Assignment 2.4 CFD

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1 Theory

In this paper we will be using finite element method to solve the Stokes flow problem on the unit square $\Omega = [0,1] \times [0,1]$. The problem is given by the following set of equations:

$$\begin{aligned} -\nabla p + \nabla^2 \mathbf{v} &= \mathbf{f}, \quad \text{on } \Omega \\ \nabla \cdot \mathbf{v} &= 0, \quad \text{on } \Omega \\ \mathbf{v} &= \mathbf{v}_\partial \quad \text{on } \partial\Omega \end{aligned} \quad . \tag{1}$$

Here $\mathbf{v}_\partial$ is the velocity on the boundary and $\mathbf{f}$ is the forcing term.

Weak Formulation:

The weak formulation of the problem is to find $\omega \in \mathcal{W}, \mathbf{v} \in \mathcal{V}^1_{v_{1,\partial}} \otimes \mathcal{V}^2_{v_{2,\partial}}$ and $p \in \mathcal{P}$ such that the following equation holds for $\forall \alpha \in \mathcal{W}, \forall \mathbf{w} \in \mathcal{V}^1_0 \otimes \mathcal{V}^2_0$ and $\forall q \in \mathcal{P}$:

$$\begin{aligned} \int_\Omega \alpha\omega d\Omega + \int_\Omega (-\alpha_{,2}, \alpha_{,1}) \cdot \mathbf{v} d\Omega &= \int_0^1 \alpha(x,0)v_{1,\partial}(x,0) - \alpha(x,1)v_{1,\partial}(x,1)dx \\ &\quad + \int_0^1 \alpha(1,y)v_{2,\partial}(1,y) - \alpha(0,y)v_{2,\partial}(0,y)dy, \\ \int_\Omega w_1\omega_{,2} - w_2\omega_{,1} d\Omega - \int_\Omega p\nabla \cdot \mathbf{w} d\Omega &= \int_\Omega \mathbf{w} \cdot \mathbf{f} d\Omega, \\ \int_\Omega q\nabla \cdot \mathbf{v} d\Omega &= 0. \end{aligned} \tag{2}$$

Finite element formulation:

To facilitate the discretization the unit square will be subdivided into a $n \times n$ grid of evenly spaced squares where the width and height of each of the squares is $h = \frac{1}{n}$. The x coordinates of the vertices of the squares are given as $x_i = hi$ and the y coordinates are given as $y_j = hj$. Furthermore, we use the notation $x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}$ and $y_{j+\frac{1}{2}} = \frac{y_j + y_{j+1}}{2}$.

In 1D, we will use the following finite element functions. Namely, linear functions that are one on a vertex and zero on the neighboring vertices. A depiction of this can be seen in Figure 1. We also use constant functions on some domain. This function can be seen in Figure 2.

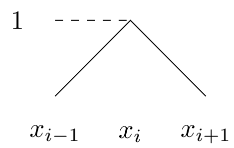

Figure 1: Example of the basis functions B^1_i

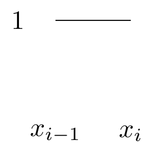

Figure 2: Example of the basis functions B^0_i

Using these functions, we define the following discrete element spaces:

$$
\begin{aligned}
\mathcal{W}_h &= \left\{ f : f(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{n} f_{ij} B_i^1(x) B_j^1(y) \right\}, \\
\mathcal{V}_h^1 &= \left\{ f : f(x,y) = \sum_{i=0}^{n} \sum_{j=1}^{n} f_{ij} B_i^1(x) B_j^0(y) \right\}, \\
\mathcal{V}_h^2 &= \left\{ f : f(x,y) = \sum_{i=1}^{n} \sum_{j=0}^{n} f_{ij} B_i^0(x) B_j^1(y) \right\}, \\
\mathcal{P}_h &= \left\{ f : f(x,y) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} B_i^0(x) B_j^0(y) \right\}.
\end{aligned}
\tag{3}
$$

Furthermore, we introduce a new element space that incorporates the boundary conditions as

$$
\begin{aligned}
\mathcal{V}_{g_1,h}^1 &= \left\{ f \in \mathcal{V}_h^1 : f(x, y_{j-\frac{1}{2}}) = g_1(x, y_{j-\frac{1}{2}}), x \in \{0,1\}, 1 \le j \le n \right\}, \\
\mathcal{V}_{g_2,h}^2 &= \left\{ f \in \mathcal{V}_h^2 : f(x_{i-\frac{1}{2}}, y) = g_2(x_{i-\frac{1}{2}}, y), y \in \{0,1\}, 1 \le i \le n \right\}.
\end{aligned}
\tag{4}
$$

The discrete formulation is then stated as: find $\omega \in \mathcal{W}_h, \mathbf{v} \in \mathcal{V}_{v_{1,\partial},h}^1 \otimes \mathcal{V}_{v_{2,\partial},h}^2$ and $p \in \mathcal{P}_h$ such that the following equation holds for $\forall \alpha \in \mathcal{W}_h, \forall \mathbf{w} \in \mathcal{V}_{0,h}^1 \otimes \mathcal{V}_{0,h}^2$ and $\forall q \in \mathcal{P}_h$:

$$
\begin{aligned}
\int_\Omega \alpha \omega d\Omega + \int_\Omega (-\alpha_{,2}, \alpha_{,1}) \cdot \mathbf{v} d\Omega &= \int_0^1 \alpha(x,0) v_{1,\partial}(x,0) - \alpha(x,1) v_{1,\partial}(x,1) dx \\
&\quad + \int_0^1 \alpha(1,y) v_{2,\partial}(1,y) - \alpha(0,y) v_{2,\partial}(0,y) dy, \\
\int_\Omega w_1 \omega_{,2} - w_2 \omega_{,1} d\Omega - \int_\Omega p \nabla \cdot \mathbf{w} d\Omega &= \int_\Omega \mathbf{w} \cdot \mathbf{f} d\Omega, \\
\int_\Omega q \nabla \cdot \mathbf{v} d\Omega &= 0.
\end{aligned}
\tag{5}
$$

2 Questions

(a) $\mathcal{W}_h$:
This space consists of functions that are 1st degree linear in both dimensions on each mesh element, and C^0 continuous (continuous but not differentiable) on all element boundaries.

$\mathcal{V}_h^1$:
The space consists of functions that are 1st degree linear in the x dimension, and consist of constant functions in the y dimension on each mesh element. The functions are C^0 continuous on the element boundaries that have a normal in the x direction, *i.e.* $f(x, y) \in \mathcal{V}_h^1$ is a continuous functions in x for a fixed y. On the element boundaries with a normal in the y direction the functions are discontinuous.

$\mathcal{V}_h^2$:
The space consists of functions that are 1st degree linear in the y dimension, and consist of constant functions in the x dimension on each mesh element. The functions are C^0 continuous on the element boundaries that have a normal in the y direction, *i.e.* $f(x, y) \in \mathcal{V}_h^2$ is a continuous functions in y for a fixed x. On the element boundaries with a normal in the x direction the functions are discontinuous.

$\mathcal{P}_h$:
These functions are constant on each mesh element with no continuity constraints.

(b) Clearly, by definition, both $\mathcal{V}_{0,h}^1 \subset \mathcal{V}_h^1$ and $\mathcal{V}_{0,h}^2 \subset \mathcal{V}_h^2$. These functions have the further constraint that they are zero on some part of the domain boundary of the domain Ω. Namely:
$\mathcal{V}_{0,h}^1$:
These functions are zero on the domain boundary with normal pointing in the x direction. This means that all coefficients corresponding to vertices lying on this boundary are necessarily zero.
$\mathcal{V}_{0,h}^2$:
These functions are zero on the domain boundary with normal pointing in the y direction. This means that all coefficients corresponding to vertices lying on this boundary are necessarily zero.

(c) We start by characterizing what an element $f \in \mathcal{V}_h^1 \otimes \mathcal{V}_h^2$ looks like. This is a vector of two functions. The first functions, $f_1 \in \mathcal{V}_h^1$ and the second, $f_2 \in \mathcal{V}_h^2$. This gives us:

$$f(x, y) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix}$$

And by definition:

$$f_1(x, y) = \sum_{i=0}^{n} \sum_{j=1}^{n} f_{ij}^1 B_i^1(x) B_j^0(y)$$

$$f_2(x, y) = \sum_{i=1}^{n} \sum_{j=0}^{n} f_{ij}^2 B_i^0(x) B_j^1(y)$$

We will first note that the derivative on a function B_l^1, turns this functions in a sum of two constant functions, so:

$$\frac{\partial}{\partial x} B_l^1(x) = cB_l^0(x) - cB_{l+1}^0(x)$$

Here l on the left side corresponds to one of the $n + 1$ mesh boundaries, and the l on the right side corresponds to one of the n mesh interiors. Lets now investigate how the derivative acts on f_1:

$$\begin{aligned}\frac{\partial f_1}{\partial x} &= \sum_{i=0}^{n}\sum_{j=1}^{n} f^1_{ij}\frac{\partial}{\partial x}B^1_i(x)B^0_j(y)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \hat{f}^1_{ij}B^0_i(x)B^0_j(y)\end{aligned}$$

Here $\hat{f}^1_{ij}$ are chosen so that:

$$f^1_{lj}\frac{\partial}{\partial x}B^1_l(x)B^0_j(y) = f^1_{lj}cB^0_l(x)B^0_j(y) - f^1_{lj}cB^0_{l+1}(x)B^0_j(y) = \hat{f}^1_{lj}B^0_l(x)B^0_j(y) - \hat{f}^1_{lj}B^0_{l+1}(x)B^0_j(y)$$

Now note that this means that $\frac{\partial f_1}{\partial x} \in \mathcal{P}_h$, and similarly, $\frac{\partial f_2}{\partial y} \in \mathcal{P}_h$. Thus we find that $\nabla\cdot\left(\mathcal{V}^1_h \otimes \mathcal{V}^2_h\right) \subset \mathcal{P}_h$. Note that we have only shown the inclusion one way.

(d) From (c), we know that $\nabla\cdot\mathbf{v} \in \mathcal{P}_h$ and is thus constant on each mesh element/subdomain. Now choose $q \in \mathcal{P}_h$ to be 1 points wise on a subdomain, and zero everywhere else. Then $q\nabla\cdot\mathbf{v}$ is the restriction of $\nabla\cdot\mathbf{v}$ to that sub domain and must be constant. Now the last equation of (5) states that this must be zero:

$$0 = \int_\Omega q\nabla\cdot\mathbf{v}d\Omega = \int_{\Omega_i} q\nabla\cdot\mathbf{v}d\Omega = \int_{\Omega_i}\nabla\cdot\mathbf{v}d\Omega = \nabla\cdot\mathbf{v}\mu\left(\Omega_i\right)$$

Here $\mu\left(\Omega_i\right)$ is the measure of the sub domain Ω_i and is strictly positive. We thus conclude that $\nabla\cdot\mathbf{v} = 0$ on the sub-domain Ω_i. Since we have chosen Ω_i arbitrary, we conclude that $\nabla\cdot\mathbf{v} = 0$ on the whole of Ω.

(e) In the previous question we have seen that the spaces for the velocity and pressure make sense. Firstly, each velocity functions $\mathbf{v} \in \mathcal{V}^1_h \otimes \mathcal{V}^2_h$ satisfies the incomprehensibility condition almost every where on the domain Ω. Secondly, as we have seen before, the gradient and divergence operators can be linked when democratizing the problem. We will even see this later in when we actually implement this problem. However, if $D \propto G^T$ as was the case for finite volume methods, the space of velocity and pressure must be linked. This is precisely what we have shown in question (c) where we have seen that $\nabla\cdot\left(\mathcal{V}^1_h \otimes \mathcal{V}^2_h\right) \subset \mathcal{P}_h$.

This means that the only space left to discuss, is $\mathcal{W}_h$ which is the space of vorticity. Firstly, it makes sense to use similar functions as the other spaces. Thus we can use B^0 or B^1 functions as basis functions where the difference is whether the derivatives are non trivial (where I mean if they are non zero functions). If we observe the second equation of the weak formulation:

$$\int_\Omega w_1\omega_{,2} - w_2\omega_{,1}d\Omega - \int_\Omega p\nabla\cdot\mathbf{w}d\Omega = \int_\Omega \mathbf{w}\cdot\mathbf{f}d\Omega. \tag{6}$$

Here $\mathbf{w} \in \mathcal{V}^1_h \otimes \mathcal{V}^2_h, \omega \in \mathcal{W}_h, p \in \mathcal{P}_h$. Now, the first term would drop out if the basis functions spanning $\mathcal{W}_h$ were B^0. This means that we would be losing the connection between the vorticity and the Stokes equation. This means that we want the derivatives of the vorticity to exist. For this reason it makes sense to use B^1 as basis functions.

(f) In the following section, we will be deriving the stencils for the discretized system. For this we introduce the notation $\Omega_{i-\frac{1}{2},j-\frac{1}{2}} = [x_{i-1}, x_i) \times [y_{j-1}, y_j)$.

- We will start of deriving the stencil for the first sub equation of (5) by taking $\alpha = B^1_i(x)B^1_j(y)$. We start with the integral:

$$
\begin{aligned}
\int_{\Omega} \alpha\omega d\Omega = &\int_{\Omega} B_i^1(x)B_j^1(y)\sum_{k=0}^{n}\sum_{l=0}^{n}\omega_{kl}B_k^1(x)B_l^1(y)d\Omega \\
=&\int_{\Omega_{i-\frac{1}{2},j-\frac{1}{2}}} B_i^1(x)B_j^1(y)\Big[\omega_{i-1j-1}B_{i-1}^1(x)B_{j-1}^1(y)+\omega_{ij-1}B_i^1(x)B_{j-1}^1(y)+ \\
&\qquad \omega_{i-1j}B_{i-1}^1(x)B_j^1(y)+\omega_{ij}B_i^1(x)B_j^1(y)\Big]d\Omega \\
&+\int_{\Omega_{i-\frac{1}{2},j+\frac{1}{2}}} B_i^1(x)B_j^1(y)\Big[\omega_{i-1j}B_{i-1}^1(x)B_j^1(y)+\omega_{ij}B_i^1(x)B_j^1(y)+ \\
&\qquad \omega_{i-1j+1}B_{i-1}^1(x)B_{j+1}^1(y)+\omega_{ij+1}B_i^1(x)B_{j+1}^1(y)\Big]d\Omega \\
&+\int_{\Omega_{i+\frac{1}{2},j+\frac{1}{2}}} B_i^1(x)B_j^1(y)\Big[\omega_{ij}B_i^1(x)B_j^1(y)+\omega_{i+1j}B_{i+1}^1(x)B_j^1(y)+ \\
&\qquad \omega_{ij+1}B_i^1(x)B_{j+1}^1(y)+\omega_{i+1j+1}B_{i+1}^1(x)B_{j+1}^1(y)\Big]d\Omega \\
&+\int_{\Omega_{i+\frac{1}{2},j-\frac{1}{2}}} B_i^1(x)B_j^1(y)\Big[\omega_{ij-1}B_i^1(x)B_{j-1}^1(y)+\omega_{i+1j-1}B_{i+1}^1(x)B_{j-1}^1(y)+ \\
&\qquad \omega_{ij}B_i^1(x)B_j^1(y)+\omega_{i+1j}B_{i+1}^1(x)B_j^1(y)\Big]d\Omega \\
=&h^2\left[\frac{1}{36}\omega_{i-1j-1}+\frac{1}{18}\omega_{ij-1}+\frac{1}{18}\omega_{i-1j}+\frac{1}{9}\omega_{ij}\right]^{\Omega_{i-\frac{1}{2},j-\frac{1}{2}}} \qquad \text{(using a change of variables to unit square)} \\
&+h^2\left[\frac{1}{18}\omega_{i-1j}+\frac{1}{9}\omega_{ij}+\frac{1}{36}\omega_{i-1j+1}+\frac{1}{18}\omega_{ij+1}\right]^{\Omega_{i-\frac{1}{2},j+\frac{1}{2}}} \\
&+h^2\left[\frac{1}{9}\omega_{ij}+\frac{1}{18}\omega_{i+1j}+\frac{1}{18}\omega_{ij+1}+\frac{1}{36}\omega_{i+1j+1}\right]^{\Omega_{i+\frac{1}{2},j+\frac{1}{2}}} \\
&+h^2\left[\frac{1}{18}\omega_{ij-1}+\frac{1}{36}\omega_{i+1j-1}+\frac{1}{9}\omega_{ij}+\frac{1}{18}\omega_{i+1j}\right]^{\Omega_{i+\frac{1}{2},j-\frac{1}{2}}},
\end{aligned}
$$

where the superscripts denote which element the sums originate from. Away from the domain boundaries the stencil is now

$$
h^2\left\{\begin{matrix} \frac{1}{36} & \frac{1}{9} & \frac{1}{36} \\ \frac{1}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{36} & \frac{1}{9} & \frac{1}{36} \end{matrix}\right\}_{\omega}^{ij}
$$

which can be interpreted as follows:

$$
b\left\{\begin{matrix} a_{-1,1} & a_{01} & a_{11} \\ a_{-1,0} & a_{00} & a_{10} \\ a_{-1,-1} & a_{0,-1} & a_{1,-1} \end{matrix}\right\}_{\omega}^{ij} = b\Big(\sum_{k=-1}^{1}\sum_{l=-1}^{1} a_{kl}\omega_{i+k,j+l}\Big). \tag{7}
$$

Near the domain boundaries the stencil can be adjusted accordingly by simply removing the sums corresponding to non-existing elements (elements outside of the boundary) from the stencil.

Next, we take a look at the second integral. For this we split $\mathbf{v}$ into its components v_1 and v_2 and start with the prior to obtain

$$\begin{aligned}
\int_\Omega (-\alpha_{,2}, \alpha_{,1}) \cdot (v_1, 0) d\Omega = &\int_\Omega -B_i^1(x)\partial_y B_j^1(y) \sum_{k=0}^{n}\sum_{l=1}^{n} v_{kl}^1 B_k^1(x) B_l^0(y) d\Omega \\
=&\frac{1}{h}\int_{\Omega_{i-\frac{1}{2}j-\frac{1}{2}}} -B_i^1(x)\left[v_{i-1j}^1 B_{i-1}^1(x)B_j^0(y) + v_{ij}^1 B_i^1(x) B_j^0(y)\right] d\Omega \\
+&\frac{1}{h}\int_{\Omega_{i-\frac{1}{2}j+\frac{1}{2}}} B_i^1(x)\left[v_{i-1j+1}^1 B_{i-1}^1(x)B_{j+1}^0(y) + v_{ij+1}^1 B_i^1(x) B_{j+1}^0(y)\right] d\Omega \\
+&\frac{1}{h}\int_{\Omega_{i+\frac{1}{2}j+\frac{1}{2}}} B_i^1(x)\left[v_{ij+1}^1 B_i^1(x)B_{j+1}^0(y) + v_{i+1j+1}^1 B_{i+1}^1(x) B_{j+1}^0(y)\right] d\Omega \\
+&\frac{1}{h}\int_{\Omega_{i+\frac{1}{2}j-\frac{1}{2}}} -B_i^1(x)\left[v_{ij}^1 B_i^1(x)B_j^0(y) + v_{i+1j}^1 B_{i+1}^1(x) B_j^0(y)\right] d\Omega.
\end{aligned}$$

By doing a change of variables such that each mesh element is mapped onto the unit square, we obtain a factor h^2 to find:

$$\begin{aligned}
=&h\left[-\frac{1}{6}v_{i-1j}^1 - \frac{1}{3}v_{ij}^1\right]^{\Omega_{i-\frac{1}{2},j-\frac{1}{2}}} \\
+&h\left[\frac{1}{6}v_{i-1j+1}^1 + \frac{1}{3}v_{ij+1}^1\right]^{\Omega_{i-\frac{1}{2},j+\frac{1}{2}}} \\
+&h\left[\frac{1}{3}v_{ij+1}^1 + \frac{1}{6}v_{i+1j+1}^1\right]^{\Omega_{i+\frac{1}{2},j+\frac{1}{2}}} \\
+&h\left[-\frac{1}{3}v_{ij}^1 - \frac{1}{6}v_{i+1j}^1\right]^{\Omega_{i+\frac{1}{2},j-\frac{1}{2}}} \\
\rightarrow &h\begin{Bmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{2}{3} & -\frac{1}{6} \\ 0 & 0 & 0 \end{Bmatrix}_{v^1}^{ij},
\end{aligned}$$

where in the end we again put the equations into stencil notation that is valid away from the boundaries. Near the boundaries we have that for a known boundary velocity coefficient, *e.g.* v_{0j}^1, we simply take

$$v_{0j}^1 = v_{1,\partial}(0, y_{j-\frac{1}{2}})$$

and move its occurrence in the integral to the right hand side (RHS) of the final linear system (9) which contains the known terms, more specifically to vector $\mathbf{t}$.

For the v_2 component we note the symmetry with the previous problem to obtain

$$\begin{aligned}
\int_\Omega (-\alpha_{,2}, \alpha_{,1}) \cdot (0, v_2) d\Omega = &\int_\Omega \partial_x B_i^1(x) B_j^1(y) \sum_{k=1}^{n} \sum_{l=0}^{n} v_{kl}^2 B_k^0(x) B_l^1(y) d\Omega \\
=&h \left[\frac{1}{6} v_{ij-1}^2 + \frac{1}{3} v_{ij}^2\right]^{\Omega_{i-\frac{1}{2},j-\frac{1}{2}}} \\
+&h \left[-\frac{1}{6} v_{i+1j-1}^2 - \frac{1}{3} v_{i+1j}^2\right]^{\Omega_{i+\frac{1}{2},j-\frac{1}{2}}} \\
+&h \left[-\frac{1}{3} v_{i+1j}^2 - \frac{1}{6} v_{i+1j+1}^2\right]^{\Omega_{i+\frac{1}{2},j+\frac{1}{2}}} \\
+&h \left[\frac{1}{3} v_{ij}^2 + \frac{1}{6} v_{ij+1}^2\right]^{\Omega_{i-\frac{1}{2},j+\frac{1}{2}}} \\
\to&h \begin{Bmatrix} 0 & \frac{1}{6} & -\frac{1}{6} \\ 0 & \frac{2}{3} & -\frac{2}{3} \\ 0 & \frac{1}{6} & -\frac{1}{6} \end{Bmatrix}_{v^2}^{ij}.
\end{aligned}$$

where the stencil is valid away from the boundaries. Known boundary coefficients, *e.g* v_{i0}^2 which becomes

$$v_{i0}^2 = v_{2,\partial}(x_{i-\frac{1}{2}}, 0),$$

are again moved to the RHS, into vector $\mathbf{t}$.

Lastly, we look into the RHS integrals. These simplify to

$$\begin{aligned}
\int_0^1 \alpha(x,0) v_{1,\partial}(x,0) - \alpha(x,1) v_{1,\partial}(x,1) dx &= \int_{x_{i-1}}^{x_{i+1}} \delta_{j0} B_i^1(x) v_{1,\partial}(x,0) - \delta_{jn} B_i^1(x) v_{1,\partial}(x,1) dx \\
\int_0^1 \alpha(1,y) v_{2,\partial}(1,y) - \alpha(0,y) v_{2,\partial}(0,y) dy &= \int_{y_{j-1}}^{y_{j+1}} \delta_{in} B_j^1(y) v_{2,\partial}(1,y) - \delta_{i0} B_j^1(y) v_{2,\partial}(0,y) dy,
\end{aligned}$$

where δ represents the kronecker delta.

- Next, we look at the second sub equation of (5). We start by taking $\mathbf{w} = (B_i^1(x) B_j^0(y), 0)$ for $1 \leq i \leq n-1$ (from the fact that $w_1 = 0$ for $x \in \{0, 1\}$ due to the Dirichlet boundary condition). The first integral becomes:

$$
\begin{aligned}
\int_\Omega w_1\omega_{,2} - w_2\omega_{,1} d\Omega = & \int_\Omega B_i^1(x)B_j^0(y)\sum_{k=0}^{n}\sum_{l=0}^{n}\omega_{kl}B_k^1(x)\partial_y B_l^1(y)d\Omega \\
=& \frac{1}{h}\int_{\Omega_{i-\frac{1}{2}j-\frac{1}{2}}} B_i^1(x)B_j^0(y)\left[\omega_{i-1j}B_{i-1}^1(x)+\omega_{ij}B_i^1(x)\right]d\Omega \\
+& \frac{1}{h}\int_{\Omega_{i+\frac{1}{2}j-\frac{1}{2}}} B_i^1(x)B_j^0(y)\left[\omega_{ij}B_i^1(x)+\omega_{i+1j}B_{i+1}^1(x)\right]d\Omega \\
-& \frac{1}{h}\int_{\Omega_{i-\frac{1}{2}j-\frac{1}{2}}} B_i^1(x)B_j^0(y)\left[\omega_{i-1j-1}B_{i-1}^1(x)+\omega_{ij-1}B_i^1(x)\right]d\Omega \\
-& \frac{1}{h}\int_{\Omega_{i+\frac{1}{2}j-\frac{1}{2}}} B_i^1(x)B_j^0(y)\left[\omega_{ij-1}B_i^1(x)+\omega_{i+1j-1}B_{i+1}^1(x)\right]d\Omega \\
=& h\left[\frac{1}{6}\omega_{i-1j}+\frac{1}{3}\omega_{ij}\right]^{\Omega_{i-\frac{1}{2},j-\frac{1}{2}}} \qquad \text{(using a change of variables to unit square)} \\
+& h\left[\frac{1}{3}\omega_{ij}+\frac{1}{6}\omega_{i+1j}\right]^{\Omega_{i+\frac{1}{2},j-\frac{1}{2}}} \\
+& h\left[-\frac{1}{6}\omega_{i-1j-1}-\frac{1}{3}\omega_{ij-1}\right]^{\Omega_{i-\frac{1}{2},j-\frac{1}{2}}} \\
+& h\left[-\frac{1}{3}\omega_{ij-1}-\frac{1}{6}\omega_{i+1j-1}\right]^{\Omega_{i+\frac{1}{2},j-\frac{1}{2}}} \\
\rightarrow & h\left\{\begin{matrix} 0 & 0 & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{2}{3} & -\frac{1}{6}\end{matrix}\right\}_\omega .
\end{aligned}
$$

Now for the other test function $\mathbf{w} = (0, B_i^0(x)B_j^1(y))$, again taking $1 \le j \le n-1$ (from the fact that $w_2 = 0$ for $y \in \{0,1\}$ due to the Dirichlet boundary condition), we can immediately find the stencil by using symmetry and an extra minus. We now obtain

$$
\begin{aligned}
\int_\Omega w_1\omega_{,2} - w_2\omega_{,1} d\Omega = & \int_\Omega -B_i^0(x)B_j^1(y)\sum_{k=0}^{n}\sum_{l=0}^{n}\omega_{kl}\partial_x B_k^1(x)B_l^1(y)d\Omega \\
=& h\left[-\frac{1}{6}\omega_{ij-1}-\frac{1}{3}\omega_{ij}\right]^{\Omega_{i-\frac{1}{2},j-\frac{1}{2}}} \\
+& h\left[-\frac{1}{3}\omega_{ij}-\frac{1}{6}\omega_{ij+1}\right]^{\Omega_{i-\frac{1}{2},j+\frac{1}{2}}} \\
+& h\left[\frac{1}{6}\omega_{i-1j-1}+\frac{1}{3}\omega_{i-1j}\right]^{\Omega_{i-\frac{1}{2},j-\frac{1}{2}}} \\
+& h\left[\frac{1}{3}\omega_{i-1j}+\frac{1}{6}\omega_{i-1j+1}\right]^{\Omega_{i-\frac{1}{2},j+\frac{1}{2}}} \\
\rightarrow & h\left\{\begin{matrix} \frac{1}{6} & -\frac{1}{6} & 0 \\ \frac{2}{3} & -\frac{2}{3} & 0 \\ \frac{1}{6} & -\frac{1}{6} & 0\end{matrix}\right\}_\omega^{ij} .
\end{aligned}
$$

Next, we discretize the second integral dependent on the pressure. For this we again start with $\mathbf{w} =$

$(B_i^1(x)B_j^0(y), 0)$ for $1 \leq i \leq n-1$. This means that the second integral is:

$$
\begin{aligned}
-\int_\Omega p\nabla\cdot\mathbf{w}d\Omega &= -\int_\Omega \left(\sum_{k=1}^{n}\sum_{l=1}^{n} p_{kl}B_k^0(x)B_l^0(y)\right)\partial_x B_i^1(x)B_j^0(y)d\Omega \\
&= h\left[-p_{ij}\right]^{\Omega_{i-\frac{1}{2},j-\frac{1}{2}}} \\
&+ h\left[p_{i+1j}\right]^{\Omega_{i+\frac{1}{2},j-\frac{1}{2}}} \\
&\to h\begin{Bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{Bmatrix}_p^{ij}
\end{aligned}
$$

Here we skipped some in between steps as the constant functions act as delta functions and can be easily calculated. Now we can also, due to symmetry, write down the second integral, for $\mathbf{w} = (0, B_i^0(x)B_j^1(y))$ with $1 \leq j \leq n-1$, as a stencil:

$$
\begin{aligned}
-\int_\Omega p\nabla\cdot\mathbf{w}d\Omega &= -\int_\Omega \left(\sum_{k=1}^{n}\sum_{l=1}^{n} p_{kl}B_k^0(x)B_l^0(y)\right)B_i^0(x)\partial_y B_j^1(y)d\Omega \\
&= h\left[-p_{ij}\right]^{\Omega_{i-\frac{1}{2},j-\frac{1}{2}}} \\
&+ h\left[p_{ij+1}\right]^{\Omega_{i-\frac{1}{2},j+\frac{1}{2}}} \\
&\to h\begin{Bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{Bmatrix}_p^{ij}.
\end{aligned}
$$

Lastly, we discretize the source function integral. This is simply

$$
\begin{aligned}
\text{For } \mathbf{w} = (B_i^1(x)B_j^0(y), 0),\ 1 \leq i \leq n-1: &\int_{x_{i-1}}^{x_{i+1}}\int_{y_{j-1}}^{y_j} B_i^1(x)B_j^0(y)f^1(x,y)dxdy \\
\text{For } \mathbf{w} = (0, B_i^0(x)B_j^1(y)),\ 1 \leq j \leq n-1: &\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_{j+1}} B_i^0(x)B_j^1(y)f^2(x,y)dxdy.
\end{aligned}
$$

- The last sub equation of (5) only has one integral. This integral is similarly discretized as before. We take $q = B_i^0(x)B_j^0(y)$ and split the integral into two integrals depending on either v_1 or v_2 to find:

$$
\begin{aligned}
\int_\Omega q\nabla\cdot(v_1, 0)d\Omega &= \int_\Omega B_i^0(x)B_j^0(y)\sum_{k=0}^{n}\sum_{l=1}^{n} v_{kl}^1\partial_x B_i^1(x)B_j^0(y)d\Omega \\
&= h\left[-v_{i-1j}^1 + v_{ij}^1\right]^{\Omega_{i-\frac{1}{2},j-\frac{1}{2}}} \\
&\to h\begin{Bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{Bmatrix}_{v^1}^{ij},
\end{aligned}
$$

where the stencil is only valid away from the boundaries, as we move the known velocity coefficients to the RHS vector $\mathbf{g}_h$. By symmetry we find that the second integral gives us the following stencil:

$$
\begin{aligned}
\int_\Omega q\nabla\cdot(0, v_2)d\Omega &= \int_\Omega B_i^0(x)B_j^0(y)\sum_{k=1}^{n}\sum_{l=0}^{n} v_{kl}^2 B_i^0(x)\partial_y B_j^0(y)d\Omega \\
&= h\left[-v_{ij-1}^2 + v_{ij}^2\right]^{\Omega_{i-\frac{1}{2},j-\frac{1}{2}}} \\
&\to h\begin{Bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{Bmatrix}_{v^2}^{ij},
\end{aligned}
$$

where the stencil is only valid away from the boundaries, as we move the known velocity coefficients to the RHS vector $\mathbf{g}_h$. The RHS of this part of (5) is always zero so we have now discretized the entire weak form.

(g) Now that we have all the stencils in place we can transform the weak form in (5) into the set of linear equations

$$\begin{aligned} \mathbf{M}[\boldsymbol{\omega}] + \mathbf{C}[\mathbf{v}] &= \mathbf{t} \\ \mathbf{A}[\boldsymbol{\omega}] + \mathbf{G}[\mathbf{p}] &= \mathbf{f}_h \\ \mathbf{D}[\mathbf{v}] &= \mathbf{g}_h, \end{aligned} \tag{8}$$

where the vectors $[\omega]$, $[\mathbf{v}]$, and $[p]$ contain all the discrete vorticity, velocity, and pressure unknowns, respectively. The derived stencils are incorporated in the matrices $\mathbf{M}$, $\mathbf{C}$, $\mathbf{A}$, $\mathbf{G}$, $\mathbf{D}$. The integrals corresponding to known velocity coefficients on $\partial\Omega$, due to the dirichlet boundary condition, are moved to the RHS, *i.e.* to vector $\mathbf{t}$ and $\mathbf{g}_h$. This means that we have in total $(n+1)^2$ vorticity unknowns, $2n(n-1)$ velocity unknowns ($n(n-1)$ for each velocity component), and n^2 pressure unknowns. The complete linear system now boils down to

$$\begin{bmatrix} \mathbf{M} & \mathbf{C} & \\ \mathbf{A} & & \mathbf{G} \\ & \mathbf{D} & \\ & & \mathbf{1}^T \end{bmatrix}^T \begin{bmatrix} \mathbf{M} & \mathbf{C} & \\ \mathbf{A} & & \mathbf{G} \\ & \mathbf{D} & \\ & & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} [\omega] \\ [\mathbf{v}] \\ [p] \end{bmatrix} = \begin{bmatrix} \mathbf{M} & \mathbf{C} & \\ \mathbf{A} & & \mathbf{G} \\ & \mathbf{D} & \\ & & \mathbf{1}^T \end{bmatrix}^T \begin{bmatrix} \mathbf{t} \\ \mathbf{f}_h \\ \mathbf{g}_h \\ 0 \end{bmatrix}, \tag{9}$$

where an additional row is added to the matrix and RHS vector to ensure that $\sum_{i,j=1}^{n} p_{ij} = 0$, as the pressure is only defined up to a constant. Subsequently, least-squares is applied to solve the system.

Additionally, we can easily see that we must have that $\mathbf{A} = -\mathbf{C}^T$ (I think there was a mistake in the assignment description that said: $\mathbf{A} = +\mathbf{C}^T$). For this we first note that $\mathbf{C}$ corresponds to

$$\int_\Omega (-\alpha_{,2}, \alpha_{,1}) \cdot \mathbf{v} d\Omega = \int_\Omega \alpha_{,1} v_2 - \alpha_{,2} v_1 d\Omega$$

with $\alpha \in \mathcal{W}_h$ and $\mathbf{v} \in \mathcal{V}^1_{v_{1,\partial},h} \otimes \mathcal{V}^1_{v_{2,\partial},h}$and $\mathbf{A}$ corresponds to

$$\int_\Omega \omega_{,2} w_1 - \omega_{,1} w_2 d\Omega$$

with $\mathbf{w} \in \mathcal{V}^1_{0,h} \otimes \mathcal{V}^1_{0,h}$ and $\omega \in \mathcal{W}_h$. Besides the swapping of test and trial function spaces these integrals are of exactly the same form, besides the '-' sign and excluding the boundary conditions on $\mathbf{v}$ and $\mathbf{w}$. However, these boundary conditions do not affect the matrices $\mathbf{A}$ and $\mathbf{C}$. This, because in the case of $\mathbf{C}$, α, and $\mathbf{v}$ the integrals containing trial basis functions centred on the boundary, are moved to the RHS into $\mathbf{t}$ and weighted by the corresponding value of $\mathbf{v}_\partial$. Additionally, in the case of $\mathbf{A}$, $\mathbf{w}$, and ω the integrals containing test basis functions centred on the boundary are always weighted by 0. The swapping of test and trial function spaces gives rise to the transpose and the '-' sign is self-explanatory. This means $\mathbf{A} = -\mathbf{C}^T$.

We can follow the same reasoning to find $\mathbf{G} = -\mathbf{D}^T$, namely we have that $\mathbf{D}$ corresponds to

$$\int_\Omega q \nabla \cdot v d\Omega$$

with $q \in \mathcal{P}_h$ and $\mathbf{v} \in \mathcal{V}^1_{v_{1,\partial},h} \otimes \mathcal{V}^1_{v_{2,\partial},h}$ and $\mathbf{G}$ corresponds to

$$-\int_\Omega p \nabla \cdot w d\Omega$$

with $\mathbf{w} \in \mathcal{V}^1_{0,h} \otimes \mathcal{V}^1_{0,h}$ and $p \in \mathcal{P}_h$. We again observe that the integrals are of exactly the same form besides the swapping of the test and trial function spaces, the '-' sign, and the boundary conditions. However, following the same reasoning as before, we again find that the boundary conditions have no effect on the matrices. Once again we have that the swapping of test and trial function spaces gives rise to the transpose and the '-' sign is again self-explanatory. This means $\mathbf{G} = -\mathbf{D}^T$.

The described numerical solver is implemented in the **methods.py** script. Running the methods in the script is done using the **Final_project.ipynb** which also contains the visualization code. For visualization purposes each of the quantities is evaluated on the element centers such that all heat maps have dimension $n \times n$. The reported values are thus

$$\xi(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}}) \text{ for } i = 1, \dots n \text{ and } j = 1, \dots, n, \tag{10}$$

where $\xi = \{\omega, v_1, v_2, p, \nabla \cdot \mathbf{v}\}$. Simulation results for lid-driven cavity flow on $\Omega = [0,1] \times [0,1]$, *i.e*

$$\mathbf{v}_\partial(\mathbf{x}) = \begin{cases} (1,0), & \mathbf{x} = (x,1) \\ (0,0), & \text{otherwise} \end{cases} \tag{11}$$

and

$$\mathbf{f} = \mathbf{0}, \tag{12}$$

taking $n = 8, 16, 32, 64$ are shown in figures 3-6.

(h) $\nabla \cdot \mathbf{v}$ can be easily determined using the definition of the basis functions. On element $\Omega_{i-\frac{1}{2},j-\frac{1}{2}}$ we have that

$$v_{1,1} = \sum_{k=i-1}^{i} v^1_{kj} \partial_x B^1_k(x) B^0_j(y) = \frac{1}{h}[v^1_{ij} - v^1_{i-1j}]$$
$$v_{2,2} = \sum_{l=j-1}^{j} v^2_{il} B^0_i(x) \partial_y B^1_l(y) = \frac{1}{h}[v^2_{ij} - v^2_{ij-1}].$$

This means we have

$$\nabla \cdot \mathbf{v}(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}}) = v_{1,1} + v_{2,2}.$$

Looking at the results for $\nabla \cdot \mathbf{v}$ (figures 3-6) for $n = 8, 16, 32, 64$ we find that the $\nabla \cdot \mathbf{v} = 0$ condition seems sufficiently satisfied away from the boundary for all n. However, moving towards the boundary we observe increased artifacting for larger n.

(i) Let us now focus on obtaining a finite element solution for the stream $\psi(x,y)$ which is defined as

$$\mathbf{v} = (\psi_{,2}, -\psi_{,1}). \tag{13}$$

From assignment 1.1 we know that the stream function can be found by solving the following boundary value problem:

$$\begin{cases} -\nabla^2 \psi & = v_{2,1} - v_{1,2} = \omega \\ \nabla \psi \cdot \mathbf{n} & = (-v_2, v_1) \cdot \mathbf{n} \end{cases} \tag{14}$$

with Neumann boundary condition. In order to find the finite element solution of ψ, using our obtained ω, we must first find the weak form of the boundary value problem. We do this by multiplying both sides of the PDE by test function λ and applying integration by parts:

$$\int_\Omega -\lambda \nabla^2 \psi d\Omega = \int_\Omega \lambda \omega d\Omega \xrightarrow{\text{Integration by parts}}$$
$$\int_\Omega \nabla \lambda \cdot \nabla \psi d\Omega = \int_\Omega \lambda \omega d\Omega + \int_{\partial\Omega} \lambda \nabla \psi \cdot \mathbf{n} ds \xrightarrow{\text{Fill in Neumann boundary condition}}$$
$$\int_\Omega \nabla \lambda \cdot \nabla \psi d\Omega = \int_\Omega \lambda \omega d\Omega + \int_0^1 \lambda(x,1) v_{1,\partial}(x,1) - \lambda(x,0) v_{1,\partial}(x,0) dx$$
$$+ \int_0^1 \lambda(0,y) v_{2,\partial}(0,y) - \lambda(1,y) v_{2,\partial}(1,y) dy.$$

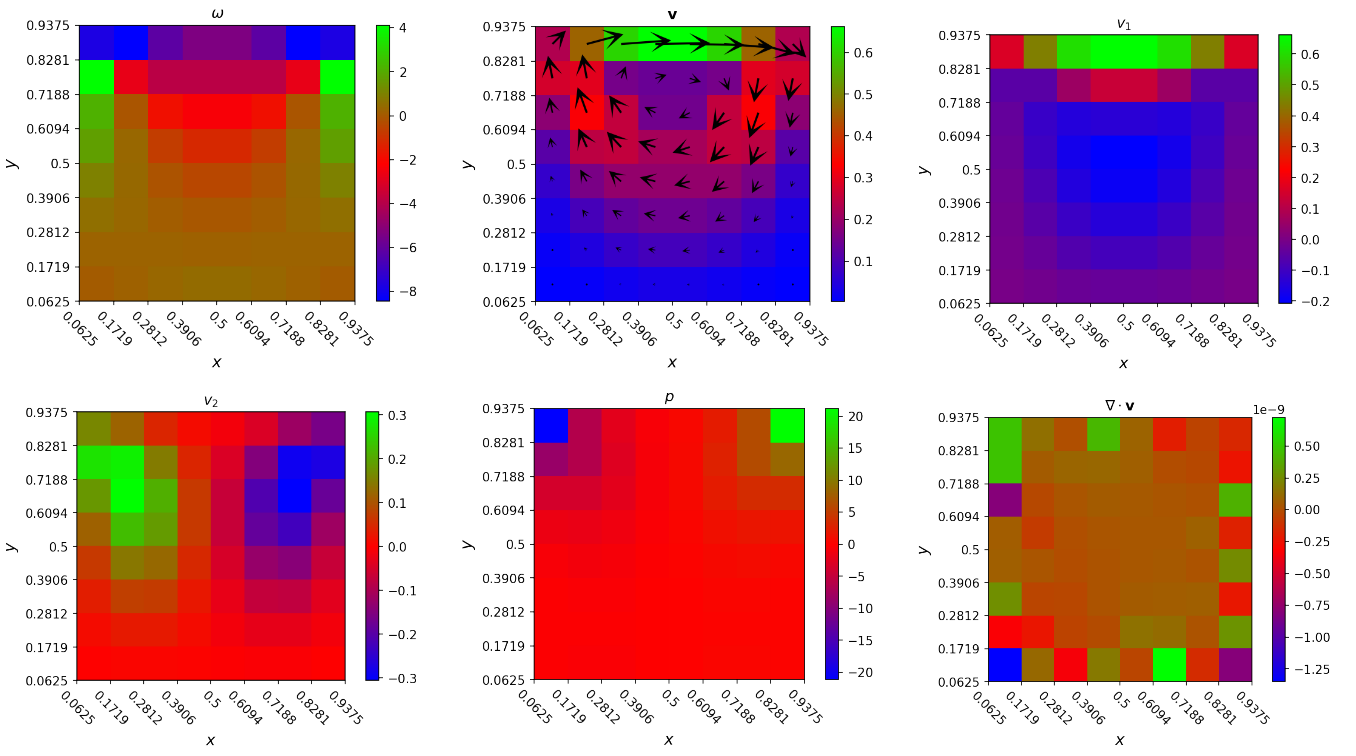

Figure 3: ω, $\mathbf{v}$, v_1, v_2, p, $\nabla \cdot \mathbf{v}$ obtained from our numerical solver evaluated at the element centers for $n = 8$.

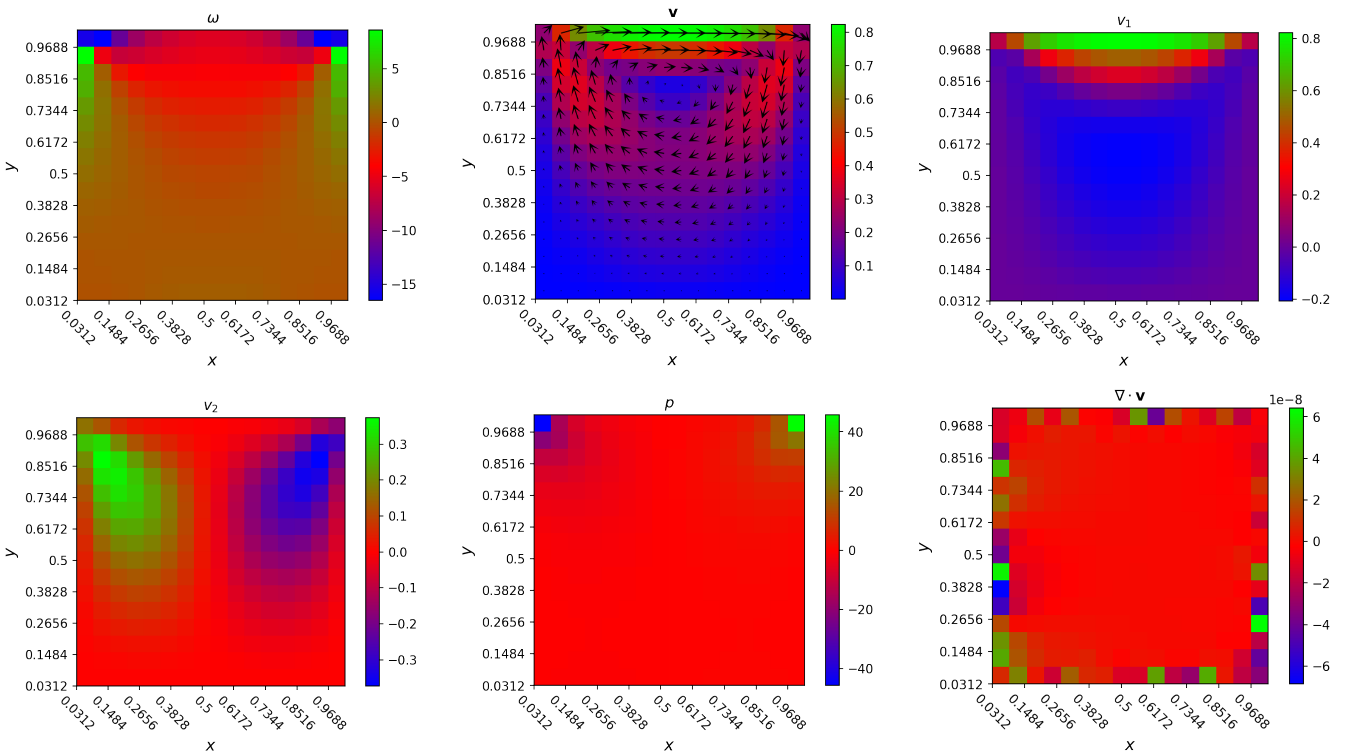

Figure 4: ω, $\mathbf{v}$, v_1, v_2, p, $\nabla \cdot \mathbf{v}$ obtained from our numerical solver evaluated at the element centers for $n = 16$.

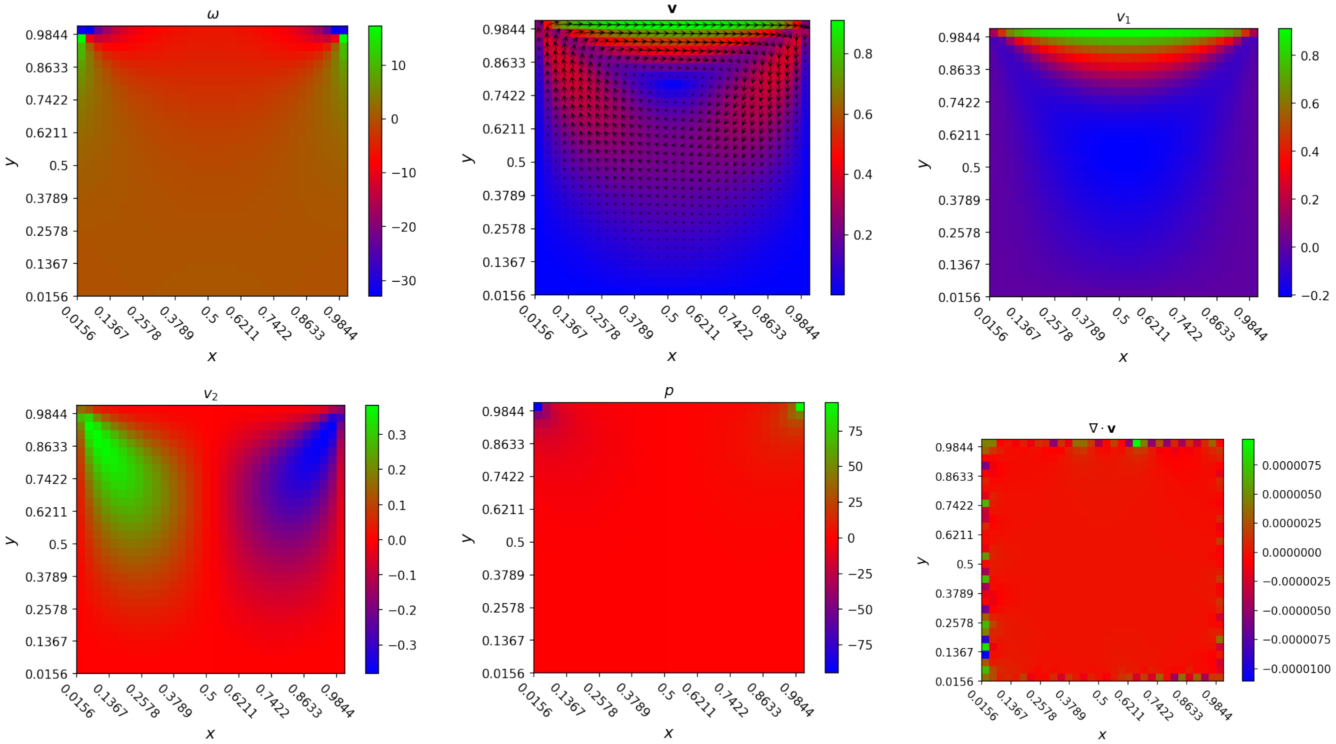

Figure 5: ω, $\mathbf{v}$, v_1, v_2, p, $\nabla \cdot \mathbf{v}$ obtained from our numerical solver evaluated at the element centers for $n = 32$.

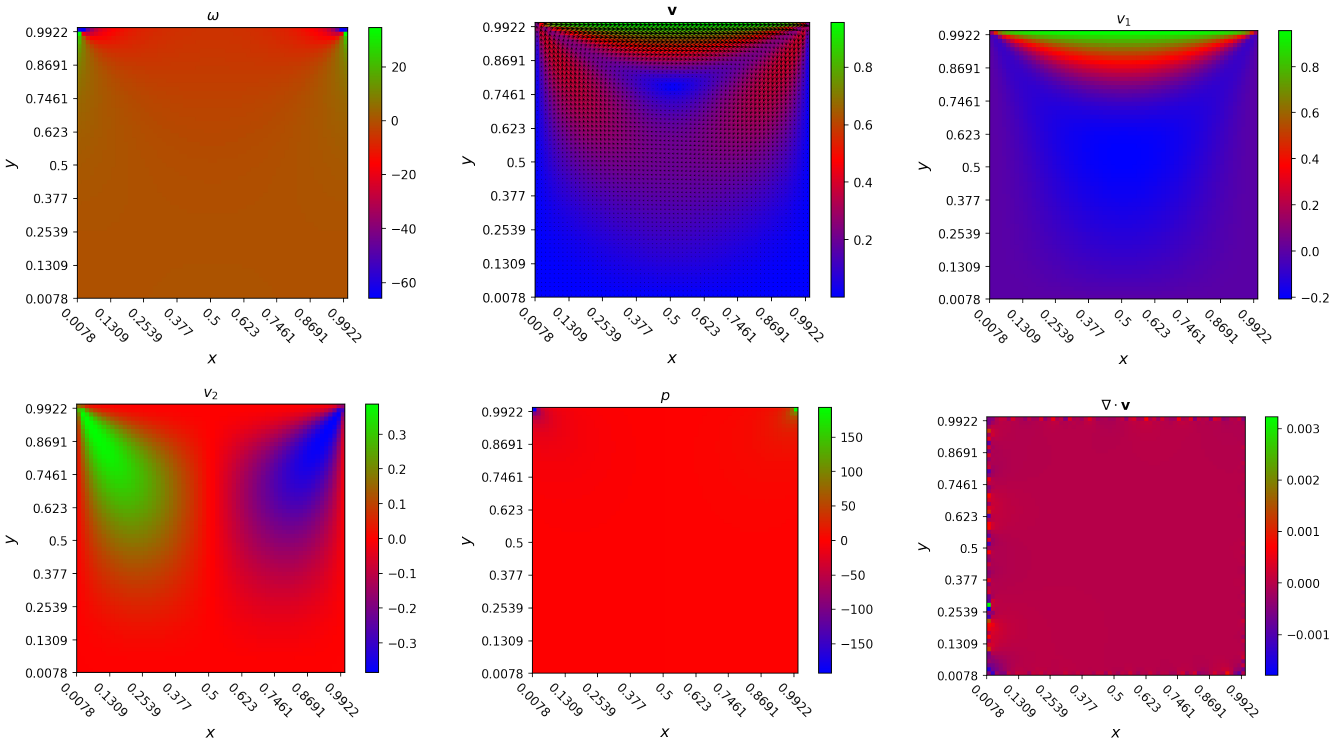

Figure 6: ω, $\mathbf{v}$, v_1, v_2, p, $\nabla \cdot \mathbf{v}$ obtained from our numerical solver evaluated at the element centers for $n = 64$.

The weak form is now: find $\psi \in \mathcal{W}_h$ such that $\forall \lambda \in \mathcal{W}_h$

$$\begin{aligned}\int_\Omega \nabla\lambda \cdot \nabla\psi d\Omega = \int_\Omega \lambda\omega d\Omega + \int_0^1 \lambda(x,1)v_{1,\partial}(x,1) - \lambda(x,0)v_{1,\partial}(x,0)dx \\ + \int_0^1 \lambda(0,y)v_{2,\partial}(0,y) - \lambda(1,y)v_{2,\partial}(1,y)dy.\end{aligned} \tag{15}$$

Similarly as for (5) we now discretize (15).

- We start off with the first part of the LHS integral, taking $\lambda = B_i^1(x)B_j^1(y)$ and obtain:

$$\begin{aligned}\int_\Omega \lambda_{,1}\psi_{,1} d\Omega &= \int_\Omega \partial_x B_i^1(x)B_j^1(y) \sum_{k=0}^n \sum_{l=0}^n \psi_{kl}\partial_x B_k^1(x)B_l^1(y)d\Omega \\ &= [-\frac{1}{6}\psi_{i-1j-1} + \frac{1}{6}\psi_{ij-1} - \frac{1}{3}\psi_{i-1j} + \frac{1}{3}\psi_{ij}]^{\Omega_{i-\frac{1}{2},j-\frac{1}{2}}} \\ &+ [-\frac{1}{6}\psi_{i-1j+1} + \frac{1}{6}\psi_{ij+1} - \frac{1}{3}\psi_{i-1j} + \frac{1}{3}\psi_{ij}]^{\Omega_{i-\frac{1}{2},j+\frac{1}{2}}} \\ &+ [-\frac{1}{6}\psi_{i+1j+1} + \frac{1}{6}\psi_{ij+1} - \frac{1}{3}\psi_{i+1j} + \frac{1}{3}\psi_{ij}]^{\Omega_{i+\frac{1}{2},j+\frac{1}{2}}} \\ &+ [-\frac{1}{6}\psi_{i+1j-1} + \frac{1}{6}\psi_{ij-1} - \frac{1}{3}\psi_{i+1j} + \frac{1}{3}\psi_{ij}]^{\Omega_{i+\frac{1}{2},j-\frac{1}{2}}} \\ &\to \begin{Bmatrix} -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \end{Bmatrix}_\psi^{ij}.\end{aligned}$$

For the second integral we obtain

$$\begin{aligned}\int_\Omega \lambda_{,2}\psi_{,2} d\Omega = &= \int_\Omega B_i^1(x)\partial_y B_j^1(y) \sum_{k=0}^n \sum_{l=0}^n \psi_{kl} B_k^1(x)\partial_y B_l^1(y)d\Omega \\ &= [-\frac{1}{6}\psi_{i-1j-1} - \frac{1}{3}\psi_{ij-1} + \frac{1}{6}\psi_{i-1j} + \frac{1}{3}\psi_{ij}]^{\Omega_{i-\frac{1}{2},j-\frac{1}{2}}} \\ &+ [-\frac{1}{6}\psi_{i-1j+1} - \frac{1}{3}\psi_{ij+1} + \frac{1}{6}\psi_{i-1j} + \frac{1}{3}\psi_{ij}]^{\Omega_{i-\frac{1}{2},j+\frac{1}{2}}} \\ &+ [-\frac{1}{6}\psi_{i+1j+1} - \frac{1}{3}\psi_{ij+1} + \frac{1}{6}\psi_{i+1j} + \frac{1}{3}\psi_{ij}]^{\Omega_{i+\frac{1}{2},j+\frac{1}{2}}} \\ &+ [-\frac{1}{6}\psi_{i+1j-1} - \frac{1}{3}\psi_{ij-1} + \frac{1}{6}\psi_{i+1j} + \frac{1}{3}\psi_{ij}]^{\Omega_{i+\frac{1}{2},j-\frac{1}{2}}} \\ &\to \begin{Bmatrix} -\frac{1}{6} & -\frac{2}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ -\frac{1}{6} & -\frac{2}{3} & -\frac{1}{6} \end{Bmatrix}_\psi^{ij}.\end{aligned}$$

The combined stencil is therefore

$$\begin{aligned}\int_\Omega \lambda_{,1}\psi_{,1} + \lambda_{,2}\psi_{,2} d\Omega &= \int_\Omega \partial_x B_i^1(x)B_j^1(y) \sum_{k=0}^n \sum_{l=0}^n \psi_{kl}\partial_x B_k^1(x)B_l^1(y) + B_i^1(x)\partial_y B_j^1(y) \sum_{k=0}^n \sum_{l=0}^n \psi_{kl} B_k^1(x)\partial_y B_l^1(y)d\Omega \\ &\to \begin{Bmatrix} -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{Bmatrix}_\psi^{ij}.\end{aligned}$$

Note here that the stencils are, once again, only valid away from the domain boundary and the stencils require adjustment by simply disregarding the sums that originate from integrals over non-existing elements.

- For the first RHS integral we obtain

$$\begin{aligned}
&\int_\Omega B_i^1(x)B_j^1(y)\sum_{k=0}^n\sum_{l=0}^n \omega_{kl}B_k^1(x)B_l^1(y)d\Omega \\
&= h^2\left[\frac{1}{36}\omega_{i-1j-1}+\frac{1}{18}\omega_{ij-1}+\frac{1}{18}\omega_{i-1j}+\frac{1}{9}\omega_{ij}\right]^{\Omega_{i-\frac{1}{2},j-\frac{1}{2}}} \\
&+ h^2\left[\frac{1}{18}\omega_{i-1j}+\frac{1}{9}\omega_{ij}+\frac{1}{36}\omega_{i-1j+1}+\frac{1}{18}\omega_{ij+1}\right]^{\Omega_{i-\frac{1}{2},j+\frac{1}{2}}} \\
&+ h^2\left[\frac{1}{9}\omega_{ij}+\frac{1}{18}\omega_{i+1j}+\frac{1}{18}\omega_{ij+1}+\frac{1}{36}\omega_{i+1j+1}\right]^{\Omega_{i+\frac{1}{2},j+\frac{1}{2}}} \\
&+ h^2\left[\frac{1}{18}\omega_{ij-1}+\frac{1}{36}\omega_{i+1j-1}+\frac{1}{9}\omega_{ij}+\frac{1}{18}\omega_{i+1j}\right]^{\Omega_{i+\frac{1}{2},j-\frac{1}{2}}}.
\end{aligned}$$

  This is also only valid away from the boundary and is simply adjusted to account for the boundaries by removing the sums corresponding to non-existing elements.

- The remaining two integrals are

$$\begin{aligned}
\int_0^1 \lambda(x,1)v_{1,\partial}(x,1)-\lambda(x,0)v_{1,\partial}(x,0)dx &= \int_{x_{i-1}}^{x_{i+1}} \delta_{jn}B_i^1(x)v_{1,\partial}(x,1)-\delta_{j0}B_i^1(x)v_{1,\partial}(x,0)dx \\
\int_0^1 \lambda(0,y)v_{2,\partial}(0,y)-\lambda(1,y)v_{2,\partial}(1,y)dy &= \int_{y_{j-1}}^{y_{j+1}} \delta_{i0}B_j^1(y)v_{2,\partial}(0,y)-\delta_{in}B_j^1(y)v_{2,\partial}(1,y)dy.
\end{aligned}$$

Now that we have discretized (15) we find that we can write it as the linear system, namely

$$\mathbf{L}[\psi]=\mathbf{r}([\omega]). \tag{16}$$

As the vorticity unknowns $[\omega]$ were already determined when solving (9) the $\mathbf{r}$ vector is known. However, before solving the system we add a row of ones to $\mathbf{L}$ and a single zero to $\mathbf{r}$ after which we apply least-squares to solve the system. This, because ψ is only defined up to a constant. This looks as follows:

$$\begin{bmatrix}\mathbf{L}\\ \mathbf{1}^T\end{bmatrix}^T\begin{bmatrix}\mathbf{L}\\ \mathbf{1}^T\end{bmatrix}[\psi]=\begin{bmatrix}\mathbf{L}\\ \mathbf{1}^T\end{bmatrix}^T\begin{bmatrix}\mathbf{r}([\omega])\\ 0\end{bmatrix}. \tag{17}$$

The obtained ψ using this method for $n=64$ for lid driven cavity flow, described in (11) and (12), is shown in figure 7.

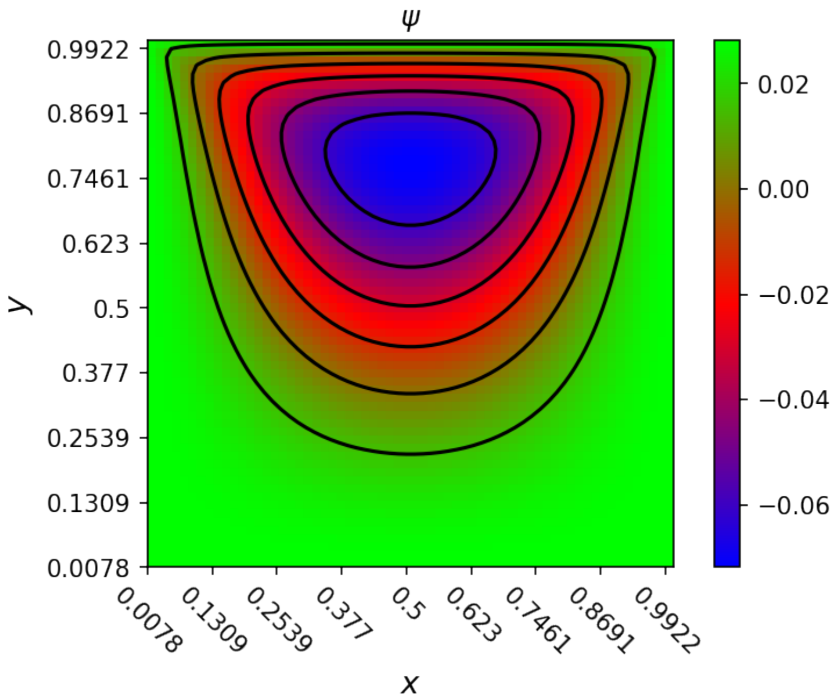

Figure 7: ψ obtained from our numerical solver evaluated at the element centers for $n = 64$. Black lines represent isolines.