## MATH341-011, Spring 2007 Exam 1: March 9

Write all solutions on these sheets. Please clearly erase or cross out irrelevant work; otherwise it will be part of the graded material. You must justify answers to receive full credit. You may not use calculators or notes.

1. (15 points) Find the position function x = x(t) given the acceleration  $a(t) = 6 + \cos t$ , and v(0) = 2, x(0) = -1.

$$v(t) = \int a(t) dt = 6t + \sin t + C$$

$$2 = v(0) = C$$

$$x(t) = \int v(t) dt = 3t^{2} - \cos t + 2t + B$$

$$-1 = x(0) = -1 + B \implies B = 0$$

$$x = 3t^{2} - \cos t + 2t$$

2. (15 points) For each initial-value problem, determine whether the existence and uniqueness theorem guarantees a local solution.

(a) 
$$\frac{dy}{dx} = x^{-1}y^{1/2}$$
,  $y(1) = 1$ 

(b) 
$$\frac{dy}{dx} = x^{-1}y^{1/2}$$
,  $y(1) = 0$ 

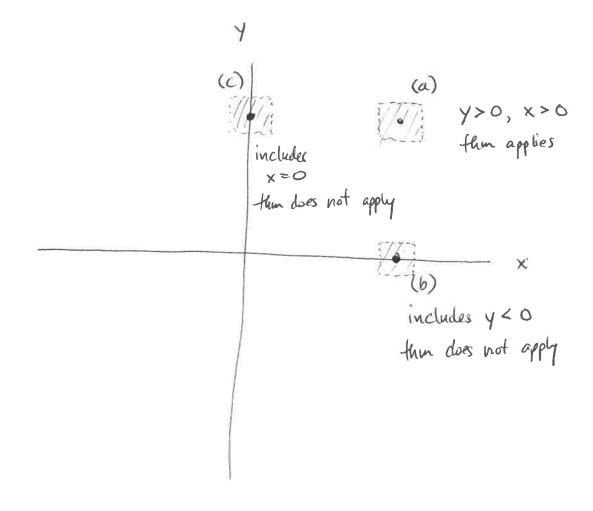
(c) 
$$\frac{dy}{dx} = x^{-1}y^{1/2}$$
,  $y(0) = 1$ 

$$f(x,\lambda) = \frac{x}{\sqrt{\lambda}}$$

fails to exist if y<0 or x=0 continuous elsewhere

$$\frac{2f}{3y} = \frac{1}{2} \frac{1}{x\sqrt{y}}$$

fails for y ≤ 0 or x=0



3. (20 points) Find the general solution of  $y \frac{dy}{dx} = x(y^2 + 1)$ .

$$\int \frac{y}{y^2 + 1} dy = \int x dx$$

$$\frac{1}{2} \log |y^2 + 1| = \frac{1}{2}x^2 + C$$

$$|y^2 + 1| = e^2 e^{x^2} = A e^{x^2} \qquad (A > 0)$$

$$|y^2 = -1 + B e^{x^2} \qquad (B \neq 0)$$

We do not need to consider B=0, since  $y^2=-1$  is not a valid solution (real-valued).

4. (15 points) Use the substitution v = y/x to change  $xy' = \sqrt{x^2 + y^2}$  into a separable, first-order equation for v. Show clearly that the equation for v is separable, but **do not try to solve the new equation.** 

$$y=VX$$
,  $y'=XV'+V$ 

$$(xv'+v) = \sqrt{x^2+v^2x^2} = x\sqrt{1+v^2}$$

$$xv' = \sqrt{1+v^2} - v$$

$$\frac{dv}{\sqrt{1+v^2'}-v} = \frac{dx}{x}$$
 Separable

5. (20 points) Solve  $xy' - 3y = x^3$ , y(1) = 10.

$$y' - \frac{3}{x}y = x^{2} \qquad \text{linear}$$
I.f.  $g(x) = \exp\left[\int -\frac{3}{x} dx\right] = e^{-3\log x} = x^{-3}$ 

$$\left[x^{-3}y\right]' = x^{-3} \cdot x^{2} = x^{-1}$$

$$x^{-3}y = \log x + C \qquad \text{(note } x > 0 \text{ initially)}$$

$$y = x^{3}(\log x + C)$$

$$10 = 1^{3}(\log 1 + C) = C$$

$$y = x^{3}(\log x + 10)$$

6. (15 points) Suppose the fish population P(t) in a lake is afflicted with a disease starting at t=0, such that the birth rate  $\beta$  is zero and the death rate  $\delta$  is inversely proportional to  $\sqrt{P}$ . Show that the fish population dies out in a finite amount of time.

$$\frac{dP}{dt} = \beta P - SP = O - \frac{k}{\sqrt{P}}P = -kP^{1/2}$$
(k>0)

$$\frac{dP}{P''z} = -kdt$$

$$P = \left(-\frac{k}{2}t + P_0^{1/2}\right)^2$$