Spectral Stability, Banded Matrices, and Fast Inverses

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Abstract

A first result (not involving banded matrices) is the stability of the fully discrete spectral method for the wave equation $u_t = cu_x$. The difference equation $U(t+\Delta t, x) = \sum a_k U(t, x + k\Delta x)$ is the spectral interpolation of the exact solution $u(t, x + c\Delta t)$. *Iterated interpolation is stable when* $c\Delta t \leq \Delta x$ (but unstable for larger Δt, which the CFL condition would allow for this infinite grid).

The inverse of a banded matrix A has a special form with low rank submatrices except at the main diagonal. That form comes directly from the "Nullity Theorem." The the inverse of that matrix A^{-1} is the original A—which can be found by a remarkable "local" inverse formula. This formula uses only the banded part of A^{-1} and it offers a very fast algorithm to produce A.

That fast algorithm has a potentially valuable application. Start now with a banded matrix B. (Possible B is the positive definite beginning of a covariance matrix C—but covariances outside the band are unknown or too expensive to compute). It is a poor idea to assume that those unknown covariances are zero. Much better to complete B to C by a rule of maximum entropy—for Gaussians this means maximum determinant.

As statisticians and linear algebraists discovered, the optimally completed matrix C is the inverse of a banded matrix. Best of all, the matrix actually needed in computations is that banded C^{-1} (which is not B!). And C^{-1} comes quickly and efficiently from the local inverse formula.