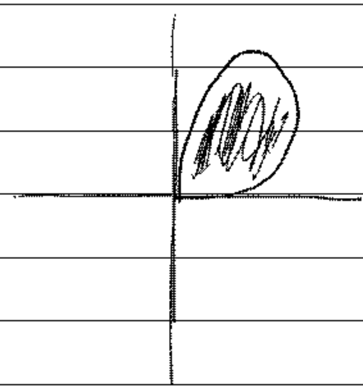
10-15

Note Title 10/15/2007

Ex Find the area inside one loop of $r = \sin 2\theta$.

Sketch :



(previous Ex, or Maple)

Curve is at origin when $r = 0$

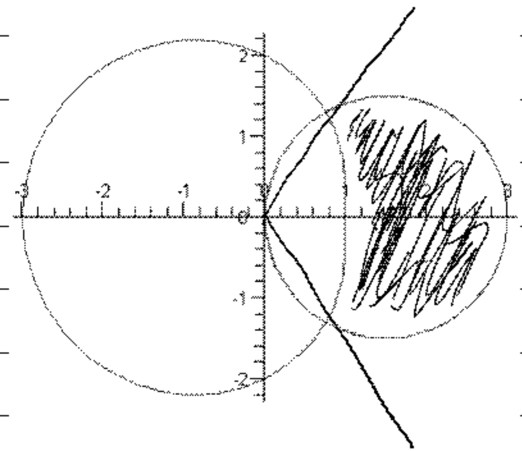
$$\Rightarrow \underbrace{\theta = 0,\ \theta = \frac{\pi}{2},}_{\text{loop in 1st quadrant}}\ \theta = \pi,\ \theta = \frac{3\pi}{2}, \ldots$$

$$A = \int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta = \frac{1}{2}\int_0^{\pi/2} (\sin 2\theta)^2 \, d\theta = \frac{1}{2}\int_0^{\pi} \frac{1}{2}(1 - \cos 4\theta)\, d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{1}{4}\left[\left(\frac{\pi}{2} - 0\right) - (0 - 0)\right] = \frac{\pi}{8}$$

Ex Find area inside $r = 3\cos\theta$ and outside $r = 2 - \cos\theta$.

1.



2. Find intersection points.
(Both equations simultaneously)

$$\left.\begin{array}{l} r = 3\cos\theta \\ r = 2 - \cos\theta \end{array}\right\} \quad 3\cos\theta = 2 - \cos\theta$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3} \text{ or } \theta = -\frac{\pi}{3}$$

3. $$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (3\cos\theta)^2 \, d\theta - \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2 - \cos\theta)^2 \, d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(8\cos^2\theta + 4\cos\theta - 4\right) d\theta$$

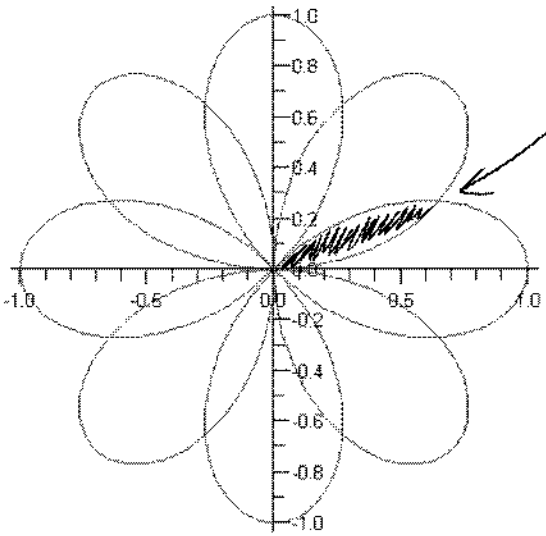
$$= 4\int_{-\pi/3}^{\pi/3} \frac{1}{2}(1 + \cos 2\theta)\, d\theta + 2\Big[\sin\theta - \theta\Big]_{-\pi/3}^{\pi/3} = 2\left[\frac{2\pi}{3}\right] + \left[\sin\frac{2\pi}{3} - \sin\left(-\frac{2\pi}{3}\right)\right]$$

$$+ 2\left[\sin\frac{\pi}{3} - \sin\left(-\frac{\pi}{3}\right)\right] - 2\left[\frac{2\pi}{3}\right]$$

$$= 3\sqrt{3}$$

$\underline{\text{Ex}}$ Find the area inside both $r = \sin 2\theta$ and $r = \cos 2\theta$.

1.

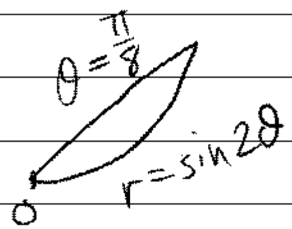


1/8th of total area
can divide in half by symmetry.

2. Find intersections

$$\left.\begin{aligned} r &= \sin 2\theta \\ r &= \cos 2\theta \end{aligned}\right\} \qquad \begin{aligned} \sin 2\theta &= \cos 2\theta \\ \tan 2\theta &= 1 \end{aligned}$$

$$2\theta = \frac{\pi}{4} \quad \text{or} \quad \theta = \frac{\pi}{8}$$



$$A = 16\left[\frac{1}{2}\int_0^{\pi/8} \sin^2 2\theta \, d\theta\right] = 4\int_0^{\pi/8} (1 - \cos 4\theta)\, d\theta$$

$$= 4\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/8} = 4\left[\left(\frac{\pi}{8} - \frac{1}{4}\right) - (0 - 0)\right] = \frac{\pi}{2} - 1$$