Exam 3 Fall 2006

① $a_n = \cos(2/n)$

$$\lim_{n\to\infty} \cos\left(\frac{2}{n}\right) = \cos(0) = 1$$ ← yes it converges since the limit exists.

② a) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$

Alternating, b_n is positive, decreasing and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+1} = 0$$

By alt. series test it converges.

b) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}(n-1)} = \sum_{n=2}^{\infty} \frac{1}{n^{3/2}-n^{1/2}}$

Limit comparison: $b_n = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{1}{n^{3/2}-n^{1/2}}}{\frac{1}{n^{3/2}}} = \frac{n^{3/2}}{n^{3/2}-n^{1/2}} = 1 > 0$$

therefore by the limit comparison test both series converge, since $\frac{1}{n^{3/2}}$ is a p-series with $p>1$.

(3) Find the sum

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

$$= e^{-1} \quad \text{since} \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (\text{put in } -1 \text{ for } x)$$

(4) Radius and interval of convergence for $\sum_{n=0}^{\infty} \sqrt{n}\,(x-2)^n$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\sqrt{n+1}\,(x-2)^{n+1}}{\sqrt{n}\,(x-2)^n}\right| = \lim_{n\to\infty} \left|\sqrt{\frac{n+1}{n}}\,(x-2)\right|$$

$$= \lim_{n\to\infty} \left|\sqrt{1+\tfrac{1}{n}}\,(x-2)\right| = |x-2| < 1$$

$$\Rightarrow \quad -1 < x-2 < 1$$

$$\Rightarrow \quad 1 < x < 3 \qquad \Rightarrow \boxed{R = 1}$$

Check endpts:

$x = 1$: $\sum_{n=0}^{\infty} (-1)^n \sqrt{n}$ ← diverges (test for divergence)

$x = 3$: $\sum_{n=0}^{\infty} \sqrt{n}$ ← diverges (test for divergence)

$$\Rightarrow \boxed{I = (1,3)}$$

(5) $$\frac{1}{1-x} = 1+x+x^2+\dots = \sum_{n=0}^{\infty} x^n$$

$$\Rightarrow \frac{x}{1+x^3} = x \cdot \frac{1}{1+x^3} = x \cdot \frac{1}{1-(-x^3)} = x \cdot \sum_{n=0}^{\infty} (-x^3)^n$$

$$= x \cdot \sum_{n=0}^{\infty} (-1)^n x^{3n}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n x^{3n+1}}$$

(6) $f(x) = \sqrt{x}$, $a = 4$

$f(4) = 2$

$f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow f'(4) = \frac{1}{4}$

$f''(x) = -\frac{1}{4}x^{-3/2} \Rightarrow f''(4) = -\frac{1}{32}$

$f'''(x) = \frac{3}{8}x^{-5/2} \Rightarrow f'''(4) = \frac{3}{256}$

So $T_3(x)$ for $f(x) = \sqrt{x}$ is

$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$

$$\Rightarrow 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$$