Math 242 Homework Set #13
Due: 12/5/07

Section 10.5

3. a.) $\frac{dy}{dt} = ky(1-\frac{y}{K}) \Rightarrow y(t) = \frac{K}{1+Ae^{-kt}}$ with $A = \frac{k-y(0)}{y(0)}$. With $K = 8\times 10^7$, $k = 0.71$ and $y(0) = 2\times 10^7$ we get the model $y(t) = \frac{8\times 10^7}{1+3e^{-0.71t}}$, so $y(1) = \frac{8\times 10^7}{1+3e^{-0.71}} \approx 3.23\times 10^7\,kg$.

b.) $y(t) = 4\times 10^7 \Rightarrow \frac{8\times 10^7}{1+3e^{-0.71t}} = 4\times 10^7 \Rightarrow 2 = 1+3e^{-0.71t} \Rightarrow e^{-0.71t} = \frac{1}{3} \Rightarrow$ $-0.71t = \ln(1/3) \Rightarrow t = \frac{\ln 3}{0.71} \approx 1.55\,years$.

7. a.) Our assumption is that $\frac{dy}{dt} = ky(1-y)$, where y is the fraction of the population that has heard the rumor.

b.) Using the logistic equation (1), $\frac{dP}{dt} = kP(1-\frac{P}{k})$, we substitute $y = \frac{P}{k}, P = Ky$, and $\frac{dP}{dt} = K\frac{dy}{dt}$, to obtain $K\frac{dy}{dt} = k(Ky)(1-y) \Leftrightarrow \frac{dy}{dt} = ky(1-y)$, our equation in part (a). Now the solution to (1) is $P(t) = \frac{K}{1+Ae^{-kt}}$, where $A = \frac{K-P_0}{P_0}$. We use the same substitution to obtain $Ky = \frac{K}{1+\frac{K-Ky_0}{Ky_0}e^{-kt}} \Rightarrow y = \frac{y_0}{y_0+(1-y_0)e^{-kt}}$.

c.) Let t be the number of hours since 8 AM. Then $y_0 = y(0) = \frac{80}{1000} = 0.08$ and $y(4) = \frac{1}{2}$, so $\frac{1}{2} = y(4) = \frac{0.08}{0.08+0.92e^{-4k}}$. Thus,

$0.08+0.92e^{-4k} - 0.16,\ e^{-4k} = \frac{0.08}{0.92} = \frac{2}{23},\ and\ e^{-k} = \left(\frac{2}{23}\right)^{1/4}$, so

$y = \frac{0.08}{0.08+0.92(2/23)^{t/4}} = \frac{2}{2+23(2/23)^{t/4}}$. Solving this equation for t, we get

$$2y+23y\left(\frac{2}{23}\right)^{t/4} = 2 \Rightarrow \left(\frac{2}{23}\right)^{t/4} = \frac{2-2y}{23y} \Rightarrow \left(\frac{2}{23}\right)^{t/4} = \frac{2}{23}\cdot\frac{1-y}{y} \Rightarrow \left(\frac{2}{23}\right)^{t/4-1} = \frac{1-y}{y}.$$

It follows that $\frac{t}{4} - 1 = \frac{\ln[(1-y)/y]}{\ln(\frac{2}{23})}$, so $t = 4\left[1 + \frac{\ln((1-y)/y)}{\ln(\frac{2}{23})}\right]$. When y=0.9,

$\frac{1-y}{y} = \frac{1}{9}$, so $t = 4\left(1 - \frac{\ln 9}{\ln(\frac{2}{23})}\right) \approx 7.6hrs$ or 7hrs36min. Thus, 90% of the population will have heard the rumor by 3:36 PM.

8. a.) $P(0) = P_0 = 400$, $P(1) = 1200$ and $K = 10{,}000$. From the solution to the logistic differential equation $P(t) = \frac{P_0 K}{P_0 + (K - P_0)e^{-kt}}$, we get

$P = \frac{400(10{,}000)}{400 + (9600)e^{-kt}} = \frac{10{,}000}{1 + 24e^{-kt}}$. $P(1) = 1200 \Rightarrow 1 + 24e^{-k} = \frac{100}{12} \Rightarrow e^{k} = \frac{288}{88} \Rightarrow$

$k = \ln\frac{36}{11}$. So $P = \frac{10{,}000}{1 + 24e^{-t\ln(36/11)}} = \frac{10{,}000}{1 + 24 \cdot (11/36)^t}$.

b.) $5{,}000 = \frac{10{,}000}{1 + 24(11/36)^t} \Rightarrow 24\left(\frac{11}{36}\right)^t = 1 \Rightarrow t\ln\left(\frac{11}{36}\right) = \ln\left(\frac{1}{24}\right) \Rightarrow t \approx 2.68\,years$.