NAME ______________________________

MATH 611, Fall 2007
Midterm Exam 2

Remember to justify your answers to receive full credit.

1. Let $A \in \mathbb{C}^{m \times m}$, $b \in \mathbb{C}^m$, and k be a positive integer. To solve the linear system $A^k x = b$, you could form the matrix $C = A^k$ and then use pivoted LU factorization on C. Describe another algorithm starting with pivoted LU factorization on A. (You **may** express your algorithm as MATLAB code, but do not have to.) Which of these two methods is faster?

$$A = P^T L U \longleftrightarrow A^{-1} = U^{-1} L^{-1} P$$

1. Factor. $\sim \frac{2}{3} m^3$ flops
2. $x = b$
3. for $j = 1, \dots, k$

   $x = Px$ no flops

   Solve $Ly = x$ for y $O(m^2)$

   Solve $Ux = y$ for x $O(m^2)$

   end

   $O(km^2)$ flops

$k-1$ matrix multiplications to get A^k $\approx (k-1)(2m^3)$ flops

$= O(km^3)$ flops

This is the slower method.

2. (a) Prove that if X is a hermitian positive definite $m \times m$ matrix such that $X^2 = I$, then $X = I$. (Hint: Use an SVD.)

(b) Prove that if A is any real SPD matrix, then there is another real SPD matrix X such that $X^2 = A$. (Hint: Use an SVD.)

(a) Write $X = U\Sigma U^*$ as SVD. Then

$$I = X \cdot X = U \Sigma U^* U \Sigma U^* = U \Sigma^2 U^*$$

this is an SVD of I, so $\Sigma^2 = I$. Since X is pos. def., $\Sigma = I$

(a) Write $X = U \Lambda U^*$, an eigenvalue decomposition of X. Then $I = X \cdot X = U \Lambda^2 U^*$, which is an SVD of I. Therefore, $\Lambda^2 = I$, or $\lambda_j^2 = 1$ for all j. But X is pos. def., so $\lambda_j = 1$ for all j and $\Lambda = I$. So $X = UU^* = I$.

(b) Let $A = R^T R$ be a Cholesky factorization. Write an SVD, $R = U \Sigma V^*$. Then $A = V \Sigma U^T U \Sigma V^T = V \Sigma^2 V^T$

$= V \Sigma V^T V \Sigma V^T = X^2$ if $X = V \Sigma V^T$. This is an eigenvalue decomposition, showing that X is positive def.

3. Suppose A is a real symmetric matrix with eigenvalues $-4, -1, 2, 12$. In each column below are eigenvalue estimates that result from running one of these four iterations on A: power iteration, inverse iteration, shifted inverse iteration, or Rayleigh quotient iteration. In each case state which iteration was used, explaining **quantitatively** why your answer is the most reasonable one.

(a)	(b)	(c)
8.310261519629201	11.993354154339482	−2.411923455369100
10.586563234632390	11.999262069508562	−3.996594223136526
11.958782543542418	11.999918025646743	−3.999996465301037
11.999999316281780	11.999990892287636	−3.999999996325349
12.000000000000000	11.999998988047848	−3.999999999996177

(a) errors are about 3.7, 1.4, 0.041, 7×10^{-8}, $< 10^{-16}$.

This is clearly superlinear, and consistent with cubic convergence : R.Q.I.

(b) errors are 6.6×10^{-3}, 7.4×10^{-4}, 8.1×10^{-5}, 9.1×10^{-6}, etc.

They decrease by a factor of about 9 at each step, which is equal to $\left|\frac{\lambda_2}{\lambda_1}\right|^2 = \left|\frac{12}{-4}\right|^2 = 3^2$. Power iter.

(c) Converge to an eigenvalue other than largest or smallest in magnitude. Error decreases by a factor of about 1000 at each step, which is linear convergence. Shifted inverse iter.

4. Let $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

(a) Use symmetric pivoting to find a symmetric tridiagonal T that is unitarily similar to A. (The standard Hessenberg reduction method is not necessary.)

(b) Describe what happens when the "pure" (unshifted) QR iteration is applied to A. Explain this convergence behavior in terms of eigenvalues.

(a) Swap rows 2 and 3, then columns 2 and 3.

$$PAP^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} P^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = T$$

(b) A is orthogonal, so $A = A \cdot I$ is the QR factorization, and $I \cdot A = A$ again. No change! (Same for T.)

Eigenvalues are $\{1, 1, -1, -1\}$ all have magnitude 1, so no progress for power iteration.