Trefftz methods for the Helmholtz equation and best approximation estimates for plane and circular waves

Andrea Moiola

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At medium and high frequencies, the numerical simulation of the propagation and interaction of linear waves with standard finite element methods requires an excessive computational effort. This is partly due to the oscillatory nature of the solutions which can not be approximated cheaply by piecewise polynomial spaces. **Trefftz methods** offer a possible way to deal with this problem: trial and test functions are particular solutions of the underlying PDE inside each element, thus the information about the frequency is directly incorporated in the discretisation. The most common Trefftz methods for time-harmonic wave problems employ discrete spaces made of (piecewise) plane and circular/spherical wave.

For some of these schemes, for example the Trefftz-discontinuous Galerkin (TDG) method, it is possible to prove quasi-optimality for any Trefftz discrete space. To obtain convergence estimates, it is then necessary to prove **best approximation estimates** for the desired discrete spaces. We consider the approximation of solutions of the homogeneous Helmholtz equation by finite dimensional spaces of **plane, circular and spherical waves** (Fourier–Bessel functions). The Vekua transform, a bijective integral operator that maps Helmholtz solutions into harmonic functions defined on the same domain, allows to reduce this problem to the approximation of harmonic functions by harmonic polynomials. For arbitrary two-dimensional star-shaped elements, these bounds are fully explicit; the domain shape comes into play only through few simple geometric parameters. In two and three dimensions, we obtain best approximation estimates with high orders of convergence in the element size and in the dimension of the discrete space used (hp-estimates).

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