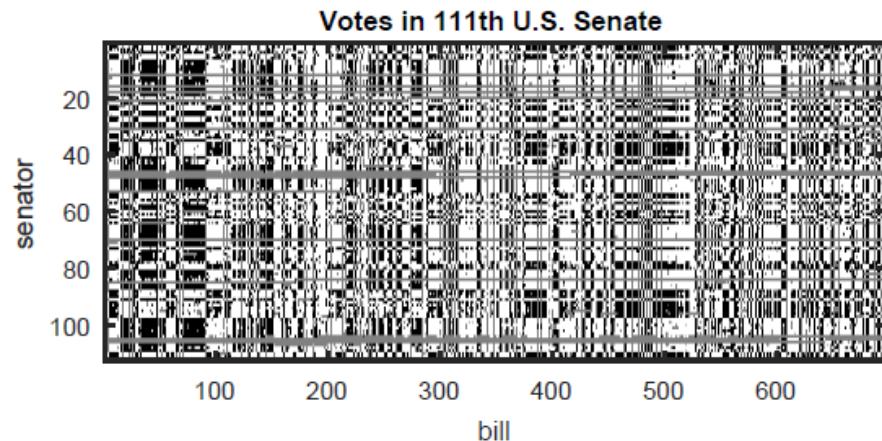


Chapter 7

Matrix Analysis



Section 7.1

From matrix to insight

From matrix to insight

- Any two-dimensional array of numbers may be interpreted as a matrix
- These may come from widely disparate tasks
- Examples:
 - Text/document search
 - Voting patterns
 - Preferences/ratings *Netflix recommendations*
 - Graphs
 - Networks: social, political, co-authorship, casting in movies,...
 - Images

Examples of matrices

- A *term-document matrix* may be used for analyzing a body of documents (or *corpus*)
- Each column may be a document; each row a term
- E.g, your textbook may have words like “numerical,” “discretization,” “matrix,” “integration” and “function”
- An analysis textbook may have words like “integration,” “function,” “continuous” and so forth
- The occurrence of “function” may be often in both books, but the other terms are likely to be much different in frequency
- Meaning could be inferred from this kind of approach: latent semantic analysis

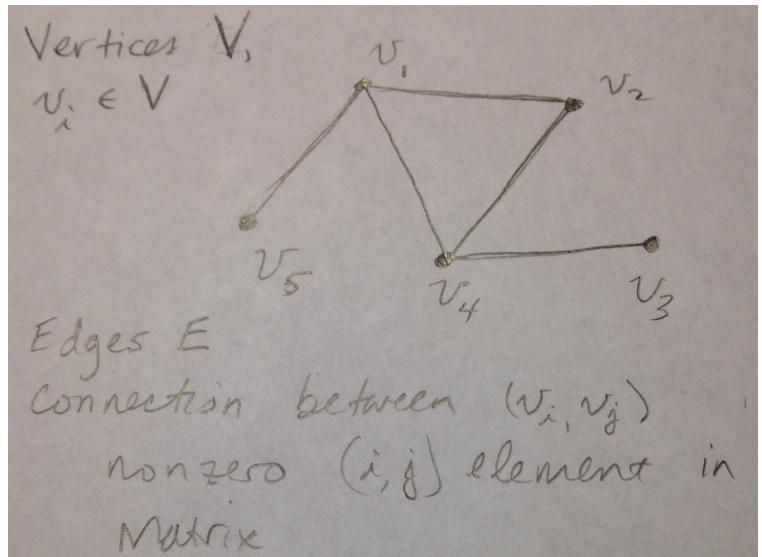
Examples of matrices

- A *term-document matrix* example: matrix in green

Term	FNC by TAD+RJB	Analysis by Rudin	SM by Trefethen	NYT coffee table book
Numerical	251	2	179	0
Integration	37	275	33	18
Function	175	345	123	0
Matrix	151	11	87	0
Continuous	15	212	11	0
Spectral	15	0	124	0
Citizen	0	0	0	13

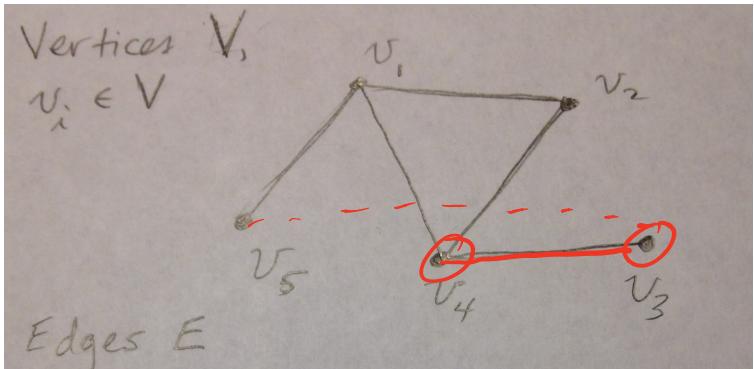
Graphs as matrices

- Could be social network, internet (or subset of it), the web, etc
- Consider graphs first
- A graph is a set of nodes V connected by set of edges E
- Sometimes the graph is denoted $G(V, E)$



Adjacency matrix

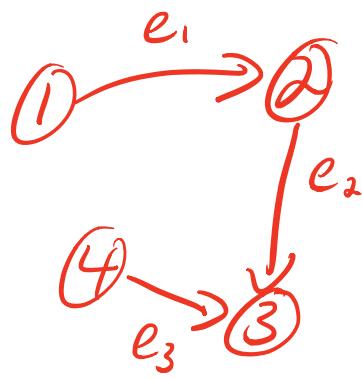
- We want adjacency matrix $A = \{a_{ij}\}$ that represents this graph
- Edges are unweighted and undirected
- If an edge between (v_i, v_j) , then a one is placed in both a_{ij} and a_{ji} (symmetric)
- No self-connections
- Corresponding matrix at right



	1	2	3	4	5
1	0	1	0	1	1
2	1	0	0	1	0
3	0	0	0	1	0
4	1	1	1	0	0
5	1	0	0	0	0

Incidence matrix:

$$\begin{matrix} & e_1 & e_2 & e_3 \\ 1 & -1 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 3 & 0 & 1 & 1 \\ 4 & 0 & 0 & -1 \end{matrix}$$



Adjacency matrix: Buckyball

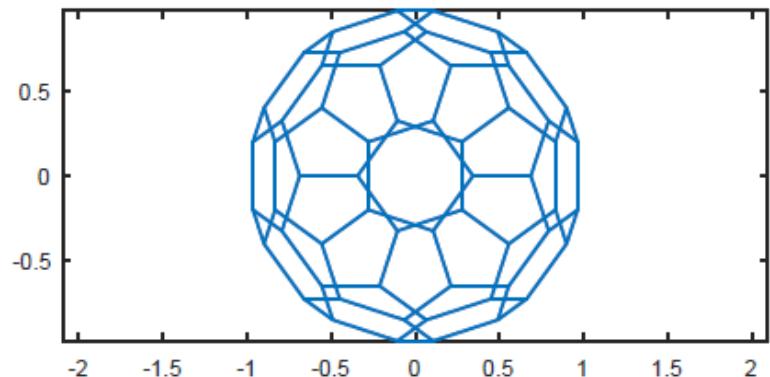
- Matlab has a built-in example of a graph representing the arrangements of carbon atoms in a C_{60} molecule, a.k.a. the buckyball:

```
[A,v] = bucky;  
size(A)
```

```
ans =  
60    60
```

- The output has the adjacency matrix A and the vertex locations v
- Plotting the graph:

```
gplot(A,v), axis equal
```

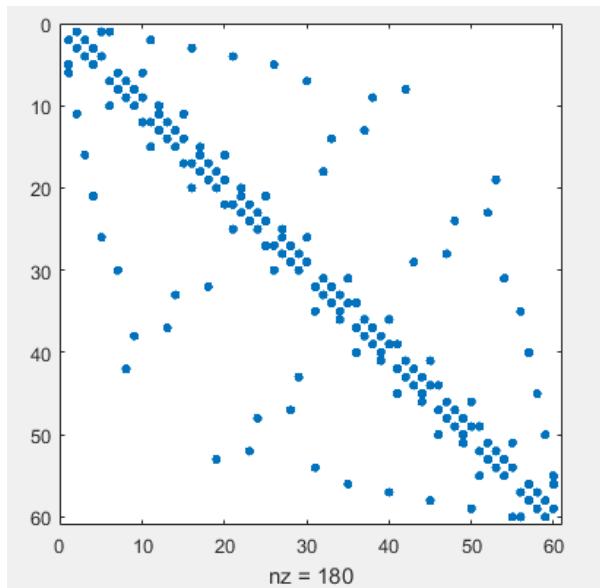


Adjacency matrix: Buckyball

- We know there are 60 nodes; how many edges?
- There is more than one way to compute this
- For undirected, you could use the spy command:

```
>> [A, v]=bucky;  
>> spy(A)  
>>
```

- Is nnz it? Not quite
- You could use the triu and sum commands; how?



[Example 7.1.4]

Images

- Pictures and images are matrices in matlab
- Use `imread` and `imshow` to display them

```
A = imread('peppers.png');  
size(A)
```

```
ans =  
    384    512     3
```

- Three “layers” are RGB components
- To display: `imshow(A)`



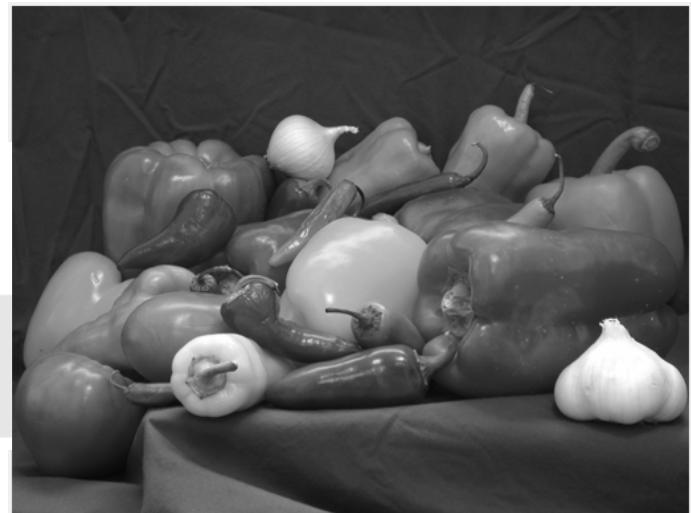
Images

- We can convert to grayscale to get only a 2D matrix
- Use `rgb2gray` and `double`:

```
A = rgb2gray(A); % collapse from 3 dimensions to 2  
A = double(A); % convert to floating point  
[m,n] = size(A)
```

```
m =  
    384  
n =  
    512
```

- To display: `imshow(A, [0, 255])`
- What if different range spec'd?



[Example 7.1.5]

Section 7.2

Eigenvalue decomposition

The eigenvalue decomposition

- We can decompose a matrix A , under some assumptions, into a useful set of other matrices using its eigenvalues and eigenvectors.
- Recall that the eigenvalue problem is $Ax = \lambda x$, for the eigenvalues λ and eigenvectors x $x \neq 0$
- (Sometimes the eigenvectors are referred to as the eigenspaces.)
- We need to review some results and terminology from linear algebra before proceeding
- In particular we need to recall the complex-valued case

$$S_\lambda = \{x \in \mathbb{C}^n \mid Ax = \lambda x\} \leftarrow \text{eigenspace}$$

Complex vectors and matrices

- Recall complex numbers $x = a + ib$, where $i^2 = -1$ and a, b are real-valued.
- The complex conjugate is $\bar{x} = a - ib$
- For a matrix A with complex-valued elements, the transpose needs to use complex conjugation at the same time: $A^* = (\bar{A})^T = \overline{A^T}$
- The hermitian will keep certain properties analogous with the real-valued case
- e.g., $(A^T A)^T = A^T A$ so that $A^T A$ is symmetric. So is $(A^* A)^* = A^* A$
- Essentially, hermitian replaces transpose for complex-valued matrices and vectors

Complex vectors and matrices

- Essentially, hermitian replaces transpose for complex-valued matrices and vectors
- For inner products with complex \mathbf{u} and \mathbf{v} : $\mathbf{u}^* \mathbf{v} = \sum_{k=1}^n \bar{u}_k v_k,$
- This defines the two-norm for vectors, $\|\mathbf{u}\|_2 = \sqrt{\mathbf{u}^* \mathbf{u}}$, and this will in turn define the two norm for matrices
- Definition of orthogonal
- And orthonormal set of vectors $\mathbf{u}_j, j = 1, 2, \dots, n$: $\mathbf{u}_i^* \mathbf{u}_j = 1$ if $i = j$, otherwise $\mathbf{u}_i^* \mathbf{u}_j = 0$

$$\mathbf{U} = [\mathbf{u}_1 \ \cdots \ \mathbf{u}_n]$$

$$\mathbf{U}^* \mathbf{U} = \mathbf{I}$$

Complex vectors and matrices

- A square **real-valued matrix** with orthonormal columns is an orthogonal matrix
- A square **complex matrix** with orthonormal columns is unitary
- An $n \times n$ unitary matrix \mathbf{U} satisfies $\mathbf{U}^{-1} = \mathbf{U}^*$ and $\|\mathbf{U}\mathbf{x}\|_2 = \|\mathbf{x}\|_2$ for any complex vector $\mathbf{x} \in \mathbb{C}^n$

$$\mathbf{U}^* \mathbf{U} = \mathbf{I}$$

The eigenvalue problem

- The eigenvalue problem is $Ax = \lambda x$, for the eigenvalues λ and eigenvectors x
- It can also be written $0 = (\lambda I - A)x$
- We find eigenvalues λ_k such that $\lambda_k I - A$ is singular, and we find the corresponding eigenvectors x_k for $k = 1, 2, \dots, n$
- When doing this by hand, one calculates the roots of the characteristic polynomial $\det(\lambda I - A)$
- There are thus n eigenvalues for an $n \times n$ matrix, counting multiplicity
- Note: eigenvalues and eigenvectors not done that way in computer

The eigenvalue decomposition

- Suppose we know the eigenvalues λ_k and eigenvectors \mathbf{v}_k so that $A\mathbf{v}_k = \lambda_k \mathbf{v}_k$, for $k = 1, 2, \dots, n$
- For each k , we have a column vector on each side of the equation
- We can assemble each of those n column vectors into a matrix:

$$\rightarrow [A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n] = [\lambda_1 \mathbf{v}_1 \quad \lambda_2 \mathbf{v}_2 \quad \cdots \quad \lambda_n \mathbf{v}_n]$$

• Rewrite: $\rightarrow A [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

• Matrix form: $\rightarrow \boxed{AV = VD}$

(7.2.3)

The eigenvalue decomposition

- If we know that there is a complete set of linearly independent (LI) eigenvectors v_k , then V^{-1} exists, and we can postmultiply by it to get:

$$A = VDV^{-1}$$

- This is the eigenvalue decomposition, or EVD, of A
- If A is diagonalizable, it will have an EVD
- When does that happen?

Theorem 7.2.1

If the $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Computing EVD in Matlab

- We can use the `eig` command to compute the EVD
- The eigenvalues arrive as the diagonal elements of D
- The eigenvectors are the columns of V
- A is singular, but has distinct eigenvalues, and because of that, it has a complete set of LI eigenvectors.
Because of that, V is nonsingular

```
A = pi*ones(2, 2);  
lambda = eig(A)
```

```
lambda =  
    0  
  6.2832
```

```
[V, D] = eig(A)
```

```
V =  
    -0.7071    0.7071  
    0.7071    0.7071  
D =  
    0         0  
    0     6.2832
```

Computing EVD in Matlab

- We can easily check that the EVD is equivalent to A:

```
>> norm( A-V*D/V ) % /V is equivalent to *inv(V)  
  
ans =  
  
8.8818e-16  
  
>> |
```

- What if A is not diagonalizable?

Computing EVD in Matlab

- If A is not diagonalizable, it will not have all eigenvalues distinct, and it will not have a full set of eigenvectors
- `eig` still works, but V^{-1} doesn't exist:
- V is only rank 1, indicating a single LI column
- You should get in the habit of critically evaluating output to check whether it is consistent with theory

[Example 7.2.1]

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

```
[V, D] = eig([1 1; 0 1])
rankV = rank(V)
```

```
V =
    1.0000    -1.0000
            0     0.0000
D =
    1         0
    0         1
rankV =
    1
```

The EVD and similarity

$$A = VDV^{-1}$$

- There is an important relationship between A and D
- For any nonsingular matrix S , then $B = SAS^{-1}$ is said to be similar to \underline{A}
- In many basic linear algebra texts, one can find a proof of this result:

Theorem 7.2.2

If X is an nonsingular matrix, then XAX^{-1} has the same eigenvalues as A .

- There is a really nice interpretation of the similarity transformation

$$\begin{aligned} \det(\lambda I - XAX^{-1}) &= \det(\lambda XX^{-1} - XAX^{-1}) \\ &= \det(X(\lambda I - A)X^{-1}) \\ &= \cancel{\det(X)} \det(\lambda I - A) \cancel{\det(X^{-1})} \\ &= \det(\lambda I - A). \end{aligned}$$

Similarity transformations

• Consider the ~~product~~ of a nonsingular matrix X with any vector:

$$\mathbf{y} = X\mathbf{z} = z_1\mathbf{x}_1 + \cdots + z_n\mathbf{x}_n$$

$$[\mathbf{y}]_X = \mathbf{z}$$

- The \mathbf{x}_i are the columns of X
- The columns of X are LI because it is invertible, so it is a basis for \mathbb{C}^n
- The z_i are the *coordinates* of \mathbf{y} using the columns of X as a basis
- But, we could left multiply by X^{-1} and then $\mathbf{z} = X^{-1}\mathbf{y}$
- The elements of \mathbf{y} are now coordinates for \mathbf{z} using the columns of X^{-1} as a basis
- Thus: *multiplication by the inverse of a matrix performs a change of basis into the coordinates associated with the matrix*

Similarity transformations

- Now consider the EVD
- Let $\mathbf{u} = \mathbf{A}\mathbf{x}$, or

$$\mathbf{u} = \mathbf{A}\mathbf{x} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{x}$$

- Premultiply by \mathbf{V}^{-1} to get

$$\mathbf{V}^{-1}\mathbf{u} = \mathbf{D}(\mathbf{V}^{-1}\mathbf{x})$$

- This equation says that using the columns of \mathbf{V} for a basis, that there is a diagonal relation between the two vectors \mathbf{u} and \mathbf{x}
- Said another way, the EVD finds a basis for \mathbb{C}^n so that the map is diagonal:

$$\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$$

- In that case, each coordinate is just multiplied by its own scalar

Similarity transformations

- This can be really convenient for matrix powers; these will be an important operation for us
- Consider \mathbf{A}^2 , and use the EVD:

$$\mathbf{A}^2 = (\mathbf{V}\mathbf{D}\mathbf{V}^{-1})(\mathbf{V}\mathbf{D}\mathbf{V}^{-1}) = \mathbf{V}\mathbf{D}(\mathbf{V}^{-1}\mathbf{V})\mathbf{D}\mathbf{V}^{-1} = \mathbf{V}\mathbf{D}^2\mathbf{V}^{-1},$$

- If we ~~knew~~ the EVD, we could just square the diagonal elements of \mathbf{D} , then reconstruct \mathbf{A}^2
- Higher powers? Then

$$\mathbf{A}^k = \mathbf{V}\mathbf{D}^k\mathbf{V}^{-1}$$

- Raise each eigenvalue to the k -th power to get \mathbf{D}^k !

Conditioning of EVD computation

- There are theorems around to tell us how much eigenvalues change in response to changes in the matrix
- One is the Bauer-Fike theorem:

Theorem 7.2.3

Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ be diagonalizable, $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$, with eigenvalues $\lambda_1, \dots, \lambda_n$. If μ is an eigenvalue of $\mathbf{A} + \mathbf{E}$ for a complex matrix \mathbf{E} , then

$$\min_{j=1, \dots, n} |\mu - \lambda_j| \leq \kappa(\mathbf{V}) \|\mathbf{E}\|, \quad (7.2.6)$$

where $\|\cdot\|$ and κ are in the 2-norm.

- Eigenvalues by as much as a factor of the condition number $\kappa(\mathbf{V})$
- If \mathbf{V} is unitary, $\kappa(\mathbf{V}) = 1$, \mathbf{A} is *normal*, eigenvalue calculation robust
- If \mathbf{V} nearly singular, $\kappa(\mathbf{V}) \gg 1$, eigenvalues can change significantly

From Wikipedia "Bauer-Fike theorem":

Alternate formulation:

Let (λ^a, v^a) be an approximate eigenvalue - eigenvector pair and

$$r = Av^a - \lambda^a v^a.$$

Then there is an eigenvalue λ of A such that

$$|\lambda - \lambda^a| \leq \kappa(V) \frac{\|r\|}{\|v^a\|}.$$

Conditioning of EVD computation

- Example: triangular matrix with $n = 15$
- The bound on the change in the condition number is around $1e7$
- Test the theorem with $1e-7$ size perturbations
- The max change in these few cases is about 25% to 50% of the possible change

[Example 7.2.2]

```
lambda = (1:n)';
A = triu( ones(n,1)*lambda' );
```

The Bauer-Fike theorem provides an upper bound these eigenvalues.

```
[V,D] = eig(A);
kappa = cond(V)
```

```
kappa =
7.1978e+07
```

```
for k = 1:3
E = randn(n); E = 1e-7*E/norm(E);
mu = eig(A+E);
max_change = norm( sort(mu)-lambda, inf )
end
```

```
max_change =
0.2407
max_change =
0.4492
max_change =
0.2737
```

Method of EVD computation

- The practical methods for computing the EVD are beyond the scope of this class (and book)
- It is worth pointing out that the methods often use the idea of matrix powers as part of the computation.
- If the eigenvalues are distinct, then raising them to a power separates them for large powers k
- There is an easy and elegant way to accomplish this separation

Method of EVD computation

- Example: create a matrix with known eigenvalues:

```
D = diag([-6 -1 2 4 5]);  
[V, R] = qr(randn(5));  
A = V*D*V'; % note that V' = inv(V)
```

The qr function takes a random 5 by 5 matrix, then returns orthogonal matrix V and upper triangular matrix R

A and D are similar

- Now do QR factorization and reverse it:

$$A = QR$$
$$Q^T A = R$$

```
[Q, R] = qr(A);  
A = R*Q;
```

- We have same eigenvalues!

$$RQ = Q^T A Q \leftarrow \text{similar to } A$$

eig(A)

```
ans =  
-6.0000  
-1.0000  
5.0000  
4.0000  
2.0000
```

Method of EVD computation

- It turns out that we can repeat this and not change the eigenvalues!
- This is *Francis QR iteration*
- For this example:
- Look at diagonal elements, and off diagonal elements are getting small

```
for k = 1:15
    [Q,R] = qr(A);
    A = R*Q;
end
A
```

A =	-5.9984	-0.1336	0.0100	-0.0000	0.0000
	-0.1336	4.9960	-0.0491	0.0000	-0.0000
	0.0100	-0.0491	4.0024	-0.0001	-0.0000
	-0.0000	0.0000	-0.0001	2.0000	-0.0001
	0.0000	-0.0000	-0.0000	-0.0001	-1.0000

[Example 7.2.3]

Section 7.3

Singular value decomposition

Singular value decomposition

- Here is another matrix factorization; it is widely used in many fields:

Theorem 7.3.1

Let $\mathbf{A} \in \mathbb{C}^{m \times n}$. Then \mathbf{A} can be written as

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^*, \quad (7.3.1)$$

where $\mathbf{U} \in \mathbb{C}^{m \times m}$ and $\mathbf{V} \in \mathbb{C}^{n \times n}$ are unitary and $\mathbf{S} \in \mathbb{R}^{m \times n}$ is real and diagonal with nonnegative entries. If \mathbf{A} is real, then so are \mathbf{U} and \mathbf{V} (which are then orthogonal matrices).

- This is the singular value decomposition, or SVD
- Cols of \mathbf{U} : left singular vectors; cols of \mathbf{V} : right singular vectors
- Diagonal elements of \mathbf{S} are the singular values $\sigma_1, \dots, \sigma_r, r = \min\{m, n\}$

Singular value decomposition

- The SVD is

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^*$$

- $\mathbf{A} \in \mathbb{C}^{m \times n}$ is $m \times n$, with complex entries
- Columns of (unitary) $\mathbf{U} \in \mathbb{C}^{m \times m}$: left singular vectors
- Columns of (unitary) $\mathbf{V} \in \mathbb{C}^{n \times n}$: right singular vectors
- Diagonal elements of $\mathbf{S} \in \mathbb{R}^{m \times n}$ are the singular values

$$\sigma_1, \dots, \sigma_r, r = \min\{m, n\}$$

- Usually the ordering of the singular values is

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq 0$$

- σ_1 is the principal singular value
- \mathbf{u}_1 and \mathbf{v}_1 are the principal singular vectors

Singular value decomposition

- What about real-valued A ?
- Consider the case with $A \in \mathbb{R}^{m \times n}$

$$A = USV^T$$

- Columns of orthogonal $U \in \mathbb{R}^{m \times m}$: left singular vectors
- Columns of unitary $V \in \mathbb{R}^{n \times n}$: right singular vectors
- Diagonal elements of $S \in \mathbb{R}^{m \times n}$ are the singular values
$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq 0, \quad r = \min\{m, n\}$$
- This is still an SVD because it satisfies the requirements:
 - the first and last matrices are orthogonal,
 - and the middle matrix is diagonal with non negative entries

Interpreting the SVD

- Let's rewrite the SVD

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^*$$

- As follows (postmultiply by V):

$$\mathbf{A}\mathbf{V} = \mathbf{U}\mathbf{S}$$

- Considering one column at a time, we have

$$\mathbf{A}\mathbf{v}_k = \sigma_k \mathbf{u}_k, \quad k = 1, 2, \dots, r, \quad r = \min\{m, n\}$$

- This means that each right singular vector \mathbf{v}_k is mapped by \mathbf{A} to a scalar multiple (σ_k) of the corresponding left singular vector \mathbf{u}_k

Interpreting the SVD: example

- Compute an SVD:

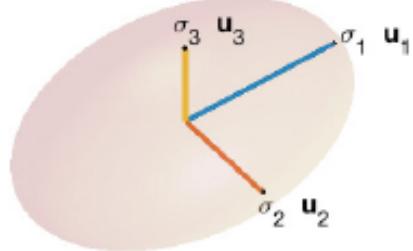
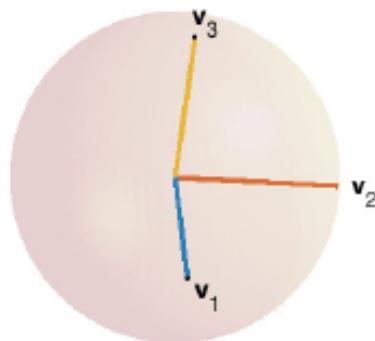
```
>> A = [2 1;3 4; 5 6]
A =
    2     1
    3     4
    5     6
>> [U,S,V] = svd(A)
U =
   -0.2164    0.9497   -0.2265
   -0.5258   -0.3088   -0.7926
   -0.8226   -0.0524    0.5661
S =
    9.4939      *    0
        0    0.9303
        0        0
V =
   -0.6450    0.7642
   -0.7642   -0.6450
```

- Compare the first column of AV with the first column of U : this should be just σ_1

```
>> AV = A*V
AV =
   -2.0541    0.8834
   -4.9917   -0.2872
   -7.8101   -0.0488
>> AV(:,1)../U(:,1)
ans =
    9.4939
    9.4939
    9.4939
>> S(1,1)
ans =
    9.4939
```

Interpreting the SVD: visualizing it

- The SVD can be visualized for 3×3 real-valued matrices
- Unit vectors in the directions of the columns of V (left) are distorted by multiplication by A (right) to



Contrasting the EVD and the SVD

Table 7.1. *Differences between the EVD and the SVD.*

EVD	SVD
most square matrices	all rectangular and square matrices
$\mathbf{A}\mathbf{x}_k = \lambda_k \mathbf{x}_k$	$\mathbf{A}\mathbf{v}_k = \sigma_k \mathbf{u}_k$
same basis for domain and range of \mathbf{A}	two orthogonal bases
may have poor conditioning	perfectly conditioned

Connection between the SVD and 2-norm (Thrm 7.3.2)

- Let $\mathbf{A} \in \mathbb{C}^{m \times n}$ have an SVD with $\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^*$ and $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq 0, r = \min\{m, n\}$

- Then:

1. The 2-norm satisfies $\|\mathbf{A}\|_2 = \sigma_1$.
2. The rank of \mathbf{A} is the number of nonzero singular values.
3. The condition number satisfies

$$\kappa(\mathbf{A}) = \|\mathbf{A}\|_2 \|\mathbf{A}^+\|_2 = \sigma_1/\sigma_r.$$

Division by zero here implies that \mathbf{A} does not have full rank.

Recall that $\mathbf{A}^+ = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$ is the pseudoinverse (chapter 3).

SVD example 7.3.2: try it

```
➤ A=vander(1:5);      % Vandermonde matrix
➤ A = A(:,1:4)         % 5 by 4 now
➤ [U,S,V] = svd(A);
➤ norm(U'*U-eye(5))  % check U is orthogonal
➤ norm(V'*V-eye(4))  % check V is orthogonal
➤ sigma = diag(S)
➤ [norm(A) sigma(1)]    % Thrm 7.3.2, no. 1
➤ [cond(A) sigma(1)/sigma(end)] % Thrm 7.3.2, no. 3
```

Connections between SVD and EVD

- Let $\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^*$
- Create the square hermitian matrix $\mathbf{B} = \mathbf{A}^*\mathbf{A}$
- Then $\mathbf{B} = (\mathbf{V}\mathbf{S}^*\mathbf{U}^*)(\mathbf{U}\mathbf{S}\mathbf{V}^*) = \mathbf{V}\mathbf{S}^*\mathbf{S}\mathbf{V}^* = \mathbf{V}(\mathbf{S}^T\mathbf{S})\mathbf{V}^{-1}$.
- $\mathbf{S}^T\mathbf{S}$ is a diagonal $n \times n$ matrix
- There are two cases:

$m \geq n$:

$$\mathbf{S}^T\mathbf{S} = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix},$$

$m < n$:

$$\mathbf{S}^T\mathbf{S} = \begin{bmatrix} \sigma_1^2 & & & \\ & \ddots & & \\ & & \sigma_m^2 & \\ & & & 0 \end{bmatrix},$$

- The lower right zero is $n - m \times n - m$

Connections between SVD and EVD

- There are two cases:

$m \geq n$:

$$S^T S = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix},$$

$m < n$:

$$S^T S = \begin{bmatrix} \sigma_1^2 & & & \\ & \ddots & & \\ & & \sigma_m^2 & \\ & & & 0 \end{bmatrix},$$

- The squares of the singular values of \mathbf{A} are the eigenvalues of $\mathbf{B} = \mathbf{A}^* \mathbf{A}$!
- Conversely, the EVD of \mathbf{B} yields the singular values of \mathbf{A} and the right singular vectors of \mathbf{A}
- We could get the left singular vectors from $\mathbf{AV} = \mathbf{US}$ by doing one column at a time

Connections between SVD and EVD

- We could also create

which is $m + n \times m + n$

for \mathbf{A} $m \times n$

$$\mathbf{C} = \begin{bmatrix} 0 & \mathbf{A}^* \\ \mathbf{A} & 0 \end{bmatrix}.$$

- If σ is a singular value of \mathbf{B} , then σ and $-\sigma$ are eigenvalues of \mathbf{C}
- The associated eigenvector immediately gives a left and right singular vector of \mathbf{A}
- This connection is implicitly exploited by software to compute the SVD

Thin form of the SVD

- Like the QR factorization, we can have both full and thin versions of the factorization
- Let $A = USV^*$ where A is $m \times n$ with $m > n$
- Then we can write

$$US = [u_1 \ \cdots \ u_n \ u_{n+1} \ \cdots \ u_m] \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & & 0 \end{bmatrix}$$

Last $m - n$ rows are zero here

$$= [u_1 \ \cdots \ u_n] \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} = \hat{U} \hat{S},$$

\hat{U} is $m \times n$, and \hat{S} is $n \times n$, but still same info about A !

- Use $\text{svd}(A, 0)$ to get this in Matlab

Section 7.4

Symmetry and definiteness

Symmetry and Definiteness

- When we had real matrices \mathbf{A} , we found in chapter 2 that there could be specializations of the factorizations there: e.g., $\mathbf{A} = \mathbf{LDL}^T$, where \mathbf{D} is diagonal and \mathbf{L} is lower triangular
- We now study the analogous hermitian case for $\mathbf{A}^* = \mathbf{A}$
- Let $\mathbf{A} = \mathbf{USV}^*$ where \mathbf{A} is $n \times n$
- Since \mathbf{S} is real and square,

$$\mathbf{A}^* = \mathbf{VS}^*\mathbf{U}^* = \mathbf{VSU}^*$$

- In this case \mathbf{A} is diagonalizable with

$$\mathbf{A} = \mathbf{VDV}^{-1} = \mathbf{VDV}^*$$

- Here \mathbf{V} is unitary, and \mathbf{D} is diagonal and real
- This means that a hermitian matrix has a complete set of orthonormal eigenvectors (a unitary diagonalization), with real eigenvalues

Symmetry and Definiteness: example

Example 7.4.1

The following matrix is not hermitian.

```
A = [0 2; -2 0]
```

```
A =  
     0      2  
    -2      0
```

so it is normal.

```
[V,D] = eig(A);  
norm( V'*V - eye(2) )  
  
ans =  
    2.2204e-16
```

The eigenvalues are pure imaginary.

```
lambda = diag(D)
```

```
lambda =  
    0.0000 + 2.0000i  
    0.0000 - 2.0000i
```

The singular values are

```
svd(A)
```

```
ans =  
    2  
    2
```

Symmetry and Definiteness

- Now we have a theorem that says that the condition number for the eigenvalues is bounded above by $\kappa(V)$
- Here V is the eigenvector matrix
- But, it is unitary (or orthogonal), which means that $\kappa = 1$!
- So, our last theorem implies that the condition number for a Hermitian or normal matrix is one, which is as good as it gets!
- In that case, that is for Hermitian or normal matrices, the eigenvalues can be changed by no more than the norm of the perturbation of the matrix!
- Can we verify this?

Symmetry and Definiteness: example

Example 7.4.2

We construct a real symmetric matrix with known eigenvalues by using the QR factorization to produce a random orthogonal set of eigenvectors.

```
n = 30;  
lambda = (1:n)';  
D = diag(lambda);  
[V,R] = qr(randn(n)); % get a random orthogonal V  
A = V*D*V';
```

The orthonormal columns of V become the associated eigenvectors for the eigenvalues in matrix D, which are the same for A. Why?

Eigenvalues are 1,2,...,30 and put on diagonal of matrix D

The qr function takes a random 30 by 30 matrix here, then returns an orthogonal matrix V and upper triangular matrix R

Symmetry and Definiteness: example

Example 7.4.2

The condition number of these eigenvalues is one. Thus bounded by the norm of the perturbation to A .

```
for k = 1:3
    E = randn(n); E = 1e-4*E/norm(E);
    mu = sort(eig(A+E));
    max_change = norm(mu-lambda, inf)
end
```

Create perturbation matrix E that is random and has norm $1e-4$

```
max_change =
2.5564e-05
max_change =
2.0501e-05
max_change =
2.3712e-05
```

Symmetry and Definiteness

- This is great, but it is not quite an SVD
- Why? The sign of the diagonal elements could be anything.
- Can we make it into an SVD? Yes!
- The trouble is with \mathbf{D} ; lets rewrite it as a product of two diagonal matrices: one is \mathbf{T} with the sign(d_{ii}) on the diagonal; the other is $|\mathbf{D}|$, which has $|d_{ii}|$ on the diagonal. We still have

$$\mathbf{D} = \mathbf{T}|\mathbf{D}|$$

- Substitute to get

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^* = \mathbf{V}\mathbf{T}|\mathbf{D}||\mathbf{V}^* = (\mathbf{V}\mathbf{T})|\mathbf{D}|(\mathbf{V}^*)$$

- This *is* an SVD, because the diagonal matrix has nonnegative entries and the left and right matrices are unitary

Symmetry and definiteness

- Recall the quadratic form $\mathbf{x}^* \mathbf{A} \mathbf{x}$ where \mathbf{A} is $n \times n$ and \mathbf{x} is $n \times 1$.
- For real entries, $\mathbf{x}^T \mathbf{A} \mathbf{x}$ is the quadratic form of interest
- For $n=2$, we can easily write it out:

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2$$

- For some matrices, this quantity remains positive for any nonzero vector \mathbf{x}
- For example, if $a_{11} > 0$, $a_{22} > 0$, and $a_{21} = -a_{12}$, then the quadratic form reduces to $a_{11}x_1^2 + a_{22}x_2^2 > 0$ for nonzero \mathbf{x}
- This matrix is called positive definite
- If the matrix were symmetric and this were true, then it is symmetric positive definite or SPD

Symmetry and definiteness

- The SPD matrix is important: it has all positive eigenvalues
- Complex case: a matrix is Hermitian positive definite (HPD) when

$$A = A^* \text{ and } x^* A x > 0$$

for any nonzero compatible x

- It turns out that we can prove some equivalent statements:

Theorem 7.4.3

If $A^* = A$, then the following statements are equivalent.

1. A is HPD.
2. The eigenvalues of A are positive numbers.
3. Any unitary EVD of A is also an SVD of A .

- This means, e.g., that all positive eigenvalues implies HPD (or SPD)

Rayleigh Quotient and eigenvalues

- In your linear algebra class, you may have discussed the Rayleigh quotient
- For $A n \times n$ and $x n \times 1$ with complex entries, the Rayleigh quotient is

$$R_A(x) = \frac{x^* A x}{x^* x}$$

- In the special case that x is an eigenvector, say v , then $A v = \lambda v$, and substitution easily gives that $R_A(v) = \lambda$
- We can conclude that: *the Rayleigh quotient maps an eigenvector into its associated eigenvalue*

Rayleigh Quotient and eigenvalues

- Consider the Hermitian case $\mathbf{A} = \mathbf{A}^*$ and the Rayleigh quotient

$$R_A(\mathbf{x}) = \frac{\mathbf{x}^* \mathbf{A} \mathbf{x}}{\mathbf{x}^* \mathbf{x}}$$

- Put in a vector that is close to an eigenvector: $\mathbf{x} = \mathbf{v} + \epsilon \mathbf{z}$ for $\epsilon \rightarrow 0$
- Using a multidimensional Taylor expansion gives that
$$R_A(\mathbf{v} + \epsilon \mathbf{z}) = R_A(\mathbf{v}) + 0 + O(\epsilon^2) = \lambda + O(\epsilon^2)$$
- This happens because, for an eigenvector, $\nabla R_A(\mathbf{v}) = \mathbf{0}$
- This means that if the input vector is within $O(\epsilon)$ of an eigenvector then the Rayleigh quotient is within $O(\epsilon^2)$ of the eigenvalue: R_A does a good job of approximating the eigenvalue! Let's explore this...

Rayleigh Quotient and eigenvalues

Example 7.4.3

We construct a symmetric matrix with a known EVD.

```
n = 20;
lambda = (1:n)'; D = diag(lambda);
[V,~] = qr(randn(n)); % get a random orthogonal V
A = V*D*V';
```

The Rayleigh quotient of an eigenvector is its eigenvalue.

```
R = @(x) (x'*A*x)/(x'*x);
format long, R(V(:,7))
```

```
ans =
7.000000000000001
```

Rayleigh Quotient and eigenvalues

Example 7.4.3

Now let's try different vectors that are closer and closer to an eigenvector

```
delta = 1./10.^ (1:4)';  
quotient = 0*delta;  
for k = 1:4  
    e = randn(n,1); e = delta(k)*e/norm(e);  
    x = V(:,7)+e;  
    quotient(k) = R(x);  
end  
table(delta,quotient)
```

Every time the input vector gets a factor of 10 closer to the eigenvector, there is a factor of 100 improvement in the eigenvalue approximation (another two zeros after the decimal point)

```
ans =  
      delta          quotient  
-----  
      0.1      7.05738940427937  
     0.01      7.00066684894918  
    0.001      7.00000278235035  
   0.0001      7.00000005557751
```

Section 7.5

Dimension reduction

Dimension Reduction

- We now return to the SVD
- We want to use it to approximate the information in a matrix with much less storage, or many less numbers
- We will see a few examples of this
- Consider matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $m > n$ (for now)
- The thin SVD is then

$$\mathbf{A} = \hat{\mathbf{U}} \hat{\mathbf{S}} \mathbf{V}^T = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n] \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix}$$

- We truncated the $n+1$ to m columns of \mathbf{U} and rows of \mathbf{S} to get $\hat{\mathbf{U}}, \hat{\mathbf{S}}$

Dimension Reduction

- Now let's rewrite the thin SVD *carefully*
- Note that $\widehat{\mathbf{U}}$ has orthonormal *columns* \mathbf{u}_i
- \mathbf{V} has orthonormal *rows* \mathbf{v}_i^T

$$\begin{aligned} \mathbf{A} &= \widehat{\mathbf{U}} \widehat{\mathbf{S}} \mathbf{V}^T = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n] \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_n & \\ & & & \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \\ &= [\sigma_1 \mathbf{u}_1 \quad \cdots \quad \sigma_n \mathbf{u}_n] \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \\ &= \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T = \sum_{i=1}^r \sigma_i \mathbf{u}_i \mathbf{v}_i^T, \end{aligned}$$

$\mathbf{u}_i \mathbf{v}_i^T$ is an outer product here, with each forming an $m \times n$ matrix

m × n here *n × n here*

Dimension Reduction

- Each of those outer products is weighed with the singular value, then adding them *all* up recovers the original matrix

$$\mathbf{A} = \sum_{i=1}^r \sigma_i \mathbf{u}_i \mathbf{v}_i^T,$$

- But, it so happens that in many matrices, there are only a few singular values that are sizable, and the rest may be quite small.
- By convention we ordered the singular values:

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$$

- So, if the largest few singular values are say $1, 2, \dots, k$, then maybe we need to keep only the first k terms to get a good approximation to the content of \mathbf{A} !
- Let's make this more precise.

Dimension Reduction

- We can think of the process of keeping more singular values as generating a sequence of matrices for increasing k , with $1 \leq k \leq r$
- Then, we only do a partial sum using the first k terms:

$$A_k = \sum_{i=1}^k \sigma_i u_i v_i^T = U_k S_k V_k^T.$$

- U_k and V_k are the first k columns of U and V
- S_k is the upper left $k \times k$ submatrix of S
- Because each $u_i v_i^T$ is a matrix with unit norm, the overall size of each matrix added is given by the singular value
- Because of this, there are many cases where stopping at a small value of k relative to r will give a v

Dimension Reduction

- What do we know about the A_k ?
- The rank of a sum of matrices is less than or equal to the sum of the ranks of each: thus, the rank of A_k is at most k
- And, it turns out that A_k is the *best rank k approximation* to A !

Theorem 7.5.1

Suppose A has rank r and let $A = USV^T$ be an SVD. Let A_k be as in (7.5.2) for $1 \leq k < r$. Then

$$A_k = \sum_{i=1}^k \sigma_i u_i v_i^T = U_k S_k V_k^T.$$

1. $\|A - A_k\|_2 = \sigma_{k+1}, \quad k = 1, \dots, r-1.$

2. If the rank of B is k or less, then $\|A - B\|_2 \geq \sigma_{k+1}.$

Dimension Reduction: example

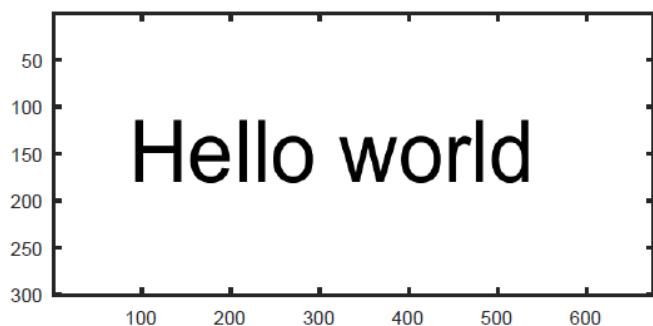
- For many matrices, k need not be very large to get a good approximation to the original matrix
- Example 7.5.1 gives an example with text
- Demo of that example and/or others...

Dimension Reduction: example

Example 7.5.1

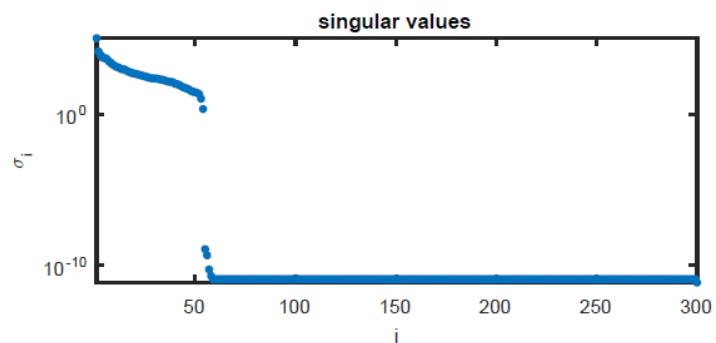
```
tobj = text(0,0,'Hello world','fontsize',44);
saveas(gcf,'hello.png')
A = imread('hello.png');
A = double(rgb2gray(A));
imagesc(A), colormap gray
[m,n] = size(A)
```

```
m =
300
n =
675
```



```
[U,S,V] = svd(A);
sigma = diag(S);
semilogy(sigma,'.')
r = find(sigma/sigma(1) > 10*eps,1,'last')
```

```
r =
56
```



Dimension Reduction: example

Example 7.5.1

```
for i = 1:4
    subplot(2,2,i)
    k = 2*i;
    Ak = U(:,1:k)*S(1:k,1:k)*V(:,1:k)';
    imshow(Ak,[0 255])
    title(sprintf('rank = %d',k))
end
```

rank = 2



rank = 4



rank = 6



rank = 8



Look how few singular values are needed to get a decent looking image! Less than 5% of the original storage to get the rank 8 approximation!

Dimension Reduction: example

- These low rank approximations can be used to get at the essence of data
- One measure of how much content is contained in each added rank is the fractional “energy” given by

$$\tau_k = \frac{\sum_{i=1}^k \sigma_i^2}{\sum_{i=1}^r \sigma_i^2}, \quad k = 1, \dots, r.$$

- Consider a different matrix of information now: Example 7.5.2
- Here we look at the voting pattern for the Senate in the 111th session of the US Congress

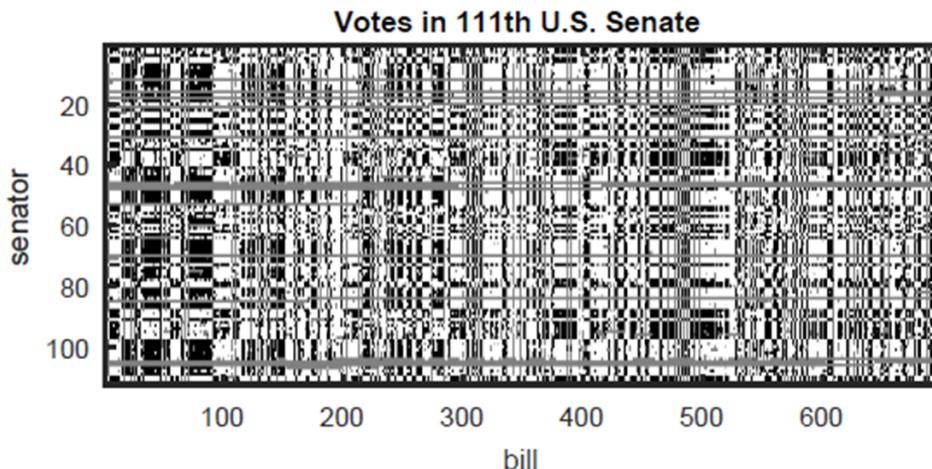
Dimension Reduction: example

Example 7.5.2

```
load voting    % get from the book's website  
[m,n] = size(A);
```

If we visualize the votes (white is “yea,” black is “nay,” and gray is anything else), we can see great similarity between many rows, reflecting party unity.

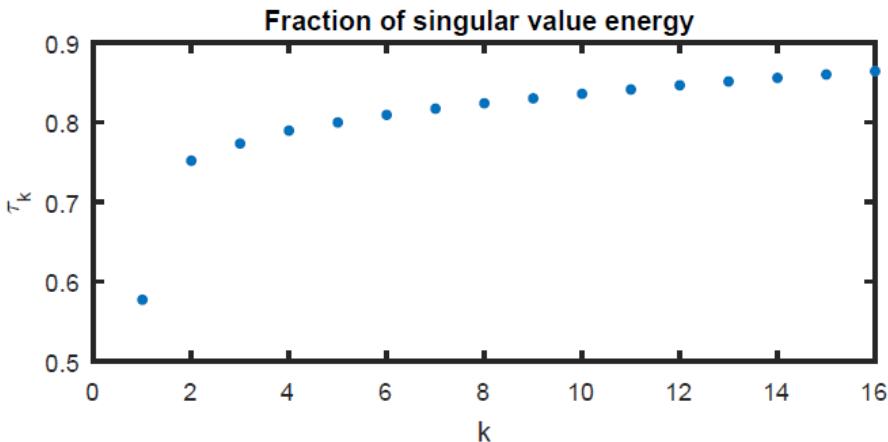
```
imagesc(A)  
colormap gray
```



Dimension Reduction: example

Example 7.5.2

```
[U,S,V] = svd(A);  
sigma = diag(S);  
tau = cumsum(sigma.^2) / sum(sigma.^2);  
plot(tau(1:16),'.')
```

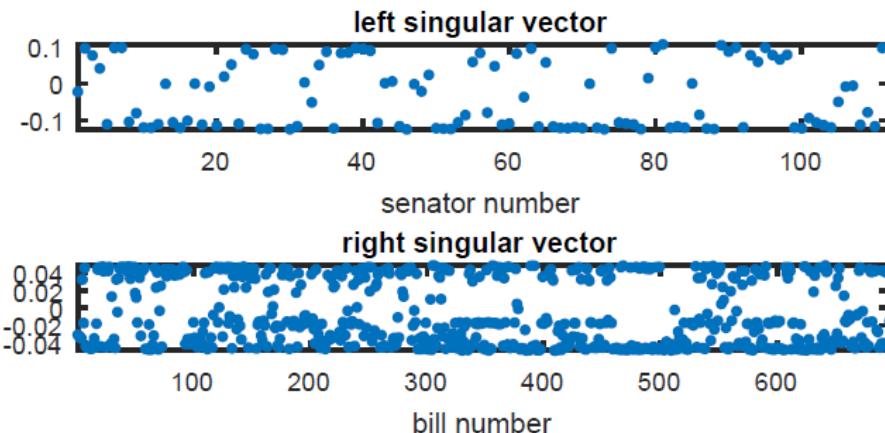


- The first two singular values account for 75% of the energy
- The remaining ones each account for relatively little of the energy
- Maybe just those first two are enough to capture the essence of the data?

Dimension Reduction: example

Example 7.5.2

```
subplot(211), plot(U(:,1),'.')
subplot(212), plot(V(:,1),'.')
```

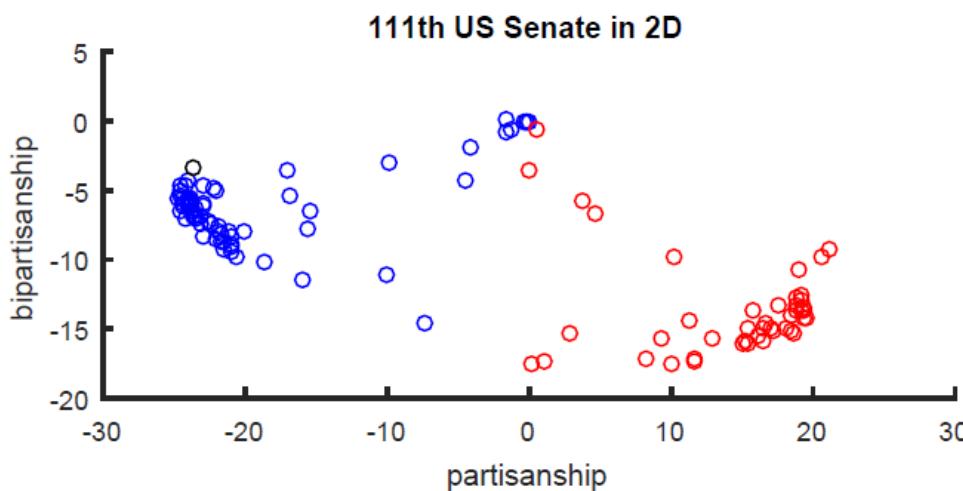


- Note the different sizes of the two vectors
- Most of the values in these vectors are at $\pm C$: there is not much in the middle

Dimension Reduction: example

Example 7.5.2

```
clf
x1 = V(:,1) '*A';    x2 = V(:,2) '*A';
scatter(x1(Dem),x2(Dem),20, 'b'), hold on
scatter(x1(Rep),x2(Rep),20, 'r')
scatter(x1(Ind),x2(Ind),20, 'k')
title('111th US Senate in 2D')
```



- Projecting each senator's votes in first two **V** coordinates, the right singular vectors: (1) is partisanship; (2) is bipartisanship
- Those coordinates are then plotted against each other
- Red: Republican
- Blue: Democrat
- Black: Independent
- The scatter plot suggests that there is a clear separation between parties!