1.

(a) $\hat{Q} = Q \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} I$ (b) $\frac{\sigma_1}{\sigma_n} = 1$

2. $A^+ = \left[\begin{array}{ccc|c} 1/a_{11} & & & \\ & 1/a_{22} & & 0 \\ & & \ddots & \\ & & 1/a_{nn} & \end{array}\right]$

(a)

(b) $\|A^+A - I\|_F = \|0\|_F = 0$

$\|AA^+ - I\|_F = \left\| \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} - I \right\|_F = \sqrt{m-n}$

3. (a) $\|u\|_D = u^*Du = (D^{1/2}u)^*(D^{1/2}u) = \|D^{1/2}u\|_2$, where

$D^{1/2} = \text{diag}\left(d_{11}^{1/2}, \cdots, d_{mm}^{1/2}\right)$.

(b) $\|Ax-b\|_D = \|D^{1/2}Ax - D^{1/2}b\|_2 \Rightarrow$

$x = \left(A^*(D^{1/2})^*D^{1/2}A\right)^{-1}A^*(D^{1/2})^*D^{1/2}b$

$= (A^*DA)^{-1}A^*Db$

4. $f(x,y) = e^{x+y}$

$J(x,y) = \begin{bmatrix} e^{x+y} & e^{x+y} \end{bmatrix}$ $\qquad \|J\|_1 = \max\{e^{x+y}, e^{x+y}\} = e^{x+y}$

$$\kappa = \frac{\|J\|_1 \, \left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\|_1}{\|f(x,y)\|_1} = \frac{e^{x+y}(|x|+|y|)}{e^{x+y}} = |x| + |y|$$

5. $\tilde{f}(x) = (1+\varepsilon_2)\sqrt{x(1+\varepsilon_1)} = \sqrt{\tilde{\tilde{x}}}$, where $\tilde{\tilde{x}} = x(1+\varepsilon_1)(1+\varepsilon_2)^2$

Then $\dfrac{\|\tilde{\tilde{x}} - x\|}{\|x\|} = \left|(1+\varepsilon_1)(1+\varepsilon_2)^2 - 1\right|$

$$= \left|\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_1\varepsilon_2 + \varepsilon_2^2 + \varepsilon_1\varepsilon_2^2\right|$$

$$= O(\varepsilon_{machine})$$

Backward stable