Gyroscopes

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Contents

1   Introduction

In rotational mechanics, there are phenomena analog to translational mechanics such as momentum of inertia, torque and angular velocity, acceleration and momentum. However, certain phenomena in rotational mechanics such as precession and nutation do not have such analogs and are thus inherently unintuitive. To explore these mechanics and to build the intuition, a mathematical derivation is required.

1.1   Precession

Precession is the motion where the spinning object rotations about a vertical axis, in addition to the rotation about its own axis.

In its inertial reference frame, the total torque of a spinning object is related to its angular momentum,

$$\vec{\tau} = \frac{d\vec{L}}{dt}. \tag{1}$$

However from a fixed reference frame, there must be a second term that accounts for the rotation of the inertial reference frame, i.e

$$\vec{\tau} = \frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L}. \tag{2}$$

This second term suggests that there is a torque that acts perpendicular to the angular momentum and velocity of the spiining object.

$$\vec{\tau}_p = \vec{\omega}_p \times \vec{L}_s. \tag{3}$$

1.2   Nutation

Nutation refers to the oscillating motion of a spinning object's own axis with respect to its original axis' position. Nutation occurs when a transient force acts on the spinning object in any direction perpendicular to its angular momentum. Hence,

$$\vec{\tau}_n = \vec{\omega}_n \times \vec{L}_s. \tag{4}$$

2   Aim

To investigate the rotational mechanics phenomena of moment of inertia, angular momentum and velocity and precession and nutation. Ultimately, to verify Equation (3) and provide some insights into Eqn (4).

3   Method

3.1   Experimental Setup

To measure the moment of inertia of the disk, the gyroscope was kept fixed, only allowed to spin along the axis of the arm. Otherwise, the gyroscope arm was free and allowed the disk to precess and undergo nutation. Throughout both experiments, a bunch of constants were used shown in the Table 1 below. The

distance from the mass to the fulcrum is denoted by x. The radius of the disk is denoted by R and the radius of the small disk is denoted by r. The mass of the disk is denoted by M.

Constant	Measurement
M	1.336 ± 0.0005 kg
R	0.1225 ± 0.0005m
r	0.025 ± 0.0005m
x	0.275 ± 0.0005m

Table 1: Table displaying measured constants in the experiment

3.1.1 Locked Gyroscope

Firstly, the gyroscope arm was fixed as shown in Figure 1 and a mass was attached to the cord. The string was wound around the small disk attached to the big disk so that the mass carrier, carrying mass 150g, would fall from a desired height, h. The full extension of the string was measured to be 58cm and so the mass was released from heights 8cm, 18cm, 28cm, 38cm and 48cm to measure the height fell by 0.1m, 0.2m, 0.3m, 0.4m and 0.5m. The time it took to fall was recorded on a stopwatch which began as the mass carrier was released and stopped when the string was at its full extension.
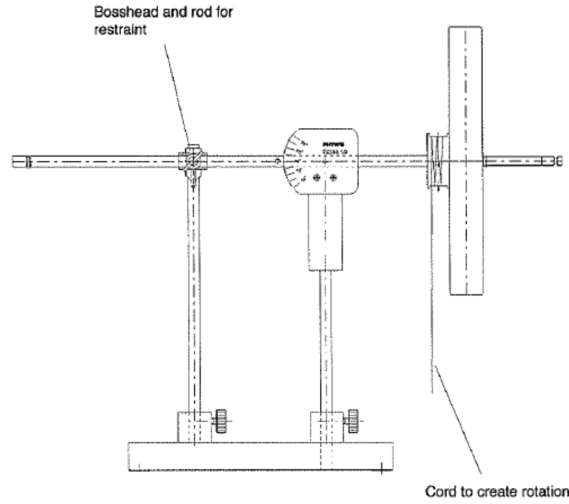

Figure 1: Diagram of the locked configuration of the gyroscope.

In this situation, the force on the mass downwards is driven by the difference between the acceleration due to gravity downwards and the acceleration due to the tension in the string upwards multiplied by the mass of the object,

$$F = m(g - a). \tag{5}$$

The acceleration due to the tension in the string is related to the angular acceleration of the small disk. This is because as the string's tensions increases, it will push the small disk, causing an angular acceleration.

$$a = r\alpha. \tag{6}$$

The angular acceleration of the small disk can be assumed to be the same as the angular acceleration of the bigger disk which is related to its moment of inertia. (The angle of the incoming force is $90°$).

$$\alpha = \frac{rF}{I_s}. \tag{7}$$

Combining Eqns. (6) and (7) gives,

$$
\begin{aligned}
a &= \frac{r^2 m(g-a)}{I_s} \\
&= \frac{r^2 mg - r^2 ma}{I_s} \\
aI_s + r^2 ma &= mgr^2 \\
a &= \frac{mgr^2}{I_s + mr^2}
\end{aligned}
\tag{8}
$$

Using kinematic equations, the acceleration of the mass is related to the height and time to fall,

$$
\begin{aligned}
h &= \frac{1}{2} a t_F^2 \\
a &= \frac{2h}{t_F^2}
\end{aligned}
\tag{9}
$$

Combining Eqns. (8) and (9) gives,

$$
\begin{aligned}
\frac{2h}{t_F^2} &= \frac{mgr^2}{I_s + mr^2} \\
t_F^2 &= \frac{2I_s + 2mr^2}{mgr^2} h
\end{aligned}
\tag{10}
$$

3.1.2 Free Gyroscope

To measure the angular velocity of precession and nutation, the support beam was removed and the gyroscope arm was allowed to move freely, shown in Figure 2. The procedure for the measurement of precession and the measurement of nutation with precession was the same whereas the measurement of nutation without precession was differed slightly.
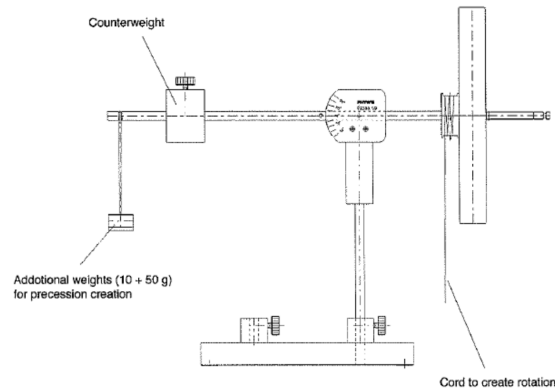

Figure 2: Diagram of free configuration of the gyroscope.

In all three measurements, the angular velocity of the spinning disk was calculated by using an optical gate counter, counting the amount of revolutions, n, in 10 seconds. To account for frictional effects on the spinning disk, two counts were done, one before the precession or nutation and one after and the average of the two counts was used,

$$\omega_s = \frac{2\pi n_{avg}}{10}.$$ (11)

For precession without nutation, the disk was spun by pulling the cord and its angular velocity of spin was initially measured. A mass carrier containing weights of 20g, 40g, 60g, 90g and 140g, combining for a total mass of 30g, 50g, 70g, 100g and 150g was added to the opposite end of the gyroscope arm as displayed in Figure 2. The spinning disk now began to process and its half period was recorded in order to calculate its angular velocity of precession,

$$\omega_p = \frac{2\pi}{0.5T}.$$ (12)

With reference to Figure 2, this added mass creates a torque acting into the page as now there was a force due to gravity acting on the left side of the arm. When the mass was added, the gyroscope arm began to tilt some degrees, the reason for the volatility of the tilt can be attributed to the external force from placing down the mass but should be explored in further experiments to accurately determine what caused the tilt. Nevertheless, the angle of tilt, θ, was measured using a protractor attached onto the gyroscope arm and the torque due to the added weight can be calculated,

$$\tau_{mass} = mgx\sin\theta.$$ (13)

The total torque of the gyroscope can be related to its angular momentum,

$$\vec{\tau} = \frac{\Delta\vec{L}}{\Delta t}.$$ (14)

From Figure 2, it can be seen that the direction of torque, into the page, is perpendicular to the direction of angular momentum. This means that the total torque will only change the direction of angular momentum, not its magnitude and so Eqn. (13) can be modified,

$$\vec{\tau} = \frac{\vec{L}\Delta\phi}{\Delta t}$$ (15)

where ϕ is the horizontal angle that angular momentum makes with its vertical axis. Therefore the $\frac{\Delta\phi}{\Delta t}$ can be replaced with ω_p where p denotes the motion of precession.

$$\vec{\tau} = \vec{\omega}_p \times \vec{L}_s.$$ (16)

This is the same as Eqn. (3) so the 'total torque' in this situation will be just the torque due to precession, ω_p. This is because Eqn. (13) should have included a change in angular momentum due to the torque created by an unstable centre of mass however $L_s \gg L_{CM}$ so this can be ignored experimentally.

As ω_p and $\vec{L}_s$ are perpendicular and L_s can be rewritten as the product of its angular velocity and momentum of inerita,

$$\begin{aligned}\tau_p &= |\omega_p||L_s| \\ &= \omega_p I_s \omega_s\end{aligned}$$ (17)

Rearranging and substituting in Eqn. (12),

5

$$\omega_p = \frac{gx}{I_s\omega_s}m. \tag{18}$$

For measuring nutation with precession, the same method was done except as the mass was added on, a small knock was also conducted to provide a transient force perpendicular to the angular momentum. The purpose of this was not to measure the oscillating motion of nutation but compare the two motions of precession, one with nutation and one without.

For measuring nutation without precession, the disk was spun on its axis shown in Figure 2. The angular velocity was measured and then varying sized knocks were stuck on the end of the gyroscope arm opposite the disk. These knocks were best kept controlled by using one finger for 'soft' knock, two fingers for 'medium' knock and three fingers for 'hard' knock. As the disk underwent nutation, its number of oscillations, n, were counted in the span of 10 seconds. The angular velocity of nutation would then be given as,

$$\omega_n = \frac{2\pi n}{10}. \tag{19}$$

Lastly, nutation was measured by varying the angular velocity of spin. To do this, the disk was spun and its angular velocity was measured. After the ten seconds, a knock was struck and the oscillations due to nutation was counted. The nutation would eventually dampen to zero which is when the angular velocity of spin was measured again, after another ten seconds, another similar knock was dealt and the oscillations were counted again.
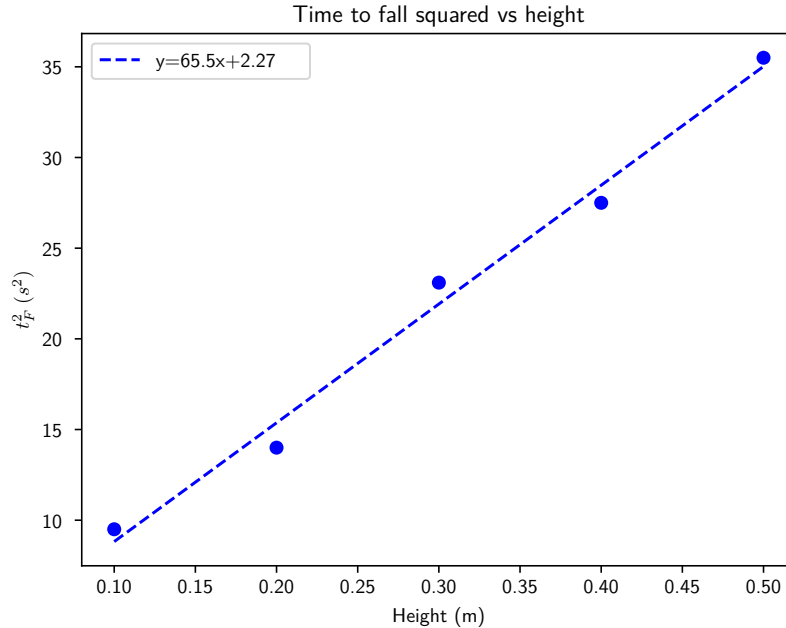
4   Results

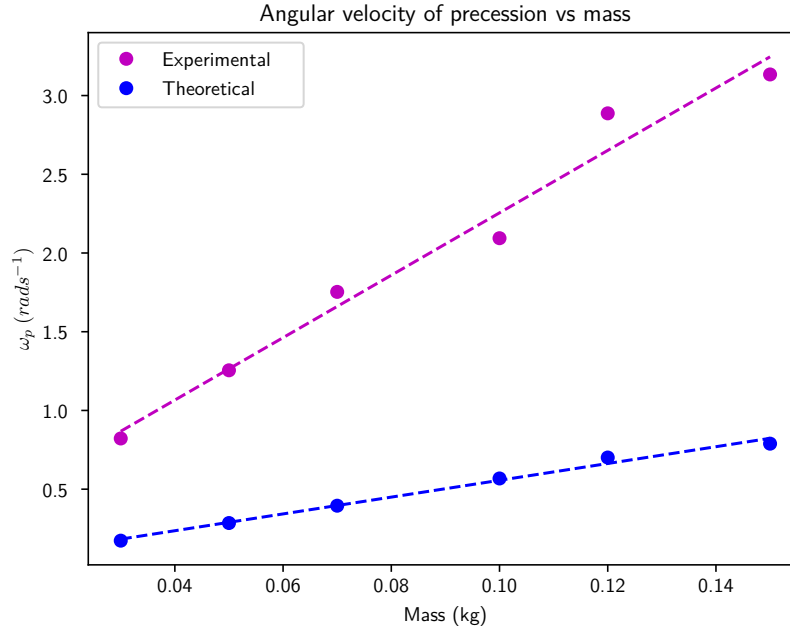Figure 3: Plot displaying time to fall squared by height dropped.

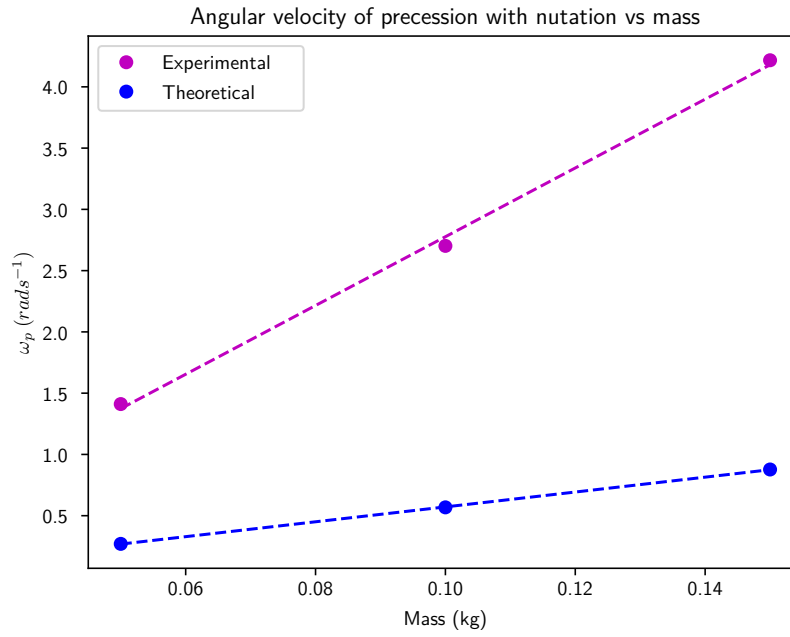Figure 4: Plot displaying experimental and theoretical relationship between ω_p and m.

Figure 5: Plot displaying experimental and theoretical relationship between ω_p with τ_n and m.
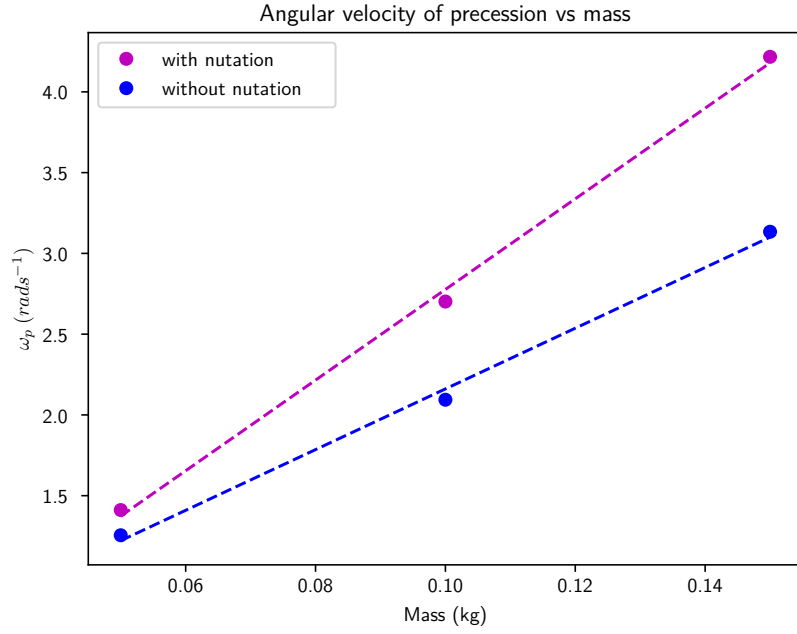
Figure 6: Plot comparing the ω_p with nutation and without nutation.
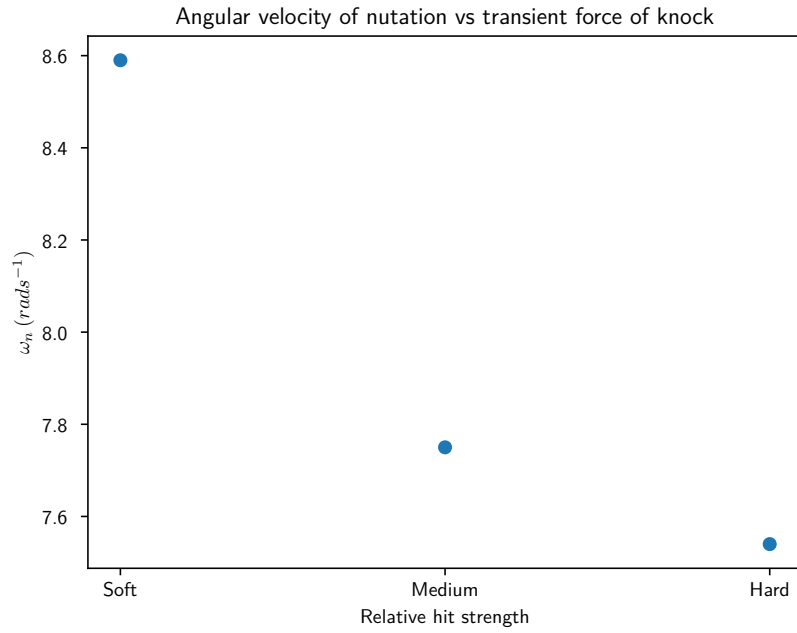

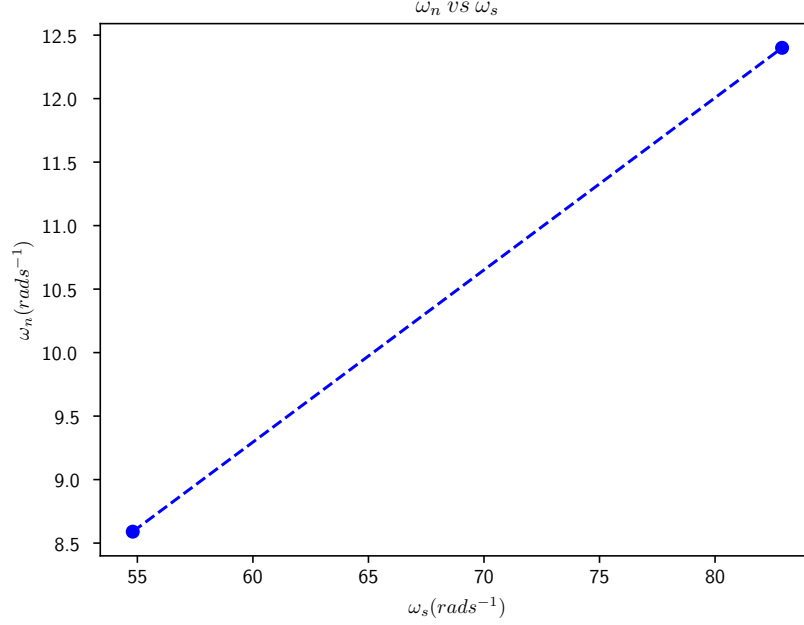Figure 7: Plot qualitatively describing ω_n with respect to different strengths of knocks.

Figure 8: Plot displaying the relationship between $\omega_n \, and \, \omega_s$.

5   Analysis

5.1   Moment of Inertia

From Figure 3, the gradient of the graph is 65.5 and using Eqn. (10),

$$65.5 = \frac{2I_s + 2mr^2}{mgr^2}. \tag{20}$$

Rearranging for I_s gives,

$$I_s = \frac{mr^2}{2}(65.5g - 2). \tag{21}$$

Substituting related constants,

$$
\begin{aligned}
I_s &= \frac{(0.15(0.025)^2)}{2}((65.5)(9.8) - 2) \\
&= 0.010 \, kgm^2.
\end{aligned}
\tag{22}
$$

To calculate the uncertainty, the sum of all percentage uncertainties of each measurement is required. The total percentage uncertainty is given as,

$$
\begin{aligned}
\sigma_{total} &= \Sigma\sigma_i \\
&= 2(\frac{0.0005}{0.025}) \\
&= 4\%.
\end{aligned}
\tag{23}
$$

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Thus, the measurement for I_s is,

$$I_s = 0.010 \pm 0.0004 \; kgm^2. \tag{24}$$

Finding the moment of inertia using the disk's geometry requires the relationship,

$$I_s = \frac{1}{2}MR^2. \tag{25}$$

Substituting known quantities,

$$\begin{aligned} I_s &= \frac{1}{2}(1.336)(0.1225)^2 \\ &= 0.010 \; kgm^2. \end{aligned} \tag{26}$$

The uncertainty of the measurement is given as,

$$\begin{aligned} \sigma &= \frac{0.0005}{1.336} + 2(\frac{0.0005}{0.1225}) \\ &= 0.9\%. \end{aligned} \tag{27}$$

With such a small uncertainty, it will be omitted but nonetheless, both results from Eqn. (21) and (23) agree.

5.2   Precession

From Figure 4, there is a difference in trendline and values from the experimental and theoretical results. The theoretical results were calculated using the moment of inertia result from Eqn. (21) and the relationship in Eqn. (17). To quantify this difference, the experiment results will be used in conjunction with Eqn. (17) to produce a value for the moment of inertia.

The gradient of the experimental trendline is 19.8, substituting this into Eqn. (17),

$$19.8 = \frac{gx}{I_s\omega_s}. \tag{28}$$

Rearranging and substituting constants,

$$\begin{aligned} I_s &= \frac{(9.8)(0.275)}{(49)(19.8)} \\ &= 0.0028 \; kgm^2 \end{aligned} \tag{29}$$

where ω_s is the average angular velocity across all trials.

The uncertainty can be calculated as,

$$\begin{aligned} \sigma &= \frac{0.0005}{0.275} + \frac{4}{49} \\ &= 8.3\% \end{aligned} \tag{30}$$

Thus, the moment of inertia is given as,

$$I_s = 0.0028 \pm 0.0003 \; kgm^2. \tag{31}$$

This value represents as a 257% error from the agreed results found in the previous section.

5.3 Nutation

Comparing the graphs in Figure 4 and Figure 5, the trendline is parallel and remains unchanged despite the addition of nutation to the motion of precession. This indicates that the torque of nutation acts perpendicular to the torque of precession. However in Figure 6, it can be seen that nutation has an influence on the angular velocity of precession,

$$\omega_p = k_n \frac{gx}{I_s \omega_s} m \tag{32}$$

where k_n is the constant of nutation that increases the angular velocity of precession, i.e $k_n \geq 1$.

Figure 7 shows that the angular velocity of nutation is inversely proportional to the strength of the knock,

$$\omega_n \propto \frac{1}{F_{knock}}. \tag{33}$$

From Figure 8, the following relationship can be somewhat inferred,

$$\omega_n \propto \omega_s. \tag{34}$$

6 Discussion

Overall the experiment was successful in achieving its aim and verifying the hypothesis presented by Eqn. (3). The moment of inertia measurements were completed with high validity resulting in a value that is highly confident in being its true value. However this is not the case with the precession and nutation experiments.

The gyroscope being in its free configuration created more chaotic elements and so some motions were not aligned with what was expected in theory. For example, theoretically, for oblate bodies, the angular velocity of precession should be twice as great as the angular velocity of spin[1],

$$\omega_p \approx 2\omega_s. \tag{35}$$

However, it was found experimentally that $\omega_s >> \omega_p$. One cause of this discrepancy could be the introduction of air resistance. Due to the large surface area of the gyroscope, its motion was slowed greatly by air resistance and thus could not reach the theoretical levels. This would explain the significant inconsistency between the moment of inertia calculated in Eqn. (30), achieving a 257% error.

Nutation was particularly hard to measure because it could only be done without the motion of precession. Even then, there were too many uncontrolled factors such as the strength of the knock or angular velocity of spin.

To improve the experiment, it should be done in a complete vacuum, removing any particles of air that would slow down its precessing motion. This would also lower, but not eliminate, the dampening effect of friction on the angular velocity of spin. There also needs to be a better or at least more consistent method of attaching the mass onto the gyroscope arm to cause it to precess. Depending on external force applied, the angle of tilt changes and thus affects the total motion of precession. Although with more repetitions, this random error would be eliminated, given the limited data, this would have played a significant role in the volatility of data recorded.

To improve the measurement of nutation, there just needs to be a consistent method of applying an external force. Some contraption with a knocking motion should be used rather than a human to improve consistency.

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[1]E. Butikov, "Inertial rotation of a rigid body", European Journal of Physics 27, 913 (2006).

There also needs to be a way to measure the oscillations due to nutation as the spinning object precesses to better gauge the relationship between nutation and precession.

7 Conclusion

This was a insightful experiment in terms of exploring the mechanics of angular momentum, precession and nutation to a somewhat indepth degree. The once unintuitive relationship between the phenomena has now been understood mathematically and revealed the more, now intuitive, physical intrepretation. However due to real world limitations, the value for moment of inertia as predicted using the motion of precession deviated 257% from its expected value.