CFD Assignment 2.2

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1 Upwind Galerkin finite elements: (a)

1.1 Problem statement

Here we consider the 1-dimensional weak form of the advection diffusion equation, with $\epsilon > 0$, on $\Omega = [0, L]$, namely: find $\phi_h \in \mathcal{S}_h$ such that

$$\forall w_h \in \mathcal{W}_h, \quad \int_\Omega -w_{h,1} u \phi_h + w_{h,1} \epsilon \phi_{h,1} dx = \int_\Omega w_h f dx \tag{1}$$

with $\phi|_{\partial\Omega} = \phi_0 \to w|_{\partial\Omega} = 0$. For this we generate a mesh on Ω using nodes $0 = x_0 < x_1 < \ldots < x_n = L$, such that $h = L/n$. We define the trial and test space ($\mathcal{S}_h$ and $\mathcal{W}_h$) using C^0 piecewise-linear finite elements on the mesh. The linear basis functions $B_i(x)$ are defined as

$$B_i = \begin{cases} (x - x_{i-1})/h, \ x \in [x_{i-1}, x_i] \\ (x_{i+1} - x)/h, \ x \in [x_i, x_{i+1}] \\ 0, \ \text{elsewhere} \end{cases} . \tag{2}$$

In the previous assignment we saw that this boils down to solving

$$\sum_{k=i-1}^{i} \int_{x_k}^{x_{k+1}} -B_{i,1} u (\sum_{j=k}^{k+1} \phi_j B_j) + B_{i,1} \epsilon (\sum_{j=k}^{k+1} \phi_j B_{j,1}) dx = 0 \text{ for } i = 1, \ldots, n-1. \tag{3}$$

for $f = 0$. Here, we used the properties of the basis functions to reduce the sum indices from $j \in \{0, \ldots, n\}$ to $j \in \{k, k+1\}$.

1.2 Applying a one-point quadrature scheme and relating to the modified equation method

We will now apply the one-point quadrature scheme,

$$\int_{x_k}^{x_{k+1}} g(x) dx \approx h g(x_k + \xi), \tag{4}$$

to (3). This, with the goal of obtaining a value for ξ such that the formulation (3) is equivalent to the modified equation method defined as

$$\begin{aligned} &u \frac{\phi_{i+1} - \phi_{i-1}}{2h} - (\epsilon + \bar{\epsilon}) \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2} = 0 \\ &\bar{\epsilon} := \frac{uh}{2} [\coth(\mathrm{Pe}_h) - \frac{1}{\mathrm{Pe}_h}] \\ &\mathrm{Pe}_h := \frac{uh}{2\epsilon}. \end{aligned} \tag{5}$$

Applying the quadrature scheme to (3) we get

$$h[-\epsilon \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2} + \frac{u}{h} (\phi_{i+1} B_{i+1}(x_i + \xi) - \phi_{i-1} B_{i-1}(x_{i-1} + \xi) - \phi_i B_i(x_{i-1} + \xi) + \phi_i B_i(x_i + \xi)) = 0 \tag{6}$$

for $i = 1, \ldots, n-1$. Using the definition of B_i (given in (2)) and assuming $0 \le \xi \le h$ we get

$$\begin{aligned} &h[\frac{u}{h}(\phi_{i+1}B_{i+1}(x_i+\xi) - \phi_{i-1}B_{i-1}(x_{i-1}+\xi) - \phi_i B_i(x_{i-1}+\xi) + \phi_i B_i(x_i+\xi))] \\ &= h[\frac{u}{h}(\phi_{i+1}\frac{\xi}{h} - \phi_{i-1}(1-\frac{\xi}{h}) + \phi_i(1-\frac{\xi}{h}) - \phi_i\frac{\xi}{h})]. \end{aligned} \tag{7}$$

Filling in $\xi = \frac{h}{2} - \frac{\bar{\epsilon}}{u}$ gives us

$$\begin{aligned} &h[\frac{u}{h}(\phi_{i+1}\frac{\xi}{h} - \phi_{i-1}(1-\frac{\xi}{h}) + \phi_i(1-\frac{\xi}{h}) - \phi_i\frac{\xi}{h})] \\ &= h[u\frac{\phi_{i+1}-\phi_{i-1}}{2h} - \bar{\epsilon}\frac{\phi_{i+1}-2\phi_i+\phi_{i-1}}{h^2}]. \end{aligned} \tag{8}$$

Filling this into (6) gives us

$$\begin{aligned} &h[-\epsilon\frac{\phi_{i+1}-2\phi_i+\phi_{i-1}}{h^2} + \frac{u}{h}(\phi_{i+1}B_{i+1}(x_i+\xi) - \phi_{i-1}B_{i-1}(x_{i-1}+\xi) - \phi_i B_i(x_{i-1}+\xi) + \phi_i B_i(x_i+\xi)) \\ &= h[-\epsilon\frac{\phi_{i+1}-2\phi_i+\phi_{i-1}}{h^2} + u\frac{\phi_{i+1}-\phi_{i-1}}{2h} - \bar{\epsilon}\frac{\phi_{i+1}-2\phi_i+\phi_{i-1}}{h^2}] = 0 \rightarrow \\ &u\frac{\phi_{i+1}-\phi_{i-1}}{2h} - (\epsilon+\bar{\epsilon})\frac{\phi_{i+1}-2\phi_i+\phi_{i-1}}{h^2} = 0 \end{aligned} \tag{9}$$

for $i = 1, \ldots, n-1$ which is exactly equal to the modified equation method (5), as intended.

For our obtained expression for ξ, namely

$$\xi = \frac{h}{2} - \frac{\bar{\epsilon}}{u}, \tag{10}$$

we have that $\xi \sim -\bar{\epsilon}$. We now use the fact that

$$\bar{\epsilon} = \frac{uh}{2}[\frac{e^{2\mathrm{Pe}_h}+1}{e^{2\mathrm{Pe}_h}-1} - \frac{1}{\mathrm{Pe}_h}] \tag{11}$$

and that, trough Taylor expanding, we can write

$$e^{2\mathrm{Pe}_h} = 1 + 2\mathrm{Pe}_h + \mathcal{O}(\mathrm{Pe}_h^2). \tag{12}$$

This helps us identify the limits

$$\begin{aligned} &\lim_{\mathrm{Pe}_h\to 0^+}\bar{\epsilon} = \lim_{\mathrm{Pe}_h\to 0^+}\frac{uh}{2}[\frac{1+2\mathrm{Pe}_h+\mathcal{O}(\mathrm{Pe}_h^2)+1}{1+2\mathrm{Pe}_h+\mathcal{O}(\mathrm{Pe}_h^2)-1} - \frac{1}{\mathrm{Pe}_h}] = \lim_{\mathrm{Pe}_h\to 0^+}\frac{uh}{2}[\frac{2+2\mathrm{Pe}_h+\mathcal{O}(\mathrm{Pe}_h^2)}{2\mathrm{Pe}_h+\mathcal{O}(\mathrm{Pe}_h^2)} - \frac{1}{\mathrm{Pe}_h}] = 0 \\ &\lim_{\mathrm{Pe}_h\to\infty}\bar{\epsilon} = \frac{uh}{2}, \end{aligned} \tag{13}$$

as

$$\frac{2+2\mathrm{Pe}_h+\mathcal{O}(\mathrm{Pe}_h^2)}{2\mathrm{Pe}_h+\mathcal{O}(\mathrm{Pe}_h^2)} \sim \frac{1}{x}. \tag{14}$$

This means that the following applies to ξ:

$$\begin{aligned}\lim_{\mathrm{Pe}_h \to 0^+} \xi &= \frac{h}{2} \\ \lim_{\mathrm{Pe}_h \to \infty} \xi &= \frac{h}{2} - \frac{uh}{u2} = 0.\end{aligned} \tag{15}$$

We can see that for small Pe_h the singe-point quadrature scheme is simply applied to the middle of the element, leaving the method unchanged, as compared to the standard Galerkin method. However for large enough Pe_h the ξ term effectively moves the region to which the quadrature scheme is applied to the edge of the element. This is why the name **upwind** Galerkin finite elements is justified.

2 Upwind Galerkin finite elements: (b)

From the problem described in the previous section we will move to a simplified two-dimensional setting with $\Omega = [0, L_x] \times [0, L_y]$. The weak form is now: find $\phi_h \in \mathcal{S}_h$ such that

$$\forall w_h \in \mathcal{W}_h, \quad \int_\Omega -\nabla w_h \cdot \mathbf{u}\phi_h + \nabla w_h \epsilon \nabla \phi_h dx = \int_\Omega w_h f dx \tag{16}$$

with $\epsilon > 0$, incompressible flow field $\mathbf{u}$, and $\phi|_{\partial\Omega} = \phi_0 \to w|_{\partial\Omega} = 0$. The mesh on Ω now consists of $n \times n$ nodes (x_i, y_j), where $0 \leq i, j \leq n$ and

$$\begin{aligned}x_i &:= ih_x, \ h_x := \frac{L_x}{n} \\ y_j &:= ih_y, \ h_y := \frac{L_y}{n}.\end{aligned} \tag{17}$$

The C^0 finite element piecewise-bilinear basis functions B_{ij} are now defined as

$$B_{ij}(x, y) = B_i(x) B_j(y), \tag{18}$$

where B_i is defined as before (equation (2)). The approximate solution ϕ_h is now

$$\phi_h = \sum_{\mu=0}^{n} \sum_{\nu=0}^{n} \phi_{\mu\nu} B_{\mu\nu}(x, y). \tag{19}$$

2.1 Rewriting the weak form using the definition of the basis functions

Similarly to (3), again taking $f = 0$, we can write (16) as

$$\sum_{k=i-1}^{i} \sum_{l=j-1}^{j} \int_{x_k}^{x_{k+1}} \int_{y_l}^{y_{l+1}} -\nabla B_{ij} \cdot \mathbf{u}\left(\sum_{\mu=k}^{k+1} \sum_{\nu=l}^{l+1} \phi_{\mu\nu} B_{\mu\nu}\right) + \epsilon \nabla B_{ij} \cdot \left(\sum_{\mu=k}^{k+1} \sum_{\nu=l}^{l+1} \phi_{\mu\nu} \nabla B_{\mu\nu}\right) dy dx \tag{20}$$

for $1 \leq i, j \leq n - 1$. Here, we used the properties of the basis functions to reduce the sum indices from $\mu \in \{0, \dots, n\}$ to $\mu \in \{k, k + 1\}$ and $\nu \in \{0, \dots, n\}$ to $\nu \in \{l, l + 1\}$.

2.2 Evaluating the integrals

The integrals in (20) can be approximated using the one-point quadrature scheme

$$\int_{x_k}^{x_{k+1}} \int_{y_l}^{y_{l+1}} g(x,y)dydx = h_x h_y g(x_k + \xi_x, y_l + \xi_y), \tag{21}$$

where

$$\begin{aligned} \xi_x &= \frac{h_x}{2} - \frac{\bar{\epsilon}_x}{u_{1,kl}} \\ \xi_y &= \frac{h_y}{2} - \frac{\bar{\epsilon}_y}{u_{2,kl}}, \end{aligned} \tag{22}$$

as derived in the previous sections, with

$$\begin{aligned} \mathrm{Pe}_{h,x} &= \frac{u_{1,kl}h_x}{2\epsilon} \\ \mathrm{Pe}_{h,y} &= \frac{u_{2,kl}h_y}{2\epsilon}, \end{aligned} \tag{23}$$

$$\begin{aligned} \bar{\epsilon}_x &:= \frac{u_{1,kl}h_x}{2}[\coth(\mathrm{Pe}_{h,x}) - \frac{1}{\mathrm{Pe}_{h,x}}] \\ \bar{\epsilon}_y &:= \frac{u_{2,kl}h_y}{2}[\coth(\mathrm{Pe}_{h,y}) - \frac{1}{\mathrm{Pe}_{h,y}}], \end{aligned} \tag{24}$$

and flow field $\mathbf{u} = (u_1(x,y), u_2(x,y))$ with $u_{i,kl} := u_i(x_k + \frac{h_x}{2}, y_l + \frac{h_y}{2})$.

Let us now evaluate the integrals in (20) one by one using the scheme in (21):

$$\begin{aligned} k = i-1,\ l = j-1:\ & h_x h_y[-(\frac{\xi_y}{h_x h_y}u_1(x_{i-1}+\xi_x) + \frac{\xi_x}{h_x h_y}u_2(y_{j-1}+\xi_y))\cdot \\ & (\phi_{i-1,j-1}(1-\frac{\xi_x}{h_x})(1-\frac{\xi_y}{h_y}) + \phi_{i,j-1}\frac{\xi_x}{h_x}(1-\frac{\xi_y}{h_y}) + \phi_{i-1,j}(1-\frac{\xi_x}{h_x})\frac{\xi_y}{h_y} + \phi_{i,j}\frac{\xi_x}{h_x}\frac{\xi_y}{h_y}) \\ & + \epsilon\frac{\xi_y}{h_x h_y}(-\phi_{i-1,j-1}\frac{1}{h_x}(1-\frac{\xi_y}{h_y}) + \phi_{i,j-1}\frac{1}{h_x}(1-\frac{\xi_y}{h_y}) - \phi_{i-1,j}\frac{1}{h_x}\frac{\xi_y}{h_y} + \phi_{i,j}\frac{1}{h_x}\frac{\xi_y}{h_y}) \\ & + \epsilon\frac{\xi_x}{h_x h_y}(-\phi_{i-1,j-1}\frac{1}{h_y}(1-\frac{\xi_x}{h_x}) + \phi_{i-1,j}\frac{1}{h_y}(1-\frac{\xi_x}{h_x}) - \phi_{i,j-1}\frac{1}{h_y}\frac{\xi_x}{h_x} + \phi_{i,j}\frac{1}{h_y}\frac{\xi_x}{h_x})] \end{aligned} \tag{25}$$

$$\begin{aligned} k = i,\ l = j-1:\ & h_x h_y[-(-\frac{\xi_y}{h_x h_y}u_1(x_i+\xi_x) + (1-\frac{\xi_x}{h_x})\frac{1}{h_y}u_2(y_{j-1}+\xi_y))\cdot \\ & (\phi_{i,j-1}(1-\frac{\xi_x}{h_x})(1-\frac{\xi_y}{h_y}) + \phi_{i+1,j-1}\frac{\xi_x}{h_x}(1-\frac{\xi_y}{h_y}) + \phi_{i,j}(1-\frac{\xi_x}{h_x})\frac{\xi_y}{h_y} + \phi_{i+1,j}\frac{\xi_x}{h_x}\frac{\xi_y}{h_y}) \\ & - \epsilon\frac{\xi_y}{h_x h_y}(-\phi_{i,j-1}\frac{1}{h_x}(1-\frac{\xi_y}{h_y}) + \phi_{i+1,j-1}\frac{1}{h_x}(1-\frac{\xi_y}{h_y}) - \phi_{i,j}\frac{1}{h_x}\frac{\xi_y}{h_y} + \phi_{i+1,j}\frac{1}{h_x}\frac{\xi_y}{h_y}) \\ & + \epsilon(1-\frac{\xi_x}{h_x})\frac{1}{h_y}(-\phi_{i,j-1}\frac{1}{h_y}(1-\frac{\xi_x}{h_x}) + \phi_{i,j}\frac{1}{h_y}(1-\frac{\xi_x}{h_x}) - \phi_{i+1,j-1}\frac{1}{h_y}\frac{\xi_x}{h_x} + \phi_{i+1,j}\frac{1}{h_y}\frac{\xi_x}{h_x})] \end{aligned} \tag{26}$$

$$
\begin{aligned}
k=i-1,\ l=j:\ & h_xh_y[-((1-\frac{\xi_y}{h_y})\frac{1}{h_x}u_1(x_{i-1}+\xi_x)-\frac{\xi_x}{h_xh_y}u_2(y_j+\xi_y))\cdot \\
&(\phi_{i-1,j}(1-\frac{\xi_x}{h_x})(1-\frac{\xi_y}{h_y})+\phi_{i,j}\frac{\xi_x}{h_x}(1-\frac{\xi_y}{h_y})+\phi_{i-1,j+1}(1-\frac{\xi_x}{h_x})\frac{\xi_y}{h_y}+\phi_{i,j+1}\frac{\xi_x}{h_x}\frac{\xi_y}{h_y}) \\
&+\epsilon(1-\frac{\xi_y}{h_y})\frac{1}{h_x}(-\phi_{i-1,j}\frac{1}{h_x}(1-\frac{\xi_y}{h_y})+\phi_{i,j}\frac{1}{h_x}(1-\frac{\xi_y}{h_y})-\phi_{i-1,j+1}\frac{1}{h_x}\frac{\xi_y}{h_y}+\phi_{i,j+1}\frac{1}{h_x}\frac{\xi_y}{h_y}) \\
&-\epsilon\frac{\xi_x}{h_xh_y}(-\phi_{i-1,j}\frac{1}{h_y}(1-\frac{\xi_x}{h_x})+\phi_{i-1,j+1}\frac{1}{h_y}(1-\frac{\xi_x}{h_x})-\phi_{i,j}\frac{1}{h_y}\frac{\xi_x}{h_x}+\phi_{i,j+1}\frac{1}{h_y}\frac{\xi_x}{h_x})]
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
k=i,\ l=j:\ & h_xh_y[((1-\frac{\xi_y}{h_y})\frac{1}{h_x}u_1(x_i+\xi_x)+(1-\frac{\xi_x}{h_x})\frac{1}{h_y}u_2(y_j+\xi_y))\cdot \\
&(\phi_{i,j}(1-\frac{\xi_x}{h_x})(1-\frac{\xi_y}{h_y})+\phi_{i+1,j}\frac{\xi_x}{h_x}(1-\frac{\xi_y}{h_y})+\phi_{i,j+1}(1-\frac{\xi_x}{h_x})\frac{\xi_y}{h_y}+\phi_{i+1,j+1}\frac{\xi_x}{h_x}\frac{\xi_y}{h_y}) \\
&-\epsilon(1-\frac{\xi_y}{h_y})\frac{1}{h_x}(-\phi_{i,j}\frac{1}{h_x}(1-\frac{\xi_y}{h_y})+\phi_{i+1,j}\frac{1}{h_x}(1-\frac{\xi_y}{h_y})-\phi_{i,j+1}\frac{1}{h_x}\frac{\xi_y}{h_y}+\phi_{i+1,j+1}\frac{1}{h_x}\frac{\xi_y}{h_y}) \\
&-\epsilon(1-\frac{\xi_x}{h_x})\frac{1}{h_y}(-\phi_{i,j}\frac{1}{h_y}(1-\frac{\xi_x}{h_x})+\phi_{i,j+1}\frac{1}{h_y}(1-\frac{\xi_x}{h_x})-\phi_{i+1,j}\frac{1}{h_y}\frac{\xi_x}{h_x}+\phi_{i+1,j+1}\frac{1}{h_y}\frac{\xi_x}{h_x})].
\end{aligned} \tag{28}
$$

2.3 Limits of Pe_h

We now assume that $h_x = h_y = h$ and that $\mathbf{u} = (u_1, u_2)$ (with u_1 and u_2 being constant in space). We can now reuse the evaluated limits on ξ from (15) and apply this to ξ_x and ξ_y to investigate the cases of $\mathrm{Pe}_{h,x} = \mathrm{Pe}_{h,y} = 0$ and $\mathrm{Pe}_{h,x} = \mathrm{Pe}_{h,y} \to \infty$. We will first focus on the case of $\mathrm{Pe}_{h,x} = \mathrm{Pe}_{h,y} = 0$ which according to (15) gives $\xi_x = \xi_y = \frac{h}{2}$. For this situation we have that when we add (25)-(28) we get

$$
\begin{aligned}
&h^2[u_1\frac{-\phi_{i-1,j-1}-2\phi_{i-1,j}+2\phi_{i+1,j}+\phi_{i+1,j+1}}{8h}+u_2\frac{-\phi_{i-1,j-1}-2\phi_{i,j-1}+2\phi_{i,j+1}+\phi_{i+1,j+1}}{8h} \\
&-\epsilon\frac{\phi_{i-1,j-1}+\phi_{i+1,j+1}-4\phi_{i,j}}{2h^2}]=0
\end{aligned} \tag{29}
$$

for $1 \le i, j \le n-1$. Next, we focus on the case of $\mathrm{Pe}_{h,x} = \mathrm{Pe}_{h,y} \to \infty$ which according to (15) gives $\xi_x = \xi_y = 0$. Similarly to the previous case, we fill this into (25)-(28) and add to get

$$
\begin{aligned}
&h^2[u_1(\frac{\phi_{i,j}-\phi_{i-1,j}}{h})+u_2(\frac{\phi_{i,j}-\phi_{i,j-1}}{h})-\epsilon(\frac{\phi_{i+1,j}+\phi_{i-1,j}-2\phi_{i,j}}{h^2}+\frac{\phi_{i,j+1}+\phi_{i,j-1}-2\phi_{i,j}}{h^2})]=0\to \\
&u_1(\frac{\phi_{i,j}-\phi_{i-1,j}}{h})+u_2(\frac{\phi_{i,j}-\phi_{i,j-1}}{h})-\epsilon(\frac{\phi_{i+1,j}+\phi_{i-1,j}-2\phi_{i,j}}{h^2}+\frac{\phi_{i,j+1}+\phi_{i,j-1}-2\phi_{i,j}}{h^2})=0
\end{aligned} \tag{30}
$$

for $1 \leq i, j \leq n-1$. This last expression is the same as the finite difference scheme with a backward-difference approximation for the advection term (first derivative of ϕ).

3 Petrov Galerkin Finite elements

Here we will consider the streamline upwind Petrov Galerkin (SUPG) method. We will again consider $\Omega = [0, L]$, and let $0 = x_0 < x_1 < \ldots < x_n = L$ be the nodes of the n-element mesh on Ω. The SUPG weak form is now: Find $\phi_h \in \mathcal{S}_h$ such that

$$\begin{aligned}
&\forall w_h \in \mathcal{W}_h, \ \int_\Omega -w_{h,1} u \phi_h + w_{h,1} \epsilon \phi_{h,1} dx + \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} p_h[w_h](u\phi_{h,1} - \epsilon \phi_{h,11} - f) dx \\
&= \int_\Omega w_h f dx
\end{aligned} \tag{31}$$

with $\phi|_{\partial\Omega} = \phi_0 \to w|_{\partial\Omega} = 0$. Here, p_h is a linear operator which gives $p_h[w_h] = \tau u w_{h,1}$, with τ being the SUPG parameter. For this problem we again employ the C^0 piecewise-linear basis functions B_i.

3.1 Deriving the SUPG parameter

Using the result from **assignment 2.1**, taking into account the definition of the basis functions (equation (2)), we can readily write (31) as

$$h[u\frac{\phi_{i+1} - \phi_{i-1}}{2h} - \epsilon \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2}] + \sum_{k=i-1}^{i} \int_{x_k}^{x_{k+1}} \tau u B_{i,1}(u\phi_{h,1} - \epsilon\phi_{h,11} - f) dx = \int_{x_{i-1}}^{x_{i+1}} B_i f dx \tag{32}$$

for $i = 1, \ldots, n-1$. Noting that $\phi_{h,11} = 0$, due to using linear basis functions, we can now write

$$\begin{aligned}
&h[u\frac{\phi_{i+1} - \phi_{i-1}}{2h} - \epsilon \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2} + \tau \frac{u^2}{h^2}(2\phi_i - \phi_{i+1} - \phi_{i-1})] \\
&= \int_{x_{i-1}}^{x_{i+1}} B_i f dx + \frac{\tau u}{h}(\int_{x_{i-1}}^{x_i} f dx - \int_{x_i}^{x_{i+1}} f dx)
\end{aligned} \tag{33}$$

for $i = 1, \ldots, n-1$. Taking $f = 0$ this gives

$$\begin{aligned}
&h[u\frac{\phi_{i+1} - \phi_{i-1}}{2h} - (\epsilon + \tau u^2)\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2}] = 0 \to \\
&u\frac{\phi_{i+1} - \phi_{i-1}}{2h} - (\epsilon + \tau u^2)\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2} = 0
\end{aligned} \tag{34}$$

for $i = 1, \ldots, n-1$. This is equivalent to the modified equation method for

$$\tau u^2 = \bar{\epsilon} \to \tau = \frac{\bar{\epsilon}}{u^2}. \tag{35}$$

Taking this value of τ we have that $\tau \sim \bar{\epsilon}$, again making it so that for large Pe_h we have a larger deviation from the standard Galerkin method. This can easily be seen from the previously derived limits of $\bar{\epsilon}$ in (13), making it so that

$$\begin{aligned} &\lim_{\mathrm{Pe}_h \to 0^+} \tau = 0 \\ &\lim_{\mathrm{Pe}_h \to \infty} \tau = \frac{uh}{2u^2} = \frac{h}{2u}. \end{aligned} \tag{36}$$

3.2 Numerical simulations

Next, we carry out a set of numerical simulations setting $L = 4$ and $n = 20$ with boundary conditions (BCs) $\phi(0) = 0$ and $\phi(4) = 1$, where we take τ as defined in (35) (SUPG), $\tau = 0$ (Bubnov Galerking = BG), and $\tau = 0$ and $n = 1000$ ("Exact") for the linear system described by (33). The results are show in figure 1 for both $f = 0$ and

$$f = \begin{cases} 1 - x, \ 0 \leq x < 1.5 \\ x - 2, \ 1.5 \leq x < 2 \\ 0, \ \text{elsewhere} \end{cases} . \tag{37}$$

Similarly to the modified equation method, we find the SUPG method to be nodally exact for $f = 0$, while BG displays increasingly erratic behavior for smaller values of ϵ, keeping $u = 1$. For $f \neq 0$ we find that the SUPG method is no longer nodally exact, however, especially if we look at $\epsilon = 0.01$, the solution lies much closer to the "exact" solution, as compared to the modified equation method (**assignment 2.1**).

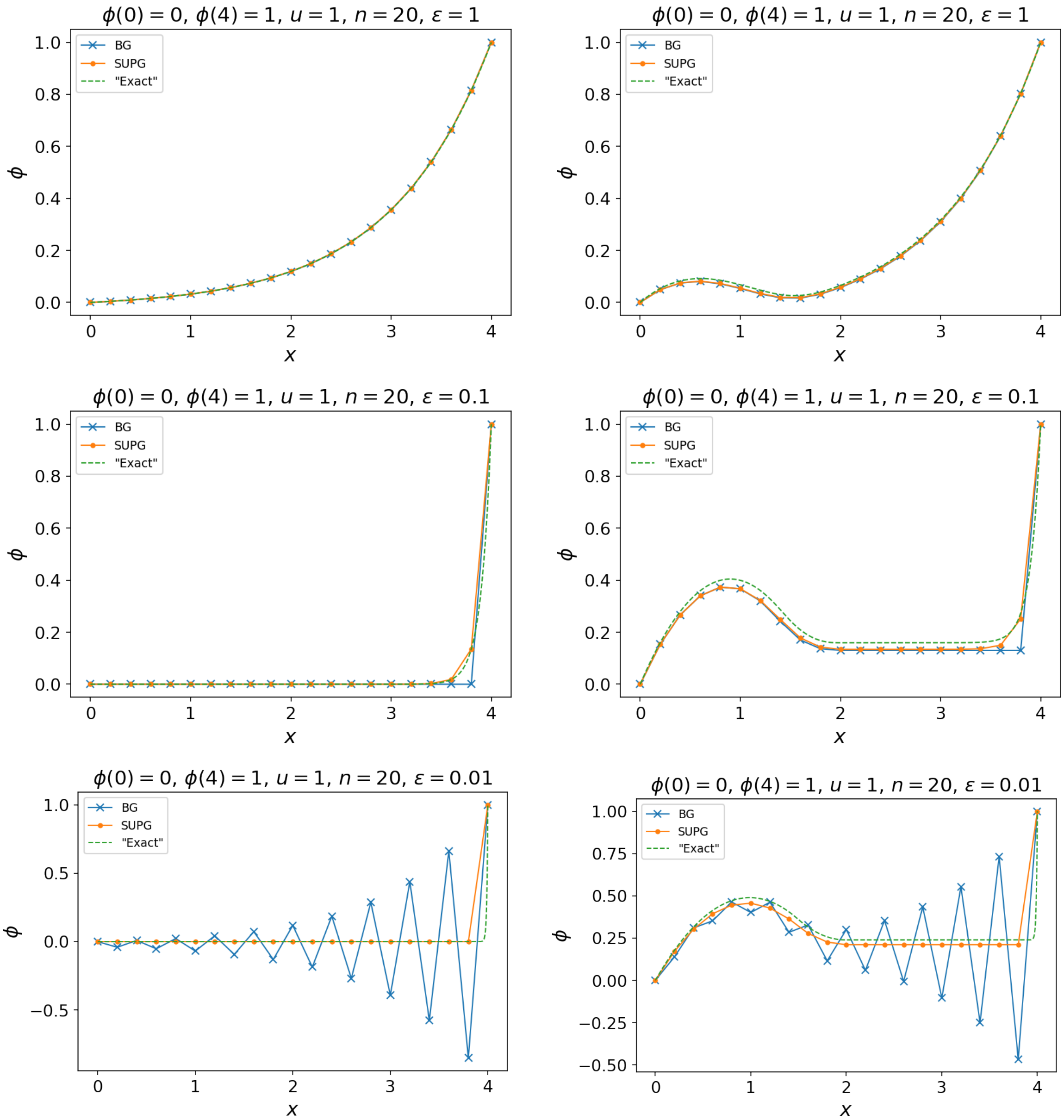

Figure 1: ((left column) Numerical solutions to the advection diffusion equation obtained using different methods, taking $f = 0$. (right column) Numerical solutions to the advection diffusion equation, taking f as defined in (37).

4 Discontinuous Galerkin finite elements

4.1 Problem statement

In this section we consider discontinuous Galerkin finite elements. For this we assume that all elements Ω_i that make up Ω are open sets. The local finite elements space $\mathcal{V}_h$ on the mesh is defined as

$$\mathcal{V}_h = \{f : f|_{\Omega_i} \in \mathcal{V}_{h,i}\}, \tag{38}$$

where $\mathcal{V}_{h,i}$ is the local finite elements space on Ω_i. No inter-elements smoothness conditions are imposed. Let us now consider some definitions

$$\begin{aligned} \Gamma_{i,\text{in}} &:= \{\mathbf{x} \in \Gamma_i : \mathbf{u} \cdot \mathbf{n}_i < 0\} \\ \Gamma_{i,\text{out}} &:= \{\mathbf{x} \in \Gamma_i : \mathbf{u} \cdot \mathbf{n}_i \geq 0\}, \end{aligned} \tag{39}$$

where Γ_i is the boundary of Ω_i and $\mathbf{n}_i$ the outward pointing normal. We also define

$$u_n(\mathbf{x}) := |\mathbf{u}(\mathbf{x}) \cdot \mathbf{n}_i(\mathbf{x})| \text{ for } \mathbf{x} \in \Gamma_i \tag{40}$$

and

$$f(\mathbf{x}^{\pm}) = \lim_{\delta \to 0} \begin{cases} f(\mathbf{x} \pm \delta \mathbf{n}_i), & \mathbf{x} \in \Gamma_{i,\text{out}} \\ f(\mathbf{x} \mp \delta \mathbf{n}_i), & \mathbf{x} \in \Gamma_{i,\text{in}} \end{cases} \tag{41}$$

for $\delta > 0$. Similarly we have that

$$\begin{aligned} \Gamma_{\text{in}} &:= \{\mathbf{x} \in \Gamma : \mathbf{u} \cdot \mathbf{n} < 0\} \\ \Gamma_{\text{out}} &:= \{\mathbf{x} \in \Gamma : \mathbf{u} \cdot \mathbf{n} \geq 0\}, \end{aligned} \tag{42}$$

where Γ is the boundary of Ω and $\mathbf{n}$ the outward pointing normal. We also define

$$u_n(\mathbf{x}) := |\mathbf{u}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})| \text{ for } \mathbf{x} \in \Gamma \tag{43}$$

and

$$f(\mathbf{x}^{\pm}) = \lim_{\delta \to 0} \begin{cases} f(\mathbf{x} \pm \delta \mathbf{n}), & \mathbf{x} \in \Gamma_{\text{out}} \\ f(\mathbf{x} \mp \delta \mathbf{n}), & \mathbf{x} \in \Gamma_{\text{in}} \end{cases} \tag{44}$$

for $\delta > 0$. We now have that the weak form of the discontinuous Galerkin problem, for only advection, for each element is given by: find $\phi_h|_{\Omega_i} \in \mathcal{V}_{h,i}$ such that.

$$\begin{aligned} &\forall w_h \in \mathcal{V}_{h,i}, \int_{\Omega_i} -\nabla w_h \cdot \mathbf{u} \phi_h dx + \int_{\Gamma_{i,\text{out}}} u_n(\mathbf{x}) w_h(\mathbf{x}^-) \phi_h(\mathbf{x}^-) ds \\ &= \int_{\Omega_i} w_h f dx + \int_{\Gamma_{i,\text{in}}} u_n(\mathbf{x}) w_h(\mathbf{x}^+) \phi_h(\mathbf{x}^-) ds, \end{aligned} \tag{45}$$

where we have BC

$$\phi_h(\mathbf{x}^-) = \phi_0(\mathbf{x}), \;\; \mathbf{x} \in \Gamma_{\text{in}}. \tag{46}$$

4.2 Euler-Lagrange equations

Applying integration by parts to the first integral of (45) we get

$$\begin{aligned}\int_{\Omega_i} -\nabla w_h \cdot \mathbf{u}\phi_h dx &= \int_{\Gamma_i} -w_h\phi_h \mathbf{u}\cdot\mathbf{n} ds + \int_{\Omega_i} w_h \mathbf{u}\cdot\nabla\phi_h dx \\ = \int_{\Omega_i} w_h \mathbf{u}\cdot\nabla\phi_h dx &- \int_{\Gamma_{i,\text{in}}} w_h(\mathbf{x}^+)\phi_h(\mathbf{x}^+)\mathbf{u}\cdot\mathbf{n} ds - \int_{\Gamma_{i,\text{out}}} w_h(\mathbf{x}^-)\phi_h(\mathbf{x}^-)\mathbf{u}\cdot\mathbf{n} ds,\end{aligned} \tag{47}$$

where we made use of the fact that we approach the boundary from the inside of the control volume such that on $\Gamma_{i,\text{in}}$ we evaluate w_h and ϕ_h at $\mathbf{x}^+$ and on $\Gamma_{i,\text{out}}$ at $\mathbf{x}^-$. Noting that

$$\begin{aligned}\mathbf{u}\cdot\mathbf{n} &= -u_n(\mathbf{x}),\ \mathbf{x}\in\Gamma_{i,\text{in}} \\ \mathbf{u}\cdot\mathbf{n} &= u_n(\mathbf{x}),\ \mathbf{x}\in\Gamma_{i,\text{out}}\end{aligned} \tag{48}$$

and filling (47) into (45) we get the required Euler-Lagrange equations:

$$\int_{\Omega_i} w_h(\mathbf{u}\cdot\nabla\phi_h - f)dx + \int_{\Gamma_{i,\text{in}}} w_h(\mathbf{x}^+)(u_n(\mathbf{x})\phi_h(\mathbf{x}^+) - u_n(\mathbf{x})\phi_h(\mathbf{x}^-))ds = 0. \tag{49}$$

This can be viewed as the weighted residual form of the problem, where we identify the two residuals r_1 and r_2 as

$$\begin{aligned}r_1 &= \mathbf{u}\cdot\nabla\phi_h - f \\ r_2 &= u_n(\mathbf{x})\phi_h(\mathbf{x}^+) - u_n(\mathbf{x})\phi_h(\mathbf{x}^-).\end{aligned} \tag{50}$$

r_1 simply takes the form of the advection equation which we aim to satisfy as best as possible and r_2 is a measure of discontinuity in the approximate solution.

4.3 Deriving the global weak formulation

To obtain the global weak formulation we sum (45) over all the elements Ω_i for $1 \leq i \leq n$ which gives: find $\phi_h|_\Omega \in \mathcal{V}_h$ such that

$$\begin{aligned}\forall w_h \in \mathcal{V}_h,\ &\sum_{i=1}^{n}\int_{\Omega_i} -\nabla w_h\cdot\mathbf{u}\phi_h dx + \sum_{i=1}^{n}\int_{\Gamma_{i,\text{out}}} u_n(\mathbf{x})w_h(\mathbf{x}^-)\phi_h(\mathbf{x}^-)ds \\ &= \sum_{i=1}^{n}\int_{\Omega_i} w_h f dx + \sum_{i=1}^{n}\int_{\Gamma_{i,\text{in}}} u_n(\mathbf{x})w_h(\mathbf{x}^+)\phi_h(\mathbf{x}^-)ds \xrightarrow{\text{BC (46)}} \\ \forall w_h \in \mathcal{V}_h,\ &\int_{\Omega} -\nabla w_h\cdot\mathbf{u}\phi_h dx + \sum_{i=1}^{n}\int_{\Gamma_{i,\text{out}}} u_n(\mathbf{x})w_h(\mathbf{x}^-)\phi_h(\mathbf{x}^-)ds \\ &= \int_{\Omega} w_h f dx + \sum_{i\notin\mathcal{Q}}^{n}\int_{\Gamma_{i,\text{in}}} u_n(\mathbf{x})w_h(\mathbf{x}^+)\phi_h(\mathbf{x}^-)ds + \int_{\Gamma_{\text{in}}} u_n(\mathbf{x})w_h(\mathbf{x}^+)\phi_0(\mathbf{x})ds,\end{aligned} \tag{51}$$

where $\mathcal{Q} = \{i|\Gamma_{i,\text{in}}\cap\Gamma_{\text{in}}\}$, from $\Gamma_{i,\text{in}} = (\Gamma_{i,\text{in}}\setminus\Gamma_{\text{in}})\cup(\Gamma_{i,\text{in}}\cap\Gamma_{\text{in}})$.

4.4 Local conservation statement

Here we assume $\mathcal{V}_{h,i} \in P_1$ and $f = 0$. In particular we choose to study the case where w_h is constant in Ω_i and zero everywhere else, i.e.

$$w_h = \begin{cases} C, & \text{in } \Omega_i \\ 0, & \text{elsewhere} \end{cases} . \tag{52}$$

Filling this into (45), using the fact that $\nabla w_h = \nabla C = \mathbf{0}$, we get

$$\begin{aligned}
&\int_{\Gamma_{i,\text{out}}} u_n(\mathbf{x}) C \phi_h(\mathbf{x}^-) ds - \int_{\Gamma_{i,\text{in}}} u_n(\mathbf{x}) C \phi_h(\mathbf{x}^-) ds = 0 \rightarrow \\
&\int_{\Gamma_{i,\text{out}}} u_n(\mathbf{x}) C \phi_h(\mathbf{x}^-) ds - \int_{\Gamma_{i,\text{in}}} u_n(\mathbf{x}) C \phi_h(\mathbf{x}^+) ds = \int_{\Gamma_{i,\text{in}}} u_n(\mathbf{x}) C \phi_h(\mathbf{x}^-) ds - \int_{\Gamma_{i,\text{in}}} u_n(\mathbf{x}) C \phi_h(\mathbf{x}^+) ds \rightarrow \\
&C[\int_{\Gamma_{i,\text{out}}} \phi_h(\mathbf{x}^-) \mathbf{u} \cdot \mathbf{n} ds + \int_{\Gamma_{i,\text{in}}} \phi_h(\mathbf{x}^+) \mathbf{u} \cdot \mathbf{n} ds] = C[\int_{\Gamma_{i,\text{in}}} u_n(\mathbf{x}) \phi_h(\mathbf{x}^-) ds - \int_{\Gamma_{i,\text{in}}} u_n(\mathbf{x}) \phi_h(\mathbf{x}^+) ds] \\
&\xrightarrow{\text{Divergence theorem}} \int_{\Omega_i} \nabla \cdot (\mathbf{u} \phi_h(\mathbf{x})) dx = \int_{\Gamma_{i,\text{in}}} u_n(\mathbf{x}) (\phi_h(\mathbf{x}^-) - \phi_h(\mathbf{x}^+)) ds
\end{aligned} \tag{53}$$

which implies the conservation statement

$$\int_{\Omega_i} \nabla \cdot (\mathbf{u} \phi) dx = 0. \tag{54}$$

4.5 Global conservation statement

To arrive at a global conservation statement we sum the obtained result at (53) over all the elements Ω_i for $1 \leq i \leq n$ to obtain

$$\begin{aligned}
&\sum_{i=1}^{n} \int_{\Omega_i} \nabla \cdot (\mathbf{u} \phi_h(\mathbf{x})) dx = \sum_{i \notin \mathcal{Q}} \int_{\Gamma_{i,\text{in}}} u_n(\mathbf{x}) (\phi_h(\mathbf{x}^-) - \phi_h(\mathbf{x}^+)) ds + \sum_{i \in \mathcal{Q}} \int_{\Gamma_{i,\text{in}}} u_n(\mathbf{x}) (\phi_0(\mathbf{x}) - \phi_h(\mathbf{x}^+)) ds \\
&\rightarrow \int_{\Omega} \nabla \cdot (\mathbf{u} \phi_h(\mathbf{x})) dx = \sum_{i \notin \mathcal{Q}} \int_{\Gamma_{i,\text{in}}} u_n(\mathbf{x}) (\phi_h(\mathbf{x}^-) - \phi_h(\mathbf{x}^+)) ds + \int_{\Gamma_{\text{in}}} u_n(\mathbf{x}) (\phi_0(\mathbf{x}) - \phi_h(\mathbf{x}^+)) ds,
\end{aligned} \tag{55}$$

after applying BC (46). Here, we again made use of $\mathcal{Q} = \{i | \Gamma_{i,\text{in}} \cap \Gamma_{\text{in}}\}$, from $\Gamma_{i,\text{in}} = (\Gamma_{i,\text{in}} \setminus \Gamma_{\text{in}}) \cup (\Gamma_{i,\text{in}} \cap \Gamma_{\text{in}})$.

References