MATH 611, Fall 2008
First midterm exam
October 23, 2008

Please start each problem on a new page. Remember to justify your answers to receive full credit.

1. (20 points) Prove true, or give a counterexample: For any $u \in \mathbf{C}^m$ and $v \in \mathbf{C}^n$, $\|uv^*\|_F = \|u\|_F \, \|v\|_F$.

2. (20 points) Show that the solution x of the least squares problem $\min \|b - Ax\|_2$ for $A \in \mathbf{C}^{m \times n}$ and $b \in \mathbf{C}^m$ satisfies
$$\begin{bmatrix} I & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$
What are the dimensions of each symbol appearing in this system?

3. For this problem consider only real numbers. A Givens rotation $G(i,j;a,b)$ for $1 \le i < j \le m$ is an $m \times m$ matrix that equals the identity except for the four elements $g_{ii} = g_{jj} = c$, $g_{ij} = -g_{ji} = s$, where
$$c = \frac{a}{\sqrt{a^2+b^2}}, \qquad s = \frac{b}{\sqrt{a^2+b^2}}.$$
For any vector x, the vector $G(i,j;x_i,x_j)x$ is zero in the jth row.

   (a) (15 points) Show that $G(i,j;a,b)$ is orthogonal.

   (b) (15 points) This algorithm sketches how to use Givens rotations to reduce A orthogonally to upper triangular R:

   for k from 1 to n do
       for i from $k+1$ to m do
           Compute the c and s of $G(k,i;a_{kk},a_{ik})$.
           $A_{[k,i],k:n} := \begin{bmatrix} c & s \\ -s & c \end{bmatrix} A_{[k,i],k:n}$
       end do
   end do

   Find an asymptotic flop count for the algorithm.

4. (a) (10 points) Find the relative 2-norm condition number for computing $x^2 - y^2$ for real x and y.

   (b) (10 points) If $x = 1 + 10^{-6}$ and $y = 1$, about how many accurate decimal digits can you expect when computing $x^2 - y^2$ in IEEE double precision? (Answer to the nearest integer.)

   (c) (10 points) Suppose now x and y are any **floating point** numbers. Which computer algorithm is more accurate,
$$(x \otimes x) \ominus (y \otimes y) \qquad \text{or} \qquad (x \oplus y) \otimes (x \ominus y)?$$
   Justify your response mathematically.