Ex. $0.9999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots = \sum_{n=1}^{\infty} \frac{9}{10^n}$

$= 9 \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n = 9 \cdot \frac{1}{10} \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^{n-1}$

$= \frac{9}{10} \left(\dfrac{1}{1-\frac{1}{10}}\right)$ geometric series

$= \frac{9}{10} \left(\frac{10}{9}\right)$

$= 1$

Ex

$$\sum_{n=1}^{\infty} \frac{e^n}{(-2)^{n+1}} = -\frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{e}{-2}\right)^n = -\frac{1}{2} \cdot \left(-\frac{e}{2}\right) \sum_{n=1}^{\infty} \left(-\frac{e}{2}\right)^{n-1}$$

geometric series with $r = -\frac{e}{2}$

$$|r| > 1 \implies \text{diverges}$$

Ex

$$\sum_{n=1}^{\infty} \left[(0.8)^{n-1} - (0.4)^n \right] = \sum_{n=1}^{\infty} (0.8)^{n-1} - \sum_{n=1}^{\infty} (0.4)^n \quad \text{if both converge}$$

$$= \frac{1}{1 - \frac{4}{5}} - \frac{2}{5} \left(\frac{1}{1 - \frac{2}{5}} \right)$$

$$= 5 - \frac{2}{5} \left(\frac{5}{3} \right) = \frac{13}{3}$$

✓

Ex

$$\sum_{n=1}^{\infty} 1 \qquad a_n = 1 \text{ for all } n$$

$$\{a_n\} \to 1 \text{ not zero} \quad \therefore \text{ series diverges}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \qquad a_n = (-1)^{n-1}$$

$$\{a_n\} \text{ does not converge} \quad \therefore \text{ series diverges}$$

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+3}\right) \qquad a_n = \ln\left(\frac{n}{2n+3}\right)$$

$$\{a_n\} \to \ln\left(\frac{1}{2}\right) \neq 0$$

$$\therefore \text{ series diverges}$$

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2+1}$

$$a_n = \frac{1}{4n^2+1} \longrightarrow 0 \qquad \text{Divergence Test} \atop \text{inconclusive}$$

$$f(x) = \frac{1}{4x^2+1}$$

$a_n = f(n)$ ✓

$f(x)$ continuous ✓

$f(x)$ decreasing ✓ $\quad f'(x) = -(4x^2+1)^{-2}(8x)$

$$< 0 \text{ for } x \geq 1$$

$f(x) > 0$ ✓

Apply Integral Test:

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{4x^2+1}\,dx = \int_2^{\infty} \frac{1}{u^2+1}\cdot\frac{1}{2}\,du = \frac{1}{2}\left[\tan^{-1}(u)\right]_2^{\infty}$$

$$u = 2x$$
$$du = 2\,dx$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(2)\right] < \infty$$

$$\text{Series} \atop \text{converges}$$