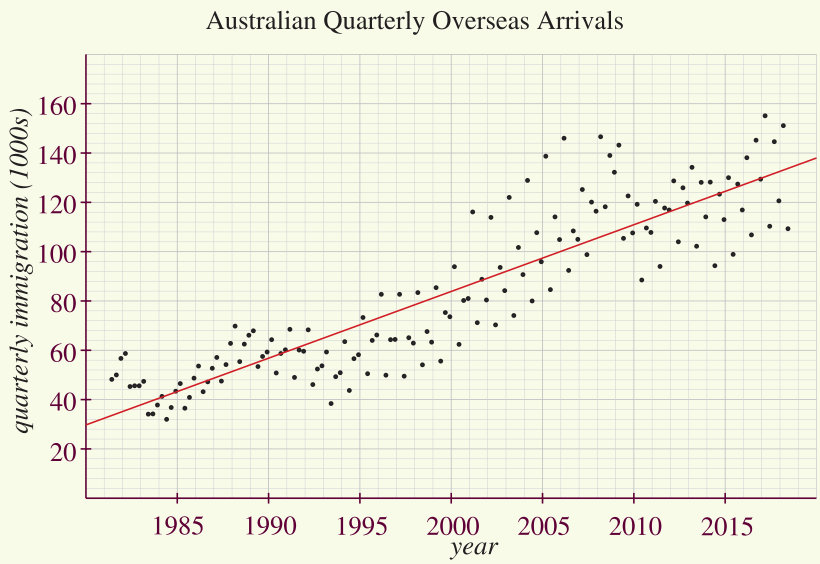
Review

7 The scatterplot below shows the immigration of people coming to live in Australia, measured in 1000s. Measurements are made quarterly (March, June, September, December).



a How many people came to live in Australia in the first quarter of 2011?

b Estimate the arrivals in the four quarters of 2000.

c What was the total number of annual arrivals and the average quarterly arrival in 2000?

d Why can it be misleading to examine only the arrivals in one quarter?

e Read the estimate predicted by the line of best fit in the first quarter of 2000.

f What information is missing to make the data useful to someone investigating immigration into Australia?

g The line of best fit shown on the diagram is $y = 2.7x - 5328.8$, correct to one decimal place, where x is the year and y is the quarterly immigration in 1000s.

- **i** Estimate the immigration rate for the first quarter of 2000 using this formula with $x = 2000.16$ (March 2000).
- **ii** Compare your estimate in part **i** with your estimate from part **e** and explain any discrepancy between these results.
- **iii** Repeat your calculation from part **i** using the formula $y = 2.70633x - 5328.8$.
- **iv** Estimate the immigration rate for the year 2030, by using this formula with $x = 2030$ and then multiplying by 4.
- **v** Use your estimate from part **c** to estimate further the percentage increase in immigration over the 31 years from 2000 to 2030 inclusive.

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10 Continuous probability distributions

Last year we moved from calculating individual probabilities in Chapter 10 to calculating in Chapter 11 a whole set of closely related probabilities grouped together as a *discrete probability distribution*. *Random variables* became part of our machinery, and we calculated expected values (or means) and standard deviations of these random variables in theoretical probability distributions.

The final section of that chapter, Section 11D, only just began to combine the theoretical probabilities in a discrete probability distribution with the data collected from experiments. This present chapter now builds on the techniques of organising and displaying data in Chapter 9 to bring probabilities and data into a closer relationship. The key to this is *relative frequency*, because relative frequencies obtained from data are estimates of theoretical probabilities.

These methods, combined with grouping, allow us finally to give a coherent account of a *continuous random variable*, defining it in terms of a *probability density function*. Integration is needed to understand this material, because a probability is now interpreted as an area under a curve — a dramatic and unexpected idea.

The *normal distribution* is the most important of all continuous distributions, and indeed of all distributions. Sections 10D–10G develop the basic theory of normal distributions, and explain how to apply them to practical problems.

There are many calculations in this chapter, as in Chapter 9. Your pen-and-paper work can be assisted in several ways, all of which can be found online:

- a scientific calculator and a table of values of the standard normal,
- a statistics calculator,
- a spreadsheet,
- specialised statistics software.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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10A Relative frequency

Relative frequencies were introduced in Chapter 11 of the Year 11 book because they are estimates of the probabilities of the outcomes of an experiment. They are needed again in this chapter, where probability theory is central once more.

A brief review of the mean and variance of a discrete distribution

In Year 11, Sections 11B and 11C introduced the expected value and the variance of a discrete random variable X. Let $p(x) = P(X = x)$. Then the expected value $\mu = \mathrm{E}(X)$ is

$$\mu = \mathrm{E}(X) = \sum x\,p(x), \quad \text{summing over all values of the distribution.}$$

This is the *weighted mean of the values, weighted according to their probabilities*.

The variance $\mathrm{Var}(X) = \sigma^2$ is the square of the standard deviation σ. It is the expected value of the squared deviation from the mean,

$$\mathrm{Var}(X) = \mathrm{E}\Big((X - \mu)^2\Big) = \sum (x - \mu)^2 p(x).$$

We proved also that the variance has an alternative form that is preferable when the mean is not an integer, and is therefore particularly suited to data,

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2 = \sum x^2 p(x) - \mu^2.$$

Setting out the calculations

Our model experiment in the Year 11 book was, 'throw four coins and record the number of heads'. The second rows in the tables below show the theoretical probabilities of obtaining 0, 1, 2, 3 or 4 heads.

A Here is the way we set out the calculations of mean and variance when we use the definition of variance as $\mathrm{E}(X - \mu)^2$.

x	0	1	2	3	4	Sum	
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	1	(a check)
$x\,p(x)$	0	$\frac{4}{16}$	$\frac{12}{16}$	$\frac{12}{16}$	$\frac{4}{16}$	2	(the mean μ)
$(x - \mu)$	-2	-1	0	1	2	—	
$(x - \mu)^2$	4	1	0	1	4	—	
$(x - \mu)^2 p(x)$	$\frac{4}{16}$	$\frac{4}{16}$	0	$\frac{4}{16}$	$\frac{4}{16}$	1	(the variance)

The mean is $\mu = 2$, the variance is $\sigma^2 = 1$, and the standard deviation is $\sigma = 1$.

B Here is our setting-out using the other variance formula $E(X^2) - \mu^2$. The calculations are more straightforward, whether or not μ is a whole number.

x	0	1	2	3	4	Sum	
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	1	(a check)
$x\,p(x)$	0	$\frac{4}{16}$	$\frac{12}{16}$	$\frac{12}{16}$	$\frac{4}{16}$	2	(the mean μ)
$x^2 p(x)$	0	$\frac{4}{16}$	$\frac{24}{16}$	$\frac{36}{16}$	$\frac{16}{16}$	5	(this is $\mathrm{E}(X^2)$)

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From the third row, $\quad E(X) = 2,$ which is the mean μ.

From the last row, $\quad \text{Var}(X) = E(X^2) - \mu^2$

$$= 5 - 2^2$$

$$= 1, \text{ which is } \sigma^2.$$

Cumulative distribution function

In Year 11 we did not discuss the cumulative distribution function, but it is easily defined. With the four coins, it is the function $F(x)$ obtained by adding all the probabilities up to a certain point:

x	0	1	2	3	4
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$
$F(x)$	$\frac{1}{16}$	$\frac{5}{16}$	$\frac{11}{16}$	$\frac{15}{16}$	1

For example, $\quad F(3) = p(0) + p(1) + p(2) + p(3),$

and in general, $\; F(x) = p(0) + p(1)) + \cdots + p(x),$ for $x = 0, 1, 2, 3, 4$.

Relative frequencies and the histogram and polygon

Relative frequency is the key connection between the probabilities in the theoretical distributions studied in Year 11 and the analysis of the datasets in Chapter 9 this year. In Year 11 we performed the experiment of tossing four coins 100 times, and obtained these results.

x	0	1	2	3	4	Sum
f	7	29	34	21	9	100
f_r	0.07	0.29	0.34	0.21	0.09	1

- The first line lists the *values* — the number x of heads can be 0, 1, 2, 3 or 4.
- The second line lists the *frequencies* f — the experiment was run 100 times.
- The third line lists the *relative frequencies* $f_r = \dfrac{f}{100}$ — divide by 100 trials.

These relative frequencies are *estimates of the probabilities* of tossing 0, 1, 2, 3 or 4 heads — they are often referred to as 'experimental probabilities'. Unless the experiments are biased in some way, these estimates will almost certainly be closer and closer to the theoretical probabilities as the number of trials increases.

Setting out the calculations using relative frequencies

Calculating the sample mean and variance of this dataset was explained in Section 11D of the Year 11 book.

x	0	1	2	3	4	Sum
f_r	0.07	0.29	0.34	0.21	0.09	1
xf_r	0	0.29	0.68	0.63	0.36	1.96
$x^2 f_r$	0	0.29	1.36	1.89	1.44	4.98

The sample mean $\bar{x}$, variance s^2 and standard deviation s can then be calculated as follows. These are *estimates* of the expected value $\mu = E(X)$, variance $\sigma^2 = \text{Var}(X)$ and standard deviation σ of the probability distribution.

$$\bar{x} = \sum x f_r \qquad s^2 = \sum x^2 f_r - \bar{x}^2 \qquad s = \sqrt{s^2}$$

$$= 1.96 \qquad = 4.98 - (1.96)^2 \qquad \doteqdot 1.07 \text{ heads (compare with } \sigma = 1)$$

$$\doteqdot 1.14 \quad (\text{compare with } \sigma^2 = 1)$$

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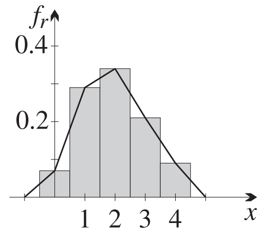
Histograms and polygons using relative frequencies

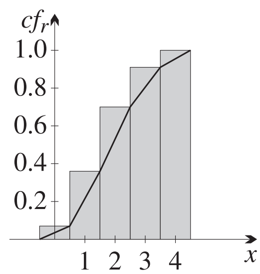


We can graph the relative frequencies in a *relative frequency histogram* and a *relative frequency polygon*. Look at the total area of the histogram rectangles, and the area under the polygon — the two areas are equal because each interval of the polygon cuts a triangle off one rectangle, and adds a triangle of the same area to an adjacent rectangle.

This particular histogram has a significant property — *the sum of the areas of the rectangles is* 1. This will always happen when the rectangles have width 1 because the sum of probabilities is 1. The Challenge Question 12 in Exercise 10A deals with the situation when the rectangles have width different from 1.

These remarks about areas are made in preparation for using integration in Section 10B.

Cumulative relative frequencies



In the fourth line of the table below, we have calculated the *cumulative relative frequencies*.

x	0	1	2	3	4	Sum
f	7	29	34	21	9	100
f_r	0.07	0.29	0.34	0.21	0.09	1
$c\,f_r$	0.07	0.36	0.70	0.91	1	

The cumulative relative frequencies have been graphed in a *cumulative relative frequency histogram*. A *cumulative relative frequency polygon* or *ogive* can also be drawn, starting with accumulation zero at $x = -\frac{1}{2}$, and finishing with accumulation 1 at $x = 4\frac{1}{2}$.

Each cumulative frequency estimates the probabilities of obtaining a particular number of heads or fewer. For example,

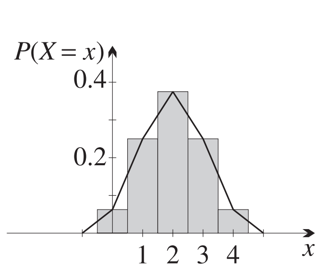
$$\text{estimated probability of tossing 3 or fewer heads} = 0.91.$$

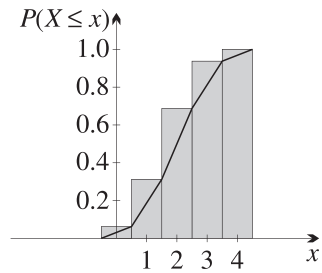
Discrete probability distributions and estimates from data

Relative frequencies obtained from data are estimates of probabilities. If we also know the theoretical probabilities, as we do for the four tossed coins, we can draw the same histograms and polygons as we have just done for data, but using probabilities and cumulative probabilities instead.

Here again is the theoretical probability distribution for the four tossed coins, and the two histograms and polygons — using decimals for easy comparison with the 100-trial data above. Compare these two diagrams with the diagrams above.

x	0	1	2	3	4
$P(X = x)$	0.0625	0.25	0.375	0.25	0.0625
$P(X \le x)$	0.0625	0.3125	0.6875	0.9375	1





Probability estimates obtained from data are rarely exactly the same as the theoretical probabilities.

1 RELATIVE FREQUENCIES AND CUMULATIVE RELATIVE FREQUENCIES

- For a dataset, the relative frequencies and cumulative relative frequencies are obtained by dividing through by the total frequency.
- The relative frequencies of a dataset are estimates of probabilities. For this reason, they are often referred to as ‘experimental probabilities’.
- The cumulative distribution function $F(x)$ of any numeric probability distribution is the probability that the score is less than or equal to x,

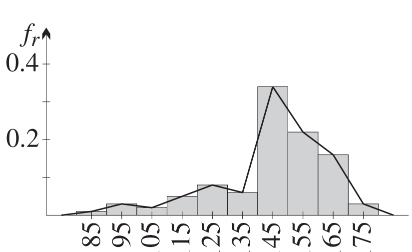
  $$F(x) = P(X \le x), \text{ for all } x \text{ in the domain.}$$

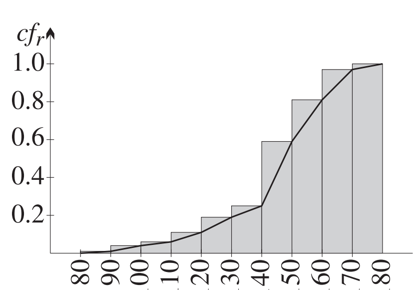
- The cumulative relative frequencies are estimates of the cumulative distribution function.
- The histograms and polygons of these relative frequencies and cumulative relative frequencies are drawn in the usual way.

Grouping data from a continuous random variable

When the underlying random variable is continuous, grouping is usually required. Here are the histograms and polygons for the heights x of 100 people from Section 9B, drawn this time using relative frequencies. The values of x are the class centres of the 10 cm intervals that were used in the grouping.

Class	80–90	90–100	100–110	100–110	100–110	100–110	100–110	100–110	100–110	100–110
x	85	95	105	115	125	135	145	155	165	175
f	1	3	2	5	8	6	34	22	16	3
f_r	0.01	0.03	0.02	0.05	0.08	0.06	0.34	0.22	0.16	0.03
F_r	0.01	0.04	0.06	0.11	0.19	0.25	0.59	0.81	0.97	1.00





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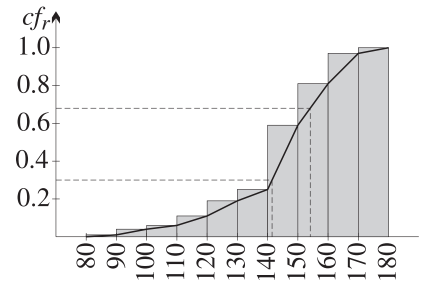
Deciles and percentiles

We have seen that quartiles divide the scores into four equal lists. They can be read approximately from the cumulative relative frequency polygon by drawing horizontal lines at height 0.25 for the lower quartile Q_1, height 0.5 for the median Q_2, and height 0.75 for the upper quartile Q_3.

The graph to the right shows how *deciles* and *percentiles* can be found similarly.

- To find the 3rd decile, draw a horizontal line of height 0.3.
- To find the 68th percentile, draw a horizontal line of height 0.68.

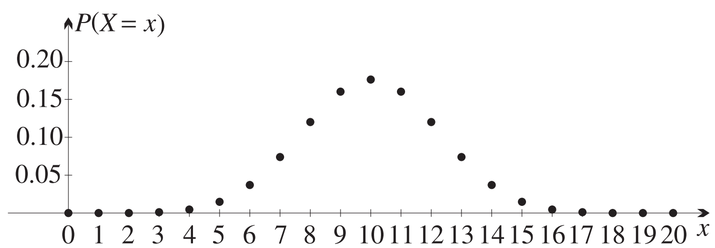


From the graph to the right, the 3rd decile is about 142 and the 68th percentile is about 154.

Note: This graphical method of finding medians and quartiles will give slightly different results from the method used in Sections 9A and 9C.
Once the cumulative histogram is drawn, it may seem more natural to use the graph, and graphical methods fit better with the integrals that are needed in the continuous distributions of this chapter.

When does a discrete distribution begin to look continuous?

As the number of values of a discrete distribution increases, the graph of the distribution may suggest a curve. For example, when 20 coins are thrown and the number of heads recorded, the diagram below shows the graph of the resulting probability distribution (the calculations are omitted). There definitely seems to be a curve involved here.



This chapter is about continuous distributions. The more coins there are, the more difficult the calculations become, and the more attractive it is to work out some way to approximate the discrete distribution by a continuous distribution. This is one of many ways in which continuous distributions are useful.

Probability and area

Here is a rather simple probability problem that requires area and cannot possibly be reduced to a discrete probability distribution.

Example 1 10A

A point-chook is wandering randomly around a 20 m × 20 m square enclosure. It is just as likely to be at any one place in the enclosure as any other. A circle 10 metres in radius has been inscribed in the square. If Farmer Brown looks out at the enclosure, what is the probability that the chook is inside the circle?

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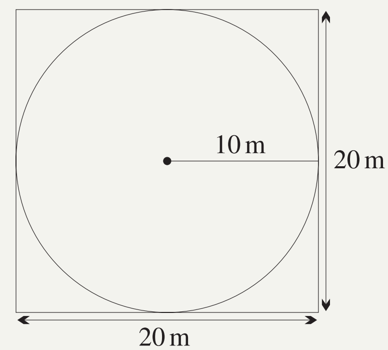
SOLUTION

$$\begin{aligned}
\text{Here} \qquad \text{area of enclosure} &= 20^2 \\
&= 400\ \text{m}^2, \\
\text{and} \qquad \text{area of circle} &= \pi r^2 \\
&= 100\pi\ \text{m}^2, \\
\text{so} \quad P\,(\text{chook is inside the circle}) &= \frac{\text{area of circle}}{\text{area of square}} \\
&= \frac{\pi}{4}.
\end{aligned}$$



In this problem it is completely obvious that we take the ratio of areas. Yet the calculations have nothing to do with the discrete sample spaces that we have spent so much time analysing. The answer $\frac{\pi}{4}$ is not even a rational number. The association of probability with area is fundamental to the way we shall deal with continuous probability distributions.

Exercise 10A

FOUNDATION

Questions 1–5 are a short review of ideas from Chapter 11 of the Year 11 book

1 The probabilities in a discrete probability distribution must all be non-negative and add to 1. Which of the following are valid discrete probability distributions?

A

x	1	2	3	4
$P(X = x)$	0.2	0.3	0.3	0.2

B

x	1	2	3	4
$P(X = x)$	1.4	0	−0.5	0.1

C

x	1	2	3	4
$P(X = x)$	0.15	0.2	0.4	0.25

D

x	1	2	3	4
$P(X = x)$	0.35	0.2	0.3	0.1

2 Two four-sided tetrahedral dice are thrown. The apex number on each die is read, and the sum of these two numbers is recorded. Let the random variable X be the outcome of this experiment.

a Record the possible outcomes and their probabilities in a probability table.

b Find:

i $P(X < 5)$ **ii** $P(X > 7)$

iii $P(X < 2)$ **iv** $P(X \leq 10)$

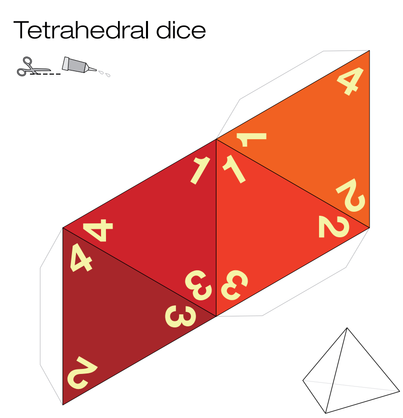
c Find the probability:

i that the sum is less than 4,

ii that the sum is odd,

iii that the sum is not 2,

iv that the sum is at least 6.



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3 A certain weighted spinner has five outcomes 1, 2, 3, 4 and 5. The spinner is known to be biased, and after a large number of trials, the relative frequencies of each outcome x were:

Score x	1	2	3	4	5	Total
f_r	0.1	0.2	0.45	0.15	0.1	
$x f_r$						
$x^2 f_r$						

a Complete the table.
b What is the significance of your total of row 2?
c Sum row 3, then use the formula $\bar{x} = \sum x f_r$ to calculate the sample mean $\bar{x}$.
d In your own words, what is the sample mean $\bar{x}$ measuring?
e Sum row 4, then use the formula $s^2 = \sum x^2 f_r - \bar{x}^2$ to calculate the sample variance.
f Hence calculate the sample standard deviation s, correct to two decimal places.
g In your own words, what is the sample standard deviation s measuring?
h What are the sample mean $\bar{x}$, and the sample standard deviation s, estimates of in this experiment?
i The spinner is thrown 100 times, and the outcome is recorded for each throw. State a reasonable estimate for the sum of these outcomes.

4 Data are recorded in the following table.

score x	3	4	5	6	7	Total
relative frequency f_r	0.04	0.21	0.35	0.25	0.15	

a Complete the table as in Question 3 to calculate the sample mean and sample standard deviation, correct to two decimal places.
b There are 5 values in this dataset and in the dataset in Question 3. Comment on the centre and spread of the data in this set, compared with Question 3.

5 These are the results from a class quiz where the maximum possible score is six marks.

score x	1	2	3	4	5	6	Total
frequency f	2	4	4	8	2	0	
$P(X = x)$							
$x \times P(X = x)$							
$x^2 \times P(X = x)$							

a Find the median and mode.
b Use the relative frequencies as probabilities to fill in the row for $P(X = x)$. Then complete the table.
c Use the formula $\mathrm{E}(X) = \sum xP(X = x)$ to estimate the expected value (also called the mean).
d Use the formula $\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \sum x^2 P(X = x) - \mu^2$ to estimate the variance.
e Find the standard deviation.
f Comment on the class's performance in this quiz by reference to the distribution of these quiz scores.
g The teacher likes to record all quiz results out of 30, so he multiplies all these results by 5 before recording them in his markbook. What will be the mean and standard deviation of this new set of marks? You may find it helpful to remember the formulae

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b \qquad \text{and} \qquad \mathrm{Var}(aX + b) = a^2\mathrm{Var}(X).$$

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Start of Foundation for Section 10A of this chapter

6 A simple experiment has generated the following table of discrete data:

score x	1	2	3
frequency f	2	5	3

a **i** Construct a frequency histogram for the data. Add the frequency polygon to your diagram by joining the centres of the data points. Remember to join the ends of the polygon back to the horizontal axis.

ii Calculate the total area of the histogram rectangles.

iii Calculate the area under the frequency polygon, bounded by the horizontal axis.

iv What do you notice?

b **i** Copy the table, and add a row showing the relative frequency, obtained by dividing the frequencies by the total number of scores, which is 10.

ii Construct a relative frequency histogram for the data, including the relative frequency polygon.

iii Calculate the total area of the histogram rectangles.

iv Calculate the area under the relative frequency polygon, bounded by the horizontal axis.

v What do you notice?

vi What is the relationship between the relative frequencies and the probabilities $P(X = x)$ of the experiment's probability distribution?

7 **a** Copy and complete the following table by filling in the relative frequencies, cumulative frequencies and cumulative relative frequencies.

x	1	2	3	4	5	6	7	Total
f	3	1	4	3	1	3	1	
f_r								
cf								—
cf_r								—

b Construct a cumulative relative frequency histogram and polygon (ogive). Mark your vertical axis in divisions of $\frac{1}{8} = 0.125$.

c Use the ogive to read off the three quartiles Q_1, Q_2 and Q_3.

8 Repeat question 7 for the following dataset. Mark your vertical axis in divisions of 0.1.

x	5	6	7	8	9	10	11
f	5	4	1	1	1	3	5

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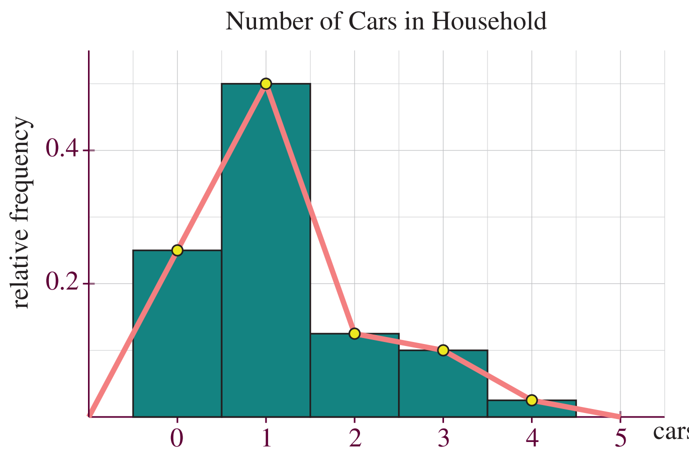
DEVELOPMENT

9 For town-planning purposes, the number of cars owned by each household in a suburb was recorded from census data. The results are displayed in the relative frequency histogram and polygon below. (This is a population, so the relative frequencies are probabilities.)



a What fraction of the households have no cars?

b What fraction of the households have fewer than 2 cars?

c What is the probability that a household chosen at random has three cars?

d Town planners will advise that additional on-street parking be provided if more than 40% of the households have 3 or more cars. Will they be advising that additional parking be provided? Explain your answer.

e Copy and complete the following table for this probability distribution.

x	0	1	2	3	4
$P(X = x)$					

f Show that the sum of the probabilities is 1. How is this related to the area of the rectangles of the histogram?

g Explain in your own words, and with reference to the graph above, why the area bounded by the relative frequency polygon and the horizontal axis will be the same as the area of the relative frequency histogram.

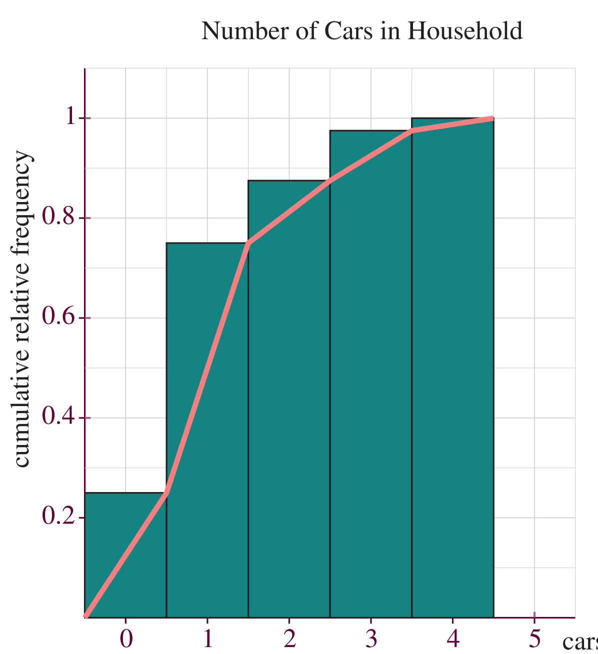
h Use your table to show that the mean number of cars per household is 1.15. What do you understand by this answer — how can a household have a fraction of a car?

i A street in the suburb is selected at random. If there are 100 households in the street, how many cars would you expect to belong to the households in the street in total? Are your assumptions for this estimate reasonable?

j Copy and complete the following table for the cumulative relative frequencies of this probability distribution.

x	0	1	2	3	4
$P(X = \leq x)$					

k A town planner constructs a cumulative relative frequency polygon and histogram from these data. His graph is shown to the right. Confirm that your data agree with this graph.



l By drawing horizontal lines at heights 0.25, 0.5 and 0.75, find the three quartiles Q_1, Q_2 and Q_3.

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10 a Construct a cumulative relative frequency histogram and polygon (ogive) for the following data.

x	0	1	2	3	4
f_r	0.1	0.3	0.2	0.1	0.3

b Estimate the 70th percentile (also called the 7th decile) by intersecting the horizontal line at height 0.7 with the ogive.

c Similarly estimate the first quartile Q_1 and the median Q_2 using horizontal lines and the ogive.

d Similarly estimate the third quartile Q_3 using the ogive.

e [Challenge] Use ratios on the last segment of the ogive to calculate the quartile Q_3 using the formula $3.5 + 0.05 \times \dfrac{4.5 - 3.5}{1 - 0.7}$. Compare your answer with part **d** .

11 To raise funds, a school running a musical performance also runs a set of stalls selling cheap items. The total amount spent at the stalls by each person attending was recorded.

Amount spent (\$)	0–1	1–2	2–3	3–4	4–5	Total
class centre x	0.50	1.50	2.50	3.50	4.50	—
frequency f	20	5	15	40	20	

(Any value in the range $0 \le$ price < 1 is recorded in the class 0–1 etc.)

a Find the median and mode.

b Copy the table and add a row for the relative frequency.

c Calculate the expected value $\mathrm{E}(X)$ and the variance $\mathrm{Var}(X)$.

d Construct a relative frequency histogram, including the relative frequency polygon.

e Find the probability that an attendee spends between:

i \$0–\$1 **ii** \$1–\$2 **iii** \$2–\$3 **iv** \$3–\$4 **v** \$4–\$5.

f Find the sum of the probabilities in part **e**. What area does this represent?

g An attendee is chosen at random and asked how much they spent.

i Is the amount spent more likely to be \$0–\$3, or \$3–\$4?

ii Is the amount spent more likely to be \$0–\$1, or \$3–\$4?

h If the school also charged an entry fee of \$2, find the expected value and variance of this new distribution $Y = X + 2$. What does the value $E(Y)$ represent?

CHALLENGE

12 We have seen several times that when the width of the rectangles is 1, the rectangles of the relative frequency histogram have total area 1. When the rectangles have a width w different from 1, however, then the total area is w. We can restore the area to 1 by using instead a scale of 'relative frequency per unit' on the vertical axis, as in this question.

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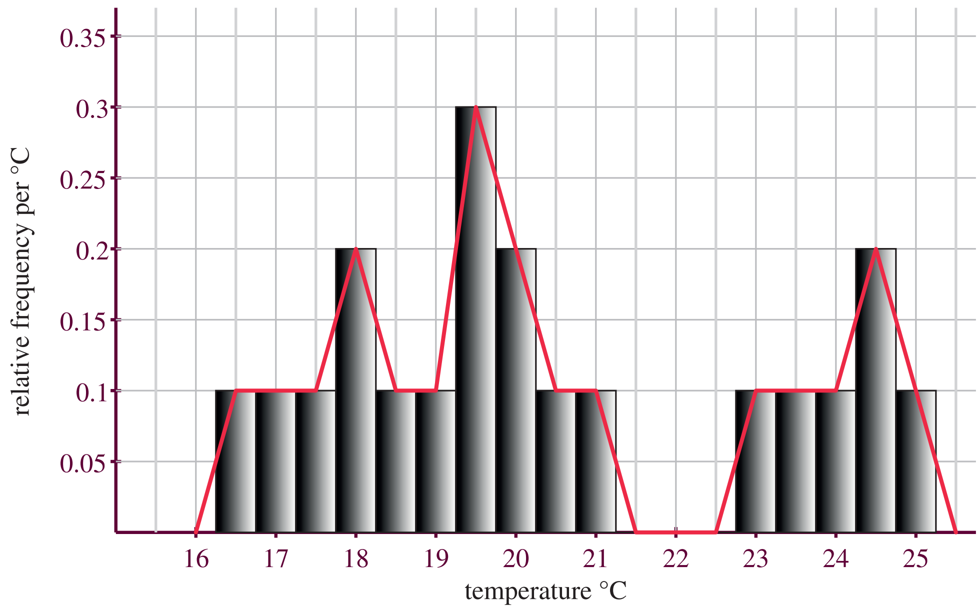
The maximum temperature for the day over a period of twenty days at a local weather station is recorded. These temperatures are displayed in the relative frequency histogram and polygon below, where the rectangles each have width 0.5°C.



In this histogram, temperature (more accurately, the temperature class) is shown on the horizontal axis, and the *relative frequency per unit of temperature* is shown on the vertical axis. Temperature has been grouped in classes of 0.5°.

a Show that the total area under the histogram is 1. (Hint: Each box on the grid has an area of $0.025 = \frac{1}{40}$.)

b With this adjustment, the probability that the temperature will lie in a given class (or classes) is the area of the corresponding rectangle (or rectangles).

  i Find the probability that the maximum temperature is between 19.25°C and 19.75°C.

  ii Find the probability that the maximum temperature is between 16.25°C and 17.25°C.

  iii Find the probability that a day chosen at random is warm, if a warm day is defined to be one with a maximum of more than 22°C.

c The probability of a given temperature is proportional to the height of the frequency polygon at that point.

  i Estimate the relative likelihood of the maximum temperature being 17°C as compared with 20°C.

  ii What is the mode, that is, the most likely maximum temperature?

d The frequency polygon gives an estimate of the shape of the continuous probability distribution that would be obtained by successively grouping the data in narrower and narrower classes of temperatures. Use the area under the frequency polygon to estimate the probability that the maximum temperature on a given day is between:

  i 16.5°C and 17.5°C,   **ii** 19°C and 20.5°C.

e Comment on the validity of using this histogram to decide on the probability of a given temperature at any time of the year.

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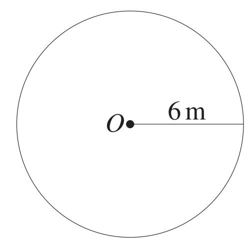
10B Continuous distributions

In a *continuous probability distribution*, the domain of the values is typically from a closed interval on the number line, such as $[0, 6]$. There are thus infinitely many values, which cannot even be listed. The probability of any one particular value is zero, and we want to talk instead about a probability such as $P(2 \le X \le 5)$, which is the probability that the value lies in the subinterval $[2, 5]$ of $[0, 6]$.

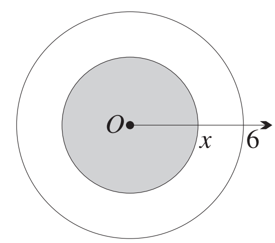
A cumulative distribution function

A point-chook is wandering randomly around a circular enclosure of radius 6 metres. It is just as likely to be in any one place in the enclosure as any other. Farmer Brown wants to know how far the chook is from the water at the centre O of the circle.

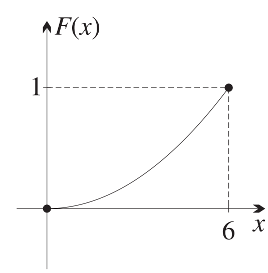


There are infinitely many distances from the centre within the enclosure. The probability that the chook is say exactly 2 metres from the centre is zero. Thus the tabular methods used with discrete probability distributions are useless here.

We can, however, approach the situation using cumulative frequency. Let $F(x)$ be the probability that when Farmer Brown looks out, the chook is no more than x metres from the centre.

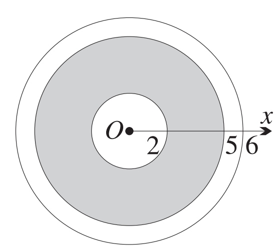


$$\begin{aligned} F(x) &= \frac{\text{area of inner circle}}{\text{area of whole circle}} \\ &= \frac{\pi x^2}{\pi \times 6^2} \\ &= \tfrac{1}{36}x^2, \quad \text{where } 0 \le x \le 6. \end{aligned}$$



This function is a *cumulative distribution function* or *CDF*. It is continuous, and increases from $F(0) = 0$ on the left to $F(6) = 1$ on the right. A cumulative function is always non-decreasing. It can also be used to solve many more problems. For example, we can find the probability that the chook is between 2 metres and 5 metres from the centre by subtraction,

$$\begin{aligned} P(\text{chook is 2–5 metres from the centre}) &= F(5) - F(4) \\ &= \tfrac{1}{36}(25 - 4) \\ &= \tfrac{21}{25}. \end{aligned}$$



We can also calculate the median and the quartiles of the probability distribution in the obvious way.

For the first quartile,
put $F(x) = \frac{1}{4}$

$$\begin{aligned} \tfrac{1}{36}x^2 &= \tfrac{1}{4} \\ x^2 &= 9 \\ x &= 3. \end{aligned}$$

For the median,
put $F(x) = \frac{1}{2}$

$$\begin{aligned} \tfrac{1}{36}x^2 &= \tfrac{1}{2} \\ x^2 &= 18 \\ x &\doteqdot 4.24. \end{aligned}$$

For the third quartile,
put $F(x) = \frac{3}{4}$

$$\begin{aligned} \tfrac{1}{36}x^2 &= \tfrac{3}{4} \\ x^2 &= 27 \\ x &\doteqdot 5.20. \end{aligned}$$

We still have not precisely defined continuous probability distributions, but let us nevertheless summarise the discussion above.

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2 THE CUMULATIVE DISTRIBUTION FUNCTION

Let a continuous random variable X have values from a closed interval $[a, b]$.

- The *cumulative distribution function* or *CDF* for X is the function

$$F(x) = P(a \le X \le x), \text{ for all } x \text{ in the interval } [a, b].$$

- The cumulative distribution function is continuous and non-decreasing, with

$$F(a) = 0 \qquad \text{and} \qquad F(b) = 1.$$

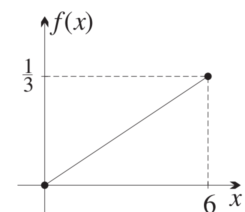
- It can be used to calculate medians, quartiles and percentiles.

A probability density function

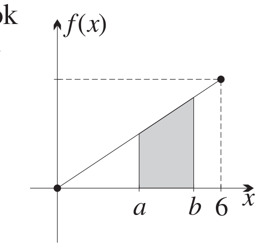
With a discrete distribution, the cumulative frequencies were obtained by adding all the probabilities up to a certain point — the same process produces the cumulative frequencies of a dataset. The continuous analogue of addition is integration, so we should expect the cumulative distribution function $F(x) = \frac{1}{36}x^2$ to be an *integral* over the values up to a certain point.

The fundamental theorem of calculus tells us that $F(x)$ is the integral of its derivative $F'(x)$. So we differentiate $F(x)$ to obtain what is called the *probability density function* $f(x)$,

$$\begin{aligned} f(x) &= \frac{d}{dx}\left(\tfrac{1}{36}x^2\right) \\ &= \tfrac{1}{18}x, \text{ where } 0 \le x \le 6. \end{aligned}$$



This linear graph of $f(x)$ is sketched above. It does not tell us the probability that the chook is x metres from the centre, because that probability is zero. Instead, it allows us to find by integration the probability that the chook is in some range of distances from centre.



The probability of the chook being in some range of positions is the area under the curve, which is found by integration.

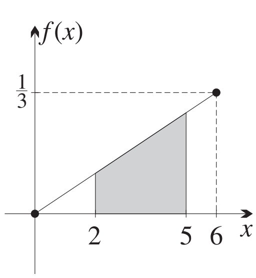
$$P(a \le x \le b) = \int_a^b f(x)\,dx.$$

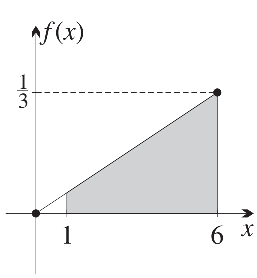
For example,

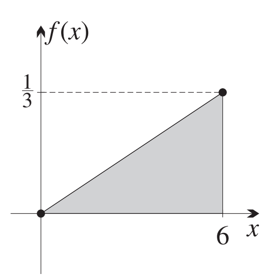
$$\begin{aligned} P(2 \le x \le 5) &= \int_2^5 \tfrac{1}{18}x\,dx \\ &= \left[\tfrac{1}{36}x^2\right]_2^5 \\ &= \tfrac{1}{36}(25 - 4) \\ &= \tfrac{21}{25}, \end{aligned}$$

$$\begin{aligned} P(x \ge 1) &= \int_1^6 \tfrac{1}{18}x\,dx \\ &= \left[\tfrac{1}{36}x^2\right]_1^6 \\ &= \tfrac{1}{36}(36 - 1) \\ &= \tfrac{35}{36}, \end{aligned}$$

$$\begin{aligned} P(0 \le X \le 6) &= \int_0^6 \tfrac{1}{18}x\,dx \\ &= \left[\tfrac{1}{36}x^2\right]_0^6 \\ &= \tfrac{1}{36}(36 - 0) \\ &= 1. \end{aligned}$$







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This probability density function has two important properties:

1 $f(x)$ is never negative, because $F(x)$ is cumulative and never decreases.

2 $\int_0^6 f(x)\,dx = 1$, because the chook is somewhere in the enclosure.

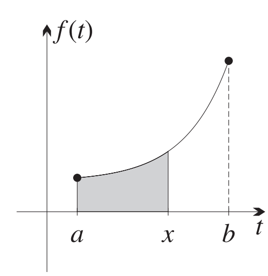
It turns out that the probability density function is more important than the cumulative distribution function when characterising a continuous distribution and working with it. It is also the best way to give a formal definition of a continuous probability distribution.

3 PROBABILITY DENSITY FUNCTIONS

- A *probability density function*, or *PDF*, is a function defined on a closed interval $[a, b]$ and satisfying two properties:
  1 $f(x) \geq 0$, for $a \leq x \leq b$.
  2 $\int_a^b f(x)\,dx = 1$.
- A *continuous probability distribution* is defined to be a probability distribution described by a probability density function.
- A global maximum of the probability density function is called a *mode*.
- Probability is area under the curve. That is, for all closed subintervals $[h, k]$,
$$P(h \leq X \leq k) = \int_h^k f(x)\,dx.$$
- Later, we will allow a to be replaced by $-\infty$, and b to be replaced by ∞.

The probability of any particular value h, however, is always zero. That is,
$$P(X = h) = \int_h^h f(x)\,dx = 0.$$
We remarked on such integrals in Box 8 of Section 4C. Because of this, it doesn't really matter whether $\leq$ or $<$ is used for intervals, or whether two adjacent intervals, say $[2, 4]$ and $[4, 5]$, overlap at the endpoints.

The CDF is the signed area function of the PDF



Suppose again that $f(x)$ is a probability density function defined on the closed domain $[a, b]$. Using the language of Section 4D, we can now use integration to define the cumulative distribution function $F(x)$ of $f(x)$ as the signed area function of the PDF,
$$F(x) = \int_a^x f(t)\,dt, \text{ for } a \leq x \leq b.$$

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4 THE CUMULATIVE DISTRIBUTION FUNCTION AS THE SIGNED AREA FUNCTION

The *cumulative distribution function* $F(x)$, or *CDF* for short, of a probability density function, or PDF, is the signed area function

$$F(x) = P(X \le x) = \int_a^x f(t)\,dt, \quad \text{for } a \le x \le b,$$

and conversely $f(x) = F'(x)$ (apart possibly from isolated sharp corners).

Be pedantic and say 'probability density function' and 'cumulative distribution function'. 'Density' means at a point, and 'distribution' means over a range.

Uniform continuous distributions

An important special case of continuous probability distributions is a *uniform continuous distribution*. This is a distribution whose PDF is a constant function.

Example 2 10B

Tran does not know the times when trains leave Lakeside Station, but he does know that they leave precisely every fifteen minutes.

a He wants to know about the probability distribution of his waiting time if he arrives at the station at a random time, and what the PDF and CDF are.

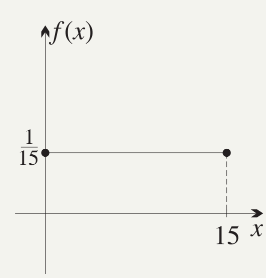
b He also wants to know the median and the 45th percentile, and the probability that he will wait between 5 and 10 minutes.

SOLUTION

a The waiting time x is anything from 0 to 15 minutes, and we have no reason to prefer any waiting time from any other waiting time. This means that the probability density function $f(x)$ is a constant function in the interval [0, 15], and the probability distribution is therefore a *uniform continuous distribution* with values from the closed interval [0, 15].

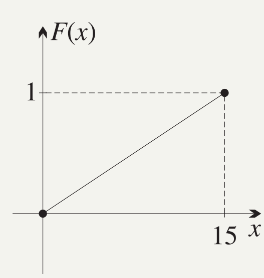
Because the area under the PDF is exactly 1 (the total probability),

$$f(x) = \tfrac{1}{15}, \quad \text{for } 0 \le x \le 15.$$



The CDF $F(x)$ is then found by integrating,

$$\begin{aligned} F(x) &= \int_0^x \tfrac{1}{15}\,dt \\ &= \left[\tfrac{1}{15}t\right]_0^x \\ &= \tfrac{1}{15}x. \end{aligned}$$



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b To find the probability that he waits between 5 and 10 minutes, either integrate the PDF or use the CDF.

$$\begin{aligned} P(5 \le X \le 10) &= \int_0^5 \tfrac{1}{15}\,dt \\ &= \left[\tfrac{1}{15}x\right]_5^{10} \\ &= \tfrac{2}{3} - \tfrac{1}{3} \\ &= \tfrac{1}{3}, \end{aligned}$$

OR

$$\begin{aligned} P(5 \le X \le 10) &= F(10) - F(5) \\ &= \tfrac{2}{3} - \tfrac{1}{3} \\ &= \tfrac{1}{3}. \end{aligned}$$

For the median, put $F(x) = \tfrac{1}{2}$

$$\begin{aligned} \tfrac{1}{15}x &= \tfrac{1}{2} \\ x &= 7\tfrac{1}{2}. \end{aligned}$$

For the 45th percentile, put $F(x) = \tfrac{45}{100}$

$$\begin{aligned} \tfrac{1}{15}x &= \tfrac{9}{20} \\ x &= 6\tfrac{3}{4}. \end{aligned}$$

5 UNIFORM CONTINUOUS DISTRIBUTIONS

- A continuous distribution is called *uniform* if its density function is constant.
- Because the area under the graph is 1, a uniform continuous distribution defined on an interval $[a, b]$ has probability density function $= \dfrac{1}{b-a}$.

Example 3 10B

Find the value of k that makes each function a probability density function. Then find the corresponding CDF $F(x)$. Hence find the median and quartiles.

a $f(x) = k$, where $0 \le x \le 10$,

b $f(x) = kx$, where $0 \le x \le 10$.

SOLUTION

a Put

$$\begin{aligned} \int_0^{10} k\,dx &= 1 \\ \left[kx\right]_0^{10} &= 1 \\ 10k - 0 &= 1 \\ k &= \tfrac{1}{10}, \end{aligned}$$

so $f(x) = \tfrac{1}{10}$.

Hence

$$\begin{aligned} F(x) &= \int_0^x \tfrac{1}{10}\,dt \\ &= \left[\tfrac{1}{10}t\right]_0^x \\ &= \tfrac{1}{10}x. \end{aligned}$$

When $F(x) = \tfrac{1}{2}, x = 5$,
when $F(x) = \tfrac{1}{4}, x = 2\tfrac{1}{2}$,
when $F(x) = \tfrac{3}{4}, x = 7\tfrac{1}{2}$,
so $Q_1 = 2\tfrac{1}{2}$, $Q_2 = 5$ and $Q_3 = 7\tfrac{1}{2}$.

b Put

$$\begin{aligned} \int_0^{10} kx\,dx &= 1 \\ \left[\tfrac{1}{2}kx^2\right]_0^{10} &= 1 \\ 50k - 0 &= 1 \\ k &= \tfrac{1}{50}, \end{aligned}$$

so $f(x) = \tfrac{1}{50}x$.

Hence

$$\begin{aligned} F(x) &= \int_0^x \tfrac{1}{50}t\,dt \\ &= \left[\tfrac{1}{100}t^2\right]_0^x \\ &= \tfrac{1}{100}x^2. \end{aligned}$$

When $F(x) = \tfrac{1}{2},\ x = 5\sqrt{2}$,
when $F(x) = \tfrac{1}{4},\ x = 5$,
when $F(x) = \tfrac{3}{4},\ x = 5\sqrt{3}$,
so $Q_1 = 5$, $Q_2 = 5\sqrt{2}$ and $Q_3 = 5\sqrt{3}$.

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Piecewise-defined probability density functions

The next worked example shows how to deal with a probability density function that is piecewise defined.

Example 4 10B

A probability density function is defined piecewise by

$$f(x) = \begin{cases} k(4 + x), & \text{for } -4 \le x \le 0, \\ k(4 - x), & \text{for } 0 \le x \le 4. \end{cases}$$

a Find the value of the constant k. Hence write the equation of $f(x)$ and sketch it.

b What is the probability that $0 \le X \le 2$?

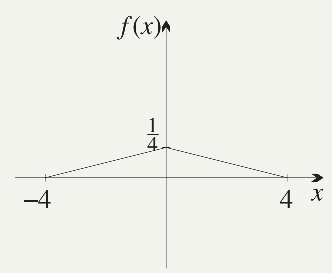
c Why is the median zero, and what is the mode?

d Find the CDF for $-4 \le x \le 0$, and the CDF for $0 \le x \le 4$. Then sketch the whole CDF.

SOLUTION

a The integral over the domain $[-4, 4]$ must be 1. Using areas of triangles is easier, but here is how to integrate piecewise over the domain by dissection.

$$\begin{aligned}\int_{-4}^{4} f(x)\,dx &= \int_{-4}^{0} k(4 + x)\,dx + \int_{0}^{4} k(4 - x)\,dx \\ &= k\Big[4x + \tfrac{1}{2}x^2\Big]_{-4}^{0} + k\Big[4x - \tfrac{1}{2}x^2\Big]_{0}^{4} \\ &= k\Big((0 + 0) - (-16 + 8) + (16 - 8) - (0 - 0)\Big) \\ &= 16k,\end{aligned}$$



so for the integral to be 1, the value of k must be $k = \frac{1}{16}$.

The function is therefore $f(x) = \begin{cases} \frac{1}{16}(4 + x), & \text{for } -4 \le x \le 0, \\ \frac{1}{16}(4 - x), & \text{for } 0 \le x \le 4. \end{cases}$

b
$$\begin{aligned}P(0 \le X \le 2) &= \int_0^2 f(x)\,dx \\ &= \tfrac{1}{16}\int_0^2 (4 - x)\,dx && \text{(only the right-hand branch is relevant)} \\ &= \tfrac{1}{16}\Big[4x - \tfrac{1}{2}x^2\Big]_0^2 \\ &= \tfrac{1}{16}\Big((8 - 2) - (0 - 0)\Big) \\ &= \tfrac{3}{8} && \text{(or use the area of a trapezium).}\end{aligned}$$

c The areas to the left and right of $x = 0$ are equal, so the median is 0.
The mode is also $x = 0$, because there is a global maximum there.

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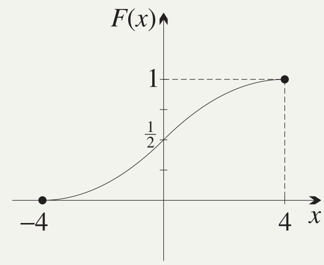
d For $-4 \le x \le 0$,

$$F(x) = \frac{1}{16}\int_{-4}^{x}(4 + t)\,dt$$
$$= \frac{1}{32}\Big[(4 + t)^2\Big]_{-4}^{x}$$
$$= \frac{1}{32}\Big((4 + x)^2 - 0\Big)$$
$$= \frac{1}{32}(4 + x)^2.$$

Hence $F(0) = \frac{1}{2}$, so for $0 \le x \le 4$,

$$F(x) = \frac{1}{2} + \frac{1}{16}\int_{0}^{x}(4 - t)\,dt$$
$$= \frac{1}{2} - \frac{1}{32}\Big[(4 - t)^2\Big]_{0}^{x}$$
$$= \frac{1}{2} - \frac{1}{32}\Big((4 - x)^2 - 16\Big)$$
$$= 1 - \frac{1}{32}(4 - x)^2.$$



Distributions with unbounded domains

We have been using integrals (and sometimes area formulae) to find areas. In many important situations, the probability density function has a horizontal asymptote, however, which means that the possible values extend to infinity. For example, the diagram in Section 10A involving 20 tossed coins suggested approximating that discrete distribution by a continuous curve with asymptotes on the left and right.

The radioactive isotope iodine-131 is often used in medicine for the treatment of thyroid cancer. It has a half-life of about 8 days. Suppose that we isolate a single nucleus of iodine-131, observe it constantly, and record the time X in days before it decays. Then using the fact that the isotope has a half-life of 8 days,

$$P(X > 8) = \tfrac{1}{2}, \qquad P(X > 16) = \tfrac{1}{4}, \qquad P(X > 24) = \tfrac{1}{8}, \qquad \ldots$$

and taking the complementary events,

$$P(X \le 8) = \tfrac{1}{2}, \qquad P(X \le 16) = \tfrac{3}{4}, \qquad P(X \le 24) = \tfrac{7}{8}, \qquad \ldots$$

In general, $\quad P(X \le 8n) = 1 - 2^{-n}$.

This formula holds for all real values of $n \ge 0$, not just for whole numbers,
and to find $P(X \le x)$, put $\quad x = 8n$,

then $n = \frac{1}{8}x$, giving $\quad P(X \le x) = 1 - 2^{-\frac{1}{8}x}$.

This last formula is the cumulative distribution function $F(x)$ for the experiment. The next worked example continues the story. We first change to base e, and write

$$F(x) = 1 - e^{-kx}, \qquad \text{where } k = \tfrac{1}{8}\,\text{In}\,2 = 0.08664\ldots \qquad \text{(store in memory).}$$

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Example 5 10B

Let $f(x)$ and $F(x) = e^{-kx}$, where $k = \frac{1}{8}\,\text{In}\,2$, be the PDF and CDF respectively for the experiment described above, observing the time x days that an iodine-131 nucleus survives before decaying.

a Explain why the domain of possible values is the unbounded interval $[0, \infty)$

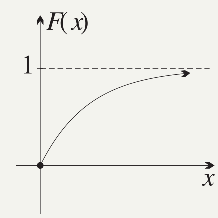
b Find the formula for the PDF, and sketch the CDF and PDF.

c Find the median, and show that it is the half-life.

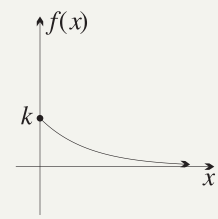
d Find the probabilities that it decays on the first day and after the first day.

SOLUTION

a The experiment is extremely unlikely to last beyond a month or two, but it is minutely possible that it will continue for 10 years or even more. Thus we use the unbounded interval $[0, \infty)$ for the domain of possible values.



b The CDF is $F(x) = 1 - e^{-kx}$.
Differentiating, $f(x) = ke^{-kx}$, which is the PDF.



c To find the median, put

$$\begin{aligned} F(x) &= 0.5 \\ 1 - e^{-kx} &= \tfrac{1}{2} \\ e^{-kx} &= \tfrac{1}{2} \\ kx &= \text{In}\,2, \end{aligned}$$

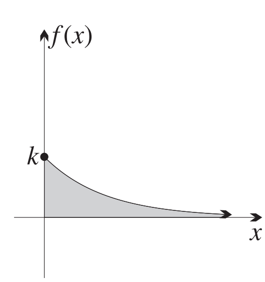
and using calculator or logs, $x = 8$ days, which is the half-life.

d
$$\begin{aligned} P(X \le 1) &= F(1) \\ &= 1 - e^{-k} \\ &\doteqdot 0.083 \end{aligned} \qquad \begin{aligned} P(X > 1) &= 1 - P(X \le 1) \\ &= e^{-k} \\ &\doteqdot 0.917 \end{aligned}$$

Challenge — improper integrals

Worked Example 4 above is quite sufficient preparation for the normal distribution later in the chapter. Readers may ask, however, how we can reasonably say that the area under the curve in the unbounded interval $[0, \infty)$ is 1 square unit, when it runs off to infinity! The integral involved here is called an *improper integral* — here is how to deal with it. The PDF is $y = ke^{-kx}$.

$$\begin{aligned} \text{area under curve over the interval } [0, \infty) &= \int_0^\infty ke^{-kx}\,dx \\ &= \Big[-e^{-kx}\Big]_0^\infty. \end{aligned}$$



Substituting $x = 0$ gives a value -1 for the primitive.
We cannot substitute $x = \infty$ because ∞ is not a number,
but we can take the limit of $-e^{-kx}$ as $x \to \infty$, which is 0,

$$\begin{aligned} \text{so} \qquad &= \int_0^\infty ke^{-kx}\,dx = 0 - (-1) \\ &= 1. \end{aligned}$$

Thus we can reasonably say that the unbounded shaded area is 1 square unit.

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Example 6 10B

a Find the shaded area under the curve $y = \frac{1}{x^2}$ over the interval $(1, \infty)$.

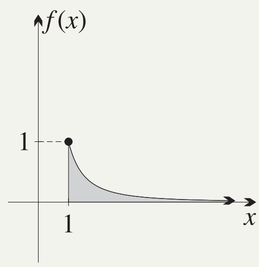
b Hence show that $y = \frac{1}{x^2}$, for $x \geq 1$, is a PDF, and find the CDF.

SOLUTION

a The improper integral over the closed interval $[1, \infty)$ is

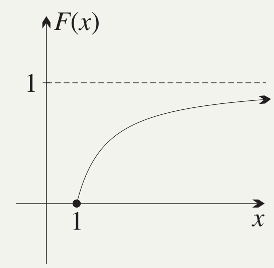
$$\int_1^\infty \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^\infty$$

When $x = 1$, the primitive is -1.
We cannot substitute ∞ because ∞ is not a number, but we can take the limit as $x \to \infty$, which is 0.



so
$$\int_1^\infty \frac{1}{x^2}\,dx = 0 - (-1) = 1.$$

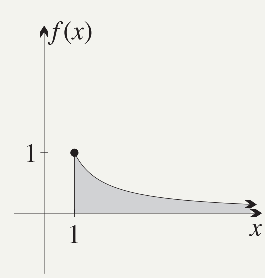
b Thus the area is 1 square unit, and the function $y = \frac{1}{x^2}$ is always positive in the interval $[1, \infty)$, so it is a PDF.



For the CDF,
$$F(x) = \int_1^x \frac{1}{t^2}\,dt = \left[-\frac{1}{t}\right]_1^x = 1 - \frac{1}{x}.$$

Example 7 10B

Show that the improper integral $\int_1^\infty \frac{1}{x}\,dx$ does not converge to a limit.



SOLUTION

Using the same procedure as before,

$$\int_1^\infty \frac{1}{x}\,dx = \int_1^\infty \left[\log_e x\right]_1^\infty.$$

Substituting $x = 1$ gives a value $\log_e 1 = 0$ for the primitive.
We cannot substitute $x = \infty$ because ∞ is not a number, but neither can we take the limit, because as $x \to \infty$, $\log_e x \to \infty$.
The conclusion is that the region has infinite area, and that the improper integral does not converge.

This example is rather striking because if we look only at the graph, the curves $y = e^{-x}$, $y = \frac{1}{x^2}$ and $y = \frac{1}{x}$ all look so similar in their asymptotic behaviour.

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Exercise 10B

FOUNDATION

1 **a** Sketch $f(x) = \frac{1}{2}$, where $0 \le x \le 2$. Then show that it satisfies the two conditions for a probability density function:

i Check from the graph that $f(x) \ge 0$, for all x in the domain.

ii Check that the area under the curve is 1, that is, that $\int_a^b f(x)\,dx = 1$.

b Repeat part **a** for $f(x) = \frac{1}{2}x$, where $0 \le x \le 2$.

c Repeat part **a** for $f(x) = \frac{1}{42}x$, where $4 \le x \le 10$.

2 Recall that a function $f(x)$ with domain the closed interval $[a, b]$ is called *a probability density function*, or *PDF* for short, if

$$f(x) \ge 0, \text{ for all } x \text{ in the domain} \qquad \text{and} \qquad \int_b^a f(x)\,dx = 1.$$

Determine whether or not each function is a probability density function. If it is a PDF, find its mode (look for global maxima).

a $f(x) = 3x^2$, where $0 \le x \le 1$

b $f(x) = \frac{1}{4}x$, where $1 \le x \le 5$

c $f(x) = \dfrac{4 - 2x}{3}$, where $0 \le x \le 3$

d $f(x) = (n + 1)x^n$, where $0 \le x \le 1$

e $f(x) = \frac{1}{2}\sin x$, where $0 \le x \le \pi$

f $f(x) = \frac{1}{12}(3x^2 + 2x)$, where $0 \le x \le 2$

3 Let $f(x) = \frac{3}{4}(x^2 - 4x + 3)$ be a function defined on the closed interval $\rightleftharpoons 0, 4]$.

a Show that $\int_0^4 f(x)\,dx = 1$.

b Show nevertheless that $f(x)$ is not a valid probability density function. (Hint: Sketch the graph of $y = f(x)$.)

4 For a distribution defined by a probability density function $f(x)$, the probability that x lies in the interval $[h, k]$ is the area given by $P(h \le X \le k) = \int_h^k f(x)\,dx$.

a Sketch the uniform probability density function $f(x) = \frac{1}{4}$, where $0 \le x \le 4$.

b Confirm that it satisfies the two requirements for a probability density function.

c By calculating areas, find:

i $P(0 \le X \le 1)$

ii $P(1 \le X \le 3)$

iii $P(X \le 2)$

iv $P(X) = 2$

v $P(X \le 3)$

vi $P(X \ge 1)$

d Confirm that $P(2 \le X \le 3) = P(x \le 3) - P(x \le 2)$.

5 Recall that for a probability density function, or PDF, defined on the interval $[a, b]$, the cumulative distribution function, or CDF, is $F(x) = \int_a^x f(t)\,dt$, for $a \le x \le b$.

For each PDF, calculate the CDF $F(x)$ and confirm that $F(b) = 1$.

a $f(x) = \frac{1}{32}x$, where $0 \le x \le 8$. (In this case $a = 0$ and $b = 8$.)

b $f(x) = \frac{3}{16}x^2$, where $-2 \le x \le 2$.

c $f(x) = \frac{3}{2}(1 - x^2)$, where $0 \le x \le 1$.

d $f(x) = \frac{1}{e}(e^x + 1)$, where $0 \le x \le 1$.

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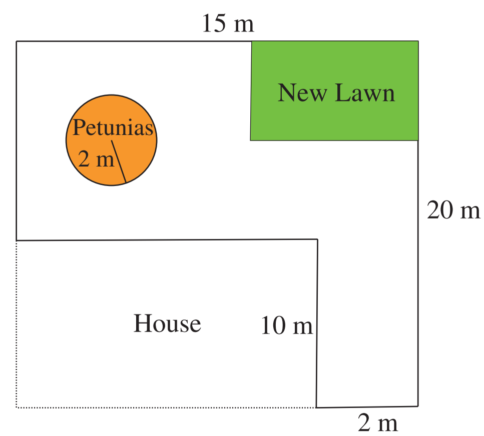
6 For the PDFs in parts **a** and **b** of Question **5**, use the CDF $F(x)$ to calculate:

a the median Q_2, by finding the value x such that $F(x) = 0.5$,

b the quartiles Q_1, by solving $F(x) = 0.25$, and Q_3, by solving $F(x) = 0.75$.

DEVELOPMENT

7 When Jack is at work, he shuts his dog Bud in the L-shaped backyard of his house. This is shown in the diagram to the right. Bud wanders around at random during the day, waiting for Jack to come home. Bud is the only cause of stress in Jack's quiet neighbourhood.



a When Bud is in the area directly to the right of the house, he will be anxious and howl for Jack to come home, to the distress of the neighbours, who shout at him. What is the probability that the neighbours will be stressed?

b If Bud is in the petunia patch, it will stress Jack's mother. What is the probability that Jack's mother will be stressed?

c If the neighbours or Jack's mother are shouting at Bud, Sally the cat cannot have a quiet sleep. What is the probability that Sally will be stressed?

d If the neighbours and Jack's mother aren't complaining, Jack's father is worried that Bud might be digging up his piece of new lawn. What is the probability that Jack's father will be stressed?

8 The function $y = 2x$, for $0 \le x \le 1$, is a probability density function.

a Sketch the graph and check that $f(x) \ge 0$.

b Use your diagram to show that the area bounded by the function and the x-axis is 1. Then check your result by integration.

c **i** Mark a point x between 0 and 1 on your diagram, and use area formulae to show that the cumulative distribution function is $P(X \le x) = x^2$.

ii Confirm your result by calculating the integral $P(X \le x) = \int_0^x 2t\,dt$.

d Use your expression for the cumulative distribution function to calculate the three quartiles Q_1, Q_2 and Q_3.

9 Find the value of the unknown constant c, given that each function $f(x)$ is a probability density function on the given domain.

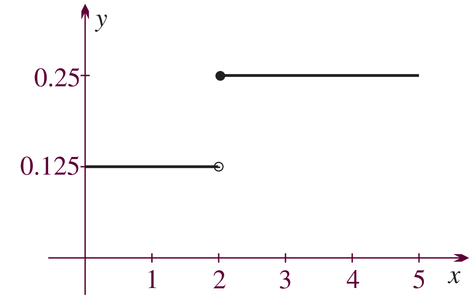
a $y = cx^4$, with domain $[0, 3]$

b $y = c$, with domain $[0, 6]$

c $y = c$, with domain $[-5, 5]$

d $f(x) = \frac{8}{3}(1 - x)$, with domain $[0, c]$

10 A function is graphed to the right.



a Verify that the function forms a valid PDF.

b Fill in the following table of values for the cumulative probabilities $P(X \le x)$.

x	0	1	2	3	4	5
$P(X \le x)$						

c Use your table to plot these points. Hence graph the cumulative probability function $P(X \le x)$ for $0 \le x \le 5$.

d Write down a formula for the CDF, writing your answer in piecewise notation.

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11 A probability density function is defined by:

$$f(x) = \begin{cases} c, & \text{for } 0 \le x \le 5, \\ 2c, & \text{for } 5 < x \le 10. \end{cases}$$

a Sketch the probability density function.
b Find the value of c.
c Find an expression for the cumulative distribution function.
d Use your cumulative distribution function to find $P(1 < X < 7)$.

12 Define a probability density function by $f(x) = \frac{3}{32}x(4 - x), 0 \le x \le 4$.
a Sketch the probability density function, and state its mode.
b Confirm that $\int_0^4 f(x)\, dx = 1$.
c Write down $P(X \le 2)$. What other property of the curve enables us to do this without calculating an integral?
d Evaluate $P(X \le 1)$ and $P(X > 1)$. Explain why your two results add to 1.
e Evaluate $P(X \le 0.5)$ and $P(X \ge 3.5)$. What do you notice about your answers?
f Determine the cumulative distribution function, defined by $F(x) = P(X \le x)$, using the formula $F(x) = \int_0^x f(t)\, dt$.
g Use your cumulative distribution function (CDF) to evaluate:
i $P(X < 1.5)$
ii $P(1 < X < 1.5) = P(X < 1.5) - P(X < 1)$
iii $P(3 < X < 3.5)$
iv $P(2 < X < 2.5)$
h Graph the CDF in your book.
i By evaluating $P(X < 2)$ using your CDF, confirm that 50% of the data lie to the left of the line $x = 2$.
j [Technology] Plot the cumulative distribution function and determine the upper and lower quartiles, defined by $P(X < Q_1) = 0.25$ and $P(X < Q_3) = 0.75$.

13 Define $f(x) = ce^{-x}$, where $0 \le x \le 1$.
a Sketch the curve $y = f(x)$.
b Find c, given that $f(x)$ is a probability density function.
c Find the cumulative distribution function.
d Find the quartiles Q_1, Q_2 and Q_3.

14 Grouping approximates a continuous distribution by a discrete distribution. A few trials of an experiment generated data in the interval 1.5–4.5, and the data were grouped in class intervals of width 1.

Class	1.5–2.5	2.5–3.5	3.5–4.5
Class centre	2	3	4
Relative frequency	0.3	0.4	0.3

a Use this dataset to draw a relative frequency histogram and a relative frequency polygon.
b Find the total area of the histogram and the area under the polygon.
c On a new set of axes, draw the cumulative relative frequency histogram and polygon.
d Estimate the three quartiles Q_1, Q_2 and Q_3 by reading off the corresponding values on the horizontal axis for the relative frequencies 0.25, 0.5 and 0.75.

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e After running more trials and taking finer intervals, the experimenter decides that the data can best be modelled by the curve

$$f(x) = \tfrac{3}{32}(x - 1)(5 - x), \text{ where } 1 \le x \le 5.$$

i Check that this curve is a probability density function.

ii Tabulate $f(x)$ for $x = 2$, 3 and 4, then graph it on top of the relative frequency polygon and compare the two (this would be easier with suitable technology).

iii Find the cumulative distribution function $F(X) = \int_1^x f(t)\,dt$, for $1 \le x \le 5$.

iv Substitute your three estimates for the quartiles into the cumulative distribution function. How close are your answers to 25%, 50% and 75%?

v [Technology] Graph the cumulative distribution function found by integration and read off the resulting estimates for the quartiles.

CHALLENGE

15 This question and the next both involve improper integrals where the upper limit is ∞. Infinity is not a number, so you cannot substitute ∞. Instead, take the limit of the primitive as $x \to \infty$.

Let $f(x) = \dfrac{1}{x^2}$, where $x \ge 1$. Notice that this function is defined on an unbounded domain.

a Show that $f(x) > 0$ and $\displaystyle\int_1^\infty f(x)\,dx = 1$.

b Evaluate the cumulative distribution function $F(x)$.

c Confirm that $F(x) \to 1$ as $x \to \infty$. Why is this significant?

d Evaluate the three quartiles Q_1, Q_2 and Q_3.

16 Repeat the previous question for the function $f(x) = e^{-x}$, where $x \ge 0$, changing the limits 1 and ∞ of the integral to 0 and ∞.

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10C Mean and variance of a distribution

The expected value (or mean) and the variance of a continuous probability distribution are obtained in almost the same ways as with discrete probability distributions. The main difference is that we replace the sum with its sigma notation $\sum$ by the integral with its integral notation $\int$. The Greek sigma and this early form of the letter S both correspond to the Latin letter S for 'sum'.

The expected value or mean of a continuous distribution

The *mean* or *expected value* of a discrete distribution is

$$\mathrm{E}(X) = \sum xp(x), \text{ summer over the whole distribution.}$$

The continuous analogy of addition is integration, so the continuous version is

$$\mathrm{E}(X) = \int_a^b xf(x)\,dx, \text{ integrating over the whole interval } [a, b].$$

The variance and standard deviation of a continuous distribution

The *variance* of a discrete distribution has two equivalent forms:

$$\mathrm{Var}(X) = E\left((X - \mu)^2\right) = \sum (x - \mu)^2 p(x),$$
$$\mathrm{Var}(X) = E(X^2) - \mu^2 = \sum x^2 p(x) - \mu^2.$$

The continuous analogies of these two forms are

$$\mathrm{Var}(X) = E\left((X - \mu)^2\right) = \int_a^b (x - \mu)^2 f(x)dx,$$
$$\mathrm{Var}(X) = E(X^2) - \mu^2 = \int_a^b x^2 f(x)dx - \mu^2.$$

The equality of these two expressions is proven in Q7 of Exercise 10C.

Example 8 10C

Apply all this to the chook at the start of Section 10B, where $f(x) = \frac{1}{18}x$.

SOLUTION

$$\begin{aligned}\mu &= \int_0^6 x \times \tfrac{1}{18}x\,dx \\ &= \tfrac{1}{54}\Big[x^3\Big]_0^6 \\ &= \tfrac{216}{54} - 0 \\ &= 4,\end{aligned}$$

so the chook's mean or expected distance from the centre is 4 metres.

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$$\sigma^2 = \int_0^6 (x-4)^2 \times \tfrac{1}{18}x\,dx$$
$$= \tfrac{1}{18}\int_0^6 (x^3 - 8x^2 + 16x)\,dx$$
$$= \tfrac{1}{18}\Big[\tfrac{1}{4}x^4 - \tfrac{8}{3}x^3 + 8x^2\Big]_0^6$$
$$= \tfrac{1}{18}(324 - 576 + 288)$$
$$= 2,$$

OR

$$\sigma^2 = \int_0^6 x^2 \times \tfrac{1}{18}x\,dx - 4^2$$
$$= \tfrac{1}{72}\Big[x^4\Big]_0^6 - 16$$
$$= (18 - 0) - 16$$
$$= 2.$$

As with discrete distributions, the second form is usually easier for calculations.

The *standard deviation* is the square root of the variance, and has the same units as the values, so here $\sigma = \sqrt{2}$ metres.

6 MEAN OR EXPECTED VALUE, AND VARIANCE

Let $f(x)$ be a probability density function on a closed interval $[a, b]$.

- The *mean* or *expected value* $\mu = \mathrm{E}(X)$ is
$$\mathrm{E}(X) = \int_a^b x\,f(x)\,dx.$$
- The *variance* $\sigma^2 = \mathrm{Var}(X)$ is the expected value of the squared deviation from the mean,
$$\mathrm{Var}(X) = E\big((X-\mu)^2\big) = \int_a^b (x-\mu)^2 f(x)\,dx$$
- Alternatively, and usually easier in calculations, the variance is the expected value of the square, minus the square of the mean,
$$\mathrm{Var}(X) = E(X^2) - \mu^2 = \int_a^b x^2 f(x)\,dx - \mu^2.$$
- The standard deviation σ is the square root of the variance.

Example 9 10C

Find the mean and standard deviation of each PDF.

a $y = \frac{1}{8}$, for $0 \le x \le 8$

b $y = \frac{1}{50}x$, for $0 \le x \le 10$

SOLUTION

a
$$\mu = \int_0^8 \tfrac{1}{8}x\,dx$$
$$= \Big[\tfrac{1}{16}x^2\Big]_0^8$$
$$= 4 - 0$$
$$= 4,$$

$$\sigma^2 = \int_0^8 \tfrac{1}{8}x^2\,dx - 4^2$$
$$= \Big[\tfrac{1}{24}x^3\Big]_0^8 - 16$$
$$= \frac{512}{24} - 0 - 16$$
$$= \frac{16}{3},$$
$$\sigma = \tfrac{4}{3}\sqrt{3}.$$

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b $y = \frac{1}{50}x$, for $0 \le x \le 10$

$$\mu = \int_0^{10} \tfrac{1}{50}x^2\,dx$$
$$= \left[\tfrac{1}{150}x^3\right]_0^{10}$$
$$= \tfrac{1000}{150} - 0$$
$$= 6\tfrac{2}{3},$$

$$\sigma^2 = \int_0^{10} \tfrac{1}{50}x^3\,dx - \left(6\tfrac{2}{3}\right)^2$$
$$= \left[\tfrac{1}{200}x^4\right]_0^{10} - \tfrac{400}{9}$$
$$= \frac{10\,000}{200} - 0 - \tfrac{400}{9}$$
$$= \frac{50}{9},$$
$$\sigma = \tfrac{5}{3}\sqrt{2}.$$

Exercise 10C

FOUNDATION

1 A function is defined by $f(x) = \frac{1}{10}$, where $0 \le x \le 10$.

a Show that $f(x)$ is a valid PDF (probability density function).

b Calculate the expected value using the formula $\mathrm{E}(X) = \displaystyle\int_a^b x\,f(x)\,dx$.

c Does your answer for the expected value agree with your understanding of expected value as an average value?

d Calculate the variance using the formula $\mathrm{Var}(X) = \displaystyle\int_a^b (x - \mu)^2 f(x)\,dx$, then find the standard deviation σ.

e Use the alternative formula for variance $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$ and confirm that your answer agrees with the previous result.

2 The previous question provides a mathematical model for selecting a random real number in the interval $[0, 10]$.

Use your calculator (or a spreadsheet) to generate a random number between 0 and 10 to as many decimal places as possible. Many calculators return a random number between 0 and 1, and you will need to multiply this answer by 10.

a Generate 20 such numbers, recording them in a table.

b Calculate the mean and standard deviation using your calculator.

c Do your results agree with the theoretical probabilities in the previous question?

d Our model includes the possibility of selecting a 10, but it is virtually certain that 10 will not be returned by the calculator's random number function. Does this affect the validity of our model and your results?

3 Define the function $f(x)$ by $f(x) = \frac{3}{2}x^2$, with domain $[-1, 1]$.

a Confirm that it is a valid PDF.

b Find the expected value $\mu = \mathrm{E}(X)$.

c Find the variance $\mathrm{Var}(X)$ and the standard deviation σ.

d Calculate $\displaystyle\int_{\mu-\sigma}^{\mu+\sigma} f(x)\,dx$ to determine what percentage of the population defined by this distribution lies within one standard deviation of the mean.

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4 Repeat Question 3 for:

a $f(x) = 2x$, with domain $[0, 1]$

b $f(x) = |x|$, with domain $[-1, 1]$

c $f(x) = \frac{3}{64}x^2$, with domain $[0, 4]$ (final answer correct to three decimal places)

DEVELOPMENT

5 Consider the function defined by $f(x) = \dfrac{1}{c}$, for $0 \le x \le c$, where $c > 0$.

a Is this function a valid PDF?

b Calculate $\mathrm{E}(X)$. Is your answer as expected?

c Calculate $\mathrm{Var}(X)$.

d Compare your answer with the special case in Question 1.

e Use the results $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$ and $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$ to find the expected value for the translated uniform probability distribution with density function $g(x) = \dfrac{1}{c}$, for $h \le x \le h + c$.

f Find the expected value and variance of the uniform probability distribution defined on the interval $h \le x \le k$.

6 a Show that the function $f(x)$ is a valid PDF,

$$f(x) = \begin{cases} \frac{1}{8}, & \text{for } 0 \le x < 2, \\ \frac{1}{4}, & \text{for } 2 \le x \le 5, \end{cases}$$

b Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

CHALLENGE

7 At the start of this chapter, we claimed that two expressions for the variance of a continuous distribution are equal,

$$\int_a^b (x - \mu)^2 f(x)\, dx = \int_a^b x^2 f(x)\, dx - \mu^2.$$

Prove this identity, starting with the LHS and expanding the integrand.

8 In Question 6 of Exercise 10A, we demonstrated that the area under a relative frequency polygon equals the area under a relative frequency histogram, and that both are equal to the total probability 1.

a Confirm that the relative frequency polygon in that question may be written piecewise as:

$$f(x) = \begin{cases} \frac{2}{10}x, & \text{for } 0 \le x \le 1, \\ \frac{1}{10}(3x - 1), & \text{for } 1 \le x \le 2, \\ \frac{1}{10}(-2x + 9), & \text{for } 2 \le x \le 3, \\ \frac{1}{10}(-3x + 12), & \text{for } 3 \le x \le 4. \end{cases}$$

b By integration, calculate $\mathrm{E}(X) = \displaystyle\int_0^4 x\, f(x)\, dx$ for the probability distribution using the PDF $f(x)$ defined above.

c Compare the answer obtained by calculating the expected value for the discrete distribution using the table of values in Question 6 of Exercise 10A.

d What is your conclusion about the PDF as a generalisation of the frequency polygon?

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9 This question and the next both involve improper integrals where the upper limit is ∞. Infinity is not a number, so you cannot substitute ∞. Instead, take the limit of the primitive as $x \to \infty$.

Define $f(x) = \dfrac{3}{x^4}$, where $x \geq 1$.

a Show that $f(x)$ is a valid PDF.

b Evaluate $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

c Calculate each probability.

i $P(X \leq 4)$ **ii** $P(X \geq 2)$ **iii** $P(2 \leq X \leq 5)$

d Find the cumulative distribution function $F(x) = P(X \leq x)$.

10 Consider the function $f(x) = e^{-x}$, where $x \geq 0$. In Question 15 of Exercise 10B, we showed that $f(x)$ is a valid PDF.

a Differentiate xe^{-x}, and hence integrate xe^{-x}.

b Evaluate $\mathrm{E}(X) = \displaystyle\int_0^\infty xe^{-x}\,dx$.

c Differentiate $x^2e^{-x} + 2xe^{-x} + 2e^{-x}$, and hence integrate x^2e^{-x}.

d Evaluate $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$ for the distribution with PDF $f(x)$.

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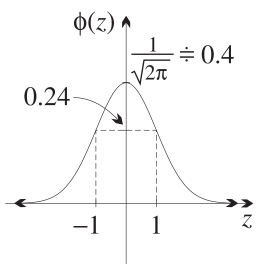
10D The standard normal distribution

Gauss and the early statisticians realised that one particular group of continuous distributions — the *Gaussian* or *normal distributions* — are particularly important. They occur in a wide variety of situations, and for reasons that we shall explain later, are involved in the study of every distribution, continuous or discrete.

Their graphs are generally referred to as *bell-shaped curves*, and every normal distribution can be obtained from every other normal distribution by shifting and stretching. We saw the shape of such a curve emerging when 20 coins were tossed at the end of Section 10A.



The graph of the *standard normal distribution* is sketched to the right. The equation of its probability density function is

$$\phi(z) = \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} \quad \text{(the Greek letter } \phi \text{ is phi, corresponding to Latin f).}$$

It is standard practice to use Z rather than X for the standard normal random variable, and z rather than x for its values, so that the PDF is $\phi(z)$.

Sketching the curve

We already have all the tools for sketching this function from its equation. The first thing is to stop looking at the complicated denominator $\sqrt{2\pi}$, which is just a constant. Use your calculator to find that $\dfrac{1}{\sqrt{2\pi}} \doteqdot 0.4$, and start thinking of the formula as $\phi(z) \doteqdot \frac{2}{5}e^{-\frac{1}{2}z^2}$, which looks much more friendly.

- The y-intercept is $\phi(0) = \dfrac{1}{\sqrt{2\pi}} \doteqdot 0.4$, because $e^0 = 1$.
- When z is non-zero, the index $-\frac{1}{2}z^2$ is negative, so $e^{-\frac{1}{2}z^2} < e^0 = 1$. Hence the value at $z = 0$ is a global maximum, and the mode is therefore $z = 0$.
- The function is defined for all values of z, and is positive for all values of z.
- The function is even, with line symmetry in the y-axis, because replacing z by $-z$ leaves the equation unchanged.
- As $z \to \infty$, and as $z \to \infty$, the index $-\frac{1}{2}z^2$ quickly becomes a large negative number, so $e^{-\frac{1}{2}z^2}$ quickly becomes an extremely small positive number. Thus the z-axis is a horizontal asymptote in both directions.
- There are points of inflection at $z = -1$ and $z = 1$ (both have y-coordinate $\dfrac{e^{-\frac{1}{2}}}{\sqrt{2\pi}} \doteqdot 0.24$). We have left the proof of this to the first of two Challenge paragraphs at the end of this section.

Why is $\phi(z)$ a probability density function?

We have now established that $\phi(z)$ has the graph shown above, but why is it a probability density function? Certainly we can see that it is always positive. But we also need to establish that

$$\int_{-\infty}^{\infty} \phi(z)\,dz = 1.$$

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Unfortunately, this integral cannot be established using the techniques in this course. This fact is one of a small number of things that readers will have to accept for now, and perhaps prove in later years.

Making the total area under the curve have the value 1 is the reason why the denominator $\sqrt{2\pi}$ has been put there, and proving the result requires a proof that $\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz = \sqrt{2\pi} \doteqdot 2.5$. The best that can be done is to confirm this result by trapezoidal rule approximations along the lines of Question 19 in Exercise 10D.

The mean and variance of the standard normal distribution

- The mean of the standard normal distribution is $\mu = 0$.
- Its variance is $\sigma^2 = 1$, and its standard deviation is therefore 1.

The fact that the mean is 0 is clear from the graph, because $y = \phi(z)$ is even, with line symmetry about the y-axis.

Establishing that the variance is 1, however, is Challenge (end of this section) because of some fancy integration.

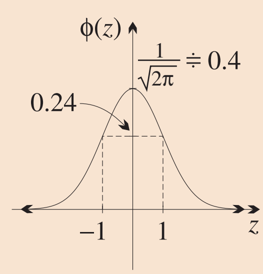
These results for the mean and standard deviation are closely tied to the turning point and inflections, so that μ and σ can be seen clearly on the graph.

- The mean $\mu = 0$ coincides with the maximum turning point at $z = 0$, which is the mode.
- The two inflections at $z = -1$ and $z = 1$ are each one standard deviation from the mean in opposite directions.

7 THE STANDARD NORMAL DISTRIBUTION

Let Z be the *standard normal random variable*.

- The probability density function of Z is $\phi(z) = \dfrac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}$.
- The graph of the PDF is a *bell-shaped curve*, with global maximum at $z = 0$ (the mode) and points of inflection at $z = 1$ and $z = -1$.
- The mean is $\mu = 0$ and the standard deviation is $\sigma = 1$.
- The points of inflection are each one standard deviation from the mean.



In Section 10E we will be shifting and stretching the standard normal distribution. Whenever you see a curve that looks even vaguely normal, always look first at the turning point, then look at the two inflections and quickly estimate the standard deviation by eye.

Integrating to find probabilities

The probability that a standard normal random variable Z lies within one standard deviation of the mean is

$$P(-1 \le Z \le 1) = \int_{-1}^{1} \phi(z)\,dz.$$

Now we have a major inconvenience — the primitive of the function $\phi(z)$ cannot be written in terms of our usual range of exponential, trigonometric and algebraic functions.

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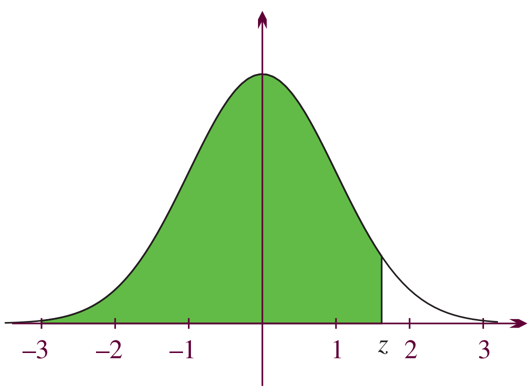
You have several options, all of which can be found online:

- Use a table of values for this integral — a short version is at the bottom of this page.
- Use a statistics calculator that has the values of these integrals built in.
- Use a spreadsheet that has these integrals amongst its functions.
- Use specialised statistics software.

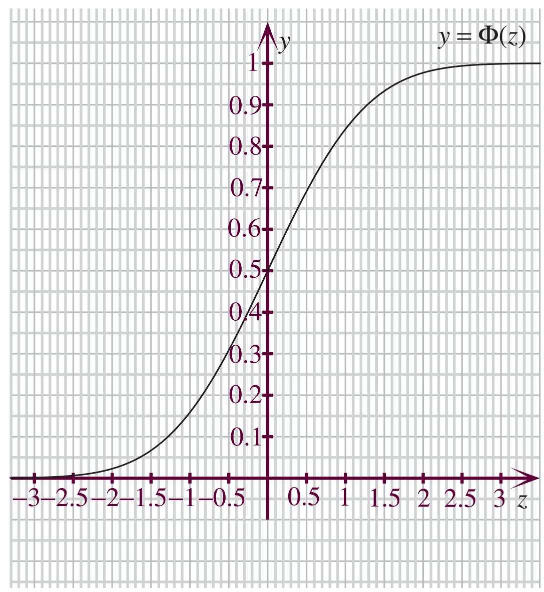
The cumulative distribution function

All these approaches will normally use the cumulative distribution function of the standard normal distribution. This CDF is usually denoted by $\Phi(z)$, using the uppercase version Φ of the Greek letter ϕ.

$$\Phi(z) = \int_{-\infty}^{z} \phi(t)\,dt.$$



This is the graph of the standard normal probability density function $\phi(z)$.



This is the graph of the standard normal cumulative distribution function $\Phi(z)$.

The new curve $y = \Phi(z)$ has two horizontal asymptotes, $y = 0$ on the left, and $y = 1$ on the right. Because also $\phi(z)$ is even, $\Phi(z)$ has point symmetry in $(0, 0.5)$.

Here then are some further details about finding values of $\Phi(z)$.

- Below is a short table of values of $\Phi(z)$ for $0 \le z < 4$, in steps of 0.1.
- Statistical calculators should have this function built in.
- In Excel, the function is `NORM.S.DIST`. The function has two arguments.
  - The first argument is the value of z (or the cell containing that value).
  - Set the second argument to true to obtain the value of the CDF $\Phi(z)$, and set it to `false` for the value of the PDF $\phi(z)$.
- Other spreadsheets, and online resources, will have their own rules.

The following short table of values of $\Phi(z)$ will be quite sufficient for most purposes in this course. Because of the even symmetry of the PDF $\phi(z)$, there is no need to give values of the CDF $\Phi(z)$ for negative values of z.

	first decimal place									
z	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0.	0.5000	0.5398	0.5793	0.6179	0.6554	0.6915	0.7257	0.7580	0.7881	0.8159
1.	0.8413	0.8643	0.8849	0.9032	0.9192	0.9332	0.9452	0.9554	0.9641	0.9713
2.	0.9772	0.9821	0.9861	0.9893	0.9918	0.9938	0.9953	0.9965	0.9974	0.9981
3.	0.9987	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

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Calculating probabilities of a normally distributed random variable

Calculating other probabilities for Z requires juggling integrals, preferably while looking at a graph of the PDF $y = \phi(z)$. Always keep two things in mind.

- The total area under the curve $y = \phi(z)$ is 1.
- The curve $y = \phi(z)$ is even — it has line symmetry in the y-axis.

The next worked example demonstrates all the methods required.

Example 10 10D

a Look up $P(Z \le 2)$ and illustrate it as an area under $y = \phi(z)$.

b Illustrate each probability as an area under $y = \phi(z)$. Then calculate it using the value of $\Phi(2)$ found in part **a**. Keep looking back to the graph in part **a** while you juggle the intervals.

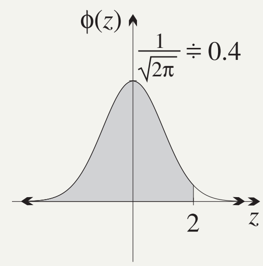
i $P(Z \ge 2)$ **ii** $P(Z \le -2)$ **iii** $P(0 \le Z \le 2)$ **iv** $P(-2 \le Z \le 2)$

SOLUTION

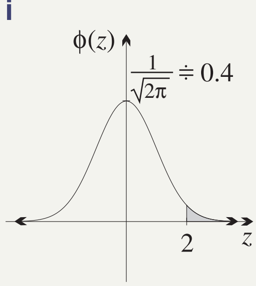
a From the table,

$$P(Z \le 2) = \Phi(2) = 0.9772.$$

The area under $y = \phi(z)$ corresponding to $\Phi(2)$ is shaded in the diagram to the right.

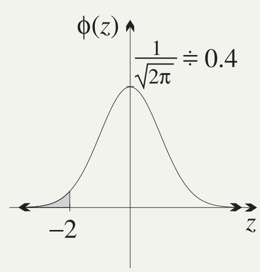


b **i**



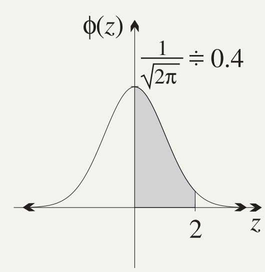
$P(Z \ge 2)$

ii



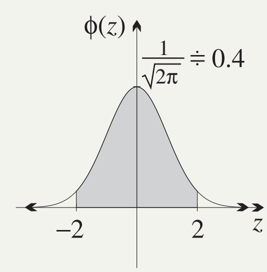
$P(Z \le -2)$

iii



$P(0 \le Z \le 2)$

iv



$P(-2 \le Z \le 2)$

i $P(Z \ge 2) = 1 - P(Z \le 2)$, because the total area is 1,
$\doteqdot 1 - 0.9772$
$\doteqdot 0.0228.$

The probability that $Z = 2$ exactly is zero, so there is no need to distinguish between $\le$ and $<$ or between $\ge$ and $>$.

ii $P(Z \le -2) = P(Z \ge 2)$, because $\phi(z)$ is even,
$\doteqdot 0.0228$, from part **a**.

iii $P(0 \le Z \le 2)$
$= \Phi(2) - \Phi(0)$, using subtraction of areas,
$\doteqdot 0.9772 - 0.5$, because exactly half the scores are below the mean,
$\doteqdot 0.4772.$

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iv $P(-2 \le Z \le 2)$
$= \Phi(2) - \Phi(-2)$
$\doteqdot 0.9772 - 0.0228$, by part **b**,
$\doteqdot 0.9544$,

OR

$P(-2 \le Z \le 2)$
$= 2 \times P(0 \le Z \le 2)$, by symmetry,
$\doteqdot 2 \times 0.4772$, by part **c**,
$\doteqdot 0.9544$.

Example 11 10D

a Explain how to find $\Phi(0.7)$ from the table, and illustrate it.

b Use symmetry and the table of values of $\Phi(z)$ to find:

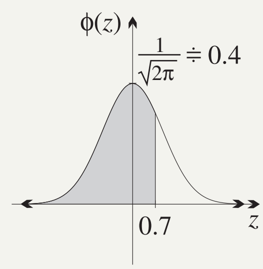
i $P(-2.5 \le Z \le -0.3)$ **ii** $P(-2.9 \le Z \le 0.6)$

SOLUTION

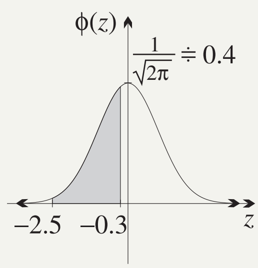
a To find $\Phi(0.7)$ from the table (as illustrated to the right):

- Look at the first row because 0.7 starts with '0.'.
- Then go to the column headed '.7'.

$P(Z \le 0.7) = \Phi(0.7) \doteqdot 0.7580$.

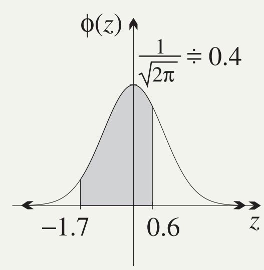


b i



$P(-2.5 \le Z \le -0.3)$
$= P(0.3 \le Z \le 2.5)$,
because $\phi(z)$ is even,
$= \Phi(2.5) - \Phi(0.3)$
$\doteqdot 0.9938 - 0.6179$
$\doteqdot 0.3759$.

ii



$P(-1.7 \le Z \le 0.6)$
$= P(-1.7 \le Z \le 0) + P(0 \le Z \le 0.6)$
$= P(0 \le Z \le 1.7) + P(0 \le Z \le 0.6)$
$= \Phi(1.7) - \Phi(0) + \Phi(0.6) - \Phi(0)$
$= \Phi(1.7) + \Phi(0.6) - 1$
$\doteqdot 0.6811$.

The empirical rule or the 68–95–99.7 rule

Sometimes statistics requires accurate results, and sometimes it uses very approximate methods. It turns out that in practical use, we constantly need to know the probabilities that a normally distributed variable is within 1, 2 or 3 standard deviations of the mean. That is intuitively straightforward, because

- Z has standard deviation $\sigma = 1$, and
- the two inflections make the region within one standard deviation of the mean stand out on the graph.

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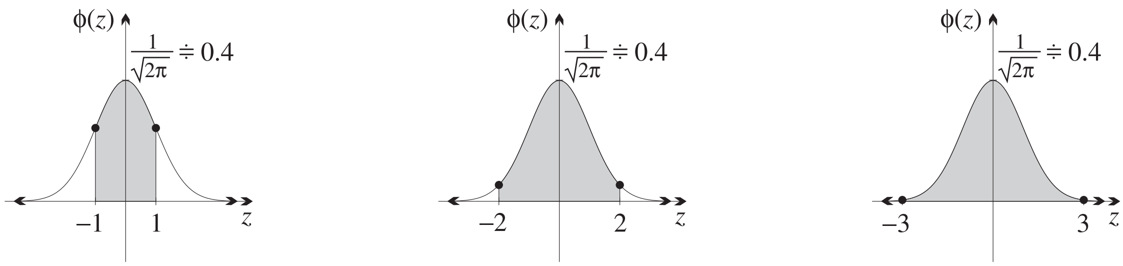
Here are those three results, derived from the table of values of $\Phi(z)$ and converted to the rounded percentages conventionally used in the empirical rule.

$$P(-1 \le Z \le 1) \doteq 0.6827 \doteq 68\%,$$
$$P(-2 \le Z \le 2) \doteq 0.9545 \doteq 95\%,$$
$$P(-3 \le Z \le 3) \doteq 0.9973 \doteq 99.7\%.$$



The percentages here are probabilities, but they are also predictions of what percentages of a normally distributed sample lie within 1, 2 or 3 standard deviations of the mean. These three results are so important that memorising them is part of learning to use the normal distribution, and together they are called the *empirical rule* or the *68–95–99.7 rule*.

8 THE EMPIRICAL RULE OR THE 68–95–99.7 RULE

In a normal distribution, the proportion of scores lying:

- within 1 standard deviation of the mean is 68%,
- within 2 standard deviations of the mean is 95%,
- within 3 standard deviations of the mean is 99.7%.

Example 12 10D

An experiment is run 1000 times. Its random variable is the standard normal variable Z. Answer these questions using the empirical rule only.

a How many scores greater than 2 would you expect?

b Find b if we would expect about 840 scores greater than b.

SOLUTION

a By the empirical rule, we expect $1000 \times 95\% = 950$ scores within $[-2, 2]$.
Because $\phi(z)$ is even, we expect $950 \div 2 = 475$ scores within $[0, 2]$.
Because 500 scores should be positive, we expect 25 scores greater than 2.

b We therefore expect 160 scores less than b, so in particular, b is negative.
Because $\phi(z)$ is even, we also expect 160 scores greater than $-b$.
Hence we expect $1000 - 160 - 160 = 680$ scores between $-b$ and b, so by the empirical rule, $b \doteq -2$.

Quartiles and percentiles

The graph of $y = \Phi(z)$ was drawn above on page 549. Approximation of the quartiles and percentiles can be found from this graph by drawing the appropriate horizontal lines. (We can also use interpolation on the table of values for $\Phi(z)$.)

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Example 13 10D

a Find the 9th decile of the standard normal distribution.

b Find the third quartile Q_3 and the first quartile Q_1 of $\Phi(z)$.

c Using the IQR criterion, what proportion of the scores of a standard normal random variable would be expected to be outliers?

SOLUTION

a For the 9th decile, we need to solve $\Phi(z) \doteqdot 0.9$.
From the table, $\Phi(1.2) \doteqdot 0.8849$ and $\Phi(1.3) \doteqdot 0.9032$, with difference 0.0183 making the 9th decile about 1.28.
This agrees with the horizontal line with height 0.9 on the graph of $\Phi(z)$.

b For the upper quartile, we need to solve $\Phi(z) \doteqdot 0.75$.
From the table, $\Phi(0.6) \doteqdot 0.7257$ and $\Phi(0.7) \doteqdot 0.7580$, with difference 0.0323.
Interpolation gives $\Phi(0.675) \doteqdot 0.7257 + \frac{3}{4} \times 0.0323 \doteqdot 0.7499$.
This agrees with the horizontal line with height 0.75 on the graph of $\Phi(z)$.

c From part **b**, the IQR is about 1.35, so $Q_3 + 1.5 \times \text{IQR} \doteqdot 2.70$.
The IQR criterion is that an outlier lies outside $-2.70 \le z \le 2.70$.

$$\begin{aligned} \text{Hence } P(Z \text{ is an outlier}) &= P(Z > 2.70) + P(Z < -2.70) \\ &= 2 \times P(Z > 2.70) \\ &= 2\big(1 - \Phi(2.70)\big) \\ &\doteqdot 0.007, \end{aligned}$$

so roughly 7 in 1000 scores would be expected to be outliers.

Note: Correct to five significant figures, $Q_3 \doteqdot 0.67449$. It is standard practice to round percentiles and quartiles of the normal to two decimal places, giving

$$Q_1 \doteqdot -0.67 \qquad \text{and} \qquad Q_3 \doteqdot 0.67 \qquad \text{and} \qquad \text{IQR} \doteqdot 1.35,$$

and using the interquartile range criterion, an outlier is a score outside the interval $-2.70 \le z \le 2.70$.
In a normal distribution, the IQR criterion makes just under 1% of scores outliers.

Challenge — the points of inflection

Showing that $y = \phi(z)$ has inflections at $z = -1$ and $z = 1$ requires the second derivative of $\phi(z)$, and is reasonably straightforward.

Differentiation of $y = e^{-\frac{1}{2}z^2}$ requires the chain rule,

$$\begin{aligned} \frac{dy}{dz} &= \frac{dy}{du} \times \frac{du}{dz} \\ &= e^{-\frac{1}{2}z^2} \times (-z) \\ &= -ze^{-\frac{1}{2}z^2}. \end{aligned}$$

Let $u = -\frac{1}{2}z^2$.
Then $y = e^u$.
Hence $\dfrac{du}{dz} = -z$,
and $\dfrac{dy}{du} = e^u$.

The function $\phi(z) = \dfrac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}$ is a multiple of $e^{-\frac{1}{2}z^2}$,
so $\phi'(z) = -z\phi(z)$.

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Hence $\phi(z)$ has a stationary point at $z = 0$, which is a maximum turning point, because $\phi(z)$ is increasing for $z < 0$ and decreasing for $z > 0$.

z	-1	0	1
$\phi'(z)$	$\phi(-1)$	0	$-\phi(1)$
sign	$+$	0	$-$
	/	—	\

Notice in both the tables to the right that $\phi(z)$ is always positive.

For the second derivative, $\phi''(z) = \dfrac{d}{dz}\big(\phi'(z)\big)$

$$= \frac{d}{dz}\big(-z\phi(z)\big),$$

and applying the product rule with $u = -z$ and $v = \phi(z)$,

$$\begin{aligned}\phi''(z) &= -\phi(z) - z\phi'(z)\\ &= -\phi(z) + z^2\phi(z)\\ &= \phi(z)(z^2 - 1).\end{aligned}$$

z	-2	-1	0	1	2
$\phi''(z)$	$3\phi(-2)$	0	$-\phi(0)$	0	$3\phi(2)$
sign	$+$	0	$-$	0	$+$
	$\smile$	$\cdot$	$\frown$	$\cdot$	$\smile$

So there are points of inflection at $z = 1$ and at $z = -1$.

We have cut corners and omitted the actual values of the first and second derivatives from our two tables. This is sloppy, and the reader should fill in the details.

Challenge — the mean and standard deviation

The integrals involved in the calculation of mean and standard deviation require some rather sophisticated techniques. The mean is given by the integral

$$\mathrm{E}(Z) = \int_{-\infty}^{\infty} z\phi(z)\,dz.$$

The integrand $z\phi(z)$ is an odd function, because it is the product of an odd function z and an even function $\phi(z)$. Hence the integral is zero.

This argument assumes that the integral converges. To avoid this assumption,

$$\begin{aligned}\mathrm{E}(Z) &= \int_{-\infty}^{\infty} z\phi(z)\,dz\\ &= \Big[-\phi(z)\Big]_{-\infty}^{\infty} &&\text{because we showed above that } \phi'(z) = -z\phi(z)\\ &= 0 - 0 &&\text{because } \phi(z) \to 0 \text{ as } z \to \infty \text{ and as } z \to -\infty.\end{aligned}$$

Because the mean is zero, the variance is given by the integral

$$\mathrm{Var}(Z) = \int_{-\infty}^{\infty} z^2\phi(z)\,dz.$$

We showed above while finding the second derivative of $\phi(z)$ that

$$\phi''(z) = \phi(z)(z^2 - 1),$$

and rearranging, $z^2\phi(z) = \phi''(z) + \phi(z)$.

Hence

$$\begin{aligned}\mathrm{Var}(Z) &= \int_{-\infty}^{\infty} z^2\phi(z)dz\\ &= \int_{-\infty}^{\infty} \phi''(z)dz + \int_{-\infty}^{\infty} \phi(z)dz.\\ &= \Big[\phi'(z)\Big]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \phi(z)dz.\\ &= 0 + 1\\ &= 1.\end{aligned}$$

The first integral above is zero, because the integrand $\phi'(z) = -z\phi(z)$ is odd, as we saw before. The second integral above is 1 because $\phi(z)$ is a probability density function.

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Exercise 10D

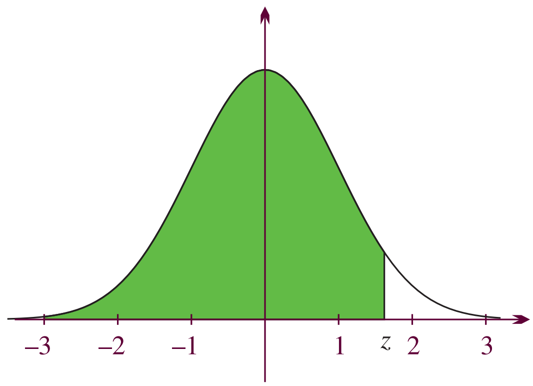
FOUNDATION

The purpose of this exercise is to build familiarity with the symmetry of the standard normal curve. It is not intended that any technology be used for the values of the standard normal distribution in the early questions, because it is important to maximise interaction with the curve and its shape.

The summary below is repeated as an appendix at the end of this chapter.

A brief summary of the standard normal probability distribution

The graph to the right is the *standard normal probability density function* $y = \phi(z)$.



The shaded area represents the value of the corresponding *cumulative distribution function*

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \phi(t)\,dt.$$

The table below gives some values of the probabilities $\Phi(z) = P(Z \le z)$. For example,

$$P(Z \le 1.6) = \Phi(1.6) = \int_{-\infty}^{1.6} \phi(z)\,dz \doteqdot 0.9452.$$

	first decimal place									
z	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0.	0.5000	0.5398	0.5793	0.6179	0.6554	0.6915	0.7257	0.7580	0.7881	0.8159
1.	0.8413	0.8643	0.8849	0.9032	0.9192	0.9332	0.9452	0.9554	0.9641	0.9713
2.	0.9772	0.9821	0.9861	0.9893	0.9918	0.9938	0.9953	0.9965	0.9974	0.9981
3.	0.9987	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

For many purposes, all that is required is the *empirical rule*, or *68–95–99.7 rule*,

$P(-1 \le Z \le 1) \doteqdot 68\%$
$P(-2 \le Z \le 2) \doteqdot 95\%$
$P(-3 \le Z \le 3) \doteqdot 99.7\%$

1 Use the table above to look up the following probabilities for the standard normal distribution. Record your answers correct to four decimal places.

a $P(Z \le 0)$ **b** $P(Z \le 1)$ **c** $P(Z \le 2)$ **d** $P(Z \le 1.5)$
e $P(Z < 0.4)$ **f** $P(Z \le 2.3)$ **g** $P(Z < 1.2)$ **h** $P(Z \le 5)$

2 Explain from the graph above why $P(Z > a) = 1 - P(Z \le a)$. Then use the standard normal table with this complementary result to find:

a $P(Z > 0)$ **b** $P(Z > 1)$ **c** $P(Z > 2)$ **d** $P(Z \ge 2.4)$
e $P(Z > 1.3)$ **f** $P(Z > 0.7)$ **g** $P(Z \ge 1.6)$ **h** $P(Z > 8)$

3 **a** Use the symmetry of the standard normal graph above to explain why if $a > 0$, then $P(Z < -a) = 1 - P(Z \le a)$. (You need not memorise this result).

b Use this result to find:

i $P(Z < -1.2)$ **ii** $P(Z \le -2.3)$ **iii** $P(Z < -0.2)$
iv $P(Z < -3.2)$ **v** $P(Z < -5)$ **vi** $P(Z \le -0.7)$
vii $P(Z < -1.6)$ **viii** $P(Z \le -1.4)$ **ix** $P(Z < -0)$

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4 a Use symmetry to explain why $P(Z \le 0) = 0.5$.
b Hence use symmetry and the standard normal table to find:
i $P(0 < Z \le 1.3)$
ii $P(0 < Z \le 2.4)$
iii $P(0 < Z \le 0.7)$
iv $P(-2.4 \le Z < 0)$
v $P(-1.1 \le Z < 0)$
vi $P(-0.7 \le Z < 0)$
vii $P(0 < Z \le 1.6)$
viii $P(-1.3 \le Z \le 0)$
ix $P(0 < Z \le 5)$
c Find:
i $P(-1.3 \le Z < 1.3)$
ii $P(-2.4 < Z \le 2.4)$
iii $P(-0.8 < Z < 0.8)$
iv $P(-2.9 < Z \le 2.9)$
v $P(-0.4 \le Z < 0.4)$
vi $P(-1.5 < Z \le 1.5)$

DEVELOPMENT

5 Match these eight probabilities into four pairs with equal values.
a $P(Z \le 2)$
b $P(Z \le -1)$
c $P(Z \le 1.2)$
d $P(Z = 4)$
e $P(Z < 2)$
f $P(Z = 2.3)$
g $P(Z \ge 1)$
h $P(Z > -1.2)$

6 Repeat the previous question for these eight values:
a $P(Z \le 5)$
b $P(Z > -1.7)$
c $P(Z < 5)$
d $P(Z \ge 2)$
e $P(Z = 3)$
f $P(Z \le -2)$
g $P(Z \le 1.7)$
h $P(Z = 1.2)$

7 a Explain why $P(a \le Z \le b) = P(Z \le b) - P(Z < a)$.
b Use this result to find:
i $P(1.2 \le Z < 1.5)$
ii $P(0.2 \le Z < 2.3)$
iii $P(0.6 \le Z < 1.7)$
iv $P(-2 \le Z < -1.2)$
v $P(-4 \le Z < -0.2)$
vi $P(-2.7 \le Z < -1)$
c Similarly find:
i $P(-1.5 \le Z < 2.2)$
ii $P(-0.9 \le Z < 1.2)$
iii $P(-2.9 \le Z < 1.3)$

8 Use just the two values $\Phi(1.2) = 0.8849$ and $\Phi(1.8) = 0.9641$, and the symmetry of $\phi(z)$, and your knowledge of its properties as a PDF, to find:
a $P(Z \le 0)$
b $P(Z = 4)$
c $P(Z > 1.8)$
d $P(Z \le 1.2)$
e $P(Z \ge 1.2)$
f $P(0 \le Z \le 1.2)$
g $P(Z \le -1.8)$
h $P(Z \ge -1.2)$
i $P(1.2 \le Z \le 1.8)$
j $P(-1.8 \le Z \le 1.2)$

9 Use the standard normal table to find:
a $P(Z \le 1.3)$
b $P(Z = 2.4)$
c $P(Z > 0.4)$
d $P(Z \le 1.7)$
e $P(Z \ge -1.3)$
f $P(0 \le Z \le 1.5)$
g $P(Z \le -0.8)$
h $P(Z \ge 0.2)$
i $P(1.1 \le Z \le 1.5)$
j $P(-1.3 \le Z \le 2.2)$

10 Use the standard normal table to find these probabilities. Recall from the probability chapter of the Year 11 book that 'and' and 'or' correspond to intersection and union.
a $P(Z \le 1.2 \text{ or } Z \ge 1.8)$
b $P(Z \le 1.8 \text{ and } Z \ge 1.2)$
c $P(Z \le 0.2 \text{ or } Z \ge 1.6)$
d $P(Z \le 2.4 \text{ and } Z \ge 1.7)$

11 Repeat any of the previous questions using a calculator or other technology in place of the standard normal tables.

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12 Use the empirical rule (also called the 68–95–99.7 rule) to find:

a $P(Z \le 0)$ **b** $P(Z \le 1)$ **c** $P(Z \le 2)$
d $P(Z < -1)$ **e** $P(0 \le Z \le 3)$ **f** $P(0 \le Z < 1)$
g $P(-2 \le Z \le 0)$ **h** $P(-3 < Z \le -2)$ **i** $P(-1 \le Z \le 1)$
j $P(-3 < Z \le 1)$ **k** $P(-2 \le Z < 1)$ **l** $P(-2 \le Z \le 7)$

13 Use the empirical rule to find the value of b in each case.

a $P(-b \le Z \le b) = 0.68$ **b** $P(0 \le Z \le b) = 0.475$
c $P(Z \ge b) = 84\%$ **d** $P(-2b \le Z \le b) = 0.815$
e $P(-3b \le Z \le 3b) = 0.997$ **f** $P(Z^2 \le b) = 0.95$

14 Use the standard normal table in reverse to find the value of a, given that:

a $P(Z < a) = 0.7257$ **b** $P(Z \le a) = 0.9893$
c $P(Z < -a) = 0.1151$ **d** $P(Z < a) = 0.2119$
e $P(-a \le Z < a) = 0.7286$ **f** $P(-a < Z \le a) = 0.9906$

15 A professional bowler discovers that when he bowls at a central target, his results form a standard normal distribution, where Z is the distance in centimetres from the target to where the ball hits on each bowl.

a Use the empirical rule to find the probability that his result lies:
  i within 1 centimetre of the central target,
  ii further to the left than 2 centimetres to the right of the target,
  iii more than 3 centimetres from the target.
b In how many centimetres either side of the target do 50% of the bowls strike? You will need to use the standard normal table in reverse for this question.

16 Give a mathematical explanation, and also a practical explanation and example, for the result $P(Z = a) = 0$ for any a.

17 Consider the standard normal curve, $\phi(z) = \dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$.

a Test your knowledge of this curve:
  i What is the domain?
  ii Is it odd, even or neither?
  iii Write down the equation of any axis of symmetry.
  iv What is the area under the curve and above the horizontal axis?
  v What are the z-coordinates of the points of inflection?
  vi What are the coordinates of the maximum turning point?
  vii What are the z-intercepts?
b Test your knowledge of the associated standard normal distribution:
  i What is its mean? **ii** What is its mode?
  iii What is its median? **iv** What is its standard deviation?
c Without looking, write down its probability density function.

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18 [Graphing the standard normal distribution]
The purpose of this question is to use our calculus and curve-sketching skills to draw a graph of $y = f(x)$, where

$$f(x) = e^{-\frac{1}{2}x^2},$$

and then use this graph to sketch the standard normal density function $y = \phi(x)$.

a Show that $f(x)$ is an even function.

b Show that $f'(x) = -xe^{-\frac{1}{2}x^2}$ and $f''(x) = (x^2 - 1)e^{-\frac{1}{2}x^2}$.

c Show that there is a unique stationary point. Find its coordinates and determine its nature.

d Show that there are two points of inflection, and that they occur one standard deviation either side of the mean. Find their coordinates.

e Explain what happens to $f(x)$ as $x \to \infty$ and $x \to -\infty$.

f Graph $y = f(x)$.

g Now use stretching to draw the graph of $y = \phi(x)$.

19 **a** **i** Use the trapezoidal rule with five function values (that is, four intervals) to estimate the integral $\int_0^1 \phi(z)dz$.

ii Double this value to estimate the probability that a value will lie within one standard deviation of the mean on the standard normal curve.

iii Why do you know that this will be an underestimate of the true result?

iv Is this in good agreement with the empirical rule and the standard normal table?

b Use the trapezoidal rule with five function vaues to determine the probability that:

i a value will lie within two standard deviations of the mean,

ii a value will lie within three standard deviations of the mean.

c Use a spreadsheet to increase your number of intervals to say 10, 20, 50, and 100, and observe the convergence.

CHALLENGE

20 In this question, you may assume the result $\int_{-\infty}^{\infty} \phi(z)\,dz = 1$, where $\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ is the PDF of the standard normal distribution.

a Write down the integral for $\mathrm{E}(Z)$ and use the symmetry of the integrand to explain why $\mathrm{E}(Z) = 0$.

b Differentiate $ze^{-\frac{1}{2}z^2}$ and hence integrate $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^2e^{-\frac{1}{2}z^2}\,dz$.

c Evaluate $\mathrm{Var}(Z)$.

Calculators

Once you are fluent with the diagrams for the standard normal and the associated calculations in this exercise, you should check on your calculator to see whether it has the function $\Phi(z)$. If it does, then as suggested in Question 11, practise until you can use the calculator confidently. The same questions will be quite adequate.

It is possible that the calculator also has the inverse function, denoted by $\Phi^{-1}(z)$ or something similar. This would allow you, for example, to find the value of z for which $\Phi(z) = 0.3$, so that there would be no need to use interpolation.

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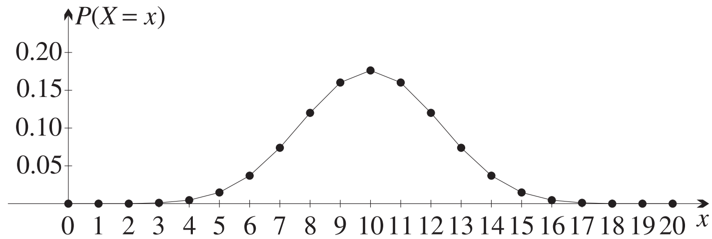
10E General normal distributions



In Section 10A we graphed the probabilities of obtaining x heads when 20 coins are thrown, and joined the 21 points to form a polygon. The polygon suggests very much that we should be approximating it with a bell-shaped normal curve. But the curve suggested by the graph is certainly not the standard normal curve, for two reasons:

- The mean is not zero.
- A glance at the inflections shows that the standard deviation is not 1.

This section extends the normal distribution to bell-shaped curves in general.

Shifting and stretching the standard normal distribution

We can estimate the means and the standard deviation from the graph and from the experiment.

- We know that the polygon above is symmetric about $x = 10$, because, for example, the probabilities of obtaining 7 heads from 20 throws, and 13 heads from 20 throws, are equal. Thus the mean is exactly 10.
- We can roughly estimate the standard deviation by looking at where the points of inflection would be if the points were joined up by a curve. The steepest intervals are the interval from $x = 7$ to $x = 8$, and the interval from $x = 12$ to $x = 13$. Let us estimate the points of inflection to be at $x = 7.5$ and $x = 12.5$. That would give a standard deviation of about 2.5.

Some further theory in the Extension 1 course (the *binomial distribution*) tells us that the true standard deviation is $\sigma = \sqrt{5}$, which is approximately 2.236. We now need to stretch and then shift the standard normal distribution to get a curve that may help understand the graph above. That is, we need to produce a normal distribution with $\mu = 10$ and $\sigma = \sqrt{5} \doteqdot 2.236$.

For the rest of this chapter, we will drop the approximately equals sign $\doteqdot$ because nearly all our numbers are estimates or approximations.

Stretching to accommodate the standard deviation

First, stretch the standard normal distribution horizontally by a factor of σ. This is done by replacing x by $\dfrac{x}{\sigma}$, as discussed in Section 2G. The standard normal is $y = \phi(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, so the result is

$$y = \phi\left(\frac{x}{\sigma}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \quad \left(\text{always look at } \frac{1}{\sqrt{2\pi}} \text{ and think } \tfrac{2}{5} \text{ or } 0.4\right).$$

When this stretching is done, the inflections at $x = 1$ and $x = -1$ become inflections at $x = \sigma$ and $x = -\sigma$. This is because stretching transforms concave-up pieces of curve to concave-up pieces, and concave-down pieces of curve to concave-down pieces.

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This function, however, is not a probability density function, because the stretching has increased the area under the curve by a factor of σ, so that the area is now σ and not 1. To correct this, we have to stretch vertically by a factor of $\frac{1}{\sigma}$, giving what is once again a probability density function,

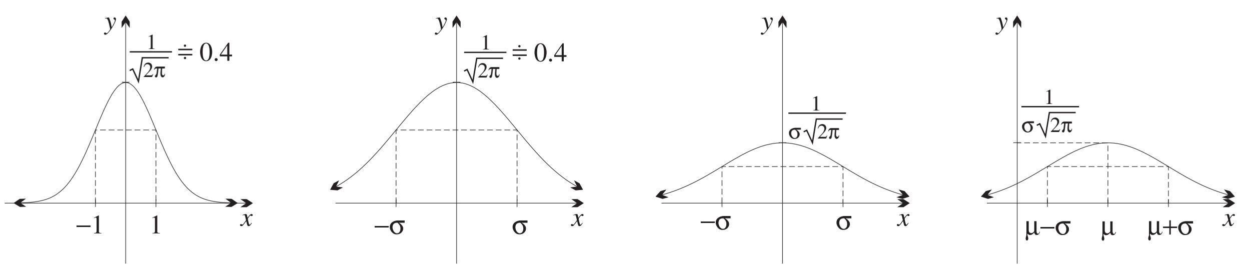
$$y = \frac{1}{\sigma}\phi\left(\frac{x}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{x^2}{2\sigma^2}}.$$

Shifting to accommodate the mean

Once the standard deviation has been sorted out, shift the curve μ to the right to make the mean μ instead of 0. This is done by replacing x by $x - \mu$, giving

$$y = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

This time the area does not need to be adjusted, and the inflections continue to be one standard deviation from the mean, that is, at $x = \mu - \sigma$ and $x = \mu + \sigma$.



$y = \phi(x)$ $\quad y = \phi\left(\frac{x}{\sigma}\right)$ $\quad y = \frac{1}{\sigma}\phi\left(\frac{x}{\sigma}\right)$ $\quad y = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)$

Summarising the transformation

Let $f(x)$ be the new stretched and shifted probability density function. Taking account of the horizontal and the vertical stretches, and then the horizontal shift, we can write $f(x)$ in terms of $\phi(x)$,

$$f(x) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right).$$

The last diagram above shows this transformed normal distribution. The continuous probability distribution described by the new function $f(x)$ is called the *normal distribution with mean μ and standard deviation σ*.

These are the important things to notice about the curves above.

- The first, third and fourth graphs are all normal distribution functions. In particular, all have area 1 under the curve.
- The third and fourth graphs both have standard deviation σ — look at the two points of inflection.
- The fourth graph has mean μ — look at the symmetry about $x = \mu$.
- The fourth graph has standard deviation σ — look at the two points of inflection σ to the right of the mean μ, and σ to the left of μ.

The four successive sketches above were drawn using the numerical values $\mu = 3$ and $\sigma = 2$. You can see in the fourth graph that if you take $\mu = 3$, then the inflections are at $x = 3 - 2 = 1$ and at $x = 3 + 2 = 5$.

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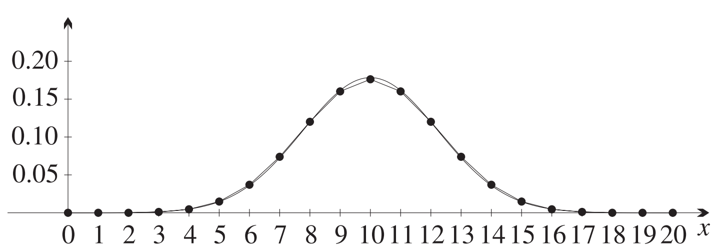
9 THE GENERAL NORMAL DISTRIBUTION

Let $f(x)$ be the probability density function describing a normal distribution with mean μ and standard deviation σ.

- The PDF $f(x)$ is obtained from the standard normal PDF by:
  - stretching horizontally with factor σ and vertically with factor $\frac{1}{\sigma}$,
  - then shifting right by μ units.
- The transformed PDF is therefore $f(x) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)$.
- Thus the transformed PDF has equation $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.
- The points of inflection of $f(x)$ are each one standard deviation from the mean, that is, at $x = \mu - \sigma$ and at $x = \mu + \sigma$.
- In any normal distribution, the mean, the median and the mode coincide.

Comparison with the 20 coin tosses

Graphed below is the polygon of the 20 coin tosses, together with the normal PDF with $\mu = 10$ and $\sigma = \sqrt{5}$. The vertical scale is the same for both graphs, but there is no name on the vertical axis. This is because for the discrete distribution the name is $P(X = x)$, and for the continuous distribution the name is $f(x)$ or y.



The fit is a very good approximation, but it is not exact. It looks very much as if the fit would get better with more and more coin tosses. The example clearly shows how useful the normal is in approximating complicated probability distributions — in this case a discrete distribution. Historically, this coin-tossing experiment was the first use of the normal to approximate another distribution.

Working with the general normal distribution — *z*-scores

We have seen that every normal distribution is obtained from the standard normal distribution by transformations. In order to work with any normal distribution, we need to convert back to the standard normal. The key to this is *z-scores*.

In any distribution, normal or not, the z-score of a score x is the number of standard deviations above the mean. This is easily calculated by the formula

$$z\text{-score} = \frac{x-\mu}{\sigma}.$$

We need to be able to convert from values of x to z-scores, and back from z-scores to values of x. The two equations are

$$z = \frac{x-\mu}{\sigma} \quad \text{and} \quad x = \mu + \sigma z.$$

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For example, check conversions both ways for this table of scores and corresponding z-scores for a distribution with mean $\mu = 10$ and standard deviation $\sigma = 2$.

x	4	5	6	7	8	9	10	11	12	13	14	15	16
z-score	-3	-2.5	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2	2.5	3

10 THE z-SCORES OF A RANDOM VARIABLE

Suppose that X is a random variable, normal or not, with mean μ and standard deviation σ.

- The *z-score* of a score x is the number of standard deviations that x lies above the mean. If the z-score is negative, then x lies below the mean.
- Thus the conversions between z-scores and values of x are given by
$$z = \frac{x - \mu}{\sigma} \quad \text{and} \quad x = \mu + \sigma z.$$
- If the distribution is normal, the z-scores allow the values and features of the standard normal distribution to be applied.
- For sample data (not population data), use $\bar{x}$ for the mean and s for the standard deviation.

Example 14 10E

A dataset has mean $\bar{x} = 12$ and standard deviation $s = 3.60$. Answer these questions correct to two decimal places.

a What scores would be 1, 2 and 3 standard deviations from the mean?

b How many standard deviations from the mean are scores of 24, 11 and 7.7?

SOLUTION

a We can do part **a** either using the formula for conversion from z scores to x-values, or working verbally with 'the number of standard deviations from the mean'.

For $z = 1$,
$$\begin{aligned} x &= \bar{x} + sz \\ &= 12 + 3.60 \\ &= 15.60, \end{aligned}$$
and for $z = -1$,
$$\begin{aligned} x &= 12 - 3.60 \\ &= 8.40. \end{aligned}$$

For $z = 2$,
$$\begin{aligned} x &= \bar{x} + 2sz \\ &= 12 + 7.20 \\ &= 19.20, \end{aligned}$$
and for $z = -1$,
$$\begin{aligned} x &= 12 - 7.20 \\ &= 4.80. \end{aligned}$$

For $z = 3$,
$$\begin{aligned} x &= \bar{x} + 3sz \\ &= 12 + 10.80 \\ &= 22.80, \end{aligned}$$
and for $z = -1$,
$$\begin{aligned} x &= 12 - 10.80 \\ &= 1.20. \end{aligned}$$

OR

The scores one SD from $\bar{x}$ are $\bar{x} + s = 12 + 3.60 = 15.60$, and $\bar{x} - s = 12 - 3.60 = 8.40$.

The scores two SDs from $\bar{x}$ are $\bar{x} + 2s = 12 + 7.20 = 19.20$, and $\bar{x} - 2s = 12 - 7.20 = 4.80$.

The scores three SDs from $\bar{x}$ are $\bar{x} + 3s = 12 + 10.80 = 22.80$, and $\bar{x} - 3s = 12 - 10.80 = 1.20$.

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b For $x = 24$, $z = \dfrac{x - \bar{x}}{s}$
$= \dfrac{24 - 12}{3.6}$
$= 3.33$,
3.33 SDs above the mean.

For $x = 11$, $z = \dfrac{x - \bar{x}}{s}$
$= \dfrac{11 - 12}{3.6}$
$= -0.28$,
0.28 SDs below the mean.

For $x = 7.7$, $z = \dfrac{x - \bar{x}}{s}$
$= \dfrac{7.7 - 12}{3.6}$
$= -1.19$,
1.19 SDs below the mean.

Example 15 — 10E

A normally distributed random variable X has mean 100 and standard deviation 20.

a Write down the two conversion formulae between z-scores and values of x.

b Find: **i** $P(X \le 110)$ **ii** $P(X \ge 90)$

c Find (nearest whole number) the value of a such that $P(X \le a) = 0.98$.

SOLUTION

a $z = \dfrac{x - 100}{20}$ and $x = 100 + 20z$.

b **i** $P(X \le 110)$
$= P(Z \le 0.5)$
$= 0.69$

ii $P(X \ge 90) = P(Z \ge \; - 0.5)$
$= P(Z \le 0.5)$ ($\phi(x)$ is even)
$= 0.69$

c From the table, $\Phi(2.0) = 0.9772$ and $\Phi(2.1) = 0.9821$,
so by interpolation, $\Phi(2.06) = 0.98$.
Converting back to x-values, $a = 100 + 20 \times 2.06 = 141$.

Quartiles, the empirical rule, and the IQR criterion for outliers

If a distribution is normal, we can use z-scores to apply results already calculated about the standard normal distribution. Suppose then that we have a normally distributed random variable X with mean μ and standard deviation σ.

The empirical rule, or 68–95–99.7 rule:

When the experiment is run a large number of times, these are the expectations.

- 68% lie within one standard deviation of the mean,
  — that is, 68% lie within the interval $\mu - \sigma \le x \le \mu + \sigma$.
- 95% lie within two standard deviations of the mean,
  — that is, 95% lie within the interval $\mu - 2\sigma \le x \le \mu + 2\sigma$.
- 99.7% lie within three standard deviations of the mean,
  — that is, 99.7% lie within the interval $\mu - 3\sigma \le x \le \mu + 3\sigma$.

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The first and third quartile:

- We saw in Section 10D that the third quartile of the standard normal is $z = 0.67$.
  This is 0.67 standard deviations above the mean,
  Hence the third quartile of the transformed distribution is $Q_3 = \mu + 0.67\sigma$.
  Alternatively, using the formula, $x = \mu + z\sigma = \mu + 0.67\sigma$.
- We saw in Section 10D that the first quartile of the standard normal is $z = -0.67$.
  This is 0.67 standard deviations below the mean,
  Hence the first quartile of the transformed distribution is $Q_1 = \mu - 0.67\sigma$.
  Alternatively, using the formula, $x = \mu + z\sigma = \mu - 0.67\sigma$.
- The standard normal has interquartile range 1.35, so for the transformed distribution, IQR $= 1.35\sigma$.

The IQR criterion for outliers:

- We showed below worked Example 13 that the IQR criterion characterises as outliers any scores lying outside the interval $-2.70 \le x \le 2.70$.
  Hence for the transformed distribution, we characterise as outliers any scores lying outside the interval $\mu - 2.70\sigma \le x \le \mu + 2.70\sigma$.

Example 16 10E

A dataset with 1000 scores is known to be a sample from a normally distributed variable X with mean $\mu = -32.6$ and standard deviation $\sigma = 5.7$.

a Describe what the empirical rule predicts about the data.

b Using z-scores, and finding $\Phi(x)$ from a table or using technology, predict roughly how many scores will:

i lie in $[-30, \infty)$, **ii** lie in $(-\infty, -40]$, **iii** lie in $[-40, -30]$.

c Using the IQR criterion $\mu - 2.70\sigma \le x \le \mu + 2.70\sigma$ for scores that are not outliers, roughly how many outliers would you expect?

SOLUTION

a About 680 scores will lie within one SD from the mean, that is, in $[-38.3, -26.9]$.
About 950 scores will lie within two SDs from the mean, that is, in $[-44.0, -21.2]$.
About 997 scores will lie within three SDs from the mean, that is, in $[-49.7, -15.5]$.

b i For -30,

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ &= \frac{-30+32.6}{5.7} \\ &= 0.456, \\ \text{so } P(X \ge -30) &= P(Z \ge 0.456) \\ &= 0.324, \end{aligned}$$

predicting roughly 324 such scores.

ii For -40,

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ &= \frac{-40+32.6}{5.7} \\ &= -1.298, \\ \text{so } P(X \le -40) &= P(Z \le -1.298) \\ &= P(Z \ge 1.298) \\ &= 1 - P(Z \le 1.298) \\ &= 1 - 0.903 \\ &= 0.097, \end{aligned}$$

predicting roughly 97 such scores.

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iii $P(-40 \le X \le -30)$
$= 1 - (P(X \le -40) + P(X \ge -30))$ (no need for z-scores)
$= 1 - (0.324 + 0.097)$ (using parts **i** and **iii**)
$= 0.579$, predicting roughly 579 such scores.

c This does not need to be recalculated. Worked Example 10**c** showed that roughly 7 in 1000 scores are outliers by the IQR criterion for the standard normal, and because the calculation depended only on z-scores, this is valid for any normally distributed variable.

Exercise 10E

FOUNDATION

Note: There is a brief summary of the normal distribution, including a graph, a table and the empirical rule, in the Appendix at the end of this chapter.

1 In each part, calculate the z-scores corresponding to the given value of x, and state how many standard deviations each value of x lies above or below the mean.

a $\mu = 4, \sigma = 1, x = 5$
b $\mu = 13, \sigma = 3, x = 7$
c $\mu = 0.5, \sigma = 0.25, x = 0.75$
d $\mu = 1, \sigma = 3, x = -5$
e $\mu = 114, \sigma = 1.2, x = 120$
f $\mu = 2.35, \sigma = 0.05, x = 2.20$

2 **a** Use the formula $z = \dfrac{x - \mu}{\sigma}$ to find the z-score when:

i $\mu = 50, \sigma = 4$ and $x = 60$,
ii $\mu = 450, \sigma = 25$ and $x = 375$,
iii $\mu = 3.19, \sigma = 0.12$ and $x = 3.85$,
iv $\mu = 23, \sigma = 8$ and $x = 25$.

b Which of the results in part **a** are:

i furthest from the mean,
ii above the mean,
iii below the mean,
iv within 2 standard deviations from the mean,
v not within the middle 68% of the data?

3 Use z-scores to convert these probability statements for the normal random variable X with mean 4 and standard deviation 2 into probability statements on the standard normal random variable Z. For example, $P(X \le 7) = P(Z \le 1.5)$.

a $P(X \le 5)$
b $P(X > 4.5)$
c $P(X \le 2)$
d $P(X \ge 1)$
e $P(0 \le X \le 3)$
f $P(0.5 \le X \le 4.5)$

4 A certain quantity is normally distributed with mean 5 and standard deviation 2. Convert the following probabilities to probabilities involving the standard normal distribution, and then use the empirical rule to find them.

a $P(X \ge 5)$
b $P(3 \le X \le 7)$
c $P(X \le 9)$
d $P(X \ge 1)$
e $P(-1 \le X \le 7)$
f $P(1 \le X \le 3)$

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5 Use the empirical rule to find the following probabilities for a normally distributed random variable with the given parameters.

a $P(10 \le X \le 18)$, given mean $\mu = 12$ and standard deviation $\sigma = 2$.

b $P(X \ge 42)$, given mean $\mu = 37$ and standard deviation $\sigma = 5$.

c $P(X \ge 4.5)$, given mean $\mu = 4$ and standard deviation $\sigma = 0.25$.

6 Find each probability for a normally distributed random variable X with the given parameters. You will need to use the table of values for the standard normal distribution, or a statistics calculator, or other technology such as a spreadsheet, or online resources.

a $P(3 \le X \le 7)$, given mean $\mu = 5$ and standard deviation $\sigma = 0.8$.

b $P(X \ge 20)$, where $\mu = 4$ and $\sigma = 10$.

c $P(X \le 8)$, where $\mu = 12$ and $\sigma = 5$.

d $P(X \ge -39)$, where $\mu = 0$ and $\sigma = 30$.

e $P(X < 36)$, where $\mu = 20$ and $\sigma = 10$.

f $P(3 < X \le 5)$, where $\mu = 8$ and $\sigma = 2$.

7 Explain what it means for a score x if the corresponding z-score is:

a positive, **b** negative, **c** zero.

DEVELOPMENT

8 A distribution is known to be normal with mean $\mu = 73$ and $\sigma = 8$. A researcher records the following data values from this distribution:

69, 80, 95, 50, 43, 90, 52, 98, 45

a Write down the data values that lie within one standard deviation of the mean.

b Write down the data values that lie within three standard deviations of the mean.

c Write down the data values that lie more than two standard deviations below the mean.

d Write down the data values that are more than two and a half standard deviations above the mean.

e The researcher believes that these data values were obtained randomly. Do they seem to fit the expected distribution for a normal random variable? Construct a stem-and-leaf plot for the data and comment on the shape of the data.

9 The results of an English examination and a mathematics examination are approximately normally distributed with these parameters:

English:	$\mu = 65\%$	$\sigma = 10$
Mathematics:	$\mu = 62\%$	$\sigma = 15$

a For each student below, determine the z-scores for the two results and state which is more impressive:

i Student A's result in English (90%), or their result in mathematics (92%),

ii Student B's result in English (57%), or their result in mathematics (53%),

iii Student C's result in English (80%), or their result in mathematics (77%).

b What is the probability that a mathematics student obtains over 95%?

c What is the probability that a student's English mark is greater than the mean of the mathematics marks?

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10 The results of an experiment are known to be normal, with mean 50 and standard deviation 10. The experiment is run 600 times.

a Describe what the empirical rule predicts about the data.

b Using z-scores, and using a table of the standard normal, or a calculator, or statistics software, predict roughly how many scores will:

i lie in $[\infty, 55]$, **ii** lie in $[35, 50]$, **iii** lie in $[38, 62]$.

c Using the IQR criterion $\mu - 2.70\sigma \le x \le \mu + 2.70\sigma$ for scores that are not outliers, roughly how many outliers would you expect?

CHALLENGE

11 At a certain school, Biology has four assessments. The mean and standard deviation for these assessments are recorded in the table below.

Assessment	Mean	SD
1	60	10
2	65	8
3	75	4
4	63	12

a Jack obtained 50, 53 and 67 for the first three assessments, but was absent for the fourth assessment due to a fall.

i Find the z-score for each of Jack's results.

ii Use these z-scores to find Jack's average deviation from the mean.

iii Hence estimate a mark for Jack in the fourth assessment.

iv What are the advantages of this method over simply giving Jack the average for his scores in the first three assessments?

v Are there any disadvantages to the method?

b Jill obtains 64, 70 and 79 for the first three assessments, but due to a tumble could not attend the final assessment. Use the same method to estimate Jill's missing result.

Calculators

Once you are confident with z-scores and their use with probability calculations, you may want to check whether your calculator can handle z-scores, and if so, practise until you can do things quickly. Be careful, however, because automating these transformations gets in the way of understanding 'the number of standard deviations from the mean'.

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10F Applications of the normal distribution

A great number of common situations follow a normal distribution, or follow it approximately enough for practical purposes. The questions in Exercise 10F are self-explanatory, given the previous theory, and one worked example should be sufficient introduction.

Example 17 10F

The Happy time Chocolate Company manufactures a 100 g chocolate–nougat bar. As with any manufacturing process, these chocolate bars do not all have precisely the same weight, and these bars are known to be normally distributed with standard deviation 2 g. To reduce the number of complaints, the company has adjusted its machinery so that the mean is 102 g (such an adjustment does not affect the standard deviation).

a Using the empirical rule where possible, and tables or technology otherwise, find the percentage of chocolate bars:

i of weight less than the stated weight of 100 g,

ii of weight greater than 105 g.

b What would the mean weight need to be for there to be less than 1 chocolate bar in 1000 under 100 g?

SOLUTION

a i A weight of 100 g is 1 standard deviation below the mean of 102 g.

By the empirical rule. $P(-1 \le Z \le 1) = 68\%$,

and using the complement, $P(Z < -1 \text{ or } Z > 1) = 32\%$,

so by the even symmetry, $P(Z < -1) = 32\% \div 2$,

$= 16\%$.

ii A weight of 105 g is 1.5 standard deviations above the mean of 102 g.

From the table. $P(Z \le 1.5) = 93\%$,

so using complements, $P(Z > 1.5) = 7\%$.

b Reading the table backwards, $P(Z \le 3.1) = 0.999$,

so by symmetry and complements, $P(Z \le -3.1) = 0.001$.

Hence we need to make the mean 3.1σ above 100 g ,

meaning that we set the controls so that $\mu = 100 + 3.1 \times 2 = 106.2$ g.

Exercise 10F

FOUNDATION

The first four questions of this exercise should be completed using the empirical rule (or the 68–95–97.7 rule) rather than the standard normal probability table or technology.

There is a brief summary of the normal distribution, including a graph, a table and the empirical rule, in the Appendix at the end of this chapter.

1 The results of a school's English examination are found to be normally distributed with mean 70 and standard deviation 10.

a What percentage of the pupils score over 50?

b What percentage of the pupils score under 80?

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2 The results in an examination are approximately normally distributed with mean 68 and standard deviation 9. In a cohort of 2000, how many students will be expected to score:

a more than 95, **b** less than 50, **c** between 59 and 86?

3 A machine produces screws that are an average of 2 cm long, with a standard deviation of 0.1 cm. The screw lengths are approximately normally distributed.

a What is the probability that a screw will be undersized, if this is taken to mean more than 2 standard deviations below the mean?

b In a batch of 2400, use z-scores to find how many screws are longer than 2.3 cm?

4 Apples of a certain variety are to be sold in packages in a supermarket. Their diameters are normally distributed with mean 68 mm and standard deviation 2 mm. Apples are discarded if their diameter is more than 72 mm or less than 64 mm. What percentage are discarded?

5 The IQ (*Intelligence Quotient*) test is designed to give a qualitative measure of a person's intelligence. In Australia, IQ is approximately normally distributed with mean 98 and standard deviation 15.

a According to one definition, a genius is defined to be someone with an IQ over 140. What percentage of the Australian population would this be?

b In a population of 25 million, how many geniuses would you expect?

DEVELOPMENT

6 A very famous and early experiment into cholesterol levels, called the Framingham study, found that the average cholesterol level in the population of adult males who did not go on to develop heart disease was 219 mg/mL, with standard deviation 41 mg/mL. Assuming that doctors call a reading of above 240 mg/mL *high*, what percentage of this population could be said to have high cholesterol?

7 In Australian adult males, height is found to be normally distributed with mean 176 cm and standard deviation 7.5 cm. A doorway is designed so that 90% of this population can enter without ducking.

a Read the supplied standard normal distribution table backwards to find the z-score such that $P(Z < z) = 90\%$, correct to 2 decimal places.

b Hence find the minimum height of the doorway.

c In the Dinaric alps, the mean and standard deviation of the heights of adult males are respectively 185 and 7.5 centimetres. A customer orders a special design for the doorway so that 95% of adult males can enter without ducking.

i Explain why a reasonable estimate from the table such that $P(Z < z) = 0.95$ is 1.65.

ii Find the minimum design height of the door.

8 A company has a machine designed to fill cereal boxes. It dispenses cereal according to a normal distribution with mean 500 g and standard deviation 2 g. To ensure that boxes are above the advertised weight at least 95% of the time, what weight should be recorded as the weight on each box?

9 The length of gestation (pregnancy) in human females is approximately normally distributed with mean 266 days and standard deviation 16 days.

a Nine months is about $0.75 \times 365 \doteqdot 274$ days. What percentage of females give birth before 274 days?

b If 266 days is considered 'on time', what percentage of females give birth more than:

i 1 week early, **ii** one week late?

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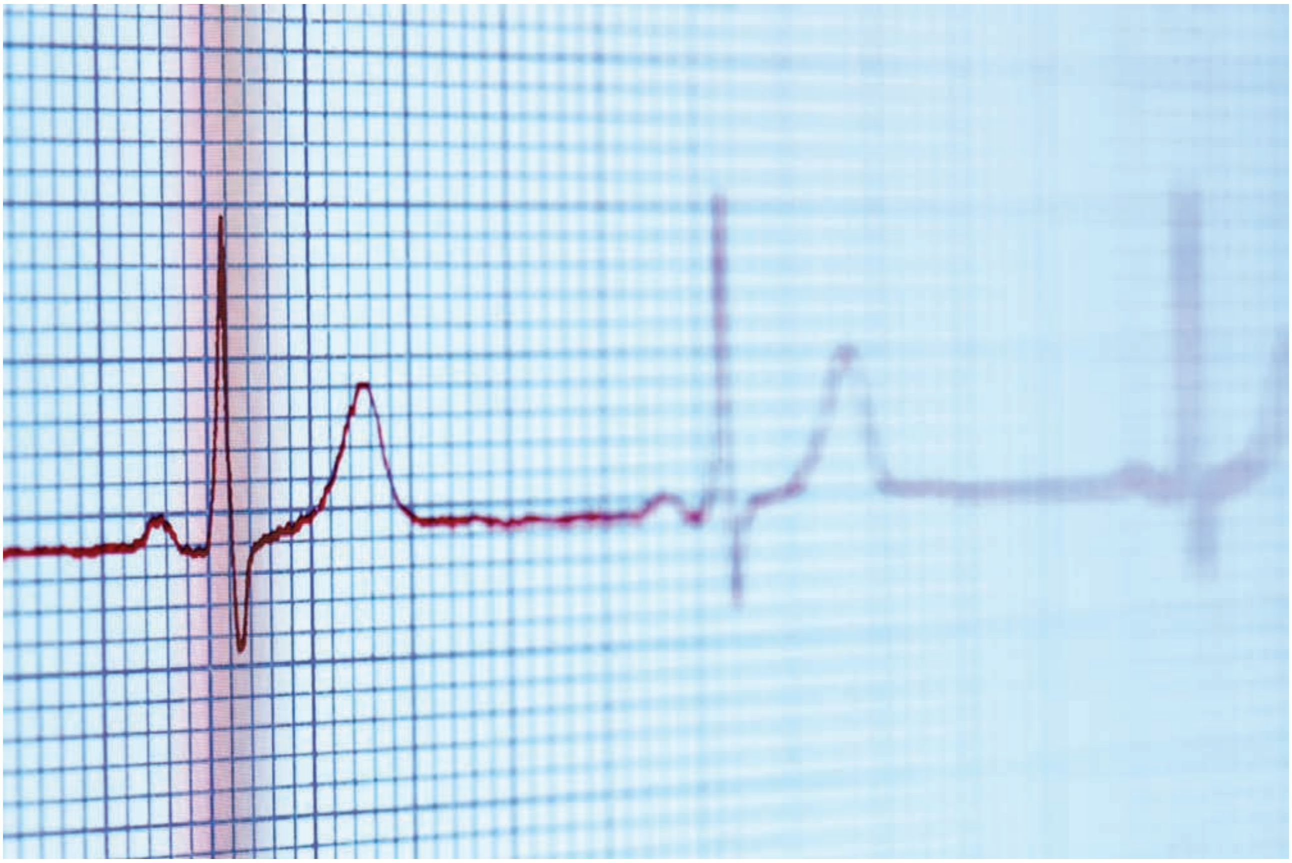
10 A certain study indicates that the pulse rate of an adult male aged 20–39 is about 71 with standard deviation about 9. The data are approximately normally distributed.

- **a** What percentage of this population would be expected to have *bradycardia*, which is defined to be a slow pulse rate below 60 beats/minute?
- **b** *Tachycardia* is defined to be a pulse rate greater than 100 beats/minute. What percentage of the population might be expected to fall in this category?
- **c** Repeat part (a)–(b) for females aged 20–39, whose mean is about 76 and standard deviation is about 9.5.

CHALLENGE

11 The apples in Question 4 must also fit within regulation weight guidelines. Suppose that the weights are normally distributed, with 97.7% of the apples weighing more than 100 g and 69.1% weighing less than 115 g. Find the mean and standard deviation of the weights of the apples. Use the supplied normal distribution table.



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10G Investigations using the normal distribution

These questions are intended to be investigations using technology — statistical calculators, spreadsheets, statistics software, or online resources. Many of the investigations can be broadened or extended into projects. The exercise is long, and it is certainly not intended that all questions be attempted.

Many of the investigations use *sampling of the mean*. This concept is the reason why the normal distribution plays such a central role in all statistics. The underlying theorem is the *central limit theorem*.

The following Challenge paragraphs explain a particular case of the theorem very briefly. It would perhaps be better read after the idea has been encountered in one or more investigations.

Challenge — Sampling of the mean

Suppose that we have a random variable X. The distribution may be discrete, or continuous and normal, or continuous and not normal. Suppose that this distribution has mean μ and standard deviation σ, neither of which we know.

We want to find the mean of this distribution, so we do the obvious thing — we *sample* the variable X. That is, we run n independent trials of the experiment and take the average of these results as our estimate for the mean μ. What this procedure has actually done is generate a new random variable Y. The procedure described is:
- Take n independent samples of the random variable X, thus generating n values of X.
- Find the mean of the n samples, and assign this mean to a random variable Y.

The central limit theorem says that in most situations, this new random variable Y:
- has the same mean μ as X (which is obvious),
- has variance $\dfrac{\sigma^2}{n}$, that is, its standard deviation is $\dfrac{\sigma}{\sqrt{n}}$ (nearly obvious),
- *tends towards a normal distribution as the number n of samples increases.*

The significance of this theorem is that sampling any random variable to find its mean generates approximately a normal distribution as more and more samples are taken. Thus the normal distribution is involved in the study of every distribution, continuous or discrete. The normal approximation of the 'toss 20 coins' polygon in Section 10E is historically one of the first examples of the theorem.

Exercise 10G

INVESTIGATION

Some of the questions below involve the use of a spreadsheet such as Excel, LibreOffice Calc or GoogleDocs. The instructions below are directly relevant for a recent version of Excel on Windows, but may be adapted depending on available software. More serious investigations could use a general programming language such as Python, or a statistical programming language such as $\mathbb{R}$.

1 [Sampling of the mean] The data from many common experiments, with any distribution, can be displayed as normally distributed data using the following technique called *sampling from the mean*. This is the reason why the normal distribution is so important.

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A student generated three real-valued random numbers between 0 and 15. The mean of these three numbers was recorded and the original three numbers discarded. This was repeated 1000 times, and the results were recorded as grouped data in the table below.

class	0–1	1–2	2–3	3–4	4–5	5–6	6–7	7–8
class centre x	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
frequency f	2	5	24	46	88	116	138	158
relative frequency f_r								

class	8–9	9–10	10–11	11–12	12–13	13–14	14–15
class centre x	8.5	9.5	10.5	11.5	12.5	13.5	14.5
frequency f	144	113	78	47	27	10	4
relative frequency f_r							

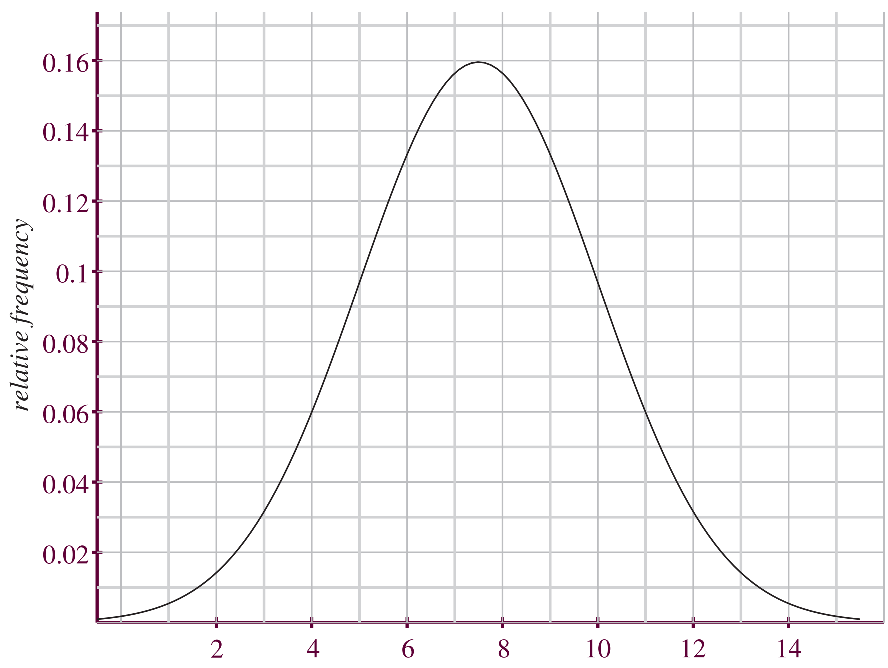
a Use your calculator or a spreadsheet to evaluate the mean and standard deviation of this data, using the given class centres and frequencies. Round your answers correct to 1 decimal place.

b Complete the table by filling in the relative frequencies. Now take the class centres x as the values, and take the relative frequencies as estimates of the probabilities $P(x)$, and evaluate the mean and standard deviation using the formulae

$$\begin{aligned} \mathrm{E}(X) &= \sum xP(x), \quad \text{and} \\ \mathrm{Var}(X) &= E(X^2) - \mathrm{E}(X)^2, \quad \text{where } \mathrm{E}(X^2) = \sum x^2 P(x). \end{aligned}$$

Check that your answers agree with the previous results.

c



In the graph above, the normal probability density curve with the mean and standard deviation you just found has been plotted. Photocopy the graph and construct a relative frequency histogram for your grouped data on the same diagram. Add the relative frequency polygon by joining the top centres of the histogram rectangles. What do you notice?

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d How do you think that you might improve the match between the normal curve and the relative frequency polygon?

e Using integration, calculate the mean $\mathrm{E}(Y)$ and variance $\mathrm{Var}(Y)$ for a uniform continuous random variable with $P(Y) = \frac{1}{15}$ on the interval $0 \leq Y \leq 15$. Confirm that the mean and variance obtained in part **a** above are related by the formulae:

$$\mathrm{E}(X) = \mathrm{E}(Y) \qquad \text{and} \qquad \mathrm{Var}(X) = \frac{\mathrm{Var}(Y)}{n},$$

where $n = 3$ (because three random numbers were averaged at each stage).

2 In this question we will replicate the experiment discussed in the previous question using a spreadsheet. In this experiment we will generate 3 real-valued random numbers between 0 and 12 and calculate their mean. This step will be carried out 100 times, giving a set of 100 means that form a distribution called *sampling of the mean*.

a A fragment of a spreadsheet is shown below. In each cell `A2:C2` we have entered `=RAND()*12`. In cell `E2` we have entered a formula to calculate the mean of the three numbers. Enter all this in your spreadsheet.

	A	B	C	D	E	F
1					Average	
2	1.234	4.578	6.914		`=AVERAGE(A2:C2)`	

b Fill down row 2 one hundred times, finishing on row 101.

c Type the formula `=AVERAGE(E2:E101)` in cell `I2` and `=STDEV.P(E2:E101)` in cell `J2` to calculate the mean and standard deviation of this sample of means.

d We will now group the data in intervals of 1 unit. In cells `K3:V3` enter the starting value of each class, that is 0, 1, . . . , 11. Also record a final 12 in cell `W3` to record the end of the data. Cells `K4:V4` will calculate the class centre and cells `K5:V5` will calculate the frequency for each class. Finally cells `K6:V6` will divide the frequency by 100 (the number of times we ran the experiment) to calculate the relative frequency. The formulae are shown in the spreadsheet fragment below for cells `K4:K6`. You should fill the formulae from cells `K4:K6` across to cells `V4:V6`.

	⋯	I	J	K	L	M	N	O
1	⋯	Mean	SD					
2	⋯	5.901	1.897					
3	⋯			0	1	2	3	4
4	⋯			`=(L3+K3)/2`				
5	⋯			`=countif($E:$E, “ < ”&L3)− countif($E:$E,” < ”&K3)`				
6	⋯			`=K5/100`				

(When using the code `$E:$E`, make sure that there are no other entries in column `E`.)

e Construct a histogram using the data from cells `K6:V6`. If your program allows, you should label the horizontal axis using your class centres and ensure that there are no gaps between the rectangles (Excel: Click on bars and select `FORMAT DATA SERIES`), because there should not be gaps between the rectangles of a histogram.

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f If your data do not generate a distribution that looks normal, you might like to recalculate with a fresh set of random numbers. In Excel this option is available under the 'Formulas' tab, option `Calculate Now` (shortcut `F9`).

g If random numbers are generated from the interval $0 \leq x \leq c$, then theory claims that

$$\mu \doteqdot \frac{c}{2} \qquad \text{and} \qquad \sigma^2 \doteqdot \frac{c^2}{12n}.$$

Test these two formulae agree with your results obtained in part **c**.

h If your distribution is normal, then approximately 68% of the data should be within 1 standard deviation of the mean. Theory predicts that the mean was 6 and the standard deviation was 2. Check that:

i approximately 68 of the 100 numbers fell in the interval [4, 8],

ii approximately 95 of the 100 numbers fell in the interval [2, 10].

i A more correct experimental approach to improve the normality of the distribution would be to run the experiment with more trials. Adapt the spreadsheet for 1000 trials. Remember to adjust cells `K6:V6` for the new experiment. Test your improved experiment by repeating part **h**.

3 In the experiment in Questions 1 and 2 we took means of a continuous distribution, but you can also take the mean of discrete data and approximate it by a normal distribution.

a In Question 2, replace cells `A1:C101` by `RANDBETWEEN(1,6)+RANDBETWEEN(1,6)`, simulating the result of throwing two dice and recording their sum. Note that this is not a uniform distribution.

b This could also be done as a practical experiment using a pair of dice.

4 [Normal approximation to the binomial] If we throw 10 coins, what is the probability of obtaining exactly 5 heads? This is called a *binomial probability*, because each coin produces one of only two possible outcomes, 0 and 1 (the prefix 'bi-' means 'two'). It is another discrete probability distribution that may be modelled by a continuous normal probability distribution.

a Throw 10 coins and record the number of heads (if you are short of cash, you could throw 10 dice and record the number of dice showing an even number).

b Repeat this 100 times, recording your results in a frequency table as you go:

Heads	0	1	2	3	4	5	6	7	8	9	10
Tally											
Freq											

c Calculate the mean and standard deviation of your distribution.

d Draw a histogram of your experiment and add the frequency polygon. Does it look normal?

e Assuming a mean of 5 and a standard deviation of about 1.5, one standard deviation either side of the mean should represents an outcome of 4, 5 or 6 heads. (Why?) Do you find 68% of the numbers fall within one standard deviation of the mean?

f To improve your results, repeat the experiment more times. If this is done as a class exercise in groups, groups could collate their data into one frequency table.

g [Technology] A spreadsheet is a good tool to record your data and construct a histogram.

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h [Technology] You may wish to try simulating the whole experiment in a spreadsheet. Here is a fragment of a spreadsheet to generate 10 coin flips and record the number of heads. The formula in cell `A1` is duplicated in cells `B1:J1` and returns a 1 for a head and a zero for a tail. The formula in cell `K1` counts the number of heads.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	`=randbetween(0,1)`				4	7	5	2	6	4	`=sum(A1:J1)`		

5 [For further investigation of Question 4]

a Theory indicates that if I count the number of heads on n throws and if p is the probability of a head, then

$$\bar{x} = \frac{n}{2} \qquad \text{and} \qquad s^2 = np(1 - p).$$

Check this for the experiment above.

b Try varying n.

c Try varying p — throw 15 dice and count the number of dice showing more than 4 $\left(\text{so that } p = \frac{1}{3}\right)$. Or, on your spreadsheet use the code `=if(randbetween(1,6)>4,1,0)`.

d Statisticians have a rule of thumb — for a fairly good approximation to a normal distribution, np and $n(1 - p)$ should be at least 5. Test this out.

6 [For further investigation] In Question 1 and 2 we generated a new set of data by averaging the means of three random numbers. If real numbers are generated from the interval $[0, c]$, the *central limit theorem* predicts that the mean and standard deviation of these data will be

$$\mu \doteqdot \frac{c}{2} \qquad \text{and} \qquad \sigma^2 \doteqdot \frac{c^2}{12n},$$

and that the approximation to normal should be increasingly good as $n \to \infty$. Investigate what happens as you use larger and larger values of n. Does your distribution look increasingly normal?

Note that if you wish to use an interval length not equal to 1 when grouping your data, then you will need to graph relative frequency per unit of width on the vertical axis.

7 It is reasonable to suppose that height follows a bell-shaped distribution, because most of a fairly homogeneous population cluster around the mean height and rapidly tails off further from the mean.

Collect the height of students in your year group. This could be done in classes and results shared, or results could be entered in an online survey.

a Using the techniques of this section, group the results, then graph the histogram and frequency polygon. You will need to choose your interval width so that enough students lie in the central classes to generate a good histogram.

b Does the curve look normal?

c Calculate the mean and standard deviation.

d Assuming that the results are approximately normal, test whether the expected number of students lie within one and two standard deviations of the mean.

e Can you improve the normality of your results by restricting your population to a certain age group or ethnic group?

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8 Using DESMOS, or other graph sketching program, we can investigate the normal distribution curve with mean μ and standard deviation σ,

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Your curve sketching program may recognise entering *pi* for 3.14159265. . . and *e* for 2.718281828. . ., but many programs will not recognise *sigma* or *mu* and you will need to replace them with *s* and *m* (or other pronumerals):

$$y = \frac{1}{s\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2s^2}}.$$

- **a** Create sliders for m and s, if possible restricting $-6 \le s \le 6$ and $-3 \le m \le 3$.
- **b** Adjust the vertical scale (say $-0.1 \le y \le 1.1$) so that the graph with its 'bell shape' is clearly displayed.
- **c** Set $s = 1$ and verify that adjusting m shifts the central mean and median of the symmetric curve.
- **d** With $s = 1$ and $m = 0$, determine the highest point of the curve. Using the equation above, what are the exact coordinates of this maximum point on the curve?
- **e** What is the effect of adjusting s to the 'fatness' and height of the curve?
- **f** What are the heights of the curve when $s = 1$, $s = 2$ and $s = 4$? Comment.
- **g** In DESMOS, define the function

  $$f(x) = \frac{1}{s\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2s^2}} \qquad \text{and then enter} \qquad \int_{-s}^{s} f(x)\,dx$$

  Check the claim that 68% of the data lie within one standard deviation of the mean.
- **h** Adjust the limits of the integral and check that 95.4% of the data lie within two standard deviations of the mean, and 99.7% of the data lie within three standard deviations of the mean.
- **i** Graph-sketching programs are not always good at handling ∞. Use the table for the standard normal curve and check the value obtained for $P(X \le 1)$ obtained by DESMOS with

  $$\int_{-3}^{1} f(x)\,dx.$$

  How small a value is needed for the lower limit to get a value accurate to 9 decimal places, assuming that the answer should be $P(X \le 1) \doteqdot 0.841344746$?

9 [Investigation] The *Galton board* is a machine designed to generate a bell-shaped curve by means of a set of steel balls falling through a triangular array of pegs. It is possible to buy Galton boards, or there are animations of this device on the web. The original Galton board was designed by Sir Francis Galton (1822–1911).

- **a** Test this device or view animations of the device in action. How good a normal approximation does it produce?
- **b** How does it work?

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Chapter 10 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 10 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

Note: There is a brief summary of the normal distribution, including a graph, a table and the empirical rule, in the Appendix at the end of this chapter.

1 A simple experiment measures the length of time in hours that a certain drug is retained in a patient's system. The following preliminary data were recorded:

0.9	1.4	2.1	2.3	2.6	2.2	2.4	2.7	3.6	3.7
4.1	4.3	4.4	4.4	4.7	5.1	5.2	6.1	6.3	7.1

a Complete the following table for these data.

x	0–1	1–2	2–3	3–4	4–5	5–6	6–7	7–8	Sum
cc	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5	—
Tally									—
f									
cf									—
f_r									
cf_r									—

b Draw a relative frequency histogram and polygon for the dataset.

c Draw a cumulative relative frequency histogram and polygon (ogive) for the dataset.

d By adding appropriate horizontal lines to your graph, find:

i the median Q_2,

ii the quartiles Q_1 and Q_3,

iii the ninth decile,

iv the eighty-fifth percentile.

e The dataset appears (almost) bimodal, with many data points falling in two specific intervals. Advise the medical researcher how to proceed next.

2 State whether each of these sentences is true or false.

a The ogive is joined to the top centre of each rectangle of the cumulative frequency histogram.

b The area under the frequency polygon is 1.

c The probability density function of a continuous probability distribution is the analogue of the relative frequency polygon of a discrete distribution.

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d A probability density function $f(x)$ defined on the interval $a \leq x \leq b$ satisfies the two conditions $f(x) \geq 0$, for all x in the interval, and $\int_a^b f(x)\,dx = 1$.

e Every normal distribution is related to the standard normal distribution by stretches and a horizontal shift.

f Approximately 99% of all data lie within three standard deviations of the mean.

3 Let $f(x) = \frac{1}{20}$, where $-10 \leq x \leq 10$.

a Show that $f(x)$ is a probability density function.

b What special name is given to this type of distribution, where the density function takes the same value across its domain?

c Calculate its expected value.

d Calculate its variance and standard deviation.

4 Let $f(x) = \frac{3}{16}(4 - x^2)$, $0 \leq x \leq 2$.

a Show that $f(x)$ is a probability density function (PDF).

b Find its cumulative density function (CDF).

c The CDF is graphed to the right. Use this graph to estimate:

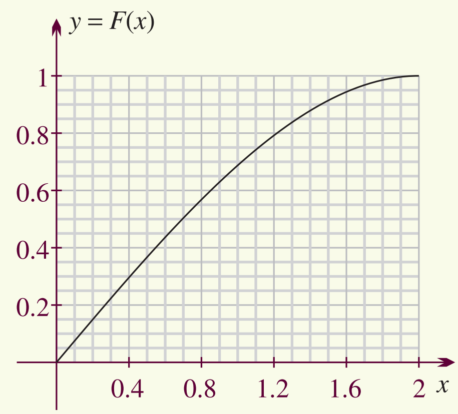
i the three quartiles Q_1, Q_2 and Q_3,

ii the sixth decile,

iii $P(X \leq 1.2)$,

iv $P(X \geq 0.3)$,

v $P(0.2 \leq X \leq 0.4)$.



5 Use your standard normal table and a knowledge of the symmetry of the curve to find:

a $P(Z < 0)$ **b** $P(Z < 1.3)$ **c** $P(-1.8 < Z < 1.8)$

d $P(Z > 0.5)$ **e** $P(Z < -0.2)$ **f** $P(-0.1 < Z < 1.2)$

6 Find the given probability for the normal distribution with given mean and standard deviation. Use the empirical rule (the 68–95–99.7 rule) to estimate:

a $P(X \leq 16)$ if $\mu = 10, \sigma = 3$ **b** $P(X \geq 3.5)$ if $\mu = 5, \sigma = 1.5$

c $P(1.85 \leq X \leq 2.3)$ if $\mu = 2, \sigma = 0.15$ **d** $P(13.65 \leq X \leq 14.1)$ if $\mu = 15, \sigma = 0.45$

7 Repeat the previous question, but this time use your standard normal distribution tables.

a $P(X \leq 22.5)$ if $\mu = 20, \sigma = 5$ **b** $P(X \geq 62)$ if $\mu = 50, \sigma = 10$

c $P(3.96 \leq X \leq 4.3)$ if $\mu = 4, \sigma = 0.2$ **d** $P(6.79 \leq X \leq 8.09)$ if $\mu = 5.75, \sigma = 1.3$

8 A washing machine manufacturer has tested the design of its machines and found them to have an expected life of 6 years 4 months with a standard deviation of 15 months.

a If a family buys one of their machines, what is the probability that it will last more than eight years?

b The manufacturer is deciding on whether to launch a promotion and advertise a five-year warranty on its machines. How many machines could they expect to come to the end of their life within the five-year period?

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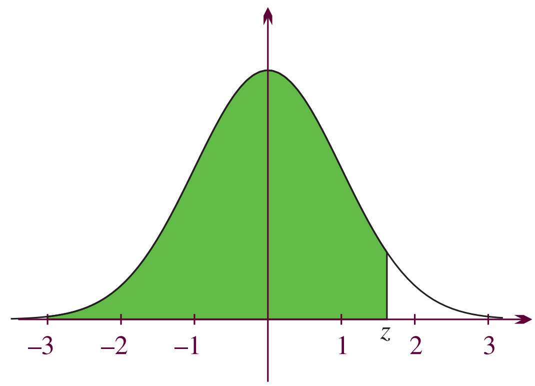
Appendix: The standard normal distribution

A brief summary of the standard normal probability distribution

The graph to the right is the *standard normal probability density function* $y = \phi(z)$.



The shaded area represents the value of the corresponding *cumulative distribution function*

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \phi(t)\, dt.$$

The table below gives some values of the probabilities $\phi(z) = P(Z \le z)$. For example,

$$P(Z \le 1.6) = \Phi(1.6) = \int_{-\infty}^{1.6} \phi(z)\, dz \doteqdot 0.9452.$$

	first decimal place									
z	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0.	0.5000	0.5398	0.5793	0.6179	0.6554	0.6915	0.7257	0.7580	0.7881	0.8159
1.	0.8413	0.8643	0.8849	0.9032	0.9192	0.9332	0.9452	0.9554	0.9641	0.9713
2.	0.9772	0.9821	0.9861	0.9893	0.9918	0.9938	0.9953	0.9965	0.9974	0.9981
3.	0.9987	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

For many purposes, all that is required is the *empirical rule*, or *68–95–99.7 rule*,

$$P(-1 \le Z \le 1) \doteqdot 68\%$$
$$P(-2 \le Z \le 2) \doteqdot 95\%$$
$$P(-3 \le Z \le 3) \doteqdot 99.7\%$$

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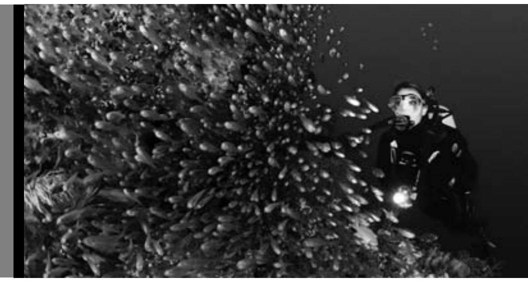
Answers



Answers are not provided for certain questions of the type 'show that' or 'prove that'. Please see worked solutions in these cases for a model.

Chapter 1

Exercise 1A

1 a 850, 1000, 1150, 1300, 1450, 1600, 1750, 1900, 2050, 2200, 2350, 2500, . . .
b 9 months

2 a 20, 25, 30, 35 **b** 36, 46, 56, 66
c 16, 32, 64, 128 **d** 24, 48, 96, 192
e 26, 22, 18, 14 **f** 12, 3, −6, −15
g $3, 1\frac{1}{2}, \frac{3}{4}, \frac{3}{8}$ **h** $3, 1, \frac{1}{3}, \frac{1}{9}$
i 1, −1, 1, −1 **j** 16, 25, 36, 49
k $\frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}$ **l** $-2, 1, -\frac{1}{2}, \frac{1}{4}$

3 a 6, 12, 18, 24 **b** 3, 8, 13, 18
c 2, 4, 8, 16 **d** 5, 25, 125, 625
e 19, 18, 17, 16 **f** 4, 2, 0, −2
g 6, 12, 24, 48 **h** 70, 700, 7000, 70 000
i 1, 8, 27, 64 **j** 2, 6, 12, 20
k −1, 1, −1, 1 **l** −3, 9, −27, 81

4 a 6, 8, 10, 12 **b** 11, 61, 111, 161
c 15, 12, 9, 6 **d** 12, 4, −4, −12
e 5, 10, 20, 40 **f** $\frac{1}{3}, 1, 3, 9$
g $18, 9, 4\frac{1}{2}, 2\frac{1}{4}$ **h** $-100, -20, -4, -\frac{4}{5}$

5 7, 12, 17, 22, 27, 32, 37, 42, 47, 52, 57, 62
a 5 **b** 11 **c** 4 **d** 8 **e** 52
f 7th term **g** Yes, 17th term.
h No, they all end in 2 or 7.
i 47, the 9th term
j 42, the 8th term

6 $\frac{3}{4}, 1\frac{1}{2}$, 3, 6, 12, 24, 48, 96, 192, 384, 768, 1536
a 6 **b** 10 **c** 3 **d** 10 **e** 384
f 9th term **g** Yes, 8th term.
h No **i** 384, the 10th term
j 48, the 7th term

7 a 13, 14, 15, 16, 17. Add 1.
b 9, 14, 19, 24, 29. Add 5.
c 10, 5, 0, −5, −10. Subtract 5.
d 6, 12, 24, 48, 96. Multiply by 2.
e −7, 7, −7, 7, −7. Multiply by −1.
f $40, 20, 10, 5, 2\frac{1}{2}$. Divide by 2.

8 c $100 = T_{33}$, 200 is not a term, $1000 = T_{333}$.

9 a 16 is not a term, $35 = T_{20}$, $111 = T_{58}$.
b $44 = T_5$, 200 and 306 are not terms.
c 40 is not a term, $72 = T_6$, $200 = T_{10}$.
d $8 = T_3$, 96 is not a term, $128 = T_7$.

10 c 49 **d** $T_{20} = 204$

11 a 52 **b** 73
c $T_{41} = 128$ **d** $T_{21} = 103$

12 a 3, 5, 7, 9 **b** 5, 17, 29, 41
c 6, 3, 0, −3 **d** 12, 2, −8, −18
e 5, 10, 20, 40 **f** 4, 20, 100, 500
g $20, 10, 5, 2\frac{1}{2}$ **h** 1, −1, 1, −1

13 a 1, 0, −1, 0, T_n where n is even.
b 0, −1, 0, 1, T_n where n is odd.
c −1, 1, −1, 1. No terms are zero.
d 0, 0, 0, 0. All terms are zero.

14 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, . . . The sum of two odd integers is even, and the sum of an even and an odd integer is odd.

Exercise 1B

1 a 14, 18, 22 **b** 18, 23, 28
c 5, −5, −15 **d** −7, −13, −19
e $9, 10\frac{1}{2}, 12$ **f** $6\frac{1}{2}, 6, 5\frac{1}{2}$

2 a 3, 5, 7, 9 **b** 7, 9, 11, 13
c 7, 3, −1, −5 **d** 17, 28, 39, 50
e 30, 19, 8, −3 **f** −9, −5, −1, 3
g $4\frac{1}{2}, 4, 3\frac{1}{2}, 3$ **h** $3\frac{1}{2}, 1\frac{1}{2}, -\frac{1}{2}, -2\frac{1}{2}$
i 0.9, 1.6, 2.3, 3

3 a AP: $a = 3, d = 4$ **b** AP: $a = 11, d = -4$
c AP: $a = 10, d = 7$ **d** not an AP
e AP: $a = 50, d = -15$ **f** AP: $a = 23, d = 11$
g AP: $a = -12, d = 5$ **h** not an AP
i not an AP **j** AP: $a = 8, d = -10$
k AP: $a = -17, d = 17$
l AP: $a = 10, d = -2\frac{1}{2}$

4 a 67 **b** −55 **c** $50\frac{1}{2}$

5 a 29 **b** 51 **c** 29

6 a $a = 6, d = 10$
b 86, 206, 996
c $T_n = 10n - 4$

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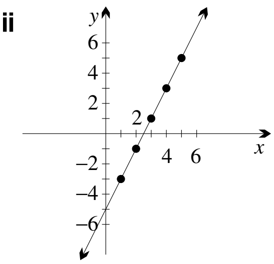
7 a $a = -20, d = 11$ **b** 57, 310, 2169
c $T_n = 11n - 31$
8 a $a = 300, d = -40$ **b** 60, −1700, −39 660
c $T_n = 340 - 40n$
9 a $d = 3, T_n = 5 + 3n$
b $d = -6, T_n = 27 - 6n$
c not an AP
d $d = 4, T_n = 4n - 7$
e $d = 1\frac{1}{4}, T_n = \frac{1}{4}(2 + 5n)$
f $d = -17, T_n = 29 - 17n$
g $d = \sqrt{2}, T_n = n\sqrt{2}$
h not an AP
i $d = 3\frac{1}{2}, T_n = \frac{1}{2}(7n - 12) = \frac{7}{2}n - 6$
10 a $T_n = 170 - 5n$ **b** 26 terms
c $T_{35} = -5$
11 a 11 terms **b** 34 terms **c** 16 terms
d 13 terms **e** 9 terms **f** 667 terms
12 a $T_n = 23 - 3n, T_8 = -1$
b $T_n = 55 - 5n, T_{12} = -5$
c $T_n = 74 - 7n, T_{11} = -3$
d $T_n = 85 - 3n, T_{29} = -2$
e $T_n = 353 - 8n, T_{45} = -7$
f $T_n = 25 - \frac{1}{2}n, T_{51} = -\frac{1}{2}$
13 a 11, 15, 19, 23, $a = 11, d = 4$
b $T_{50} + T_{25} = 314, T_{50} - T_{25} = 100$
d $815 = T_{202}$
e $T_{248} = 999, T_{249} = 1003$
f $T_{49} = 203, \ldots, T_{73} = 299$ lie between 200 and 300, making 25 terms.
14 a i $T_n = 8n$
ii $T_{63} = 504, T_{106} = 848$
iii 44 terms
b $T_{91} = 1001, T_{181} = 1991$, 91 terms
c $T_{115} = 805, T_{285} = 1995$, 171 terms
15 a $d = 3$. 7, 10, 13, 16
b $d = -18$. 100, 82, 64, 46, 28
c $d = 8$. $T_{20} = 180$
d $d = -2$. $T_{100} = -166$
16 a \$500, \$800, \$1100, \$1400, . . .
b $a = 500, d = 300$ **c** \$4700
d cost $= 200 + 300n$ **e** 32
17 a 180, 200, 220, . . . **b** $a = 180, d = 20$
c 400 km **d** length $= 160 + 20n$
e 19 months
18 a 9, 6, 3, 0, −3, . . . and $T_n = 12 - 3n$
b i $T_n = 2n - 5, f(x) = 2x - 5$
ii



19 a $d = 4, x = 1$ **b** $d = 6x, x = \frac{1}{3}$
20 a $d = \log_3 2, T_n = n\log_3 2$
b $d = -\log_a 3, T_n = \log_a 2 + (4 - n)\log_a 3$
c $d = x + 4y, T_n = nx + (4n - 7)y$
d $d = -4 + 7\sqrt{5}, T_n = 9 - 4n + (7n - 13)\sqrt{5}$
e $d = -1.88, T_n = 3.24 - 1.88n$
f $d = -\log_a x, T_n = \log_a 3 + (3 - n)\log_a x$
21 a $a = m + b, d = m$
b gradient $= d$, y-intercept $= m - a$

Exercise 1C

1 a 8, 16, 32 **b** 3, 1, $\frac{1}{3}$
c −56, −112, −224 **d** −20, −4, $-\frac{4}{5}$
e −24, 48, −96 **f** 200, −400, 800
g −5, 5, −5 **h** 1, $-\frac{1}{10}$, $\frac{1}{100}$
i 40, 400, 4000
2 a 1, 3, 9, 27 **b** 12, 24, 48, 96
c 5, −10, 20, −40 **d** 18, 6, 2, $\frac{2}{3}$
e 18, −6, 2, $-\frac{2}{3}$ **f** 50, 10, 2, $\frac{2}{5}$
g 6, −3, $1\frac{1}{2}$, $-\frac{3}{4}$ **h** −13, −26, −52, −104
i −7, 7, −7, 7
3 a GP: $a = 4, r = 2$ **b** GP: $a = 16, r = \frac{1}{2}$
c GP: $a = 7, r = 3$ **d** GP: $a = -4, r = 5$
e not a GP
f GP: $a = -1000, r = \frac{1}{10}$
g GP: $a = -80, r = -\frac{1}{2}$
h GP: $a = 29, r = 1$
i not a GP
j GP: $a = -14, r = -1$
k GP: $a = 6, r = \frac{1}{6}$
l GP: $a = -\frac{1}{3}, r = -3$
4 a 40 **b** $\frac{3}{10}$ **c** −56
d −8 **e** −88 **f** 120
5 a 3^{69} **b** 5×7^{69}
c $8 \times (-3)^{69} = -8 \times 3^{69}$
6 a $a = 7, r = 2$
b $T_6 = 224, T_{50} = 7 \times 2^{49}$
c $T_n = 7 \times 2^{n-1}$

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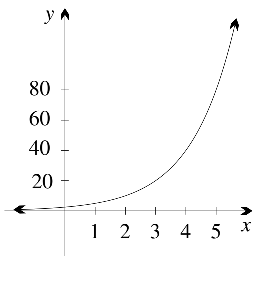
7 a $a = 10, r = -3$
b $T_6 = -2430, T_{25} = 10 \times (-3)^{24} = 10 \times 3^{24}$
c $T_n = 10 \times (-3)^{n-1}$
8 a $a = -80, r = \frac{1}{2}$
b $T_{10} = -\frac{5}{32}, T_{100} = -80 \times \left(\frac{1}{2}\right)^{99}$
c $T_n = -80 \times \left(\frac{1}{2}\right)^{n-1}$
9 a $T_n = 10 \times 2^{n-1}, T_6 = 320$
b $T_n = 180 \times \left(\frac{1}{3}\right)^{n-1}, T_6 = \frac{20}{27}$
c not a GP **d** not a GP
e $T_n = \frac{3}{4} \times 4^{n-1}, T_6 = 768$
f $T_n = -48 \times \left(\frac{1}{2}\right)^{n-1}, T_6 = -1\frac{1}{2}$
10 a $r = -1, T_n = (-1)^{n-1}, T_6 = -1$
b $r = -2, T_n = -2 \times (-2)^{n-1} = (-2)^n, T_6 = 64$
c $r = -3, T_n = -8 \times (-3)^{n-1}, T_6 = 1944$
d $r = -\frac{1}{2}, T_n = 60 \times \left(-\frac{1}{2}\right)^{n-1}, T_6 = -\frac{15}{8}$
e $r = -\frac{1}{2}, T_n = -1024 \times \left(-\frac{1}{2}\right)^{n-1}, T_6 = 32$
f $r = -6, T_n = \frac{1}{16} \times (-6)^{n-1}, T_6 = -486$
11 a $T_n = 2^{n-1}$, 7 terms **b** $T_n = -3^{n-1}$, 5 terms
c $T_n = 8 \times 5^{n-1}$, 7 terms
d $T_n = 7 \times 2^{n-1}$, 6 terms
e $T_n = 2 \times 7^{n-1}$, 5 terms
f $T_n = 5^{n-3}$, 7 terms
12 a $r = 2$. 25, 50, 100, 200, 400
b $r = 2$. 3, 6, 12, 24, 48, 96
c Either $r = 3$, giving 1, 3, 9, 27, 81, or $r = -3$, giving 1, −3, 9, −27, 81.
13 a $r = \frac{1}{9}$ or $-\frac{1}{9}$ **b** $r = 0.1$ or -0.1
c $r = -\frac{3}{2}$ **d** $r = \sqrt{2}$ or $-\sqrt{2}$
14 a 50, 100, 200, 400, 800, 1600, $a = 50, r = 2$
b $6400 = T_8$
c $T_{50} \times T_{25} = 5^4 \times 2^{75}, T_{50} \div T_{25} = 2^{25}$
e The six terms $T_6 = 1600, \ldots, T_{11} = 51\,200$ lie between 1000 and 100000.
15 The successive thicknesses form a GP with 101 terms, and with $a = 0.1$ mm and $r = 2$. Hence thickness $= T_{101} = \frac{2^{100}}{10}$ mm $\doteqdot 1.27 \times 10^{23}$ km $\doteqdot 1.34 \times 10^{10}$ light years, which is close to the present estimate of the distance to the Big Bang.
16 a $\frac{4}{5}$, 4, 20, 100, 500, . . . and $T_n = \frac{4}{25} \times 5^n$
b i $T_n = \frac{5}{2} \times 2^n, f(x) = \frac{5}{2} \times 2^x$
ii



17 a $r = \sqrt{2}, T_n = \sqrt{6} \times \left(\sqrt{2}\right)^{n-1} = \sqrt{3} \times \left(\sqrt{2}\right)^n$
b $r = ax^2, T_n = a^n x^{2n-1}$
c $r = \frac{y}{x}, T_n = -x^{2-n}y^{n-2}$
18 a $T_n = 2x^n, x = 1$ or -1
b $T_n = x^{6-2n}, x = \frac{1}{3}$ or $-\frac{1}{3}$
c $T_n = 2^{-16} \times 2^{4n-4}x = 2^{4n-20}x, x = 6$
19 a $a = cb, r = b$ **b** $f(x) = \frac{a}{r} \times r^x$

Exercise 1D

1 a 11 **b** 23 **c** −31
d −8 **e** 12 **f** 10
2 a 6 or −6 **b** 12 or −12 **c** 30 or −30
d 14 or −14 **e** 5 **f** −16
3 a 10. 8 or −8 **b** 25. 7 or −7
c $20\frac{1}{2}$. 20 or −20 **d** $-12\frac{1}{2}$. 10 or −10
e −30. 2 **f** 0. 6
g −3. 1 **h** 24. −3
i 40. 45 **j** 84. −16
k $-5\frac{3}{4}$. −36 **l** −21. 7
4 a 7, 14, 21, 28, 35, 42
b 27, 18, 12, 8
c 40, $36\frac{1}{2}$, 33, $29\frac{1}{2}$, 26, $22\frac{1}{2}$, 19, $15\frac{1}{2}$, 12, $8\frac{1}{2}$, 5
d 1, 10, 100, 1000, 10000, 100000, 1000000 or 1, −10, 100, −1000, 10000, −100000, 1000000
e 3, $14\frac{1}{4}$, $25\frac{1}{2}$, $36\frac{3}{4}$, 48
f 3, 6, 12, 24, 48 or 3, −6, 12, −24, 48
5 a $d = 3, a = -9$ **b** $d = 4, a = -1$
c $d = -9, a = 60$ **d** $d = 3\frac{1}{2}, a = -4\frac{1}{2}$
6 a $r = 2, a = 4$ **b** $r = 4, a = \frac{1}{16}$
c $r = 3$ and $a = \frac{1}{9}$, or $r = -3$ and $a = -\frac{1}{9}$
d $r = \sqrt{2}$ and $a = \frac{3}{2}$, or $r = -\sqrt{2}$ and $a = \frac{3}{2}$
7 a $T_8 = 37$ **b** $T_2 = 59$ **c** $T_2 = \frac{3}{8}$
8 a $n = 13$ **b** $n = 8$
c $n = 11$ **d** $n = 8$
9 b $n = 19$ **c** $n = 29$
d $n = 66$ **e** 10 terms
f 37 terms
10 a $T_n = 98 \times \left(\frac{1}{7}\right)^{n-1}$, 10 terms
b $T_n = 25 \times \left(\frac{1}{5}\right)^{n-1} = \left(\frac{1}{5}\right)^{n-3}$, 11 terms
c $T_n = (0.9)^{n-1}$, 132 terms
11 a 78% **b** 152 sheets
12 a $a = 28, d = -1$
b $a = \frac{1}{3}$ and $r = 3$, or $a = \frac{2}{3}$ and $r = -3$
c $T_6 = -2$

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13 a $x = 10$. 9, 17, 25

b $x = -2$. $-2, -6, -10$

c $x = 2$. $-1, 5, 11$

d $x = -4$. $-14, -4, 6$

14 a $x = -\frac{1}{2}$. $-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$

b $x = 1$. 1, 2, 4 or $x = 6$. $-4, 2, -1$

15 a **i** $x = -48$ **ii** $x = 6$

b **i** $x = 0.10001$

ii $x = 0.002$ or $x = -0.002$

c **i** $x = 0.398$ **ii** $x = 20$

d **i** They can't form an AP. **ii** $x = 9$

e **i** $x = 2$ **ii** $x = 4$ or $x = 0$

f **i** $x = \sqrt{5}$ **ii** $x = 2$ or $x = -2$

g **i** $x = \frac{3}{2}\sqrt{2}$ **ii** $x = 2$ or $x = -2$

h **i** $x = 40$ **ii** 2^5 or -2^5

i **i** $x = 0$ **ii** They can't form a GP.

16 a $a = 2$ and $r = 4$

b $a = \log_2 3$ and $d = 1$

17 a $a = 6\frac{1}{4}$ and $b = 2\frac{1}{2}$, or $a = 4$ and $b = -2$

b $a = 1, b = 0$

18 a $T_n = 2^{8-3n}$

19 b $\frac{T_8}{T_1} = \left(\frac{1}{2}\right)^{\frac{7}{12}} \doteqdot 0.6674 \doteqdot \frac{2}{3}$

c $\frac{T_5}{T_1} = \left(\frac{1}{2}\right)^{\frac{4}{12}} \doteqdot 0.7937 \doteqdot \frac{4}{5}$

d $\frac{T_6}{T_1} = \left(\frac{1}{2}\right)^{\frac{5}{12}} \doteqdot 0.7491 \doteqdot \frac{3}{4}$,

$\frac{T_4}{T_1} = \left(\frac{1}{2}\right)^{\frac{3}{12}} \doteqdot 0.8409 \doteqdot \frac{5}{6}$

e $\frac{T_3}{T_1} = \left(\frac{1}{2}\right)^{\frac{2}{12}} \doteqdot 0.8908 \doteqdot \frac{8}{9}$,

$\frac{T_2}{T_1} = \left(\frac{1}{2}\right)^{\frac{1}{12}} \doteqdot 0.9439 \doteqdot \frac{17}{18}$

Exercise 1E

1 a 24 **b** 80 **c** 0 **d** $3\frac{3}{4}$

2 a 450 **b** 24 **c** -54 **d** 15.6

3 a 10, 30, 60, 100, 150 **b** $1, -2, 7, -20, 61$

c 1, 5, 14, 30, 55 **d** 3, $7\frac{1}{2}$, $13\frac{1}{2}$, 21, 30

4 a $-2, 3, -3$

b 120, 121, $121\frac{1}{3}$

c 60, 50, 30

d 0.1111, 0.11111, 0.111111

5 S_n: 2, 7, 15, 26, 40, 57, 77

S_n: 40, 78, 114, 148, 180, 210, 238

S_n: $2, -2, 4, -4, 6, -6, 8$

S_n: 7, 0, 7, 0, 7, 0, 7

6 T_n: 1, 3, 5, 7, 9, 11, 13

T_n: 2, 4, 8, 16, 32, 64, 128

T_n: $-3, -5, -7, -9, -11, -13, -15$

T_n: $8, -8, 8, -8, 8, -8, 8$

7 T_n: 1, 1, 1, 2, 3, 5, 8, 13

T_n: 3, 1, 3, 4, 7, 11, 18, 29

8 a 42 **b** 75 **c** 15 **d** 174

e 100 **f** 63 **g** 117 **h** -1

i 0 **j** 404 **k** 7 **l** -7

10 a $\sum_{n=1}^{40} n^3$ **b** $\sum_{n=1}^{40} \frac{1}{n}$ **c** $\sum_{n=1}^{20} (n + 2)$ **d** $\sum_{n=1}^{12} 2^n$

e $\sum_{n=1}^{10} (-1)^n n$ **f** $\sum_{n=1}^{10} (-1)^{n+1} n$ or $\sum_{n=1}^{10} (-1)^{n-1} n$

11 a $T_1 = 2, T_n = 2^{n-1}$ for $n \ge 2$

b 2, 2, 4, 8, 16, . . .

c The derivative of e^x is the original function e^x. Remove the initial term 2 from the sequence in part **b**, and the successive differences are the original sequence.

12 b $T_1 = 1$ and $T_n = 3n^2 - 3n + 1$ for $n \ge 2$

c $U_1 = 1$ and $U_n = 6n$ for $n \ge 2$

d 1, 7, 19, 37, 61, 91, . . . and 1, 6, 12, 18, 24, 30, . . .

e The derivative of x^3 is the quadratic $3x^2$, and its derivative is the linear function $6x$. Taking successive differences once gives a quadratic, and taking them twice gives a linear function.

Exercise 1F

1 77

2 a $n = 100$, 5050 **b** $n = 50$, 2500

c $n = 50$, 2550 **d** $n = 100$, 15150

e $n = 50$, 7500 **f** $n = 9000$, 49504500

3 a 180 **b** 78 **c** -153 **d** -222

4 a $a = 2, d = 4$, 882

b $a = 3, d = 7$, 1533

c $a = -6, d = 5$, 924

d $a = 10, d = -5$, -840

e $a = -7, d = -3$, -777

f $a = 1\frac{1}{2}, d = 2$, $451\frac{1}{2}$

5 a 222 **b** -630 **c** 78400

d 0 **e** 65 **f** 30

6 a 101 terms, 10100 **b** 13 terms, 650

c 11 terms, 275 **d** 100 terms, 15250

e 11 terms, 319 **f** 10 terms, $61\frac{2}{3}$

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7 a 500 terms, 250 500 **b** 2001 terms, 4 002 000
c 3160 **d** 1440

8 a $S_n = \frac{1}{2}n(5 + 5n)$ **b** $S_n = \frac{1}{2}n(17 + 3n)$
c $S_n = n(1 + 2n)$ **d** $\frac{1}{2}n(5n - 23)$
e $S_n = \frac{1}{4}n(21 - n)$ **f** $\frac{1}{2}n\left(2 + n\sqrt{2} - 3\sqrt{2}\right)$

9 a $\frac{1}{2}n(n + 1)$ **b** n^2
c $\frac{3}{2}n(n + 1)$ **d** $100n^2$

10 a 450 legs. No creatures have the mean number of 5 legs.
b 16 860 years **c** \$352 000

11 a $a = 598, \ell = 200, 79\,800$
b $a = 90, \ell = -90, 0$
c $a = -47, \ell = 70, 460$
d $a = 53, \ell = 153, 2163$

12 a $\ell = 22$ **b** $a = -7.1$
c $d = 11$ **d** $a = -3$

13 b i 16 terms **ii** more than 16 terms
c 5 terms or 11 terms
d $n = 18$ or $n = -2$, but n must be a positive integer.
e $n = 4, 5, 6, \ldots, 12$
f Solving $S_n > 256$ gives $(n - 8)^2 < 0$, which has no solutions.

14 a $S_n = n(43 - n)$, 43 terms
b $S_n = \frac{3}{2}n(41 - n)$, 41 terms
c $S_n = 3n(n + 14)$, 3 terms
d $\frac{1}{4}n(n + 9)$, 6 terms

15 a 20 rows, 29 logs on bottom row
b $S_n = 5n^2$, 7 seconds
c 11 trips, deposits are 1 km apart.

16 a $d = -2, a = 11, S_{10} = 20$
b $a = 9, d = -2, T_2 = 7$
c $d = -3, a = 28\frac{1}{2}, T_4 = 19\frac{1}{2}$

17 a 10 terms, $55\log_a 2$
b 11 terms, 0
c 6 terms, $3(4\log_b 3 - \log_b 2)$
d $15(\log_x 2 - \log_x 3)$

Exercise 1G

1 728

2 2801 kits, cats, sacks, wives and man

3 a 1093 **b** 547

4 a $1023, 2^n - 1$ **b** $242\,, 3^n - 1$
c $-11\,111, -\frac{1}{9}\left(10^n - 1\right)$ **d** $-781, -\frac{1}{4}\left(5^n - 1\right)$
e $-341, \frac{1}{3}\left(1 - (-2)^n\right)$ **f** $122, \frac{1}{2}\left(1 - (-3)^n\right)$
g $-9091, -\frac{1}{11}\left(1 - (-10)^n\right)$
h $-521, -\frac{1}{6} = \left(1 - (-5)^n\right)$

5 a $\frac{1023}{64}, 16\left(1 - \left(\frac{1}{2}\right)^n\right)$ **b** $\frac{364}{27}, \frac{27}{2}\left(1 - \left(\frac{1}{3}\right)^n\right)$
c $\frac{605}{9}, \frac{135}{2}\left(1 - \left(\frac{1}{3}\right)^n\right)$ **d** $\frac{211}{24}, \frac{4}{3}\left(\left(\frac{3}{2}\right)^n - 1\right)$
e $\frac{341}{64}, \frac{16}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right)$ **f** $\frac{182}{27}, \frac{27}{4}\left(1 - \left(\frac{1}{3}\right)^n\right)$
g $-\frac{305}{9}, -\frac{135}{4}\left(1 - \left(-\frac{1}{3}\right)^n\right)$ **h** $\frac{55}{24}, \frac{4}{15}\left(1 - \left(-\frac{3}{2}\right)^n\right)$

6 a $5\left((1.2)^n - 1\right)$, 25.96
b $20\left(1 - (0.95)^n\right)$, 8.025
c $100\left((1.01)^n - 1\right)$, 10.46
d $100\left(1 - (0.99)^n\right)$, 9.562

7 a i 2^{63} **ii** $2^{64} - 1$
b $615\,\text{km}^3$

8 a $S_n = \left(\left(\sqrt{2}\right)^n - 1\right)\left(\sqrt{2} + 1\right)$,
$S_{10} = 31\left(\sqrt{2} + 1\right)$
b $S_n = \frac{1}{2}\left(1 - \left(-\sqrt{5}\,\right)^n\right)\left(\sqrt{5} - 1\right)$,
$S_{10} = -1562\left(\sqrt{5} - 1\right)$

9 a $a = 6, r = 2, 762$ **b** $a = 9, r = 3, 3276$
c $a = 12, r = \frac{1}{2}, \frac{765}{32}$

10 a $\frac{1}{8} + \frac{3}{4} + \frac{9}{2} + 27 + 162 = 194\frac{3}{8}$ or
$\frac{1}{8} - \frac{3}{4} + \frac{9}{2} - 27 + 162 = 138\frac{7}{8}$
b $15\frac{3}{4}$ **c** 1562.496 **d** 7 **e** 640

11 a i 0.01172 tonnes
ii 11.99 tonnes
b 4.9×10^{-3}g
c i $S_n = 10P(1.1^{10} - 1)$
ii \$56.47

12 a 6 terms **b** 8 terms **c** 5 terms **d** 7 terms

13 b $n = 8$ **c** 14 terms
d $S_{14} = 114\,681$

14 a 41 powers of 3 **b** 42 terms

Exercise 1H

1 a $18, 24, 26, 26\frac{2}{3}, 26\frac{8}{9}, 26\frac{26}{27}$
b $S_\infty = 27$
c $S_\infty - S_6 = 27 - 26\frac{26}{27} = \frac{1}{27}$

2 a $24, 12, 18, 15, 16\frac{1}{2}, 15\frac{3}{4}$ **b** $S_\infty = 16$
c $S_\infty - S_6 = 16 - 15\frac{3}{4} = \frac{1}{4}$

3 a $a = 1, S_\infty = 2$ **b** $a = 8, S_\infty = 16$
c $a = -4, S_\infty = -8$

4 a $a = 1, S_\infty = \frac{3}{4}$ **b** $a = 36, S_\infty = 27$
c $a = -60, S_\infty = -45$

5 a $r = \frac{1}{4}, S_\infty = 80$ **b** $r = -\frac{1}{2}, S_\infty = 40$
c $r = -\frac{1}{5}, S_\infty = 50$

6 a $r = -\frac{1}{2}, S_\infty = \frac{2}{3}$ **b** $r = \frac{1}{3}, S_\infty = \frac{3}{2}$
c $r = -\frac{2}{3}, S_\infty = \frac{3}{5}$ **d** $r = \frac{3}{5}, S_\infty = 2\frac{1}{2}$

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e $r = -\frac{3}{2}$, no limiting sum
f $r = \frac{1}{3}, S_\infty = 18$ **g** $r = \frac{1}{10}, S_\infty = 1111\frac{1}{9}$
h $r = -\frac{1}{10}, S_\infty = 909\frac{1}{11}$
i $r = -1$, no limiting sum
j $r = \frac{9}{10}, S_\infty = 1000$ **k** $r = -\frac{1}{5}, S_\infty = -\frac{5}{3}$
l $r = \frac{1}{5}, S_\infty = -\frac{5}{6}$

7 a The successive down-and-up distances form a GP with $a = 12$ and $r = \frac{1}{2}$.
b $S_\infty = 24$ metres

8 a T_n: 10, 10, 10, 10, 10, 10. S_n: 10, 20, 30, 40, 50, 60. $S_n \to \infty$ as $n \to \infty$.
b T_n: $10, -10, 10, -10, 10, -10$.
S_n: 10, 0, 10, 0, 10, 0. S_n oscillates between 10 and 0 as $n \to \infty$.
c T_n: 10, 20, 40, 80, 160, 320.
S_n: 10, 30, 70, 150, 310, 630. $S_n \to \infty$ as $n \to \infty$.
d T_n: $10, -20, 40, -80, 160, -320$.
S_n: $10, -10, 30, -50, 110, -210$. S_n oscillates between larger and larger positive and negative numbers as $n \to \infty$.

9 a $S_\infty - S_4 = 160 - 150 = 10$
b $S_\infty - S_4 = 111\frac{1}{9} - 111\frac{1}{10} = \frac{1}{90}$
c $S_\infty - S_4 = 55\frac{5}{9} - 32\frac{4}{5} = 22\frac{34}{45}$

10 a $a = 2000$ and $r = \frac{1}{5}$
b $S_\infty = 2500$
c $S_\infty - S_4 = 4$

11 a $S_\infty = 10\,000$ **b** $S_\infty - S_{10} \doteqdot 3487$

12 a $S_\infty = \frac{5}{1 - x}, x = \frac{1}{2}$ **b** $S_\infty = \frac{5}{1 - x}, x = -\frac{2}{3}$
c $S_\infty = \frac{5}{1 + x}, x = -\frac{2}{3}$ **d** $S_\infty = \frac{3x}{2}, x = \frac{4}{3}$
e $S_\infty = \frac{3x}{4}, x = \frac{8}{3}$ **f** $S_\infty = 3x, x = \frac{2}{3}$

13 a $-1 < x < 1, \frac{7}{1 - x}$ **b** $-\frac{1}{3} < x < \frac{1}{3}, \frac{2x}{1 - 3x}$
c $0 < x < 2, \frac{1}{2 - x}$ **d** $-2 < x < 0, -\frac{1}{x}$

14 a $r = 1.01$, no limiting sum
b $r = -0.99, S_\infty = \frac{100}{199}$
c $r = (1.01)^{-1}, S_\infty = 101$
d $r = -\frac{1}{6}, S_\infty = \frac{108}{175}$

15 a $r = \frac{1}{4}, S_\infty = \frac{64}{3}\sqrt{5}$
b $r = -\frac{1}{3}, S_\infty = 81\sqrt{7}$ **c** $\frac{7}{6}\left(7 + \sqrt{7}\right)$
d $4\left(2 - \sqrt{2}\right)$ **e** $5\left(5 - 2\sqrt{5}\right)$
f $r = \frac{1}{3}\sqrt{10} > 1$, so there is no limiting sum.
g $\frac{1}{3}\sqrt{3}$ **h** $\frac{1}{2}\left(\sqrt{3} + 1\right)$

16 a $a = \frac{1}{3}, r = \frac{1}{3}, S_\infty = \frac{1}{2}$
b $a = \frac{7}{2}, r = \frac{1}{2}, S_\infty = 7$
c $a = -24, r = -\frac{3}{5}, S_\infty = -15$

17 a $r = \frac{4}{5}$
b $18 + 6 + 2 + \cdots$ or $9 + 6 + 4 + \cdots$
c $r = \frac{5}{6}$
d i $r = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$ ($r = -\frac{1}{2} - \sqrt{5} < -1$, so it is not a possible solution.)
ii $r = \frac{1}{2}$ **iii** $r = \frac{1}{2}\sqrt{2}$ or $-\frac{1}{2}\sqrt{2}$

18 a $-\sqrt{2} < x < \sqrt{2}$ and $x \neq 0, S_\infty = \frac{1}{2 - x^2}$
b $x \neq 0, S_\infty = \frac{1 + x^2}{x^2}$

19 a $w = \frac{1}{1 - v}$ **b** $v = \frac{w}{1 + w}$ **c** v

Exercise 1I

1 a $0.3 + 0.03 + 0.003 + \cdots = \frac{1}{3}$
b $0.1 + 0.01 + 0.001 + \cdots = \frac{1}{9}$
c $0.7 + 0.07 + 0.007 + \cdots = \frac{7}{9}$
d $0.6 + 0.06 + 0.006 + \cdots = \frac{2}{3}$

2 a $0.27 + 0.0027 + 0.000027 + \cdots = \frac{3}{11}$
b $\frac{81}{99} = \frac{9}{11}$ **c** $\frac{1}{11}$ **d** $\frac{4}{33}$ **e** $\frac{26}{33}$
f $\frac{1}{37}$ **g** $\frac{5}{37}$ **h** $\frac{5}{27}$

3 a $12 + (0.4 + 0.04 + \cdots) = 12\frac{4}{9}$
b $7 + 0.81 + 0.0081 + \cdots = 7\frac{9}{11}$
c $8.4 + (0.06 + 0.006 + \cdots) = 8\frac{7}{15}$
d $0.2 + (0.036 + 0.00036 + \cdots) = \frac{13}{55}$

4 a $0.\dot{9} = 0.9 + 0.09 + 0.009 + \cdots = \frac{0.9}{1 - 0.1} = 1$
b $2.7\dot{9} = 2.7 + (0.09 + 0.009 + 0.0009 + \cdots)$
$= 2.7 + \frac{0.09}{1 - 0.1} = 2.7 + 0.1 = 2.8$

5 a $\frac{29}{303}$ **b** $\frac{25}{101}$ **c** $\frac{3}{13}$ **d** $\frac{3}{7}$
e $0.25 + (0.0057 + 0.000057 + \cdots) = \frac{211}{825}$
f $1\frac{14}{135}$ **g** $\frac{1}{3690}$ **h** $7\frac{27}{35}$

6 If $\sqrt{2}$ were a recurring decimal, then we could use the methods of this section to write it as a fraction.

Chapter 1 review exercise

1 $14, 5, -4, -13, -22, -31, -40, -49$
a 6 **b** 4 **c** -31
d T_8 **e** No **f** $T_{11} = -40$

2 a $52, -62, -542, -5\,999\,942$
b 20 no, $10 = T_8, -56 = T_{19}, -100$ no
c $T_{44} = -206$
d $T_{109} = -596$

3 a $4, 7, 7, 7, 7, 7, \ldots$
b $0, 1, 2, 3, 4, 5, 6, \ldots$
c $T_1 = 5, T_n = 2n - 1$ for $n > 1$
d $T_1 = 3, T_n = 2^{3n-1}$ for $n > 1$

4 a 82 **b** -15 **c** 1 **d** $\frac{63}{64}$

5 a $-5, 5, -5, 5, -5, 5, -5, 5$
b $-5, 0$

6 a AP, $d = 7$ **b** AP, $d = -121$
c neither **d** GP, $r = 3$
e neither **f** GP, $r = -\frac{1}{2}$

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7 a $a = 23, d = 12$
b $T_{20} = 251, T_{600} = 7211$
d $143 = T_{11}$, 173 is not a term.
e $T_{83} = 1007, T_{165} = 1991$
f 83 (Count both T_{83} and T_{165}.)

8 a $a = 20, d = 16$ **b** $T_n = 4 + 16n$
c 12 cases, \$4 change **d** 18

9 a $a = 50, r = 2$
b $T_n = 50 \times 2^{n-1}$ (or 25×2^n)
c $T_8 = 6400, T_{12} = 102\,400$
d $1600 = T_6$, 4800 is not a term.
e 320 000 **f** 18 terms

10 a $a = 486, r = \frac{1}{3}$
b 486, 162, 54, 18, 6, 2 (no fractions)
c 4 **d** $S_6 = 728$ **e** 729

11 a 75 **b** 45 or -45

12 a 11 111 **b** $-16\,400$ **c** 1025

13 a $n = 45, S_{45} = 4995$ **b** $n = 101, S_{101} = 5050$
c $n = 77, S_{77} = 2387$

14 a 189 **b** -1092 **c** $-157\frac{1}{2}$

15 a 300
b $r = -\frac{3}{2} < -1$, so there is no limiting sum.
c $-303\frac{3}{4}$

16 a $-3 < x < -1$ **b** $S_\infty = -\frac{2 + x}{1 + x}$

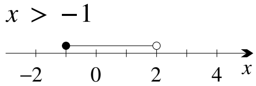
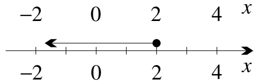
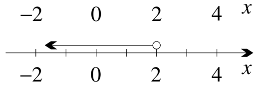
17 a $\frac{13}{33}$ **b** $\frac{52}{111}$ **c** $12\frac{335}{1100} = 12\frac{67}{220}$

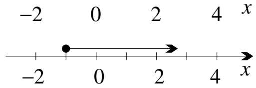
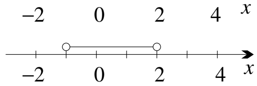
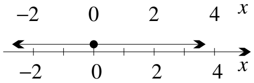
18 a $d = 5$, 511 **b** -1450 **c** $r = -2, -24$
d $d = -5$ **e** $n = 2$ or $n = 8$
f $r = -\frac{1}{3}$ **g** 16

Chapter 2

Exercise 2A

1 a i $-1 \le x \le 2$ **ii** $[-1, 2]$
b i $-1 < x \le 2$ **ii** $(-1, 2]$
c i $x > -1$ **ii** $(-1, \infty)$

2 a i 
ii $[-1, 2)$
b i 
ii $(-\infty, 2]$
c i 
ii $(-\infty, 2)$

3 a i 
ii $x \ge -1$
b i 
ii $-1 < x < 2$
c i 
ii $\mathbb{R}$ (There is no way of writing the interval using inequalities.)

4 a i $2^{15} = 32\,768$ **ii** $5 \times 8 = 40$
iii $2^8 = 256$ **iv** $5 \times 15 = 75$
b i 2^{5x} **ii** 5×2^x
iii 2^{2^x} **iv** $25x$

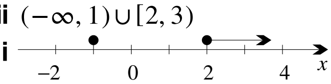
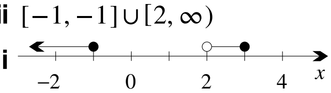
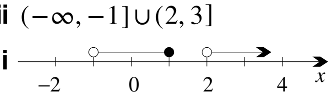
5 a i $x < -1$ or $0 < x < 1$
ii $-1 < x < 0$ or $x > 1$
b i $-5 < x < -2$ or $x > 1$
ii $x < -5$ or $-2 < x < 1$

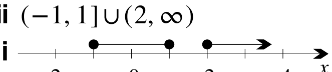
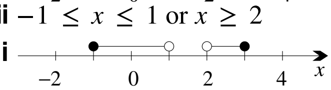
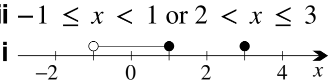
6 a i $x = -3, -1$ or 2
ii $-3 < x < -1$ or $x > 2$
iii $x < -3$ or $-1 < x < 2$
b i $x = -2, 3$ or 4
ii $-2 < x < 3$ or $x > 4$
iii $x < -2$ or $3 < x < 4$

7 a $(-\infty, 1)$, one-to-one
b $(0, 2)$, many-to-one
c $(0, 1)$, one-to-one
d $(4, \infty)$, one-to-one

8 a $x \le 0$ or $1 \le x \le 2$
b $-2 < x < 0$ or $2 < x < 4$
c $0 < x < 3$ or $x > 3$
d $x = 0$ or $x \ge 4$
e $x = -3$ or $x = 3$
f $x = -3$ or $x \ge 0$

9 a i $-1 < x < 1$ or $2 \le x \le 3$
ii $(-1, 1) \cup [2, 3]$
b i $x < 1$ or $x \ge 2$
ii $(-\infty, 1) \cup [2, \infty)$
c i $x < 1$ or $2 \le x < 3$
ii $(-\infty, 1) \cup [2, 3)$

10 a i 
ii $[-1, -1] \cup [2, \infty)$
b i 
ii $(-\infty, -1] \cup (2, 3]$
c i 
ii $(-1, 1] \cup (2, \infty)$

11 a i 
ii $-1 \le x \le 1$ or $x \ge 2$
b i 
ii $-1 \le x < 1$ or $2 < x \le 3$
c i 
ii $-1 < x \le 1$ or $x = 3$

12 a $(-\infty, 0] \cup [1, 2]$ **b** $(-2, 0) \cup (2, 4)$
c $(0, 3) \cup (3, \infty)$ **d** $[0, 0] \cup (4, \infty]$
e $[-3, -3] \cup [3, 3]$ **f** $[-3, -3] \cup [0, \infty)$

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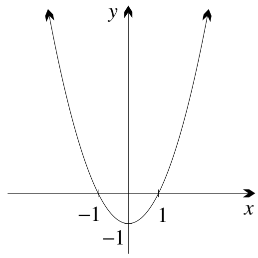
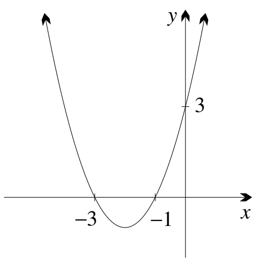
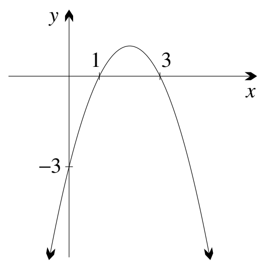
13 y: 3, 0, -1, 0, 3
sign: +, 0, $-$, 0, +



14 a



b



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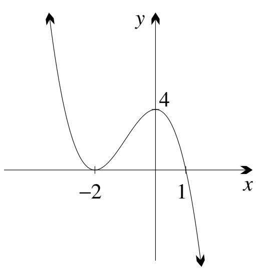
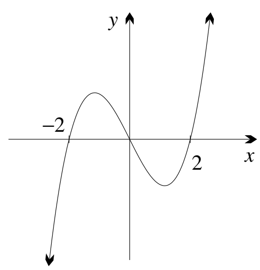


d



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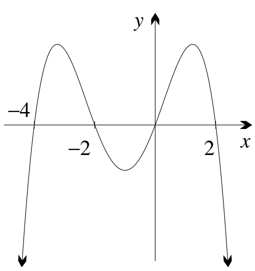

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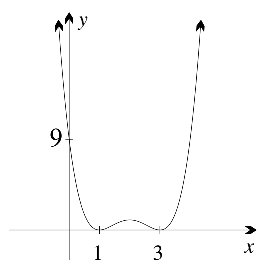
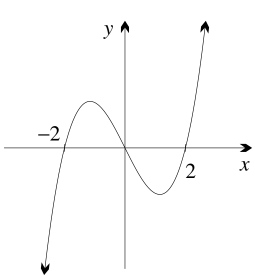
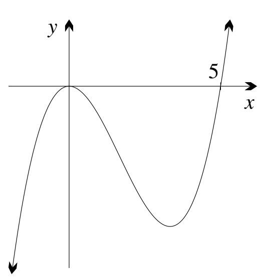


15 a



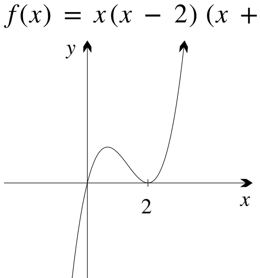
$f(x) = x(x - 2)(x + 2)$

b



$f(x) = x^2(x - 5)$

c

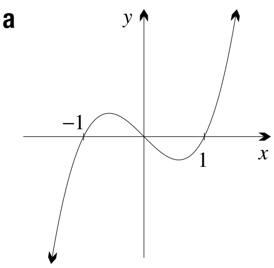


$f(x) = x(x - 2)^2$

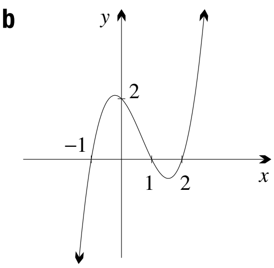
16 a $-2 < x < 0$ or $x > 2$
b $x < 0$ or $0 < x < 5$
c $x \leq 0$ or $x = 2$

17 a $y = x(x + 1)(x - 1)$, $x = -1, 0$ or 1
b $y = (x - 2)(x - 1)(x + 1)$, $x = -1, 1$ or 2
c $y = (x + 2)^2(x - 2)$, $x = -2$ or 2

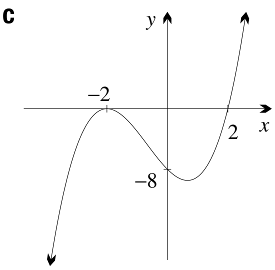
18 a



b



c



19 a zero for $x = 0$, undefined at $x = 3$, positive for $x < 0$ or $x > 3$, negative for $0 < x < 3$
b zero for $x = 4$, undefined at $x = -2$, positive for $x < -2$ or $x > 4$, negative for $-2 < x < 4$
c zero for $x = -3$, undefined at $x = -1$, positive for $x < -3$ or $x > -1$, negative for $-3 < x < -1$
d never zero, undefined at $x = -1$ and at $x = 1$, positive for $x < -1$ or $x > 1$, and negative for $-1 < x < 1$
e zero for $x = -2$ and for $x = 2$, undefined at $x = 0$, positive for $-2 < x < 0$ or $x > 2$, negative for $x < -2$ or $0 < x < 2$
f zero for $x = -2$ and for $x = 2$, undefined at $x = -4$ and at $x = 4$, positive for $x < -4$ or $-2 < x < 2$ or $x > 4$, negative for $-4 < x < -2$ or $2 < x < 4$

20 LHS $= (f \circ g)h(x) = f\big(g(h(x))\big)$,
RHS $= f(g \circ h(x)) = f\big(g(h(x))\big) =$ LHS.

21 a $x < 1$ or $3 < x < 5$
b $-3 \leq x \leq 1$ or $x \geq 4$
c $x \neq 1$ and $x \neq 3$ (alternatively, $x < 1$ or $1 < x < 3$ or $x > 3$)
d $x < -2$ or $0 < x < 2$ or $x > 4$
e $-3 < x < 0$ or $x > 3$
f $x \leq 0$ or $x \geq 5$

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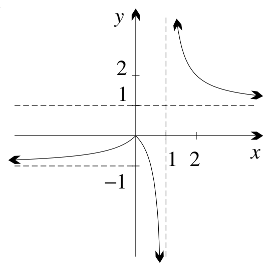
22 a **i** $x \neq 1$

ii $x = 0$

iii $[0, 0] \cup (1, \infty)$

iv

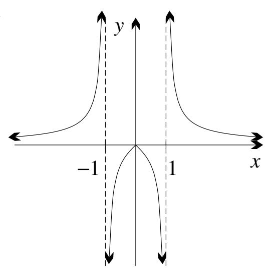


b **i** $x \neq -1$ or 1

ii $x = 0$

iii $(-\infty, -1) \cup [0, 0] \cup (1, \infty)$

iv



23 a It has one endpoint 5, which it contains.

b It contains all its endpoints (there are none).

c It does not contain any of its endpoints (there are none).

Exercise 2B

1 In each case $y \to 0$ as $x \to \infty$ and as $x \to -\infty$.

a **i** $x \neq 1$ **ii** $(0, -1)$

iii Dividing top and bottom by x gives $y = \dfrac{\frac{1}{x}}{1 - \frac{1}{x}}$, which has limit zero as $x \to \infty$ and as $x \to -\infty$. Alternatively, there is a constant on the top, so it is clear that $y \to 0$ as $x \to \infty$ and as $x \to -\infty$.

iv

x	-1	0	1	2	3
y	$-\frac{1}{2}$	-1	*	1	$\frac{1}{2}$

v vertical asymptote: $x = 1$, as $x \to 1^+$, $y > 0$ so $y \to +\infty$, and as $x \to 1^-$, $y < 0$ so $y \to -\infty$

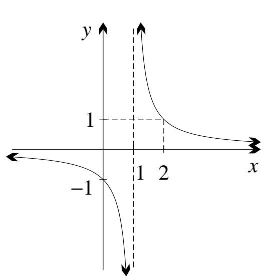
b **i** $x \neq 3$ **ii** $\left(0, \frac{2}{3}\right)$

iv

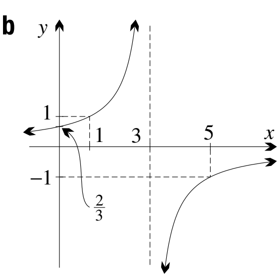
x	0	1	3	5	6
y	$\frac{2}{3}$	1	*	-1	$-\frac{2}{3}$

v vertical asymptote: $x = 3$, as $x \to 3^+$, $y < 0$ so $y \to -\infty$, and as $x \to 3^-$, $y > 0$ so $y \to +\infty$

a



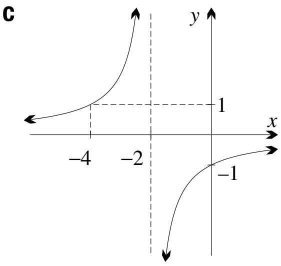
b



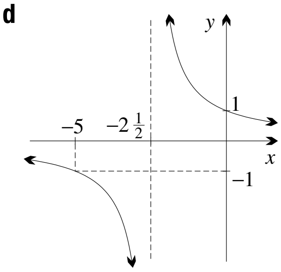
c domain: $x \neq -2$, vertical asymptote: $x = -2$, as $x \to -2^+$, $y < 0$ so $y \to -\infty$, and as $x \to -2^-$, $y > 0$ so $y \to +\infty$

d domain: $x \neq -2\frac{1}{2}$, vertical asymptote: $x = -2\frac{1}{2}$, as $x \to -2\frac{1}{2}^+$, $y > 0$ so $y \to +\infty$, and as $x \to -2\frac{1}{2}^-$, $y < 0$ so $y \to -\infty$

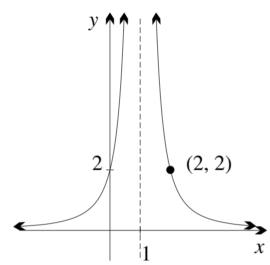
c



d

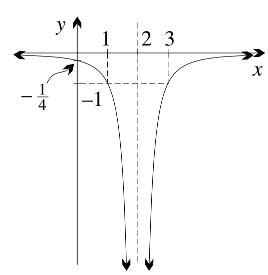


2



domain: $x \neq 1$,
$y \to 0$ as $x \to \infty$ and as $x \to -\infty$
vert'l asymptote $x = 1$, as $x \to 1^+$, $y > 0$ so $y \to \infty$ and as $x \to 1^-$, $y > 0$ so $y \to \infty$

3



domain: $x \neq 2$,
$y \to 0$ as $x \to \infty$ and as $x \to -\infty$
vert'l asymptote $x = 2$, as $x \to 2^+$, $y < 0$ so $y \to -\infty$, as $x \to 2^-$, $y < 0$ so $y \to -\infty$

4 a $f(x) \to 0$ as $x \to \infty$ and $x \to -\infty$

b $f(x) \to 1$ as $x \to \infty$ and $x \to -\infty$

c $f(x) \to -2$ as $x \to \infty$ and $x \to -\infty$

d $f(x) \to \frac{1}{2}$ as $x \to \infty$ and $x \to -\infty$

e $f(x) \to 0$ as $x \to \infty$ and $x \to -\infty$

f $f(x) \to 0$ as $x \to \infty$ and $x \to -\infty$

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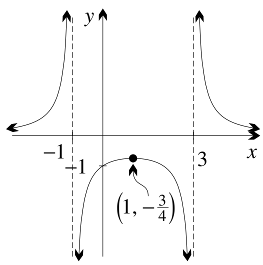
5



a $x \neq -1, 3$

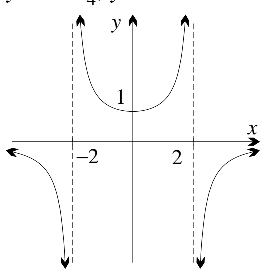
b $(0, -1)$

c $y \to 0$ as $x \to \infty$ and as $x \to -\infty$.

e as $x \to 3^+$, $y > 0$ so $y \to \infty$, as $x \to 3^-$, $y < 0$ so $y \to -\infty$, as $x \to 1^+$, $y < 0$ so $y \to -\infty$, and as $x \to 1^-$, $y > 0$ so $y \to \infty$

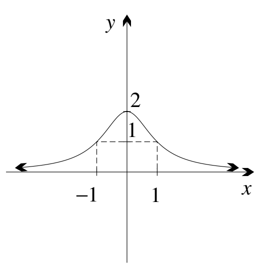
f $y \leq -\frac{3}{4}$, $y > 0$

6 a



b $y < 0$, $y \geq 1$

7



a $y \to 0$ as $x \to \infty$ and as $x \to -\infty$.

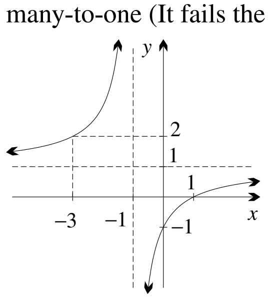
b $x^2 + 1$ is never 0.

c $y' = -4x(x^2 + 1)^{-2}$

e $0 < y \leq 2$

f many-to-one (It fails the horizontal line test.)

8



a $x \neq -1$

b $(0, -1)$

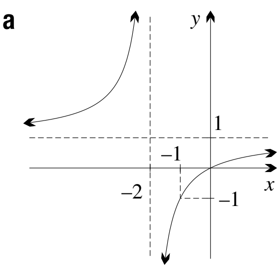
c $y \to 1$ as $x \to \infty$ and as $x \to -\infty$.

d as $x \to 1^+$, $y < 0$ so $y \to -\infty$, as $x \to 1^-$, $y > 0$ so $y \to \infty$

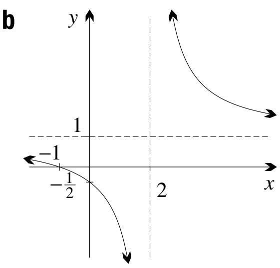
f $y \neq 1$

g one-to-one

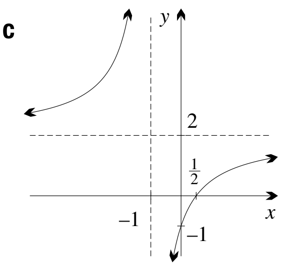
9 a



b



c



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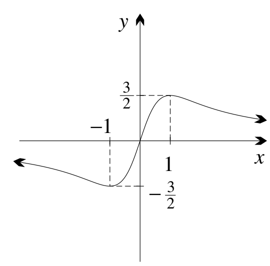

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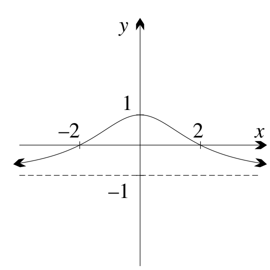


12 a **i** $y = 2$ **ii** $y = \frac{(x + 2)(x + 3)}{(x - 1)(x - 3)}$

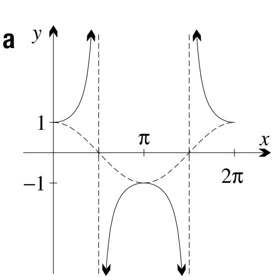
iii $x = 1, x = 3$

b **i** $y = \frac{(x - 1)^2}{(x + 1)(x + 4)}$, $x = -1$, $x = -4$ and $y = 1$

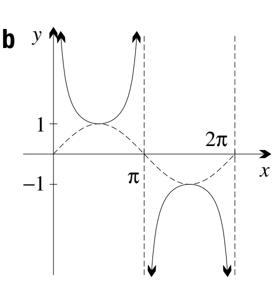
ii $y = \frac{x - 5}{(x - 2)(x + 5)}$, $x = -5$, $x = 2$ and $y = 0$

iii $y = \frac{(1 - 2x)(1 + 2x)}{(1 - 3x)(1 + 3x)}$, $x = \frac{1}{3}$, $x = -\frac{1}{3}$ and $y = \frac{4}{9}$

13 a



b



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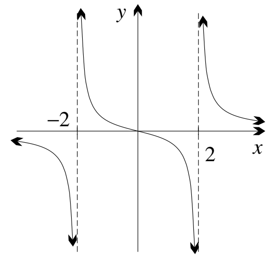
14



a point symmetry in the origin

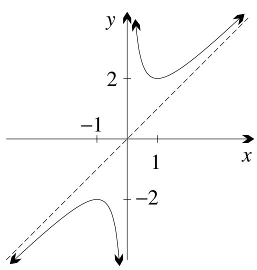
b domain: $x \neq 2$ and $x \neq -2$,
asymptotes: $x = 2$ and $x = -2$

d $y = 0$

f $f'(x) < 0$ for $x \neq 2$ & $x \neq -2$

h all real y

15



a point symmetry in the origin

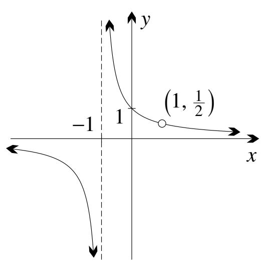
b domain: $x \neq 0$,
asymptote: $x = 0$

e $(-1, -2)$ and $(1, 2)$

g $y \geq 2$ or $y \leq -2$

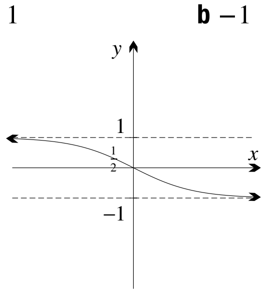
16 a Both numerator and denominator are zero, so y is undefined at $x = 1$. For $x \neq 1$, $y = \frac{1}{x+1}$

b



18 a 1 **b** -1 **c** $(0, 0)$

d



e odd

Exercise 2C

1a $y = \frac{9}{(x-3)(x+3)}$

b $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

c symmetry in the y-axis

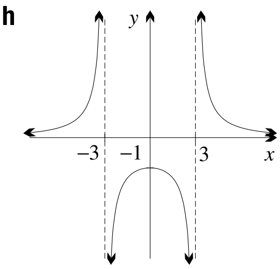
d $(0, -1)$

e $-3 < x < 3$

f $x = -3, x = 3$

g $y = 0$

h



i $y'(0) = 0$

2 a $y = \frac{x}{(2-x)(2+x)}$

b $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

c point symmetry in the origin

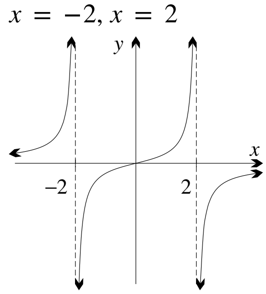
d $(0, 0)$

e $x < -2$ or $0 \leq x < 2$

f $x = -2, x = 2$

g $y = 0$

h



i $y' > 0$ in the domain.

3 a $y = x(x-2)(x+2)$

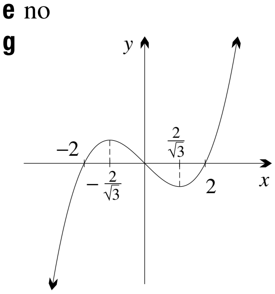
b $(-\infty, \infty)$

c $(-2, 0)$, $(0, 0)$, $(2, 0)$

d point symmetry in the origin

e no

g



4 a $\frac{2(x-2)}{(x-1)(x-4)}$

b $x \neq 1$ and $x \neq 4$

c The domain is not symmetric about $x = 0$.

d $\left(0, -\frac{5}{4}\right)$ and $\left(2\frac{1}{2}, 0\right)$

e $1 < x < 2\frac{1}{2}$ or $x > 4$

f $x = 1$ and $x = 4$

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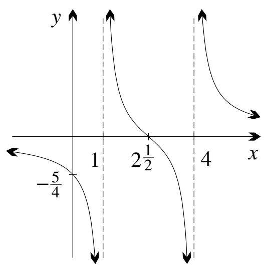
g $y = 0$

h



5 a $y = \frac{3(x-1)}{(x-3)(x+1)}$

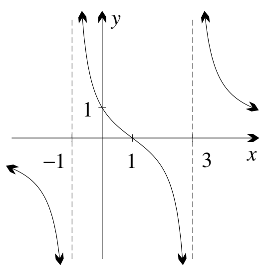
b domain: $x \neq -1$ and $x \neq 3$

intercepts: $(1, 0)$ and $(0, 1)$

c The domain is not symmetric about $x = 0$.

d $x = -1$, $x = 3$, and $y = 0$

e



6 a $y = \frac{(x+1)^2}{(x-1)(x+3)}$

b domain: $x \neq -3$ and $x \neq 1$

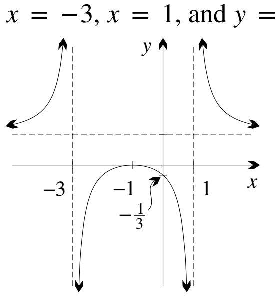
intercepts: $(-1, 0)$ and $\left(0, -\frac{1}{3}\right)$

c The domain is not symmetric about $x = 0$.

d $y < 0$ either side of $x = -1$.

e $x = -3$, $x = 1$, and $y = 1$

f



g $y \leq 0$ or $y > 1$

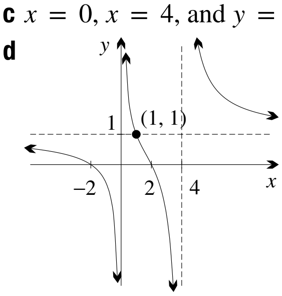
7 a $f(x) = \frac{(x-2)(x+2)}{x(x-4)}$

b domain: $x \neq 0$ and $x \neq 4$

intercepts: $(-2, 0)$ and $(2, 0)$

c $x = 0$, $x = 4$, and $y = 1$

d



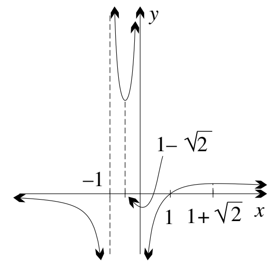
e all real y

8 $y = \frac{x-1}{x(x+1)}$

$y' = \frac{1+2x-x^2}{x^2(x+1)^2}$

so $y' = 0$ at $x = 1 + \sqrt{2}$ and $x = 1 - \sqrt{2}$

The graph on the right is not to scale.



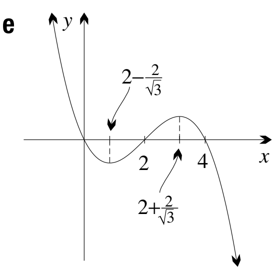
9 $y = x(x-2)(x-4)$

a $-\infty < x < \infty$

b $(0, 0)$, $(2, 0)$, $(4, 0)$

c no

e



10 a all real x

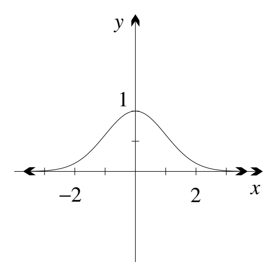
b even

c $(0, 1)$

d $y = 0$

e $(0, 1)$

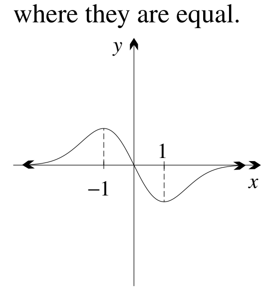
f $0 < y \leq 1$



11 b When $x = 0$, $y' = 0$.

c $e^{-\frac{1}{2}} < 2^{-\frac{1}{2}}$ so $y = 2^{-\frac{1}{2}x^2}$ is higher, except at $x = 0$ where they are equal.

12

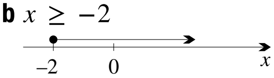


The graph shows $f(x)$ is greatest at $x = -1$ and least at $x = 1$.

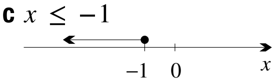
Exercise 2D

1a $x < 5$

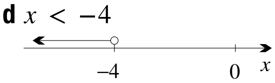
b $x \geq -2$



c $x \leq -1$



d $x < -4$



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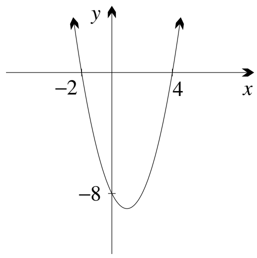
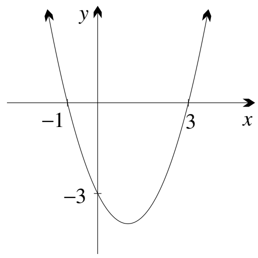
2 a $x > -3$, $(-3, \infty)$
b $x \le 10$, $(-\infty, 10]$
c $x < -2$, $(-\infty, -2)$
d $x \ge -5$, $[-5, \infty)$
e $x \ge -4$, $[-4, \infty)$
f $x < 6$, $(-\infty, 6)$

3 a $0 < x < 4$
b $x \le -1$ or $x \ge 3$
c $x \le 0$ or $x \ge 2$

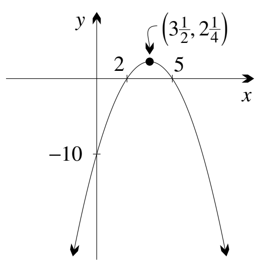
4 a $-2 < x < 4$



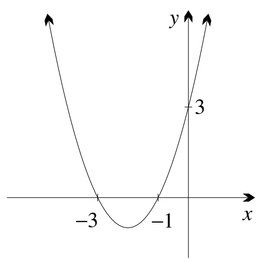
b $x < -1$ or $x > 3$



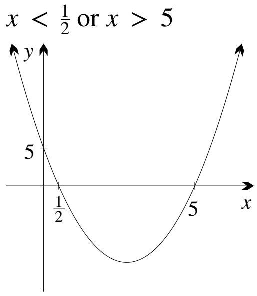
c $2 \le x \le 5$



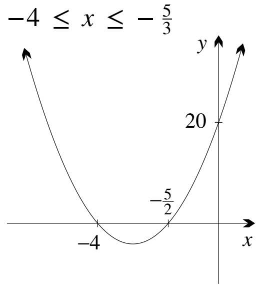
d $x \le -3$ or $x \ge -1$



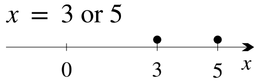
e $x < \frac{1}{2}$ or $x > 5$



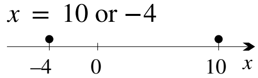
f $-4 \le x \le -\frac{5}{3}$



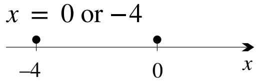
5 a $x = 3$ or 5



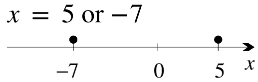
b $x = 10$ or -4



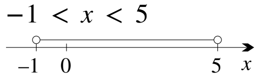
c $x = 0$ or -4



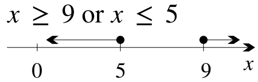
d $x = 5$ or -7



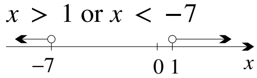
e $-1 < x < 5$



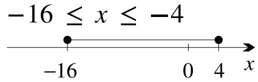
f $x \ge 9$ or $x \le 5$



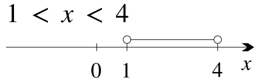
g $x > 1$ or $x < -7$



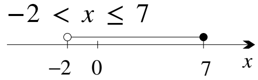
h $-16 \le x \le -4$



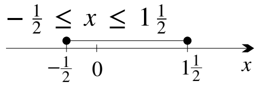
6 a $1 < x < 4$



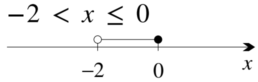
b $-2 < x \le 7$



c $-\frac{1}{2} \le x \le 1\frac{1}{2}$



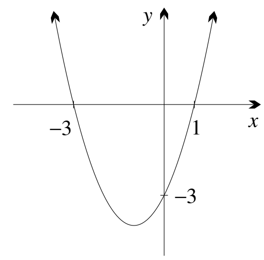
d $-2 < x \le 0$



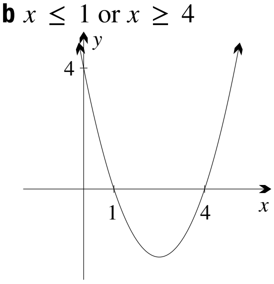
7 a $-4 < x < 2$, $(-4, 2)$
b $-1 \le x \le 2$, $[-1, 2]$
c $\frac{1}{3} < x \le 4$, $\left(-\frac{1}{3}, 4\right]$
d $-6 \le x < 15$, $[-6, 15)$

8 a $x > -10$
b $x \le 4$
c $x \ge -1$
d $x < -4\frac{2}{3}$

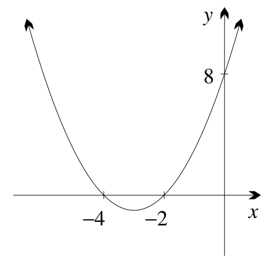
9 a $-3 < x < 1$



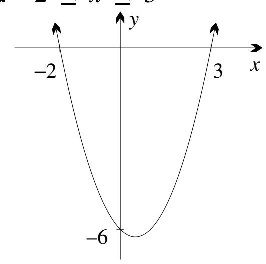
b $x \le 1$ or $x \ge 4$



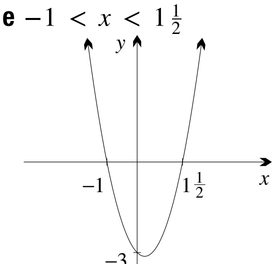
c $x < -4$ or $x > -2$



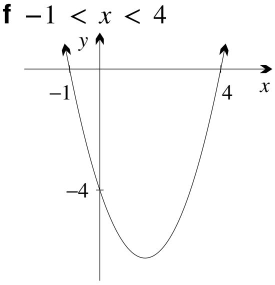
d $-2 \le x \le 3$



e $-1 < x < 1\frac{1}{2}$



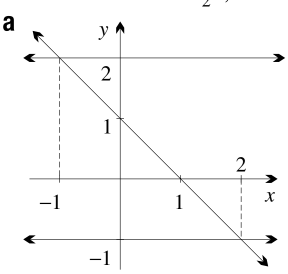
f $-1 < x < 4$



10 a $-1 \le x \le 1$
b $x < 0$ or $x > 3$
c $x \le -12$ or $x \ge 12$
d $x < 0$ or $x > 0$ (or simply $x \ne 0$)
e $x = 3$
f $1 \le x \le 3$

11 $5x - 4 < 7 - \frac{1}{2}x$, with solution $x < 2$

12 a



b $-1 \le x < 2$. The solution of the inequation is where the diagonal line lies between the horizontal lines.

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13 a $x = 5$ or -5 **b** $x = 6$ or -5
c $x = 2$ or $-\frac{8}{7}$ **d** $x = 2$ or $-3\frac{1}{3}$
e $\frac{1}{3} \le x \le 3$ **f** $x > 2$ or $x < \frac{1}{3}$
g $-2 < x < 1$ **h** $x \ge \frac{2}{5}$ or $x \le -2$

14 a $x = 5$ or -5 **b** $x = 8$ or -4
c $x = 2$ or $-3\frac{1}{3}$ **d** $x = \frac{7}{5}$ or $-\frac{11}{5}$
e $-1 < x < 5$ **f** $\frac{1}{3} \le x \le 3$
g $x \ge \frac{2}{5}$ or $x \le -2$ **h** $x > 2$ or $x < \frac{1}{3}$

15 a i $x = 4$ **ii** $2x - 3$ **iii** 5
b i $x = -3$ **ii** 4 **iii** $-2x - 2$
c i $x = -2$ **ii** $x + 9$ **iii** $-3x + 1$
d i $x = 1$ **ii** $4x - 4$ **iii** $-2x + 2$

16 a i $3x - 1 = 0, x = \frac{1}{3}$
ii $-x - 1 = 0, x = -1$
b i $4x - 6 = 4, x = 2\frac{1}{2}$
ii $6 - 2x = 4, x = 1$
c i $\frac{1}{2}x + 1 = 3, x = 4$
ii $-\frac{3}{2}x - 1 = 3, x = -\frac{8}{3}$
d i $3x - 2 = x + 6, x = 4$
ii $-3x + 2 = x + 6, x = -1$

17 a $x \le -2$ or $x \ge 2$ **b** $x < -2$ or $x > 2$

18 a $-2 \le x \le 2$ **b** $-2 < x < 2$
c $x \le -2$ or $x \ge 2$ **d** $x < -2$ or $x > 2$

Exercise 2E

1a 1 **b** 2 **c** 3 **d** 2 **e** 2 **f** 3

2 a $x = \frac{1}{2}$ **b** $x = -\frac{3\pi}{4}$ or $\frac{\pi}{4}$
c $x \doteqdot -2.1$ or 0.3 or 1.9 **d** $x = 1$ or $x \doteqdot 3.5$
e $x = 1$ or $x \doteqdot -1.9$
f $x = 0$ or $x \doteqdot -1.9$ or 1.9

3 a i $x > 1$ **ii** $x < 1$
b i $x < -3$ or $x > 2$ **ii** $-3 < x < 2$

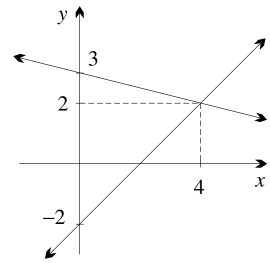
4 a i $x = 0$ or 3 **ii** $0 < x < 3$
iii $x < 0$ or $x > 3$
b i $x = -2$ or 1 **ii** $x < -2$ or $x > 1$
iii $-2 < x < 1$

5 a $x \le -3$ **b** $0 \le x \le 2$ **c** $x = 1$

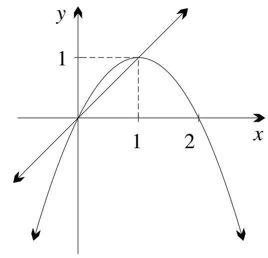
6 a $x < -2$ or $x > 1$ **b** $0 \le x \le 1$
c $-1 < x < 0$ or $x > 1$

7 a $\sqrt{2} \doteqdot 1.4, \sqrt{3} \doteqdot 1.7$
b $x = -1$ or $x = 2$
c $x < -1$ or $x > 2$
d $x = -2$ or $x = 1, -2 \le x \le 1$
e $x \doteqdot 1.62$ or $x \doteqdot -0.62$
f i Draw $y = -x$; $x = 0$ or $x = -1$.
ii Draw $y = x + \frac{1}{2}$; $x \doteqdot 1.37$ or $x \doteqdot -0.37$.
iii Draw $y = \frac{1}{2}x + \frac{1}{2}$; $x = 1$ or $x = -\frac{1}{2}$.

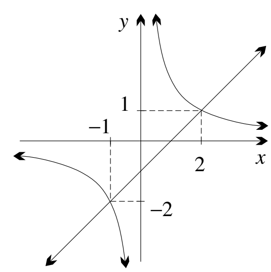
8 a $(4, 2)$, $x - 2 = 3 - \frac{1}{4}x$



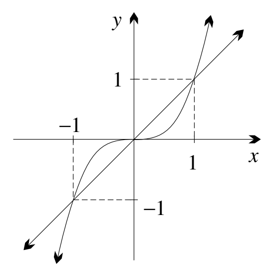
b $(0, 0)$ and $(1, 1)$, $x = 2x - x^2$



c $(-1, -2)$ and $(2, 1)$, $\frac{2}{x} = x - 1$



d $(-1, -1)$, $(0, 0)$ and $(1, 1)$, $x^3 = x$



9 a $x \ge 4$
b $0 < x < 1$
c $x < -1$ or $0 < x < 2$
d $-1 < x < 0$ or $x > 1$

10 a Divide by e^x to get $e^x = e^{1-x}$
b Multiply by $\cos x$ to get $\sin x = \cos x$
c Subtract 1 then divide by x to get $x^2 - 4 = -\frac{1}{x}$

11 a The table below traps the solution between $x = -1.690$ and $x = -1.6905$, so it is $x = -1.690$, correct to three decimal places.

x	-2	-1.7	-1.6	-1.68
2^x	0.25	0.3078	0.3299	0.3121
$x + 2$	0	0.3	0.4	0.32

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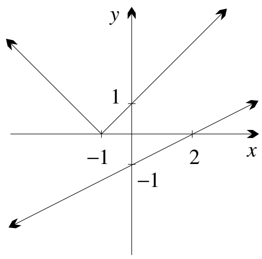
x	-1.69	-1.691	-1.6905
2^x	0.3099	0.3097	0.3098
$x + 2$	0.31	0.309	0.3095

b Part **c**: $x \doteqdot -2.115$. Part **e**: $x \doteqdot -1.872$.

12 a

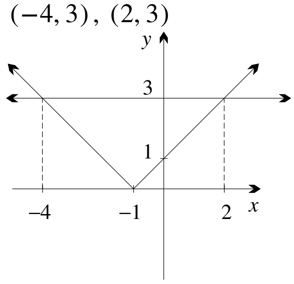


b The graph of $y = |x + 1|$ is always above the graph of $y = \frac{1}{2}x - 1$.

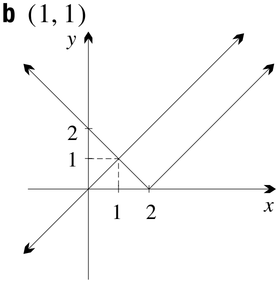
13 a The curve is always above the line.

b The two lines are parallel and thus the first is always below the second.

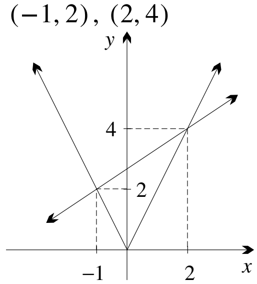
14 a $(-4, 3)$, $(2, 3)$



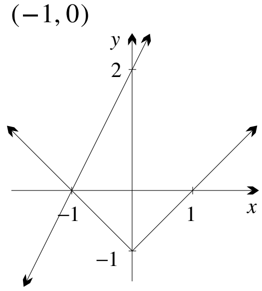
b $(1, 1)$



c $(-1, 2)$, $(2, 4)$



d $(-1, 0)$



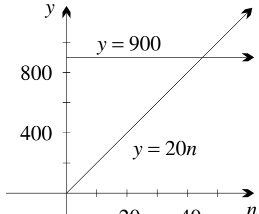
15 a $-4 \le x \le 2$

b $x < 1$

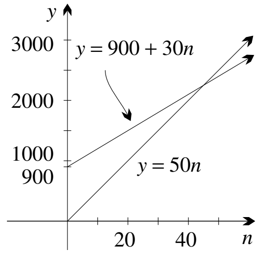
c $x \le -1$ or $x \ge 2$

d $x < -1$

16 a

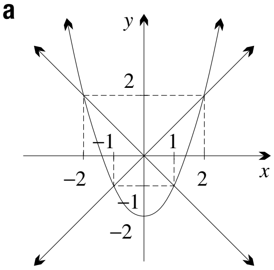


b



In both cases the break-even point is $n = 45$. Total sales are $2250 at that point.

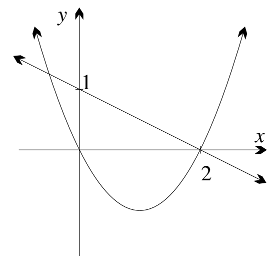
17 a



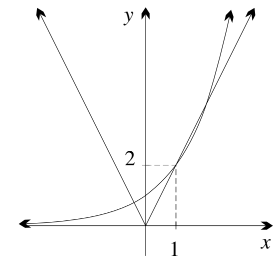
b $x = 2$ or -2

c $x < -2$ or $x > 2$

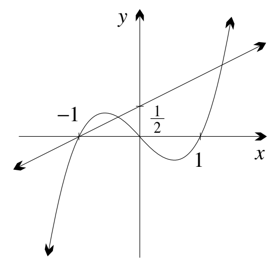
18 a 2 solutions



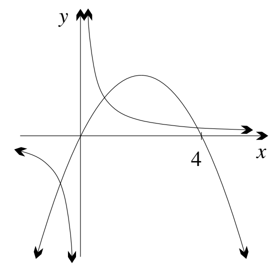
b 3 solutions



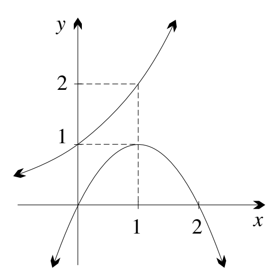
c 3 solutions



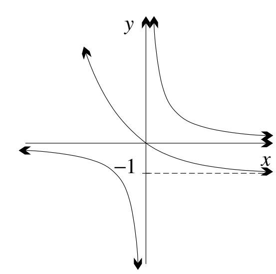
d 3 solutions



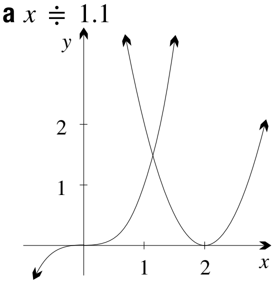
e no solutions



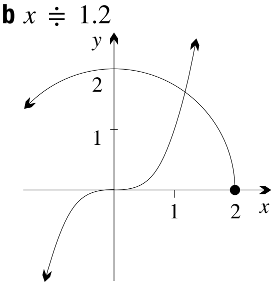
f no solutions



19 a $x \doteqdot 1.1$



b $x \doteqdot 1.2$



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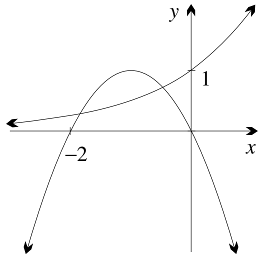
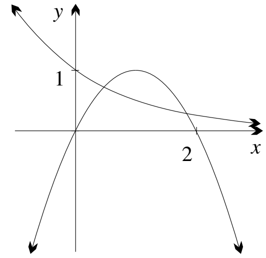
c $x \doteqdot -0.5$ or $x \doteqdot -1.9$

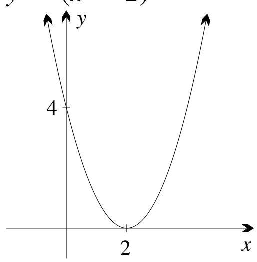


d $x \doteqdot 0.5$ or $x \doteqdot 1.9$.

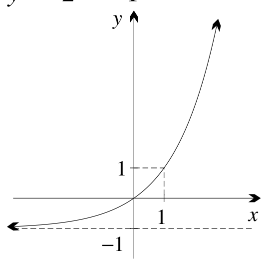


Exercise 2F

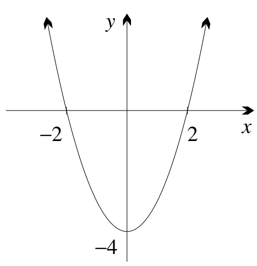
1a $y = (x - 2)^2$



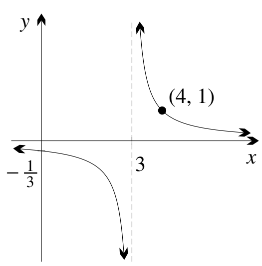
b $y = 2^x - 1$



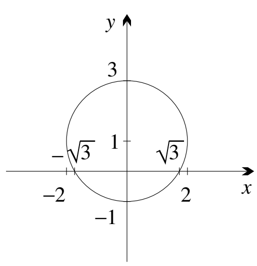
c $y = x^2 - 4$



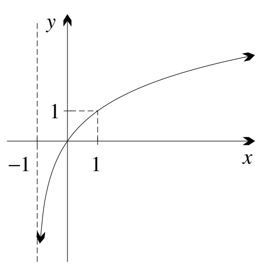
d $y = \frac{1}{x - 3}$



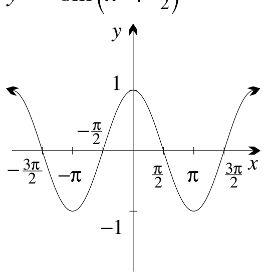
e $x^2 + (y - 1)^2 = 4$



f $y = \log_2 (x + 1)$

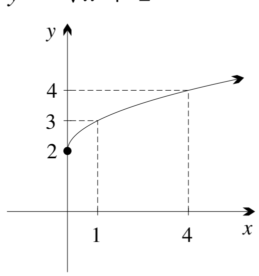


g $y = \sin\left(x + \frac{\pi}{2}\right)$

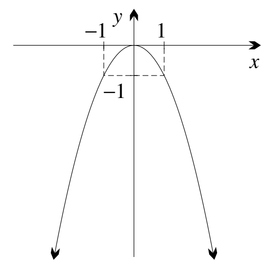


This is also $y = \cos x$.

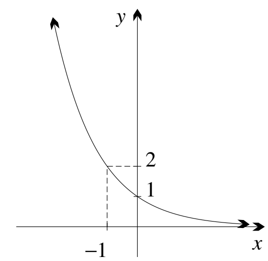
h $y = \sqrt{x} + 2$



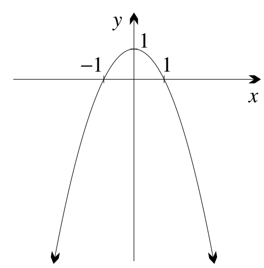
2 a $y = -x^2$



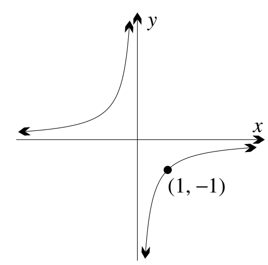
b $y = 2^{-x}$



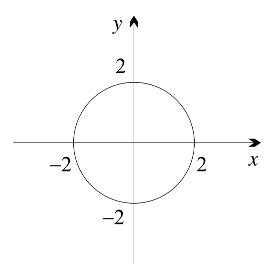
c $y = 1 - x^2$



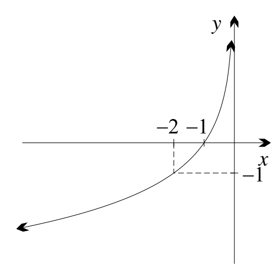
d $y = -\frac{1}{x}$



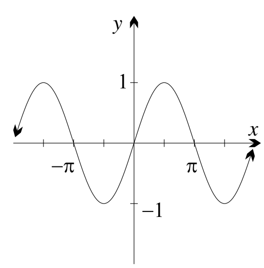
e $x^2 + y^2 = 4$



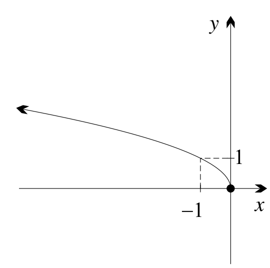
f $y = -\log_2 (-x)$



g $y = \sin x$



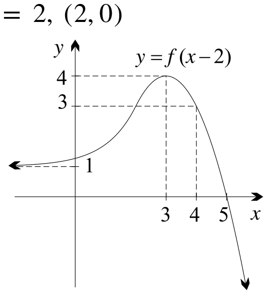
h $y = \sqrt{-x}$



3 In part **e** the circle is symmetric in the y-axis.
In part **g** $y = \sin x$ is an odd function, and so is unchanged by a rotation of 180°.

4 a $r = 2,\ (-1, 0)$ **b** $r = 1,\ (1, 2)$
c $r = 2,\ (2, 0)$ **d** $r = 5,\ (0, 3)$

5 a i



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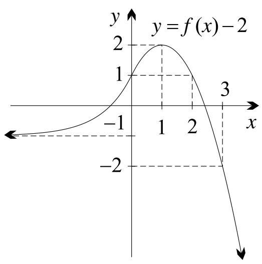
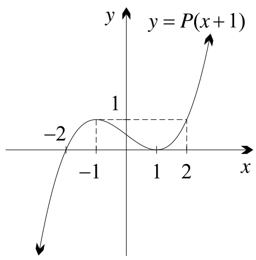
ii



b i



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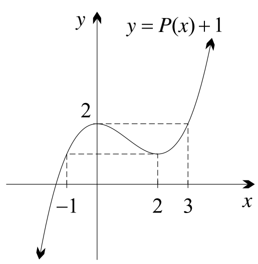
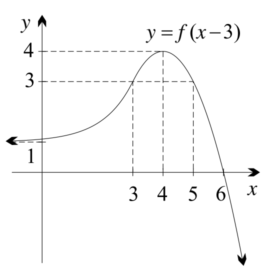


6 a $y = (x + 1)^2 + 2$ **b** $y = \frac{1}{x - 2} + 3$

c $y = \cos\left(x - \frac{\pi}{3}\right) - 2$ **d** $y = e^{x+2} - 1$

7 a i



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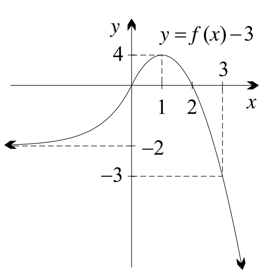
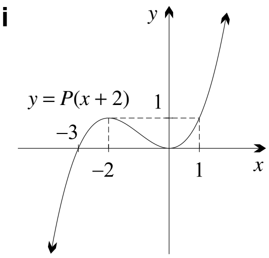


b i



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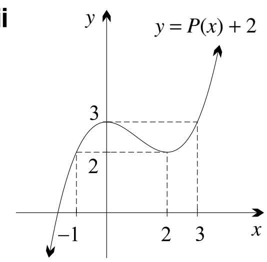
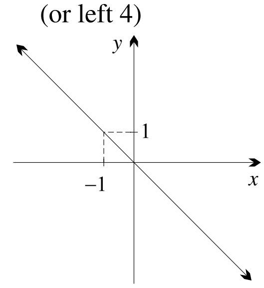


8 a From $y = -x$:

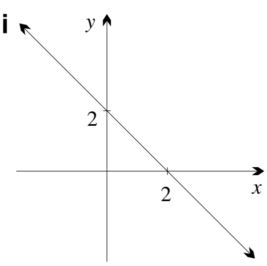
i shift up 2 (or right 2)

ii shift down 2 (or left 2)

iii reflect in x-axis (or y-axis) and shift up 4 (or left 4)



i



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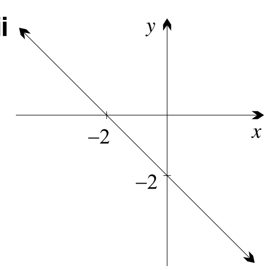
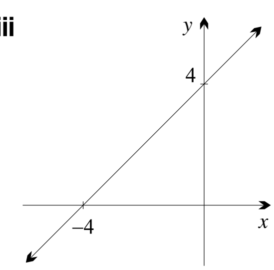


iii

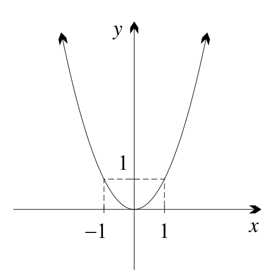


b From $y = x^2$:

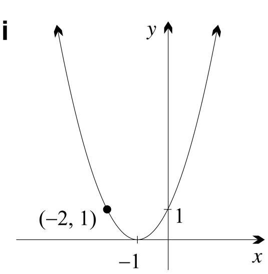
i shift 1 left

ii shift 1 left and reflect in x-axis

iii shift 1 left and shift down 1



i



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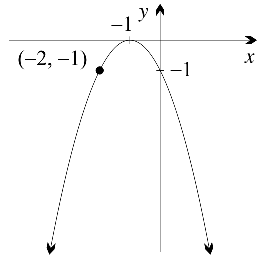

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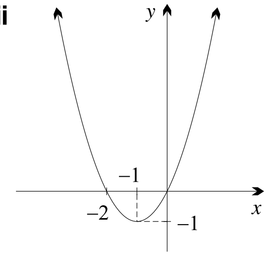
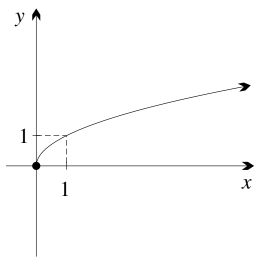


c From $y = \sqrt{x}$:

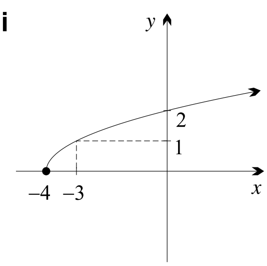
i shift 4 left

ii reflect in x-axis

iii shift 4 left and reflect in x-axis



i



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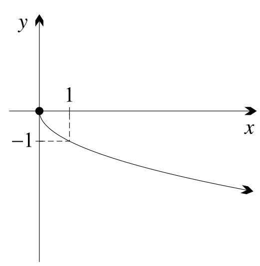

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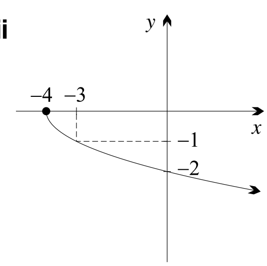
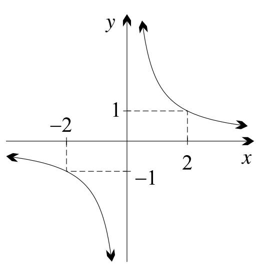


d From $y = \dfrac{2}{x}$:

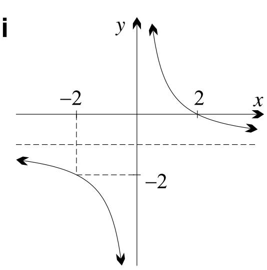
i shift down 1

ii shift down 1, left 2

iii reflect in the x-axis or in the y-axis



i



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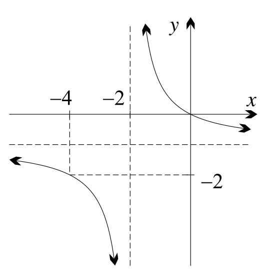

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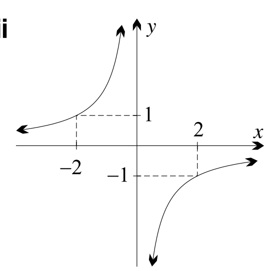
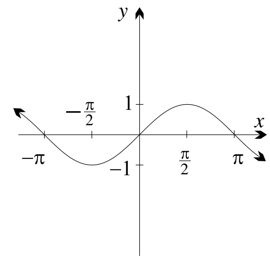


e From $y = \sin x$:

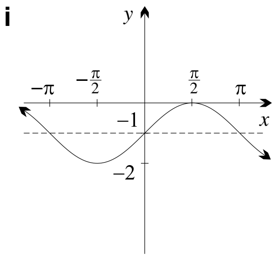
i shift down 1

ii shift down 1, right $\frac{\pi}{4}$

iii reflect in the x-axis or in the y-axis



i



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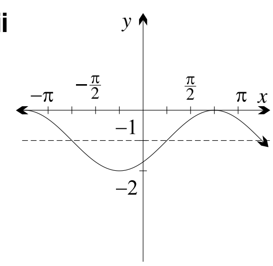

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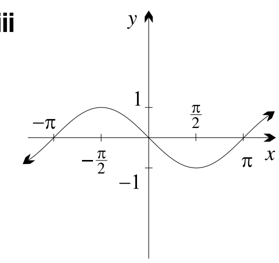
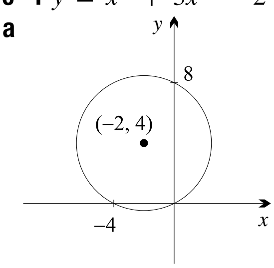


9 a $(1, -2)$ and $(-1, 2)$

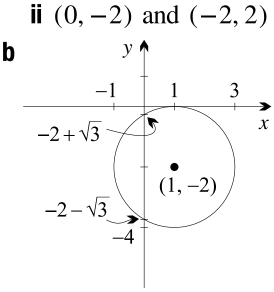
b **i** $y = x^3 - 3x + 1$ **ii** $(1, -1)$ and $(-1, 3)$

c **i** $y = x^3 + 3x^2 - 2$ **ii** $(0, -2)$ and $(-2, 2)$

10 a



b



11 a The parabola $y = x^2$ shifted left 2, down 1. $y + 1 = (x + 2)^2$

b The hyperbola $xy = 1$ shifted right 2, down 1. $y + 1 = \frac{1}{x - 2}$

c The exponential $y = 2^x$ reflected in the x-axis, shifted 1 up. $y = 1 - 2^x$

d The curve $y = \cos x$ reflected in the x-axis and shifted 1 up. $y = 1 - \cos x$

12 a The parabola $y = x^2$ reflected in the x-axis, then shifted 3 right and 1 up. $y - 1 = -(x - 3)^2$

b The curve $y = \log_2 x$ reflected in the y-axis, then shifted right 2, down 1. $y + 1 = -\log_2 (x - 2)$

c The half parabola $y = \sqrt{x}$ reflected in the x-axis, then shifted left 4 and 2 up.

$y - 2 = -\sqrt{x + 4}$

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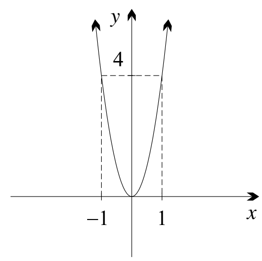
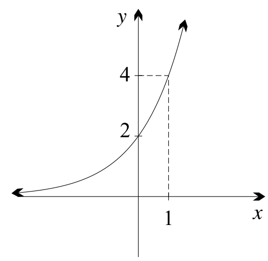
Exercise 2G

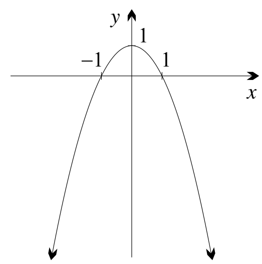
1a $y = 4x^2$



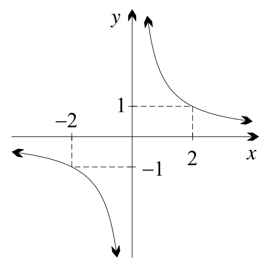
b $y = 2 \times 2^x = 2^{x+1}$



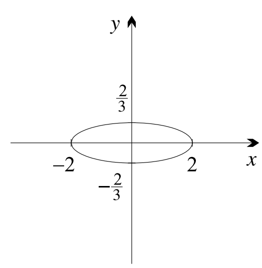
c $y = 1 - x^2$



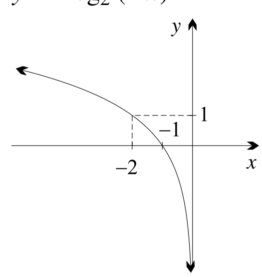
d $y = \frac{2}{x}$



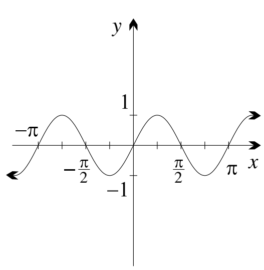
e $x^2 + 9y^2 = 4$



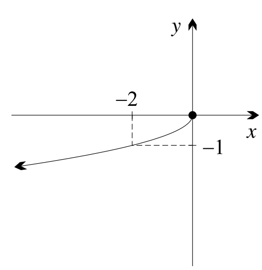
f $y = \log_2(-x)$



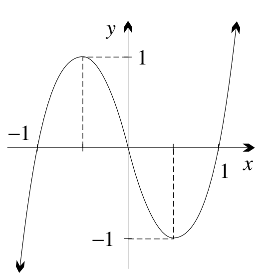
g $y = \sin 2x$



h $y = -2\sqrt{x}$



2 a i



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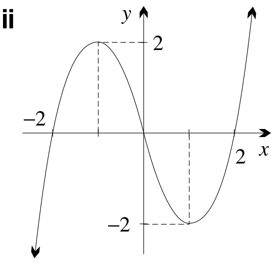
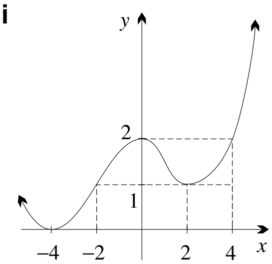


b i



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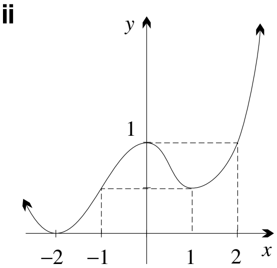
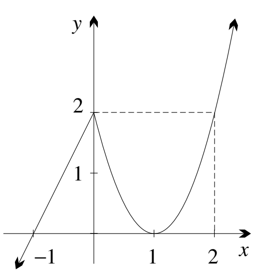


c i



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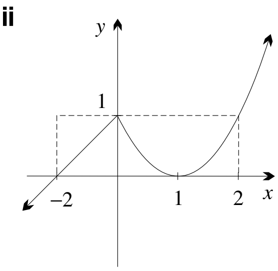
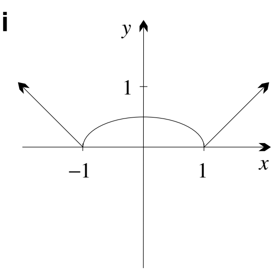


d i



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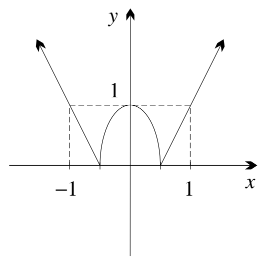
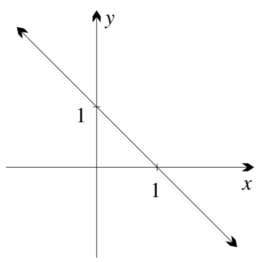
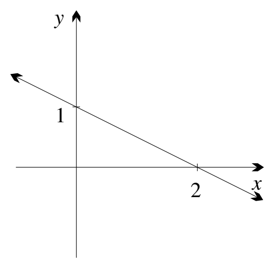


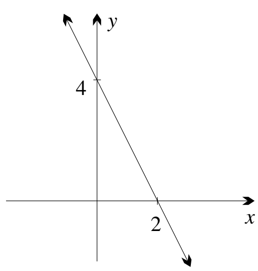
3 a stretch horizontally by factor 2

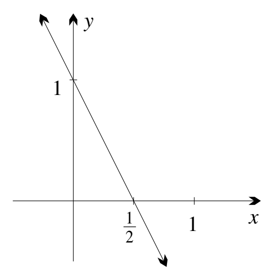
b stretch horizontally by factor 2, vertically by factor 4

c stretch horizontally by factor $\frac{1}{2}$

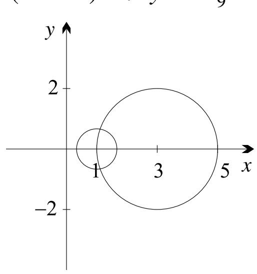


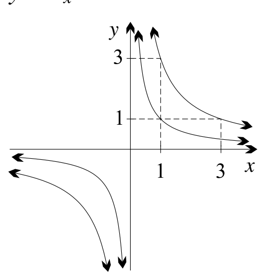




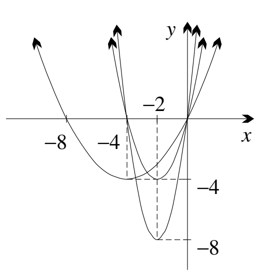


4 a $(x - 1)^2 + y^2 = \frac{4}{9}$ **b** $y = \frac{3}{x}$

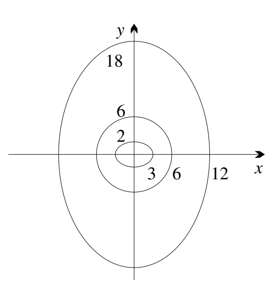




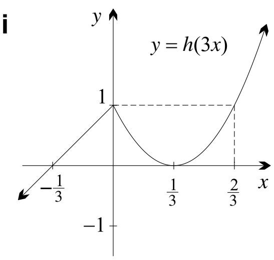
5 a



b



6 a i



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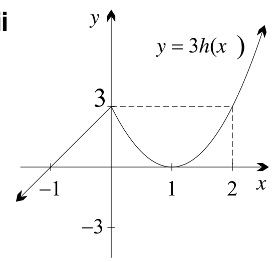
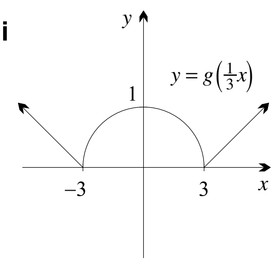
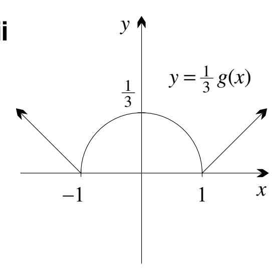


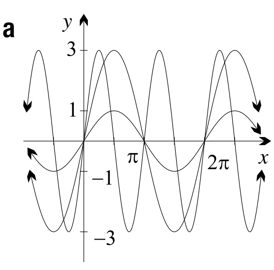
b i



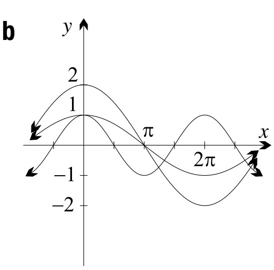
ii



7 a



b



8 a $(1, -2)$ and $(-1, 2)$

b i $y = 2x^3 - 6x$ **ii** $(1, -4)$ and $(-1, 4)$

c i $y = \frac{1}{27}x^3 - x$ **ii** $(3, -2)$ and $(-3, 2)$

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9 a vertical factor 3 **b** horizontal factor $\frac{1}{2}$

c horizontal factor 4 **d** vertical factor 2

10 a $y = \frac{2}{x}$ **b** $y = \frac{2}{x}$

c Both dilations give the same graph.

d yes: by factor $\sqrt{2}$

11 a $y = 4x^2$ **b** $y = 4x^2$

c Both dilations give the same graph.

d no

12 a $M(0) = 3$ **b** 53 years

c i The mass has been dilated by factor 2, so $M = 6 \times 2^{-\frac{1}{53}t}$

14 a The unit circle $x^2 + y^2 = 1$, horizontally by 3, vertically by 2. $\frac{x^2}{9} + \frac{y^2}{4} = 1$

b The exponential $y = 3^x$, vertically by -2. $y = -2 \times 3^x$

c The curve $y = \tan x$, horizontally by 3, vertically by 2. $y = 2\tan\frac{x}{3}$

15 a i stretch vertically by factor 2, $\frac{y}{2} = 2^x$, or translate left by 1, $y = 2^{(x+1)}$

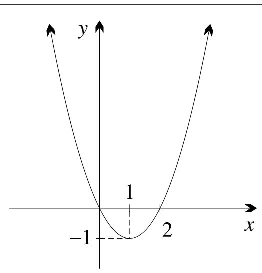
ii stretch along both axes by k, $\frac{y}{k} = \frac{1}{\frac{x}{k}}$, or stretch horizontally by k^2, $y = \frac{1}{\frac{x}{k^2}}$

iii reciprocal, $y = \frac{1}{3^x}$, or reflect in the y-axis, $y = 3^{-x}$

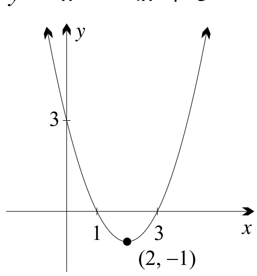
16 vertically by factor a^2

Exercise 2H

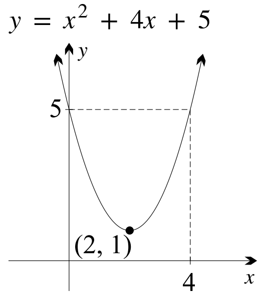
1



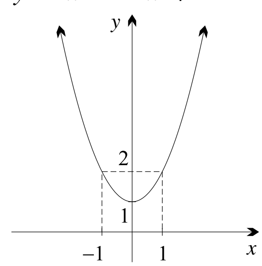
a i $y = x^2 - 4x + 3$



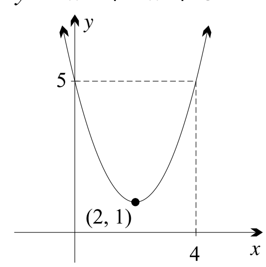
ii $y = x^2 + 4x + 5$



b i $y = x^2 - 2x + 2$

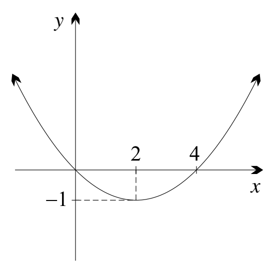


ii $y = x^2 + 4x + 5$

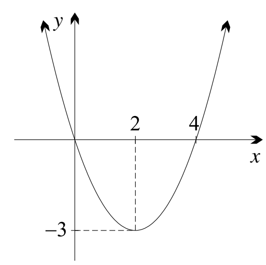


c yes

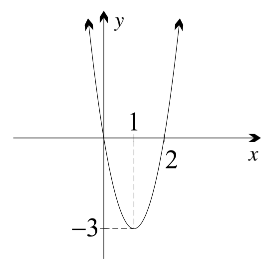
2 a i $y = \frac{1}{4}x^2 - x$



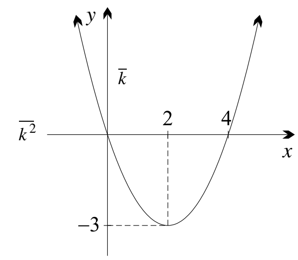
ii $y = \frac{3}{4}x^2 - 3x$



b i $y = 3x^2 - 6x$

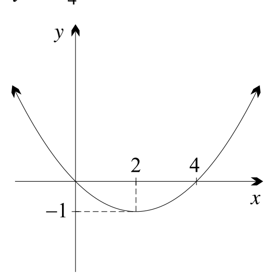


ii $y = \frac{3}{4}x^2 - 3x$

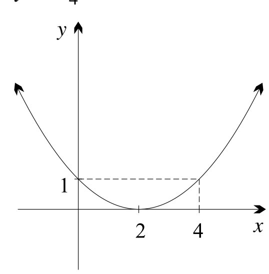


c yes

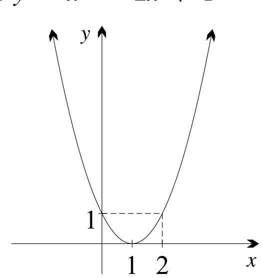
3 a i $y = \frac{1}{4}x^2 - x$



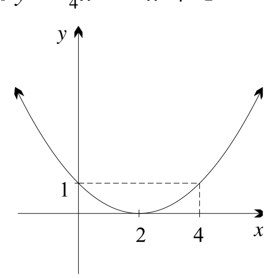
ii $y = \frac{1}{4}x^2 - x + 1$



b i $y = x^2 - 2x + 1$



ii $y = \frac{1}{4}x^2 - x + 1$



c yes

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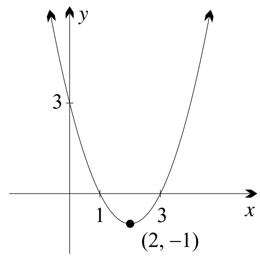
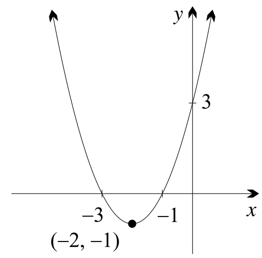
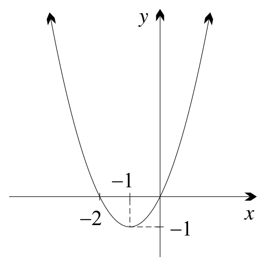
4 a **i** $y = x^2 - 4x + 3$



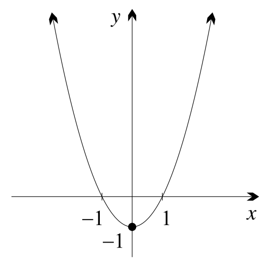
ii $y = x^2 + 4x + 3$



b **i** $y = x^2 + 2x$



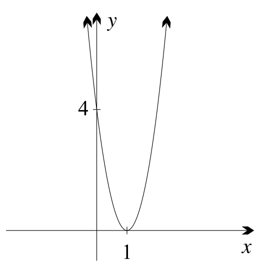
ii $y = x^2 - 1$



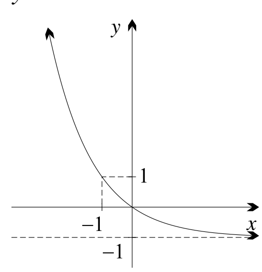
c no: order matters

5 a no **b** no **c** yes **d** yes **e** no **f** yes

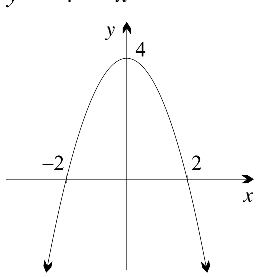
6 a $y = 4(x - 1)^2$



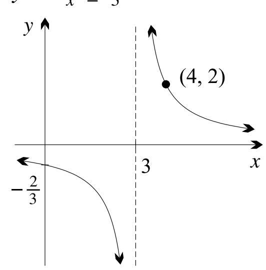
b $y = 2^{-x} - 1$



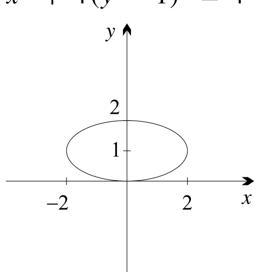
c $y = 4 - x^2$



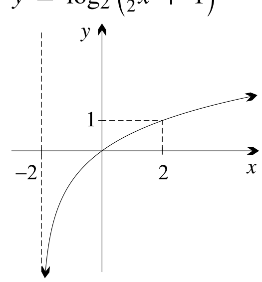
d $y = \frac{2}{x - 3}$



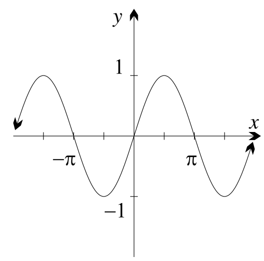
e $x^2 + 4(y - 1)^2 = 4$



f $y = \log_2\left(\frac{1}{2}x + 1\right)$

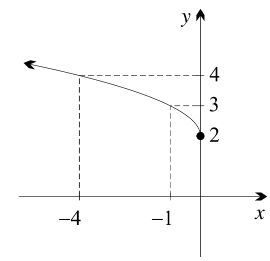


g $y = -\sin(x + \pi)$

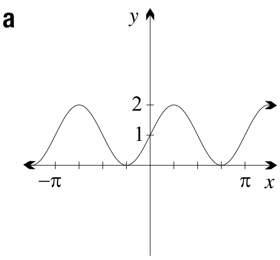


This is also $y = \sin x$.

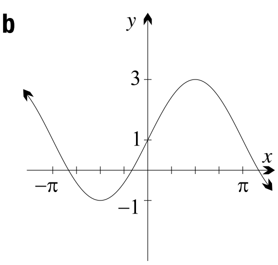
h $y = -\sqrt{x} + 2$



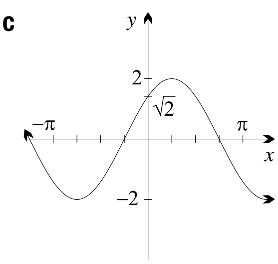
7 a



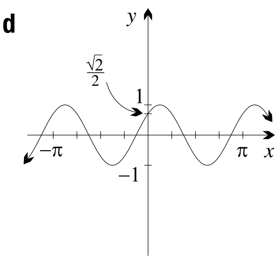
b



c



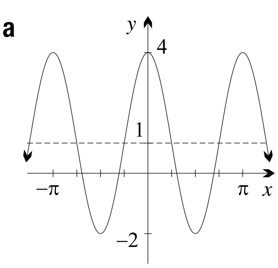
d



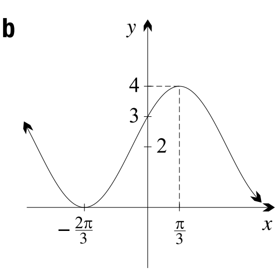
8 a $y = \frac{1}{4}(x + 2)^2 - 4$ **b** $y = \frac{1}{4}(x + 1)^2 - 4$

c $y = 2 - 2^x$ **d** $y = \frac{2}{x - 2} + 1$

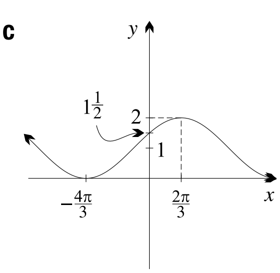
9 a



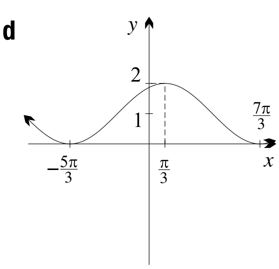
b



c



d



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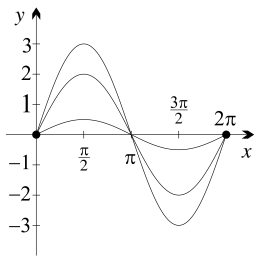
Exercise 2I

1a

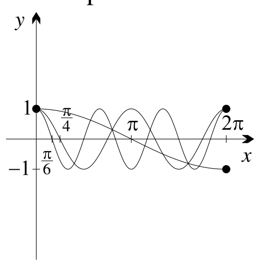


i $\frac{1}{2}$ **ii** 2 **iii** 3

b The graph $y = \sin x$ is stretched vertically by a factor of k.

c The amplitude increases. The bigger the amplitude, the steeper the wave.

2 a

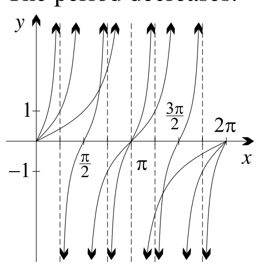


i 4π **ii** π **iii** $\frac{2\pi}{3}$

b The graph $y = \cos x$ is stretched horizontally by a factor of $\frac{1}{n}$.

c The period decreases.

3 a

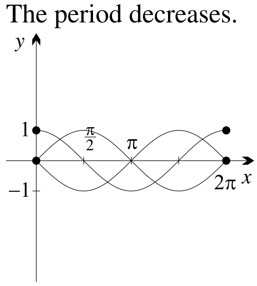


i π **ii** 2π **iii** $\frac{\pi}{2}$

b The graph $y = \tan x$ is stretched horizontally by a factor of $\frac{1}{a}$.

c The period decreases.

4 a

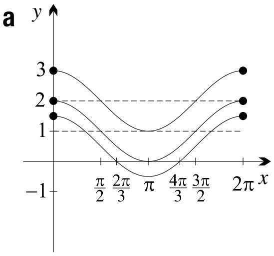


i $\frac{\pi}{2}$ **ii** π **iii** 2π or 0

b The graph $y = \sin x$ is shifted α units to the left.

c The graph stays the same, because $y = \sin x$ has period 2π.

5 a



i Range: $0 \le y \le 2$ or $[0, 2]$, mean value: 1

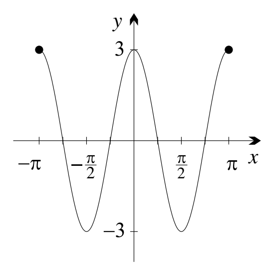
ii Range: $[1, 3]$, mean value: 2

iii Range: $\left[-\frac{1}{2}, \frac{3}{2}\right]$, mean value: $\frac{1}{2}$

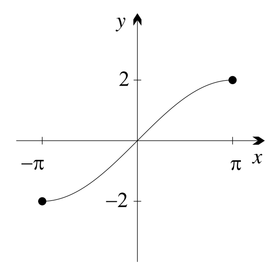
b The graph $y = \cos x$ is shifted c units up, and the mean value is c.

c It moves up.

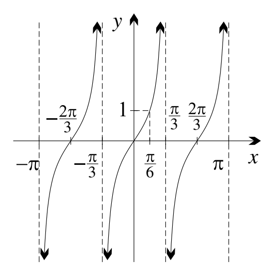
6 a period $= \pi$, amplitude $= 3$



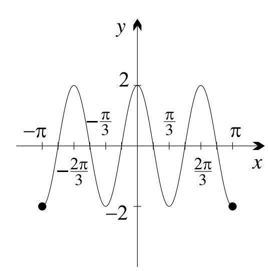
b period $= 4\pi$, amplitude $= 2$



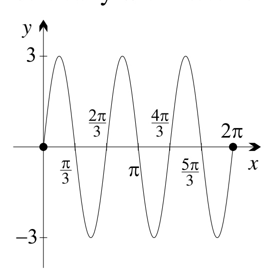
c period $= \frac{2\pi}{3}$, no amplitude



d period $= \frac{2\pi}{3}$, amplitude $= 2$



7 a Stretch horizontally by a factor of $\frac{1}{3}$, then stretch vertically with factor of 3.



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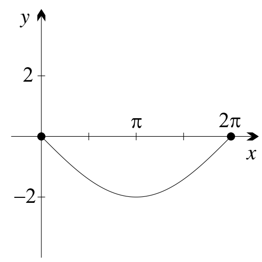
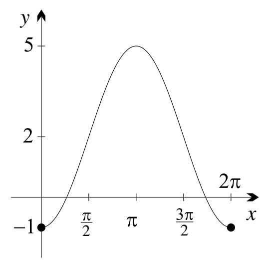
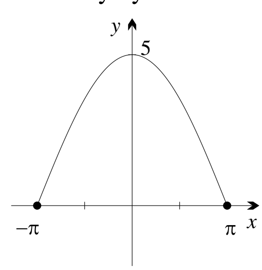
b Stretch horizontally with factor 2, then stretch vertically with factor 2, then reflect in the x-axis.



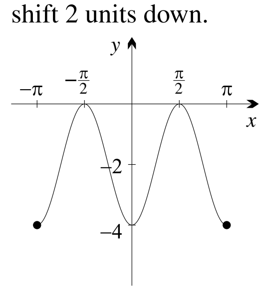
c Shift $\frac{\pi}{2}$ units right, then stretch vertically by a factor of 3, then shift 2 units up.



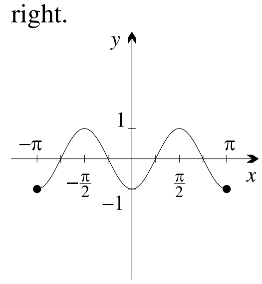
8 a Stretch horizontally by a factor 2, then stretch vertically by a factor of 5.



b Stretch horizontally by a factor of $\frac{1}{2}$, then stretch vertically by a factor 2 then reflect in the x-axis, then shift 2 units down.



c Stretch horizontally by a factor of $\frac{1}{2}$, then shift $\frac{\pi}{2}$ units right.



9 a Stretch horizontally by a factor of $\frac{1}{3}$, then shift $\frac{\pi}{6}$ units left.

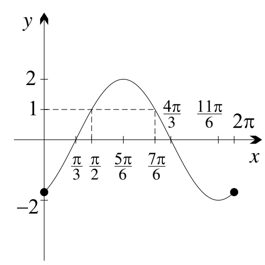
b Stretch horizontally by a factor of $\frac{1}{4}$, then shift $\frac{\pi}{4}$ units right, then stretch vertically by a factor of $\frac{1}{4}$, then shift 4 units down.

c Stretch horizontally by a factor of 2, then shift $\frac{\pi}{2}$ units left, then stretch vertically by a factor of 6, then reflect in the x-axis.

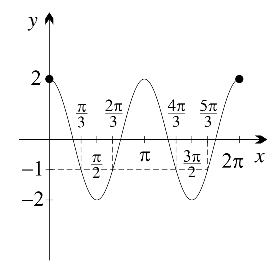
10 a Part **a**: period $= \frac{2\pi}{3}$, phase $= 0 + \frac{\pi}{2} = \frac{\pi}{2}$.
Part **b**: period $= \frac{2\pi}{4} = \frac{\pi}{2}$, phase $= -\pi$ (but this is twice the period, so we can also say that phase $= 0$).
Part **c**: period $= 4\pi$, phase $= \frac{\pi}{4}$.

b **i** period $= \pi$, phase $= 2\left(0 - \frac{\pi}{3}\right) = -\frac{2\pi}{3}$
ii period $= 6\pi$, phase $= \frac{\pi}{3}$
iii period $= \frac{\pi}{3}$, phase $= \frac{3\pi}{8}$

11 a $x = \frac{\pi}{2}$ or $\frac{7\pi}{6}$



b $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$ or $\frac{5\pi}{3}$

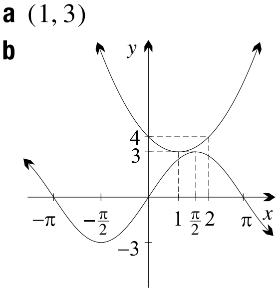


12 a $x \doteqdot 1.675$

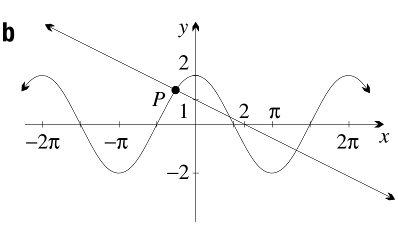
b $x \doteqdot 0.232$ or 1.803

13 a $(1, 3)$

b



14 a, b



c 3
d P is in the second quadrant

15 a 4 **c** the origin **d** $m > \frac{1}{4}$ *or* $m = 0$

16 a 10 metres **b** 6 metres
c 2 pm **d** 9:20 am

17 a **ii** 1 **iii** $0 < k < 1$
b **ii** 1.3 **iii** $\angle AOB = 2\theta \doteqdot 2.6$ radians
c **ii** $\ell > 300$

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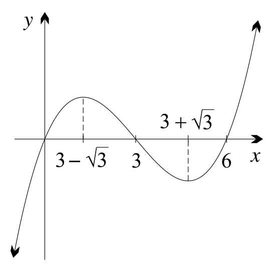
Chapter 2 review exercise

1a **i** $-1 < x < 2$ **ii** $(-1, 2)$
b **i** $-1 \le x < 2$ **ii** $[-1, 2)$
c **i** $x \le 2$ **ii** $(-\infty, 2]$

2 a **i** 0 **ii** 4 **iii** 8 **iv** 0
b **i** $x^2 + 2x$ **ii** x^2
iii $x^4 - 2x^2$ **iv** $x + 2$

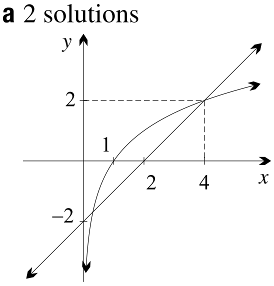
3 a $f(x) \to 0$ as $x \to \infty$ and $x \to -\infty$
b $f(x) \to \frac{1}{2}$ as $x \to \infty$ and $x \to -\infty$
c $f(x) \to 0$ as $x \to \infty$ and $x \to -\infty$

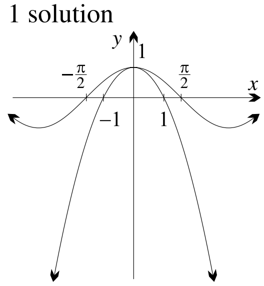
4 $y = x(x - 3)(x - 6)$
a $-\infty < x < \infty$ **b** $(0, 0)$, $(3, 0)$, $(6, 0)$
c no asymptotes
e $y' = 3(x^2 - 6x + 6)$
so $y' = 0$ at $x = 3 - \sqrt{3}$ or $x = 3 + \sqrt{3}$



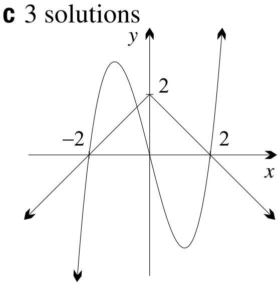
5 a $-4 \le x < 2$, $\left[-4, 2\right)$ **b** $-\frac{3}{2} < x < 0$, $\left(-\frac{3}{2}, 0\right)$
c $-3 \le x < \frac{1}{2}$, $\left[-3, \frac{1}{2}\right)$ **d** $-4 \le x \le 10$, $[-6, 10]$

6 a 2 solutions **b** 1 solution

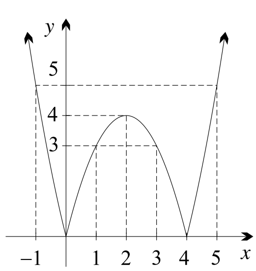




c 3 solutions



7 a **i**



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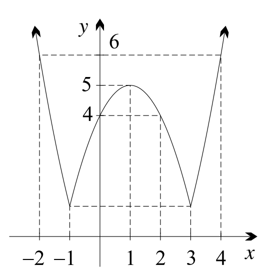
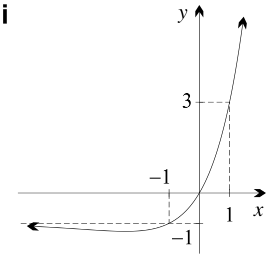


b **i**



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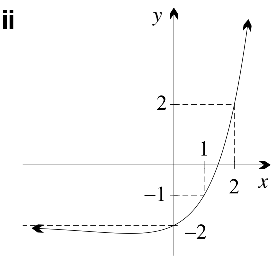
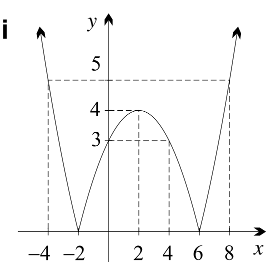
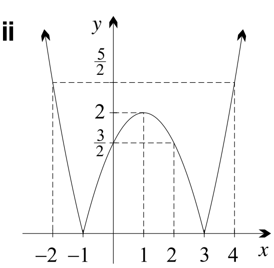


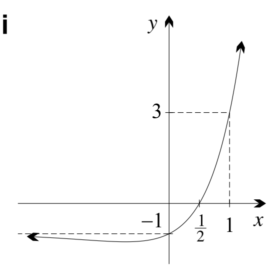
8 a **i**



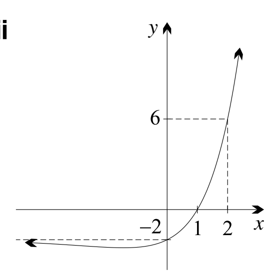
ii



b **i**



ii



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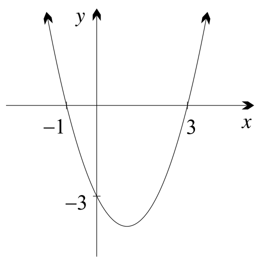
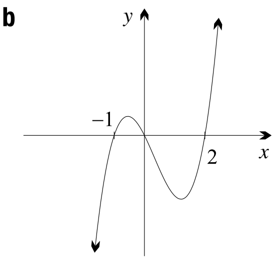
9 a



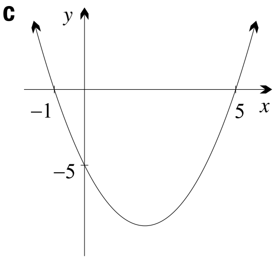
$-1 \le x \le 3$

b



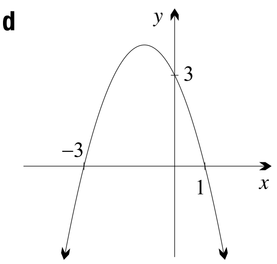
$x \le -1$ or $0 \le x \le 2$

c



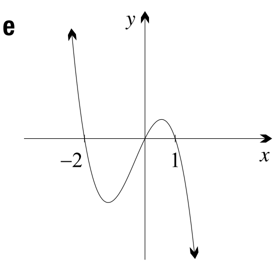
$-1 \le x \le 5$

d



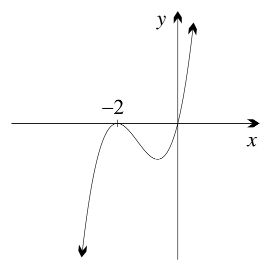
$x \le -3$ or $x \ge 1$

e



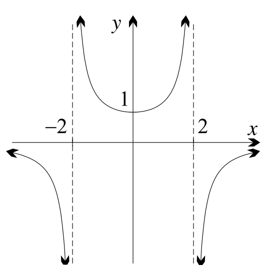
$x \le -2$ or $0 \le x \le 1$

f



$x \le 0$

10



a $x \ne -2, 2$ **b** $(0, 1)$

c $y \to 0$ as $x \to \infty$ and as $x \to -\infty$

e As $x \to 2^+$, $y < 0$ so $y \to -\infty$, as $x \to 2^-$, $y > 0$ so $y \to \infty$, as $x \to -2^+$, $y > 0$ so $y \to \infty$, and as $x \to -2^-$, $y < 0$ so $y \to -\infty$

f $(-\infty, 0) \cup [1, \infty)$

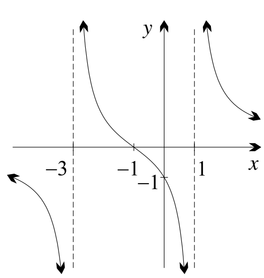
11 a $y = \frac{3(x+1)}{(x+3)(x-1)}$

b domain: $x \ne 1$ and $x \ne -3$
intercepts: $(-1, 0)$ and $(0, -1)$

c The domain is not symmetric about $x = 0$.

d $x = -3$, $x = 1$, and $y = 0$

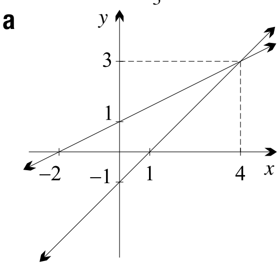
e



12 a $x = -3\frac{1}{2}$ or $3\frac{1}{2}$ **b** $x = \frac{1}{3}$ or 1

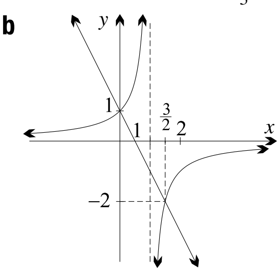
c $-3 \le x \le \frac{1}{3}$ **d** $x < -2$ or $x > -\frac{1}{3}$

13 a



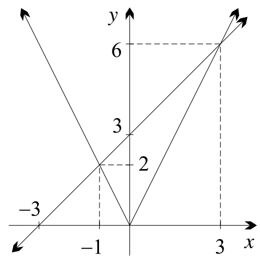
$x \ge 4$

b



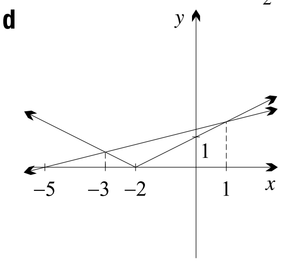
$0 < x < 1$ or $x > 1\frac{1}{2}$

c



$-1 \le x \le 3$

d



$x < -3$ or $x > 1$

14 a $y = (x-2)^2 + 1$ **b** $y = \frac{1}{x+2} - 3$

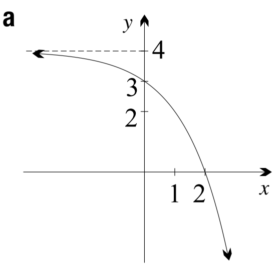
c $y = \sin\left(x + \frac{\pi}{6}\right) - 1$ **d** $y = e^{x-2} + 1$

15 a horizontal factor 2 **b** vertical factor $\frac{1}{2}$

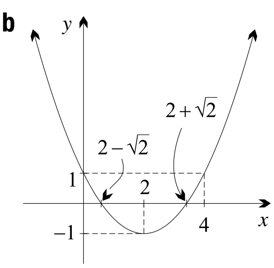
c vertical factor $\frac{1}{3}$ **d** horizontal factor 2

16 a yes **b** no **c** no **d** yes

17 a



b



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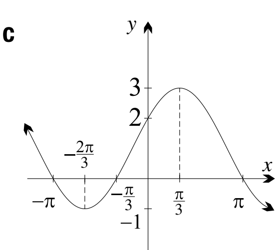
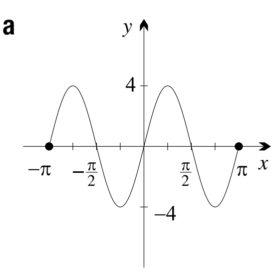
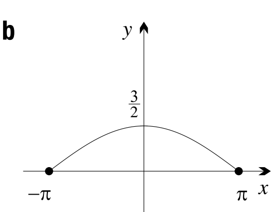


18 a



amplitude is 4,
period is π

b



amplitude is $\frac{3}{2}$,
period is 4π

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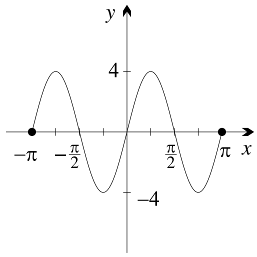
19 a Reflect in the x-axis, then shift up 1 unit.

b



20 a Reflect in the y-axis, then stretch vertically with factor 3, then shift down 2 units. Actually, the first transformation, reflect in the y-axis, is unnecessary because $y = \cos x$ is even.

b Stretch horizontally with factor $\frac{1}{4}$, and vertically with factor 4. There is no need to shift left $\frac{\pi}{2}$ units because the period is $\frac{\pi}{2}$.

c Stretch horizontally with factor $\frac{1}{2}$, then shift right $\frac{\pi}{6}$ units.

21 a 0

b $4\left(0 + \frac{\pi}{2}\right) = 2\pi$, or more simply 0

c $0 - \frac{\pi}{3} = -\frac{\pi}{3}$

Chapter 3

Exercise 3A

1a A, G and I **b** C and E

c B, D, F and H

2 a increasing **b** stationary **c** decreasing

d decreasing **e** increasing **f** stationary

3 a $2x - 6$

b i decreasing **ii** decreasing **iii** stationary

iv increasing **v** decreasing

4 a $3x^2 - 12x + 9$

b i increasing **ii** stationary **iii** decreasing

iv stationary **v** increasing

5 a $x = 1$ **b** $x = 2$

c $x = -3$ **d** $x = 4$

e $x = 0$ or $x = 2$ **f** $x = 2$ or $x = -2$

6 a The derivative is always negative.

b The derivative is always positive.

c $f'(x) = 3x^2$, which is positive except at $x = 0$.

d $f'(x) = 2x$, which is positive if $x > 0$ and negative if $x < 0$. At $x = 0$ the function is stationary.

7 a increasing **b** decreasing **c** stationary

8 a stationary **b** increasing **c** decreasing

9 a increasing **b** decreasing **c** increasing

10 a increasing **b** decreasing **c** increasing

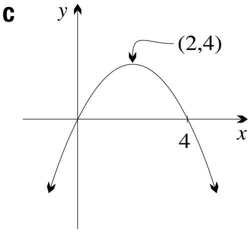
11 a $4 - 2x$

b i $x < 2$ **ii** $x > 2$ **iii** $x = 2$

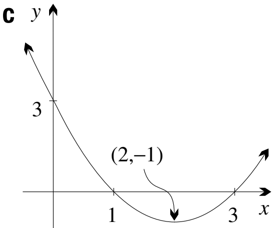
12 a $2x - 4$

b i $x > 2$ **ii** $x < 2$ **iii** $x = 2$

11 c



12 c



14 a $3x^2 + 4x + 1$ **b** $-\frac{1}{3}$ and -1

d $-1 < x < -\frac{1}{3}$

15 a $x > 2$ **b** $x < -3$

c $x > 1$ or $x < -1$ **d** $x < 0$ or $x > 2$

16 a III **b** I **c** IV **d** II

17 a $\dfrac{1}{x^2}$

b The function is not continuous at $x = 0$.

18 a $-\dfrac{6}{(x-3)^2}$

b $f'(x)$ is negative for $x \neq 3$.

19 a $\dfrac{x^2(x^2+3)}{(x^2+1)^2}$

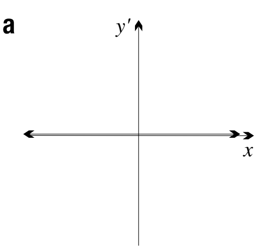
b $f'(x)$ is positive for $x \neq 0$.

20 a $x^2 + 2x + 5$

b $f'(x) > 0$ for all values of x.

c $f(-3) = -8$, $f(0) = 7$, $f(x)$ is increasing for all x. Hence the curve crosses the x-axis exactly once between $x = -3$ and $x = 0$ and nowhere else.

21 a



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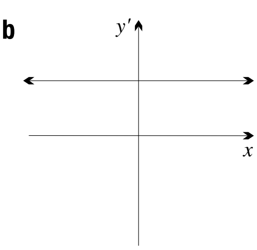
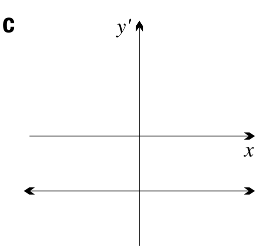
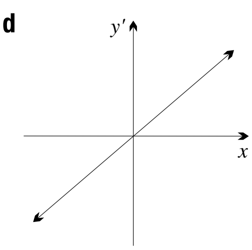


c



d



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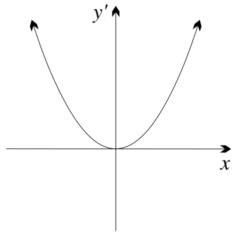
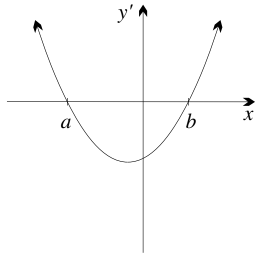
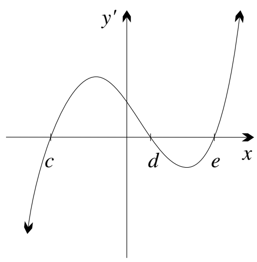
e



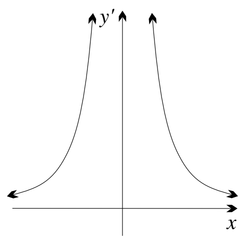
f



g



h

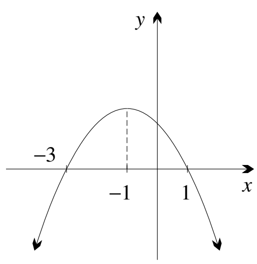


22 a $f'(x) = -3x^2 + 4x - 5$

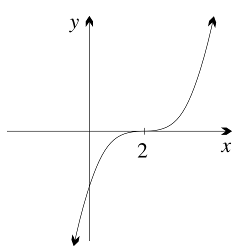
b For $f'(x)$, $\triangle = -44 < 0$, so $f'(x)$ has no zeroes. Also $f'(0) = -5 < 0$, so $f'(x)$ is negative for all values of x.

c 1

23 a



b



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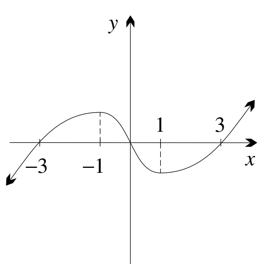
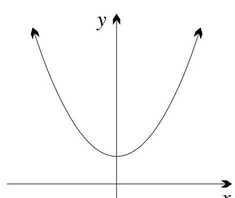


d



Exercise 3B

1a $x = 3$ **b** $x = -2$

c $x = 1$ or -1

2 a $(2, 3)$

b $(4, 0)$

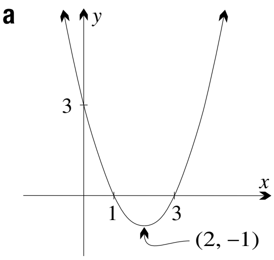
c $(1, -2)$

d $(1, 0)$

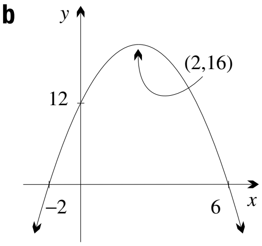
e $(0, 0)$ and $(2, -4)$

f $(1, -2)$

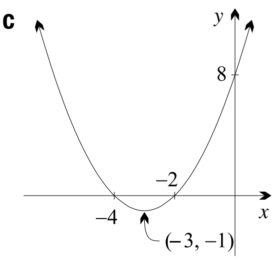
3 a



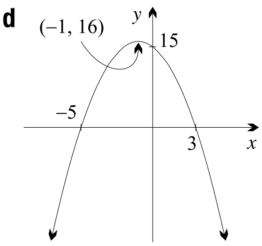
b



c



d



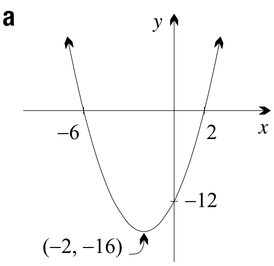
4 a minimum

b maximum

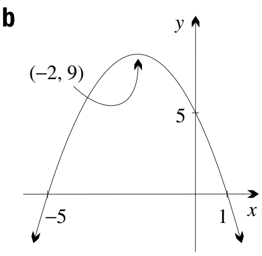
c minimum

d horizontal (or stationary) point of inflection

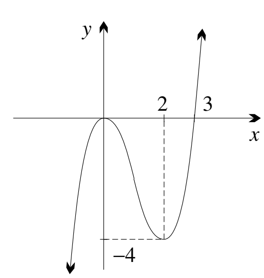
5 a



b



6



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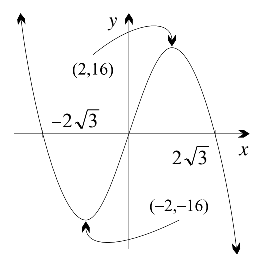
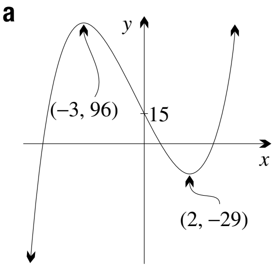


8 a



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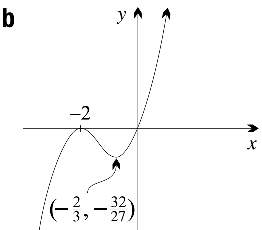

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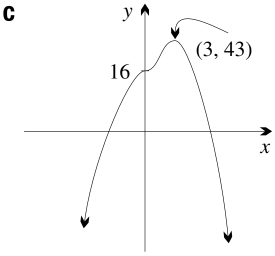
c



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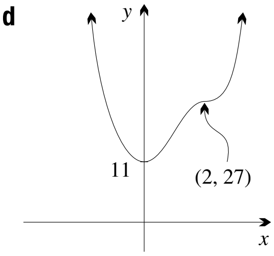

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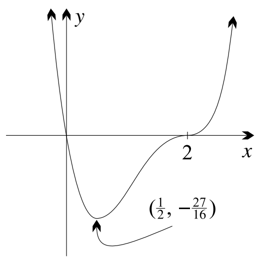

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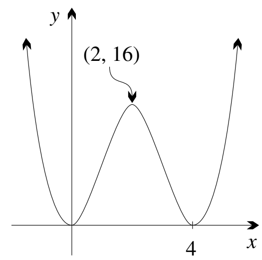

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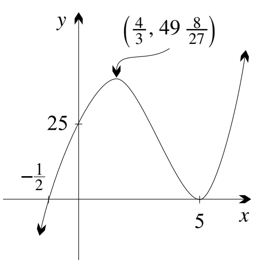


12 a $a = -8$ **b** $a = 2$

13 a $a = 2$ and $c = 3$

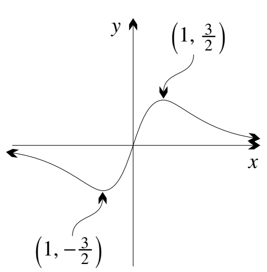
b $b = -3$ and $c = -24$

14 b $a = b = -1, c = 6$

15 a The curve passes through the origin.

c $a = -1$

16 c



d i no roots **ii** 1 root

iii 2 roots **iv** 1 root

17 b $9a - 3b + 3c = -27$

c $a = 2, b = 3, c = -12$

d $d = 7$

Exercise 3C

1a $3x^2, 6x, 6$ **b** $10x^9, 90x^8, 720x^7$

c $7x^6, 42x^5, 210x^4$ **d** $2x, 2, 0$

e $8x^3, 24x^2, 48x$ **f** $15x^4, 60x^3, 180x^2$

g $-3, 0, 0$ **h** $2x - 3, 2, 0$

i $12x^2 - 2x, 24x - 2, 24$

j $20x^4 + 6x^2, 80x^3 + 12x, 240x^2 + 12$

2 a $2x + 3, 2$

b $3x^2 - 8x, 6x - 8$

c $2x - 1, 2$

d $6x - 13, 6$

e $30x^4 - 36x^3, 120x^3 - 108x^2$

f $32x^7 + 40x^4, 224x^6 + 160x^3$

3 a $0.3x^{-0.7}, -0.21x^{-1.7}, 0.357x^{-2.7}$

b $-\frac{1}{x^2}, \frac{2}{x^3}, -\frac{6}{x^4}$ **c** $-\frac{2}{x^3}, \frac{6}{x^4}, -\frac{24}{x^5}$

d $-\frac{15}{x^4}, \frac{60}{x^5}, -\frac{300}{x^6}$ **e** $2x - \frac{1}{x^2}, 2 + \frac{2}{x^3}, -\frac{6}{x^4}$

4 a $-\frac{3}{x^4}, \frac{12}{x^5}$ **b** $-\frac{4}{x^5}, \frac{20}{x^6}$

c $-\frac{6}{x^3}, \frac{18}{x^4}$ **d** $-\frac{6}{x^4}, \frac{24}{x^5}$

5 a $2(x + 1), 2$

b $9(3x - 5)^2, 54(3x - 5)$

c $8(4x - 1), 32$

d $-11(8 - x)^{10}, 110(8 - x)^9$

6 a $\frac{-1}{(x + 2)^2}, \frac{2}{(x + 2)^3}$ **b** $\frac{2}{(3 - x)^3}, \frac{6}{(3 - x)^4}$

c $\frac{-15}{(5x + 4)^4}, \frac{300}{(5x + 4)^5}$ **d** $\frac{12}{(4 - 3x)^3}, \frac{108}{(4 - 3x)^4}$

7 a $\frac{1}{2\sqrt{x}}, \frac{-1}{4x\sqrt{x}}$ **b** $\frac{1}{3}x^{-\frac{2}{3}}, -\frac{2}{9}x^{-\frac{5}{3}}$

c $\frac{3}{2}\sqrt{x}, \frac{3}{4\sqrt{x}}$ **d** $-\frac{1}{2}x^{-\frac{3}{2}}, \frac{3}{4}x^{-\frac{5}{2}}$

e $\frac{1}{2\sqrt{x + 2}}, \frac{-1}{4(x + 2)^{\frac{3}{2}}}$ **f** $\frac{-2}{\sqrt{1 - 4x}}, \frac{-4}{(1 - 4x)^{\frac{3}{2}}}$

8 a $f'(x) = 3x^2 + 6x + 5, f''(x) = 6x + 6$

b **i** 5 **ii** 14 **iii** 6 **iv** 12

9 a **i** 15 **ii** 12 **iii** 6 **iv** 0

b **i** -8 **ii** 48 **iii** -192 **iv** 384

10 a $\frac{1}{(x + 1)^2}, \frac{-2}{(x + 1)^3}$

b $\frac{7}{(2x + 5)^2}, \frac{-28}{(2x + 5)^3}$

11 $(x - 1)^3(5x - 1), 4(x - 1)^2(5x - 2)$

12 a $1, -1$ **b** $-\frac{1}{3}$

13 a $nx^{n-1}, n(n - 1)x^{n-2}, n(n - 1)(n - 2)x^{n-3}$

b $n(n - 1)(n - 2)\ldots 1, 0$

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Exercise 3D

1

Point	A	B	C	D	E	F	G	H	I
y'	0	+	0	−	0	−	0	+	0
y''	+	0	−	0	0	0	+	0	0

2 a concave down **b** concave up
c concave up **d** concave down

3 a minimum **b** maximum
c minimum **d** minimum

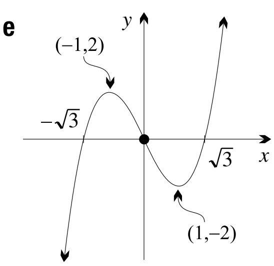
4 a $y'' = 2$, so $y'' > 0$ for all values of x.
b $y'' = -6$, so $y'' < 0$ for all values of x.

5 a $y'' = 6x - 6$
b i $x > 1$ **ii** $x < 1$

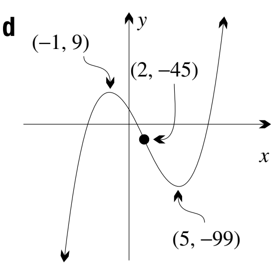
6 a $y'' = 6x - 2$
b i $x > \frac{1}{3}$ **ii** $x < \frac{1}{3}$

7 $x = 0$ and $x = 2$, but not $x = -3$.

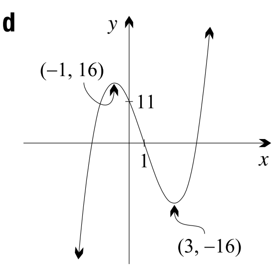
8 e



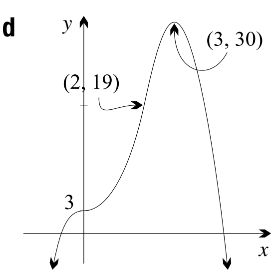
9 d



10 d



11 d



12 a $x > 2$ or $x < -1$ **b** $-1 < x < 2$
c $x > \frac{1}{2}$ **d** $x < \frac{1}{2}$

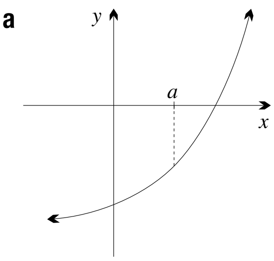
13 a $y' = 3x^2 + 6x - 72$, $y'' = 6x + 6$
d $75x + y - 13 = 0$

14 a $f'(x) = 3x^2$, $f''(x) = 6x$, $g'(x) = 4x^3$, $g''(x) = 12x^2$
b $f''(x) = g''(x) = 0$, no
c $f(x)$ has a horizontal (or stationary) point of inflection, $g(x)$ has a minimum turning point.

15 a $y'' = 6x - 2a$, $a = 6$
b $y'' = 6x + 4a$, $a > 1\frac{1}{2}$
c $y'' = 12x^2 + 6ax + 2b$, $a = -5$, $b = 6$
d $a > -\frac{2}{3}$

16 a Increasing. **b** Concave down.

17 a



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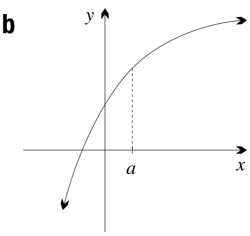

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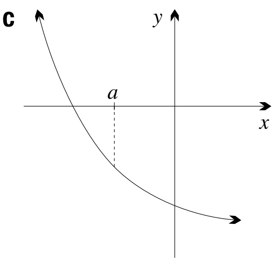
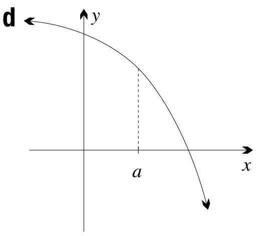


d



18 a $y' = (x - 3)^2 + 2 \geq 2$ for all real x.
b There is a point of inflection at $x = 3$.
c One, because the function is continuous and increasing for all real x.

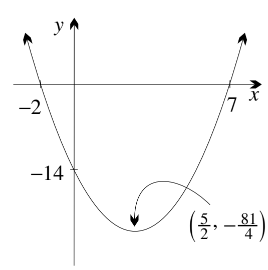
19 $a = 2$, $b = -3$, $c = 0$ and $d = 5$

Exercise 3E

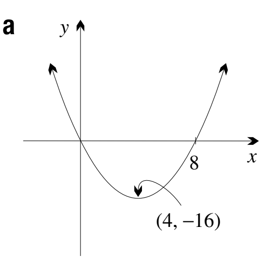
1a $(6, 0)$ **b** $(4, 32)$ **c** $(2, 16)$

2 a $x = -1$ or $x = 2$ **b** $x = 0$
c $-1 < x < 2$ **d** $x < 0$

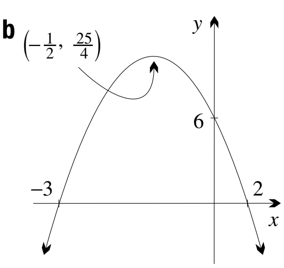
3



4 a



b



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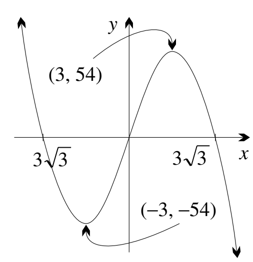

a Show that $f(-x) = -f(x)$. Point symmetry in the origin.
e When $x = 0$, $y' = 27$.

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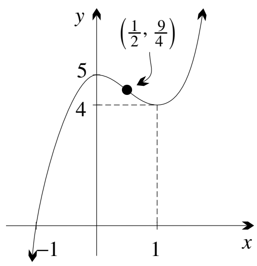
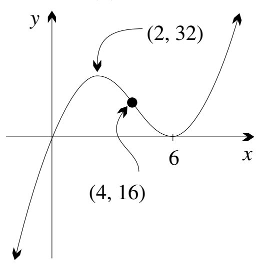
6 When $x = \frac{1}{2}$, $y' = 1\frac{1}{2}$.



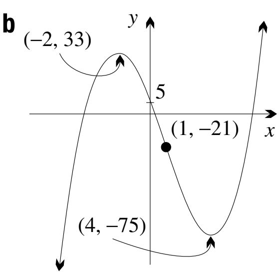
c $f''(0 = -6$, $f''\left(\frac{1}{2}\right) = 0$ and $f''(1) = 6$, so the concavity changes sign around $x = \frac{1}{2}$.

b Also $f'\left(\frac{1}{2}\right) = -1\frac{1}{2}$.

7 a



b



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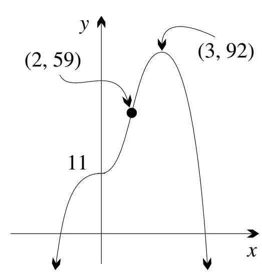

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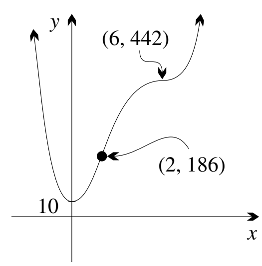

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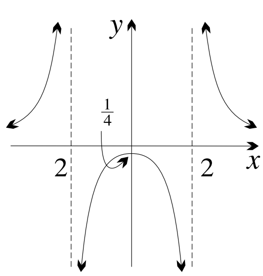
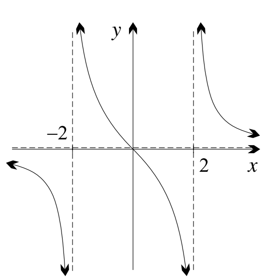


c line symmetry in the y-axis

d domain: $x \neq 2$ and $x \neq -2$, asymptotes: $x = 2$ and $x = -2$

e $y = 0$ **g** $y > 0$ or $y \leq -\frac{1}{4}$

11



c gradient $= -\frac{1}{4}$

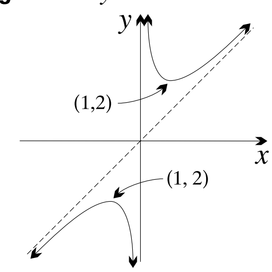
d domain: $x \neq 2$ and $x \neq -2$, asymptotes: $x = 2$ and $x = -2$

e $y = 0$

f point symmetry in the origin

g all real y

12



c point symmetry in the origin

d domain: $x \neq 0$, asymptote: $x = 0$

g $y \geq 2$ or $y \leq -2$

Exercise 3F

1a A local maximum, B local minimum

b C global maximum, D local minimum, E local maximum, F global minimum

c G global maximum, H horizontal point of inflection

d I horizontal point of inflection, J global minimum

2 a 0, 4 **b** 2, 5 **c** 0, 4 **d** 0, 5

e $0, 2\sqrt{2}$ **f** $-1, -\frac{1}{4}$ **g** $-1, 2$

3 a $-1, 8$ **b** $-49, 5$ **c** 0, 4 **d** 0, 9

4 a local maximum $y = 4$ at $x = 1$, global minimum -5, global maximum 20

b local minimum $y = 4$ at $x = 1$, global minimum -5, local maximum 11, global maximum 139

c global minimum 4, global maximum 11

Exercise 3G

1b $3\frac{1}{8}$ metres

2 b 3 **c** 18

3 b 4 **c** 32

4 c 5 **d** 25 cm^2

5 c 10 **d** 200 m^2

6 After 2 hours and 40 minutes.

7 d 24 cm

8 b $x = 30$ m and $y = 20$ m

9 c $h = 2$, $w = \frac{3}{2}$

10 a $\frac{x}{4}, \frac{10 - x}{4}$ **c** 5 **d** $\frac{25}{8}$ cm^2

11 a $R = x(47 - \frac{1}{3}x)$

b $-\frac{8}{15}x^2 + 32x - 10$

c 30

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12 c $\frac{10}{3\pi}$ **d** $\frac{1000}{27\pi}$ m^3

13 c $20\sqrt{10}\,\pi$ cm^3

14 c 4 cm by 4 cm by 2 cm

15 c $\frac{10}{3}$

16 a $S = 16x + 4y$

c 27 m by 9 m by 18 m

17 b Width $16\sqrt{3}$ cm and depth $16\sqrt{6}$ cm.

18 b 15 cm by 5 cm by 3.75 cm

19 d $r = 8$

20 c 48 cm^2

21 d $2\left(\sqrt{10} + 1\right)$ cm by $4\left(\sqrt{10} + 1\right)$ cm

22 b 80 km/h **c** $400

23 b When $x = 0$, T2.75 hours, and when $x = 20$, $T \doteqdot 2.61$ hours.

d 2.45 hours

Exercise 3H

1 a $\frac{1}{7}x^7 + C$ **b** $\frac{1}{4}x^4 + C$ **c** $\frac{1}{11}x^{11} + C$

d $\frac{3}{2}x^2 + C$ **e** $5x + C$ **f** $\frac{1}{2}x^{10} + C$

g $3x^7 + C$ **h** C

2 a $\frac{1}{3}x^3 + \frac{1}{5}x^5 + C$ **b** $x^4 - x^5 + C$

c $\frac{2}{3}x^3 + \frac{5}{8}x^8 + C$ **d** $\frac{1}{3}x^3 - \frac{1}{2}x^2 + x + C$

e $3x - 2x^2 + 2x^8 + C$ **f** $x^3 - x^4 - x^5 + C$

3 a $\frac{1}{3}x^3 - \frac{3}{2}x^2 + C$ **b** $\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + C$

c $x^3 + \frac{11}{2}x^2 - 4x + C$ **d** $\frac{5}{6}x^6 - x^4 + C$

e $x^8 + \frac{1}{2}x^4 + C$

f $\frac{1}{2}x^2 + \frac{1}{4}x^4 - 3x - x^3 + C$

4 a i $y = x^2 + 3x + 3$

ii $y = x^2 + 3x + 4$

b i $y = 3x^3 + 4x + 1$

ii $y = 3x^3 + 4x - 2$

c i $y = x^3 - 2x^2 + 7x$

ii $y = x^3 - 2x^2 + 7x - 7$

5 a $-\dfrac{1}{x} + C$ **b** $-\dfrac{1}{2x^2} + C$

c $\dfrac{1}{x^2} + C$ **d** $\dfrac{1}{x^3} + C$

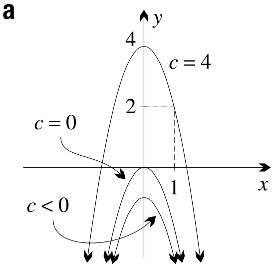
e $-\dfrac{1}{x} + \dfrac{1}{2x^2} + C$

6 a $\frac{2}{3}x^{\frac{3}{2}} + C$ **b** $2\sqrt{x} + C$ **c** $\frac{3}{4}x^{\frac{4}{3}} + C$

d $4\sqrt{x} + C$ **e** $\frac{5}{8}x^{\frac{8}{5}} + C$

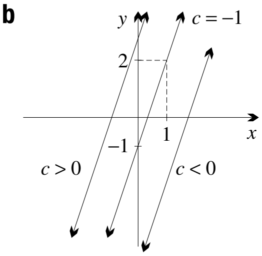
7 a $y = \frac{2}{3}x^{\frac{3}{2}} + 1$ **b** $y = \frac{2}{3}x^{\frac{3}{2}} - 16$

8 a



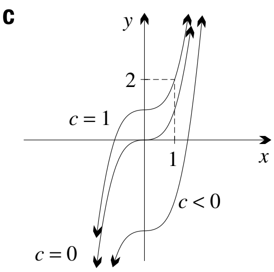
$y = -2x^2 + c$,
$y = 4 - 2x^2$

b



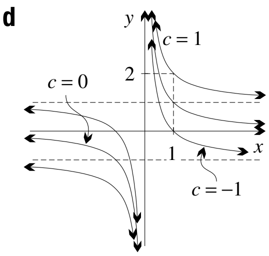
$y = 3x + c$,
$y = 3x - 1$

c



$y = x^3 + c$,
$y = x^3 + 1$

d



$y = \dfrac{1}{x} + c$,
$y = \dfrac{1}{x} + 1$

9 a $\frac{1}{4}(x + 1)^4 + C$ **b** $\frac{1}{6}(x - 2)^6 + C$

c $\frac{1}{3}(x + 5)^3 + C$ **d** $\frac{1}{10}(2x + 3)^5 + C$

e $\frac{1}{21}(3x - 4)^7 + C$ **f** $\frac{1}{20}(5x - 1)^4 + C$

g $-\frac{1}{4}(1 - x)^4 + C$ **h** $-\frac{1}{28}(1 - 7x)^4 + C$

i $\dfrac{-1}{3(x - 2)^3} + C$ **j** $\dfrac{1}{9(1 - x)^9} + C$

10 a $\frac{2}{3}(x + 1)^{\frac{3}{2}} + C$ **b** $\frac{2}{3}(x - 5)^{\frac{3}{2}} + C$

c $-\frac{2}{3}(1 - x)^{\frac{3}{2}} + C$ **d** $\frac{1}{3}(2x - 7)^{\frac{3}{2}} + C$

e $\frac{2}{9}(3x - 4)^{\frac{3}{2}} + C$

11 a $y = \frac{1}{5}(x - 1)^5$

b $\frac{1}{8}(2x + 1)^4 - \frac{9}{8}$

c $y = \frac{1}{3}(2x + 1)^{\frac{3}{2}}$

12 a $y = \frac{3}{5}x^5 - \frac{1}{4}x^4 + x$

b $y = -\frac{1}{4}x^4 + x^3 + 2x - 2$

c $y = -\frac{1}{20}(2 - 5x)^4 + \frac{21}{20}$

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13 30

14 The rule gives the primitive of x^{-1} as $\frac{x^0}{0}$, which is undefined. This problem will be addressed in Chapter 5.

15 $y = x^3 + 2x^2 - 5x + 6$

16 $y = -x^3 + 4x^2 + 3$

17 $y = \begin{cases} \frac{1}{x} + 4\frac{1}{2}, & \text{for } x > 0, \\ \frac{1}{x} + 1, & \text{for } x < 0. \end{cases}$

Chapter 3 review exercise

1a C and H **b** A and F

c B, D, E and G **d** A, B, G and H

e D **f** C, E and F

2 a $f'(x) = 3x^2 - 2x - 1$

b **i** decreasing **ii** stationary

iii increasing **iv** increasing

3 a $2x - 4$

b **i** $x > 2$ **ii** $x < 2$ **iii** $x = 2$

4 a $y' = 3x^2$, increasing

b $y' = 2x - 1$, increasing

c $y' = 5(x - 1)^4$, stationary

d $y' = -\frac{4}{(x - 3)^2}$, decreasing

5 a $7x^6, 42x^5$

b $3x^2 - 8x, 6x - 8$

c $5(x - 2)^4, 20(x - 2)^3$

d $-\frac{1}{x^2}, \frac{2}{x^3}$

6 a concave up

b concave down

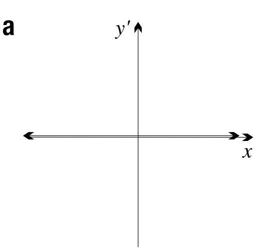
7 a $12x - 6$

b **i** $x > \frac{1}{2}$ **ii** $x < \frac{1}{2}$

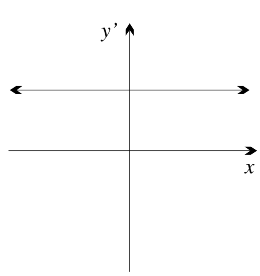
8 a $x < 1$ or $x > 3$ **b** $1 < x < 3$

c $x > 2$ **d** $x < 2$

9 a



b



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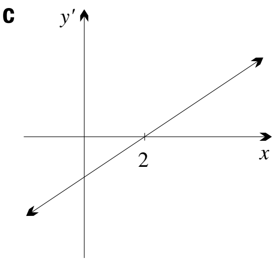

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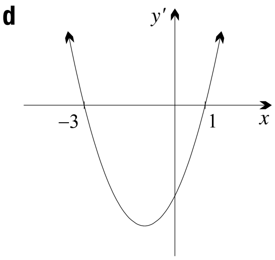
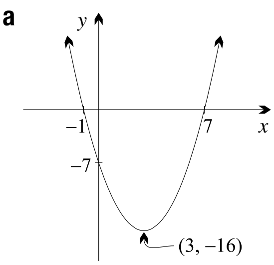


10 a $P(-1, 3)$, $Q\left(\frac{1}{3}, \frac{49}{27}\right)$ **b** $x > -\frac{1}{3}$

c $\frac{49}{27} < k < 3$

11 a



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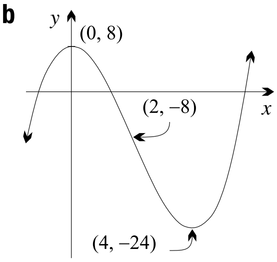

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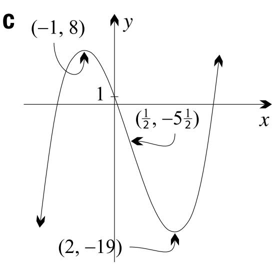
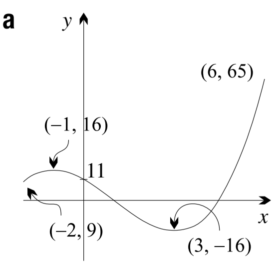


12 a



b 65 and −16

13 a $a = -2$ **b** $a = 3$ and $b = 6$

14 b −16

15 a 175 **b** 0 **c** 256

16 b $\frac{1600}{27}$ cm^3

17 b $r = 8$ m

18 a $\frac{1}{8}x^8 + C$ **b** $x^2 + C$

c $4x + C$ **d** $2x^5 + C$

e $4x^2 + x^3 - x^4 + C$

19 a $x^3 - 3x^2 + C$

b $\frac{1}{3}x^3 - 2x^2 - 5x + C$

c $\frac{4}{3}x^3 - 6x^2 + 9x + C$

20 a $\frac{1}{6}(x + 1)^6 + C$

b $\frac{1}{8}(x - 4)^8 + C$

c $\frac{1}{8}(2x - 1)^4 + C$

21 a $-\frac{1}{x} + C$ **b** $\frac{2}{3}x^{\frac{2}{3}} + C$

22 $f(x) = x^3 - 2x^2 + x + 3$

23 25

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Chapter 4

Exercise 4A

1 a $\frac{1}{2}u^2$
b The area under the curve is less than the area of the triangle.
2 a $\frac{1}{16}u^2$ **b** $\frac{5}{16}u^2$
c The area under the curve is less than the combined area of the triangle and trapezium.
3 b The gaps between the upper line segments and the curve are getting smaller.
4 a 6 **b** 12 **c** 8 **d** 9
e 2 **f** $\frac{25}{2}$ **g** 6 **h** 20
5 a 8 **b** 25 **c** 9 **d** 24
e 36 **f** 24 **g** 9 **h** 8
6 a $\frac{7}{32}u^2$ **b** $\frac{15}{32}u^2$
c The sum of the areas of the lower rectangles is less than the exact area under the curve which is less than the sum of the areas of the upper rectangles. Note that $\int_0^1 x^2\,dx = \frac{1}{3}$.
7 d As the number of rectangles increases, the interval within which the exact area lies becomes smaller. Note that $\int_0^1 2^x\,dx = \frac{1}{\ln 2} \doteqdot 1.44$.
8 d As the number of rectangles increases, the interval within which the exact area lies becomes smaller. Note that $\int_2^4 \ln x\,dx = 6\ln 2 - 2 \doteqdot 2.16$.
9 a 15 **b** 15 **c** 25 **d** 40
e $\frac{25}{2}$ **f** 12 **g** 16 **h** 24
i 8 **j** 18 **k** 4 **l** 16
m 4 **n** 16 **o** $\frac{25}{2}$ **p** $\frac{25}{2}$
10 a 8π **b** $\frac{25}{4}\pi$
11 a You should count approximately 133 squares. $\frac{133}{400} \doteqdot 0.33$. We shall see later that $\int_0^1 x^2\,dx = \frac{1}{3}$.
b The exact values are:
i $\frac{1}{24}$ **ii** $\frac{7}{24}$
12 b 0.79 **c** 3.16
13 e The interval is getting smaller.
f Yes, they appear to be getting closer and closer to the exact value.

Exercise 4B

1 a 1 **b** 15 **c** 16 **d** 84 **e** 19
f 243 **g** 62 **h** 2 **i** 1
2 a i 4 **ii** 25
iii 1 (Note that $\int_4^5 dx$ means $\int_4^5 1\,dx$.)
b Each function is a horizontal line, so each integral is a rectangle.
3 a 30 **b** 6 **c** 33 **d** 18 **e** 132
f 2 **g** 23 **h** 44 **i** 60
4 a 2 **b** 2 **c** 9
d 30 **e** 96 **f** 10
5 a $13\frac{1}{2}$ **b** $4\frac{2}{3}$ **c** $29\frac{1}{4}$
d 2 **e** $20\frac{5}{6}$ **f** 98
6 a 24 **b** 18 **c** $2\frac{2}{3}$
d 21 **e** $\frac{1}{4}$ **f** $\frac{8}{15}$
7 a 42 **b** 14 **c** 62
d $8\frac{1}{3}$ **e** $6\frac{2}{3}$ **f** 6
8 a $\frac{1}{24}$ **b** $\frac{20}{27}$ **c** $\frac{7}{8}$
9 a i $\frac{1}{10}$ **ii** $\frac{5}{36}$ **iii** 15
b i $\frac{1}{2}$ **ii** $\frac{15}{32}$ **iii** 7
10 a ii 8 **b ii** 6
11 a $k = 1$ **b** $k = 4$ **c** $k = 8$
d $k = 3$ **e** $k = 3$ **f** $k = 2$
12 a $1 + \frac{\pi}{2}$ **b** $2\frac{1}{2}$
13 a $\frac{3}{2}$ **b** $\frac{5}{8}$ **c** $42\frac{1}{3}$
14 a $13\frac{1}{3}$ **b** $8\frac{59}{120}$ **c** $\frac{1}{24}$
15 a x^2 is never negative.
b The function has an asymptote $x = 0$, which lies in the given interval. Hence the integral is meaningless and the use of the fundamental theorem is invalid.
c Part **ii** is meaningless because it crosses the asymptote at $x = 3$.
16 a i x^2 **ii** $x^3 + 3x$ **iii** $\frac{1}{x}$ **iv** $(x^3 - 3)^4$

Exercise 4C

1 The values are 6 and −6, which differ by a factor of −1.
2 a LHS = RHS = 2
b LHS = RHS = $6\frac{3}{4}$
c LHS = RHS = 0
3 a The interval has width zero.
b $y = x$ is an odd function.
4 a The area is below the x-axis.
b The area is above the x-axis.
c The areas above and below the x-axis are equal.
d The area below the x-axis is greater than the area above.
5 a The area is above the x-axis.
b The area is below the x-axis.

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c $y = 1 - x^2$ is an even function, so is symmetrical about the y-axis.

d The area under the parabola from 0 to $\frac{1}{2}$ is greater than the area from $\frac{1}{2}$ to 1.

6 a -7 **b** -5

7 a $1\frac{1}{3}$

b The integral from -1 to 1 is negative because the curve is below the x-axis and the integral runs forwards. But when the limits are reversed and the integral runs backwards, the value is the opposite and so is positive.

8 The area under the line $y = 2x$ from $x = 0$ to $x = 1$ is greater than the area under $y = x$.

9 The area below the x-axis is greater than the area above.

10 a i 6

ii -6. The integrals are opposites because the limits have been reversed.

b i 5

ii 5. The factor 20 can be taken out of the integral.

c i 45 **ii** 30

iii 15. An integral of a sum is the sum of the integrals.

d i 48 **ii** 3

iii 45. The interval $0 \le x \le 2$ can be dissected into the intervals $0 \le x \le 1$ and $1 \le x \le 2$.

e i 0

ii 0. An integral over an interval of zero width is zero.

11 a 0. The interval has zero width.

b 0. The interval has zero width.

c 0. The integrand is odd.

d 0. The integrand is odd.

e 0. The integrand is odd.

f 0. The integrand is odd.

12 a The curves meet at $(0, 0)$ and at $(1, 1)$.

b In the interval $0 \le x \le 1$, the curve $y = x^3$ is below the curve $y = x^2$.

c $\frac{1}{4}$ and $\frac{1}{3}$

13 a $\frac{\pi}{2} - \frac{1}{2}$ **b** $1 - \frac{\pi}{2}$

14 a The function is odd, so the integral is zero.

b The function is even, so its graph is symmetrical about the y-axis.

15 a false **b** true **c** false **d** false

Exercise 4D

1 a $A(x) = \frac{3}{2}x^2$

b $A'(x) = 3x$

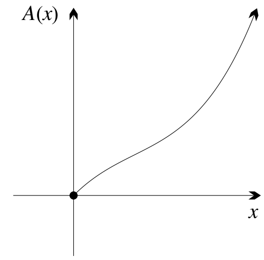
2 a $y = 3$, $A(x) = 3x$, $A'(x) = 3$

b $y = 2t$, $A(x) = x^2$, $A'(x) = 2x$

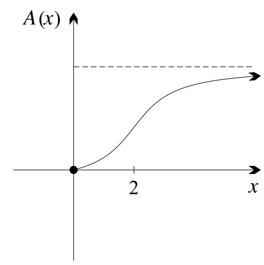
c $y = 2 + t$, $A(x) = 2x + \frac{1}{2}x^2$, $A'(x) = 2 + x$

d $y = 5 - t$, $A(x) = 5x - \frac{1}{2}x^2$, $A'(x) = 5 - x$

3 a $A(x)$ is increasing at a decreasing rate in the interval $0 \le x < 2$, and increasing at an increasing rate for $x > 2$. It has an inflection at $x = 2$.



b $A(x)$ is increasing at an increasing rate in the interval $0 \le x \le 2$, and increasing at a decreasing rate for $x \ge 2$. It has a point of inflection at $x = 2$.



4 a $A'(x) = \dfrac{1}{x}$ **b** $A'(x) = \dfrac{1}{1 + t^3}$

c $A'(x) = e^{-\frac{1}{2}x^2}$

5 a $A'(x) = 3x^2 - 12$, $A(x) = x^3 - 12x + 11$

b $A'(x) = x^3 + 4x$, $A(x) = \frac{1}{4}x^4 + 2x^2 - 12$

c $A'(x) = \frac{1}{x^2}$, $A(x) = \frac{1}{2} - \frac{1}{x}$

6 a The curve $y = A(x)$ looks like $y = e^x - 1$. The curve is increasing at an increasing rate.

b The curve is zero at $x = 1$, and is increasing at an increasing rate.

c The curve is zero at $x = 1$, and is increasing at a decreasing rate.

7 a The values are 0, 1, 0, -1, 0. The curve looks like $y = \sin x$, and this suggests that $\sin x$ has derivative $\cos x$.

b The values are 0, 1, 2, 1, 0. The graph looks like $y = 1 - \cos x$, which suggest that the derivative of $\cos x$ is $-\sin x$.

8 a $A(x)$ is increasing when $f(t)$ is positive, that is, for $t < c$, and is decreasing for $t > c$.

b $A(x)$ has a maximum turning point at $x = c$, and no minimum turning points.

c $A(x)$ has inflections when $f'(t)$ changes sign, that is, at $x = b$ and $x = d$.

d Because of the point symmetry of $f(t)$, there are two zeroes of $A(x)$ are $x = a$ and $x = e$.

e $A(x)$ is positive for $a < x < e$ and negative for $x < a$ and for $x > e$.

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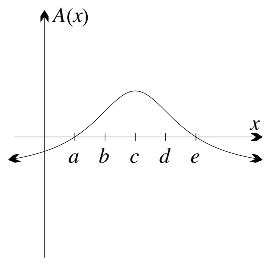
f



9 a The function is continuous at every real number, so it is a continuous function.

b The domain is $x \neq 2$, and y is continuous at every value in its domain, so it is a continuous function.

c Zero now lies in the domain, and y is not continuous at $x = 0$, so it is not a continuous function.

d The domain is $x \geq 0$, and y is continuous at every value in its domain, so it is a continuous function.

e The domain is $x > 0$, and y is continuous at every value in its domain, so it is a continuous function.

f The domain is $x \geq 0$, and y is not continuous at $x = 0$, so it is not a continuous function.

Exercise 4E

1 a $4x + C$ **b** $x + C$ **c** C
d $-2x + C$ **e** $\frac{x^2}{2} + C$ **f** $\frac{x^3}{3} + C$
g $\frac{x^4}{4} + C$ **h** $\frac{x^8}{8} + C$

2 a $x^2 + C$ **b** $2x^2 + C$ **c** $x^3 + C$
d $x^4 + C$ **e** $x^{10} + C$ **f** $\frac{x^4}{2} + C$
g $\frac{2x^6}{3} + C$ **h** $\frac{x^9}{3} + C$

3 a $\frac{x^2}{2} + \frac{x^3}{3} + C$ **b** $\frac{x^5}{5} - \frac{x^4}{4} + C$
c $\frac{x^8}{8} + \frac{x^{11}}{11} + C$ **d** $x^2 + x^5 + C$
e $x^9 - 11x + C$ **f** $\frac{x^{14}}{2} + \frac{x^9}{3} + C$
g $4x - \frac{3x^2}{2} + C$ **h** $x - \frac{x^3}{3} + \frac{x^5}{5} + C$
i $x^3 - 2x^4 + \frac{7x^5}{5} + C$

4 a $-x^{-1} + C$ **b** $-\frac{1}{2}x^{-2} + C$ **c** $-\frac{1}{7}x^{-7} + C$
d $-x^{-3} + C$ **e** $-x^{-9} + C$ **f** $-2x^{-5} + C$

5 a $\frac{2}{3}x^{\frac{3}{2}} + C$ **b** $\frac{3}{4}x^{\frac{4}{3}} + C$ **c** $\frac{4}{5}x^{\frac{5}{4}} + C$
d $\frac{3}{5}x^{\frac{5}{3}} + C$ **e** $2x^{\frac{1}{2}} + C$ **f** $\frac{8}{3}x^{\frac{3}{2}} + C$

6 a $\frac{1}{3}x^3 + x^2 + C$ **b** $2x^2 - \frac{1}{4}x^4 + C$
c $\frac{5}{3}x^3 - \frac{3}{4}x^4 + C$ **d** $\frac{1}{5}x^5 - \frac{5}{4}x^4 + C$
e $\frac{1}{3}x^3 - 3x^2 + 9x + C$ **f** $\frac{4}{3}x^3 + 2x^2 + x + C$
g $x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + C$ **h** $4x - 3x^3 + C$
i $\frac{1}{3}x^3 - \frac{1}{2}x^4 - 3x + 3x^2 + C$

7 a $\frac{1}{2}x^2 + 2x + C$ **b** $\frac{1}{2}x^2 + \frac{1}{3}x^3 + C$
c $\frac{1}{6}x^3 - \frac{1}{16}x^4 + C$

8 a $-\dfrac{1}{x} + C$ **b** $-\dfrac{1}{2x^2} + C$ **c** $-\dfrac{1}{4x^4} + C$
d $-\dfrac{1}{9x^9} + C$ **e** $-\dfrac{1}{x^3} + C$ **f** $-\dfrac{1}{x^5} + C$
g $-\dfrac{1}{x^7} + C$ **h** $-\dfrac{1}{3x} + C$ **i** $-\dfrac{1}{28x^4} + C$
j $\dfrac{1}{10x^2} + C$ **k** $\dfrac{1}{4x^4} - \dfrac{1}{x} + C$
l $-\dfrac{1}{2x^2} - \dfrac{1}{3x^3} + C$

9 a $\frac{2}{3}x^{\frac{3}{2}} + C$ **b** $\frac{3}{4}x^{\frac{4}{3}} + C$
c $2\sqrt{x} + C$ **d** $\frac{3}{5}x^{\frac{5}{3}} + C$

10 a 18 **b** 12 **c** 4 **d** $\frac{3}{5}$

11 a $\frac{1}{6}(x + 1)^6 + C$ **b** $\frac{1}{4}(x + 2)^4 + C$
c $-\frac{1}{5}(4 - x)^5 + C$ **d** $-\frac{1}{3}(3 - x)^3 + C$
e $\frac{1}{15}(3x + 1)^5 + C$ **f** $\frac{1}{32}(4x - 3)^8 + C$
g $-\frac{1}{14}(5 - 2x)^7 + C$ **h** $-\frac{1}{40}(1 - 5x)^8 + C$
i $\frac{1}{24}(2x + 9)^{12} + C$ **j** $\frac{3}{22}(2x - 1)^{11} + C$
k $\frac{4}{35}(5x - 4)^7 + C$ **l** $-\frac{7}{8}(3 - 2x)^4 + C$

12 a $\frac{3}{5}\left(\dfrac{1}{3}x - 7\right)^5 + C$ **b** $\frac{4}{7}\left(\dfrac{1}{4}x - 7\right)^7 + C$
c $-\frac{5}{4}\left(1 - \dfrac{1}{5}x\right)^4 + C$

13 a $-\dfrac{1}{2(x + 1)^2} + C$ **b** $-\dfrac{1}{3(x - 5)^3} + C$
c $-\dfrac{1}{3(3x - 4)} + C$ **d** $\dfrac{1}{4(2 - x)^4} + C$
e $-\dfrac{3}{5(x - 7)^5} + C$ **f** $-\dfrac{1}{2(4x + 1)^4} + C$
g $\dfrac{2}{15(3 - 5x)^3} + C$ **h** $\dfrac{1}{5 - 20x} + C$
i $-\dfrac{7}{96(3x + 2)^4} + C$

14 a $\frac{3}{2}x^2 - \frac{2}{5}x^{\frac{5}{2}} + C$ **b** $\frac{1}{2}x^2 - 4x + C$
c $2x^2 - \frac{8}{3}x^{\frac{3}{2}} + x + C$

15 a i $\frac{2}{3}$ **ii** 2 **iii** 12
b i $5\frac{1}{3}$ **ii** $96\frac{4}{5}$ **iii** 4

16 a 2 **b** $-\frac{13}{6}$ **c** $12\frac{1}{6}$

17 $\displaystyle\int x^{-1}\,dx = \frac{x^0}{0} + C$ is meaningless. Chapter 5 deals with the resolution of this problem.

18 a $\frac{1}{3}(2x - 1)^{\frac{3}{2}} + C$ **b** $-\frac{1}{6}(7 - 4x)^{\frac{3}{2}} + C$
c $\frac{3}{16}(4x - 1)^{\frac{4}{3}} + C$ **d** $\frac{2}{3}\sqrt{3x + 5} + C$

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19 a $\frac{242}{5}$ **b** 0 **c** $121\frac{1}{3}$
d 1 **e** $\frac{13}{6}$ **f** 2
g 0 **h** $\frac{112}{9}$ **i** $8\frac{2}{5}$

Exercise 4F

1 a $4\,u^2$ **b** $26\,u^2$ **c** $81\,u^2$ **d** $12\,u^2$
e $9\,u^2$ **f** $6\frac{2}{3}u^2$ **g** $\frac{128}{3}u^2$ **h** $6\,u^2$
i $\frac{1}{4}u^2$ **j** $57\frac{1}{6}u^2$ **k** $36\,u^2$ **l** $60\,u^2$

2 a $25\,u^2$ **b** $8\,u^2$

3 a $4\,u^2$ **b** $108\,u^2$ **c** $\frac{9}{2}u^2$
d $34\frac{2}{3}u^2$ **e** $18\,u^2$ **f** $2\,u^2$

4 a $\frac{4}{3}u^2$ **b** $\frac{27}{2}u^2$ **c** $\frac{81}{4}u^2$ **d** $46\frac{2}{5}u^2$

5 a $\frac{9}{2}u^2$ **b** $\frac{4}{3}u^2$ **c** $\frac{45}{4}u^2$ **d** $9\,u^2$

6 b $4\frac{1}{2}u^2$ **c** $2\,u^2$ **d** $6\frac{1}{2}u^2$
e $2\frac{1}{2}$. This is the area above the x-axis minus the area below it.

7 b $10\frac{2}{3}u^2$ **c** $2\frac{1}{3}u^2$ **d** $13\,u^2$
e $-8\frac{1}{3}$. This is the area above the x-axis minus the area below it.

8 b $2\frac{2}{3}u^2$ **c** $\frac{5}{12}u^2$ **d** $3\frac{1}{12}u^2$
e $-2\frac{1}{4}$. This is the area above the x-axis minus the area below it.

9 a $11\frac{2}{3}u^2$ **b** $128\frac{1}{2}u^2$ **c** $4\,u^2$
d $8\frac{1}{2}u^2$ **e** $32\frac{3}{4}u^2$ **f** $11\frac{1}{3}u^2$

10 a $13\,u^2$ **b** $2\frac{1}{2}u^2$ **c** $9\frac{1}{3}u^2$ **d** $7\frac{1}{3}u^2$

11 a i $64\,u^2$ **ii** $128\,u^2$ **iii** $64\frac{4}{5}u^2$
b i $50\,u^2$ **ii** $18\,u^2$ **iii** $\frac{32}{3}u^2$

12 $16\,u^2$

13 a $(2, 0)$, $\left(0, 4\sqrt{2}\right)$, $\left(0, -4\sqrt{2}\right)$
b $\frac{16\sqrt{2}}{3}u^2$
ii $x = 2 - \frac{y^2}{16}$

14 a $y = \frac{1}{3}x^3 - 2x^2 + 3x$
b The curve passes through the origin, $(1, 1\frac{1}{3})$ is a maximum turning point and $(3, 0)$ is a minimum turning point.
c $\frac{4}{3}u^2$

Exercise 4G

1 a $\frac{1}{6}\,u^2$ **b** $\frac{1}{4}\,u^2$ **c** $\frac{3}{10}\,u^2$ **d** $\frac{1}{12}\,u^2$
e $\frac{2}{35}\,u^2$ **f** $20\frac{5}{6}\,u^2$ **g** $36\,u^2$ **h** $20\frac{5}{6}\,u^2$

2 a $\frac{4}{3}u^2$ **b** $\frac{1}{6}u^2$ **c** $\frac{4}{3}\,u^2$ **d** $4\frac{1}{2}u^2$

3 a $5\frac{1}{3}u^2$ **b** $\frac{9}{4}u^2$

4 a $16\frac{2}{3}\,u^2$ **b** $9\frac{1}{3}u^2$

5 c $4\frac{1}{2}u^2$

6 c $\frac{4}{3}u^2$

7 c $36\,u^2$

8 a $4\frac{1}{2}u^2$ **b** $20\frac{5}{6}u^2$ **c** $2\frac{2}{3}u^2$

9 c $36\,u^2$

10 c $\frac{4}{3}u^2$

11 a $4\frac{1}{2}u^2$ **b** $20\frac{5}{6}u^2$ **c** $21\frac{1}{3}u^2$

12 c $\frac{1}{3}\,u^2$

13 b $y = x - 2$ **c** $5\frac{1}{3}u^2$

14 c $108\,u^2$

15 b The points are $(-4, -67)$, $(1, -2)$, and $(2, 5)$.
c $73\frac{5}{6}\,u^2$

Exercise 4H

1 a 40 **b** 22 **c** -26

2 a 164

3 a 30

4 a The curve is concave up, so the chord is above the curve, and the area under the chord will be greater than the area under the curve.
b The curve is concave down, so the chord is underneath the curve, and the area under the chord will be less than the area under the curve.

5 b 10
c $10\frac{2}{3}$, the curve is concave down.
d $6\frac{1}{4}\%$

6 b $10\frac{1}{10}$
c y'' is positive in the interval $1 \le x \le 5$, so the curve is concave up.

7 b 12.660
c $12\frac{2}{3}$. y'' is negative in the interval $4 \le x \le 9$, so the curve is concave down.

8 a 0.73 **b** 4.5 **c** 3.4 **d** 37

9 a 1.12 **b** 0.705 **c** 22.9 **d** 0.167

10 9.2 metres

11 $550\ m^2$

12 5900

13 a 0.7489
b $\pi \doteqdot 3.0$, the approximation is less than the integral, because the curve is concave down.

15 a 2
b Let $A = (-1, 0)$, $P = (0, 1)$ and $B = (1, 2)$. Then the curve has point symmetry in its y-intercept $P(0, 1)$, and the trapezoidal rule gives the area under the chord APB. The result now follows by symmetry.

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Exercise 4I

1 a $8(2x+3)^3$
b i $(2x+3)^4 + C$ **ii** $2(2x+3)^4 + C$

2 a $9(3x-5)^2$
b i $(3x-5)^3 + C$ **ii** $3(3x-5)^3 + C$

3 a $20(1+4x)^4$
b i $(1+4x)^5 + C$ **ii** $\frac{1}{2}(1+4x)^5 + C$

4 a $-8(1-2x)^3$
b i $(1-2x)^4 + C$ **ii** $\frac{1}{4}(1-2x)^4 + C$

5 a $-4(4x+3)^{-2}$
b i $(4x+3)^{-1} + C$
ii $-\frac{1}{4}(4x+3)^{-1} + C$

6 a $(2x-5)^{-\frac{1}{2}}$
b i $(2x-5)^{\frac{1}{2}} + C$ **ii** $\frac{1}{3}(2x-5)^{\frac{1}{2}} + C$

7 a $8x(x^2+3)^3$
b i $(x^2+3)^4 + C$ **ii** $5(x^2+3)^4 + C$

8 a $15x^2(x^3-1)^4$
b i $(x^3-1)^5 + C$ **ii** $\frac{1}{5}(x^3-1)^5 + C$

9 a $\dfrac{2x}{\sqrt{2x^2+3}}$
b i $\sqrt{2x^2+3} + C$ **ii** $\frac{1}{2}\sqrt{2x^2+3} + C$

10 a $\dfrac{3\left(\sqrt{x}+1\right)^2}{2\sqrt{x}}$
b i $\left(\sqrt{x}+1\right)^3 + C$ **ii** $\frac{2}{3}\left(\sqrt{x}+1\right)^3 + C$

11 a $12(x^2+2x)(x^3+3x^2+5)^3$
b i $(x^3+3x^2+5)^4 + C$
ii $\frac{1}{12}(x^3+3x^2+5)^4 + C$

12 a $-7(2x+1)(5-x^2-x)^6$
b i $(5-x^2-x)^7 + C$
ii $-\frac{1}{7}(5-x^2-x)^7 + C$

13 a $\frac{1}{4}(5x+4)^4 + C$ **b** $\frac{1}{6}(1-3x)^6 + C$
c $\frac{1}{8}(x^2-5)^8 + C$ **d** $\frac{1}{5}(x^3+7)^5 + C$
e $\frac{-1}{3x^2+2} + C$ **f** $2\sqrt{9-2x^3} + C$

14 a $\frac{1}{3}(5x^2+3)^3 + C$ **b** $\frac{1}{4}(x^2+1)^4 + C$
c $\frac{1}{6}(1+4x^3)^6 + C$ **d** $\frac{1}{30}(1+3x^2)^5 + C$
e $-\frac{1}{32}(1-x^4)^8 + C$ **f** $\frac{2}{3}(x^3-1)^{\frac{3}{2}} + C$
g $\frac{1}{15}(5x^2+1)^{\frac{3}{2}} + C$ **h** $2\sqrt{x^2+3} + C$
i $\frac{1}{4}\sqrt{4x^2+8x+1} + C$ **j** $-\dfrac{1}{4(x^2+5)^2} + C$

15 a $\frac{32}{15}$ **b** $\frac{7}{144}$ **c** $\frac{1}{12}$ **d** 936

Chapter 4 review exercise

1 a 1 **b** $\frac{3}{2}$ **c** 609 **d** $\frac{2}{5}$ **e** -12
f $8\frac{2}{3}$ **g** 8 **h** -10 **i** $21\frac{1}{3}$

2 a $4\frac{2}{3}$ **b** $-1\frac{2}{3}$ **c** $\frac{1}{3}$

3 a $-1\frac{1}{2}$ **b** 15 **c** $-6\frac{1}{6}$

4 a ii $k = 6$
b ii $k = 3$

5 a 0. The integral has zero width.
b 0. The integrand is odd.
c 0. The integrand is odd.

6 a 8 **b** $\frac{3}{2}$

7 a i $4x - \frac{1}{2}x^2 + 10$ **ii** $\frac{1}{2} - x^{-1}$
b i $4 - x$ **ii** x^{-2}
c i $x^5 - 5x^3 + 1$ **ii** $\dfrac{x^2+4}{x^2-1}$

8 a $\frac{x^2}{2} + 2x + C$
b $\frac{x^4}{4} + x^3 - \frac{5x^2}{2} + x + C$
c $\frac{x^3}{3} - \frac{x^2}{2} + C$
d $-\frac{x^3}{2} + \frac{5x^2}{2} - 6x + C$
e $-x^{-1} + C$
f $-\frac{1}{6x^6} + C$
g $\dfrac{2x^{\frac{2}{3}}}{3} + C$
h $\frac{1}{5}(x+1)^5 + C$
i $\frac{1}{12}(2x-3)^6 + C$

9 a $9\frac{1}{3}\,\text{u}^2$ **b** $4\,\text{u}^2$ **c** $\frac{4}{3}\,\text{u}^2$ **d** $1\,\text{u}^2$
e $\frac{1}{6}\,\text{u}^2$ **f** $\frac{4}{15}\,\text{u}^2$ **g** $\frac{1}{6}\,\text{u}^2$ **h** $4\frac{1}{2}\,\text{u}^2$

10 b $\frac{4}{3}\,\text{u}^2$

11 a 9 **b** 0.56

12 a $18(3x+4)^5$
b i $(3x+4)^6 + C$
ii $\frac{1}{2}(3x+4)^6 + C$

13 a $6x(x^2-1)^2$
b i $(x^2-1)^3 + C$
ii $\frac{1}{6}(x^2-1)^3 + C$

14 a $\frac{1}{5}(x^3+1)^5 + C$
b $-\frac{1}{2(x^2-5)^2} + C$

16 a There is an asymptote at $x = 1$.
b $\sqrt{9-3x}$ is undefined for $x > 3$.
c $\log_e(x-2)$ is undefined for $x \le 2$ (or, there is an asymptote at $x = 2$).
d There is an asymptote at $x = 1$.

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Chapter 5

Exercise 5A

1 a 2^{10} **b** e^7 **c** 2^4
d e^3 **e** 2^{12} **f** e^{30}

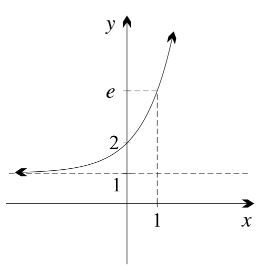
2 a e^{7x} **b** e^{2x} **c** e^{10x}
d e^{-5x} **e** e^{-3x} **f** e^{-12x}

3 a 7.389 **b** 0.04979 **c** $e^1 \doteqdot 2.718$
d $e^{-1} \doteqdot 0.3679$ **e** $e^{\frac{1}{2}} \doteqdot 1.649$ **f** $e^{-\frac{1}{2}} \doteqdot 0.6065$

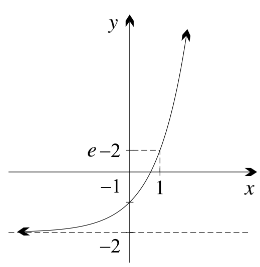
4 a $y' = e^x$ and $y'' = e^x$
b 'The curve $y = e^x$ is always concave up, and is always increasing at an increasing rate.'

5 a gradient $= e$, $y = ex$.
b $y = x + 1$
c $y = \frac{1}{e}(x + 2)$

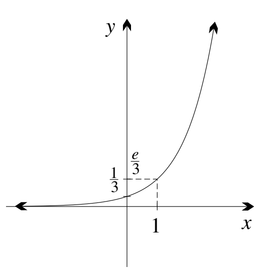
6 a $P = (1, e - 1)$
b $\dfrac{dy}{dx} = e^x$. When $x = 1$, $\dfrac{dy}{dx} = e$.
c tangent: $ex - y - 1 = 0$,
normal: $x + ey - e^2 + e - 1 = 0$

7 a


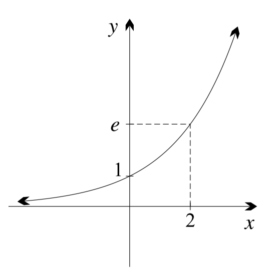
Shift e^x up 1

b


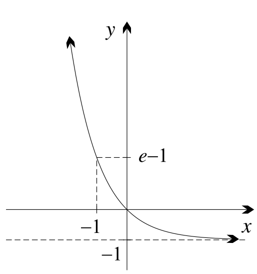
Shift e^x down 2

c


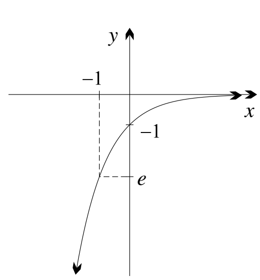
Stretch e^x vertically with factor $\frac{1}{3}$

d


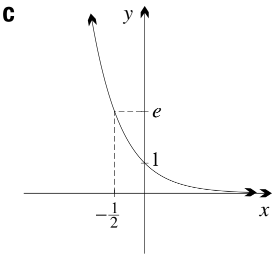
Stretch e^x horizontally with factor 2

8 a


Shift e^{-x} down 1

b


Reflect e^{-x} in x-axis

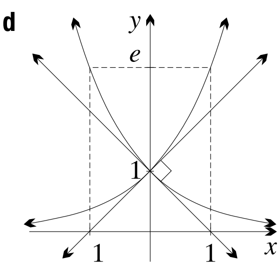
c


Stretch e^{-x} horizontally with factor $-\frac{1}{2}$

9 It is a vertical dilation of $y = e^x$ with factor $-\frac{1}{3}$. Its equation is $y = -\frac{1}{3}e^x$.

10 a $e^{2x} - 1$ **b** $e^{6x} + 3e^{4x} + 3e^{2x} + 9$
c $1 - 2e^{3x}$ **d** $e^{-4x} + 2 + e^{4x}$

11 a $e^{2x} + e^x$ **b** $e^{-2x} - e^{-x}$
c $e^{20x} + 5e^{30x}$ **d** $2e^{-4x} + 3e^{-5x}$

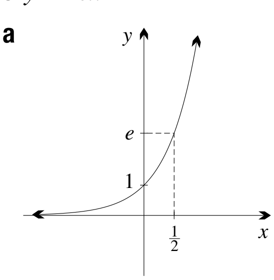
12 a 1
b Reflection in y-axis
c -1
d


e Horizontal dilation with factor -1

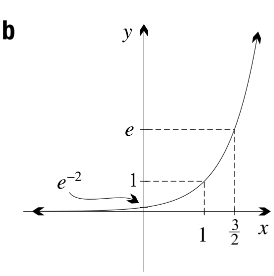
13 a e^x, e^x, e^x, e^x
b $e^x + 3x^2, e^x + 6x, e^x + 6, e^x$
c $4e^x, 4e^x, 4e^x, 4e^x$
d $5e^x + 10x, 5e^x + 10, 5e^x, 5e^x$. In part **c**, the gradient equals the height.

14 a 1, 45° **b** e, 69°48′
c e^{-2}, 7°42′ **d** e^5, 89°37′

15 a $e - 1$
b $\dfrac{dy}{dx} = e^x$. When $x = 1$, $\dfrac{dy}{dx} = e$.
c $y = ex - 1$

16 a


Stretch horizontally with factor $\frac{1}{2}$.

b


Shift right 1.

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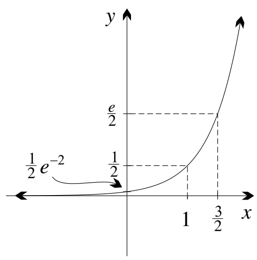
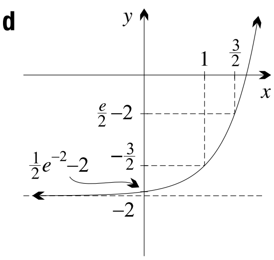
c



Stretch vertically with factor $\frac{1}{2}$.

d



Shift down 2.

17 a Shift left 2. Alternatively, $y = e^2e^x$, so it is a vertical dilation with factor e^2.

b Stretch vertically with factor 2. Alternatively, $y = e^{\log_e 2}e^x = e^{x+\log_e 2}$, so it is a shift left $\log_e 2$.

Exercise 5B

1 a $7e^{7x}$ **b** $12e^{3x}$ **c** $-5e^{5x}$ **d** $2e^{\frac{1}{3}x}$ **e** $7e^{-7x}$ **f** e^{-2x}

2 a $y' = e^{x-3}$ **b** $y' = 3e^{3x+4}$
c $y' = 2e^{2x-1}$ **d** $y' = 4e^{4x-3}$
e $y' = -3e^{-3x+4}$ **f** $y' = -2e^{-2x-7}$

3 a $e^x - e^{-x}$ **b** $2e^{2x} + 3e^{-3x}$
c $\frac{e^x + e^{-x}}{2}$ **d** $\frac{e^x - e^{-x}}{3}$
e $e^{2x} + e^{3x}$ **f** $e^{4x} + e^{5x}$

4 a $y' = 3e^{3x}$ **b** $y' = 2e^{2x}$
c $y' = 2e^{2x}$ **d** $y' = 6e^{6x}$
e $y' = 3e^{3x}$ **f** $y' = -e^{-x}$
g $y' = -3e^{-3x}$ **h** $y' = -5e^{-5x}$

5 a i $-e^{-x}, e^{-x}, -e^{-x}, e^{-x}$
ii Successive derivatives alternate in sign. More precisely, $f^{(n)}(x) = \begin{cases} e^{-x}, & \text{if } n \text{ is even,} \\ -e^{-x}, & \text{if } n \text{ is odd.} \end{cases}$

b i $2e^{2x}, 4e^{2x}, 8e^{2x}, 16e^{2x}$
ii Each derivative is twice the previous one. More precisely, $f^{(n)}(x) = 2^ne^{2x}$.

6 a $2e^{2x} + e^x$ **b** $e^{-x} - 4e^{-2x}$
c $2e^{2x} + 2e^x$ **d** $2e^{2x} + 6e^x$
e $2e^{2x} - 2e^x$ **f** $2e^{2x} - 4e^x$
g $2(e^{2x} + e^{-2x})$ **h** $10(e^{10x} + e^{-10x})$

7 a $2e^{2x+1}$ **b** $2xe^{x^2}$
c $-xe^{-\frac{1}{2}x}$ **d** $2xe^{x^2+1}$
e $-2xe^{1-x^2}$ **f** $2(x + 1)e^{x^2+2x}$
g $(1 - 2x)e^{6+x-x^2}$ **h** $(3x - 1)e^{3x^2-2x+1}$

8 a $(x + 1)e^x$ **b** $(1 - x)e^{-x}$
c xe^x **d** $(3x + 4)e^{3x-4}$
e $(2x - x^2)e^{-x}$ **f** $4xe^{2x}$
g $(x^2 + 2x - 5)e^x$ **h** $x^2e^{2x}(3 + 2x)$

9 a $y' = \frac{x-1}{x^2}e^x$ **b** $y' = (1 - x)e^{-x}$
c $y' = \frac{(x-2)e^x}{x^3}$ **d** $y' = (2x - x^2)e^{-x}$
e $y' = \frac{x}{(x+1)^2}e^x$ **f** $y' = -xe^{-x}$
g $y' = (7 - 2x)e^{-2x}$
h $y' = (x^2 - 2x - 1)e^{-x}$

10 a $2e^{2x} + 3e^x$ **b** $4e^{4x} + 2e^{2x}$
c $-2e^{-2x} - 6e^{-x}$ **d** $-6e^{-6x} + 18e^{-3x}$
e $3e^{3x} + 2e^{2x} + e^x$ **f** $12e^{3x} + 2e^{2x} + e^{-x}$

11 a $-5e^x(1 - e^x)^4$ **b** $16e^{4x}(e^{4x} - 9)^3$
c $-\frac{e^x}{(e^x-1)^2}$ **d** $-\frac{6e^{3x}}{(e^{3x}+4)^3}$

13 a $f'(x) = 2e^{2x+1}$, $f'(0) = 2e$, $f''(x) = 4e^{2x+1}$, $f''(0) = 4e$
b $f'(x) = -3e^{-3x}$, $f'(1) = -3e^{-3}$, $f''(x) = 9e^{-3x}$, $f''(1) = 9e^{-3}$
c $f'(x) = (1 - x)e^{-x}$, $f'(2) = -e^{-2}$, $f''(x) = (x - 2)e^{-x}$, $f''(2) = 0$
d $f'(x) = -2xe^{-x^2}$, $f'(0) = 0$, $f''(x) = (4x^2 - 2)e^{-x^2}$, $f''(0) = -2$

14 a $y' = ae^{ax}$ **b** $y' = -ke^{-kx}$
c $y' = Ake^{kx}$ **d** $y' = -B\ell e^{-\ell x}$
e $y' = pe^{px+q}$ **f** $y' = pCe^{px+q}$
g $y' = \dfrac{pe^{px} - qe^{-qx}}{r}$ **h** $e^{ax} - e^{-px}$

15 a $3e^x(e^x + 1)^2$
b $4(e^x - e^{-x})(e^x + e^{-x})^3$
c $(1 + 2x + x^2)e^{1+x} = (1 + x)^2e^{1+x}$
d $(2x^2 - 1)e^{2x-1}$
e $\frac{e^x}{(e^x+1)^2}$
f $-\frac{2e^x}{(e^x-1)^2}$

16 a $y' = -e^{-x}$
b $y' = e^x$
c $y' = e^{-x} - 4e^{-2x}$
d $y' = -12e^{-4x} - 3e^{-3x}$
e $y' = e^x - 9e^{3x}$
f $y' = -2e^{-x} - 2e^{-2x}$

17 a $y' = \frac{1}{2}\sqrt{e^x}$ **b** $y' = \frac{1}{3}\sqrt{e^x}$
c $y' = -\frac{1}{2\sqrt{e^x}}$ **d** $y' = -\frac{1}{3\sqrt[3]{e^x}}$
e $\frac{1}{2\sqrt{x}}e^{\sqrt{x}}$ **f** $-\frac{1}{2\sqrt{x}}e^{-\sqrt{x}}$
g $-\frac{1}{x^2}e^{\frac{1}{x}}$ **h** $\frac{1}{x^2}e^{-\frac{1}{x}}$
i $\left(1 + \frac{1}{x^2}\right)e^{x-\frac{1}{x}}$ **j** $e^xe^{e^x} = e^{x+e^x}$

20 a -5 or 2
b $-\frac{1}{2}\left(1 + \sqrt{5}\right)$ or $-\frac{1}{2}\left(1 - \sqrt{5}\right)$

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Exercise 5C

1 a $A = \left(\frac{1}{2}, 1\right)$ **b** $y' = 2e^{2x-1}$ **c** $y = 2x$

2 a $R = \left(-\frac{1}{3}, 1\right)$ **b** $y' = 3e^{3x+1}$

c $-\frac{1}{3}$ **d** $3x + 9y - 8 = 0.$

3 a $-e$ **b** $\frac{1}{e}$

c $x - ey + e^2 + 1 = 0$

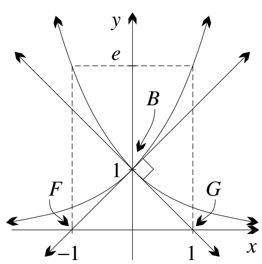
d $x = -e^2 - 1, y = e + e^{-1}$

e $\frac{1}{2}(e^3 + 2e + e^{-1})$

4 a 1 **b** $y = x + 1$

c -1 **d** $y = -x + 1$

e



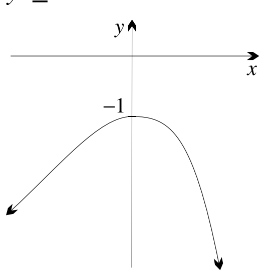
f isosceles right triangle, 1 square unit

5 a $1 - e$ **b** $y = (1 - e)x$

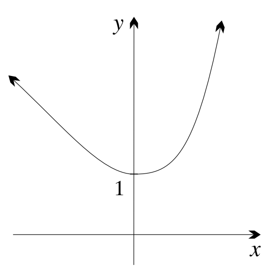
6 a $y' = 1 - e^x, y'' = -e^x$

c maximum turning point at $(0, -1)$

d $y \leq -1$



e



7 a $y' = -xe^x$

b $y = e^{-1}(x + 3)$

c -3

8 a The x-intercept is -1 and the y-intercept is e.

b Area $= \frac{1}{2}e$

9 a $y' = 3e^{3x-6}, y' = 9e^{3x-6}$

b $3e^{3x-6}$ and $9e^{3x-6}$ are always positive.

c $(2, 1)$ **d** $3e^{-6}, -\frac{1}{3}e^6$

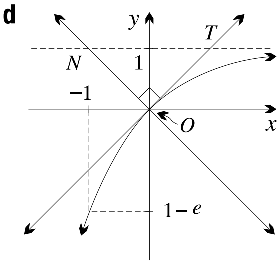
10 a $y' = -2xe^{-x^2}$

b $ex - 2y + (2e^{-1} - e) = 0$

c $1 - 2e^{-2}$

11 b $y = -x$ **c** $y = 1$

d



e 1 square unit

12 b $y = e^t(x - t + 1)$

c The x-intercept of each tangent to $y = e^x$ is 1 unit left of the x-value of the point of contact.

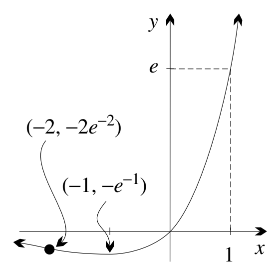
13 b $\lim\limits_{x \to -\infty} xe^x = 0$

d $\lim\limits_{x \to \infty} xe^{-x} = 0$

14 a There is a zero at $x = 0$, it is positive for $x > 0$ and negative for $x < 0$. It is neither even nor odd.

e They all tend towards ∞.

f $y \geq -e^{-1}$



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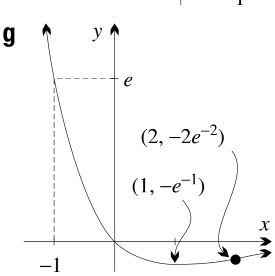
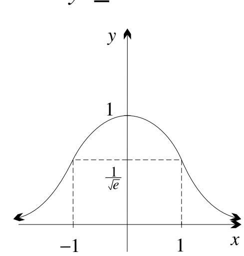


15 a It is always positive.

d $(1, e^{\frac{1}{2}})$ and $(-1, e^{\frac{1}{2}})$

f $0 < y \leq 1$



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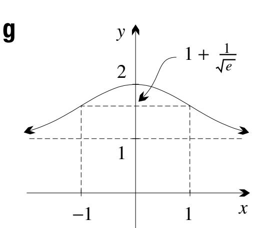

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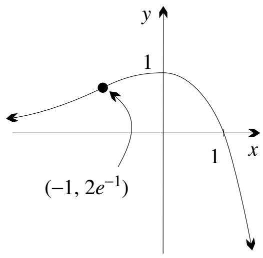
16 a

x	0	1	2
y	1	0	$-e^2$
sign	$+$	0	$-$

d They all tend to ∞.

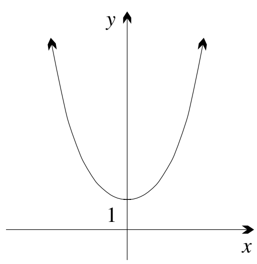
e $y \le 1$



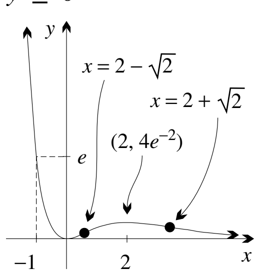
17 b $y' = \frac{e^x - e^{-x}}{2}$

d As $x \to \infty$, $y \to \infty$.

e



18 d $y \ge 0$



Exercise 5D

1 a $\frac{1}{2}e^{2x} + C$ **b** $\frac{1}{3}e^{3x} + C$

c $3e^{\frac{1}{3}x} + C$ **d** $2e^{\frac{1}{2}x} + C$

e $5e^{2x} + C$ **f** $4e^{3x} + C$

2 a $\frac{1}{4}e^{4x+5} + C$ **b** $\frac{1}{4}e^{4x-2} + C$

c $2e^{3x+2} + C$ **d** $e^{4x+3} + C$

e $-\frac{1}{2}e^{7-2x} + C$ **f** $-\frac{1}{6}e^{1-3x} + C$

3 a $e - 1$ **b** $e^2 - e$

c $e - e^{-3}$ **d** $e^2 - 1$

e $\frac{1}{2}(e^4 - 1)$ **f** $\frac{1}{3}(e^3 - e^{-3})$

g $4(e^5 - e^{-10})$ **h** $2(e^{12} - e^{-4})$

i $\frac{3}{2}(e^{18} - e^{-6})$

4 a $e - e^{-1}$ **b** $\frac{1}{2}(e^3 - e^{-1})$

c $\frac{1}{4}(e^{-3} - e^{-11})$ **d** $\frac{1}{3}(e^{-1} - e^{-4})$

e $\frac{e^2}{2}(e^2 - 1)$ **f** $\frac{e}{3}(e^2 - 1)$

g $2e^4(e^3 - 1)$ **h** $3e^3(e^4 - 1)$

i $4e^2(e^3 - 1)$

5 a $-e^{-x} + C$ **b** $-\frac{1}{2}e^{-2x} + C$

c $-\frac{1}{3}e^{-3x} + C$ **d** $e^{-3x} + C$

e $-3e^{-2x} + C$ **f** $4e^{2x} + C$

6 a $f(x) = \frac{1}{2}e^{2x} + C$, for some constant C

b $C = -2\frac{1}{2}$, so $f(x) = \frac{1}{2}e^{2x} - 2\frac{1}{2}$

c $f(1) = \frac{1}{2}e^2 - 2\frac{1}{2}$, $f(2) = \frac{1}{2}e^4 - 2\frac{1}{2}$

7 a $f(x) = x + 2e^x - 1$, $f(1) = 2e$

b $f(x) = 2 + x - 3e^x$, $f(1) = 3 - 3e$

c $f(x) = 1 + 2x - e^{-x}$, $f(1) = 3 - e^{-1}$

d $f(x) = 1 + 4x + e^{-x}$, $f(1) = 5 + e^{-1}$

e $f(x) = \frac{1}{2}e^{2x-1} + \frac{5}{2}$, $f(1) = \frac{1}{2}(e + 5)$

f $f(x) = 1 - \frac{1}{3}e^{1-3x}$, $f(1) = 1 - \frac{1}{3}e^{-2}$

g $f(x) = 2e^{\frac{1}{2}x+1} - 6$, $f(1) = 2e^{\frac{3}{2}} - 6$

h $f(x) = 3e^{\frac{1}{3}x+2} - 1$, $f(1) = 3e^{\frac{7}{3}} - 1$

8 a $\frac{1}{2}e^{2x} + e^x + C$

b $\frac{1}{2}e^{2x} - e^x + C$

c $e^{-x} - e^{-2x} + C$

d $\frac{1}{2}e^{2x} + 2e^x + x + C$

e $\frac{1}{2}e^{2x} + 6e^x + 9x + C$

f $\frac{1}{2}e^{2x} - 2e^x + x + C$

g $\frac{1}{2}e^{2x} - 4e^x + 4x + C$

h $\frac{1}{2}(e^{2x} + e^{-2x}) + C$

i $\frac{1}{10}(e^{10x} + e^{-10x}) + C$

9 a $\frac{1}{2}e^{2x+b} + C$ **b** $\frac{1}{7}e^{7x+q} + C$

c $\frac{1}{3}e^{3x-k} + C$ **d** $\frac{1}{6}e^{6x-\lambda} + C$

e $\frac{1}{a}e^{ax+3} + C$ **f** $\frac{1}{s}e^{sx+1} + C$

g $\frac{1}{m}e^{mx-2} + C$ **h** $\frac{1}{k}e^{kx-1} + C$

i $e^{px+q} + C$ **j** $e^{mx+k} + C$

k $\frac{A}{s}e^{sx-t} + C$ **l** $\frac{B}{k}e^{kx-\ell} + C$

10 a $-e^{1-x} + C$ **b** $-\frac{1}{3}e^{1-3x} + C$

c $-\frac{1}{2}e^{-2x-5} + C$ **d** $-2e^{1-2x} + C$

e $2e^{5x-2} + C$ **f** $-4e^{5-3x} + C$

11 a $x - e^{-x} + C$ **b** $e^x - e^{-x} + C$

c $\frac{1}{2}e^{-2x} - e^{-x} + C$ **d** $e^{-3x} - \frac{1}{2}e^{-2x} + C$

e $e^{-3x} - e^{-2x} + C$ **f** $e^{-x} - e^{-2x} + C$

12 a $y = e^{x-1}$, $y = e^{-1}$

b $y = e^2 + 1 - e^{2-x}$, $y = e^2 + 1$

c $f(x) = e^x + \frac{x}{e} - 1$, $f(0) = 0$

d $f(x) = e^x - e^{-x} - 2x$

13 a $e^2 - e$

b $\frac{1}{2}(e^2 - e^{-2}) + 4(e - e^{-1}) + 8$

c $e + e^{-1} - 2$

d $\frac{1}{4}(e^4 - e^{-4}) + \frac{1}{2}(e^{-2} - e^2)$

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e $e - e^{-1}$
f $e - e^{-1} + \frac{1}{2}(e^{-2} - e^{2})$

14 a **i** $2xe^{x^2+3}$ **ii** $e^{x^2+3} + C$
b **i** $2(x - 1)\,e^{x^2-2x+3}$ **ii** $\frac{1}{2}e^{x^2-2x+3} + C$
c **i** $(6x + 4)\,e^{3x^2+4x+1}$ **ii** $\frac{1}{2}e^{3x^2+4x+1} + C$
d **i** $3x^2e^{x^3}$ **ii** $\frac{1}{3}(1 - e^{-1})$

15 a $-\frac{1}{2}e^{-2x} + C$ **b** $-\frac{1}{3}e^{-3x} + C$
c $2e^{\frac{1}{2}x} + C$ **d** $3e^{\frac{1}{3}x} + C$
e $-2e^{-\frac{1}{2}x} + C$ **f** $-3e^{-\frac{1}{3}x} + C$

16 a **i** $y' = xe^{x}$ **ii** $e^2 + 1$
b **i** $y' = -xe^{-x}$ **ii** $-1 - e^2$

17 a $2e^{\frac{1}{2}x} + \frac{2}{3}e^{-\frac{3}{2}x} + C$ **b** $\frac{3}{2}e^{\frac{2}{3}x} - \frac{3}{4}e^{-\frac{4}{3}x} + C$

18 b 0

Exercise 5E

1 a **i** $e - 1 \doteqdot 1.72$ **ii** $1 - e^{-1} \doteqdot 0.63$
iii $1 - e^{-2} \doteqdot 0.86$ **iv** $1 - e^{-3} \doteqdot 0.95$
b The total area is exactly 1.

2 a $(1 - e^{-1})$ square units
b $e(e^2 - 1)$ square units
c $(e - e^{-1})$ square units
d $e - e^{-2}$ square units

3 a **i** $\frac{1}{2}(e^6 - 1) \doteqdot 201.2$ square units
ii $\frac{1}{2}(1 - e^{-6}) \doteqdot 0.4988$ square units
b **i** $1 - e^{-1} \doteqdot 0.6321$ square units
ii $e - 1 \doteqdot 1.718$ square units
c **i** $3(e - 1) \doteqdot 5.155$ square units
ii $3(1 - e^{-1}) \doteqdot 1.896$ square units

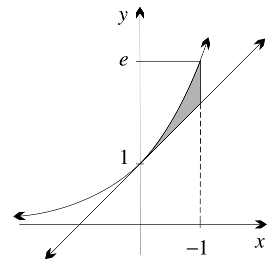
4 a $e(e^2 - 1)\,\text{u}^2$ **b** $e(e^2 - 1)\,\text{u}^2$
c $\frac{1}{2}(e - e^{-1})\,\text{u}^2$ **d** $\frac{1}{3}(e - e^{-2})\,\text{u}^2$
e $(e^2 - 1)\,\text{u}^2$ **f** $\frac{1}{2}e(e^2 - 1)\,\text{u}^2$
g $3e^2(e - 1)\,\text{u}^2$ **h** $2(1 - e^{-2})\,\text{u}^2$

5 a $(e^2 - 1)\,\text{u}^2$ **b** $2(e - e^{-\frac{1}{2}})\text{u}^2$
c $(1 - e^{-1})\,\text{u}^2$ **d** $2(e^{\frac{1}{2}} - e^{-1})\text{u}^2$

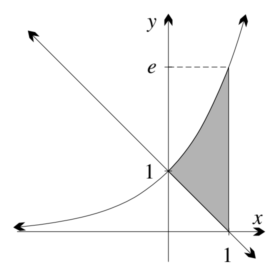
6 a $(3 - e^{-2}) \doteqdot 2.865\,\text{u}^2$ **b** $e^{-1} \doteqdot 0.3679\,\text{u}^2$
c $2(e^2 - e^{-2}) \doteqdot 14.51\,\text{u}^2$
d $18 + e^3 - e^{-3} \doteqdot 38.04\,\text{u}^2$

7 a $(1 + e^{-2})\,\text{u}^2$ **b** $1\,\text{u}^2$
c $e^{-1}\,\text{u}^2$ **d** $(3 + e^{-2})\,\text{u}^2$
e $1\,\text{u}^2$
f $(9 + e^{-2} - e)\,\text{u}^2$

8 a $\int_0^1 (e^x - 1 - x)\,dx = \left(e - 2\frac{1}{2}\right)\text{u}^2$



b $\int_0^1 (e^x - 1 + x)\,dx = \left(e - 1\frac{1}{2}\right)\text{u}^2$

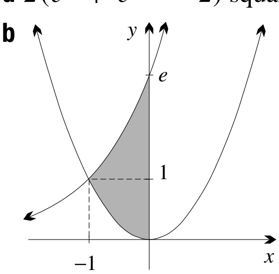


9 a The region is symmetric, so the area is twice the area in the first quadrant.
b $2 - \frac{2}{e}$ square units

10 a The region is symmetric, so the area is twice the area in the first quadrant.
b 2 square units

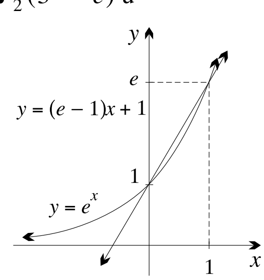
11 b 0
c The region is symmetric, so the area is twice the area in the first quadrant.
d $2(e^3 + e^{-3} - 2)$ square units

12 b



c $\left(e - 1\frac{1}{3}\right)\text{u}^2$

13 b $\frac{1}{2}(3 - e)\,\text{u}^2$



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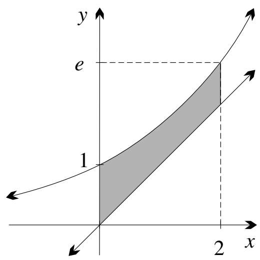
14 $(e^2 - 3)$ square units



15 a $e - 1 \doteqdot 1.7183$ **b** 1.7539

c The trapezoidal-rule approximation is greater. The curve is concave down, so all the chords are above the curve.

16 a 0.8863 square units **b** 3.5726 square units

17 a **i** $1 - e^N$ **ii** 1

b **i** $1 - e^{-N}$ **ii** 1

18 a $-\frac{1}{2}e^{-x^2}$

b From $x = 0$ to $x = 2$, area $= \frac{1}{2} - \frac{1}{2}e^{-4}$ square units. The function is odd, so the area (not signed) from $x = -2$ to $x = 2$ is $1 - e^{-4}$ square units.

Exercise 5F

1 a 2.303 **b** -2.303 **c** 11.72

d -12.02 **e** 3.912 **f** -3.912

2 a 3 **b** -1 **c** -2 **d** $\frac{1}{2}$

e 5 **f** 0.05 **g** 1 **h** e

3 b $1 = e^0$, so $\log_e 1 = \log_e e^0 = 0$.

d $e = e^1$, so $\log_e e = \log_e e^1 = 1$.

4 a $\log_e x = 6$ **b** $x = e^{-2}$ or $x = 1/e^2$

c $e^x = 24$ **d** $x = \log_e \frac{1}{3}$

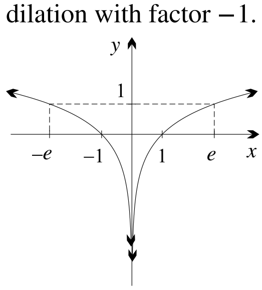
5 a $\dfrac{\log_e 7}{\log_e 2} \doteqdot 2.807$ **b** $\dfrac{\log_e 25}{\log_e 10} \doteqdot 1.398$

c $\dfrac{\log_e 0.04}{\log_e 3} \doteqdot -2.930$

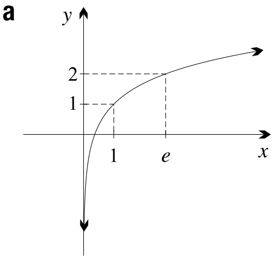
6 a Reflection in $y = x$, which reflects lines with gradient 1 to lines of gradient 1. The tangent to $y = e^x$ at its y-intercept has gradient 1, so its reflection also has gradient 1.

b Reflection in the y-axis, which is also a horizontal dilation with factor -1.

c

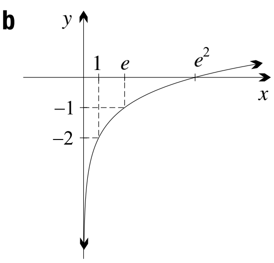


7 a



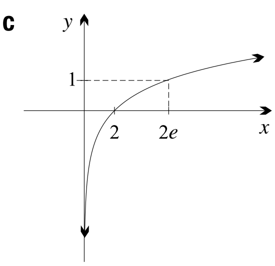
Shift $y = \log_e x$ up 1.

b



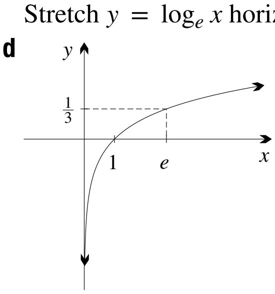
Shift $y = \log_e x$ down 2.

c



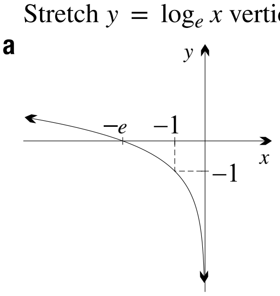
Stretch $y = \log_e x$ horizontally with factor 2.

d



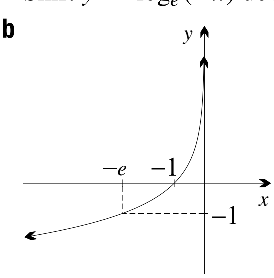
Stretch $y = \log_e x$ vertically with factor $\frac{1}{3}$.

8 a



Shift $y = \log_e(-x)$ down 1.

b



Reflect $y = \log_e(-x)$ in the x-axis.

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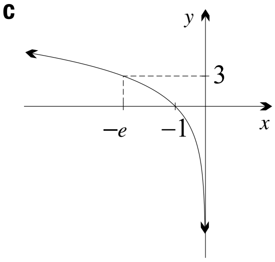
c



Stretch $y = \log_e(-x)$ vertically with factor 3.

9 It is a horizontal dilation of $y = \log_e(-x)$ with factor $\frac{1}{2}$. Its equation is $y = \log_e(-2x)$.

10 a $x = 1$ or $x = \log_2 7$

b $x = 2$ ($3^x = -1$ has no solutions.)

c i $x = 2$ or $x = 0$

ii $x = 0$ or $x = \log_3 4$

iii $x = \log_3 5$ ($3^x = -4$ has no solutions.)

iv The quadratic has no solutions because $\triangle < 0$

v $x = 2$

vi $x = 1$ or 2

11 a $x = 0$

b $x = \log_e 2$

c $x = 0$ or $x = \log_e 3$

d $x = 0$

12 a $x = e$ or $x = e^4$ **b** $x = 1$ or $x = e^3$

13 a i e **ii** 1 **iii** 0

b i $\ln 20$ **ii** $\ln 5$

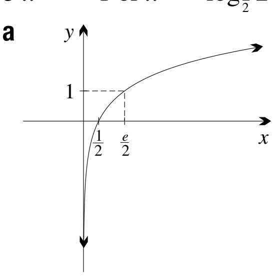
iii $\ln 8$ or $3 \ln 2$

14 a $x = 1$ or $x = \log_4 3 \doteqdot 0.792$

b $x = \log_{10} \frac{1 + \sqrt{5}}{2} \doteqdot 0.209$. $\log_{10} \frac{1 - \sqrt{5}}{2}$ does not exist because $\frac{1 - \sqrt{5}}{2}$ is negative.

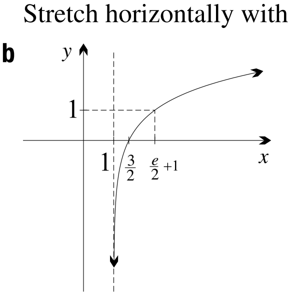
c $x = -1$ or $x = \log_{\frac{1}{2}} 2 \doteqdot -0.431$

15 a



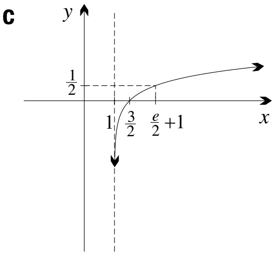
Stretch horizontally with factor $\frac{1}{2}$.

b



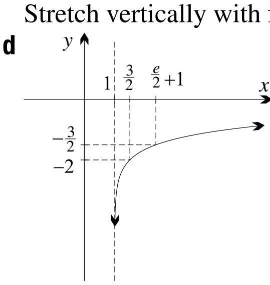
Shift right 1.

c



Stretch vertically with factor $\frac{1}{2}$.

d



Shift down 2.

16 a Stretch horizontally with factor $\frac{1}{5}$. Alternatively, $y = \log_e x + \log_e 5$, so it is a shift up $\log_e 5$.

b Shift up 2. Alternatively, $y = \log_e x + \log_e e^2 = \log_e e^2 x$, so it is a dilation horizontally with factor e^{-2}.

Exercise 5G

1 a $y' = \frac{1}{x + 2}$ **b** $y' = \frac{1}{x - 3}$

c $y' = \frac{3}{3x + 4}$ **d** $y' = \frac{2}{2x - 1}$

e $y' = \frac{-4}{-4x + 1}$ **f** $y' = \frac{-3}{-3x + 4}$

g $y' = \frac{-2}{-2x - 7} = \frac{2}{2x + 7}$

h $y' = \frac{6}{2x + 4} = \frac{3}{x + 2}$

i $y' = \frac{15}{3x - 2}$

2 a $y = \log_e 2 + \log_e x$, $y' = \frac{1}{x}$

b $y = \log_e 5 + \log_e x$, $y' = \frac{1}{x}$

c $\frac{1}{x}$ **d** $\frac{1}{x}$ **e** $\frac{4}{x}$ **f** $\frac{3}{x}$ **g** $\frac{4}{x}$ **h** $\frac{3}{x}$

3 a $y' = \frac{1}{x + 1}$, $y'(3) = \frac{1}{4}$

b $y' = \frac{2}{2x - 1}$, $y'(3) = \frac{2}{5}$

c $y' = \frac{2}{2x - 5}$, $y'(3) = 2$

d $y' = \frac{4}{4x + 3}$, $y'(3) = \frac{4}{15}$

e $y' = \frac{5}{x + 1}$, $y'(3) = \frac{5}{4}$

f $y' = \frac{12}{2x + 9}$, $y'(3) = \frac{4}{5}$

4 a $\frac{1}{x}$ **b** $\frac{-1}{x + 1}$

c $1 + \frac{4}{x}$ **d** $8x^3 + \frac{3}{x}$

e $\frac{2}{2x - 1} + 6x$ **f** $3x^2 - 3 + \frac{5}{5x - 7}$

5 a $y = 3 \ln x$, $y' = \frac{3}{x}$

b $y = 2 \ln x$, $y' = \frac{2}{x}$

c $y = -3 \ln x$, $y' = -\frac{3}{x}$

d $y = -2 \ln x$, $y' = -\frac{2}{x}$

e $y = \frac{1}{2} \ln x$, $y' = \frac{1}{2x}$

f $y = \frac{1}{2} \ln(x + 1)$, $y' = \frac{1}{2(x + 1)}$

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6 a $\frac{1}{x}$ **b** $\frac{1}{x}$ **c** $\frac{3}{x}$
d $-\frac{6}{x}$ **e** $1 + \frac{1}{x}$ **f** $12x^2 - \frac{1}{x}$

7 a $\frac{2x}{x^2 + 1}$ **b** $\frac{-2x}{2 - x^2}$ **c** $\frac{e^x}{1 + e^x}$

8 a $\frac{2x + 3}{x^2 + 3x + 2}$ **b** $\frac{6x^2}{1 + 2x^3}$ **c** $\frac{e^x}{e^x - 2}$
d $1 - \frac{2x + 1}{x^2 + x}$ **e** $2x + \frac{3x^2 - 1}{x^3 - x}$
f $12x^2 - 10x + \frac{4x - 3}{2x^2 - 3x + 1}$

9 a $1, 45°$ **b** $\frac{1}{3}, 18°26'$
c $2, 63°26'$ **d** $\frac{1}{4}, 14°2'$

10 a $1 + \log_e x$ **b** $\frac{2x}{2x + 1} + \log_e(2x + 1)$
c $\frac{2x + 1}{x} + 2\log_e x$ **d** $x^3(1 + 4\log_e x)$
e $\log_e(x + 3) + 1$ **f** $\frac{2(x - 1)}{2x + 7} + \log_e(2x + 7)$
g $e^x\left(\frac{1}{x} + \log_e x\right)$ **h** $e^{-x}\left(\frac{1}{x} - \log_e x\right)$

11 a $\frac{1 - \log_e x}{x^2}$ **b** $\frac{1 - 2\log_e x}{x^3}$ **c** $\frac{\log_e x - 1}{(\log_e x)^2}$
d $\frac{x(2\log_e x - 1)}{(\log_e x)^2}$ **e** $\frac{1 - x\log_e x}{xe^x}$ **f** $\frac{e^x(x\log_e x - 1)}{x(\log_e x)^2}$

12 a $\frac{3}{x}$ **b** $\frac{4}{x}$ **c** $\frac{1}{3x}$ **d** $\frac{1}{4x}$
e $-\frac{1}{x}$ **f** $-\frac{1}{x}$ **g** $\frac{1}{2x - 4}$ **h** $\frac{5}{10x + 4}$

13 a $f'(x) = \frac{1}{x - 1}, f'(3) = \frac{1}{2}, f''(x) = -\frac{1}{(x - 1)^2}$,
$f''(3) = -\frac{1}{4}$
b $f'(x) = \frac{2}{2x + 1}, f'(0) = 2, f''(x) = -\frac{4}{(2x + 1)^2}$,
$f''(0) = -4$
c $f'(x) = \frac{2}{x}, f'(2) = 1, f''(x) = -\frac{2}{x^2}, f''(2) = -\frac{1}{2}$
d $f'(x) = 1 + \log x, f'(e) = 2, f''(x) = \frac{1}{x}$,
$f''(e) = \frac{1}{e}$

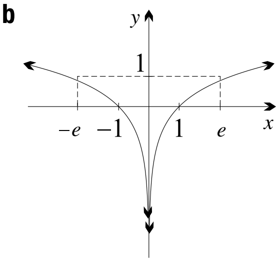
14 a $\log_e x, x = 1$
b $x(1 + 2\log_e x), x = e^{-\frac{1}{2}}$
c $\frac{1 - \log_e x}{x^2}, x = e$
d $\frac{2\log_e x}{x}, x = 1$
e $\frac{4(\log_e x)^3}{x}, x = 1$
f $\frac{-1}{x(1 + \log_e x)^2}$ is never zero.
g $\frac{8}{x}(2\log_e x - 3)^3, x = e^{\frac{3}{2}}$
h $\frac{-1}{x(\log_e x)^2}$ is never zero.
i $\frac{1}{x\log_e x}$ is never zero.

15 a $\left(\frac{1}{e}, -\frac{1}{e}\right)$ **b** $(1, 1)$

16 a $y' = \dfrac{\ln x - 1}{(\ln x)^2}$

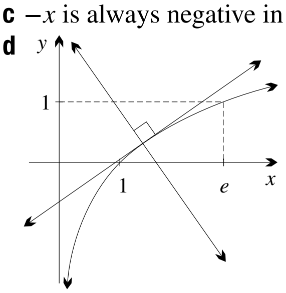
17 a $\frac{1}{x + 2} + \frac{1}{x + 1}$ **b** $\frac{1}{x + 5} + \frac{3}{3x - 4}$ **c** $\frac{1}{1 + x} + \frac{1}{1 - x}$
d $\frac{3}{3x - 1} - \frac{1}{x + 2}$ **e** $\frac{2}{x - 4} - \frac{3}{3x + 1}$ **f** $\frac{1}{x} + \frac{1}{2(x + 1)}$

18 a $y = x\log_e 2, y' = \log_e 2$
b $y = x, y' = 1$
c $y = x\log_e x, y' = 1 + \log_e x$

19 a $\log_e|x| = \begin{cases} \log_e x, & \text{for } x > 0, \\ \log_e(-x), & \text{for } x < 0. \end{cases}$
b


c For $x > 0$, $\log_e|x| = \log_e x$, so $\dfrac{d}{dx}\log_e x = \dfrac{1}{x}$.
For $x < 0$, $\log_e|x| = \log_e(-x)$, and using the standard form, $\dfrac{d}{dx}\log_e(-x) = -\dfrac{1}{-x} = \dfrac{1}{x}$.
d $\log_e 0$ is undefined. In fact, $\log_e x \to -\infty$ as $x = 0$, so $x = 0$ is an asymptote.

Exercise 5H

1 a $y' = \frac{1}{x}$ **b** $\frac{1}{e}$ **c** $y = \frac{1}{e}x$
2 a 1 **b** $y = x - 1$
3 a e **b** $y = ex - 2$
4 a 1
c $y = -x + 1$. When $x = 0, y = 1$.
5 a $y = 4x - 4, y = -\frac{1}{4}x + \frac{1}{4}$
b $y = x + 2, y = -x + 4$
c $y = 2x - 4, y = -\frac{1}{2}x - 1\frac{1}{2}$
d $y = -3x + 4, y = \frac{1}{3}x + \frac{2}{3}$
6 b $3, -\frac{1}{3}$
c $y = 3x - 3, -3, y = -\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}$
d $\frac{5}{3}$ square units
7 a $\frac{1}{2}$ **b** $y = \frac{1}{2}x$
8 a $y = -\ln 2 \times (x - 2)$ **b** $2\ln 2$
9 a $x > 0, y = \frac{1}{x}$
b $\frac{1}{x}$ is always positive in the domain.
c $-x$ is always negative in the domain.
d


e $y'' = -\frac{1}{x^2}$. It is always concave down.
10 a $(2, \log_e 2), y = \frac{1}{2}x - 1 + \log_e 2$,
$y = -2x + 4 + \log_e 2$
b $\left(\frac{1}{2}, -\log_e 2\right), y = 2x - 1 - \log_e 2$,
$y = -\frac{1}{2}x + \frac{1}{4} - \log_e 2$

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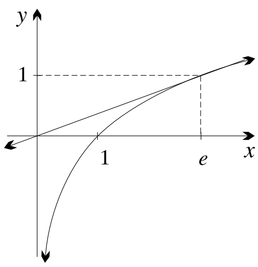
11 a As P moves left along the curve, the tangent becomes steeper, so it does not pass through the origin.
As P moves right, the angle of the tangent becomes less steep, hence it does not pass through the origin.



b There are no tangents through each point below the curve. There are two tangents through each point above the curve and to the right of the y-axis. There is one tangent through each point on the curve, and through each point on and to the left of the y-axis.

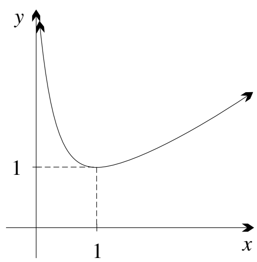
12 a $x > 0$. The domain is not symmetric about the origin, so the function is certainly not even or odd.

b $y' = 1 - \frac{1}{x}$, $y'' = \frac{1}{x^2}$

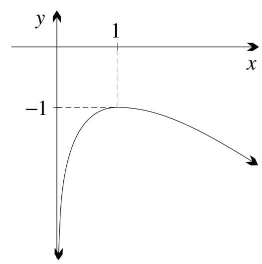
c $y'' > 0$, for all x

d $(1, 1)$

e $y \geq 1$

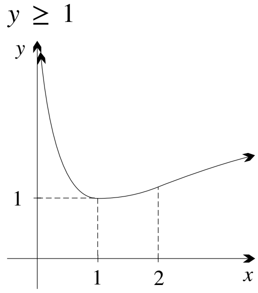


f



13 a $x > 0$

d $y \geq 1$



14 a $\lim\limits_{x \to \infty} \dfrac{\log_e x}{x} = 0$

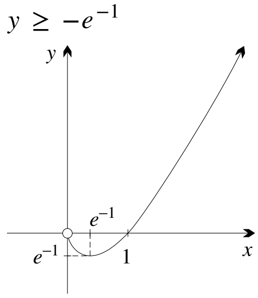
b $\lim\limits_{x \to 0^+} x \log_e x = 0$

15 a $x > 0$, $(1, 0)$

b $y'' = \frac{1}{x}$

c $(e^{-1}, -e^{-1})$ is a minimum turning point.

d $y \geq -e^{-1}$



16 a $x > 0$, $(e, 0)$

b

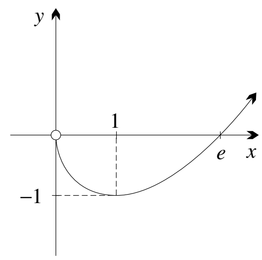
x	1	e	e^2
y	-1	0	e^2
sign	$-$	0	$+$

c $y'' = \dfrac{1}{x}$

d $(1, -1)$ is a minimum turning point.

e It is concave up throughout its domain.

f $y \geq -1$



17 a all real x

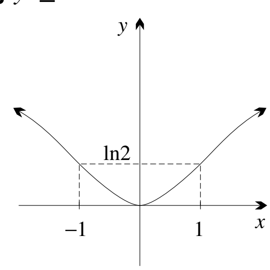
b Even

c It is zero at $x = 0$, and is positive otherwise because the logs of numbers greater than 1 are positive.

e $(0, 0)$ is a minimum turning point.

f $(1, \log_e 2)$ and $(-1, \log_e 2)$

g $y \geq 0$

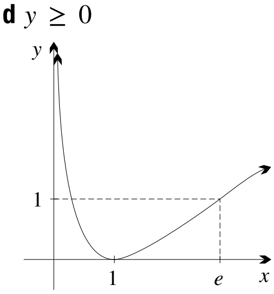


18 a $x > 0$

b It is zero at $x = 1$, and is positive otherwise because squares cannot be negative.

c $y' = \frac{2}{x} \ln x$

d $y \geq 0$



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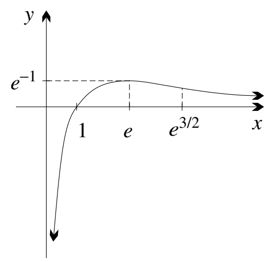
19 a $x > 0$

c (e, e^{-1}) is a maximum turning point.

d $\left(e^{\frac{3}{2}}, \frac{3}{2}e^{-\frac{3}{2}}\right)$

e $y \le e^{-1}$



Exercise 5I

1 a $2\log_e|x| + C$ **b** $5\log_e|x| + C$
c $\frac{1}{2}\log_e|x| + C$ **d** $\frac{1}{3}\log_e|x| + C$
e $\frac{4}{5}\log_e|x| + C$ **f** $\frac{3}{2}\log_e|x| + C$

2 a $\frac{1}{4}\log_e|4x + 1| + C$ **b** $\frac{1}{5}\log_e|5x - 3| + C$
c $2\log_e|3x + 2| + C$ **d** $3\log_e|5x + 1| + C$
e $\log_e|4x + 3| + C$ **f** $-\log_e|3 - x| + C$
g $-\frac{1}{2}\log_e|7 - 2x| + C$ **h** $\frac{4}{5}\log_e|5x - 1| + C$
i $-4\log_e|1 - 3x| + C$

3 a $\log_e 5$
b $\log_e 3$
c $\log_e|-2| - \log_e|-8| = -2\log_e 2$
d The integral is meaningless because it runs across an asymptote at $x = 0$.
e $\frac{1}{2}(\log_e 8 - \log_e 2) = \log_e 2$
f $\frac{1}{5}(\log_e|-75| - \log_e|-25|) = -\frac{1}{5}\log_e 3$

4 a $\log_e 2 \doteqdot 0.6931$
b $\log_e 3 - \log_e 5 \doteqdot -0.5108$
c $3\log_e 2 \doteqdot 2.079$ **d** $\frac{2}{3}\log_e 2 \doteqdot 0.4621$
e $-\frac{1}{2}\log_e 7 \doteqdot -0.9730$ **f** $\frac{3}{2}\log_e 3 \doteqdot 1.648$
g $\log_e\frac{5}{2} \doteqdot 0.9163$ **h** $\frac{3}{2}\log_e 5 \doteqdot 2.414$
i The integral is meaningless because it runs across an asymptote at $x = 5\frac{1}{2}$.

5 a 1 **b** 2 **c** 3 **d** $\frac{1}{2}$

6 a $x + \log_e|x| + C$ **b** $\frac{1}{5}x + \frac{3}{5}\log_e|x| + C$
c $\frac{2}{3}\log_e|x| - \frac{1}{3}x + C$ **d** $\frac{1}{9}\log_e|x| - \frac{8}{9}x + C$
e $3x - 2\log_e|x| + C$
f $x^2 + x - 4\log_e|x| + C$
g $\frac{3}{2}x^2 + 4\log_e|x| + \frac{1}{x} + C$
h $\frac{1}{3}x^3 - \log_e|x| - \frac{2}{x} + C$

7 a $\log_e|x^2 - 9| + C$
b $\log_e|3x^2 + x| + C$
c $\log_e|x^2 + x - 3| + C$
d $\log_e|2 + 5x - 3x^2| + C$
e $\frac{1}{2}\log_e|x^2 + 6x - 1| + C$
f $\frac{1}{4}\log_e|12x - 3 - 2x^2| + C$
g $\log_e(1 + e^x) + C$
h $-\log_e(1 + e^{-x}) + C$
i $\log_e(e^x + e^{-x}) + C$
The denominators in parts **g**–**i** are never negative, so the absolute value sign is unnecessary.

8 a $f(x) = x + 2\ln|x|, f(2) = 2 + 2\ln 2$
b $f(x) = x^2 + \frac{1}{3}\ln|x| + 1, f(2) = 5 + \frac{1}{3}\ln 2$
c $f(x) = 3x + \frac{5}{2}\ln|2x - 1| - 3,$
$f(2) = 3 + \frac{5}{2}\ln 3$
d $f(x) = 2x^3 + 5\ln|3x + 2| - 2,$
$f(2) = 14 + 5\ln 8$

9 a $y = \frac{1}{4}(\log_e|x| + 2), x = e^{-2}$
b $y = 2\log_e|x + 1| + 1$
c $y = \log_e\left|\frac{x^2 + 5x + 4}{10}\right| + 1, y(0) = \log_e\frac{4}{10} + 1$
d $y = 2\log_e|x| + x + C, y = 2\log_e|x| + x,$
$y(2) = \log_e 4 + 2$
e $f(x) = 2 + x - \log_e|x|, f(e) = e + 1$

10 a $\frac{1}{2}\log_e|2x + b| + C$ **b** $\frac{1}{3}\log_e|3x - k| + C$
c $\frac{1}{a}\log_e|ax + 3| + C$ **d** $\frac{1}{m}\log_e|mx - 2| + C$
e $\log_e|px + q| + C$ **f** $\frac{A}{S}\log_e|sx - t| + C$

11 a $\log_e|x^3 - 5| + C$
b $\log_e|x^4 + x - 5| + C$
c $\frac{1}{4}\log_e|x^4 - 6x^2| + C$
d $\frac{1}{2}\log_e|5x^4 - 7x^2 + 8| + C$
e $2\log_e 2$
f $\log_e\frac{4(e + 1)}{e + 2}$

12 a $f(x) = x + \ln|x| + \frac{1}{2}x^2$
b $g(x) = x^2 - 3\ln|x| + \frac{4}{x} - 6$

13 $\frac{1}{3}(e^3 - e^{-3}) + 2$

14 a $y' = \log_x$
b i $x\log_e x - x + C$ **ii** $\frac{\sqrt{e}}{2}$

15 b $\frac{1}{2}x^2\log_e x - \frac{1}{4}x^2$ **c** $2\log 2 - 1 - \frac{e^2}{4}$

16 a $\frac{2\log_e x}{x}$ **b** $\frac{3}{8}$

17 $\ln(\ln x) + C$

18 The key to all this is that
$\log_e|5x| = \log_e 5 + \log_e|x|,$
so that $\log_e|x|$ and $\log_e|5x|$ differ only by a constant $\log_e 5$. Thus $C_2 = C_1 - \frac{1}{5}\log_e 5$, and because C_1 and C_2 are arbitrary constants, it does not matter at all. In particular, in a definite integral, adding a constant doesn't change the answer, because it cancels out when we take $F(b) - F(a)$.

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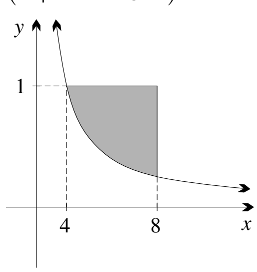
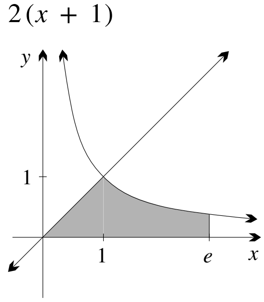
19 a i $a = e^5$ ii $a = e^{-4}$
b i $a = -e^2$ ii $a = -e^{-1}$
20 a $\log_e\left(1 + \sqrt{2}\right)$ **b** $\log_e\left(2 + \sqrt{3}\right)$

Exercise 5J

1 b $e \doteqdot 2.7$
2 a i 1u^2 ii $\log_e 5 \doteqdot 1.609\,\text{u}^2$
b i 1u^2 ii $2\log_e 2 \doteqdot 1.386\,\text{u}^2$
iii 14u^2 iv $14\log_e 5 \doteqdot 22.53\,\text{u}^2$
3 a $\log_e 2$ square units
b $(\log_e 3 - \log_e 2)$ square units
c $-\log_e \frac{1}{3} = \log_e 3$ square units
d $\log_e 2 - \log_e \frac{1}{2} = 2\log_e 2$ square units
4 a i $\frac{1}{2}(\log_e 11 - \log_e 5) \doteqdot 0.3942\,\text{u}^2$
ii $\frac{1}{2}\log_e 3 \doteqdot 0.5493\,\text{u}^2$
b i $\frac{1}{3}(\log_e 5 - \log_e 2) \doteqdot 0.3054\,\text{u}^2$
ii $\frac{1}{3}\log_e 10 \doteqdot 0.7675\,\text{u}^2$
c i $\frac{1}{2}\log_e 3 \doteqdot 0.5493\,\text{u}^2$
ii $\log_e 3 \doteqdot 1.099\,\text{u}^2$
d i 9u^2
ii $3(\log_e 11 - \log_e 2) \doteqdot 5.114\,\text{u}^2$
5 a $\log_e 2 + 1\,\text{u}^2$ **b** $2\log_e 2 + \frac{15}{8}\text{u}^2$
c $\log_e 3 + 8\frac{2}{3}\text{u}^2$
6 a $(6 - 3\log_e 3)\,\text{u}^2$ **b** $(4 - \log_e 3)\,\text{u}^2$
7 a $\left(3\frac{3}{4} - 2\log_e 4\right)\text{u}^2$ **b** $\left(3\frac{3}{4} - 2\log_e 4\right)\text{u}^2$
8 a

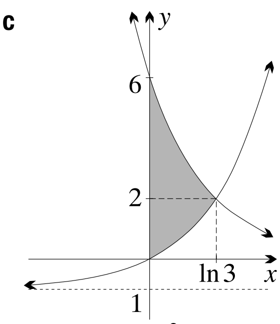


b $4(1 - \log_e 2)\,\text{u}^2$
9 a $2\log_e 2\text{u}^2$ **b** $(1 - \log_e 2)\,\text{u}^2$
10 a $(\log_e 4)\,\text{u}^2$ **b** $(6 - 3\log_e 3)\,\text{u}^2$
11 a $\frac{1}{2}\text{u}^2$ **b** $2\log_e \frac{4}{3}\text{u}^2$
12 a $\left(\frac{1}{3}, 3\right)$ and $(1, 1)$ **b** $\left(\frac{4}{3} - \log_e 3\right)\text{u}^2$
13 a $2x$ **b** $\frac{1}{2}\log_e 5 \doteqdot 0.805\,\text{u}^2$
14 a $2(x + 1)$ **b** $\frac{1}{2}\log_e 2\text{u}^2$
15 a



b $\frac{3}{2}\text{u}^2$

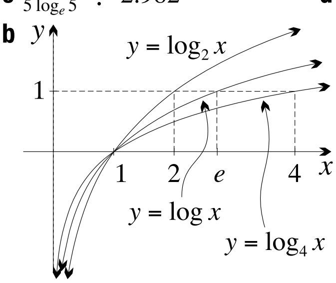
16 a $\ln 2 \doteqdot 0.693$ **b** $\frac{17}{24} \doteqdot 0.708$
17 a $\ln 3 \doteqdot 1.0986$ square units
b 1.1667 square units
18 a 3.9828 square units
19 b $(e - 1)\,\text{u}^2$ **c** $1\,\text{u}^2$
20 b $(\ln 3, 2)$
c



d $(2 + \ln 3)\,\text{u}^2$
21 a $e - 2$ square units **b** e^{-1} square units
c $e - 2 + e^{-1}$ square units

Exercise 5K

1 a 1.58 **b** 3.32 **c** 2.02 **d** −4.88
2 a $y' = \frac{1}{x\log_e 2}$ **b** $y' = \frac{1}{x\log_e 10}$ **c** $y' = \frac{3}{x\log_e 5}$
3 a $y' = \frac{1}{x\log_e 3}$ **b** $y' = \frac{1}{x\log_e 7}$ **c** $y' = \frac{5}{x\log_e 6}$
4 a $3^x\log_e 3$ **b** $4^x\log_e 4$ **c** $2^x\log_e 2$
5 a $y' = 10^x\log_e 10$ **b** $y' = 8^x\log_e 8$
c $y' = 3 \times 5^x\log_e 5$
6 a $\frac{2^x}{\log_e 2} + C$ **b** $\frac{6^x}{\log_e 6} + C$
c $\frac{7^x}{\log_e 7} + C$ **d** $\frac{3^x}{\log_e 3} + C$
7 a $\frac{1}{\log_e 2} \doteqdot 1.443$ **b** $\frac{2}{\log_e 3} \doteqdot 1.820$
c $\frac{24}{5\log_e 5} \doteqdot 2.982$ **d** $\frac{15}{\log_e 4} \doteqdot 10.82$
8 b



9 a $\frac{1}{\log_e 2}$ **b** $y = \frac{1}{\log_e 2}(x - 1)$
c i $y = \frac{1}{\log_e 3}(x - 1)$ ii $y = \frac{1}{\log_e 5}(x - 1)$
10 a $\frac{6}{\log_e 2} \doteqdot 8.6562$ **b** $2 + \frac{8}{3\log_e 3} \doteqdot 4.4273$
c $\frac{99}{\log_e 10} - 20 \doteqdot 32.9952$
11 $y = \frac{\log_e x}{\log_e 10}, y' = \frac{1}{x\log_e 10}$
a $\frac{1}{10\log_e 10}$
b $x - 10y\log_e 10 + 10(\log_e 10 - 1) = 0$
c $x = \frac{1}{\log_e 10}$

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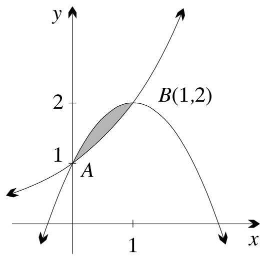
12 a $y = \frac{1}{\log_e 2}\left(\frac{x}{3} - 1 + \log_e 3\right)$, $y = \frac{x}{3} - 1 + \log_e 3$, $y = \frac{1}{\log_e 4}\left(\frac{x}{3} - 1 + \log_e 3\right)$

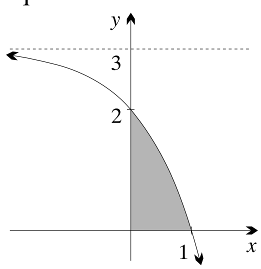
b They all meet the x-axis at $(3 - 3\log_e 3, 0)$

13 b $\left(\frac{5}{3} - \frac{1}{\log_e 2}\right)$ u^2



14 intercepts $(0, 7)$ and $(3, 0)$, area $\left(24 - \frac{7}{\log_e 2}\right)$ square units

15 a



b $\left(3 - \frac{2}{\log_e 3}\right)$ u^2

16 a $\int_{-\frac{1}{2}}^{0} x + 1 - 4^x dx$ **b** $\frac{3}{8} - \frac{1}{2\log_e 4}$

18 a $x\log_e x - x + C$ **b** $10 - \frac{9}{\log_e 10}$

19 a i $y' = \frac{1}{x\log_e 3}$ **ii** $y' = \frac{2}{(2x + 3)\log_e 7}$

iii $y' = -\frac{45}{(4 - 9x)\log_e 6}$

b i $y' = 10^x \log_e 10$

ii $y' = 4 \times 8^{4x-3}\log_e 8$

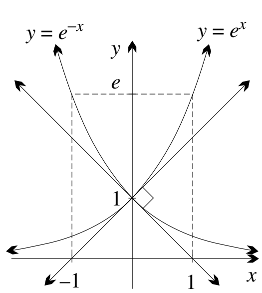
iii $y' = -21 \times 5^{2-7x}\log_e 5$

c i $\frac{3^{5x}}{5\log_e 3} + C$ **ii** $\frac{6^{2x+7}}{2\log_e 6} + C$

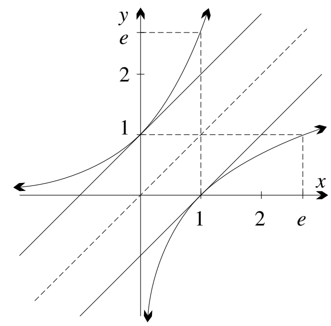
iii $-\frac{5 \times 7^{4-9x}}{9\log_e 7} + C$

Chapter 5 review exercise

1 a Each graph is reflected onto the other in the line $x = 0$. The tangents have gradients 1 and -1, and are at right angles.



b Each graph is reflected onto the other in the line $y = x$. The tangents both have gradients 1, and are thus parallel.



2 a 54.60 **b** 2.718 **c** 0.2231 **d** 0.6931
e -0.3010 **f** -5.059 **g** 130.6 **h** 0.5925

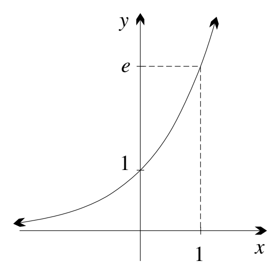
3 a 2.402 **b** 5.672 **c** 5.197 **d** 3.034

4 a e^{5x} **b** e^{6x} **c** e^{-4x} **d** e^{9x}

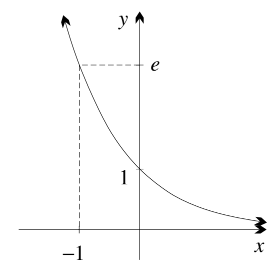
5 a $x = 2$

b $x = \log_e 4\,(= 2\log_e 2)$ or $\log_e 7$

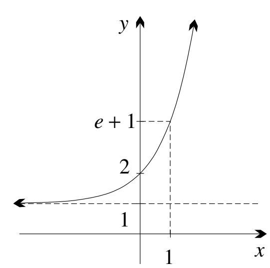
6 a $y > 0$



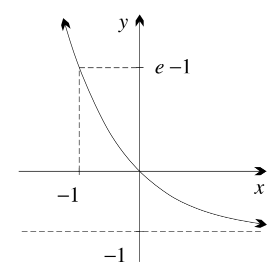
b $y > 0$



c $y > 1$

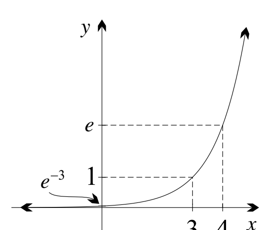


d $y > -1$



7 a i Shift $y = e^x$ right 3 units.

ii $y = e^{-3}e^x$ or $\frac{y}{e^{-3}} = e^x$, so dilate vertically with factor e^{-3}.



b i $y = \log_e \frac{x}{1/3}$, so dilate $y = \log_e x$ horizontally with factor $\frac{1}{3}$.

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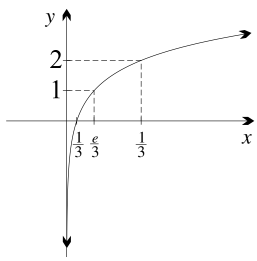
ii $y = \log_e x + \log_e 3$, or $y - \log_e 3 = \log_e x$, so shift $y = \log_e x$ up $\log_e 3$.



8 a e^x **b** $3e^{3x}$ **c** $2e^{2x+3}$ **d** $-e^{-x}$

e $-3e^{-3x}$ **f** $6e^{2x+5}$ **g** $2e^{\frac{1}{2}x}$ **h** $4e^{6x-5}$

9 a $5e^{5x}$ **b** $4e^{4x}$ **c** $-3e^{-3x}$ **d** $-6e^{-6x}$

10 a $3x^2e^{x^3}$ **b** $(2x - 3)e^{x^2-3x}$

c $e^{2x} + 2xe^{2x} = e^{2x}(1 + 2x)$

d $6e^{2x}(e^{2x} + 1)^2$

e $\frac{e^{3x}(3x - 1)}{x^2}$

f $2xe^{x^2}(1 + x^2)$

g $5(e^x + e^{-x})(e^x - e^{-x})^4$

h $\frac{4xe^{2x}}{(2x + 1)^2}$

11 a $y' = 2e^{2x+1}$, $y'' = 4e^{2x+1}$

b $y' = 2xe^{x^2+1}$, $y'' = 2e^{x^2+1}(2x^2 + 1)$

12 $y = e^2x - e^2$, x-intercept 1, y-intercept $-e^2$.

13 a $\frac{1}{3}$

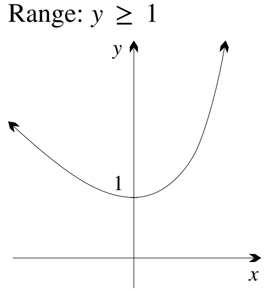
b When $x = 0$, $y'' = 9$, so the curve is concave up there.

14 a $y' = e^x - 1$, $y'' = e^x$

b $(0, 1)$ is a minimum turning point.

c $y'' = e^x$, which is positive for all x.

d Range: $y \geq 1$



15 $\left(\frac{1}{2}, \frac{1}{2e}\right)$ is a maximum turning point.

16 a $\frac{1}{5}e^{5x} + C$ **b** $-2e^{2-5x} + C$

c $5e^{\frac{1}{5}x} + C$ **d** $\frac{3}{5}e^{5x-4} + C$

17 a $e^2 - 1$ **b** $\frac{1}{2}(e^2 - 1)$

c $e - 1$ **d** $\frac{1}{3}(e^2 - 1)$

e $\frac{1}{2}e^2(e - 1)$ **f** $4(e - 1)$

18 a $-\frac{1}{5}e^{-5x} + C$ **b** $\frac{1}{4}e^{4x} + C$

c $-2e^{-3x} + C$ **d** $\frac{1}{6}e^{6x} + C$

e $-\frac{1}{2}e^{-2x} + C$ **f** $e^x - \frac{1}{2}e^{-2x} + C$

g $\frac{1}{3}e^{3x} + e^x + C$ **h** $x - 2e^{-x} - \frac{1}{2}e^{-2x} + C$

19 a $2 - e^{-1}$ **b** $\frac{1}{2}(e^4 + 3)$

c $2(1 - e^{-1})$ **d** $\frac{1}{3}(e - 2)$

e $e - e^{-1}$ **f** $\frac{1}{2}(e^2 + 4e - 3)$

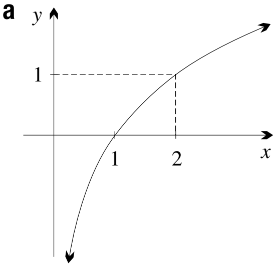
20 $f(x) = e^x + e^{-x} - x + 1$, $f(1) = e + e^{-1}$

21 a $3x^2e^{x^3}$ **b** $\frac{1}{3}(e - 1)$

22 a $3.19\ \text{u}^2$ **b** $0.368\ \text{u}^2$

23 a $\frac{1}{2}(1 + e^{-2})\ \text{u}^2$ **b** $\frac{1}{2}(3 - e)\ \text{u}^2$

24 a



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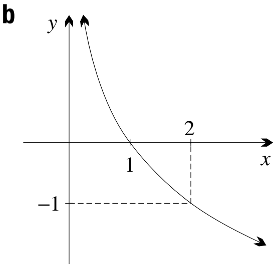

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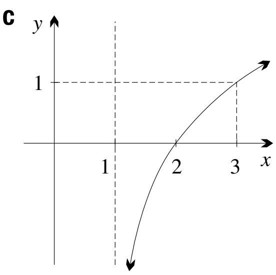

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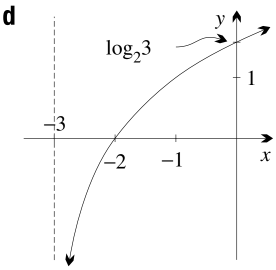
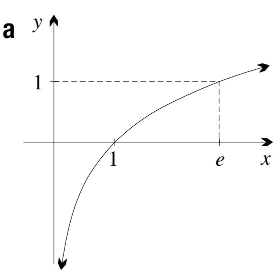


25 a



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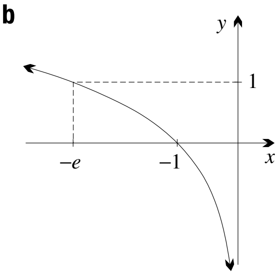

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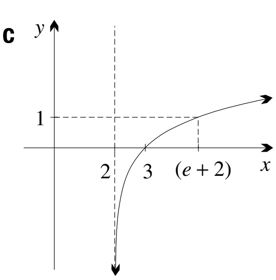
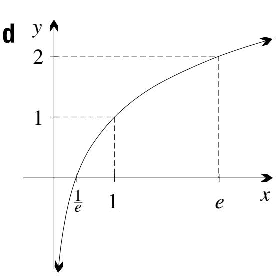


d



26 a e **b** 3 **c** -1 **d** e

27 a $\dfrac{1}{x}$ **b** $\dfrac{1}{x}$ **c** $\dfrac{1}{x + 4}$

d $\dfrac{2}{2x - 5}$ **e** $\dfrac{10}{5x - 1}$ **f** $1 + \dfrac{1}{x}$

g $\dfrac{2x - 5}{x^2 - 5x + 2}$ **h** $\dfrac{15x^4}{1 + 3x^5}$

i $8x - 24x^2 + \dfrac{2x}{x^2 - 2}$

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28 a $\dfrac{3}{x}$ **b** $\dfrac{1}{2x}$

c $\dfrac{1}{x} + \dfrac{1}{x+2}$ **d** $\dfrac{1}{x} - \dfrac{1}{x-1}$

29 a $1 + \log x$ **b** $\dfrac{e^x}{x} + e^x \log x$

c $\dfrac{\ln x - 1}{(\ln x)^2}$ **d** $\dfrac{1 - 2\ln x}{x^3}$

30 $y = 3x + 1$

32 a $\log_e |x| + C$ **b** $3\log_e |x| + C$

c $\frac{1}{5}\log_e |x| + C$ **d** $\log_e |x + 7| + C$

e $\frac{1}{2}\log_e |2x - 1| + C$ **f** $-\frac{1}{3}\log_e |2 - 3x| + C$

g $\log_e |2x + 9| + C$ **h** $-2\log_e |1 - 4x| + C$

33 a $\log_e \frac{3}{2}$ **b** $\frac{1}{4}\log_e 13$ **c** 1 **d** 1

34 a $\log_e (x^2 + 4) + C$

b $\log_e |x^3 - 5x + 7| + C$

c $\frac{1}{2}\log_e |x^2 - 3| + C$

d $\frac{1}{4}\log_e |x^4 - 4x| + C$

35 $\log_e 2\,\mathrm{u}^2$

36 a $12 - 5\log_e 5\,\mathrm{u}^2$

37 a e^x **b** $2^x \log_e 2$ **c** $3^x \log_e 3$ **d** $5^x \log_e 5$

38 a $e^x + C$ **b** $\dfrac{2^x}{\log_e 2} + C$

c $\dfrac{3^x}{\log_e 3} + C$ **d** $\dfrac{5^x}{\log_e 5} + C$

39 a $x\log_e x - x$ **b** $xe^x - e^x$

40 a $8\log_e 2$ **b** $\dfrac{1}{8\log_e 2}$

c The curves $y = 2^x$ and $y = \log_2 x$ are reflections of each other in $y = x$. This reflection exchanges A and B, and exchanges their tangents. Because it also exchanges rise and run, the gradients are reciprocals of each other.

41 a $\dfrac{7}{\ln 2}$ and $\dfrac{7}{8\ln 2}$

b When $y = 2^x$ is transformed successively by a vertical dilation with factor 8 and a shift right 3 units, the result is the same graph $y = 2^x$. The region in the second integral is transformed to the region in the first integral by this compound transformation.

Chapter 6

Exercise 6A

1 a The entries under 0.2 are 0.198669, 0.993347, 0.202710, 1.013550, 0.980067.

b 1 and 1

3 a $\frac{\pi}{90}$ **b** $\sin 2° = \sin\frac{\pi}{90} \div \frac{\pi}{90}$ **c** 0.0349

4 a The entries under 5° are 0.08727, 0.08716, 0.9987, 0.08749, 1.003, 0.9962.

b $\sin x < x < \tan x$

c i 1 **ii** 1

d $x \le 0.0774$ (correct to four decimal places), that is, $x \le 4°26'$.

6 87 metres

7 $26'$

12 a $AB^2 = 2r^2(1 - \cos x)$, arc $AB = rx$

b The arc is longer than the chord, so $\cos x$ is larger than the approximation.

Exercise 6B

1 a $\cos x$ **b** $-\sin x$ **c** $\sec^2 x$ **d** $2\cos x$

e $2\cos 2x$ **f** $-3\sin x$ **g** $-3\sin 3x$ **h** $4\sec^2 4x$

i $4\sec^2 x$ **j** $6\cos 3x$ **k** $4\sec^2 2x$ **l** $-8\sin 2x$

m $-2\cos 2x$ **n** $2\sin 2x$ **o** $-2\sec^2 2x$ **p** $\frac{1}{2}\sec^2 \frac{1}{2}x$

q $-\frac{1}{2}\sin\frac{1}{2}x$ **r** $\frac{1}{2}\cos\frac{x}{2}$ **s** $\sec^2 \frac{1}{5}x$ **t** $-2\sin\frac{x}{3}$

u $4\cos\frac{x}{4}$

2 a $2\pi\cos 2\pi x$ **b** $\frac{\pi}{2}\sec^2 \frac{\pi}{2}x$

c $3\cos x - 5\sin 5x$ **d** $4\pi\cos\pi x - 3\pi\sin\pi x$

e $2\cos(2x - 1)$ **f** $3\sec^2(1 + 3x)$

g $2\sin(1 - x)$ **h** $-5\sin(5x + 4)$

i $-21\cos(2 - 3x)$ **j** $-10\sec^2(10 - x)$

k $3\cos\left(\frac{x+1}{2}\right)$ **l** $-6\sin\left(\frac{2x+1}{5}\right)$

3 a $2\cos 2x$, $-4\sin 2x$, $-8\cos 2x$, $16\sin 2x$, amplitudes: 2, 4, 8, 16

b $-10\sin 10x$, $-100\cos 10x$, $1000\sin 10x$, $10000\cos 10x$

c $\frac{1}{2}\cos\frac{1}{2}x$, $-\frac{1}{4}\sin\frac{1}{2}x$, $-\frac{1}{8}\cos\frac{1}{2}x$, $\frac{1}{16}\sin\frac{1}{2}x$

d $-\frac{1}{3}\sin\frac{1}{3}x$, $-\frac{1}{9}\cos\frac{1}{3}x$, $\frac{1}{27}\sin\frac{1}{3}x$, $\frac{1}{81}\cos\frac{1}{3}x$, amplitudes: $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$

4 $-2\sin 2x$

a 0 **b** -1 **c** $-\sqrt{3}$ **d** -2

5 $\frac{1}{4}\cos\left(\frac{1}{4}x + \frac{\pi}{2}\right)$

a 0 **b** $-\frac{1}{4}$ **c** $\frac{1}{8}\sqrt{2}$ **d** $-\frac{1}{8}\sqrt{2}$

6 a $x\cos x + \sin x$ **b** $2(\tan 2x + 2x\sec^2 2x)$

c $2x(\cos 2x - x\sin 2x)$ **d** $3x^2(\sin 3x + x\cos 3x)$

7 a $\frac{x\cos x - \sin x}{x^2}$ **b** $\frac{-x\sin x - \cos x}{x^2}$

c $\frac{x(2\cos x + x\sin x)}{\cos^2 x}$ **d** $\frac{1 + \sin x - x\cos x}{(1 + \sin x)^2}$

8 a $2x\cos(x^2)$ **b** $-2x\cos(1 - x^2)$

c $-3x^2\sin(x^3 + 1)$ **d** $-\frac{1}{x^2}\cos(\frac{1}{x})$

e $-2\cos x\sin x$ **f** $3\sin^2 x\cos x$

g $2\tan x\sec^2 x$ **h** $\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}$

9 d $y = \cos x$

11 a $e^{\tan x}\sec^2 x$ **b** $2e^{\sin 2x}\cos 2x$ **c** $2e^{2x}\cos(e^{2x})$

d $-\tan x$ **e** $\cot x$ **f** $-4\tan 4x$

12 a $\cos^2 x - \sin^2 x$ **b** $14\sin 7x\cos 7x$

c $-15\cos^4 3x\sin 3x$ **d** $9\sin 3x(1 - \cos 3x)^2$

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e $2(\cos 2x \sin 4x + 2\sin 2x \cos 4x)$
f $15\tan^2(5x-4)\sec^2(5x-4)$

13 a $\frac{-\cos x}{(1+\sin x)^2}$ **b** $\frac{1}{1+\cos x}$
c $\frac{-1}{1+\sin x}$ **d** $\frac{-1}{(\cos x+\sin x)^2}$

14 c i The graphs are reflections of each other in the x-axis.
ii The graphs are identical.
d i $y = e^x$ **ii** $y = e^x, y = e^{-x}$
iii $y = e^x$ **iv** $y = e^x, y = e^{-x}, y = \sin x$

16 a $y' = e^x \sin x + e^x \cos x, y'' = 2e^x \cos x$
b $y' = -e^{-x}\cos x - e^{-x}\sin x, y'' = 2e^{-x}\sin x$

18 a $\log_b P - \log_b Q$

Exercise 6C

1 a 1 **b** -1 **c** $\frac{1}{2}$ **d** $-\frac{1}{2}$
e $\frac{1}{\sqrt{2}}$ **f** 1 **g** 2 **h** -2
i $\frac{\sqrt{3}}{4}$ **j** $\frac{1}{4}$ **k** 8 **l** $\sqrt{3}$

3 a $y = -x + \pi$ **b** $2x - y = \frac{\pi}{2} - 1$
c $x + 2y = \frac{\pi}{6} + \sqrt{3}$ **d** $y = -2x + \frac{\pi}{2}$
e $x + y = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$ **f** $y = -\pi x + \pi^2$

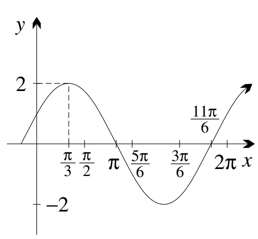
4 a $\frac{\pi}{2}, \frac{3\pi}{2}$ **b** $\frac{\pi}{3}, \frac{5\pi}{3}$ **c** $\frac{\pi}{6}, \frac{5\pi}{6}$ **d** $\frac{5\pi}{6}, \frac{7\pi}{6}$

6 b 1 and -1
c $x - y = \frac{\pi}{4} - \frac{1}{2}, x + y = \frac{\pi}{4} + \frac{1}{2}$

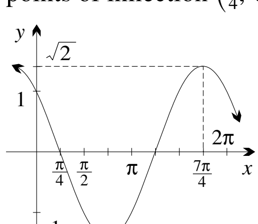
7 a $y' = \cos x e^{\sin x}$ **b** $\frac{\pi}{2}, \frac{3\pi}{2}$

8 a $y' = -\sin x e^{\cos x}$ **b** $0, \pi, 2\pi$

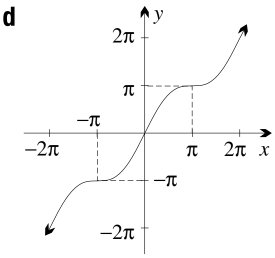
9 a $y' = -\sin x + \sqrt{3}\cos x, y'' = -\cos x - \sqrt{3}\sin x$
b maximum turning point $\left(\frac{\pi}{3}, 2\right)$, minimum turning point $\left(\frac{4\pi}{3}, -2\right)$
c $\left(\frac{5\pi}{6}, 0\right), \left(\frac{11\pi}{6}, 0\right)$
d



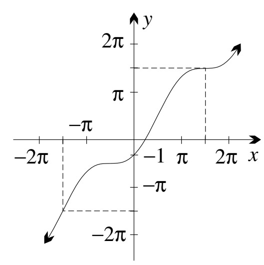
10 a $y' = -\sin x - \cos x, y'' = -\cos x + \sin x$,
minimum turning point $\left(\frac{3\pi}{4}, -\sqrt{2}\right)$,
maximum turning point $\left(\frac{7\pi}{4}, \sqrt{2}\right)$,
points of inflection $\left(\frac{\pi}{4}, 0\right), \left(\frac{5\pi}{4}, 0\right)$
b



11 a $y' = 1 + \cos x$
b $(-\pi, -\pi)$ and (π, π) are horizontal points of inflection.
c $(0, 0)$
d

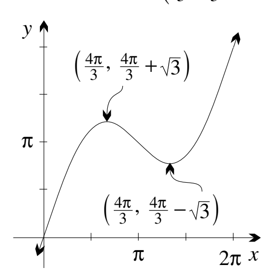


12 $y' = 1 + \sin x, y'' = \cos x$, horizontal points of inflection $\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$, points of inflection $\left(-\frac{3\pi}{2}, -\frac{3\pi}{2}\right), \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$



13 c $\sin^2\theta = \frac{1}{3}, \theta \doteqdot 19.47°$ **d** $2\pi\sqrt{3}\text{ m}^3$

14 maximum turning point $\left(\frac{2\pi}{3}, \frac{2\pi}{3} + \sqrt{3}\right)$, minimum turning point $\left(\frac{4\pi}{3}, \frac{4\pi}{3} - \sqrt{3}\right)$, inflection (π, π)



16 b minimum $\sqrt{3}$ when $\theta = \frac{\pi}{6}$, maximum 2 when $\theta = 0$

Exercise 6D

1 a $\tan x + C$ **b** $\sin x + C$
c $-\cos x + C$ **d** $\cos x + C$
e $2\sin x + C$ **f** $\frac{1}{2}\sin 2x + C$
g $\frac{1}{2}\sin x + C$ **h** $2\sin\frac{1}{2}x + C$
i $-\frac{1}{2}\cos 2x + C$ **j** $\frac{1}{5}\tan 5x + C$
k $\frac{1}{3}\sin 3x + C$ **l** $3\tan\frac{1}{3}x + C$
m $-2\cos\frac{x}{2} + C$ **n** $-5\sin\frac{1}{5}x + C$
o $2\cos 2x + C$ **p** $-\cos\frac{1}{4}x + C$
q $-36\tan\frac{1}{3}x + C$ **r** $6\sin\frac{x}{3} + C$

2 a 1 **b** $\frac{1}{2}$ **c** $\frac{1}{\sqrt{2}}$ **d** $\sqrt{3}$ **e** 1
f $\frac{3}{4}$ **g** 2 **h** 1 **i** 4

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3 a **i** $y = 1 - \cos x$ **ii** $y = 3 - \cos x$
iii Shift up 2.
b $y = \sin x + \cos 2x - 1$
c $y = -\cos x + \sin x - 3$

6 a $\sin(x + 2) + C$ **b** $\frac{1}{2}\sin(2x + 1) + C$
c $-\cos(x + 2) + C$ **d** $-\frac{1}{2}\cos(2x + 1) + C$
e $\frac{1}{3}\sin(3x - 2) + C$ **f** $\frac{1}{5}\cos(7 - 5x) + C$
g $-\tan(4 - x) + C$ **h** $-3\tan\left(\frac{1 - x}{3}\right) + C$
i $3\cos\left(\frac{1 - x}{3}\right) + C$

7 a $2\sin 3x + 8\cos\frac{1}{2}x + C$
b $4\tan 2x - 40\sin\frac{1}{4}x - 36\cos\frac{1}{3}x + C$

8 a $f(x) = \sin \pi x,\ f\left(\frac{1}{3}\right) = \frac{1}{2}\sqrt{3}$
b $f(x) = \frac{1}{2\pi} + \frac{1}{\pi}\sin \pi x,\ f\left(\frac{1}{6}\right) = \frac{1}{\pi}$
c $f(x) = -2\cos 3x + x + \left(1 - \frac{\pi}{2}\right)$

9 a $-\cos(ax + b) + C$ **b** $\pi \sin \pi x + C$
c $\frac{1}{u^2}\tan(v + ux) + C$ **d** $\tan ax + C$

10 a $1 + \tan^2 x = \sec^2 x,\ \tan x - x + C$
b $1 - \sin^2 x = \cos^2 x,\ 2\sqrt{3}$

11 a $\log_e|f(x)| + C$

12 a $\int \tan x = -\ln\cos x + C$
b $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x\,dx = \Big[\log|\sin x|\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \log 2$

13 a **i** $2x\cos x^2$ **ii** $\sin x^2 + C$
b **i** $-3x^2\sin x^3$ **ii** $-\frac{1}{3}\cos x^3 + C$
c **i** $\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}$ **ii** $2\tan\sqrt{x} + C$

14 a $5\sin^4 x\cos x,\ \frac{1}{5}\sin^5 x + C$
b $3\tan^2 x\sec^2 x,\ \frac{1}{3}\tan^3 x + C$

15 a $\cos x e^{\sin x},\ e - 1$ **b** $e^{\tan x} + C,\ e - 1$

16 a 1 **b** $\frac{5}{24}$

17 $\sin 2x + 2x\cos 2x,\ \frac{\pi - 2}{8}$

18 All three integrals are meaningless because:
a $\sec x$ has an asymptote at $x = \frac{\pi}{2}$,
b $\tan x$ has an asymptote at $x = \frac{\pi}{2}$,
c $\cot x$ has an asymptote at $x = 0$.

Exercise 6E

1 a 1 square unit **b** $\frac{1}{2}$ square unit
2 a 1 square unit **b** $\sqrt{3}$ square units
3 a $1 - \frac{1}{\sqrt{2}}$ square units **b** $1 - \frac{\sqrt{3}}{2}$ square units
4 a $\frac{1}{2}\sqrt{3}\ \text{u}^2$ **b** $\frac{1}{2}\sqrt{3}\ \text{u}^2$
5 a $\frac{1}{2}\ \text{u}^2$ **b** $\frac{1}{2}\ \text{u}^2$
c $1 - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(2 - \sqrt{3}\right)\ \text{u}^2$
d $\frac{1}{3}\left(1 - \frac{1}{\sqrt{2}}\right) = \frac{1}{6}\left(2 - \sqrt{2}\right)\ \text{u}^2$
e $\frac{2}{3}\sqrt{3}\ \text{u}^2$
f $4\ \text{u}^2$

6 a $\left(\sqrt{2} - 1\right)\ \text{u}^2$ **b** $\frac{1}{4}\ \text{u}^2$
c $\left(\frac{\pi^2}{8} - 1\right)\ \text{u}^2$ **d** $(\pi - 2)\ \text{u}^2$

7 a $\left(2 - \sqrt{2}\right)\ \text{u}^2$ **b** $1\frac{1}{2}\ \text{u}^2$

8 a $2\ \text{u}^2$ **b** $1\ \text{u}^2$

9 a $2\ \text{u}^2$ **b** $\sqrt{2}\ \text{u}^2$ **c** $2\ \text{u}^2$
d $\frac{1}{2}\ \text{u}^2$ **e** $4\ \text{u}^2$ **f** $1\ \text{u}^2$

10 b $\frac{4}{\pi}\ \text{u}^2$

11 $3.8\ \text{m}^2$

12 $4\ \text{u}^2$

14 b $\frac{1}{2}\left(3 + \sqrt{3}\right)\ \text{u}^2$

15 b $\frac{3}{4}\sqrt{3}\ \text{u}^2$

16 They are all 4u^2.

17 b The curve is below $y = 1$ just as much as it is above $y = 1$, so the area is equal to the area of a rectangle n units long and 1 unit high.

Chapter 6 review exercise

1 a $5\cos x$ **b** $5\cos 5x$
c $-25\sin 5x$ **d** $5\sec^2(5x - 4)$
e $\sin 5x + 5x\cos 5x$ **f** $\frac{-5x\sin 5x - \cos 5x}{x^2}$
g $5\sin^4 x\cos x$ **h** $5x^4\sec^2(x^5)$
i $-5\sin 5x e^{\cos 5x}$ **j** $\frac{5\cos 5x}{\sin 5x} = 5\cot 5x$

2 $-\sqrt{3}$

3 a $y = 4x + \sqrt{3} - \frac{4\pi}{3}$ **b** $y = -\frac{\pi}{2}x + \frac{\pi^2}{4}$

4 a $\frac{\pi}{2}$ **b** $\frac{3\pi}{4},\ \frac{7\pi}{4}$

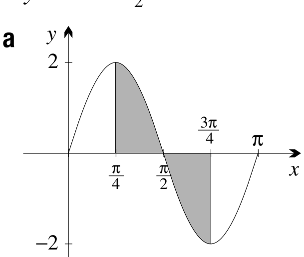
5 a $4\sin x + C$ **b** $-\frac{1}{4}\cos 4x + C$ **c** $4\tan\frac{1}{4}x + C$

6 a $\sqrt{3} - 1$ **b** $\frac{1}{2}$ **c** $\frac{1}{2}$

7 0.089

8 $y = 2\sin\frac{1}{2}x - 1$

9 a



b 2u^2

10 a $\frac{1}{2}\text{u}^2$ **b** $\frac{3\sqrt{3}}{4}\text{u}^2$

11 a $\tan x = \frac{\sin x}{\cos x}$ **b** $\frac{1}{2}\ln 2\,\text{u}^2$

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Chapter 7

Exercise 7A

1 a $x = 2$ **b** $x = 18$ **c** 4 m/s

2 a $x = 0, 20$; 10 cm/s **b** $x = 4, 0$; -2 cm/s

c $x = 3, 3$; 0 cm/s **d** $x = 1, 4$; $1\frac{1}{2}$ cm/s

3 a $x = 0, -3, 0, 15, 48$

b i -3 cm/s **ii** 3 cm/s

iii 15 cm/s **iv** 33 cm/s

4 a $x = -4, -3, 0, 5$

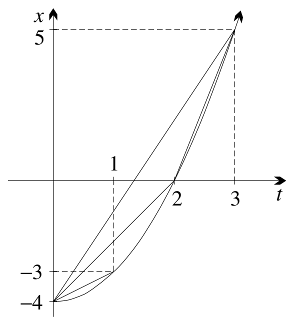
b i 1 m/s

ii 2 m/s

iii 3 m/s

iv 5 m/s

c



5 a $x = 0, 120, 72, 0$ **b** 240 metres

c 20 m/s

d i 30 m/s **ii** -15 m/s **iii** 0 m/s

6 a $x = 0, 3, 4, 3, 0$

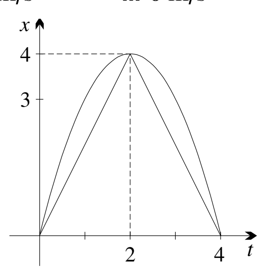
c The total distance travelled is 8 metres.

The average speed is 2 m/s.

d i 2 m/s

ii -2 m/s

iii 0 m/s



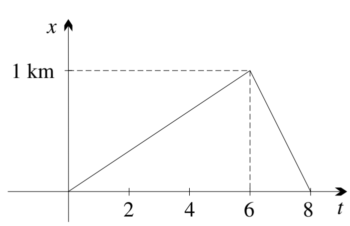
7 a i 6 minutes

ii 2 minutes

c 15 km/hr

d 20 km/hr

b



8 a $x = 0, 3, 1, 4, 2, 5, 3, 6$

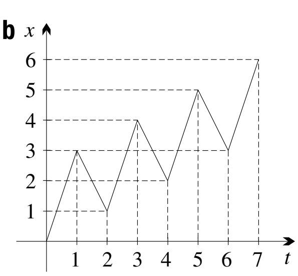
c 7 hours

d 18 metres, $2\frac{4}{7}$ m/hr

e $\frac{6}{7}$ m/hr

f Those between 1 and 2 metres high or between 4 and 5 metres high

b



9 a $t = 0, 1, 4, 9, 16$

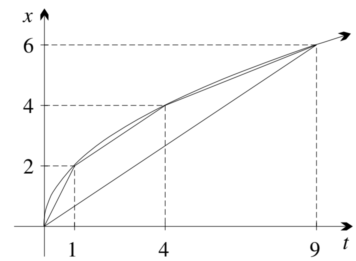
b i 2 cm/s

ii $\frac{2}{3}$ cm/s

iii $\frac{2}{5}$ cm/s

iv $\frac{2}{3}$ cm/s

c They are parallel.



10 a i -1 m/s

ii 4 m/s

iii -2 m/s

b 40 metres, $1\frac{1}{3}$ m/s

c 0 metres, 0 m/s

d $2\frac{2}{19}$ m/s

11 a i once

ii three times

iii twice

b i when $t = 4$ and when $t = 14$

ii when $0 \le t < 4$ and when $4 < t < 14$

c It rises 2 metres, at $t = 8$.

d It sinks 1 metre, at $t = 17$.

e As $t \to \infty$, $x \to 0$, meaning that eventually it ends up at the surface.

f i -1 m/s **ii** $\frac{1}{2}$ m/s **iii** $-\frac{1}{3}$ m/s

g i 4 metres **ii** 6 metres

iii 9 metres **iv** 10 metres

h i 1 m/s **ii** $\frac{3}{4}$ m/s **iii** $\frac{9}{17}$ m/s

12 b $x = 3$ and $x = -3$ **c** $t = 4$, $t = 20$

d $t = 8$, $t = 16$ **e** $8 < t < 16$

f 12 cm, $\frac{3}{4}$ cm/s

13 a amplitude: 4 metres, period: 12 seconds

b 10 times

c $t = 3, 15, 27, 39, 51$

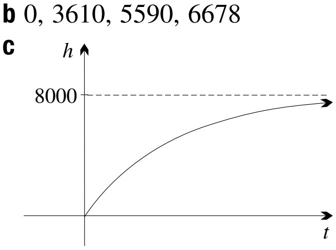
d It travels 16 metres with average speed $1\frac{1}{3}$ m/s.

e $x = 0$, $x = 2$ and $x = 4$, 2 m/s and 1 m/s

14 a When $t = 0$, $h = 0$. As $t \to \infty$, $h \to 8000$.

b 0, 3610, 5590, 6678

c



d 361 m/ min , 198 m/ min , 109 m/ min

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Exercise 7B

1 a $v = -2t$
b $a = -2$
c $x = 11$ metres, $v = -6$ m/s, $a = -2$ m/s^2
d distance from origin: 11 metres, speed: 6 m/s

2 a $v = 10t - 10$, $a = 10$. When $t = 1$, $x = -5$ metres, $v = 0$ m/s, $a = 10$ m/s^2.
b $v = 3 - 6t^2$, $a = -12t$. When $t = 1$, $x = 1$ metre, $v = -3$ m/s, $a = -12$ m/s^2.
c $v = 4t^3 - 2t$, $a = 12t^2 - 2$. When $t = 1$, $x = 4$ metres, $v = 2$ m/s, $a = 10$ m/s^2.

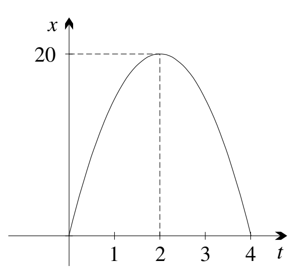
3 a $v = 2t - 10$
b displacement: -21 cm, distance from origin: 21 cm, velocity: $v = -4$ cm/s, speed: $|v| = 4$ cm/s
c When $v = 0$, $t = 5$ and $x = -25$.

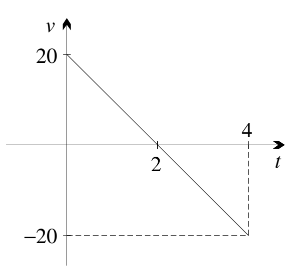
4 a $v = 3t^2 - 12t$, $a = 6t - 12$
b When $t = 0$, $x = 0$ cm, $|v| = 0$ cm/s and $a = -12$ cm/s^2.
c left $(x = -27 \text{ cm})$
d left $(v = -9 \text{ cm/s})$
e right $(a = 6 \text{ cm/s}^2)$
f When $t = 4$, $v = 0$ cm/s and $x = -32$ cm.
g When $t = 6$, $x = 0$, $v = 36$ cm/s and $|v| = 36$ cm/s.

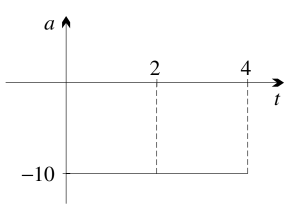
5 a $v = \cos t$, $a = -\sin t$, 1 cm, 0 cm/s, -1 cm/s^2
b $v = -\sin t$, $a = -\cos\ t$, 0 cm, -1 cm/s, 0 cm/s^2

6 a $v = e^t$, $a = e^t$, e metres, e m/s, e m/s^2
b $v = -e^{-t}$, $a = e^{-t}$, $1/e$ metres, $-1/e$ m/s, $1/e$ m/s^2

7 a $x = 5t(4 - t)$
$v = 20 - 10t$
$a = -10$







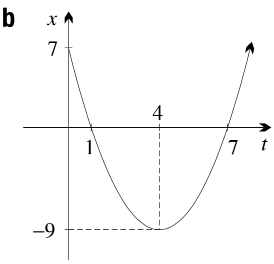
b 20 m/s
c It returns at $t = 4$; both speeds are 20 m/s.
d 20 metres after 2 seconds
e -10 m/s^2. Although the ball is stationary, its velocity is changing, meaning that its acceleration is non-zero.

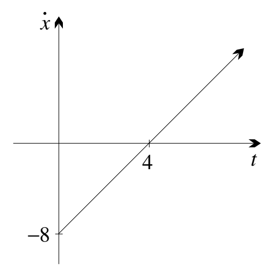
8 $\dot{x} = -4e^{-4t}$, $\ddot{x} = 16e^{-4t}$
a e^{-4t} is positive, for all t, so $\dot{x}$ is always negative and $\ddot{x}$ is always positive.
b i $x = 1$ **ii** $x = 0$
c i $\dot{x} = -4$, $\ddot{x} = 16$ **ii** $\dot{x} = 0$, $\ddot{x} = 0$

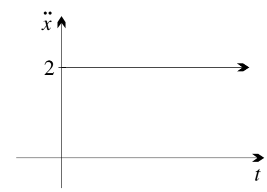
9 $v = 2\pi \cos \pi t$, $a = -2\pi^2 \sin \pi t$
a When $t = 1$, $x = 0$, $v = -2\pi$ and $a = 0$.
b i right $(v = \pi)$ **ii** left $\left(a = -\pi^2\sqrt{3}\right)$

10 a $x = (t - 7)(t - 1)$
$\dot{x} = 2(t - 4)$
$\ddot{x} = 2$

b

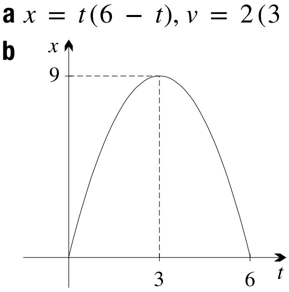


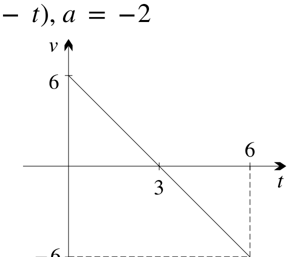




c i $t = 1$ and $t = 7$ **ii** $t = 4$
d i 7 metres when $t = 0$
ii 9 metres when $t = 4$
iii 27 metres when $t = 10$
e -1 m/s, $t = 3\frac{1}{2}$, $x = -8\frac{3}{4}$
f 25 metres, $3\frac{4}{7}$ m/s

11 a $x = t(6 - t)$, $v = 2(3 - t)$, $a = -2$
b





c i When $t = 2$, it is moving upwards and accelerating downwards.
ii When $t = 4$, it is moving downwards and accelerating downwards.
d $v = 0$ when $t = 3$. It is stationary for zero time, 9 metres up the plane, and is accelerating downwards at 2 m/s^2.
e 4 m/s. When $v = 4$, $t = 1$ and $x = 5$.
f All three average speeds are 3 m/s.

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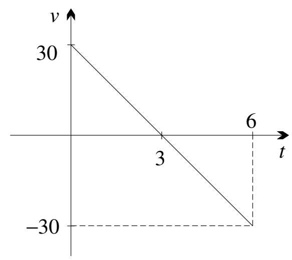
12 a 45 metres, 3 seconds, 15 m/s



b 30 m/s, 20, 10, 0, -10, -20, -30

c 0 seconds

d The acceleration was always negative.

e The velocity was decreasing at a constant rate of 10 m/s every second.

13 a 8 metres, when $t = 3$

b **i** when $t = 3$ and $t = 9$ (because the gradient is zero)

ii when $0 \le t < 3$ and when $t > 9$ (because the gradient is positive)

iii when $3 < t < 9$ (because the gradient is negative)

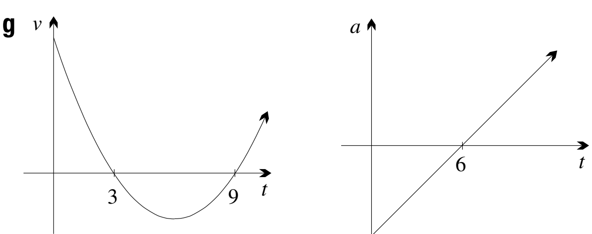
c $x = 0$ again when $t = 9$. Then $v = 0$ (because the gradient is zero) and it is accelerating to the right (because the concavity is upwards).

d at $t = 6$ (at the point of inflection the second derivative is zero), $x = 4$, moving to the left

e $0 \le t < 6$

f **i** $t \doteqdot 4, 12$ **ii** $t \doteqdot 10$

g



14 a $x = 4\cos\frac{\pi}{4}t$, $v = -\pi\sin\frac{\pi}{4}t$, $a = -\frac{1}{4}\pi^2\cos\frac{\pi}{4}t$

b maximum displacement: $x = 4$ when $t = 0$ or $t = 8$, maximum velocity: π m/s when $t = 6$, maximum acceleration: $\frac{1}{4}\pi^2$ m/s^2 when $t = 4$

c 40 metres, 2 m/s

d $1\frac{1}{3} < t < 6\frac{2}{3}$

e **i** $t = 0$, $t = 4$ and $t = 8$ **ii** $4 < t < 8$

15 a **i** $0 \le t < 8$ **ii** $0 \le t < 4$ and $t > 12$

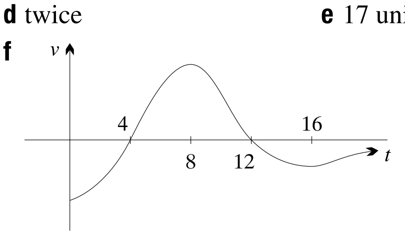
iii roughly $8 < t < 16$

b roughly $t = 8$

c **i** $t \doteqdot 5, 11, 13$ **ii** $t \doteqdot 13, 20$

d twice **e** 17 units

f



16 b **i** downwards (Downwards is positive here.)

ii upwards

c The velocity and acceleration tend to zero and the position tends to 12 metres below ground level.

d $t = 2\log_e 2$ minutes. The speed then is 3 m/min (half the initial speed of 6 m/min) and the acceleration is $-1\frac{1}{2}$ m/min^2 (half the initial acceleration of -3 m/min^2).

e When $t = 18$, $x \doteqdot 11.9985$ metres.
When $t = 19$, $x \doteqdot 11.9991$ metres.

Exercise 7C

1 a $x = t^3 - 3t^2 + 4$

b When $t = 2$, $x = 0$ metres and $v = 0$ m/s.

c $a = 6t - 6$

d When $t = 1$, $a = 0$ m/s^2 and $x = 2$ metres.

2 a $v = -3t^2$

b When $t = 5$, $v = -75$ cm/s and $|v| = 75$ cm/s.

c $x = -t^3 + 8$

d When $t = 2$, $x = 0$ cm and $a = -12$ cm/s^2.

3 a $v = 8t - 16$

b $x = 4t^2 - 16t + 16$

c When $t = 0$, $x = 16$, $v = -16$ and $|v| = 16$.

4 a $v = 6t - 30$ **b** 5 seconds **c** $a = 0$

5 a $v = 2t - 20$

b $x = t^2 - 20t$

c When $v = 0$, $t = 10$ and $x = -100$.

d When $x = 0$, $t = 0$ or 20. When $t = 20$, $v = 20$ m/s.

6 a $v = 10t$, $x = 5t^2$

b When $t = 4$, $x = 80$ metres, which is at the bottom, and $v = 40$ m/s.

c After 2 seconds, it has fallen 20 metres and its speed is 20 m/s.

d When $t = 2\sqrt{2}$, $x = 40$ metres, which is halfway down, and $v = 20\sqrt{2}$ m/s.

7 a $a = -10$, $v = -10t - 25$, $x = -5t^2 - 25t + 120$

b 3 seconds **c** 55 m/s **d** 40 m/s

8 a $\dot{x} = -4t$, $x = -2t^2$

b $\dot{x} = 3t^2$, $x = t^3$

c $\dot{x} = 2e^{\frac{1}{2}t} - 2$, $x = 4e^{\frac{1}{2}t} - 2t - 4$

d $\dot{x} = -\frac{1}{3}e^{-3t} + \frac{1}{3}$, $x = \frac{1}{9}e^{-3t} + \frac{1}{3}t - \frac{1}{9}$

e $\dot{x} = -4\cos 2t + 4$, $x = -2\sin 2t + 4t$

f $\dot{x} = \frac{1}{\pi}\sin \pi t$, $x = -\frac{1}{\pi^2}\cos \pi t + \frac{1}{\pi^2}$

g $\dot{x} = \frac{2}{3}t^{\frac{3}{2}}$, $x = \frac{4}{15}t^{\frac{5}{2}}$

h $\dot{x} = -12(t+1)^{-1} + 12$,
$x = -12\log_e(t+1) + 12t$

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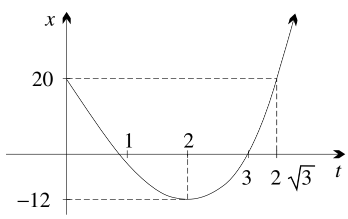
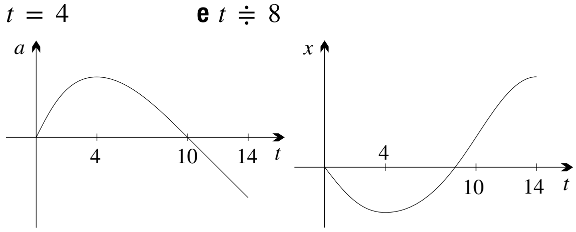
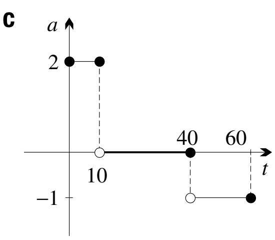
9 a $a = 0, x = -4t - 2$
b $a = 6, x = 3t^2 - 2$
c $a = \frac{1}{2}e^{\frac{1}{2}t}, x = 2e^{\frac{1}{2}t} - 4$
d $a = -3e^{-3t}, x = -\frac{1}{3}e^{-3t} - 1\frac{2}{3}$
e $a = 16\cos 2t, x = -4\cos 2t + 2$
f $a = -\pi\sin \pi t, x = \frac{1}{\pi}\sin \pi t - 2$
g $a = \frac{1}{2}t^{-\frac{1}{2}}, x = \frac{2}{3}t^{\frac{3}{2}} - 2$
h $a = -24(t + 1)^{-3}, x = -12(t + 1)^{-1} + 10$

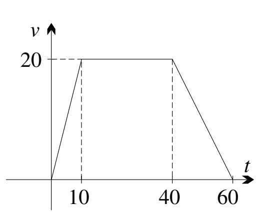
10 a $\dot{x} = 6t^2 - 24, x = 2t^3 - 24t + 20$
b $t = 2\sqrt{3}$, speed: 48 m/s
c $x = -12$ when $t = 2$.
d


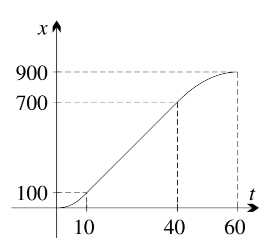
11 a $k = 6$ and $C = -9$, hence $a = 6t$ and $v = 3t^2 - 9$.
b i $x = t^3 - 9t + 2$
ii at $t = 3$ seconds (Put $x = 2$ and solve for t.)

12 a $4 < t < 14$ **b** $0 < t < 10$ **c** $t = 14$
d $t = 4$ **e** $t \doteqdot 8$
f


13 a 20 m/s

c






14 a $\ddot{x} = -4, x = 16t - 2t^2 + C$
b $x = C$ after 8 seconds, when the speed is 16 cm/s.
c $\dot{x} = 0$ when $t = 4$. Maximum distance right is 32 cm when $t = 4$, maximum distance left is 40 cm when $t = 10$. The acceleration is -4 cm/s^2 at all times.
d 104 cm, 10.4 cm/s

15 a $x = \log_e(t + 1) - 1, a = -\dfrac{1}{(t + 1)^2}$
b $e - 1$ seconds, $v = 1/e$, $a = -1/e^2$
c The velocity and acceleration approach zero, but the particle moves to infinity.

16 a $\dot{x} = -5 + 20e^{-2t}, x = -5t + 10 - 10e^{-2t}$, $t = \log_e 2$ seconds
b It rises $7\frac{1}{2} - 5\log_e 2$ metres, when the acceleration is 10 m/s^2 downwards.
c The velocity approaches a limit of 5 m/s downwards, called the *terminal velocity*.

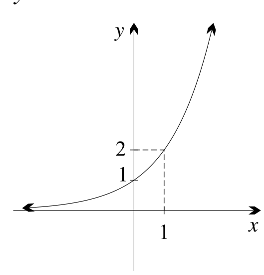
17 a $v = 1 - 2\sin t, x = t + 2\cos t$
b $\frac{\pi}{2} < t < \frac{3\pi}{2}$
c $t = \frac{\pi}{6}$ when $x = \frac{\pi}{6} + \sqrt{3}$, and $\frac{5\pi}{6}$ when $x = \frac{5\pi}{6} - \sqrt{3}$.
d 3 m/s when $t = \frac{3\pi}{2}$, and -1 m/s when $t = \frac{\pi}{2}$

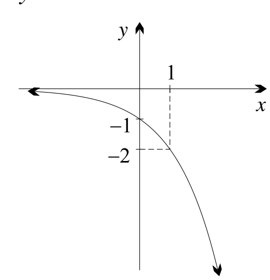
Exercise 7D

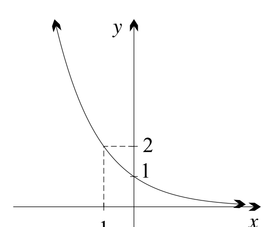
1 a 80 tonnes **b** When $t = 0, V = 0$.
c 360 tonnes **d** 20 tonnes/minute

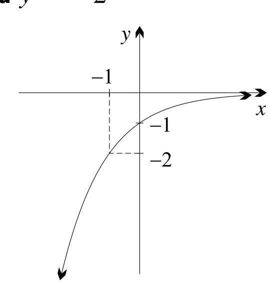
2 a 80 000 litres **b** 5000 litres
c 20 minutes, $0 \le t \le 20$ **d** 6000 L/min
e The tank is emptying, so F is decreasing.
f average rate $= \frac{80000}{20} = 4000$ L/min

3 a 1500 **b** 300 **c** 15 minutes
d average rate $= \frac{60000 - 1500}{15} = 300$ L/min (This is, of course, just the flow rate, which is constant.)

4 a $y = 2^x$


b $y = -2^x$


c $y = 2^{-x}$


d $y = -2^{-x}$


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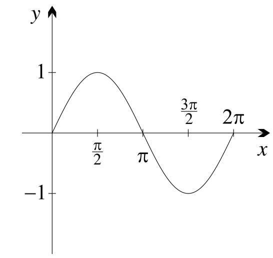
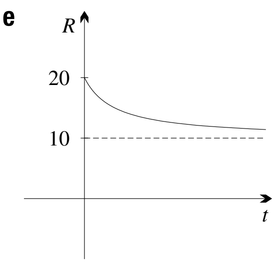


a **i** $0 \le x \le \frac{\pi}{2}$
ii $\frac{\pi}{2} \le x \le \pi$
iii $\pi \le x \le \frac{3\pi}{2}$
iv $\frac{3\pi}{2} \le x \le 2\pi$
b **i** $\pi \le x \le 2\pi$
ii $0 \le x \le \pi$

6 a $\dot{h} = 60e^{-\frac{t}{3}} - 30$
b 30 m/s upwards
c $h = 27.62$ m at $3 \ln 2 \doteqdot 2.079$ seconds
d $h \doteqdot 10.23$ m and speed is 15 m/s downwards
e 30 m/s downwards

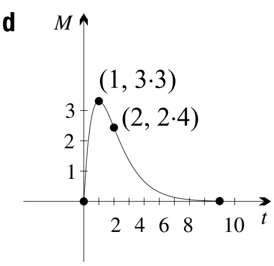
7 a **i** 12 kg/min
ii $10\frac{2}{3}$ kg/min
b 10 kg/min
c $\dot{R} = \frac{-20}{(1 + 2t)^2}$,
$\ddot{R} = \frac{80}{(1 + 2t)^3}$
d R is decreasing at a decreasing rate

e



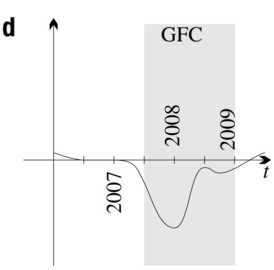
8 a $(0, 0)$ and $(9, 81e^{-9}) \doteqdot (9, 0.0)$
b $\dot{M} = 9(1 - t)e^{-t}$, $(1, 9e^{-1}) \doteqdot (1, 3.3)$
c $\ddot{M} = 9(t - 2)e^{-t}$, $(2, 18e^{-2}) \doteqdot (2, 2.4)$
e $t = 1$
f $t = 0$
g $t = 2$

d



9 a The graph is steepest in January 2008.
b It levels out in 2009?
c The LIBOR reduced at a decreasing rate. It may have been mistaken as indicating the crisis was ending.

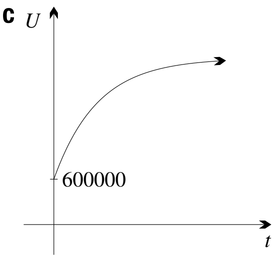
d



10 It took time for the scheme to start working with maximum pollution in the river reached in 2016. Then the level of pollution decreased at an increasing rate. But after 2017 the rate at which the pollution decreased gradually slowed down and was almost zero in 2020. A new scheme would have been required to remove the remaining pollution.

11 a Unemployment was increasing.
b The rate of increase was decreasing.

c



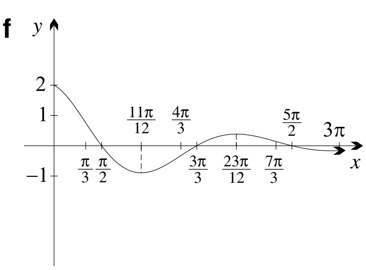
12 a $A = 9 \times 10^5$ **b** $N(1) = 380087$
c When t is large, N is close to 4.5×10^5.
d $\dot{N} = \frac{9 \times 10^5 e^{-t}}{(2 + e^{-t})^2}$

13 a $I = \dfrac{300t\left(2 - \frac{1}{5}t\right)}{200 + 3t^2 - \frac{1}{5}t^3}\%$ **b** $I(4) \doteqdot 6.12\%$
c $t = 0$ or 10. The latter is rejected because the model is only valid for 8 years.

14 b Exponentials are always positive.
c $\phi(0) = \frac{1}{\sqrt{2\pi}}$, $\lim_{x \to \infty} \phi(x) = 0$
d $\phi'(x) < 0$ for $x > 0$ (decreasing)
e At $x = 1$ and $x = -1$, where $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}}$.
g decreasing at increasing rate for $0 \le x \le 1$, decreasing at decreasing rate for $x \ge 1$.
h The curve approaches the horizontal asymptote more slowly for larger x.

15 a $y = 2$ and $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

f



Exercise 7E

1 a $y = 3t - 1$ **b** $y = 2 + t - t^2$
c $y = \sin t + 1$ **d** $y = e^t - 1$
2 b 15 min
3 a 25 minutes **c** 3145 litres
4 a **i** 3 cm^3/min **ii** 13 cm^3/min
b $E = \frac{1}{2}t^2 + 3t$
c **i** 80 cm^3 **i** 180 cm^3
5 b $t = 4$ **c** 57 **d** $t = 2$
6 a $P = 6.8 - 2\log_e(t + 1)$
b approximately 29 days
7 b $k = \frac{5}{24}$

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8 a no **b** $t \doteqdot 1.28$ **c** $x = \frac{5}{2}$

9 a 0 **b** 250 m/s

c $x = 1450 - 250(5e^{-0.2t} + t)$

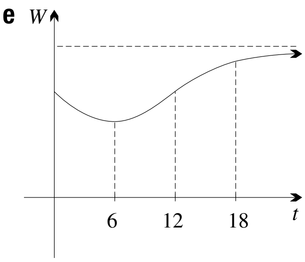
10 a It was decreasing for the first 6 months and increasing thereafter.

b after 6 months

c after 12 months

d It appears to have stabilised, increasing towards a limiting value.

e



11 a $-2\ \text{m}^3/\text{s}$ **b** 20 s

c $V = 520 - 2t + \frac{1}{20}t^2$ **d** $20\ \text{m}^3$

e 2 minutes and 20 seconds

12 a $V = \frac{1}{5}t^2 - 20t + 500$

b $t = 50 - 25\sqrt{2} \doteqdot 15$ seconds. Discard the other answer $t = 50 + 25\sqrt{2}$ because after 50 seconds the bottle is empty.

13 a $I = 18000 - 5t + \frac{48}{\pi}\sin\frac{\pi}{12}t$

b $\dfrac{dI}{dt}$ has a maximum of -1, so it is always negative.

c There will be 3600 tonnes left.

14 a $1200\ \text{m}^3$ per month at the beginning of July

b $W = 0.7t - \frac{3}{\pi}\sin\frac{\pi}{6}t$

Exercise 7F

1 a 4034 **b** 2.3 **c** 113.4 **d** 603

2 a 0.2695 **b** 2.77 **c** -12.5 **d** -2.7

3 a 20 **b** 66

c 24th **d** 5 rabbits per month

4 a 100 kg **b** 67 kg **c** 45 kg

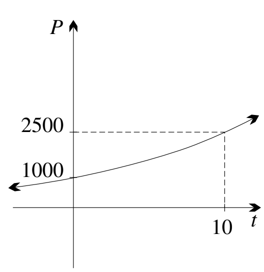
d 75 **e** 0.8 kg/year **f** 2 kg/year

5 a $k = \frac{1}{10}\log_e\frac{5}{2}$
$\doteqdot 0.092$

b 8230

c during 2020

d $\dfrac{dP}{dt} = kP$
$\doteqdot 905$



6 b 1350 **c** 135 per hour **d** 23 hours

7 c 6.30 grams, 1.46 grams per minute

d 6 minutes 58 seconds

e $20\,\text{g}$, $20e^{-k} \doteqdot 15.87\,\text{g}$, $20e^{-2k} \doteqdot 12.60\,\text{g}$,

$20e^{-3k} = 10\,\text{g}$, $r = e^{-k} = 2^{-\frac{1}{3}} \doteqdot 0.7937$

8 b $-\frac{1}{5}\log_e\frac{7}{10}$ **b** 10290

c At $t \doteqdot 8.8$, that is, some time in the fourth year from now.

9 b $h_0 = 100$ **c** $k = -\frac{1}{5}\log_e\frac{2}{5} \doteqdot 0.18$

d 6.4°C

10 b $k = \dfrac{\log_e 2}{5750} \doteqdot 1.21 \times 10^{-4}$

c $t = \frac{1}{k}\log_e\frac{100}{15} \doteqdot 16000$ years, correct to the nearest 1000 years.

11 b 30

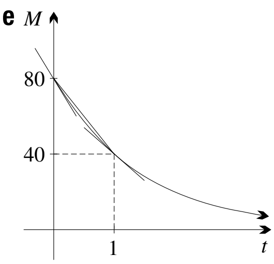
c i 26 **ii** $\frac{1}{5}\log_e\frac{15}{13}$ (or $-\frac{1}{5}\log_e\frac{13}{15}$)

12 a 80 g, 40 g, 20 g, 10 g

b 40 g, 20 g, 10 g. During each hour, the average mass loss is 50%.

c $M_0 = 80$,
$k = \log 2 \doteqdot 0.693$

e



d 55.45 g/hr, 27.73 g/hr, 13.86 g/hr, 6.93 g/hr

13 a 72% **b** 37% **c** 7%

14 a $k = \dfrac{\log_e 2}{1690} \doteqdot 4.10 \times 10^{-4}$

b $t = \dfrac{\log_e 5}{k} \doteqdot 3924$ years

15 b $\mu_1 = 1.21 \times 10^{-4}$

c $\mu_2 = 1.16 \times 10^{-4}$

d The values of μ differ so the data are inconsistent.

e i 625.5 millibars

ii 1143.1 millibars

iii 19205 metres

16 b $L = \frac{1}{2}$

17 a 34 minutes **b** 2.5%

18 b $C_0 = 20000$, $k = \frac{1}{5}\log_e\frac{9}{8} \doteqdot 0.024$

c 64946 ppm

d i 330 metres from the cylinder

ii If it had been rounded down, then the concentration would be above the safe level.

19 a $p = \frac{1}{10}\log_e\frac{5}{4} \doteqdot 0.022$, $q = \frac{1}{10}\log_e\frac{6}{5} \doteqdot 0.018$

b $t = \frac{\log_e 2}{p + q} \doteqdot 17.10$ years, that is during 2017.

Chapter 7 review exercise

1 a $x = 24$, $x = 36$, 6 cm/s

b $x = 16$, $x = 36$, 10 cm/s

c $x = -8$, $x = -8$, 0 cm/s

d $x = 9$, $x = 81$, 36 cm/s

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2 a $v = 40 - 2t, a = -2$, 175 m, 30 m/s, -2 m/s^2

b $v = 3t^2 - 25, a = 6t$, 0 m, 50 m/s, 30 m/s^2

c $v = 8(t - 3), a = 8$, 16 m, 16 m/s, 8 m/s^2

d $v = -4t^3, a = -12t^2$, -575 m, -500 m/s, -300 m/s

e $v = 4\pi \cos \pi t$ $a = -4\pi^2 \sin \pi t$, 0 m, -4π m/s, 0 m/s^2

f $v = 21\,e^{3t-15}, a = 63\,e^{3t-15}$, 7 m, 21 m/s, 63 m/s^2

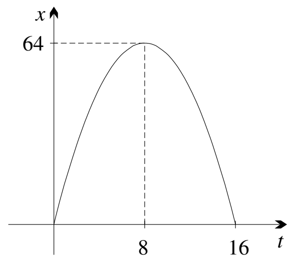
3 a $v = 16 - 2t, a = -2$

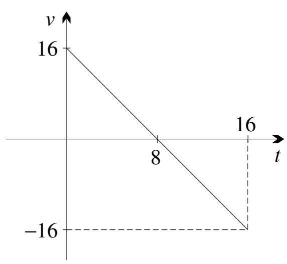
b 60 m, -4 m/s, 4 m/s, -2 m/s^2

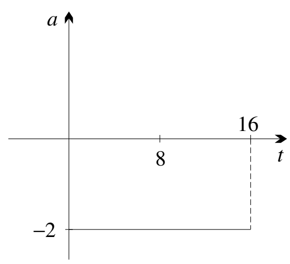
c $t = 16$ s, $v = -16$ m/s

d $t = 8$ s, $x = 64$ m

e







4 a $a = 0, x = 7t + 4$

b $a = -18t, x = 4t - 3t^3 + 4$

c $a = 2(t - 1), x = \frac{1}{3}(t - 1)^3 + 4\frac{1}{3}$

d $a = 0, x = 4$

e $a = -24 \sin 2t, x = 4 + 6 \sin 2t$

f $a = -36e^{-3t}, x = 8 - 4e^{-3t}$

5 a $v = 3t^2 + 2t, x = t^3 + t^2 + 2$

b $v = -8t, x = -4t^2 + 2$

c $v = 12t^3 - 4t, x = 3t^4 - 2t^2 + 2$

d $v = 0, x = 2$

e $v = 5 \sin t, x = 7 - 5 \cos t$

f $v = 7e^t - 7, x = 7e^t - 7t - 5$

6 a $\dot{x} = 3t^2 - 12, x = t^3 - 12t$

b When $t = 2, \dot{x} = 0$.

c 16 cm

d $2\sqrt{3}$ seconds, 24 cm/s, $12\sqrt{3}$ cm/s^2

e As $t \to \infty, x \to \infty$ and $v \to \infty$.

7 a The acceleration is 10 m/s^2 downwards.

b $v = -10t + 40, x = -5t^2 + 40t + 45$

c 4 seconds, 125 metres

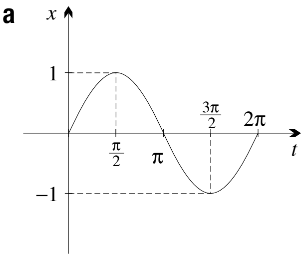
d When $t = 9, x = 0$.

e 50 m/s

f 80 metres, 105 metres

g 25 m/s

8 a



b $t = \pi$ and $t = 2\pi$

c $\dot{x} = -\cos t$

d $t = \frac{\pi}{2}$

e **i** $x = 5 - \sin t$

ii $x = 4$

9 a $v = 20$ m/s

b $20\,e^{-t}$ is always positive.

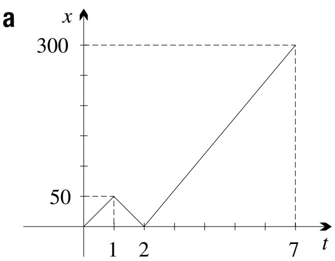
c $a = -20\,e^{-t}$

d -20 m/s^2

e $x = 20 - 20\,e^{-t}$

f As $t \to \infty, a \to 0, v \to 0$ and $x \to 20$.

10 a



b 400 km

c $57\frac{1}{7}$ km/hr

11 a $x = 20$ m, $v = 0$

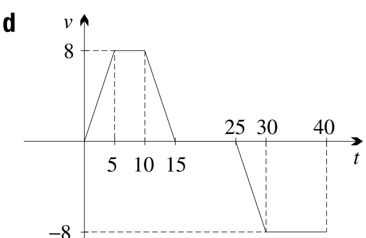
b **i** 8 m/s **ii** 0 **iii** -8 m/s

c **i** north (The graph is concave up.)

ii south (The graph is concave down.)

iii south (The graph is concave down.)

d



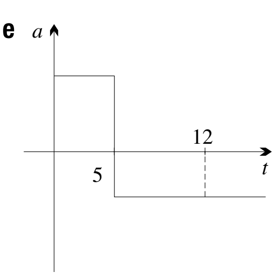
12 a at $t = 5$

b at $t = 12, 0 < t < 12, t > 12$

c $0 < t < 5, t > 5$

d at $t = 12$, when the velocity was zero

e



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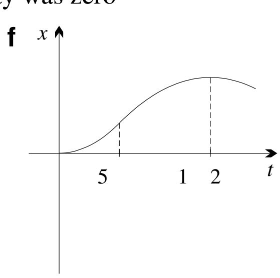

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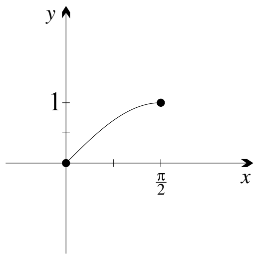
13 a i



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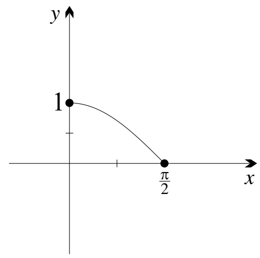
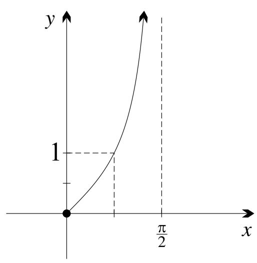
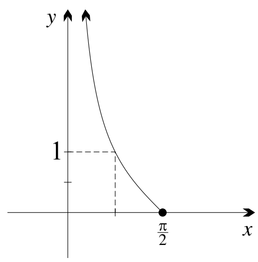


iii



iv

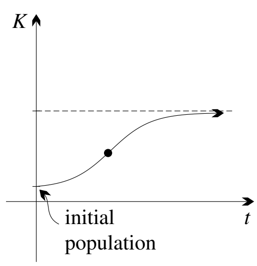


b i $y = \sin x$ **ii** $y = \cos x$

iii $y = \cot x$ **iv** $y = \tan x$

14 a Initially K increases at an increasing rate so the grpah is concave up. Then K increases at a decreasing rate so is concave down. The change in concavity coincides with the inflection point.

b



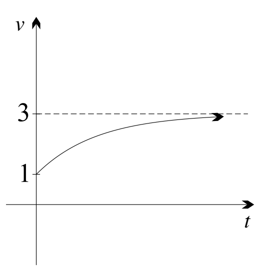
15 a 7500 L

b $\dot{V} = -12(50 - 2t)$

c $\dot{V}$ is negative in the given domain.

d $\dot{V}$ is negative and $\ddot{V} = 24$ is positive, so the outflow decreases.

16 a



b $\dot{x}$ increases so it accelerates.

c $\ddot{x} = \frac{2}{5}e^{-\frac{1}{5}t}$ which is always positive.

d $\lim\limits_{t \to \infty} \dot{x} = 3\,\text{m/s}$

$x = 3t + 10\left(e^{-\frac{1}{5}t} - 1\right)$

17 a 13 000 **b** 0.088 **c** 31 400

18 a 56 min 47 s **b** 48 grams

Chapter 8

Exercise 8A

1 a $T_3 - T_2 = T_2 - T_1 = 7$ **b** $a = 8, d = 7$

c 358 **d** 1490

2 a $a = 2, d = 2$ **b** 250 500

3 a $\dfrac{T_3}{T_2} = \dfrac{T_2}{T_1} = 2$ **b** $a = 5, r = 2$

c 320 **d** 635

4 a $\dfrac{T_3}{T_2} = \dfrac{T_2}{T_1} = \frac{1}{2}$ **b** $a = 96, r = \frac{1}{2}$

c $\frac{3}{4}$ **d** $191\frac{1}{4}$

e $-1 < r < 1, S_\infty = 192$

5 a i $a = 52, d = 6$

ii 14 **iii** 1274

b i 125 **ii** $-\dfrac{25}{49}$

c i $d = -3$

ii $T_{35} = -2$

iii $S_n = \frac{1}{2}n(203 - 3n)$

6 a i 1.01

ii $T_{20} = 100 \times 1.01^{19} \doteqdot 120.81$

iii 2201.90

b i $\frac{3}{2}$

ii 26 375

iii $|r| = \frac{3}{2} > 1$

c i $\frac{1}{3}$

ii $|r| = \frac{1}{3} < 1, S_\infty = 27$

7 a \$96 000, \$780 000 **b** the 7th year

8 a $r = 1.05$ **b** \$124 106, \$1 006 232

9 a i All the terms are the same.

ii The terms are decreasing.

b If $r = 0$, then $T_2 \div T_1 = 0$, so $T_2 = 0$. Hence $T_3 \div T_2 = T_3 \div 0$ is undefined.

c i The terms alternate in sign.

ii All the terms are the same.

iii The terms are $a, -a, a, -a, \ldots$.

iv The terms are decreasing in absolute value.

10 a \$50 000, \$55 000, \$60 000, $d = \$5000$

b \$40 000, \$46 000, \$52 900, $r = 1.15$

c For Lawrence $T_5 = \$70\,000$ and $T_6 = \$75\,000$. For Julian $T_5 \doteqdot \$69\,960.25$ and $T_6 \doteqdot \$80\,454.29$. The difference in T_6 is about \$5454.

11 a i $T_n = 47\,000 + 3000n$

ii the 18th year

b \$71 166

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12 a 12 metres, 22 metres, 32 metres
b $10n + 2$
c i 6 **ii** 222 metres
13 a 18 times **b** 1089 **c** Monday
14 a 85 000 **b** 40 000
15 a $D = 6400$ **b** $D = 7600$ **c** the 15th year
d $S_{13} = \$1\,092\,000$, $S_{14} = 1\,204\,000$
16 a 40 m, 20 m, 10 m, $a = 40$, $r = \frac{1}{2}$
b 80 m
c The GP has ratio $r = 2$ and hence does not converge. Thus Stewart would never stop running.
17 a $\left(\frac{1}{2}\right)^{\frac{1}{4}}$
b $S_\infty = \dfrac{F}{1 - \left(\frac{1}{2}\right)^{\frac{1}{4}}} \doteqdot 6.29F$
18 a i $r = \cos^2 x$ **ii** $x = 0, \pi, 2\pi$
iv When $\cos x = 0$, the series is not a GP because the ratio cannot be zero. But the series is then $1 + 0 + 0 + \cdots$, which trivially converges to 1. When $\cos x = 0$, then $\sin x = 1$ or -1, so $\operatorname{cosec}^2 x = 1$, which means that the given formula for S_∞ is still correct.
b i $r = \sin^2 x$ **ii** $x = \frac{\pi}{2}, \frac{3\pi}{2}$
iv When $\sin x = 0$, the series is not a GP because the ratio cannot be zero. But the series is then $1 + 0 + 0 + \cdots$, which trivially converges to 1. When $\sin x = 0$, then $\cos x = 1$ or -1, so $\sec^2 x = 1$, which means that the given formula for S_∞ is still correct.
19 b at $x = 16$
c i at $x = 18$, halfway between the original positions
ii 36 metres, the original distance between the bulldozers

Exercise 8B

1 a 5 **b** 14 **c** 23 **d** 3
e 9 **f** 15 **g** 4 **h** 8
i 14 **j** 2 **k** 5 **l** 11
2 a $\dfrac{T_3}{T_2} = \dfrac{T_2}{T_1} = 1.1$
b $a = 10$, $r = 1.1$
c $T_{15} = 10 \times 1.1^{14} \doteqdot 37.97$
d 19
3 a $r = 1.05$
b \$62 053, \$503 116
c the 13th year
4 the 19th year
7 a SC 50: 50%, SC 75: 25%, SC 90: 10%
c 4 **d** at least 7
8 a $T_n = 3 \times \left(\dfrac{2}{3}\right)^{n-1}$ **b** 4.5 metres
c ii 16
9 a the 10th year **b** the 7th year
10 a Increasing by 100% means doubling, increasing by 200% means trebling, increasing by 300% means multiplying by 4, and so on.
b Solve $(1.25)^n > 4$. The smallest integer solution is $n = 7$.
11 a $S_n = \dfrac{3\left(1 - \left(\frac{2}{3}\right)^n\right)}{1 - \frac{2}{3}} = 9\left(1 - \left(\frac{2}{3}\right)^n\right)$
b The common ratio is less than 1. $S = 9$
c $n = 17$

Exercise 8C

1 a i \$900 **ii** \$5900
b i \$120 **ii** \$420
c i \$3750 **ii** \$13 750
d i \$5166 **ii** \$17 166
2 a i \$5955.08 **ii** \$955.08
b i \$443.24 **ii** \$143.24
c i \$14 356.29 **ii** \$4356.29
d i \$18 223.06 **ii** \$6223.06
3 a i \$4152.92 **ii** \$847.08
b i \$199.03 **ii** \$100.97
c i \$6771.87 **ii** \$3228.13
d i \$7695.22 **ii** \$4304.78
4 a \$507.89 **b** \$1485.95
c \$1005.07 **d** \$10 754.61
5 a \$6050 **b** \$25 600 **c** 11 **d** 5.5%
6 a $A_n = 10\,000(1 + 0.065 \times n)$
b $A_{15} = \$19\,750$, $A_{16} = \$20\,400$
7 a \$101 608.52 **b** \$127 391.48
8 a Howard — his is \$21 350 and hers is \$21 320.
b Juno — hers is now \$21 360.67 so is better by \$10.67.
9 a \$1120 **b** \$1123.60 **c** \$1125.51 **d** \$1126.83
10 a \$8000 **b** \$12 000 **c** \$20 000
11 \$19 990
12 a \$7678.41 **b** \$1678.41
c 9.32% per annum
13 a \$12 209.97
b 4.4% per annum
c Solve $10\,000 \times \left(\frac{1.04}{12}\right)^n > 15\,000$. The smallest integer solution is $n = 122$ months.
14 \$1 110 000
15 a 24 **b** 14 **c** 7 **d** 9
16 $A_n = 6000 \times 1.12^n$ **a** 7 years
b 10 years **c** 13 years **d** 21 years

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17 8 years and 6 months

18 a $C = C_0 \times 1.01^t$

i $1.01^{12} - 1 \doteqdot 12.68\%$

ii $\log_{1.01} 2 \doteqdot 69.66$ months

b $k = \log_e 1.01$

i $e^{12k} - 1 \doteqdot 12.68\%$

ii $\frac{1}{k}\log_e 2 \doteqdot 69.66$ months

19 7.0%

20 a \$5250 **b** \$20 250

c 6.19% per annum

21 a \$40 988 **b** \$42 000

22 b 3 years

Exercise 8D

1 a **i** \$732.05 **ii** \$665.50 **iii** \$605

iv \$550 **v** \$2552.55

b **i** \$550, \$605, \$665.50, \$732.05

ii $a = 550, r = 1.1, n = 4$

iii \$2552.55

2 a **i** \$1531.54 **ii** \$1458.61

iii \$1389.15, \$1323, \$1260 **vi** \$6962.30

b **i** \$1260, \$1323, \$1389.15, \$1458.61, \$1531.54

ii $a = 1260, r = 1.05, n = 5$

iii \$6962.30

3 a **i** $\$1500 \times 1.07^{15}$

ii $\$1500 \times 1.07^{14}$

iii $\$1500 \times 1.07$

iv $A_{15} = (1500 \times 1.07) + (1500 \times 1.07^2) + \cdots + (1500 \times 1.07^{15})$

b \$40 332

4 a **i** $\$250 \times 1.005^{24}$

ii $\$250 \times 1.005^{23}$

iii $\$250 \times 1.005$

iv $A_{24} = (250 \times 1.005) + (250 \times 1.005^2) + \cdots + (250 \times 1.005^{24})$

b \$6390

5 a **i** $\$3000 \times 1.065^{25}$

ii $\$3000 \times 1.065^{24}$

iii $\$3000 \times 1.065$

iv $A_{25} = (3000 \times 1.065) + (3000 \times 1.065^2) + \cdots + (3000 \times 1.065^{25})$

c \$188 146 and \$75 000

6 b \$669 174.36 **c** \$429 174.36 **e** \$17 932.55

7 c iii 18

8 a \$200 000 **b** \$67 275 **c** \$630 025

d **i** $A_n = 100000 \times 1.1 \times ((1.1)^n - 1)$

iii 25

e $\frac{100000}{630025} \times 10000 \doteqdot \$15872$

9 a \$360 **b** \$970.27

10 a \$31 680 **b** \$394 772 **c** \$1 398 905

11 a \$134 338 **b** \$309 281

12 \$3086

13 a \$286 593

b **i** \$107 355 **ii** \$152 165

14 a \$27 943.29 **b** the 19th year

15 a 18

16 The function FV calculates the value just after the last premium has been paid, not at the end of that year.

17 c $A_2 = 1.01M + 1.01^2M$,

$A_3 = 1.01M + 1.01^2M + 1.01^3M$,

$A_n = 1.01M + 1.01^2M + \cdots + 1.01^nM$

e \$4350.76 **f** \$363.70

18 b $A_2 = 1.002 \times 100 + 1.002^2 \times 100$,

$A_3 = 1.002 \times 100 + 1.002^2 \times 100 + 1.002^3 \times 100$,

$A_n = 1.002 \times 100 + 1.002^2 \times 100 + \cdots + 1.002^n \times 100$

Exercise 8E

1 b **i** \$210.36 **ii** \$191.24 **iii** \$173.86

iv \$158.05 **v** \$733.51

c **i** \$158.05, \$173.86, \$191.24, \$210.36

ii $a = 158.05, r = 1.1, n = 4$

iii \$733.51

2 b **i** \$1572.21 **ii** \$1497.34

iii \$1426.04, \$1358.13, \$1293.46

iv \$7147.18

c **i** \$1293.46, \$1358.13, \$1426.04, \$1497.34, \$1572.21

ii $a = 1293.46, r = 1.05, n = 5$

iii \$7147.18

3 a **ii** $\$1646.92 \times 1.07^{14}$ **iii** $\$1646.92 \times 1.07^{13}$

iv $\$1646.92 \times 1.07$ **v** \$1646.92

vi $A_{15} = 15000 \times (1.07)^{15} - (1646.92 + 646.92 \times 1.07 + \cdots + 1646.92 \times (1.07)^{13} + 1646.92 \times (1.07)^{14})$

c \$0

4 a **i** $100000 \times 1.005^{240}$

ii $M \times 1.005^{239}$

iii $M \times 1.005^{238}$ and M

iv $A_{240} = 100000 \times 1.005^{240} - (M + 1.005M + 1.005^2M + \cdots + 1.005^{239}M)$

c The loan is repaid. **d** \$716.43

e \$171 943.20

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5 a **i** 10000×1.015^{60}

ii $M \times 1.015^{59}$

iii $M \times 1.015^{58}$ and M

iv $A_{60} = 10000 \times 1.015^n - (M + 1.015M + \cdots + 1.015^2 M + 1.015^{59} M)$

c \$254

6 a $A_{180} = 165000 \times 1.0075^{180} - (1700 + 1700 \times 1.0075 + 1700 \times 1.0075^2 + \cdots + 1700 \times 1.0075^{179})$

c −\$10012.67

7 a $A_n = 250000 \times 1.006^n - (2000 + 2000 \times 1.006 + 2000 \times 1.006^2 + \cdots + 2000 \times 1.006^{n-1})$

c \$162498, which is more than half.

d −\$16881 **f** 8 months

8 c It will take 57 months, but the final payment will only be \$5490.41.

9 a The loan is repaid in 25 years.

c \$1226.64 **d** \$367993

e \$187993 and 4.2% pa

10 b \$345

11 a \$4202 **b** $A_{10} = \$6.66$

c Each instalment is approximately 48 cents short because of rounding.

12 b \$216511

13 a \$2915.90 **b** \$84.10

14 a \$160131.55

b \$1633.21 < \$1650, so the couple can afford the loan.

15 b zero balance after 20 years

c \$2054.25

16 \$44131.77

17 b 57

18 c $A_2 = 1.005^2 P - M - 1.005M$,
$A_3 = 1.005^3 P - M - 1.005M - 1.005^2 M$,
$A_n = 1.005^n P - M - 1.005M - \cdots - 1.005^{n-1} M$

e \$1074.65 **f** \$34489.78

19 b $A_2 = 1.008^2 P - M - 1.008M$,
$A_3 = 1.008^3 P - M - 1.008M - 1.008^2 M$,
$A_n = 1.008^n P - M - 1.008M - \cdots - 1.008^{n-1} M$

d \$136262

e $n = \log_{1.008} \dfrac{125M}{125M - P}$, 202 months

Chapter 8 review exercise

1 a $a = 31, d = 13$ **b** 16 **c** 2056

2 a $\frac{1}{2}$ **b** $|r| = \dfrac{1}{2} < 1$ **c** $S_\infty = 48$

3 a $n = 11$ **b** $n = 99$ **c** $n = 228$ **d** $n = 14$

4 27000 litres

5 ‘Increasing by 2000%’ means that the profit is 21 times larger. The smallest integer solution of $(1.14)^n > 21$ is $n = 24$.

6 a $r = 1.04$ **b** \$49816, \$420214

7 a $T_n = 43000 + 4000n$ **b** 2017

8 2029

9 a \$15593.19 **b** \$3593.19 **c** 5.99%

10 a $\$25000 \times (0.88)^4 \doteqdot 14992$

b \$2502 per year

c $\$25000 \div (0,88)^4 \doteqdot \$41688$

d \$4172 per year

11 b \$224617.94

c \$104617.94

d The value is \$277419.10, with contributions of \$136000.00.

13 a $A_{180} = 159000 \times 1.005625^{180} - (1415 + 1415 \times 1.005625 + 1415 \times 1.005625^2 + \cdots + 1415 \times 1.005625^{179})$

c −\$2479.44

d \$1407.01

14 a $A_n = 1700000 \times 1.00375^n - (18000 + 18000 \times 1.00375 + 18000 \times 1.00375^2 + \cdots + 18000 \times 1.00375^{n-1})$

c \$919433, which is more than half.

d −\$57677.61

f 3 months

Chapter 9

Exercise 9A

1 a categorical

b numeric and continuous. But ‘height correct to the nearest mm’ is numeric and discrete.

c numeric and continuous. But ‘age in years’ is numeric and discrete.

d categorical by party or political code. This would need to be defined carefully — if a person can be affiliated to two parties, it would not be a function.

e categorical

f categorical

g numeric and discrete

h Shoe sizes are often arranged into categories.

i These are frequently integers from 1–100, that is, numeric and discrete. If results are reported by a grade, for example, A, B, C, . . . , this might be considered categorical.

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2 a median 14, mode 14, range 8

b median 10, every score is trivially a mode, range 12

c median 8, mode 3, range 12

d median 6.5, mode 4 & 6, range 6

e median 4, mode 4, range 7

f median 5.5, mode 2 & 3 & 9, range 8

3 a

score x	1	2	3	4	5	6	7	8
frequency f	4	3	4	2	1	1	1	6
cumulative	4	7	11	13	14	15	16	22

b 3.5

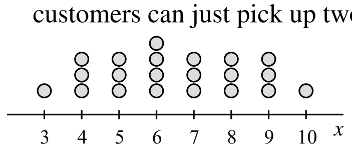
c 8

d **i** This is a median, but it might be more useful to use the mode in this case. It may be easier to develop a square box for four cupcakes rather than three.

ii See the previous comments. It is also common for sales to package a larger box to encourage customers to overbuy.

iii This is the mode, but if a box of four is marketed, customers can just pick up two boxes of four.

4 a



b

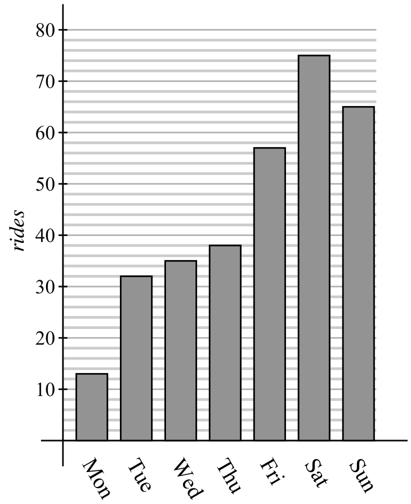
score x	3	4	5	6	7	8	9	10
frequency f	1	3	3	4	3	3	3	1
cumulative	1	4	7	11	14	18	20	21

c 6 hoops

d 6.5 hoops

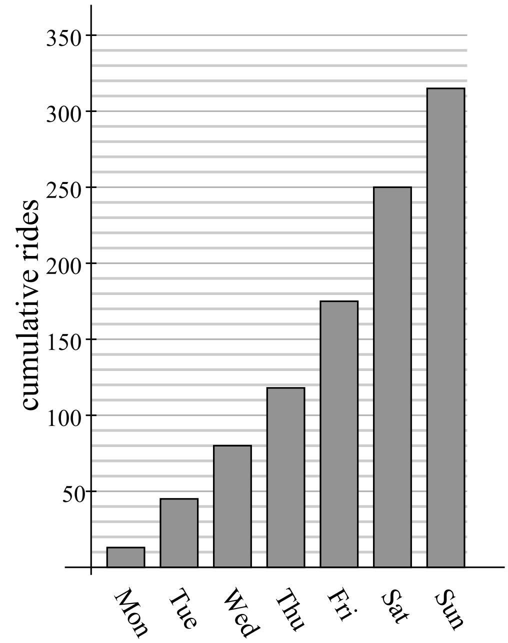
e Not really. If the scores are ordered by time, his scores improve over the sessions. This information is lost in the table and plot.

5 a



b

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
frequency	13	32	35	38	57	75	65
cumulative	13	45	80	118	175	250	315



6 a Blond hair and blue eyes. Different results might be expected in a different part of the world.

b Red hair and green eyes

c 45%

d 17%

e $25 \div 54 \doteqdot 46\%$

f $90 \div 247 \doteqdot 36\%$

g $671 \div 753 \doteqdot 89\%$

h These two results would suggest so. Geneticists link this to various pigment genes that affect both characteristics.

i The proportion of the various eye and hair colours will vary in different genetic populations and ethnic groups. Studies such as this may be done with a relatively non-diverse population to prevent the clouding effects of differing genetics.

7 a 80

b

salad	pie	soup	panini	burger
32.5%	12.5%	8.75%	20%	26.25%

c

salad	pie	soup	panini	burger
$130	$60	$70	$96	$168

d $524

e It returns more money than the more popular pie option. It is probably also important for the café to include a vegetarian option on the menu to cater for such customers or for groups with such customers.

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8 a In 2002 the price was \$400 thousand, and in 2017 it was \$1 million.

b Prices increased by 150%.

c \$40 thousand per year

d They will increase another $13 \times \$40000 = \$520000$ to around \$1.5 million.

e From 2014 to 2015, median house prices increased \$120 thousand.

f From 2010 to 2011, median house prices decreased \$40 thousand. How much did prices change?

9 a 35%, 140 dogs **b** 11%

c 75% **d** 15%

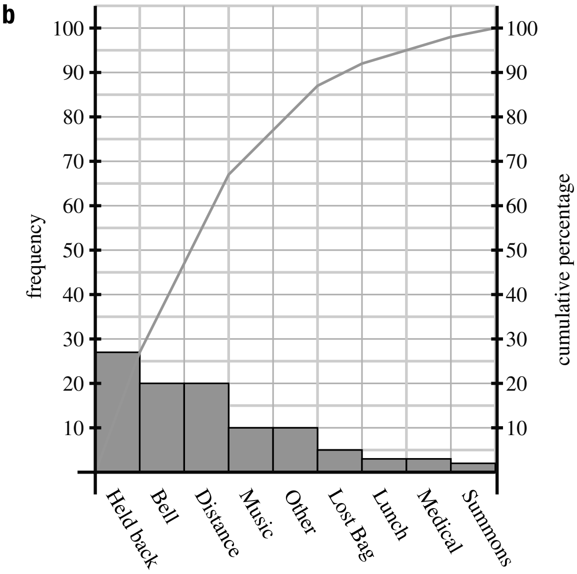
e This is quite a large category, and it may be that more investigation should be done to see if there were any other popular types of pets lumped into this category.

f Some pets may require more care and attention. For example, dogs may require frequent exercise and attention. This may give an opportunity for 'value adding' if owners are willing to pay for it. They should also consider what other pet boarding facilities are in the area, because it may be better to pick up a niche market, not covered by other pet boarding houses. Some pets may also be able to use the same types of accommodation, for example, rabbits and guinea pigs.

10 a

Reason	frequency	cumulative
Held back	27	27
Bell	20	47
Distance	20	67
Music	10	77
Other	10	87
Lost bag	5	92
Lunch	3	95
Medical	3	98
Summons	2	100

b



c The categories are arranged in descending order, so the function will be increasing (if every frequency is greater than zero), but by less at nearly every stage, causing it to curve downwards.

d 67%

e Remind teachers to release students promptly, increase the volume of the bell or the number of locations where the bell sounds, timetable students in rooms closer together where possible.

11 a 6% **b** 64% **c** 5%

d Care is needed when the graph is read in a hurry. Compare this with the Pareto chart later in this exercise where both axes are the same scale.

12 a The vertical origin is not at a 0% unemployment rate. This exaggerates the scale of the graph, which only shows a variation of 0.25%. This is still potentially significant, but it is only shown over a four-month period, so it is impossible to examine long-term trends. There are natural cycles — for example, there may be a rise when school pupils enter the employment market, and a drop when Christmas provides short-term retail employment. January may be a low point in economic indicators, before businesses return from holidays and begin to hire staff.

b There has been a significant increase over this five-year period, but more questions need to be asked by someone viewing the graph. What does the vertical scale represent — is it spending per citizen or spending per household? If it is per household, have the household structures changed over the period, such as more larger households? Is this a

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small community, in which case the data won't be very robust to changes in population? Is the data collected from sales at local shops, and does it include tourists and people passing through — has there been an increase in tourism, and was the data collected at the same time of year (more takeaways may be sold at the height of the tourist season)? What is included in the category of 'takeaway food' — if this is a health study, takeaway salads may be considered healthier than takeaway burgers (which the graphic is trying to suggest). Finally, note that the eye interprets the increase by the size of the graphic, but in fact it is the height that holds information, suggesting a greater increase than was actually the case.

c **i** People who do not have access to the internet, or do not feel as comfortable accessing and filling in an online survey, will not be represented. This may be more prevalent amongst older demographics.

ii The group should look at other hospitals, unless they particularly want to investigate the change in costs at their local hospital. Hospital costs could be influenced by government policy increasing the staffing numbers at the hospital, by purchase of new expensive diagnostic equipment, by opening and closing particular hospital wards (possibly relocating them to other hospitals), by quality control improvements, by industrial action of staff, and so on. The group likely will want to investigate the cause of any changes to overall expenses and may want to produce graphs of particular expenses, such as doctors' fees. They need to be clear what questions they actually want to ask — for example, are they concerned that medical treatment is getting more expensive for certain sections of the community who cannot afford it?

13 a 58%

b Around 3 billion

c About 0.92 billion

d 5.2%

e It may be of some use if choosing a major world language is a consideration, but there are often other considerations in deciding what language to learn. For example, you may have relatives who speak Malay, or a girl-friend who is French, or you may want to learn Japanese because of Japan's importance to Australia's economy. Others learn languages for academic reasons, such as Latin because of its historical and linguistic importance, or Russian to study Russian literature. When deciding a language on the number of speakers, it is probably more useful to consider the total number of speakers, not merely those who speak it as a first language — close to a billion people speak English, but only a third of them do so as a first language.

14 a **i** 15°C **ii** 30°C

b **i** 17°C **ii** 23°

c Around 6−7°C in December–January

d September and May

e November–February

f June–August

g Colour-blind readers may find the colours difficult to distinguish. Using dashes and colour also provides two visuals cues for the bulk of readers, making the graph easier to read.

15 a 30

b 73% and 27%

c Bill on Essay writing, Claire on Interpretation, Ellie on all sections.

d 80%

e 40%

f Aaron and Dion. Notice that Claire has not reached 50% in the Interpretation section.

16 a Provided that similar levels of postgraduates survive to the 55–64 age bracket.

b 61.1% **c** 21.6%

Exercise 9B

1 a $\bar{x} = 7$, Var $= 3.6$, $s \doteqdot 1.90$

x	f	xf	$(x - \bar{x})^2$	$(x - \bar{x})^2 f$
3	1	3	16	16
5	1	5	4	4
6	1	6	1	1
7	3	21	0	0
8	2	16	1	2
9	1	9	4	4
10	1	10	9	9
Total	10	70		36

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b $\bar{x} = 7$, Var $= 3.6$, $s \doteqdot 1.90$

x	f	xf	$x^2 f$
3	1	3	9
5	1	5	25
6	1	6	36
7	3	21	147
8	2	16	128
9	1	9	81
10	1	10	100
Total	10	70	526

2 a $\bar{x} = 18$, $s \doteqdot 3.67$ **b** $\bar{x} = 7$, $s \doteqdot 3.06$
c $\bar{x} = 55$, $s \doteqdot 7.58$ **d** $\bar{x} = 11$, $s \doteqdot 1.88$

3 a $\bar{x} \doteqdot 7.17$, $s \doteqdot 3.18$ **b** $\bar{x} = 5.7$, $s \doteqdot 1.73$
c $\bar{x} = 3.03$, $s \doteqdot 0.94$ **d** $\bar{x} \doteqdot 42.88$, $s \doteqdot 10.53$

4 a 34 **b** $\mu \doteqdot 3.26$, $\sigma \doteqdot 1.75$

c

class	0−2	3−5	6−8
centre	1	4	7
freq	12	18	4

d $\mu \doteqdot 3.29$, $\sigma \doteqdot 1.93$

e Information is lost when data are grouped, causing the summary statistics to change.

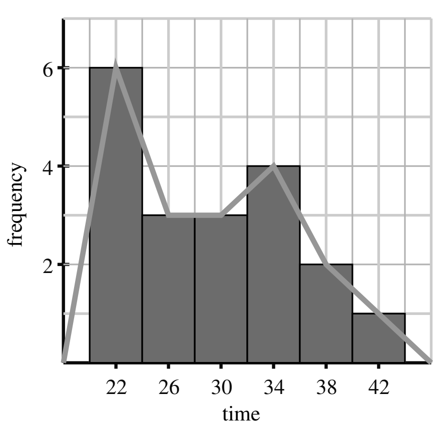
5 a 29.5

b

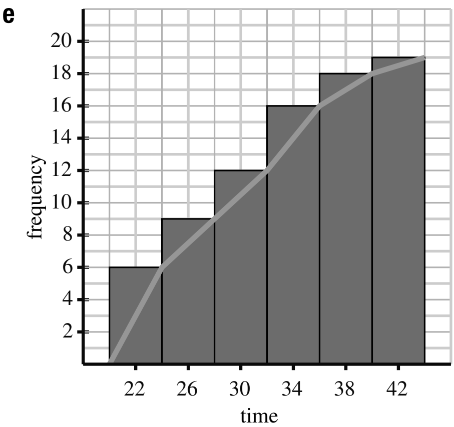
class	20−24	24−28	28−32	32−36	36−40	40−44
centre	22	26	30	34	38	40
freq	6	3	3	4	2	1
c.f.	6	9	12	16	18	20

c 30. No, because information is lost when the data are grouped.

d



e



6 a

x	152	154	155	157	158	159	162	163
f	1	2	1	1	2	3	2	3

x	164	165	166	168	170
f	2	2	3	1	1

b 162.5

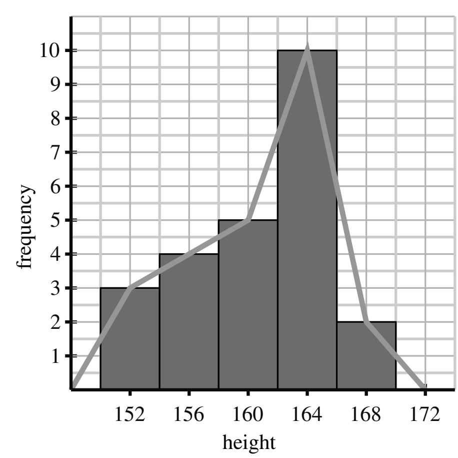
c Trends are less clear when the data are not grouped, because it is less visually clear that the data are falling in certain zones on the domain.

d

group	150–154	154–158	158–162	162–166	166–170
centre	152	156	160	164	168
freq	3	4	5	10	2

e 162

f



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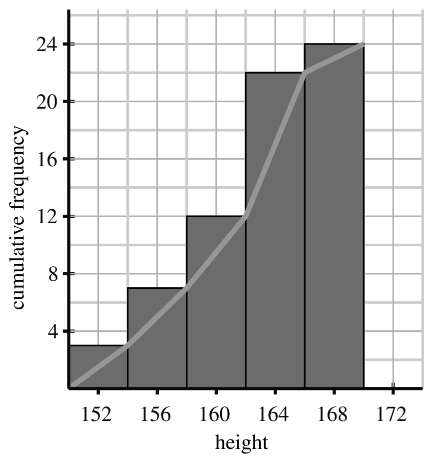
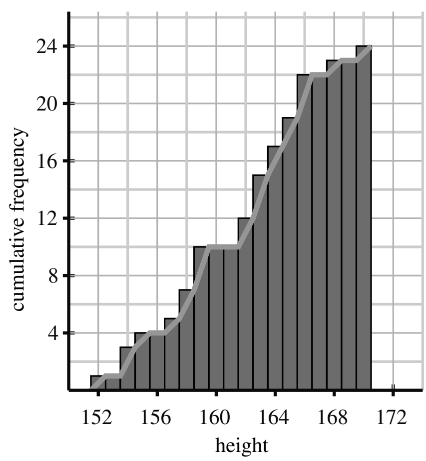


h



i The cumulative frequency polygon and ogive are much less sensitive to the grouping process than the frequency histogram and ogive. The graphs in parts **g** and **h** look very similar in shape.

7 a **i** 14.3 (1 decimal place)
ii 13.7 (1 decimal place)
iii 13.6 (1 decimal place)
b 0.005%

Exercise 9C

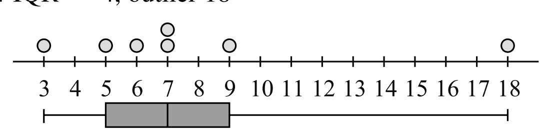
1 a mean 6.9, median 8, mode 8, range 10
b mean 21.4, median 22.5, mode 12, range 18
c mean 5.2, median 5.5, mode 7, range 5
d mean 62.3, median 61, trimodal: 54, 61, 73, range 19

2 a $Q_1 = 7, Q_2 = 13, Q_3 = 17, \text{IQR} = 10$
b $Q_1 = 12.5, Q_2 = 18.5, Q_3 = 25.5, \text{IQR} = 13$
c $Q_1 = 7.5, Q_2 = 11, Q_3 = 18, \text{IQR} = 10.5$
d $Q_1 = 5, Q_2 = 8.5, Q_3 = 13, \text{IQR} = 8$
e $Q_1 = 4, Q_2 = 7, Q_3 = 13, \text{IQR} = 9$
f $Q_1 = 10, Q_2 = 15, Q_3 = 21, \text{IQR} = 11$
g $Q_1 = 5, Q_2 = 9, Q_3 = 13.5, \text{IQR} = 8.5$
h $Q_1 = 12, Q_2 = 14, Q_3 = 18, \text{IQR} = 6$

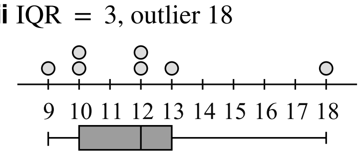
3 a $Q_1 = 4, Q_2 = 12, Q_3 = 16, \text{IQR} = 12$
b $Q_1 = 1, Q_2 = 6.5, Q_3 = 11, \text{IQR} = 10$
c $Q_1 = 7, Q_2 = 9, Q_3 = 12, \text{IQR} = 5$
d $Q_1 = 2.5, Q_2 = 5, Q_3 = 7, \text{IQR} = 4.5$
e $Q_1 = 7, Q_2 = 7, Q_3 = 10, \text{IQR} = 3$
f $Q_1 = 4, Q_2 = 5, Q_3 = 9, \text{IQR} = 5$
g $Q_1 = 2.5, Q_2 = 4, Q_3 = 9.5, \text{IQR} = 7$
h $Q_1 = 4.5, Q_2 = 9, Q_3 = 12, \text{IQR} = 7.5$

4 a Answers may differ here, but 40 and 92 are likely.
b 40, 54, 59, 69, 92
c $\text{IQR} = 15, Q_1 - 1.5 \times \text{IQR} = 31.5$ and $Q_3 + 1.5 \times \text{IQR} = 91.5$. Thus 92 is the only outlier by the IQR criterion.
d Some may identify 40 as an outlier by eye — this shows the advantage of plotting values, where it becomes evident that this score is well separated from other scores. A student receiving 40 in this cohort should be noted as someone needing extra attention and assistance.
e **i** 54, 60, 70.5, $\text{IQR} = 16.5$
ii 53.5, 58, 68.5, $\text{IQR} = 14.5$
iii 54, 59, 68.5, $\text{IQR} = 14.5$
f In this case, with a reasonably sized dataset, the middle of the data is fairly stable and removing an extreme value has only a small effect on the quartiles and IQR. With a large dataset and tightly clustered values in the middle two quarters of the data, the difference would be even smaller.
g **i** 60.8, 11.1 **ii** 61.6, 10.5
iii 59.5, 9.4 **iv** 60.3. 8.7
h 2.4 is 22% of 11.1. Any deviation from the mean is exaggerated by the standard deviation because the deviation from the mean is squared when calculating the variance.

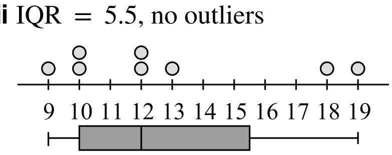
5 a **i** $\text{IQR} = 4$, outlier 18



ii $\text{IQR} = 3$, outlier 18



iii $\text{IQR} = 5.5$, no outliers



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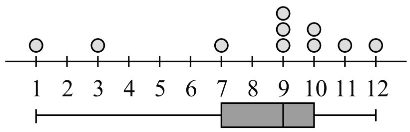
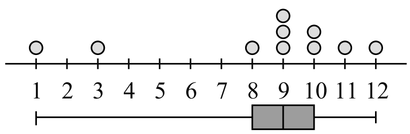
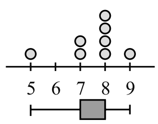
iv IQR $= 3$, outlier 1



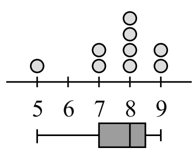
v IQR $= 2$, outliers 1, 3



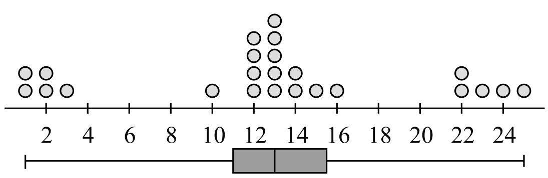
vi IQR $= 1$, outlier 5



vii IQR $= 1.5$, no outliers



viii IQR $= 4.5$, outliers 1, 1, 2, 2, 3, 23, 24, 25



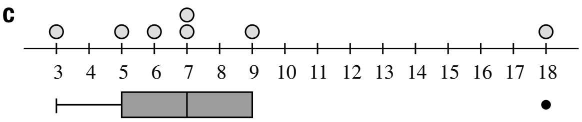
b It must be noted that some of the pathologies in these examples come about because of the small datasets. Statistics is always more accurate and reliable with a large dataset.

Generally the definition picks up the values that appear extreme on the dot plots. Notably (in these small datasets), it picks up single extreme values — if more values are a long way from the mean, they may not be marked as outliers.

Datasets with a small IQR may need a closer inspection — in part **vi** and **vii**, the value at 5 is not so extreme and the datasets are not so different, yet in one case it is marked as an outlier, but in the other it is not. The final dataset has a very tight subset of data between the Q_1 and Q_3, giving a small interquartile range. This definition of outliers gives 8 values in 24 (one third of the data) as outliers. Furthermore, 23–25 are outliers, but 22 is not. The issue here is the unusual shape of the distribution.

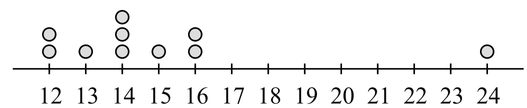
Rules such as this IQR criterion for outliers should be an invitation to inspect the values that have been flagged more closely, rather than following a rule blindly.

c



6 a $\bar{x} = 15$, $s = 3.29$

b The value 24 appears to be an outlier



c IQR $= 3$ and $Q_3 = 16$.
Because $24 > 16 + 1.5 \times 3$, this definition also labels 24 an outlier.

d $\bar{x} = 14$, $s = 1.41$

e This does not have much effect on the mean, but it has a big percentage effect on the standard deviation — removing the outlier more than halves the standard deviation. The operation of squaring $(x - \bar{x})$ means that values well separated from the mean have an exaggerated effect on the size of the variance.

f No effect at all!

g If there are significant outliers, or at least values spread far from the mean, this can have a big influence on the IQR. The IQR is a good measure if you are more interested in the spread of the central 50% of the data.

7 a Emily got less than 62

b Around 50% (and no more than 50%)

c The mathematics results were more spread out, and the centre of the data (by median) was 5 marks higher. The interquartile range of both distributions, however, was the same. Clearly the mathematics cohort has some students who perform much more strongly, and others who perform much weaker, than the majority of their peers.

d Xavier was placed in the upper half of the English cohort, but in the lower half of the mathematics cohort. The English result was thus more impressive.

e **i** 45

ii The bottom 25% of English scores show a spread of 6 marks (51–57). The bottom 25% of mathematics scores show a spread of 8 marks (53−61). The spread of the lower half is now much more comparable.

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8 a The results are not paired. Just becausey Genjo received the lowest score in the writing task does not mean that he received the lowest score in the speaking task. Thus we cannot answer the question, although we might make conjectures, given that Genjo is obviously struggling significantly with English.

b i mean 66.1, median 68, range 56

ii IQR $= 73 - 60 = 13$, 91 and 35 are outliers.

c i mean 64.4, median 65.5, range 56

ii $IQR = 71 - 57.5 = 13.5$, 37 and 93 are outliers.

d It is difficult to say. Students have found the second task more challenging, evidenced by the lower mean and median. This could be due to the construction of the task, or simply because it is a type of task that some students find more difficult.

Exercise 9D

1 a i height **ii** weight

b i radius

ii area. It is natural to think that the area of the circle is determined by the radius chosen when it is drawn, but mathematically we could write $r = \sqrt{\frac{A}{\pi}}$, reversing the natural relationship.

c i weight

ii price. Note that the price may change when meat is bought in bulk, so there is a deeper relationship between these two quantities than simply price $=$ weight $\times$ cost per kg.

d i world rank **ii** placing

e i temperature

ii power consumption. Power consumption increases with the use of air conditioners (higher temperatures) or heaters (colder weather).

f It is natural to take x as the independent variable and y as the dependent variable. Note in this case the relationship cannot naturally be reversed, because there are multiple x-values resulting from the same y-value.

2 a strong positive **b** virtually none.

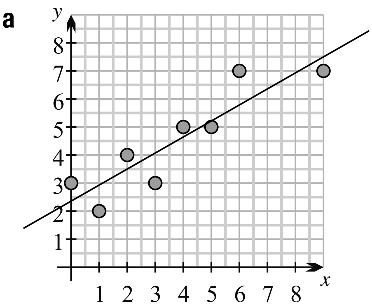
c strong negative **d** strong negative

e moderate positive **f** weak positive

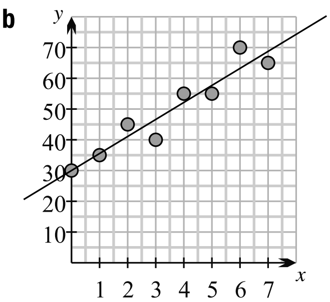
g strong negative **h** strong positive

i moderate negative

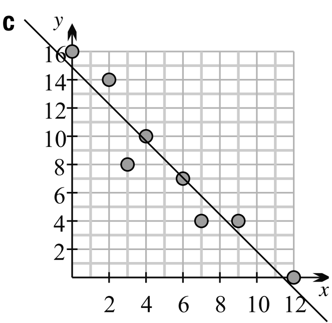
3 a



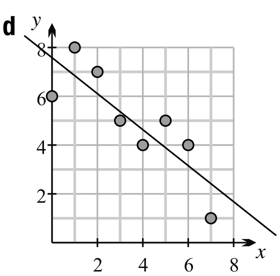
b



c

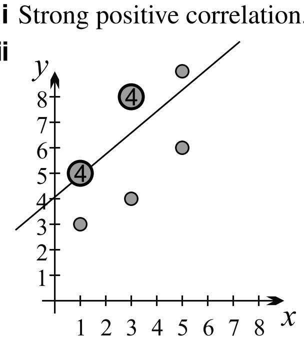


d



4 a i Strong positive correlation.

ii



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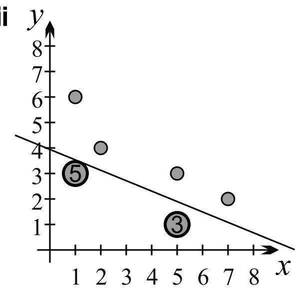
b **i** Strong negative correlation.

ii



5 a A quadratic relationship (a parabola).

b A square root.

c A hyperbola.

d A circle.

e An exponential.

f No obvious relationship.

6 a **ii** 6 L and 10 L

b $V = 2t$

c The V-intercept is zero. In no minutes, zero water will flow through the pipe.

d This is the flow rate of the water, 2 L/s.

e Negative time makes little sense here, because he cannot measure the volume of water that flowed for say -3 minutes.

f Experimental error could certainly be a factor, but it may simply be that the flow rate of water is not constant. It may vary due to factors in, for example, the pumping system.

g 60 L. The extrapolation seems reasonable provided that the half-hour chosen is at about the same time of day that he performed his experiment.

h 22.5 minutes

i Yasuf's experiments were all carried out in a period of several hours during the day. It may be that the flow rate changes at certain times of the day, for example, at peak demands water pressure may be lower and the flow rate may decrease. The flow rate may also be different at night — for example, the water pump may only operate during the day. More information and experimentation is required.

7 a 1000

b **i** $P = 0.9t + 5$

ii It looks fairly good.

iii Predicted $P = 13.1$, actual $P = 15.4$, so the error was 230 people.

c **i** The new model predicts $P = 16.4$, so it is certainly much better.

ii Population is growing very strongly in Hammonsville. Investigators should be looking into the cause of the growth, which may change over the next few years. For example, it may be due to a short-term mining boom. Eventually there may be other constraining factors, such as available land for housing.

d Extrapolation can be dangerous. Provided, however, that the independent variable is constrained to a small enough interval, linear predictions may well have validity. This is the idea behind calculus, where curves are approximated locally by a tangent.

8 a 99 in assessment 1, 98 in assessment 2. They were obtained by the same student, but another student also got 99 in assessment 1.

b 27 in assessment 1, 33 in assessment 2. They were the same student.

c Students getting below about 77 marks in assessment 1 do better in assessment 2, students above 77 marks in assessment 1 get a lower mark in assessment 2, according to the line of best fit. Perhaps the second assessment started easier, but was harder at the end.

d **i** 50 **ii** 65 **iv** 26

v A negative score! Clearly the model breaks down for small scores.

e $y = 0.74x + 20$

f A more accurate method would incorporate data from more than one assessment task in estimating their missing score. This is a question better tackled using standard deviation and the techniques of the next chapter.

9 a The maximum vertical difference between a plotted point and the line of best fit is about $0.8\,\text{s}^2$.

b It could be experimental error. For example, the string could have been twisted or released poorly, the experiment could have been incorrectly timed, or there could have been a recording error.

c They may have measured 10 periods and then divided by 10 before recording the length of one period. Errors could then arise if the motion was *damped*, that is, if the pendulum slowed down significantly over a short time period.

d By this model, $T^2 = \frac{2\pi^2}{g}L \doteqdot 4.03L$. These results are in pretty good agreement with the theory.

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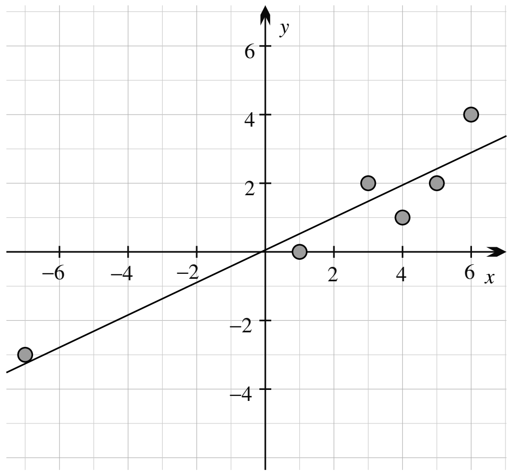
Exercise 9E

1 a There appears to be a fairly strong correlation, though note the small dataset.

b



c

							Sum
x	−7	1	3	4	5	6	12
y	−3	0	2	1	2	4	6
$x - \bar{x}$	−9	−1	1	2	3	4	0
$y - \bar{y}$	−4	−1	1	0	1	3	0
$(x - \bar{x})^2$	81	1	1	4	9	16	112
$(y - \bar{y})^2$	16	1	1	0	1	9	28
$(x - \bar{x})\,(y - \bar{y})$	36	1	1	0	3	12	53

d $(\bar{x}, \bar{y}) = (2, 1)$

e See above

f $53 \div \sqrt{112 \times 28} = 0.95$

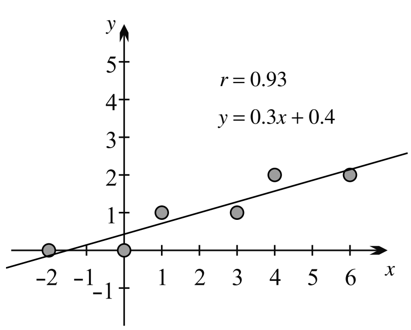
g It is a good fit.

h $53 \div 112 = 0.47$

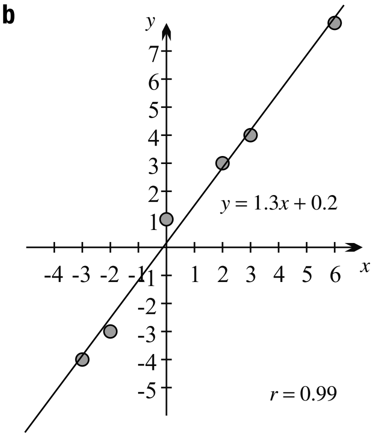
i $b = 1 - 0.47 \times 2 = 0.06$

j $y = \frac{1}{2}x + 0.$

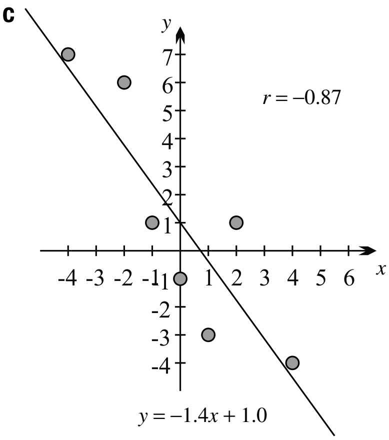
2 a



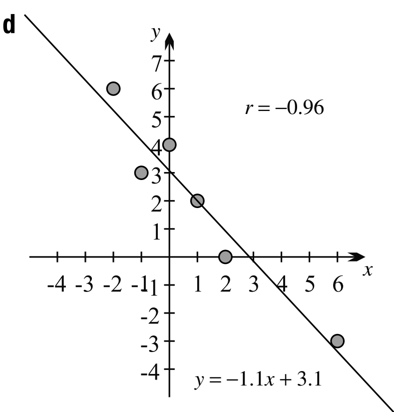
b



c



d



Exercise 9F

1 a $r = 0.96, y = 0.96x + 0.47$

b $r = 0.79, y = 0.45x + 2.6$

c $r = -0.86, y = -1.05x + 8.75$

d $r = -0.53, y = -0.41x + 4.70$

e $r = 0.96, y = 1.38x + 0.75$

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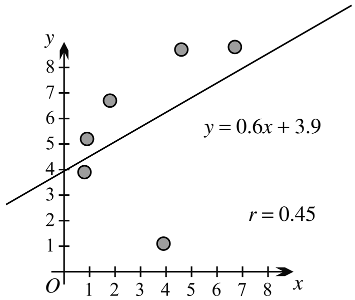
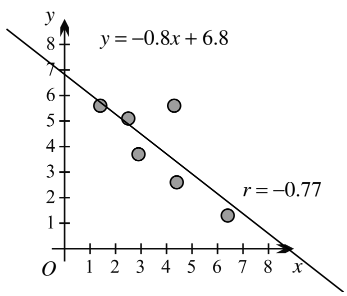
2 a $r = 0.45$, $y = 0.58x + 3.94$



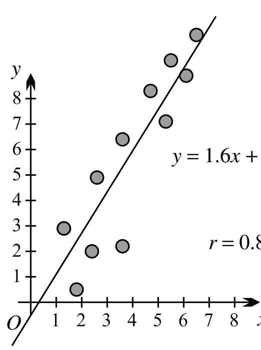
If the outlier at $(3.9, 1.1)$ is removed, then $r = 0.91$, $y = 0.75x + 4.43$.

b $r = -0.77$, $y = -0.78x + 6.83$



If the outlier at $(4.3, 5.6)$ is removed, then $r = -0.97$, $y = -0.89x + 6.79$.

c $r = 0.89$, $y = 1.62x - 0.51$



If the outlier at $(3.6, 2.2)$ is removed, then $r = 0.93$, $y = 1.61x - 0.19$.

3 Because the dataset was larger, the effect of the single outlier was mitigated by the other data points.

4 a Dataset 1:

i $y = 1x + 1.4$, $r = 0.86$

ii $y = 0.8x + 1.9$, $r = 0.79$

Dataset 2:

i $y = 0.7x + 3.0$, $r = 0.76$

ii $y = 0.7x + 2.5$, $r = 0.82$

b In all cases the correlation is strong. In part **a**, the repeated point has strengthened the correlation, but in the second example it has weakened it. Note that a strong correlation doesn't indicate that the data are correct. In part **a**, for example, leaving out 4 of the 9 points still gave a strong correlation, but a very different equation of line of best fit.

c The effect is less in the larger dataset, as expected. The gradient is unchanged (correct to one decimal place) and the y-intercept only differs by 20%, rather than by 26%. In a larger (more realistically sized) dataset, the effect would likely be less again. The effect of the repeated point will also depend on its place on the graph (central versus on the extremes of the data) and how close it is to the line of best fit.

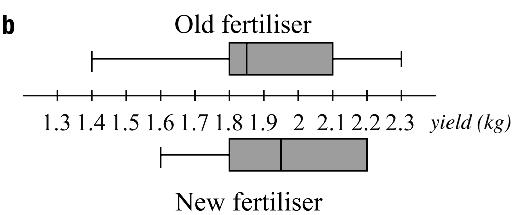
Chapter 9 review exercise

1 a mean 5, median 4.5, mode 4, range 8

b mean 15, median 15, mode 15 and 16 (it is bimodal), range 7

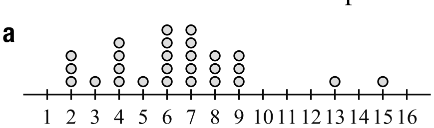
2 a Old fertiliser: 1.8, 1.85, 2.1,
New fertiliser: 1.8, 1.95, 2.2

b



c The fertiliser does appear to increase his yield — the median yield has increased by 100 g. Probably more more data are required because the lower quarter (0–25%) shows an increase, but the maximum has reduced. These claims, however, are each being made on the basis of one data point.

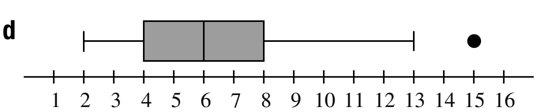
3 a



b By eye, 13 and 15 look like outliers.

c IQR $= 4$. By the IQR criterion, 15 is an outlier, but 13 is not.

d



4 a mean $\doteqdot$ 11.82 s, standard deviation $\doteqdot$ 0.537 s

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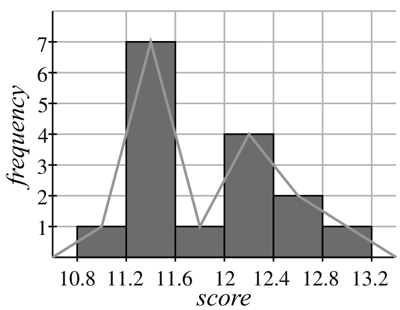
b

group	10.8–11.2	11.2–11.6	11.6–12.0
centre	11.0	11.4	11.8
freq	1	7	1

group	12.0–12.4	12.4–12.8	12.8–13.2
centre	12.2	12.6	13.0
freq	4	2	1

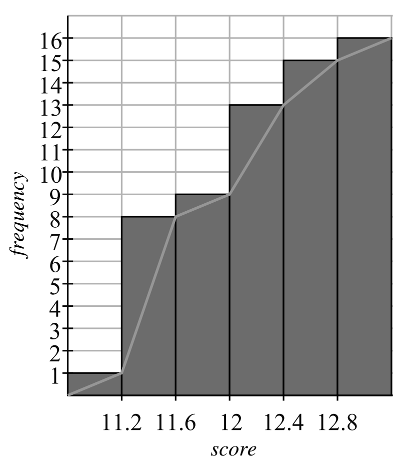
c mean ≑ 11.85 s, standard deviation ≑ 0.563 s. Agreement is reasonable, but as expected, the answers are not exactly the same.

d



e 0.5 seconds is a big difference in the time of a 100 metre sprint — the scale would be too coarse.

f



g The line at 50% of the data (frequency 8) meets the polygon where the sprint time is 11.6 seconds. You can confirm that this agrees with the result for splitting the grouped ordered data into two equal sets.

5 a

	first	second	Total
order entrée	45	42	87
no entreé	38	28	66
Total	83	70	153

b 153

c $87 \div 153 \doteqdot 57\%$

d 54%

e $P(\text{order entrée} \mid \text{attend first}) = 45 \div 83 \doteqdot 54\%$.
$P(\text{order entrée} \mid \text{attend second}) = 42 \div 70 = 60\%$.
No, it is not correct.

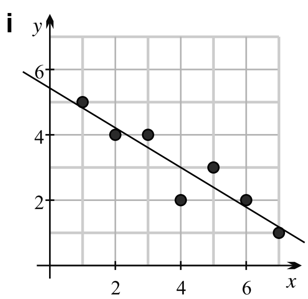
f $P(\text{attended first} \mid \text{ordered an entrée})$
$= 45 \div 87 \doteqdot 52\%$.

g $90 \times 60\% = 54$

h Those attending the first session may prefer a quick meal before heading out to the theatre or some other event. There may also be more family groups operating on a tighter budget.

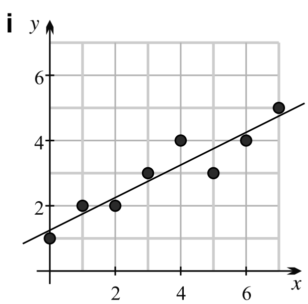
i If they can estimate the demand on certain dishes, then they may be able to prepare parts of the dish in advance, for example, preparing the garnishes or chopping the ingredients.

6 a i



ii $r = -0.93$, $y = -0.61x + 5.43$

b i



ii $r = 0.94$, $y = 0.5x + 1.25$

7 a 120 000

b 94 000, 62 000, 80 000, 80 000

c 316 000 and 79 000

d The arrivals may vary over the year because of seasonal or other effects. Government policy may consider an annual immigration quota, allowing a higher rate in one quarter to be balanced by a low rate in a subsequent quarter. As in 2000, examining the average for each quarter balances out such effects.

e 84 000

f It would be important to know the emigration rate of those leaving the country. The Net Overseas Migration (NOM) may be the better measure for

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many purposes. Other information of interest might include country of origin, destination within Australia, and whether they're intending to stay permanently or for a limited period.

g **i** 71 600

ii Rounding error has affected these calculations — a discrepancy in the second decimal place of the gradient is multiplied by 2000, resulting in an answer that is out by as much as $0.05 \times 2000 = 100$ thousand.

iii 84.300, which is in agreement with part **d**.

iv 660 000

v $660 \div 316 \times 100\% \doteqdot 209\%$, which is a 108% increase.

Chapter 10

Exercise 10A

1 **A** and **C**

2 a

x	2	3	4	5	6	7	8
$P(X = x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

b **i** $\frac{3}{8}$ **ii** $\frac{1}{16}$ **iii** 0 **iv** 1

c **i** $\frac{3}{16}$ **ii** $\frac{1}{2}$ **iii** $\frac{15}{16}$ **iv** $\frac{3}{8}$

3 a

score x	1	2	3	4	5	Total
f_r	0.1	0.2	0.45	0.15	0.1	1
xf_r	0.1	0.4	1.35	0.6	0.5	2.95
x^2f_r	0.1	0.8	4.05	2.4	2.5	9.85

b The sum of probabilities is 1.

c $\bar{x} = 2.95$

d The sample mean $\bar{x}$ is a measure of the centre of the dataset.

e $s^2 \doteqdot 1.15$ **f** $\sigma \doteqdot 1.07$

g The sample standard deviation s is a measure of the spread of the dataset.

h They are estimates of the mean $\mu = \mathrm{E}(X)$ and the standard deviation σ of the probability distribution.

i The sample mean $\bar{x}$ is 2.95 , so after 100 throws, 295 is a reasonable estimate of the sum.

4 a $\bar{x} = 5.26, x \doteqdot 1.07$

b The centre of the data is about 2.3 units greater, but the spread is about the same, according to the standard deviation.

5 a 3.5 and 4

b

score x	1	2	3	4	5	6	Total
frequency f	2	4	4	8	2	0	10
$P(X = x)$	0.1	0.2	0.2	0.4	0.1	0	1
$x \times P(x)$	0.1	0.4	0.6	1.6	0.5	0	3.2
$x^2 \times P(x)$	0.1	0.8	1.8	6.4	2.5	0	11.6

c $E(X) = 3.2$

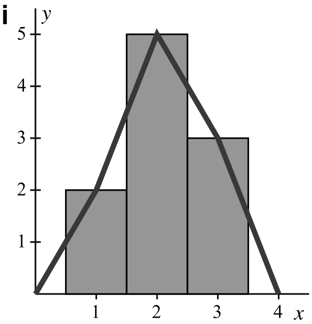
d $\mathrm{Var}(X) = 11.6 - (3.2)^2 = 1.36$

e 1.17

f It is usual to expect that for a quiz (covering recent work and including short easy questions) the marks will be high. These marks don't look impressive.

g $\mathrm{E}(X) = 16$, $\mathrm{Var}(X) = 34$, standard deviation 5.83

6 a **i**



ii 10

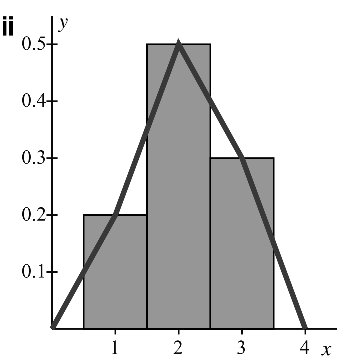
iii 10

iv Both areas are the same and equal to the total frequency, that is the number of scores.

b **i**

score x	1	2	3
frequency f	2	5	3
relative frequency f_r	0.2	0.5	0.3

ii



iii 1

iv 1

v Both areas are the same and equal to the total 1, that is the sum of the relative frequencies. (This will only happen when the rectangles have width 1.)

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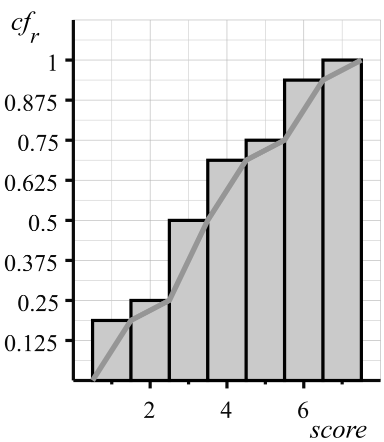
vi The relative frequencies are estimates of the probabilities. Note that both add to 1, both are non-negative, and both measure the chance that a random value will lie within the given rectangle of the histogram. A relative frequency is the *experimental* probability of an outcome, and is an *estimate* of the theoretical probability.

7 a

x	1	2	3	4	5	6	7	Total
f	3	1	4	3	1	3	1	16
f_r	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{16}$	1
cf	3	4	8	11	12	15	16	—
cf_r	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{8}{16}$	$\frac{11}{16}$	$\frac{12}{16}$	$\frac{15}{16}$	1	—

b

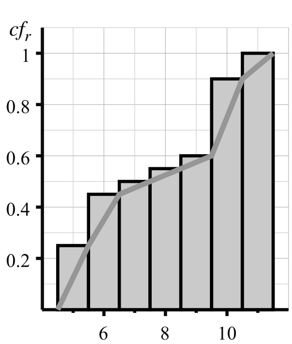


c $Q_1 = 2.5, Q_2 = 3.5, Q_3 = 5.5$

8 a

x	5	6	7	8	9	10	11	Total
f	5	4	1	1	1	6	2	16
f_r	$\frac{5}{20}$	$\frac{4}{20}$	$\frac{1}{20}$	$\frac{1}{20}$.	$\frac{1}{20}$	$\frac{6}{20}$	$\frac{2}{20}$	1
cf	5	9	10	11	12	15	20	—
cf_r	$\frac{5}{20}$	$\frac{9}{20}$	$\frac{10}{20}$	$\frac{11}{20}$	$\frac{12}{20}$	$\frac{18}{20}$	1	—

b



c $Q_1 = 5.5, Q_2 = 7.5, Q_3 = 10$

9 a $\frac{1}{4}$ **b** $\frac{3}{4}$ **c** 0.1

d 12.5% of the households have 3 or more cars, so the town planners will not recommend additional on-street parking.

e

x	0	1	2	3	4
$P(X = x)$	0.25	0.5	0.125	0.10	0.025

f The area of the histogram is exactly the sum of the probabilities, because the width of each bar is 1 in this graph.

g The triangles cut off above the polygon fit into the spaces below the polygon.

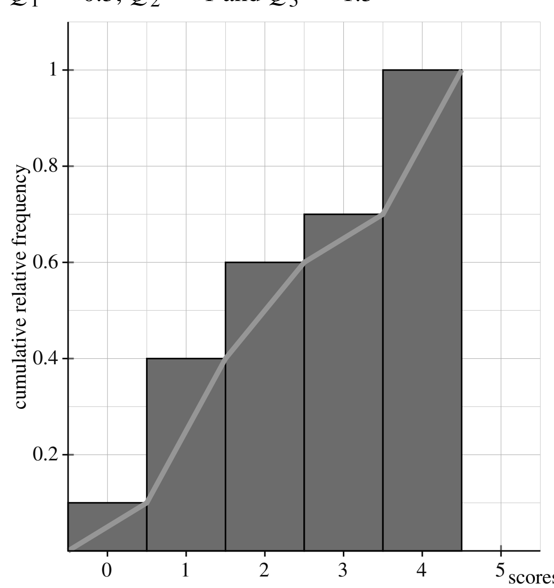
h This is an average, and is best understood by saying that for a large sample of n houses, we would expect them to have about $1.15n$ cars between them — see the next part.

i 115 cars. We are assuming that streets in the suburb are uniform with respect to car ownership. Actually, streets closer to train stations may manage with fewer cars because people catch the train to work, more affluent streets may own more cars, people may adjust car ownership to allow for availability of off-street or on-street parking.

x	0	1	2	3	4
$P(X \le x)$	0.25	0.75	0.875	0.975	1

j $Q_1 = 0.5, Q_2 = 1$ and $Q_3 = 1.5$

10 a



b 3.5 **c** 1, 2

d $Q_3 \doteqdot 3.7$ **e** 3.67. They agree.

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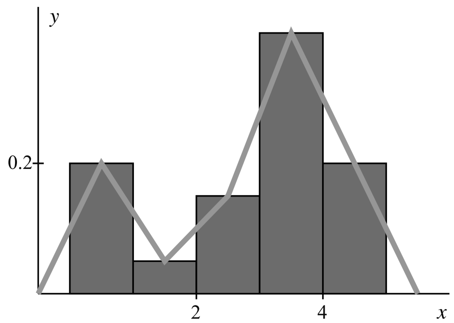
11 a median 3.5, mode 3.5

b

spent	0–1	1–2	2–3	3–4	4–5	Total
cc x	0.50	1.50	2.50	3.50	4.50	—
f	20	5	15	40	20	100
f_r	0.20	0.05	0.15	0.40	0.20	1

c mean 2.85, variance 1.9275, standard deviation 1.39

d



e **i** 0.2 **ii** 0.05 **iii** 0.15
iv 0.4 **v** 0.2

f The area of the relative frequency polygon, or the area under the frequency polygon bounded by the x-axis (they are the same). This only happens because the rectangles have width 1.

g **i** Equally likely
ii They are twice as likely to have spent between \$3–\$4.

h $E(Y) = 4.85$, same variance

12 a The histogram covers 40 grid rectangles.

b **i** $0.3 \times 0.5 = 0.15$
ii 0.1 **iii** 0.3

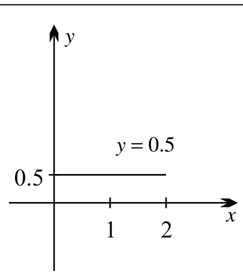
c **i** It is twice as likely to be 20°C.
ii In the class $19.25 - 19.75$

d **i** 0.1 **ii** 0.3

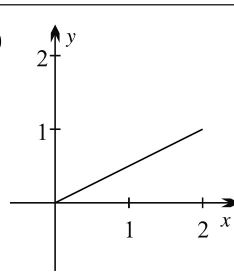
e First, the histogram only records the maximum daily temperature. Secondly, it recorded 20 consectuve days, but there will be natural variation over the year, and even within a season.

Exercise 10B

1 a



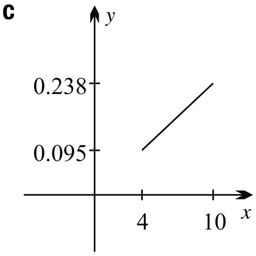
b



i $\int_0^2 f(x)\,dx = \text{area rectangle} = 1$

ii $\int_0^2 f(x)\,dx = \text{area triangle} = 1$

c



i $\int_4^{10} f(x)\,dx = \text{area trapezium}$
$= \frac{1}{2} \times 6\left(\frac{4}{42} + \frac{10}{42}\right) = 1$

2 a Yes, mode is $x = 1$

b No, the integral is 3

c No. The integral is 1, but $f(x) < 0$ if $x > 2$.

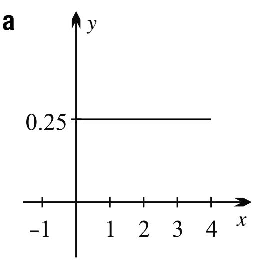
d Yes, provided that $n \geq 0$. Then mode is $x = 1$.

e Yes, mode is $x = \frac{\pi}{2}$

f Yes, mode is $x = 2$

3 b $f(x) = \frac{3}{4}(x - 3)(x - 1) < 0$, for $1 < x < 3$

4 a



b Clear from the graph

c **i** $\frac{1}{4}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{2}$
iv 0 **v** $\frac{3}{4}$ **vi** $\frac{3}{4}$

d LHS $= \frac{1}{4}$, RHS $= \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$

5 a $F(x) = \frac{1}{64}x^2$

b $F(x) = \frac{1}{16}(x^3 + 8)$

c $F(x) = \frac{x}{2}(3 - x^2)$

d $F(x) = \frac{1}{e}(e^x + x - 1)$

6 a $Q_2 = 4\sqrt{2}, Q_1 = 4, Q_3 = 4\sqrt{3}$

b $Q_2 = 0, Q_1 = -\sqrt[3]{4}, Q_3 = \sqrt[3]{4}$

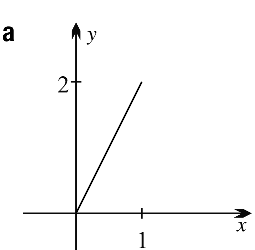
7 a $\frac{20}{170} \doteqdot 12\%$

b $\frac{4\pi}{170} \doteqdot 7\%$

c $\frac{20+4\pi}{170} \doteqdot 19\%$

d $1 - \frac{20+4\pi}{170} \doteqdot 81\%$

8 a



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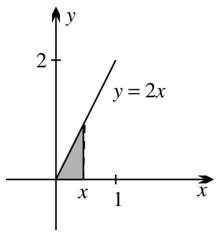
c **i** Area $= \frac{1}{2}x \times 2x$



d $Q_1 = \frac{1}{2}, Q_2 = \frac{1}{\sqrt{2}}, Q_3 = \frac{\sqrt{3}}{2}$

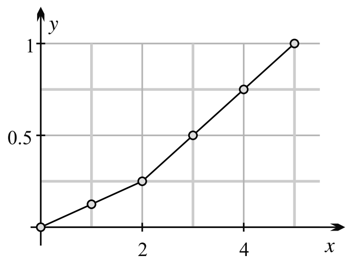
9 a $\frac{5}{243}$ **b** $\frac{1}{6}$ **c** $\frac{1}{10}$ **d** $\frac{1}{2}$

10 a Clearly $f(x) \geq 0$ for all x, and the area under the graph is $2 \times 0.125 + 3 \times 0.25 = 1$.

b

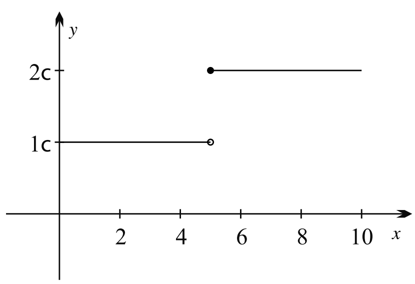
x	0	1	2	3	4	5
$P(X \leq x)$	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1

c



d $F(x) = \begin{cases} \frac{1}{8}x, & \text{for } 0 \leq x < 2, \\ \frac{1}{4}x - \frac{1}{4}, & \text{for } 2 \leq x \leq 5. \end{cases}$

11 a



b Area $= 15c$, so $c = \frac{1}{15}$

c $F(x) = \begin{cases} cx, & \text{for } 0 \leq x < 5, \\ 2cx - 5c, & \text{for } 5 \leq x \leq 10. \end{cases}$

d $P(1 < X < 7) = F(7) - F(1) = 8c = \frac{8}{15}$

12 a The mode is $x = 2$ (where the vertex is).

c Symmetric about $x = 2$, $P(X = 2) = 0$, and total area is 1.

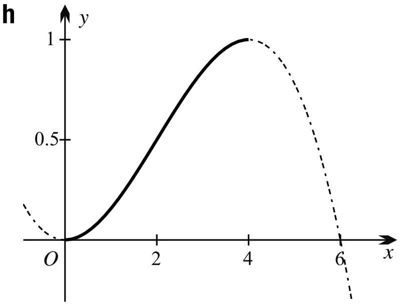
d $\frac{5}{32}$ and $\frac{27}{32}$ are complementary probabilities.

e $\frac{11}{256}$. The symmetry of the graph means that the areas are the same.

f $\frac{1}{32}x^2(6 - x)$

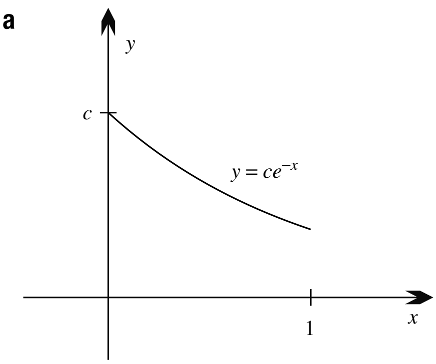
g **i** $\frac{81}{256}$ **ii** $\frac{41}{256}$ **iii** $\frac{29}{256}$ **iv** $\frac{47}{256}$

h



i $Q_1 = 1.3$, $Q_3 = 2.7$.

13 a



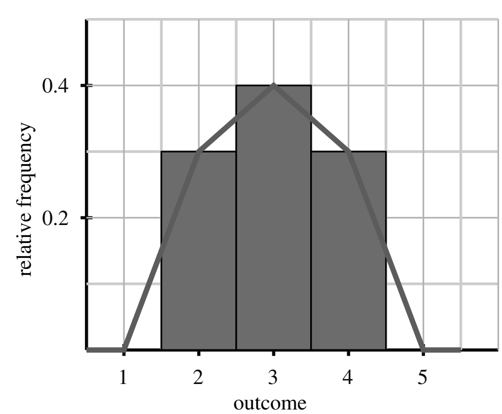
b $c = \frac{e}{e - 1}$

c $F(x) = \frac{e}{e - 1}(1 - e^{-x})$

d $Q_1 = \ln \frac{4e}{3e + 1}, Q_2 = \ln \frac{2e}{e + 1}, Q_3 = \ln \frac{4e}{e + 3}$

14 a See part **c**.

b Both areas are 1.



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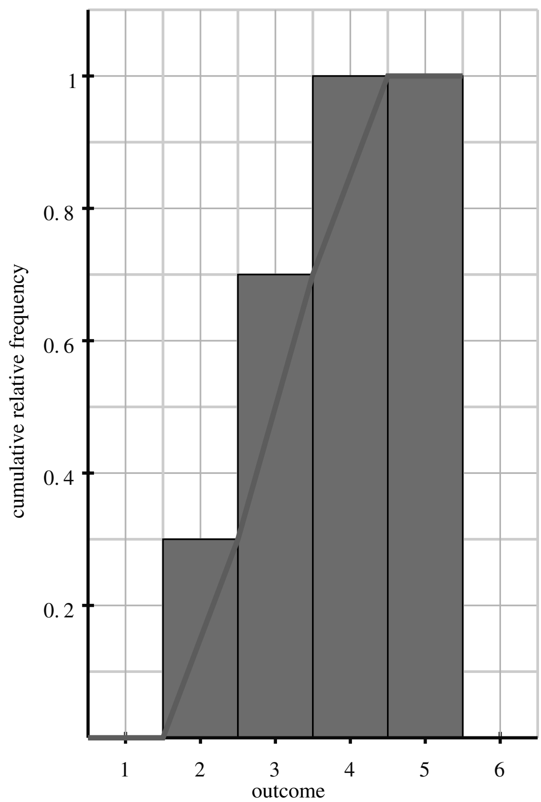
c

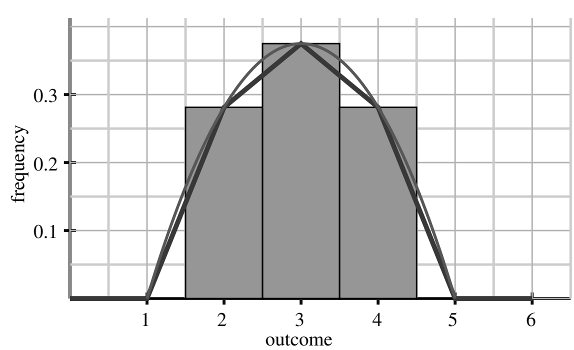


d 2.3, 3, 3.7

e i $\int_1^5 f(x)\,dx = 1$, and $f(x) \geq 0$ for $1 \leq x \leq 5$

ii

x	1	2	3	4	5
$f(x)$	0	0.28	0.375	0.28	0



iii $\frac{1}{32}(-x^3 + 9x^2 - 15x + 7)$

iv $P(X \leq 2.3) \doteqdot 0.25$, $P(X \leq 3) = 0.5$, $P(X \leq 3.7) \doteqdot 0.75$

v 2.3, 3 and 3.7 still seem good approximations.

15 b $F(x) = 1 - \frac{1}{x}$

c Total probability is 1.

d $\frac{4}{3}$, 2, 4

16 b $F(x) = 1 - e^{-x}$

c $Q_1 = \ln \frac{4}{3}$, $Q_2 = \ln 2$, $Q_3 = \ln 4$

Exercise 10C

1 a $f(x) \geq 0$ and by area formula or integration, $\int_0^{10} f(x)\,dx = 1$.

b $E(X) = 5$

c Yes — in the centre of this distribution interval $[0, 10]$

d $\text{Var} = \frac{25}{3}$, $\sigma = \frac{5}{3}\sqrt{3}$

e $E(X^2) = \frac{100}{3}$ and $\text{Var} = \frac{25}{3}$

3 a The function is never negative, and the integral over $[-1, 1]$ is 1.

b $E(X) = 0$

c $\text{Var}(X) = \frac{3}{5}$, $\sigma = \frac{\sqrt{15}}{5}$

d $\frac{3\sqrt{15}}{25} \doteqdot 0.46$

4 a $E(X) = \frac{2}{3}$, $\text{Var}(X) = \frac{1}{18}$, $\sigma = \frac{\sqrt{2}}{6}$, $P(\mu - \sigma \leq X \leq \mu + \sigma) = \frac{4\sqrt{2}}{9}$

b $E(X) = 0$, $\text{Var}(X) = \frac{1}{2}$, $\sigma = \frac{1}{\sqrt{2}}$, $P(\mu - \sigma \leq X \leq \mu + \sigma) = \frac{1}{2}$

c $E(X) = 3$, $\text{Var}(X) = \frac{3}{5}$, $\sigma = \frac{\sqrt{15}}{5}$, $P(\mu - \sigma \leq X \leq \mu + \sigma) \doteqdot 0.668$

5 a Yes.

b $E(X) = \frac{c}{2}$, as expected for a measure of the centre of this uniform distribution.

c $\frac{c^2}{12}$

d The answer agrees for this special case with $c = 10$.

e $E(X) = \dfrac{c}{2} + h = \dfrac{c + 2h}{2}$, variance unchanged.

f Put $h + c = k$ in the previous result: $E(X) = \frac{k + h}{2}$, $\text{Var}(X) = \frac{(k - h)^2}{12}$

6 b $E(X) = \frac{23}{8}$, $E(X^2) = \frac{121}{12}$, $\text{Var}(X) = \frac{349}{192} \doteqdot 1.82$

7 $\text{LHS} = \int_a^b x^2 f(x)\,dx - \int_a^b 2\mu x f(x)\,dx + \int_a^b \mu^2 f(x)\,dx$

By the definition of a PDF,

Term 3 $= \mu^2 \int_a^b f(x)\,dx = \mu^2$.

By the formula for the mean,

Term 2 $= -2\mu \int_a^b x f(x)\,dx = -2\mu^2$.

8 b $E(X) = 2.1$ **c** Agrees.

d Not only do both satisfy the condition that the area under the curve is 1, but they give the same result for the expected value.

9 b $E(X) = \frac{3}{2}$, $E(X^2) = 3$, $\text{Var}(X) = \frac{3}{4}$

c i $1 - \frac{1}{4^3}$ **ii** $\frac{1}{8}$ **iii** $\frac{117}{1000}$

d $F(x) = 1 - \frac{1}{x^3}$

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10 a $\frac{d}{dx}xe^{-x} = e^{-x} - xe^{-x}$,

so $\int xe^{-x}dx = -e^{-x} - xe^{-x}$.

b $\mathrm{E}(X) = 1$

c The derivative is $(2xe^{-x} - x^2e^{-x})$
$+ (2e^{-x} - 2xe^{-x}) - 2e^{-x} = -x^2e^{-x}$,

so $\int x^2e^{-x}dx = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

d $\mathrm{E}(X^2) = 2$ and $\mathrm{Var}(X) = 1$.

Exercise 10D

1 a 0.5 **b** 0.8413 **c** 0.9972 **d** 0.9332
e 0.6554 **f** 0.9893 **g** 0.8849 **h** 1.0000

2 The total area under the curve is 1, so the areas of regions to the right and left of $z = a$ add to 1. This identity is true for any probability distribution.
a 0.5 **b** 0.1587 **c** 0.0228 **d** 0.0082
e 0.0968 **f** 0.2420 **g** 0.0548 **h** 0.0000

3 a From the even symmetry of the graph,
$P(Z < -a) = P(Z > a) = 1 - P(Z \le a)$.
(The result also holds for $a \le 0$, but this is not useful to us.) This result is certainly not true for all probability distributions.
b **i** 0.1151 **ii** 0.0107 **iii** 0.4207
iv 0.0007 **v** 0.0000 **vi** 0.2420
vii 0.0548 **viii** 0.0808 **ix** 0.5000

4 b **i** 0.4032 **ii** 0.4918 **iii** 0.2580
iv 0.4918 **v** 0.3643 **vi** 0.2580
vii 0.4452 **viii** 0.4032 **ix** 0.5000
c **i** 0.8064 **ii** 0.9836 **iii** 0.5762
iv 0.9962 **v** 0.3108 **vi** 0.8664

5 h **a** and **e**, **b** and **g**, **c** and **h**, **d** and **f**

6 h **a** and **c**, **b** and **g**, **d** and **f**, **e** and **h**

7 a This is evident from a graph by subtraction of areas.
b **i** 0.0483 **ii** 0.4100 **iii** 0.2297
iv 0.0923 **v** 0.4207 **vi** 0.1552
c **i** 0.9193 **ii** 0.7008 **iii** 0.9013

8 a 0.5 **b** 0 **c** 0.0359
d 0.8849 **e** 0.1151 **f** 0.3849
g 0.0359 **h** 0.8849 **i** 0.0792
j 0.8490

9 a 0.9032 **b** 0 **c** 0.3446
d 0.9554 **e** 0.9032 **f** 0.4332
g 0.2119 **h** 0.4207 **i** 0.0689
j 0.8893

10 a 0.9208 **b** 0.0792 **c** 0.6341 **d** 0.0364

12 a 50% **b** 84%
c 97.5% (Note the inaccuracy here. From the tables it should be 97.72.)
d 16% **e** 49.85% **f** 34%
g 47.5% **h** 2.35% **i** 68%
j 83.85% **k** 81.5% **l** 97.5%

13 a $b = 1$ **b** $b = 2$ **c** $b = -1$
d $b = 1$ **e** $b = 1$ **f** $b = 4$

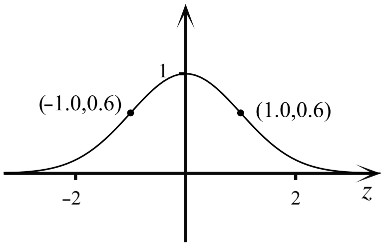
14 a 0.6 **b** 2.3 **c** 1.2
d −0.8 **e** 1.1 **f** 2.6

15 a **i** $P(-1 < Z < 1) \doteqdot 68\%$
ii $P(Z < 2) \doteqdot 97.5\%$
iii $P(Z\langle -3 \text{ or } Z\rangle 3) = 0.3\%$
b Around 0.7 centimetres.

16 Mathematically, $P(Z = a) = \int_a^a f(x)\,dx$, which is an area of zero width. Practically, this represents the probability of getting a value exactly $Z = a$ for a continuous distribution, for example a height of exactly 1.7142435345345 … metres. In a continuous distribution, all such probabilities are zero.

17 a **i** all real values **ii** Even
iii $x = 0$ **iv** 1
v $z = -1$ and $z = 1$ **vi** $\left(0, \frac{1}{\sqrt{2\pi}}\right)$
vii There are no z-intercepts.
b **i** 0 **ii** 0 **iii** 0 **iv** 1
c $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$

18 c Stationary point $(0, 1)$. It is a maximum.
d Inflections at $(1, e^{-0.5})$ and $(-1, e^{-0.5})$
e $f(x) \to 0$ as $x \to \infty$ and as $x \to -\infty$
f



g See the graph at the top of this exercise.

19 a **i** $P(0 \le Z \le 1) = 0.3401$
ii $P(-1 \le Z \le Z) = 0.6802$
iii The graph is concave up on $[0, 1]$ — the concavity changes at the point of inflection at $z = 1$. Thus the polygonal path of the trapezoidal rule will lie below the exact curve.
iv This is good agreement with the empirical rule (68) and the table (0.6826).

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b **i** $P(-2 < Z < 2) = 2 \times 0.4750 = 0.95$

ii $P(-3 < Z < 3) = 2 \times 0.4981 = 0.9962$

20 a $E(Z) = \int_{-\infty}^{\infty} z \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,dz$, which is the integral of an odd function on a symmetric domain, so $E(Z) = 0$.

b $\frac{d}{dz}\left(ze^{-\frac{1}{2}z^2}\right) = 1 \times e^{-\frac{1}{2}z^2} + z \times -ze^{-\frac{1}{2}z^2}$

$$ze^{-\frac{1}{2}z^2} = \int e^{-\frac{1}{2}z^2}\,dz - \int z \times ze^{-\frac{1}{2}z^2}\,dz$$

$$\left[ze^{-\frac{1}{2}z^2}\right]_{-\infty}^{\infty} = \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz - \int_{-\infty}^{\infty} z^2e^{-\frac{1}{2}z^2}\,dz$$

The LHS is 0, so

$$\int_{-\infty}^{\infty} z^2e^{-\frac{1}{2}z^2}\,dz = \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz$$

and $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^2e^{-\frac{1}{2}z^2}\,dz = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz$

$= 1$

Thus we have shown that $E(Z^2) = 1$.

c Using the previous part,

$$\begin{aligned}\text{Var}(Z) &= E(Z^2) - E(Z)^2 \\ &= 1 - 0 \\ &= 1\end{aligned}$$

Exercise 10E

1 a $z = 1$, 1 SD above **b** $z = -2$, 2 SD below

c $z = 1$, 1 SD above **d** $z = -2$, 2 SD below

e $z = 5$, 5 SD above **f** $z = -3$, 3 SD below

2 a **i** $+2.5$ **ii** -3 **iii** $+5.5$ **iv** $+\frac{1}{4}$

b **i** iii **ii** i, iii, iv

iii ii **iv** iv

v i, ii, iii

3 a $P(Z \le 0.5)$ **b** $P(Z > 0.25)$

c $P(Z \le -1)$ **d** $P(Z \ge -1.5)$

e $P(-2 \le Z \le -0.5)$ **f** $P(-1.75 \le Z \le 0.25)$

4 a $P(Z \ge 0) = 0.5$

b $P(-1 \le Z \le 1) = 0.68$

c $P(Z \le 2) = 0.975$

d $P(Z \ge -2) = 0.975$

e $P(-3 \le Z \le 1) = 0.8385$

f $P(-2 \le Z \le -1) = 0.1475$

5 a $P(-1 \le Z \le 3) = 0.8385$

b $P(Z \ge 1) = 0.16$

c $P(Z \ge 2) = 0.025$

6 a $P(-2.5 \le Z \le 2.5) = 0.9876$

b $P(Z \ge 1.6) = 0.0548$

c $P(Z \le -0.8) = 0.2119$

d $P(Z \ge -1.3) = 0.9032$

e $P(Z < 1.6) = 0.9452$

f $P(-2.5 < Z \le -1.5) = 0.0606$

7 a The score is above the mean.

b The score is below the mean.

c The score is equal to the mean.

8 a 69, 80

b 69, 80, 95, 50, 90, 52, 45

c 43, 45, 50, 52

d 95, 98

e It doesn't look very normal ('bell shaped').

Here is the stem-and-leaf plot:

4	3 5
5	0 2
6	9
7	
8	0
9	0 5 8

9 a **i** z-score for English (2.5) and maths (2). English is more impressive.

ii z-score for English (-0.8) and maths (-0.6). Maths is more impressive.

iii z-score for English (1.5) and maths (1). English is more impressive.

b 95 is 2.2 standard deviations above the mean.

$P(Z > 2.2) \doteqdot 1 - 0.9861 \doteqdot 1.4\%$

c The mathematics mean of 62 is 0.3 English standard deviations below the English mean 65%.

$P(Z > -0.3) = P(Z < 0.3) = 0.6179 \doteqdot 0.62$

10 a About 408 scores will lie within one SD from the mean, that is, in $[40, 60]$. About 570 scores will lie within two SDs from the mean, that is, in $[30, 70]$. About 598 scores will lie within three SDs from the mean, that is, in $[20, 80]$.

b **i** 415 **ii** 260 **iii** 462

c 4

11 a **i** $-1, -1.5, -2$

ii 1.5 standard deviations below the mean

iii 45

iv Some assessments may be harder than others — simply averaging his other results takes no account of this.

v Jack may perform better in certain types of assessments, for example, in Biology lab experiments, or he may perform better at certain times of the year. For example, his results may improve towards the end of the year.

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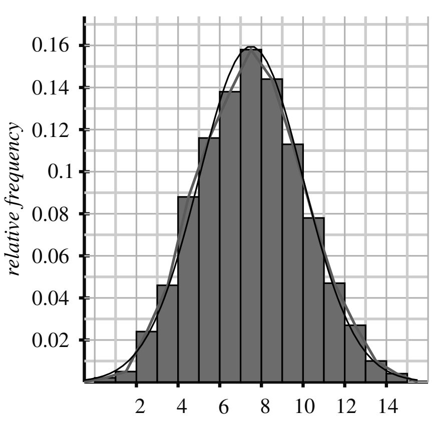
b z-scores 0.4, 0.625, 1, average 0.675.
Jill's estimate is 71.1

Exercise 10F

1 a 97.5% **b** 84%
2 a 3 **b** 50 **c** 1630
3 a 2.5%
b $2400 \times 0.15 \div 100 = 3.6$ screws (perhaps round to 4)
4 5%
5 a 0.26% **b** 65 000
6 about 31%
7 b 186 cm
c i Interpolate between 1.6 and 1.7
ii 197 cm
8 The boxes need to be marked with a mean weight of 496.7 grams — this would probably be rounded down to 496, which is actually two standard deviations below the mean, so 97.5 of boxes weigh above this value.
9 a 69%
b i 33% **ii** 33%
10 a 12% **b** about 0.07%
c i 4.5% **ii** 0.6%
11 2.5 standard deviations is 15 g, so 1 standard deviation is 6 g. The mean weight is 112 g.

Exercise 10G

1 a $\mu = 7.5, \sigma = 2.5$
c



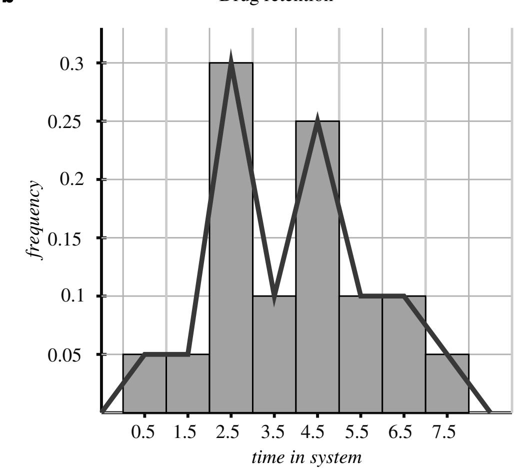
d Either perform the experiment more than 1000 times, or average more than three random numbers at each stage.
4 c The mean should be about 5 and the standard deviation about 1.6.
8 d $\left(0, \frac{1}{\sqrt{2\pi}}\right)$

Chapter 10 review exercise

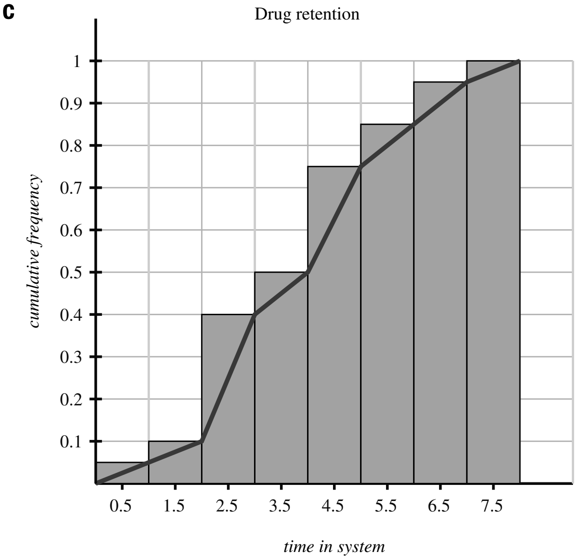
1 a

x	0–1	1–2	2–3	3–4	4–5	5–6	6–7	7–8
cc	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
f	1	1	6	2	5	2	2	1
cf	1	2	8	10	15	17	19	20
f_r	0.05	0.05	0.3	0.1	0.25	0.1	0.1	0.05
cf_r	0.05	0.1	0.4	0.5	0.75	0.85	0.95	1

b



c



d i 4.0
ii $Q_1 = 2.5, Q_3 = 5.0$
iii 6.5
iv 6.0
e This was only a preliminary experiment, and a larger dataset may resolve the unusual outcomes. It may be worth investigating any common links between patients falling in the two intervals associated with the two modes — perhaps different sexes react differently to the drug, perhaps it was administered

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differently, or perhaps the two groups are behaving differently after medication, for example, changing their levels of follow-up exercise or their food intake.

2 a False, it joins to the right end.

b False. The area under the relative frequency polygon is 1 if the rectangles each have width 1.

c True. **d** True. **e** True.

f The empirical rule says 99.7% and only applies to a normal distribution, so false in general.

3 b A uniform probability distribution with a uniform probability density function.

c $\mathrm{E}(X) = 0$

d $\mathrm{E}(X^2) = \mathrm{Var}(X) = \frac{100}{3}, \sigma = \frac{10\sqrt{3}}{3}$

4 b $F(x) = \frac{1}{16}x(12 - x^2), 0 \leq x \leq 2$

c **i** 0.34, 0.7, 1.1 **ii** 0.85

iii 0.8 **iv** 0.78

v $P(X \leq 0.4) - P(x \leq 0.2) = 0.3 - 0.15 = 0.15$

5 a 0.5 **b** 0.9032 **c** 0.9282

d 0.3085 **e** 0.4207 **f** 0.4247

6 a 97.5% **b** 84% **c** 81.5%

d 2.35%

7 a 0.6915 **b** 0.1151 **c** 0.5125

d 0.1760

8 a Let X be the lifetime in years of a machine.

$P(X > 8) = P(Z > 1.3) = 9.7\%$

b $P(X < 5) = P(Z < -1.07) = 14.2\%$. This is probably an unacceptable risk for the manufacturer, and they should increase the mean life, or decrease the standard deviation, or adjust the length of their advertised warranty.

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