A Very Incomplete Orbital Elements Tutorial II

In our last episode we were introduced to the 6 Keplerian orbital elements:

a - Semimajor axis	Ω - Longitude of the ascending node
e - Eccentricity	ω - Argument of the perihelion
i - Inclination	**M** - Mean anomaly at epoch

and the various values that were calculated from them:

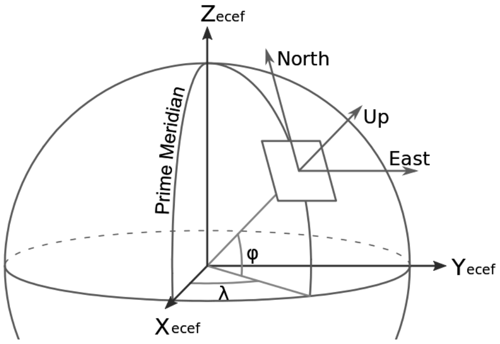
n - mean motion	**r** - orbital position vector
E - Eccentricity Anomaly	**v** - speed of the object
T - True Anomaly	

Now finally we can calculate the position (**X,Y,Z**) of the object in the three-dimensional Cartesian coordinate system, with the origin at the Sun:

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{Z} \end{pmatrix} = \mathbf{r} \cdot \begin{pmatrix} cos(\Omega)\, cos(\omega+T) - sin(\Omega)\, sin(\omega+T)\, cos(i) \\ sin(\Omega)\, cos(\omega+T) + cos(\Omega)\, sin(\omega+T)\, cos(i) \\ sin(\omega+T)\, sin(i) \end{pmatrix}$$

Remember the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the three-dimensional Cartesian coordinate system can be found by:

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



We can now finally calculate where to look of the object in the sky. To start with we have to move the coordinate system to Earth-centered, and rotate it to match the Earth's tilt.

First the tilt. The obliquity of the Earth is $\varepsilon = 23.4393°$, so we rotate the coordinates around the X-axis:

$$\begin{pmatrix} \mathbf{X}' \\ \mathbf{Y}' \\ \mathbf{Z}' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{Z} \end{pmatrix}$$

$$\mathbf{X}' = \mathbf{X}$$
$$\mathbf{Y}' = \mathbf{Y}\ \cos(\varepsilon) - \mathbf{Z}\ \sin(\varepsilon)$$
$$\mathbf{Z}' = \mathbf{Y}\ \sin(\varepsilon) + \mathbf{Z}\ \cos(\varepsilon)$$

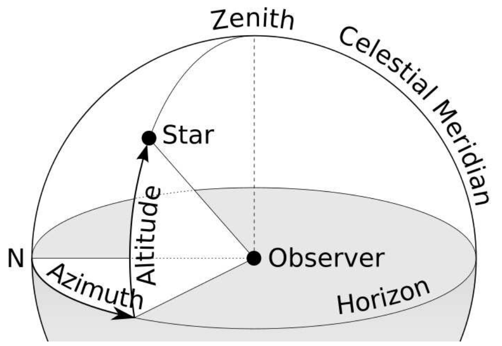
And then translate the coordinates system so the Earth is the center (Geocentric coordinates):

$$\mathbf{X}_\oplus = \mathbf{X}' - \mathbf{X}_{\mathbf{Earth}}$$
$$\mathbf{Y}_\oplus = \mathbf{Y}' - \mathbf{Y}_{\mathbf{Earth}}$$
$$\mathbf{Z}_\oplus = \mathbf{Z}' - \mathbf{Z}_{\mathbf{Earth}}$$

Note: In this coordinate system, the Sun has the coordinates (-$\mathbf{X}_{Earth}$, -$\mathbf{Y}_{Earth}$, -$\mathbf{Z}_{Earth}$)

With $\mathbf{X}_\oplus$,$\mathbf{Y}_\oplus$,$\mathbf{Z}_\oplus$ we can now calculate the Right Ascension (α) and Declination (δ).

$$\alpha = \begin{cases} \tan^{-1}(\mathbf{Y}_\oplus/\mathbf{X}_\oplus) & \text{if } \mathbf{X}_\oplus > 0 \text{ and } \mathbf{Y}_\oplus > 0 \\ \tan^{-1}(\mathbf{Y}_\oplus/\mathbf{X}_\oplus) + 180° & \text{if } \mathbf{X}_\oplus < 0 \\ \tan^{-1}(\mathbf{Y}_\oplus/\mathbf{X}_\oplus) + 360° & \text{if } \mathbf{X}_\oplus > 0 \text{ and } \mathbf{Y}_\oplus < 0 \end{cases} \qquad \delta = \tan^{-1}\left(\frac{\mathbf{Z}_\oplus}{\sqrt{\mathbf{X}_\oplus^2 + \mathbf{Y}_\oplus^2}}\right)$$



Once you have α and δ, you can calculate the altitude (Alt) and azimuth (Az) for any location on the Earth.

For Seattle latitude (ϕ) = 47.653435°, longitude = -122.311714°

$$\sin \mathrm{Alt} = \sin\phi \cdot \sin\delta + \cos\phi \cdot \cos\delta \cdot \cos H$$

$$\cos \mathrm{Az} = \frac{\cos\phi \cdot \sin\delta - \sin\phi \cdot \cos\delta \cdot \cos H}{\cos \mathrm{Alt}} \qquad \texttt{if } \frac{-\cos\delta \cdot \sin H}{\cos \mathrm{Alt}} < 0 \texttt{ then } \mathrm{Az} = 360 \text{ - } \mathrm{Az}$$

H is called the Hour Angle and is defined as: H = LST - α

Where LST is the local sidereal time. LST corresponds to the right ascension (α) of an object that is presently on the local meridian (directly overhead). LST is calculated by first finding the sidereal time at Greenwich, England (GMST) and offsetting it for your location.

GMST [hours] = 18.697374558 + 24.06570982441908 $\cdot$ D

D = (Epoch - 51544.5) days since Jan 1, 2000

You will need to (GMST `mod` 24) to get 0 < GMST < 24.

There are 15° in an hour, so GMST [deg] = GMST [hours] $\times$ 15

LST [deg] = GMST [deg] + longitude [deg]

H [deg] = LST [deg] - α [deg] Make sure 0 < H < 360