Near Normal Dilations of Nonnormal Matrices and Linear Operators

It was shown by Sz.-Nagy [Sz.-Nagy and Foias, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, 1970] that every contraction T on a Hilbert space $\mathcal{H}$ has a *unitary dilation* U: a unitary operator whose domain is a larger space $\mathcal{K} \supset \mathcal{H}$ and whose orthogonal projection onto $\mathcal{B}(\mathcal{H})$ is T. Moreover, U can be constructed so that all powers U^k, $k = 1, 2, \ldots$, are dilations of the corresponding powers of T. This can be used with a result of Paulsen [Every completely polynomially bounded operator is similar to a contraction, *J. Funct. Anal.*, 55 (1984), 1-17] and work of Crouzeix [Numerical range and functional calculus in Hilbert space, *J. Funct. Anal.*, 244 (2007), 668-690] to show that every square matrix A has a *near-normal* dilation M (one that is similar to a normal matrix via a similarity transformation with condition number at most 11.08 and probably at most 2) whose spectrum lies on the boundary of the numerical range of A: $W(A) = \{\langle Aq, q\rangle : \langle q, q\rangle = 1\}$. By taking M to be finite but sufficiently large, one can make powers of M up to any fixed limit K be dilations of the corresponding powers of A. We explicitly construct such near normal dilations for a number of matrices including Jordan blocks.

Put another way, any nonnormal matrix A can be thought of as an orthogonal projection of a near normal operator M with spectrum on $\partial W(A)$, and since powers of A are projections of the corresponding powers of M it follows that $\|f(A)\| \leq \|f(M)\|$ for any analytic function f. This point of view gives insight into why this upper bound may be a good estimate for some functions f but a large overestimate for others, where more of the action of $f(M)$ takes place outside the space onto which it is being projected. We illustrate this for $f(z) = e^{tz}$, where the upper bound may be a good estimate for small t but not so for larger t. We use numerically computed near normal dilations of matrices arising from nonnormal differential operators such as the advection diffusion equation.