Potential theoretic approach to design an optimal formula for function approximation in a weighted Hardy space

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We propose a method to design an optimal interpolation formula on $\mathbf{R}$ for function approximation in a weighted Hardy space

$$\boldsymbol{H}^{\infty}(\mathcal{D}_d, w) := \left\{ f : \mathcal{D}_d \to \mathbf{C} \;\middle|\; f \text{ is analytic in } \mathcal{D}_d \text{ and } \|f\| := \sup_{z \in \mathcal{D}_d} \left| \frac{f(z)}{w(z)} \right| < \infty \right\},$$

where $d > 0$, $\mathcal{D}_d := \{z \in \mathbf{C} \mid |\mathrm{Im} z| < d\}$, and w is a weight function satisfying $w(z) \neq 0$ for any $z \in \mathcal{D}_d$. This is a space of functions analytic in the strip region $\mathcal{D}_d$, being characterized by the decay rate of its elements (functions) in the neighborhood of infinity. Some of elementary and special functions belong to $\boldsymbol{H}^{\infty}(\mathcal{D}_d, w)$ for some d's and w's. Function approximation in $\boldsymbol{H}^{\infty}(\mathcal{D}_d, w)$ is fundamental not only in the practical context of application to numerical quadrature, numerical solution of functional equations, etc., but also in the theoretical context of discussing optimality. We regard an approximation formula as optimal if it gives the minimum worst error in $\boldsymbol{H}^{\infty}(\mathcal{D}_d, w)$.

Various formulas for approximation in $\boldsymbol{H}^{\infty}(\mathcal{D}_d, w)$ have been considered thus far. For example, the SE-Sinc formula for $w(x) = \mathrm{O}(\exp(-\alpha|x|))$ $(|x| \to \infty)$ and the DE-Sinc formula for $w(x) = \mathrm{O}(\exp(-\beta \exp(\gamma|x|)))$ $(|x| \to \infty)$ are known to be very accurate. In each case, however, the optimal formula corresponding to each weight w is more accurate than the Sinc formula. In fact, an optimal formula in the case $w(x) = \mathrm{O}(\exp(-\alpha|x|))$ $(|x| \to \infty)$ is known according to the results of Ganelius (1976) and Jang and Haber (2001). In the other cases, however, explicit forms of optimal formulas have not been given thus far.

We adopt potential theoretic approach to obtain an optimal formula in an explicit form for $\boldsymbol{H}^{\infty}(\mathcal{D}_d, w)$ in the case of a general weight function w. Let $E_N^{\min}(\boldsymbol{H}^{\infty}(\mathcal{D}_d, w))$ be the minimum error norm in $\boldsymbol{H}^{\infty}(\mathcal{D}_d, w)$, where the minimum is taken over all the N-point approximation formulas. Then, using the fact

$$E_N^{\min}(\boldsymbol{H}^{\infty}(\mathcal{D}_d, w)) = \inf_{a_i \in \mathbf{R}} \left\{ \sup_{x \in \mathbf{R}} \left| w(x) \prod_{i=1}^{N} \tanh\left(\frac{\pi}{4d}(x - a_i) \right) \right| \right\}$$

shown by Sugihara (2003), we reduce the problem of finding an optimal formula to the minimizing problem of the energy of a Green potential with an external field. We propose a numerical method to solve this problem and find an optimal formula for $\boldsymbol{H}^{\infty}(\mathcal{D}_d, w)$. Some numerical results show the validity of the method. In particular, in the case $w(x) = \mathrm{O}(\exp(-\beta \exp(\gamma|x|)))$ $(|x| \to \infty)$, the formula designed by our method surpasses the DE-Sinc formula.