Run the demo again. Explain how the values of `te`, `ye`, and `ie` are related to `tspan` and `y0`.

7.12. A classical model in mathematical ecology is the Lotka-Volterra predator-prey model. Consider a simple ecosystem consisting of rabbits that have an infinite supply of food, and foxes that prey on the rabbits for their food. This is modeled by a pair of nonlinear, first-order differential equations:

$$\frac{dr}{dt} = 2r - \alpha r f, \; r(0) = r_0,$$
$$\frac{df}{dt} = -f + \alpha r f, \; f(0) = f_0.$$

where t is time, $r(t)$ is the number of rabbits, $f(t)$ is the number of foxes, and α is a positive constant. If $\alpha = 0$, the two populations do not interact, the rabbits do what rabbits do best and the foxes die off from starvation. If $\alpha > 0$, the foxes encounter the rabbits with a probability that is proportional to the product of their numbers. Such an encounter results in a decrease in the number of rabbits and (for less obvious reasons) an increase in the number of foxes.

The solutions to this nonlinear system cannot be expressed in terms of other known functions; the equations must be solved numerically. It turns out that the solutions are always periodic, with a period that depends on the initial conditions. In other words, for any $r(0)$ and $f(0)$, there is a value $t = t_p$ when both populations return to their original values. Consequently, for all t,

$$r(t + t_p) = r(t), \; f(t + t_p) = f(t)$$

(a) Compute the solution with $r_0 = 300$, $f_0 = 150$, and $\alpha = 0.01$. You should find that t_p is close to 5. Make two plots, one of r and f as functions of t and one a phase plane plot with r as one axis and f as the other.

(b) Compute and plot the solution with $r_0 = 15$, $f_0 = 22$, and $\alpha = 0.01$. You should find that t_p is close to 6.62.

(c) Compute and plot the solution with $r_0 = 102$, $f_0 = 198$, and $\alpha = 0.01$. Determine the period t_p, either by trial and error or with an event handler.

(d) The point $(r_0, f_0) = (1/\alpha, 2/\alpha)$ is a stable equilibrium point. If the populations have these initial values, they do not change. If the initial populations are close to these values, they do not change very much. Let $u(t) = r(t) - 1/\alpha$ and $v(t) = f(t) - 2/\alpha$. The functions $u(t)$ and $v(t)$ satisfy another nonlinear system of differential equations, but if the uv terms are ignored, the system becomes linear. What is this linear system? What is the period of its periodic solutions?

7.13. Many modifications of the Lotka-Volterra predator-prey model (see previous problem) have been proposed to more accurately reflect what happens in nature. For example, the number of rabbits can be prevented from growing indefinitely by changing the first equation as follows.

$$\frac{dr}{dt} = 2\left(1 - \frac{r}{R}\right) r - \alpha r f, \; r(0) = r_0,$$