Assignment 1.4 CFD

Toby van Gastelen (5373123)

November 2020

Finite volume implementation of Stokes flow

In this section we concern ourselves with Stokes flow on the square domain $\Omega := [0,1]^2$. The general boundary value problem (BVP), with Dirichlet boundary conditions, is given by

$$\begin{aligned} -\nabla p + \nabla^2 \mathbf{v} &= \mathbf{f}, \text{ on } \Omega \\ \nabla \cdot \mathbf{v} &= 0, \text{ on } \Omega \\ \mathbf{v} &= \mathbf{v}_0, \text{ on } \partial\Omega, \end{aligned} \tag{1}$$

where $\mathbf{v}$ is the fluid velocity field, p the pressure, $\mathbf{v}_0$ the known velocity on the boundary of Ω, and $\mathbf{f}$ the known source function. To solve this boundary value problem numerically we will make use of the finite volume method along with a staggered grid, such as depicted in figure 1. In this case we divide the domain into $J_1 \times J_2$ rectangular unit cells (for the x^1 and x^2 direction respectively). We divide Ω uniformly such that each cell is of size $h^1 \times h^2$ with $h^1 = \frac{1}{J_1}$ and $h^2 = \frac{1}{J_2}$, where h^1 is the cell length in the horizontal, or x^1, direction and h^2 the cell length in the vertical, or x^2, direction. Again, we index each unit cells Ω_{j_1,j_2} according to their horizontal index j_1 ($1 \leq j_1 \leq J_1$) and vertical index j_2 ($1 \leq j_2 \leq J_2$). We now have approximations $p(\mathbf{x}_{j_1,j_2}) \approx p_{j_1,j_2}$, $v^1(\mathbf{x}_{j_1,j_2}) \approx v^1_{j_1,j_2}$, and $v^2(\mathbf{x}_{j_1,j_2}) \approx v^2_{j_1,j_2}$ with $\mathbf{x}_{j_1,j_2} = (\frac{1}{2}h^1 + (j_1-1)h^1, \frac{1}{2}h^2 + (j_2-1)h^2)$.

1. Let us now start with the divergence condition in BVP (1) on $\mathbf{v}$. We first write

$$\begin{aligned} &\int_\Omega \nabla \cdot \mathbf{v} ds = 0 \xrightarrow{\text{Divergence theorem}} \\ &\int_{\partial\Omega} \mathbf{v} \cdot \hat{\mathbf{n}} dl = 0. \end{aligned} \tag{2}$$

For a general cell Ω_{j_1,j_2}, not touching the boundary, we now get

$$\int_{\partial\Omega_{j_1,j_2}} \mathbf{v} \cdot \hat{\mathbf{n}} dl \approx -h^1 v^2_{j_1,j_2-\frac{1}{2}} + h^1 v^2_{j_1,j_2+\frac{1}{2}} - h^2 v^1_{j_1-\frac{1}{2},j_2} + h^2 v^1_{j_1+\frac{1}{2},j_2}, \tag{3}$$

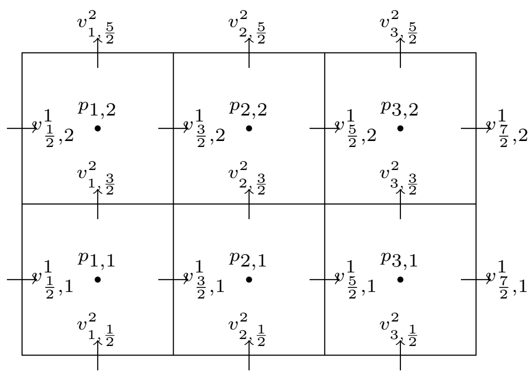

Figure 1: Staggered finite volume mesh with 3×2 unit cells. Points corresponding to pressure and velocity unknowns are labeled.

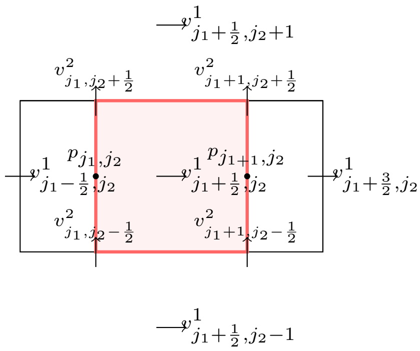

Figure 2: Unit cell $\Omega_{j_1+\frac{1}{2},j_2}$ (red).

where we applied the central-point approximation for each of the boundary integrals. Next, we consider the first equation of BVP (1) for which we do

$$\begin{gathered}
\int_\Omega -\nabla p + \nabla^2 \mathbf{v} ds = \int_\Omega \mathbf{f} ds \xrightarrow{\text{Separate equations}} \\
\int_\Omega -\frac{dp}{dx^1} + \frac{dv^1}{dx^1 dx^1} + \frac{dv^1}{dx^2 dx^2} ds = \int_\Omega f^1 ds \\
\int_\Omega -\frac{dp}{dx^2} + \frac{dv^2}{dx^1 dx^1} + \frac{dv^2}{dx^2 dx^2} ds = \int_\Omega f^2 ds \xrightarrow{\text{Divergence theorem}} \\
\int_{\partial\Omega} -p + \frac{dv^1}{dx^1} + \frac{dv^1}{dx^2} ds = \int_\Omega f^1 ds \\
\int_{\partial\Omega} -p + \frac{dv^2}{dx^1} + \frac{dv^2}{dx^2} ds = \int_\Omega f^2 ds.
\end{gathered} \tag{4}$$

Let us now consider a special type of unit cell $\Omega_{j_1+\frac{1}{2},j_2}$ such as depicted in figure 2. Employing this unit cell, away from the boundary, we can write

$$\begin{aligned}
&\int_{\partial\Omega_{j_1+\frac{1}{2},j_2}} -p + \frac{dv^1}{dx^1} + \frac{dv^1}{dx^2} ds = \int_{\partial\Omega_{j_1+\frac{1}{2},j_2}} f^1 ds \xrightarrow{\text{Central-point approx.}} \\
&h^1\left(\frac{v^1_{j_1+\frac{1}{2},j_2+1} - v^1_{j_1+\frac{1}{2},j_2}}{h^2} - \frac{v^1_{j_1+\frac{1}{2},j_2} - v^1_{j_1+\frac{1}{2},j_2-1}}{h^2}\right) \\
&+ h^2\left(p_{j_1,j_2} - p_{j_1+1,j_2} + \frac{v^1_{j_1+\frac{3}{2},j_2} - v^1_{j_1+\frac{1}{2},j_2}}{h^1} - \frac{v^1_{j_1+\frac{1}{2},j_2} - v^1_{j_1-\frac{1}{2},j_2}}{h^1}\right) = h^1 h^2 f^1(\mathbf{x}_{j_1+\frac{1}{2},j_2})
\end{aligned} \tag{5}$$

Similarly, by employing $\Omega_{j_1,j_2+\frac{1}{2}}$ (figure 3), we can write

$$\begin{aligned}
&\int_{\partial\Omega_{j_1,j_2+\frac{1}{2}}} -p + \frac{dv^2}{dx^1} + \frac{dv^2}{dx^2} ds = \int_{\partial\Omega_{j_1,j_2+\frac{1}{2}}} f^2 ds \xrightarrow{\text{Central-point approx.}} \\
&h^2\left(\frac{v^2_{j_1+1,j_2+\frac{1}{2}} - v^2_{j_1,j_2+\frac{1}{2}}}{h^1} - \frac{v^2_{j_1,j_2+\frac{1}{2}} - v^2_{j_1-1,j_2+\frac{1}{2}}}{h^1}\right) \\
&+ h^1\left(p_{j_1,j_2} - p_{j_1,j_2+1} + \frac{v^2_{j_1,j_2+\frac{3}{2}} - v^2_{j_1,j_2+\frac{1}{2}}}{h^2} - \frac{v^2_{j_1,j_2+\frac{1}{2}} - v^2_{j_1,j_2-\frac{1}{2}}}{h^2}\right) = h^1 h^2 f^2(\mathbf{x}_{j_1,j_2+\frac{1}{2}}).
\end{aligned} \tag{6}$$

2. Next, we treat the unit cells at the boundaries. With regards to the discretization presented in equation (3)

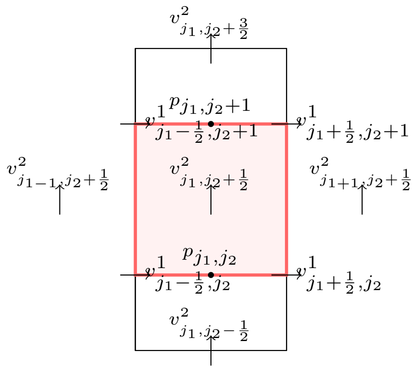

Figure 3: Unit cell $\Omega_{j_1,j_2+\frac{1}{2}}$ (red).

on Ω_{j_1,j_2} we make the following changes on the boundaries:

$$\begin{aligned}
\Omega_{j_1,1} &: \; -h^1 v^2_{j_1,j_2-\frac{1}{2}} \xrightarrow{\text{Replace by}} -h^1 v^2_0(\mathbf{x}_{j_1,\frac{1}{2}}) \\
\Omega_{J_1,j_2} &: \; h^2 v^1_{j_1+\frac{1}{2},j_2} \xrightarrow{\text{Replace by}} h^2 v^1_0(\mathbf{x}_{J_1+\frac{1}{2},j_2}) \\
\Omega_{j_1,J_2} &: \; h^1 v^2_{j_1,j_2+\frac{1}{2}} \xrightarrow{\text{Replace by}} h^1 v^2_0(\mathbf{x}_{j_1,J_2+\frac{1}{2}}) \\
\Omega_{1,j_2} &: \; -h^2 v^1_{j_1-\frac{1}{2},j_2} \xrightarrow{\text{Replace by}} -h^2 v^1_0(\mathbf{x}_{\frac{1}{2},j_2}).
\end{aligned} \tag{7}$$

For the discretization presented in equation (5) on $\Omega_{j_1+\frac{1}{2},j_2}$ we make the following changes on the boundaries:

$$\begin{aligned}
\Omega_{j_1+\frac{1}{2},1} &: h^1\Big(-\frac{v^1_{j_1+\frac{1}{2},j_2} - v^1_{j_1+\frac{1}{2},j_2-1}}{h^2}\Big) \xrightarrow{\text{Replace by}} h^1\Big(-\frac{v^1_{j_1+\frac{1}{2},1} - v^1_0(\mathbf{x}_{j_1+\frac{1}{2},\frac{1}{2}})}{\frac{1}{2}h^2}\Big) \\
\Omega_{J_1-\frac{1}{2},j_2} &: h^2\Big(\frac{v^1_{j_1+\frac{3}{2},j_2} - v^1_{j_1+\frac{1}{2},j_2}}{h^1}\Big) \xrightarrow{\text{Replace by}} h^2\Big(\frac{v^1_0(\mathbf{x}_{J_1+\frac{1}{2},j_2}) - v^1_{J_1-\frac{1}{2},j_2}}{h^1}\Big) \\
\Omega_{j_1+\frac{1}{2},J_2} &: h^1\Big(\frac{v^1_{j_1+\frac{1}{2},j_2+1} - v^1_{j_1+\frac{1}{2},j_2}}{h^2}\Big) \xrightarrow{\text{Replace by}} h^1\Big(\frac{v^1_0(\mathbf{x}_{j_1+\frac{1}{2},J_2+\frac{1}{2}}) - v^1_{j_1+\frac{1}{2},J_2}}{\frac{1}{2}h^2}\Big) \\
\Omega_{\frac{3}{2},j_2} &: h^2\Big(-\frac{v^1_{j_1+\frac{1}{2},j_2} - v^1_{j_1-\frac{1}{2},j_2}}{h^1}\Big) \xrightarrow{\text{Replace by}} h^2\Big(-\frac{v^1_{\frac{3}{2},j_2} - v^1_0(\mathbf{x}_{\frac{1}{2},j_2})}{h^1}\Big).
\end{aligned} \tag{8}$$

For the discretization presented in equation (6) on $\Omega_{j_1,j_2+\frac{1}{2}}$ we make the following changes on the boundaries:

$$\begin{aligned}
\Omega_{j_1,\frac{3}{2}} &: h^1\Big(-\frac{v^2_{j_1,j_2+\frac{1}{2}} - v^2_{j_1,j_2-\frac{1}{2}}}{h^2}\Big) \xrightarrow{\text{Replace by}} h^1\Big(-\frac{v^2_{j_1,\frac{3}{2}} - v^2_0(\mathbf{x}_{j_1,\frac{1}{2}})}{h^2}\Big) \\
\Omega_{J_1,j_2+\frac{1}{2}} &: h^2\Big(\frac{v^2_{j_1+1,j_2+\frac{1}{2}} - v^2_{j_1,j_2+\frac{1}{2}}}{h^1}\Big) \xrightarrow{\text{Replace by}} h^2\Big(\frac{v^2_0(\mathbf{x}_{J_1+\frac{1}{2},j_2+\frac{1}{2}}) - v^2_{J_1,j_2+\frac{1}{2}}}{\frac{1}{2}h^1}\Big) \\
\Omega_{j_1,J_2-\frac{1}{2}} &: h^1\Big(\frac{v^2_{j_1,j_2+\frac{3}{2}} - v^2_{j_1,j_2+\frac{1}{2}}}{h^2}\Big) \xrightarrow{\text{Replace by}} h^1\Big(\frac{v^2_0(\mathbf{x}_{j_1,J_2+\frac{1}{2}}) - v^2_{j_1,J_2-\frac{1}{2}}}{h^2}\Big) \\
\Omega_{1,j_2+\frac{1}{2}} &: h^2\Big(-\frac{v^2_{j_1,j_2+\frac{1}{2}} - v^2_{j_1-1,j_2+\frac{1}{2}}}{h^1}\Big) \xrightarrow{\text{Replace by}} h^2\Big(-\frac{v^2_{1,j_2+\frac{1}{2}} - v^2_0(\mathbf{x}_{\frac{1}{2},j_2+\frac{1}{2}})}{\frac{1}{2}h^1}\Big).
\end{aligned} \tag{9}$$

When a unit cell meets two conditions both replacements are made.

3. The derived finite volume scheme can be expressed as the following linear system

$$\begin{aligned}
A\mathbf{v}_h + G\mathbf{p}_h &= \mathbf{f}_h \\
D\mathbf{v}_h &= \mathbf{g}_h,
\end{aligned} \tag{10}$$

where the first equation and second equation are the discretized version of the first and second equation in BVP (1), respectively. The approximated velocity and pressure on Ω are now obtained by solving for $\mathbf{v}_h$ and $\mathbf{p}_h$. This can be done by first writing the problem as

$$\begin{bmatrix} A & G \\ D & \end{bmatrix} \begin{bmatrix} \mathbf{v}_h \\ \mathbf{p}_h \end{bmatrix} = \begin{bmatrix} \mathbf{f}_h \\ \mathbf{g}_h \end{bmatrix} \tag{11}$$

using the previously described discretizations. Before solving this system we add the following constraint:

$$\int_\Omega p ds = 0. \tag{12}$$

Practically, this boils down to

$$\begin{bmatrix} A & G \\ D & \\ \mathbf{0}^T & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} \mathbf{v}_h \\ \mathbf{p}_h \end{bmatrix} = \begin{bmatrix} \mathbf{f}_h \\ \mathbf{g}_h \\ 0 \end{bmatrix}. \tag{13}$$

The least-squares solution of $\mathbf{v}_h$ and $\mathbf{p}_h$ is now obtained by solving

$$\begin{bmatrix} A & G \\ D & \\ \mathbf{0}^T & \mathbf{1}^T \end{bmatrix}^T \begin{bmatrix} A & G \\ D & \\ \mathbf{0}^T & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} \mathbf{v}_h \\ \mathbf{p}_h \end{bmatrix} = \begin{bmatrix} A & G \\ D & \\ \mathbf{0}^T & \mathbf{1}^T \end{bmatrix}^T \begin{bmatrix} \mathbf{f}_h \\ \mathbf{g}_h \\ 0 \end{bmatrix}. \tag{14}$$

Our implementation of this procedure is included in our submitted **Jupyter notebook**.

4. In order to asses the accuracy of our numerical solver we employ the method of manufactured solutions. For this we take $\mathbf{v} = (\sin(x)\sin(y), \cos(x)\cos(y)$ and $p = \sin(x)\sin(y)$. This choice of $\mathbf{v}$ ensures that

$$\nabla \cdot \mathbf{v} = \cos(x)\sin(y) - \cos(x)\sin(y) = 0. \tag{15}$$

For $\mathbf{f}$ we now have

$$-\nabla p + \nabla^2 \mathbf{v} = (-\cos(x)\sin(y) - 2\sin(y)\sin(x), -\sin(x)\cos(y) - 2\cos(x)\cos(y)). \tag{16}$$

Lastly we take $\mathbf{v}_0 = \mathbf{v}$.

Let us now first look at the numerical solution for $J_1 = 35$ and $J_2 = 30$ presented in figure 4. Regarding the visualized velocities we take

$$\begin{aligned} v^1_{j_1,j_2} &= \frac{v^1_{j_1+\frac{1}{2},j_2} + v^1_{j_1-\frac{1}{2},j_2}}{2} \\ v^2_{j_1,j_2} &= \frac{v^2_{j_1,j_2+\frac{1}{2}} + v^2_{j_1,j_2-\frac{1}{2}}}{2}. \end{aligned} \tag{17}$$

From this figure we can see that the approximate solution matches the behavior of the intended manufactured solution. This serves as confirmation of the correctness of our implementation. We can now evaluate the convergence behavior of the solution as a function of $h = \max(h^1, h^2)$. The error measures are:

$$\begin{aligned} e_p &= \max_{j_1,j_2} |p(\mathbf{x}_{j_1,j_2}) - p_{j_1,j_2}| \\ e_\mathbf{v} &= \max[\max_{j_1,j_2} |v^1(\mathbf{x}_{j_1+\frac{1}{2}}) - v^1_{j_1+\frac{1}{2},j_2}|, \max_{j_1,j_2} |v^2(\mathbf{x}_{j_1,j_2+\frac{1}{2}}) - v^2_{j_1,j_2+\frac{1}{2}}|]. \end{aligned} \tag{18}$$

Note here that p is only defined up to a constant, as BVP (1) only includes its gradient. This is why, in order to attempt to make a fair comparison between $p(\mathbf{x}_{j_1,j_2})$ and p_{j_1,j_2}, we make sure both are centred around the average pressure on Ω before evaluating e_p:

$$\begin{aligned} p(\mathbf{x}_{j_1,j_2}) &\to p(\mathbf{x}_{j_1,j_2}) - \sum_{j_1,j_2=1}^{J_1,J_2} \frac{p(\mathbf{x}_{j_1,j_2})}{J_1 J_2} \\ pj_1, j_2 &\to p_{j_1,j_2} - \sum_{j_1,j_2=1}^{J_1,J_2} \frac{p_{j_1,j_2}}{J_1 J_2}. \end{aligned} \tag{19}$$

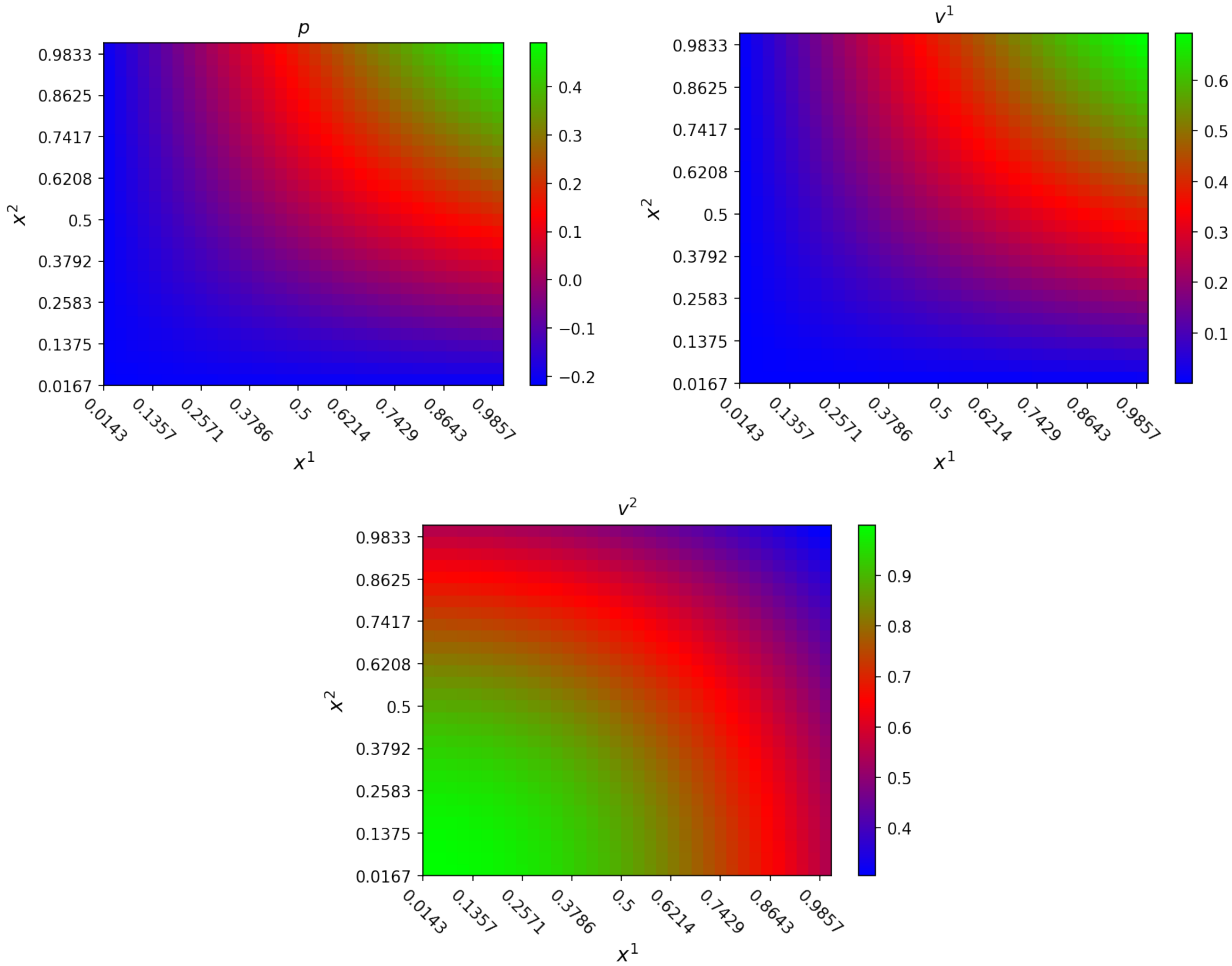

Figure 4: Finite volume solution of BVP (1) with $\mathbf{f} = (-\cos(x)\sin(y) - 2\sin(y)\sin(x), -\sin(x)\cos(y) - 2\cos(x)\cos(y))$ and $\mathbf{v}_0 = (\sin(x)\sin(y), \cos x \cos y)$ for $J_1 = 35$ and $J_2 = 30$.

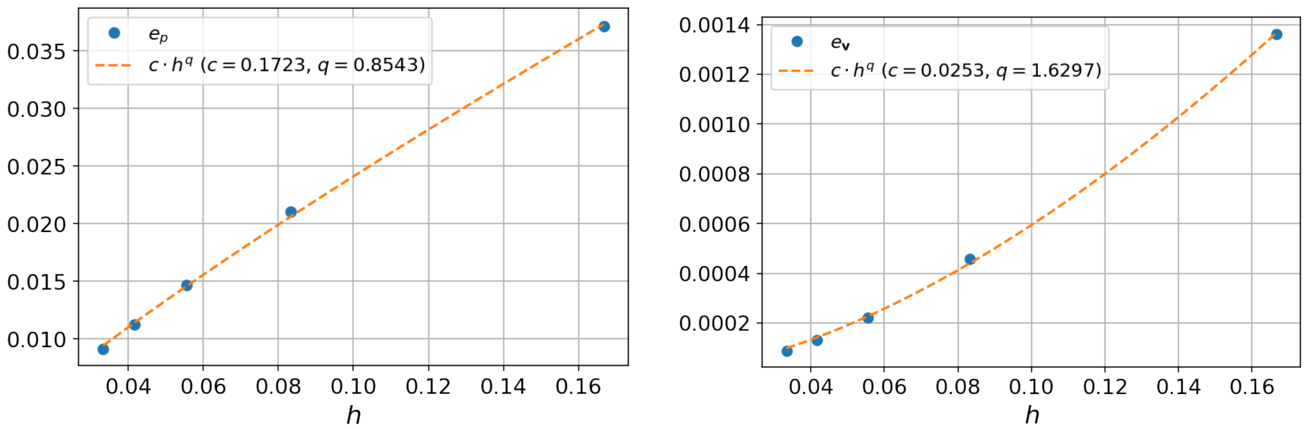

Figure 5: e_p and $e_{\mathbf{v}}$ evaluated for BVP (1) with $\mathbf{f} = (-\cos(x)\sin(y) - 2\sin(y)\sin(x), -\sin(x)\cos(y) - 2\cos(x)\cos(y))$ and $\mathbf{v}_0 = (\sin(x)\sin(y), \cos x \cos y)$. Here, $h = \max(h^1, h^2) = h^2$ for $h^1 = \frac{6}{7}h^2$.

We can finally fit the observed errors to the relation

$$e = c \cdot h^q, \tag{20}$$

where e is a general error measure and c and q the fitted parameters. For the fitting procedure we use the non-linear least-squares procedure provided in the **SciPy** package [1]. The results are shown in figure 5 for $h^1 = \frac{6}{7}h^2$ such that $J_1 = 7k$ and $J_2 = 6k$, for $k = 1, 2, \ldots, 5$. We find $q = 0.8543$ for e_p and faster convergence for $e_{\mathbf{v}}$ with $q = 1.6296$. e_p is also much larger than $e_{\mathbf{v}}$ for all tested values of h. When interpreting these results one has to take into account the correction introduced in (19).

5. After extensively testing our methodology and the convergence of the results, we finally end up at simulating the lid driven cavity flow on Ω. For this we take

$$\mathbf{v}_0 = \begin{cases} (1,0), & \mathbf{x} = (x^1, 1) \\ (0,0), & \text{otherwise} \end{cases} . \tag{21}$$

along with $\mathbf{f} = (0, 0)$ for BVP (1). The results for $J_1 = J_2 = 32$ are shown in figure 6.

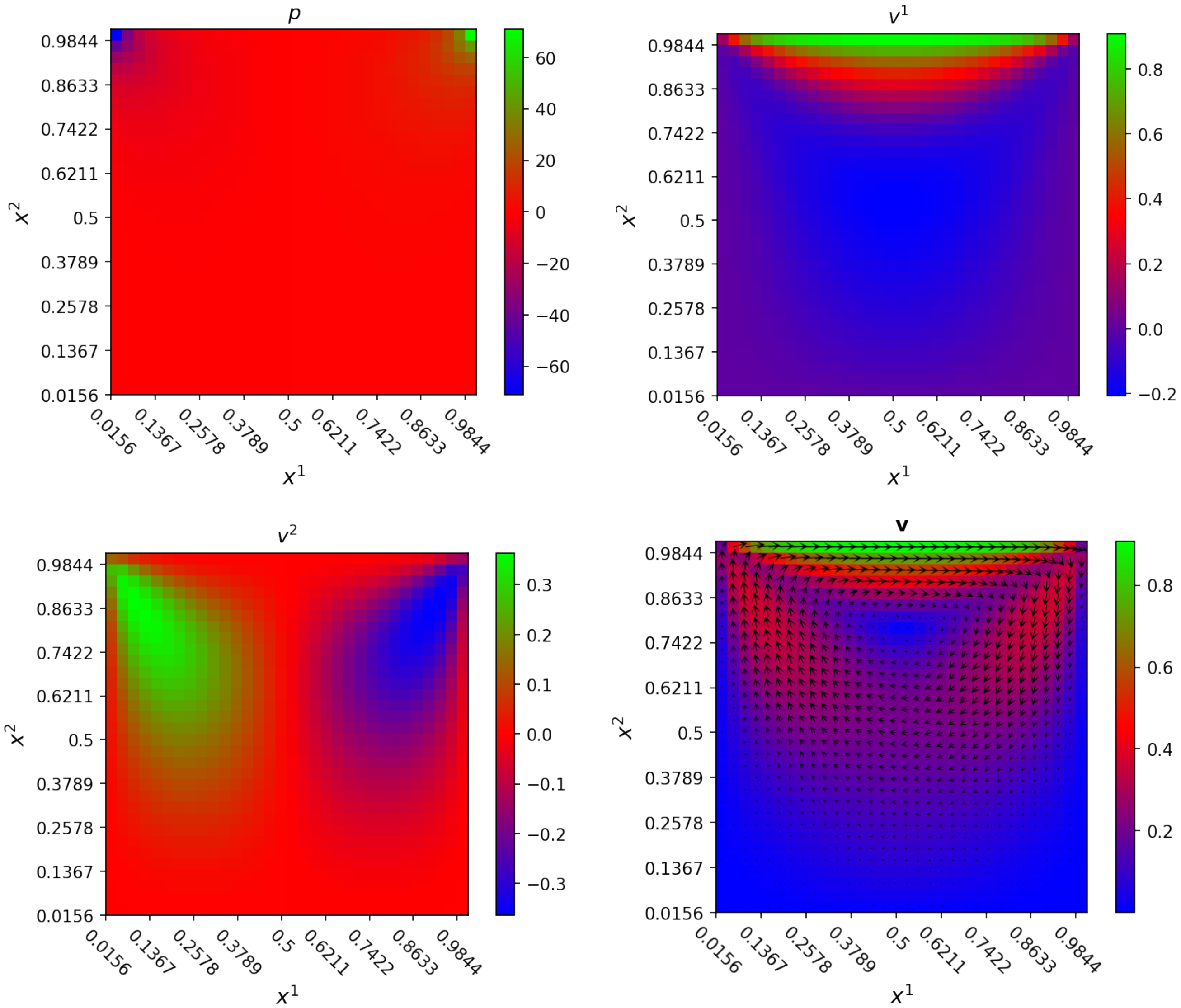

Figure 6: Finite volume solution of BVP (1) with $\mathbf{f} = (0, 0)$ and $\mathbf{v}_0$ given by (21) for $J_1 = J_2 = 32$.

References

[1] Pauli Virtanen, Ralf Gommers, Travis E. Oliphant, Matt Haberland, Tyler Reddy, David Cournapeau, Evgeni Burovski, Pearu Peterson, Warren Weckesser, Jonathan Bright, Stéfan J. van der Walt, Matthew Brett, Joshua Wilson, K. Jarrod Millman, Nikolay Mayorov, Andrew R. J. Nelson, Eric Jones, Robert Kern, Eric Larson, C J Carey, İlhan Polat, Yu Feng, Eric W. Moore, Jake VanderPlas, Denis Laxalde, Josef Perktold, Robert Cimrman, Ian Henriksen, E. A. Quintero, Charles R. Harris, Anne M. Archibald, Antônio H. Ribeiro, Fabian Pedregosa, Paul van Mulbregt, and SciPy 1.0 Contributors. SciPy 1.0: Fundamental Algorithms for Scientific Computing in Python. *Nature Methods*, 17:261–272, 2020.