EXTREMAL PROBLEMS FOR POLYNOMIALS INITIATED BY NUMERICAL COMPUTATIONS

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Abstract.
Polynomial-related topics of numerical analysis give rise to challenging theoretical optimization problems. We will discuss three examples, and will provide explicit solutions:

Problem 1 (*Condition Number of Polynomials in Power Form*): Determine, in the sense of [1], the condition number γ of real polynomials P_n of degree $\leq n$ represented on $\boldsymbol{I} = [-1, 1]$ in the power form, i.e. $P_n(x) = \sum\limits_{k=0}^{n} a_k x^k$. Let T_n with $T_n(x) = \sum\limits_{k=0}^{n} t_{n,k} x^k$ denote the n-th Chebyshev polynomial of the first kind relative to $\boldsymbol{I}$, see [10].

Solution: Gautschi [1, Theorem 3.1] provides a formula for γ which requires to determine three maxima: $\gamma = (n+1)\max\{\max_k |t_{n,k}|, \max_k |t_{n-1,k}|\}$. We economize it by avoiding to compute *all* $n+1$ nonzero absolute coefficients of $T_n + T_{n-1}$, and to search among them for the largest one. For we have identified in [6] the index $k = k^* = k^*(n)$ with $|t_{n,k^*}| = \max_k |t_{n,k}|$, so that only k^* and *one* coefficient of T_n need to be computed, since $\max_k |t_{n,k}| \geq \max_k |t_{n-1,k}|$. In this way we are able to express $\gamma = \gamma(n) = (n+1)|t_{n,k^*(n)}|$ as an explicit function of n alone. It will be specified in the talk, along with its generalization to zero-symmetric intervals $[-\omega, \omega]$ with $\omega > 0$ (replacing $\boldsymbol{I}$).

Problem 2 (*Optimal Nodes for Quadratic and Cubic Lagrange Interpolation*): Determine analytically, for the quadratic ($n = 2$) and cubic ($n = 3$) case of Lagrange interpolation with nodes $x_1 < x_2 < x_3$ resp. $x_1 < x_2 < x_3 < x_4$ in $\boldsymbol{I}$, all infinitely many (zero-symmetric and zero-asymmetric) optimal node configurations which lead to the minimal Lebesgue constant $\Lambda_3^* = 1.25$ resp. Λ_4^*.

Solution: The case $n = 2$ has been settled in [11], and alternatively in [7]. The value of Λ_4^* and the optimal canonical nodes $-1 = x_1 < x_2 < x_3 < x_4 = 1$ we have explicitly determined in [3], [4]. In the talk we will reveal, based on [8], *all* optimal node configurations $-1 \leq x_1 < x_2 < x_3 < x_4 \leq 1$.

Finally, we will turn to extremal problems initiated by the evaluation of P_n: Worst case scenarios of the naïve and Horner's algorithm lead to V. A. Markov-type [2, p. 246] extremal coefficient functional problems ([5], [9]), and the search for the most stable evaluation algorithm leads to the barycentric interpolation formula [12, p. 38]. In the talk we will amplify the result given in [9] which is referenced to in [10].

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