Math 242 Homework Set #10
Due: 11/9/07

Section 12.3

5. The function $f(x)=\dfrac{1}{3x+1}$ is continuous, positive and decreasing on $[1,\infty)$, so the Integral Test applies. So we can look at the integral of $f(x)=\dfrac{1}{3x+1}$:

$$\int_1^\infty \frac{1}{3x+1}dx = \lim_{b\to\infty}\int_1^b \frac{dx}{3x+1} = \lim_{b\to\infty}[\frac{1}{3}\ln(3x+1)]_1^b = \lim_{b\to\infty}[\frac{1}{3}\ln(3b+1)-\frac{1}{3}\ln 4] = \infty$$

Therefore, since the improper integral diverges, so does the series $\sum_{n=1}^\infty \dfrac{1}{3x+1}$.

6. The function $f(x)=e^{-x}$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies. $\int_1^\infty e^{-x}dx = \lim_{b\to\infty}\int_1^b e^{-x}dx = \lim_{b\to\infty}[-e^{-x}]_1^b = \lim_{b\to\infty}[-e^{-b}+e^{-1}] = e^{-1}$,

therefore $\sum_{n=1}^\infty e^{-n}$ converges. Note, this is a geometric series, that converges to $\dfrac{e^{-1}}{1-e^{-1}} = \dfrac{1}{e-1}$.

12. $1+\dfrac{1}{2\sqrt{2}}+\dfrac{1}{3\sqrt{3}}+\dfrac{1}{4\sqrt{4}}+\dfrac{1}{5\sqrt{5}}+\ldots = \sum_{n=1}^\infty \dfrac{1}{n\sqrt{n}} = \sum_{n=1}^\infty \dfrac{1}{n^{3/2}}$. This is a p-series with $p=\dfrac{3}{2}>1$, so it converges by definition 1.

20. $f(x)=\dfrac{\ln x}{x^2}$ is continuous and positive for $x\ge 2$, and $f'(x)=\dfrac{1-2\ln x}{x^3}<0$ for $x\ge 2$, so f is decreasing. $\int_2^\infty \dfrac{\ln x}{x^2}dx = \lim_{t\to\infty}[-\dfrac{\ln x}{x}-\dfrac{1}{x}]_2^t = \dfrac{\ln 2+1}{2}$. Therefore $\sum_{n=1}^\infty \dfrac{\ln n}{n} = \sum_{n=2}^\infty \dfrac{\ln n}{n}$ converges by the Integral Test.

Section 12.4

3. $\dfrac{1}{n^2+n+1}<\dfrac{1}{n^2}$ for all $n\ge 1$, so $\sum_{n=1}^\infty \dfrac{1}{n^2+n+1}$ converges by comparison with $\sum_{n=1}^\infty \dfrac{1}{n^2}$, which converges because it is a p-series with p=2>1.

6. $\frac{1}{n-\sqrt{n}} > \frac{1}{n}$ for all $n \geq 2$, so $\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$ diverges by comparison with the divergent (partial) harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$.

9. $\frac{\cos^2(n)}{n^2+1} \leq \frac{1}{n^2+1} \leq \frac{1}{n^2}$, so the series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+1}$ converges by comparison with the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (p=2>1).

14. $\frac{\sqrt{n}}{n-1} > \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$, so diverges by comparison with the divergent (partial) p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ (p=1/2<1).

19. $\frac{2^n}{1+3^n} < \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$. $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ is a convergent geometric series, since 2/3<1, so $\sum_{n=1}^{\infty} \frac{2^n}{1+3^n}$ converges by the Comparison Test.

Section 12.5

2. $-\frac{1}{3} + \frac{2}{4} - \frac{3}{5} + \frac{4}{6} - \frac{5}{7} + \ldots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$. Here $a_n = (-1)^n \frac{n}{n+2}$. Since $\lim_{n\to\infty} a_n \neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

3. $\frac{4}{7} - \frac{4}{8} + \frac{4}{9} - \frac{4}{10} + \frac{4}{11} - \ldots = \sum_{n=1}^{\infty} (-1)^n \frac{4}{n+6}$. Here, $b_n = \frac{4}{n+6} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the series converges by the Alternating Series Test.

6. $b_n = \frac{1}{3n-1} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n-1}$ converges by the Alternating Series Test.

13. $\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln n}$. $\lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{1/x} = \infty$, so the series diverges by the Test for Divergence.

15. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{3/4}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$. $b_n = \frac{1}{n^{3/4}}$ is decreasing and positive and $\lim_{n\to\infty} \frac{1}{n^{3/4}} = 0$, so the series converges by the Alternating Series Test.