(Sheet # 1)

Q1: Given vectors $\boldsymbol{A} = \boldsymbol{a_x} + 3\boldsymbol{a_z}$ and $\boldsymbol{B} = 5\boldsymbol{a_x} + 2\boldsymbol{a_y} - 6\boldsymbol{a_z}$

Determine:

(a) $|\boldsymbol{A} + \boldsymbol{B}|$
(b) $5\boldsymbol{A} - \boldsymbol{B}$
(c) $\boldsymbol{A} \cdot \boldsymbol{B}$
(d) A unit vector parallel to $3\boldsymbol{A} + \boldsymbol{B}$

Solution

(a) $A + B = 6a_x + 2a_y - 3a_z$ $\quad |A + B| = \sqrt{6^2 + 2^2 + (-3)^2} = \mathbf{7}$

(b) $5A - B = \boldsymbol{-2a_y + 21a_z}$

(c) $A \cdot B = (1 \times 5) + (0 \times 2) + (3 \times -6) = \mathbf{-13}$

(d) $3A + B = 8a_x + 2a_y + 3a_z$ $\quad u_{3A+B} = \frac{3A+B}{|3A+B|} = \frac{8a_x+2a_y+3a_z}{\sqrt{8^2+2^2+3^2}} = \boldsymbol{0.9117a_x + 0.228a_y + 0.342a_z}$

Q2: Given points $\boldsymbol{P}(1, 23, 5)$, $\boldsymbol{Q}(2, 4, 6)$, and $\boldsymbol{R}(0, 3, 8)$

Determine:

(a) The position vectors of P and R
(b) The distance vector r_{QR}
(c) The distance between Q and R.

Solution

(a) $P = OP = (1, 23, 5) - (0, 0, 0) = \boldsymbol{a_x + 23a_y + 5a_z}$ $\quad R = OR = (0, 3, 8) - (0, 0, 0) = \boldsymbol{3a_y + 8a_z}$

(b) $Q = OQ = (2, 4, 6) - (0, 0, 0) = 2a_x + 4a_y + 6a_z$ $\quad r_{QR} = R - Q = \boldsymbol{-2a_x - a_y + 2a_z}$

(c) $|r_{QR}| = |r_{RQ}| = |R - Q| = |Q - R| = \sqrt{(-2)^2 + (-1)^2 + 2^2} = \mathbf{3}$

Q3: If $A = a_x + 3a_z$ and $B = 5a_x + 2a_y - 6a_z$

Determine:

(a) θ_{AB} using dot product
(b) θ_{AB} using cross product

Solution

(a) $A \cdot B = 5 + 0 - 18 = -13$

$$\therefore \theta_{AB}' = \cos^{-1}\frac{A \cdot B}{|A||B|} = \cos^{-1}\frac{-13}{\sqrt{1^2+3^2}\sqrt{5^2+2^2+(-6)^2}} = 120.6573^\circ$$

$$\therefore \boldsymbol{\theta_{AB} = 180^\circ - \theta_{AB}' = 59.3427^\circ}$$

(b) $A \times B = \begin{vmatrix} a_x & a_y & a_z \\ 1 & 0 & 3 \\ 5 & 2 & -6 \end{vmatrix} = a_x(0-6) - a_y(-6-15) + a_z(2-0) = -6a_x + 21a_y + 2a_z$

$$\therefore \boldsymbol{\theta_{AB} = \sin^{-1}\frac{|A \times B|}{|A||B|} = \sin^{-1}\frac{\sqrt{(-6)^2+21^2+2^2}}{\sqrt{1^2+3^2}\sqrt{5^2+2^2+(-6)^2}} = 59.3427^\circ}$$

Q4: Three field quantities are given by

$$\boldsymbol{P = 2a_x - a_z} \qquad \boldsymbol{Q = 2a_x - a_y + 2a_z} \qquad \boldsymbol{R = 2a_x - 3a_y + a_z}$$

Determine:

(a) $(\boldsymbol{P} + \boldsymbol{Q}) \times (\boldsymbol{P} - \boldsymbol{Q})$

(b) $\boldsymbol{Q} \cdot \boldsymbol{R} \times \boldsymbol{P}$

(c) $\boldsymbol{P} \cdot \boldsymbol{Q} \times \boldsymbol{R}$

(d) $\boldsymbol{P} \times (\boldsymbol{Q} \times \boldsymbol{R})$

(e) A unit vector perpendicular to both Q and R

Solution

(a) $P + Q = 4a_x - a_y + a_z$ $\qquad$ $P - Q = a_y - 3a_z$

$$(P+Q) \times (P-Q) = \begin{vmatrix} a_x & a_y & a_z \\ 4 & -1 & 1 \\ 0 & 1 & -3 \end{vmatrix} = a_x(3-1) - a_y(-12-0) + a_z(4-0) = \boldsymbol{2a_x + 12a_y + 4a_z}$$

(b) $Q \cdot R \times P = \begin{vmatrix} 2 & -1 & 2 \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix} = 2(3-0) + 1(-2-2) + 2(0+6) = 6 - 4 + 12 = \boldsymbol{14}$

(c) $P \cdot Q \times R = \begin{vmatrix} 2 & 0 & -1 \\ 2 & -1 & 2 \\ 2 & -3 & 1 \end{vmatrix} = 2(-1+6) + 0(2-4) - 1(-6+2) = 10 + 0 + 4 = \boldsymbol{14}$

(d) $Q \times R = \begin{vmatrix} a_x & a_y & a_z \\ 2 & -1 & 2 \\ 2 & -3 & 1 \end{vmatrix} = a_x(-1+6) - a_y(2-4) + a_z(-6+2) = 5a_x + 2a_y - 4a_z$

$$\therefore P \times (Q \times R) = \begin{vmatrix} a_x & a_y & a_z \\ 2 & 0 & -1 \\ 5 & 2 & -4 \end{vmatrix} = a_x(0+2) - a_y(-8+5) + a_z(4-0) = \boldsymbol{2a_x + 3a_y + 4a_z}$$

(e) $a_N = \frac{Q \times R}{|Q \times R|} = \frac{5a_x + 2a_y - 4a_z}{\sqrt{5^2+(-4)^2+2^2}} = \boldsymbol{0.7454a_x + 0.298a_y - 0.5963a_z}$

Q5: E and F are vector fields given by

$$\boldsymbol{E} = 2x\boldsymbol{a_x} + \boldsymbol{a_y} + yz\boldsymbol{a_z} \qquad \boldsymbol{F} = xy\boldsymbol{a_x} - y^2\boldsymbol{a_y} + xyz\boldsymbol{a_z}$$

Determine:

(a) $\boldsymbol{E}$ at (1, 2, 3)

(b) $\boldsymbol{E} \cdot \boldsymbol{F}$ at (1, 2, 3)

(c) A unit vector perpendicular to both $\boldsymbol{E}$ and $\boldsymbol{F}$ at (0,1, −3)

Solution

(a) $E|_{x=1,\ y=2,\ z=3} = \boldsymbol{2a_x + a_y + 6a_z}$

(b) $F|_{x=1,\ y=2,\ z=3} = 2a_x - 4a_y + 6a_z \qquad E \cdot F = (2 \times 2) + (1 \times -4) + (6 \times 6) = \boldsymbol{36}$

(c) $E|_{x=0,\ y=1,\ z=-3} = a_y - 3a_z \qquad F|_{x=0,\ y=1,\ z=-3} = -a_y$

$$E \times F = \begin{vmatrix} a_x & a_y & a_z \\ 0 & 1 & -3 \\ 0 & -1 & 0 \end{vmatrix} = -3a_x \qquad a_N = \frac{E \times F}{|E \times F|} = \frac{-3a_x}{\sqrt{(-3)^2}} = \boldsymbol{-a_x}$$

Q6: Given the two points, $\boldsymbol{C}(-3, 2, 1)$ $\qquad$ $\boldsymbol{D}(\boldsymbol{r} = 5, \boldsymbol{\theta} = 20°, \boldsymbol{\phi} = -70°)$

Determine:

(a) The spherical coordinates of $\boldsymbol{C}$

(b) The rectangular coordinates of $\boldsymbol{D}$

(c) The distance from $\boldsymbol{C}$ to $\boldsymbol{D}$

Solution

(a) Convert point C to spherical

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{9 + 4 + 1} = \sqrt{14} \qquad \emptyset = \tan^{-1}\frac{y}{x} = -33.69^{\circ} \qquad \theta = \cos^{-1}\frac{z}{r} = \cos^{-1}\frac{1}{\sqrt{14}} = 74.5^{\circ}$$

$$\therefore \boldsymbol{C} = \left(\sqrt{\boldsymbol{14}}, \boldsymbol{-33.69^{\circ}}, \boldsymbol{74.5^{\circ}}\right)$$

(b) Convert point D to rectangular

$$x = r\sin\theta\cos\emptyset = 0.585 \qquad y = r\sin\theta\sin\emptyset = -1.607 \qquad z = r\cos\theta = 4.7$$

$$\therefore \boldsymbol{D} = (\boldsymbol{0.585}, \boldsymbol{-1.607}, \boldsymbol{4.7})$$

(c) $CD = D - C = 3.585a_x - 3.607a_y + 3.7a_z \qquad |CD| = \sqrt{3.585^2 + (-3.607)^2 + 3.7^2} = \boldsymbol{6.289}$

Q7: Given vectors $\boldsymbol{A} = 2a_x + 4a_y + 10a_z$ $\qquad$ $\boldsymbol{B} = -5\,a_x + a_y - 3\,a_z$

Determine:

(a) $\boldsymbol{A} + \boldsymbol{B}$

(b) The angle between A and B

(c) $\boldsymbol{A} \times \boldsymbol{B}$

Solution

(a) $A + B = \mathbf{-3a_x + 5a_y + 7a_z}$

(b) $A \cdot B = (2 \times -5) + (4 \times 1) + (10 \times -3) = -36$

$$\theta_{AB}{}' = \cos^{-1}\frac{A \cdot B}{|A||B|} = \cos^{-1}\frac{-36}{\sqrt{(-5)^2 + 1^2 + (-3)^2}\sqrt{2^2 + 4^2 + 10^2}} = 123.745^{\circ}$$

$$\therefore \boldsymbol{\theta_{AB} = 180^{\circ} - \theta_{AB}{}' = 56.255^{\circ}}$$

(c) $A \times B = \begin{vmatrix} a_x & a_y & a_z \\ 2 & 4 & 10 \\ -5 & 1 & -3 \end{vmatrix} = a_x(-12 - 10) - a_y(-6 + 50) + a_z(2 + 20) = \mathbf{-22a_x - 44a_y + 22a_z}$

Q8: Given vectors $A = 2a_x + ya_y + 10a_z$ $\quad$ $B = -5\,a_x + a_y - 3\,a_z$

Determine:

(a) Value of y if vector A is perpendicular to vector B

(b) Value of y if $\theta_{AB} = 60^{\circ}$

Solution

(a) *if A is perpendicular to B* $\quad \therefore A \cdot B = 0$

$A \cdot B = (2 \times -5) + (y \times 1) + (10 \times -3) = 0 \quad A \cdot B = y - 40 = 0 \quad \boldsymbol{y = 40}$

(b) $A \cdot B = |A||B| \cos 60^{\circ} = (2 \times -5) + (y \times 1) + (10 \times -3) \quad \sqrt{4 + y^2 + 100}\sqrt{25 + 1 + 9} \times 0.5 = y - 40$

$2.958\sqrt{104 + y^2} = y - 40 \quad 8.75y^2 + 910 = y^2 - 80y + 1600$

$7.75y^2 + 80y - 690 = 0 \quad \boldsymbol{y = 5.594} \quad \boldsymbol{y = -15.9163}$

Q9: Given the two points, $A(1, 5, 2)$ $\quad$ $B(\rho = 5, \phi = 30°, z = 6)$

Determine:

(a) The cylindrical coordinates of A

(b) The rectangular coordinates of B

(c) The distance vector from A to B

Solution

(a) Convert point A to cylindrical

$\rho = \sqrt{x^2 + y^2} = \sqrt{1 + 25} = \sqrt{26} \quad \emptyset = \tan^{-1}\frac{y}{x} = 78.69^{\circ} \quad z = 2 \quad \therefore \boldsymbol{A = (\sqrt{26}, 78.69^{\circ}, 2)}$

(b) Convert point B to rectangular

$x = \rho \cos \emptyset = 2.5\sqrt{3} = 4.33 \quad y = \rho \sin \emptyset = 2.5 \quad z = 6$

$$\therefore \boldsymbol{B = (4.33, 2.5, 6)}$$

(c) $AB = B - A = 3.33a_x - 2.5a_y + 4a_z \quad |AB| = \sqrt{3.33^2 + (-2.5)^2 + 4^2} = \mathbf{5.774}$

(Report Section) Q10: Given vectors $A = -5\,a_x + a_y + 10z a_z \qquad B = -5\,a_x + a_y - 3\,a_z$

Determine:

(a) Value of z if vector A is perpendicular to vector B

(b) Value of z if vector A is parallel to vector B

(c) Value of z if $\theta_{AB} = 45^{\circ}$

Solution

(a) $if\ A\ is\ perpendicular\ to\ B \qquad \therefore A \cdot B = 0$

$A \cdot B = (-5 \times -5) + (1 \times 1) + (10z \times -3) = 0 \qquad A \cdot B = -30z + 26 = 0 \qquad \boldsymbol{z = 0.8667}$

(b) $if\ A\ is\ parallel\ to\ B \qquad \therefore A \times B = 0$

$$A \times B = \begin{vmatrix} a_x & a_y & a_z \\ -5 & 1 & 10z \\ -5 & 1 & -3 \end{vmatrix} = a_x(-3-10z) - a_y(15+50z) + a_z(-5+5)$$

$$A \times B = (-3-10z)a_x - (15+50z)a_y = 0 \qquad \boldsymbol{z = -0.3}$$

Another Solution

$if\ A\ is\ parallel\ to\ B \qquad \therefore u_A = u_B \qquad \dfrac{A}{|A|} = \dfrac{B}{|B|} \qquad \dfrac{-5a_x + a_y + 10za_y}{\sqrt{25+1+100z^2}} = \dfrac{-5a_x + a_y - 3a_y}{\sqrt{25+1+9}}$

$10z = -3 \qquad 0 \qquad \boldsymbol{z = -0.3}$

(c) $A \cdot B = |A||B|\cos 45^{\circ} = (-5 \times -5) + (1 \times 1) + (10z \times -3)$

$$\sqrt{25+100z^2+1}\sqrt{25+1+9} \times \frac{1}{\sqrt{2}} = -30z + 26$$

$4.1833\sqrt{26+100z^2} = -30z + 26 \qquad 1750z^2 + 455 = 900z^2 - 1560z + 676$

$850z^2 + 1560z - 221 = 0 \qquad \boldsymbol{z = 0.13215} \qquad \boldsymbol{z = -1.96745}$

End of Sheet Manual Solution

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