Note Title 11-07 11/7/2007

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(-2)^n}$ $\qquad a_n = \frac{n}{(-2)^n}$ $\qquad a_{n+1} = \frac{n+1}{(-2)^{n+1}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-2)^n}{n} \cdot \frac{n+1}{(-2)^{n+1}}\right| = \frac{n+1}{n} \cdot \frac{\cancel{2^n}}{2^{\cancel{n+1}}} = \frac{1}{2}\left(\frac{n+1}{n}\right)$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2} < 1$$

By Ratio Test, series is absolutely convergent, hence convergent.

**Ex** $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{n^{10}}$ $\qquad a_n = \dfrac{e^{2n}}{n^{10}} \qquad a_{n+1} = \dfrac{e^{2(n+1)}}{(n+1)^{10}}$

$(e^{2n+2} = e^{2n} \cdot e^2)$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n^{10}}{e^{2n}} \cdot \frac{e^{2(n+1)}}{(n+1)^{10}}\right| = \frac{n^{10}}{(n+1)^{10}} \cdot \frac{e^{2n+2}}{e^{2n}} = \left(\frac{n}{n+1}\right)^{10} e^2$$

$$\lim_{n\to\infty} e^2\left(\frac{n}{n+1}\right)^{10} = e^2 \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{10} \to e^2(1)^{10} = e^2 > 1$$

By Radio Test, series diverges.

(Note: $\dfrac{e^{2n}}{n^{10}} \to \dfrac{2e^{2n}}{10n^9} \to \dfrac{4e^{2n}}{90n^8} \to \cdots \longrightarrow \infty$

so diverges by Divergence Test.)

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)^3}{n!}$ $\qquad a_n = \dfrac{(n-1)^3}{n!} \qquad a_{n+1} = \dfrac{(n)^3}{(n+1)!}$

[ Note: $(n+1)! \neq n! + 1!$ , $(n+1)! \neq n! \cdot 1!$

$$(n+1)! = (n+1)\underbrace{(n)(n-1)(n-2)\cdots(1)}$$

$(n+1)! = (n+1)\,n!$ important factorial identity ]

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n!}{(n-1)^3}\cdot\frac{n^3}{(n+1)!}\right| = \frac{n!}{(n+1)!}\cdot\frac{n^3}{(n-1)^3}$$

$$= \frac{\cancel{n(n-1)\cdots(1)}}{(n+1)\cancel{(n)(n-1)\cdots(1)}}\cdot\left(\frac{n}{n-1}\right)^3 = \frac{n^3}{(n+1)(n-1)^3}$$

$$\lim_{n\to\infty} \frac{n^3}{(n+1)(n-1)^3}\cdot\frac{n^{-4}}{n^{-4}} = \lim_{n\to\infty}\frac{n^{-1}}{(1+n^{-1})(1-n^{-1})^3} = 0 < 1$$

$\boxed{\text{Converges by Ratio Test}}$

<u>Ex</u> $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$ $\quad a_n = \frac{1}{(2n)!}$ $\quad a_{n+1} = \frac{1}{(2(n+1))!}$

(Note: $(2n)! \neq 2!\,n!$)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|(2n)! \cdot \frac{1}{(2n+2)!}\right| = \frac{(2n)!}{(2n+2)!} = \frac{\cancel{(2n)}\cancel{(2n-1)}\cancel{(2n-2)}\cancel{\cdots(1)}}{(2n+2)(2n+1)\cancel{(2n)}\cancel{(2n-1)}\cancel{\cdots(1)}}$$

$$\lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0 < \underline{1}$$

$\boxed{\text{converges by Ratio Test}}$

Ex $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ $\quad a_n = \frac{n^n}{n!}$ $\quad a_{n+1} = \frac{(n+1)^{(n+1)}}{(n+1)!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n!}{n^n} \frac{(n+1)^{n+1}}{(n+1)!}\right| = \frac{(n+1)^{n+1}}{n^n} \frac{\cancel{n!}}{(n+1)\cancel{(n)\cdots(1)}}$$

$$= \frac{(n+1)^{n+1}}{n^n} \frac{1}{n+1} = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n \qquad 1^\infty \text{ indeterminate}$$

$$L = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n, \quad \ln L = \lim_{n\to\infty} n \ln\left(\frac{n+1}{n}\right) \qquad (\infty \cdot 0)$$

$$\ln L = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{\left(-\frac{1}{n^2}\right)}{\left(1+\frac{1}{n}\right)}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

$\ln L = 1$, so $L = e^1 > \underline{1}$ $\quad$ $\boxed{\text{diverges by Ratio Test}}$