1. $A = uv^* \iff a_{ij} = u_i \bar{v}_j \;, \quad \|A\|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} |u_i|^2 |v_j|^2$

$$= \sum_{i=1}^{m} |u_i|^2 \sum_{j=1}^{n} |v_j|^2 = \left(\sum_{j=1}^{n} |v_j|^2\right)\left(\sum_{i=1}^{m} |u_i|^2\right) = \|u\|_F^2 \, \|v\|_F^2$$

2. $A^*Ax = A^*b \implies A^*(Ax-b) = 0$

Let $r = b - Ax$. $\begin{cases} r + Ax = b \\ A^* r \quad = 0 \end{cases}$

$r \in \mathbb{C}^m \qquad b \in \mathbb{C}^m$

$x \in \mathbb{C}^n \qquad 0 \in \mathbb{C}^n$

$$\begin{array}{cc} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \left[\begin{array}{c|c} I & A \\ \hline A^* & 0 \end{array}\right] \end{array}$$

3. (a) $G = [g_1\ g_2\ \cdots\ g_m]$, $\quad g_k = \begin{cases} ce_i - se_j & \text{if } k=i \\ ce_j + se_i & \text{if } k=j \\ e_k & \text{otherwise} \end{cases}$

Suppose $r \notin \{i,j\}$ and $s \notin \{i,j\}$. Then $g_r^T g_s = 0$ if $r \neq s$ and $g_r^T g_s = 1$ if $r=s$, simply from zero patterns.

Other cases:
$$g_i^T g_i = c^2 e_i^T e_i - sc e_i^T e_j - sc e_j^T e_i + s^2 e_j^T e_j = c^2+s^2 = 1$$
$$g_i^T g_j = c^2 e_i^T e_j + sc e_i^T e_i - sc e_j^T e_j - s^2 e_j^T e_i = sc - sc = 0$$
$$g_j^T g_j = \cdots = c^2+s^2 = 1$$

(b)
$$\sum_{k=1}^{n} \sum_{i=k+1}^{m} \underset{\substack{\uparrow \\ \text{get } c,s}}{6} + \underset{\substack{\uparrow \\ \text{two rows}}}{2} \sum_{j=k}^{n} \underset{\substack{\uparrow \\ cx+sy}}{3} \sim 6 \sum_{k=1}^{n} \sum_{i=k+1}^{m} (n-k+1)$$

$$= 6 \sum_{k=1}^{n} (m-k)(n-k+1) \sim 6 \sum_{k=1}^{n} (mn - mk - nk + k^2)$$

$$\sim 6\left(mn^2 - \frac{1}{2}mn^2 - \frac{1}{2}n^3 + \frac{1}{3}n^3\right) = 3mn^2 - n^3$$

4. (a) $f(x,y) = x^2 - y^2 \qquad J(x,y) = [2x\ \ -2y]$

J is rank-1, so $\|J\|_2 = 2(x^2+y^2)^{1/2}$.

$$\kappa_2(x,y) = \frac{2(x^2+y^2)^{1/2} \cdot (x^2+y^2)^{1/2}}{|x^2-y^2|} = 2\left|\frac{x^2+y^2}{x^2-y^2}\right|$$

(b) $K_2(1+10^{-6}, 1) = 2\left(\dfrac{1+2e{-6}+10^{-12}+1}{1+2\cancel{10}^{-6}+10^{-12}-1}\right) \approx 2\times 10^6$

Lose 6 digits from 16, so 10 should be accurate.

(c) $f_1(x,y) = (x^2(1+\varepsilon_1) - y^2(1+\varepsilon_2))(1+\varepsilon_3)$ $\quad (|\varepsilon_i| \le \varepsilon_{machine})$

$f_2(x,y) = ((x+y)(1+\varepsilon_1)\cdot(x-y)(1+\varepsilon_2))(1+\varepsilon_3)$

Note: $\left|\dfrac{f_2(x,y)}{f(x,y)} - 1\right| = (1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) - 1 \le 3\varepsilon_{machine} + O(\varepsilon_{machine}^2)$

Try it for f_1: rel. error $= \left|\dfrac{\varepsilon_1 x^2 - \varepsilon_2 y^2 + \varepsilon_3(x^2-y^2) + O(\varepsilon_{machine}^2)}{x^2-y^2}\right|$

$$\le |\varepsilon_1+\varepsilon_3| + \left|\frac{(\varepsilon_1-\varepsilon_2)y^2}{x^2-y^2}\right| + O(\varepsilon_{machine}^2)$$

$\hookrightarrow$ Not small if $|x^2-y^2| \ll y^2$

(and $|x^2-y^2| \ll x^2$)

Formula f_2 is more accurate.

(Both methods are backward stable. But in f_2, the B.S. perturbations to x and y are the same, which allows for smaller error than general perturbations of the same size.)