MATH 508 Challenge problem set #2

Turn in no later than Mar. 1 for credit

1. In 1807, Joseph Fourier announced a discovery so amazing that it was met with disbelief: For any (well, almost any) function $f(\theta)$ that is periodic with period 2π (i.e., $f(\theta + 2k\pi) = f(\theta)$ for any integer k), f has the representation

$$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos n\theta + b_n \sin n\theta\right],$$

where the coefficients are given uniquely by

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta.$$

Use this fact to show that for any integer $n \geq 1$,

$$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\cos n\theta\, d\theta = \frac{\pi}{n!},$$

without doing any integrals.

2. Let m be an integer and $\omega = \omega_m = e^{2\pi i/m}$ be the principal mth root of unity. Define the $m \times m$ matrix F by

$$F = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{m-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(m-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{m-1} & \omega^{2(m-1)} & \cdots & \omega^{(m-1)(m-1)} \end{bmatrix}.$$

Define the matrix F^* by $F^* = (\overline{F})^T$ (i.e., take the complex conjugates of the entries and then transpose the matrix). Show that

$$F^* = mF^{-1},$$

where F^{-1} is the matrix inverse of F. (Note: This result is also very closely related to Fourier series! That fact isn't necessary to solve the problem, though.)