Math 242 Homework Set #12
Due: 11/28/07

Section 12.9

3. The goal is to write the function in the form $\frac{1}{1-r}$, and then use Equation (1) to represent the function as a sum of a power series.

$f(x)=\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}(-x)^n=\sum_{n=0}^{\infty}(-1)^n x^n$ with $|-x|<1 \Leftrightarrow |x|<1$, so R=1 and I = (-1,1).

6. $f(x)=\frac{1}{1+9x^2}=\frac{1}{1-(-9x^2)}=\sum_{n=0}^{\infty}(-9x^2)^n=\sum_{n=0}^{\infty}(-1)^n 3^{2n}x^{2n}$. The series converges when $|-9x^2|<1$. That occurs when $|x|<\frac{1}{3}$, so I = (-1/3, 1/3).

13. a.) $f(x)=\frac{1}{(1+x)^2}=\frac{d}{dx}\left(\frac{-1}{1+x}\right)=-\frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]$ (from Exercise 3)

$=\sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1}$ [from Theorem 2(i)] $=\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$ with R=1. In the last step, note that we decreased the initial value of the summation variable n by 1, and then increased each occurrence of n in the term by 1 [also note that $(-1)^{n+2}=(-1)^n$].

b.) $f(x)=\frac{1}{(1+x)^3}=-\frac{1}{2}\frac{d}{dx}\left[\frac{1}{(1+x)^2}\right]=-\frac{1}{2}\frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n(n+1)x^n\right]$ (from part a)

$=-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1}=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ with R=1.

c.) $f(x)=\frac{x^2}{(1+x)^3}=x^2\cdot\frac{1}{(1+x)^3}=x^2\cdot\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ (from part b)

$=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}$. To write the power series with x^n rather than x^{n+2}, we will decrease each occurrence of n in the term by 2 and increase the initial value of the summation variable by 2. This gives us $\frac{1}{2}\sum_{n=2}^{\infty}(-1)^n(n)(n-1)x^n$.

14. a.) $\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}(-1)^n x^n$ (geometric series with R=1), so

$$f(x)=\ln(1+x)=\int\frac{dx}{1+x}=\int\sum_{n=0}^{\infty}(-1)^n x^n dx=C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$$
$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$$ [C=0 since f(0)=ln(1)=0], with R=1.

b.) $f(x)=x\ln(1+x)=x\left[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}\right]$ (by part (a))
$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{n+1}}{n}=\sum_{n=2}^{\infty}\frac{(-1)^n x^n}{n-1}$$ with R=1.

c.) $f(x)=\ln(x^2+1)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x^2)^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n}$ with R=1.

Section 12.10

6. $\ln(1+x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5+\ldots$
$$=x-\frac{1}{2}x^2+\frac{2}{6}x^3-\frac{6}{24}x^4+\frac{24}{120}x^5-\ldots=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\ldots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$$
$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right|=\lim_{n\to\infty}\frac{|x|}{1+1/n}=|x|<1$ for convergence, so R=1.

11. $f(x)=7+5(x-2)+\frac{2}{2!}(x-2)^2+\sum_{n=3}^{\infty}\frac{0}{n!}(x-2)^n=7+5(x-2)+(x-2)^2$
Since a_n=0 for large n, $R=\infty$.

13. Clearly, $f^{(n)}(x)=e^x$, so $f^{(n)}(3)=e^3$ and $e^x=\sum_{n=0}^{\infty}\frac{e^3}{n!}(x-3)^n$. If $a_n=\frac{e^3}{n!}(x-3)^n$,
then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{e^3(x-3)^{n+1}}{(n+1)!}\cdot\frac{n!}{e^3(x-3)^n}\right|=\lim_{n\to\infty}\frac{|x-3|}{n+1}=0<1$ for all x, so $R=\infty$.

27. $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\Rightarrow f(x)=x^2e^{-x}=x^2\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{n!}$, $\quad R=\infty$.

42. $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\Rightarrow e^x-1=\sum_{n=1}^{\infty}\frac{x^n}{n!}\Rightarrow\frac{e^x-1}{x}=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}\Rightarrow\int\frac{e^x-1}{x}dx=C+\sum_{n=1}^{\infty}\frac{x^n}{n\cdot n!}$, with
$R=\infty$.

51. As in example 8(a), we have $e^{-x^2} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots$ and we know that $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ from Equation (16). Therefore, $e^{-x^2}\cos(x) = (1 - x^2 + \frac{1}{2}x^4 - \ldots)(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \ldots)$. Writing only the terms with degree less than or equal to 4 we get

$$e^{-x^2}\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - x^2 + \frac{1}{2}x^4 + \frac{1}{2}x^4 + \ldots = 1 - \frac{3}{2}x^2 + \frac{25}{24}x^4 + \ldots$$

55. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!} = \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} = e^{-x^4}$, by (11).