MATH 611, Fall 2007
Final exam
Due December 14, 3:00 pm

You may not discuss the questions on this exam with any other person. Your answers are expected to be analytical in nature; supporting numerical evidence will not be considered.

1. Let A be the $m \times m$ matrix of the form $\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix}$.

    (a) Show that for the system $Ax = e_1$, where e_1 is the first column of the identity, GMRES takes m iterations to find the solution. (An elementary argument, without any reference to eigenvalues or polynomials, is available.)

    (b) Show that if the conjugate gradient iteration is applied to the normal equations for $Ax = b$, the method converges in one iteration, regardless of b.

2. Consider the tridiagonal matrix

$$D_m = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

Its eigenvalues can be shown (by Fourier analysis) to be

$$\lambda_j = 2 - 2\cos\left(\frac{j\pi}{m+1}\right), \qquad j = 1, \ldots, m.$$

    (a) Let $s(m)$ be the number of CG iterations needed to solve $A_m x = b$ to a fixed accuracy ϵ. Estimate the asymptotic behavior of $s(m)$ as $m \to \infty$. (E.g., $O(\log m)$, $O(m^p)$, etc.)

    (b) Repeat (a) for the linear system $(A_m + \alpha I)x = b$, for any fixed $\alpha > 0$.

3. Suppose that the $m \times m$ matrix A is in the form $A = I + B$, where B is real and skew-symmetric (that is, $B^T = -B$).

    (a) Show that for the Rayleigh quotient $r(x)$ for A, $r(x) \equiv 1$ for all x.

    (b) Show that the Hessenberg matrix H_n in the Arnoldi iteration is of the form

$$H_n = \begin{bmatrix} 1 & -\eta_1 & & & \\ \eta_1 & 1 & -\eta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \eta_{n-2} & 1 & -\eta_{n-1} \\ & & & \eta_{n-1} & 1 \end{bmatrix}.$$

4. The Lanczos iteration in Algorithm 36.1 requires division by β_n in all steps (except $n = m$). Hence if $\beta_n = 0$ for some $n < m$, we say that the iteration breaks down.

   Prove that if A has an eigenvalue of algebraic multiplicity greater than 1, then the Lanczos iteration must break down before the last step. (Hint: Use a result from the homework.)

5. Suppose the $m \times m$ matrix A is of the form

$$A = \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix},$$

   for a matrix Y of any size.

   (a) Write a *simple* expression for $p(A)$, where p is any polynomial.

   (b) Using exact arithmetic, how many steps of GMRES are required to get an exact solution of $Ax = b$, regardless of b? (Part (a) might suggest a clue.)

   (c) (Bonus) Generalize part (b) to the matrix

$$A = \begin{bmatrix} I & Y_1 & & & \\ & I & Y_2 & & \\ & & \ddots & \ddots & \\ & & & I & Y_{k-1} \\ & & & & I \end{bmatrix}.$$