10-22

Note Title 10/22/2007

<u>Ex</u> $a_n = \dfrac{n}{n^2+1}$, $n \geq 1$ $\left\{ \dfrac{1}{2}, \dfrac{2}{5}, \dfrac{3}{10}, \dots \right\}$ decreasing?

$a_{n+1} \leq a_n$?

$$\frac{(n+1)}{(n+1)^2+1} \leq \frac{n}{n^2+1}$$

$$(n+1)(n^2+1) \leq n\left[(n+1)^2+1\right]$$

$$n^3+n^2+n+1 \leq n^3+2n^2+2n$$

$1 \leq n^2+n$ True for all $n \geq 1$

decreasing ✓

$a_n \geq 0$ for all n bounded below

$a_n \leq 1$ since $n \leq n^2+1$ for all $n \geq 1$

bounded above

$\therefore \{a_n\}$ is monotonic, bounded

$\therefore$ convergent

Ex $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i(i+2)}$ $\qquad S_n = \displaystyle\sum_{i=1}^{n} \frac{1}{i(i+2)}$

Trick: Partial fraction decomp.

$$\frac{1}{i(i+2)} = \frac{A}{i} + \frac{B}{i+2} \qquad 1 = A(i+2) + B(i)$$

$i=-2$: $1 = -2B$ $\qquad B = -\frac{1}{2}$

$i=0$: $1 = 2A$ $\qquad A = \frac{1}{2}$

$$S_n = \sum_{i=1}^{n} \left[\frac{1}{2}\frac{1}{i} - \frac{1}{2}\frac{1}{i+2}\right]$$

First approach: Write out terms and look for pattern

$S_1 = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{3}\right)$ $\qquad S_2 = S_1 + \left(\frac{1}{2}\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{4}\right)$

$$\left(\frac{1}{2}-\frac{1}{\cancel{2}}\cancel{\frac{1}{3}}\right)+\left(\frac{1}{2}\cdot\frac{1}{2}-\frac{1}{\cancel{2}}\cdot\cancel{\frac{1}{4}}\right)+\left(\frac{1}{\cancel{2}}\cdot\cancel{\frac{1}{3}}-\frac{1}{\cancel{2}}\cdot\cancel{\frac{1}{5}}\right)+\left(\frac{1}{\cancel{2}}\cdot\cancel{\frac{1}{4}}-\frac{1}{\cancel{2}}\cdot\cancel{\frac{1}{6}}\right)$$

$$+\left(\frac{1}{\cancel{2}}\cdot\cancel{\frac{1}{5}}-\frac{1}{\cancel{2}}\cancel{\frac{1}{7}}\right)+\cdots$$

$$+\left(\frac{1}{\cancel{2}}\cancel{\frac{1}{n-1}}-\frac{1}{2}\frac{1}{n+1}\right)$$

$$+\left(\frac{1}{\cancel{2}}\cancel{\frac{1}{n}}-\frac{1}{2}\frac{1}{n+2}\right)$$

$$S_n=\frac{1}{2}+\frac{1}{4}-\frac{1}{2}\cdot\frac{1}{n+1}-\frac{1}{2}\frac{1}{n+2}$$

Second approach: Trust algebra

$$S_n=\sum_{i=1}^{n}\left[\frac{1}{2}\cdot\frac{1}{i}-\frac{1}{2}\frac{1}{i+2}\right]=\sum_{i=1}^{n}\frac{1}{2}\cdot\frac{1}{i}-\sum_{i=1}^{n}\frac{1}{2}\frac{1}{i+2}$$

$j=i+2$

$$=\frac{1}{2}\sum_{i=1}^{n}\frac{1}{i}-\frac{1}{2}\sum_{j=3}^{n+2}\frac{1}{j}$$

$$= \frac{1}{2}\sum_{i=1}^{n} \frac{1}{i} - \frac{1}{2}\sum_{j=3}^{n+2} \frac{1}{j} = \frac{1}{2}\sum_{i=1}^{n} \frac{1}{i} - \frac{1}{2}\sum_{i=3}^{n+2} \frac{1}{i}$$

$$S_n = \frac{1}{2}\left(1+\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2}\right)$$

Take limit as $n \to \infty$:

$$\sum_{i=1}^{\infty} \frac{1}{i(i+2)} = \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2}\right) = \frac{3}{4}$$

$S_n = a_1 + \cdots + a_n$ = four terms because of internal cancellation

<u>telescoping series</u>