On Operator and Matrix View to (Preconditioned) Iterative Methods

Josef Málek[a] and Zdeněk Strakoš[b]

[a]Mathematical Institute, Faculty of Mathematics and Physics, Charles University in Prague, malek@karlin.mff.cuni.cz

[b]Department of Numerical Mathematics, Faculty of Mathematics and Physics, Charles University in Prague, strakos@karlin.mff.cuni.cz

In mathematical modeling of real world phenomena, the problem of interest is often described in terms of the operator equation

$$\mathcal{A}u = f.$$

The operator $\mathcal{A}$ maps the solution space to the data space, where both are (within our setting) infinite dimensional Hilbert spaces with specific properties depending on the given problem. We will consider iterative approximation to the solution u via a sequence $u_0, u_1, u_2, \ldots$, where the iteration is stopped whenever the error of the *computed* approximation $u - u_n$ becomes acceptable. The error should be evaluated in the solution space, with including its distribution over the associated domain defined by the problem. This also naturally means, independently of the numerical process determining the computed approximations, that the rate of convergence should be linked with the (infinite dimensional) solution space.

Computation, however, can very rarely be performed in a straightforward way using the infinite dimensional operator and the objects within the solution and data spaces. *Discretization* is needed in order to reduce the original infinite dimensional problem into its tractable finite dimensional counterpart. Computation is then typically performed using algebraic matrices and vectors, which in the linear case is symbolically represented by solving a system of linear algebraic equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}\,,$$

with the algebraic approximations $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ to $\mathbf{x}$ representing the desired approximations $u_0, u_1, u_2, \ldots$. In the nonlinear case the linear algebraic system is embedded into an (algebraic) iterative framework that deals with the nonlinearity via a sequence of its linearizations. Convergence behavior, error evaluation and stopping criteria, and other important computational issues such as efficient handling sparsity in the matrix/vector representations and numerical stability analysis, are often dealt with on the algebraic level.

This contribution will address several issues in *coupling the infinite dimensional operator and algebraic matrix/vector levels.* In particular, we will focus on distribution of the computational error over the domain, various descriptions of the convergence behavior, and the interplay between discretization and acceleration of convergence in iterative solution of the resulting algebraic problems.

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References: J.Málek and Z. Strakoš, *Preconditioning of the Conjugate Gradient Method in the Context of Solving PDEs,* SIAM, Philadelphia, 2015.